電腦視覺與應用
Computer Vision and Applications

Lecture-05

Projective 3D geometry

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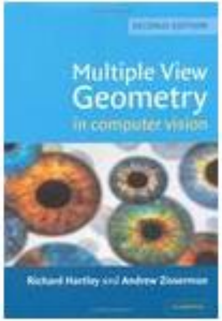
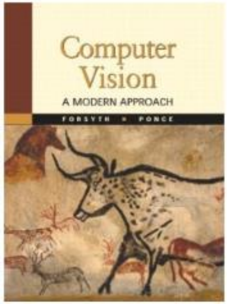
Projective 3D geometry

- Lecture Reference at:
  - Multiple View Geometry in Computer Vision, Chapter 3. (major)
  - Computer Vision A Modern Approach, (NA).





Notation description

- Capital character (大寫字母) → for 3D (4 elements in homogenous)
- Low case character (小寫字母) → for 2D (3 elements in homogenous)

- Bold (粗體字)→ vector or matrix.
- Italic (斜體)→ real, scalar or variable.

NOTE:

Notation in this lecture may differ from those in reference/textbook.

3D point representation

- In general, 3D point is written as

$$(X,Y,Z)^{\mathsf{T}} \text{ in } \mathbf{R}^3$$

$$\mathbf{X}=(X_1,X_2,X_3,X_4)^{\mathsf{T}} \text{ in } \mathbf{P}^3$$

- Homogenous representation

$$\mathbf{X}=\left(\frac{X_1}{X_4},\frac{X_2}{X_4},\frac{X_3}{X_4},1\right)^{\mathsf{T}}=(X,Y,Z,1)^{\mathsf{T}} \quad (X_4\neq 0)$$

- 3D point at infinity

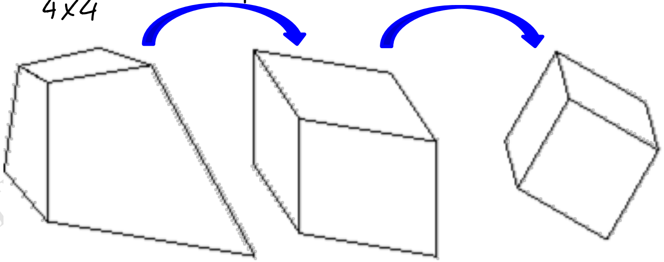
$$\mathbf{X}=(X,Y,Z,0)^{\mathsf{T}}$$

3D point transformation

- 3D point transformation is similar to 2D, projective transformation (homography)

$$\mathbf{X}' = \mathbf{H}\mathbf{X} \quad \text{(4x4-1=15 DOF, upto scale)}$$

4x4 4x1



4x1

3D point transformation—cont.

$$\mathbf{X}' = \mathbf{HX}$$

→ 3D   3D point

$$\rightarrow \begin{bmatrix} X_1' \\ X_2' \\ X_3' \\ X_4' \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ H_{31} & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}$$

15 DoF

$$\rightarrow \begin{bmatrix} X' \\ Y' \\ Z' \\ 1 \end{bmatrix} \sim \begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ H_{31} & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \rightarrow \begin{aligned} X' &= \frac{H_{11}X + H_{12}Y + H_{13}Z + H_{14}}{H_{41}X + H_{42}Y + H_{43}Z + H_{44}} \\ Y' &= \frac{H_{21}X + H_{22}Y + H_{23}Z + H_{24}}{H_{41}X + H_{42}Y + H_{43}Z + H_{44}} \\ Z' &= \frac{H_{31}X + H_{32}Y + H_{33}Z + H_{34}}{H_{41}X + H_{42}Y + H_{43}Z + H_{44}} \end{aligned}$$

3D point transformation—cont.

- Cont.

$$\rightarrow \begin{aligned} &H_{11}X + H_{12}Y + H_{13}Z + H_{14} - X'(H_{41}X + H_{42}Y + H_{43}Z + H_{44}) = 0 \\ &H_{21}X + H_{22}Y + H_{23}Z + H_{24} - Y'(H_{41}X + H_{42}Y + H_{43}Z + H_{44}) = 0 \\ &H_{31}X + H_{32}Y + H_{33}Z + H_{34} - Z'(H_{41}X + H_{42}Y + H_{43}Z + H_{44}) = 0 \end{aligned}$$

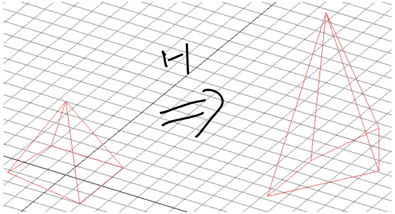
- For abbreviation, let

$$\boxed{\begin{aligned} &\tilde{\mathrm{H}}_1^{\mathrm{T}} = [H_{11} \quad H_{12} \quad H_{13} \quad H_{14}] \\ &\tilde{\mathrm{H}}_2^{\mathrm{T}} = [H_{21} \quad H_{22} \quad H_{23} \quad H_{24}] \\ &\tilde{\mathrm{H}}_3^{\mathrm{T}} = [H_{31} \quad H_{32} \quad H_{33} \quad H_{34}] \\ &\tilde{\mathrm{H}}_4^{\mathrm{T}} = [H_{41} \quad H_{42} \quad H_{43} \quad H_{44}] \\ &\mathbf{X}^{\mathrm{T}} = [X \quad Y \quad Z \quad 1] \end{aligned}}$$

$$\rightarrow \begin{bmatrix} \mathbf{X}^{\mathrm{T}} & 0^{\mathrm{T}} & 0^{\mathrm{T}} & -X'\mathbf{X}^{\mathrm{T}} \\ 0^{\mathrm{T}} & \mathbf{X}^{\mathrm{T}} & 0^{\mathrm{T}} & -Y'\mathbf{X}^{\mathrm{T}} \\ 0^{\mathrm{T}} & 0^{\mathrm{T}} & \mathbf{X}^{\mathrm{T}} & -Z'\mathbf{X}^{\mathrm{T}} \end{bmatrix}_{3\times16} \begin{bmatrix} \tilde{\mathrm{H}}_1 \\ \tilde{\mathrm{H}}_2 \\ \tilde{\mathrm{H}}_3 \\ \tilde{\mathrm{H}}_4 \end{bmatrix}_{16\times1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_{3\times1}$$

3D point transformation—cont.

- Example



(Note: need to avoid degenerated points)

$\mathbf{X}1=[0,0,40,1]^T$ $\mathbf{XP}1=[90,90,71.4392,1]^T$

$\mathbf{X}2=[-20,-20,5,1]^T$ $\mathbf{XP}2=[70,70,-24.9519,1]^T$

$\mathbf{X}3=[20,-20, 0,1]^T$ $\mathbf{XP}3=[125.1275,80.8687,0.0,1]^T$

$\mathbf{X}4=[20,20,0,1]^T$ $\mathbf{XP}4=[104.0309,116.0521,0.0,1]^T$

$\mathbf{X}5=[-20,20,0,1]^T$ $\mathbf{XP}5=[70,110,-24.9519,1]^T$

solved by DLT method

SVD

H =

$$\begin{matrix} 1.0463 & 0.8320 & 0.1572 & 90.9871 \\ 0.1791 & 2.0459 & -0.0409 & 98.9097 \\ 0.7726 & 0 & 2.3166 & -15.4527 \\ -0.0001 & 0.0119 & 0.0020 & 1.0000 \end{matrix}$$

16 DoF → 15 DoF

3D point transformation—cont.

- Example

① Dirce least

② SVD

③

(in Matlab)

```
z=[0 0 0 0]'

A=[X1' z' z'  -XP1(1).*X1';
 z' X1' z'  -XP1(2).*X1';
 z'  z' X1' -XP1(3).*X1';
 X2' z' z'  -XP2(1).*X2';
 z' X2' z'  -XP2(2).*X2';
 z'  z' X2' -XP2(3).*X2';
 X3' z' z'  -XP3(1).*X3';
 z' X3' z'  -XP3(2).*X3';
 z'  z' X3' -XP3(3).*X3';
 X4' z' z'  -XP4(1).*X4';
 z' X4' z'  -XP4(2).*X4';
 z'  z' X4' -XP4(3).*X4';
 X5' z' z'  -XP5(1).*X5';
 z' X5' z'  -XP5(2).*X5';
 z'  z' X5' -XP5(3).*X5'];
```

$$
\mathrm{inv}\left(1.0e{+}003 * \begin{matrix}
0 & 0 & 0.0400 & 0.0010 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3.6000 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.0400 & 0.0010 & 0 & 0 & 0 & 0 & 0 & 0 & -3.6000 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0400 & 0.0010 & 0 & 0 & -2.8576 \\
-0.0200 & -0.0200 & 0.0050 & 0.0010 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.4000 & 1.4000 & -0.3500 \\
0 & 0 & 0 & 0 & -0.0200 & -0.0200 & 0.0050 & 0.0010 & 0 & 0 & 0 & 0 & 1.4000 & 1.4000 & -0.3500 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.0200 & -0.0200 & 0.0050 & 0.0010 & -0.4990 & -0.4990 & 0.1248 \\
0.0200 & -0.0200 & 0 & 0.0010 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.5026 & 2.5026 & 0 \\
0 & 0 & 0 & 0 & 0.0200 & -0.0200 & 0 & 0.0010 & 0 & 0 & 0 & 0 & -1.6174 & 1.6174 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0200 & -0.0200 & 0 & 0.0010 & 0 & 0 & 0 \\
0.0200 & 0.0200 & 0 & 0.0010 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.0806 & -2.0806 & 0 \\
0 & 0 & 0 & 0 & 0.0200 & 0.0200 & 0 & 0.0010 & 0 & 0 & 0 & 0 & -2.3210 & -2.3210 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0200 & 0.0200 & 0 & 0.0010 & 0 & 0 & 0 \\
-0.0200 & 0.0200 & 0 & 0.0010 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.4000 & -1.4000 & 0 \\
0 & 0 & 0 & 0 & -0.0200 & 0.0200 & 0 & 0.0010 & 0 & 0 & 0 & 0 & 2.2000 & -2.2000 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.0200 & 0.0200 & 0 & 0.0010 & -0.4990 & 0.4990 & 0
\end{matrix} \right) * \begin{matrix}
90.0000 \\ 90.0000 \\ 71.4392 \\ 70.0000 \\ 70.0000 \\ -24.9519 \\ 125.1275 \\ 80.8687 \\ 0 \\ 104.0309 \\ 116.0521 \\ 0 \\ 70.0000 \\ 110.0000 \\ -24.9519
\end{matrix}
$$

Planes

- 3D plane

$$\pi_1 X + \pi_2 Y + \pi_3 Z + \pi_4 1 = 0$$

$$\pi_1 X_1 + \pi_2 X_2 + \pi_3 X_3 + \pi_4 X_4 = 0 \quad \rightarrow \text{homogenous}$$

$$\boldsymbol{\pi}^{\mathsf{T}} \mathbf{X} = 0 \quad \rightarrow \text{3D points } \mathbf{X} \text{ on a plane } \boldsymbol{\pi}$$

Note: $\boldsymbol{\pi}$ is a plane equation equals to $(\pi_1, \pi_2, \pi_3, \pi_4)$

$\mathbf{X}$ denotes 3D points equal to (X_1, X_2, X_3, X_4)

- Transformation

$$\mathbf{X}' = \mathbf{H}\mathbf{X} \quad \rightarrow \text{3D points mapping to another 3D points}$$

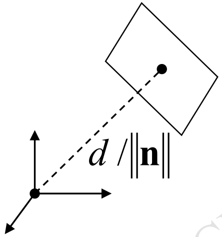
$$\boldsymbol{\pi}' = \mathbf{H}^{-\mathsf{T}} \boldsymbol{\pi} \quad \rightarrow \text{3D planes mapping to another 3D planes}$$

Planes—cont.: in Euclidean case

- Euclidean representation

$$\mathbf{n} \cdot \widetilde{\mathbf{X}} + d = 0 \qquad \mathbf{n} = (\pi_1, \pi_2, \pi_3)^{\mathsf{T}}$$ →normal of this plane

$$\pi_4 = d$$



$$\widetilde{\mathrm{X}} = (X, Y, Z)^{\mathsf{T}} \qquad X_4 = 1$$

Dual: points ↔ planes, lines ↔ lines

Planes from points

Solve $\boldsymbol{\pi}$ from $X_1^{\mathsf T}\pi = 0$, $X_2^{\mathsf T}\pi = 0$ and $X_3^{\mathsf T}\pi = 0$

$$\begin{bmatrix} \mathbf{X}_1^{\mathsf T} \\ \mathbf{X}_2^{\mathsf T} \\ \mathbf{X}_3^{\mathsf T} \end{bmatrix} \pi = 0 \quad \left(\text{solve } \pi \text{ as right nullspace of } \begin{bmatrix} X_1^{\mathsf T} \\ X_2^{\mathsf T} \\ X_3^{\mathsf T} \end{bmatrix}\right)$$

Given 3 3D-points to determine a plane in 3D space.

Known: $\mathbf{X}_1^{\mathsf T} = [(X_1)_1 \quad (X_1)_2 \quad (X_1)_3 \quad (X_1)_4]$ $\mathbf{X}_2^{\mathsf T} = [(X_2)_1 \quad (X_2)_2 \quad (X_2)_3 \quad (X_2)_4]$ $\mathbf{X}_3^{\mathsf T} = [(X_3)_1 \quad (X_3)_2 \quad (X_3)_3 \quad (X_3)_4]$

Unknown: $\boldsymbol{\pi} = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{bmatrix}$

$$\begin{bmatrix} (X_1)_1 & (X_1)_2 & (X_1)_3 & (X_1)_4 \\ (X_2)_1 & (X_2)_2 & (X_2)_3 & (X_2)_4 \\ (X_3)_1 & (X_3)_2 & (X_3)_3 & (X_3)_4 \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{bmatrix} = 0$$

Planes from points

- Or implicitly from coplanarity condition

$$\det\begin{bmatrix} \boxed{X_1} & (X_1)_1 & (X_2)_1 & (X_3)_1 \\ X_2 & (X_1)_2 & (X_2)_2 & (X_3)_2 \\ X_3 & (X_1)_3 & (X_2)_3 & (X_3)_3 \\ X_4 & (X_1)_4 & (X_2)_4 & (X_3)_4 \end{bmatrix} = 0$$

determinant of sub-matrix

$$\det[\mathbf{X}\,\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3] = 0$$

$$X_1 D_{234} - X_2 D_{134} + X_3 D_{124} - X_4 D_{123} = 0$$

$$\boldsymbol{\pi} = (D_{234}, -D_{134}, D_{124}, -D_{123})^{\mathsf{T}}$$

Determinant of matrix—review

- 3x3 matrix

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$
$$= aei + bfg + cdh - ceg - bdi - afh.$$

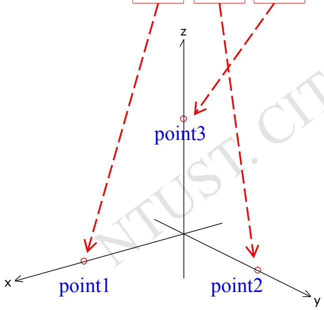
- higher order matrix

(decomposition from row or column)

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix}$$

Planes from points—example

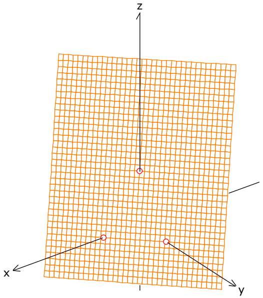
$$\det\begin{bmatrix} X_1 & 1 & 0 & 0 \\ X_2 & 0 & 1 & 0 \\ X_3 & 0 & 0 & 1 \\ X_4 & 1 & 1 & 1 \end{bmatrix} = 0$$



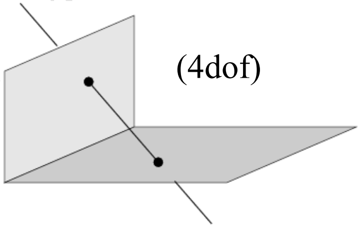
$$D_{234} = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 1 \qquad D_{124} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix}$$

$$D_{134} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -1 \qquad D_{123} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$$\boldsymbol{\pi} = \left(D_{234}, -D_{134}, D_{124}, -D_{123}\right)^{\mathsf{T}}$$
$$= (1,1,1,-1)^{\mathsf{T}}$$



Lines representation



(4dof)

$$W = \begin{bmatrix} A^T \\ B^T \end{bmatrix} \qquad \lambda A + \mu B$$

→one point & one direction (two point representation)
一點 & 一方向(或兩點表示)

or

$$W^* = \begin{bmatrix} P^T \\ Q^T \end{bmatrix} \qquad \lambda P + \mu Q$$

→intersection of two planes
(兩平面交集)

$$W^* W^T = W W^{*T} = 0_{2\times 2}$$

Example: *X*-axis

$$W = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

→ the original (原點)
→ ideal point on x axis (點在x無窮遠處)

$$W^* = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

→z plane
→y plane

Other lines representation

- Plücker matrices (4x4 skew symmetric homogenous)
- Plücker line coordinates

Quadrics and dual quadrics

Conic C =>

3D

$\mathbf{X}^{\mathsf{T}}\mathbf{Q}\mathbf{X} = 0$ ($\mathbf{Q}$: 4x4 symmetric matrix)

上三角 Matrix

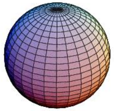
$$Q = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \circ & \bullet & \bullet & \bullet \\ \circ & \circ & \bullet & \bullet \\ \circ & \circ & \circ & \bullet \end{bmatrix}$$

1. 9 DOF
2. in general 9 points define quadric
3. det$|\mathbf{Q}|$=0 ↔ degenerate quadric
4. pole – polar $\boldsymbol{\pi} = \mathbf{QX}$
5. (plane ∩ quadric)=conic $\mathbf{C} = \mathbf{M}^{\mathsf{T}}\mathbf{Q}\mathbf{M}$ $\pi : \mathrm{X} = \mathrm{Mx}$
6. transformation (under $\mathbf{X}' = \mathbf{HX}$)

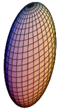
$$\mathbf{Q}' = \mathbf{H}^{-\mathsf{T}}\mathbf{Q}\mathbf{H}^{-1}$$

Quadric classification

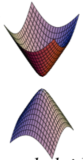
- Projective equivalent to sphere:



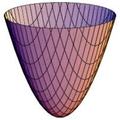
sphere



ellipsoid

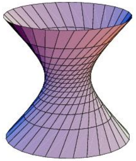


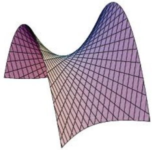
hyperboloid of two sheets



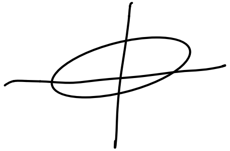
paraboloid

- Ruled quadrics:

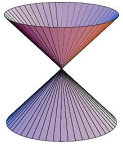




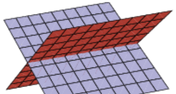
hyperboloids of one sheet



- Degenerate ruled quadrics:

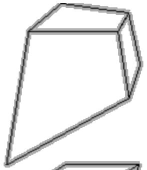
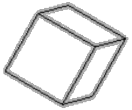
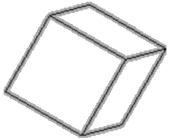


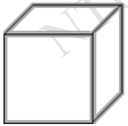
cone



two planes

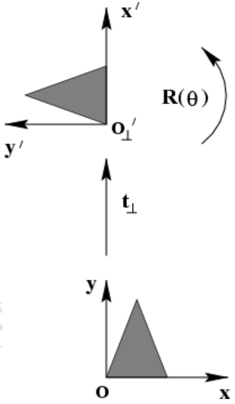
Hierarchy of transformations

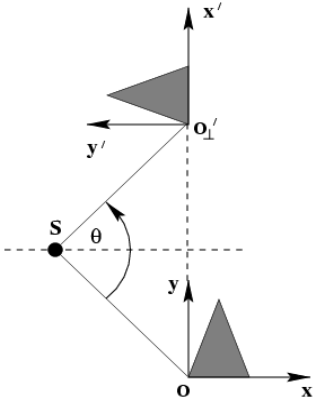
Projective 15dof	$\begin{bmatrix} A & t \\ v^T & v \end{bmatrix}$		Intersection and tangency
Affine 12dof	$\begin{bmatrix} A & t \\ 0^T & 1 \end{bmatrix}$		Parallellism of planes, Volume ratios, centroids, **The plane at infinity π_∞**
Similarity 7dof	$\begin{bmatrix} sR & t \\ 0^T & 1 \end{bmatrix}$		**The absolute conic Ω_∞**
Euclidean 6dof	$\begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix}$		Volume



Screw decomposition

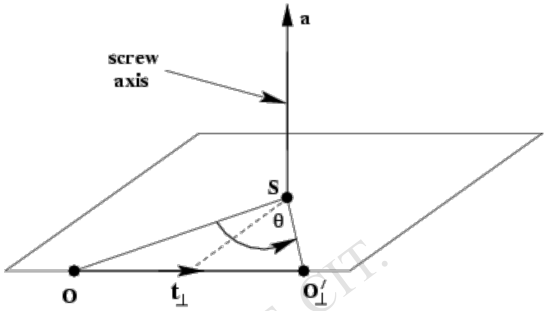
- Any particular translation and rotation is equivalent to a rotation about a screw axis and a translation along the screw axis.
- 2D case:

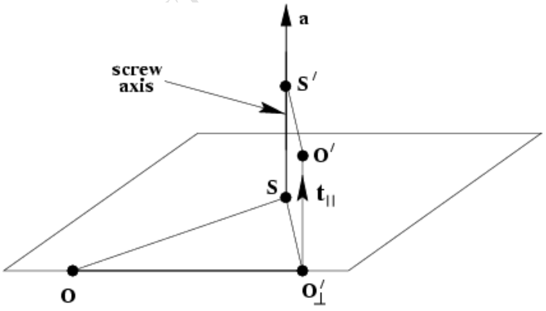




Screw decomposition

- 3D case:





screw axis // rotation axis

$$t = t_{//} + t_{\perp}$$

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