

# 電腦視覺與應用

# Computer Vision and Applications

Lecture-03 Projective 2D geometry

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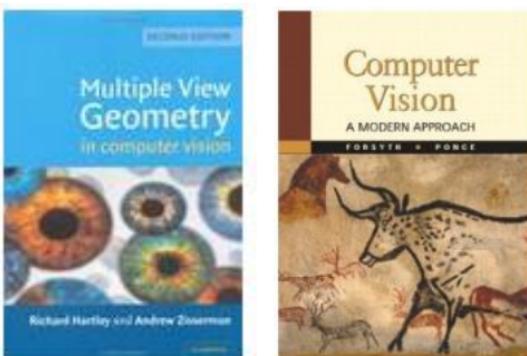




# Projective 2D geometry

Lecture Reference at:

- Multiple View Geometry in Computer Vision, Chapter 2. (major)
- Computer Vision A Modern Approach, Chapter 10.





# Projective 2D geometry

## Topics

- Points, lines & conics
- Transformations & invariants (between images)
- 1D projective geometry and the Cross-ratio



# Homogeneous coordinates

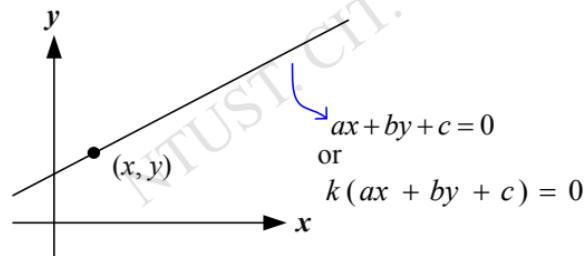
## ■ Homogeneous representation of lines

$$ax + by + c = 0 \quad (a, b, c)^T$$

$$(ka)x + (kb)y + kc = 0, \forall k \neq 0 \quad (a, b, c)^T \sim k(a, b, c)^T$$

equivalence class of vectors, any vector is representative

Set of all equivalence classes in  $\mathbf{R}^3 - (0, 0, 0)^T$  forms  $\mathbf{P}^2$



define one line as a vector format:

$$\mathbf{l} = (a, b, c)^T$$



# Homogeneous coordinates

- Homogeneous coordinates of points

$$\mathbf{x} = (x, y, 1)^T \text{ on } \mathbf{l} = (a, b, c)^T \text{ if and only if } ax + by + c = 0$$

$$(x, y, 1)(a, b, c)^T = (x, y, 1)\mathbf{l} = 0$$

$$(x, y, 1)^T \sim k(x, y, 1)^T, \forall k \neq 0$$

- The point  $\mathbf{x}$  lies on the line  $\mathbf{l}$  if and only if  $\mathbf{x}^T \mathbf{l} = \mathbf{l}^T \mathbf{x} = 0$

*Homogeneous* coordinates  $(x_1, x_2, x_3)^T$  but only 2DOF

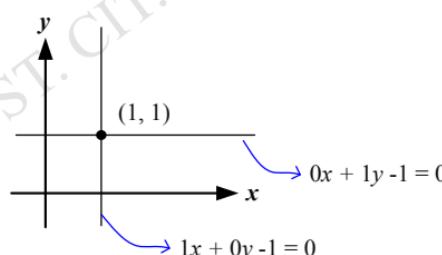
*Inhomogeneous* coordinates  $(x, y)^T$



# 2D Points from lines and vice-versa

- Intersections of lines
  - The intersection of two lines  $\mathbf{l}$  and  $\mathbf{l}'$  is  $\mathbf{x} = \mathbf{l} \times \mathbf{l}'$
- Line joining two points
  - The line through two points  $\mathbf{x}$  and  $\mathbf{x}'$  is  $\mathbf{l} = \mathbf{x} \times \mathbf{x}'$

Example: intersections of lines



$$\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

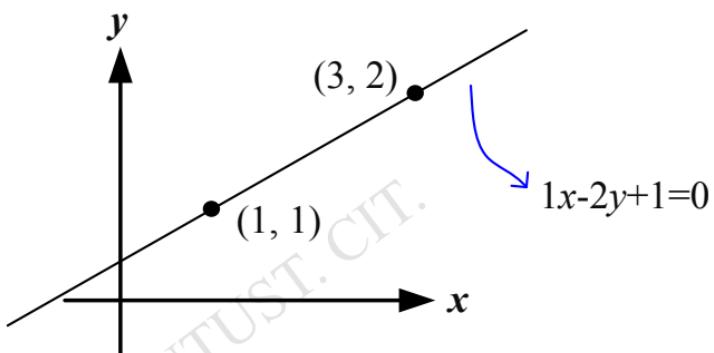
or

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$
$$\mathbf{l} \times \mathbf{l}' = \mathbf{x}$$



# 2D Points from lines and vice-versa

- Example: Line joining two points



$$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

or

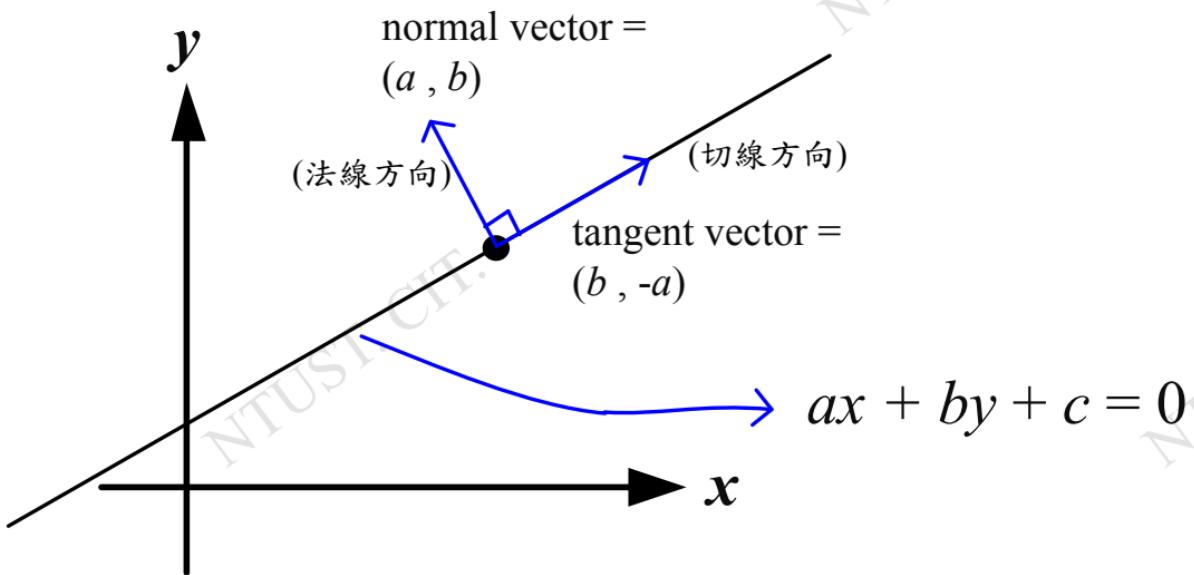
$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

$$\mathbf{x} \times \mathbf{x}' = \mathbf{l}$$



# Points from lines and vice-versa

- Normal vector and tangent vector of one line:



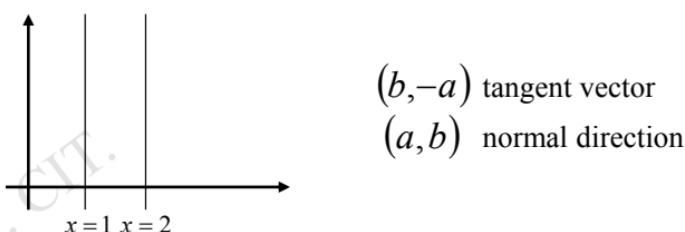


# Ideal points and the line at infinity

## ■ Intersections of parallel lines

$$\mathbf{l} = (a, b, c)^T \text{ and } \mathbf{l}' = (a, b, c')^T \quad \mathbf{x} = \mathbf{l} \times \mathbf{l}' = (b, -a, \underline{0})^T \rightarrow \text{point at infinity}$$

Example



Ideal points  $\rightarrow (x_1, x_2, 0)^T$

Line at infinity  $\rightarrow \mathbf{l}_\infty = (0, 0, 1)^T$

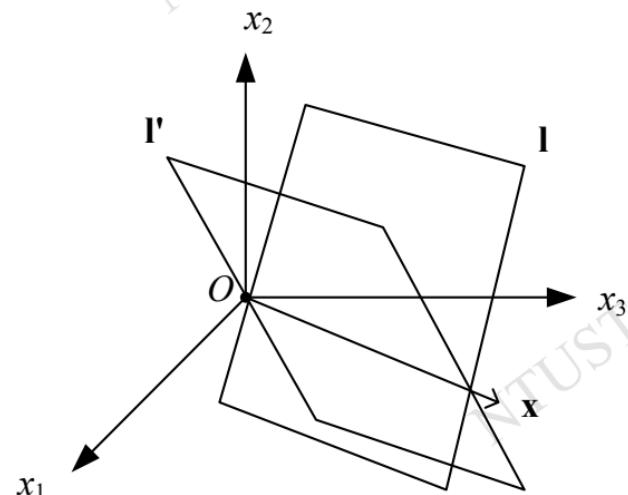
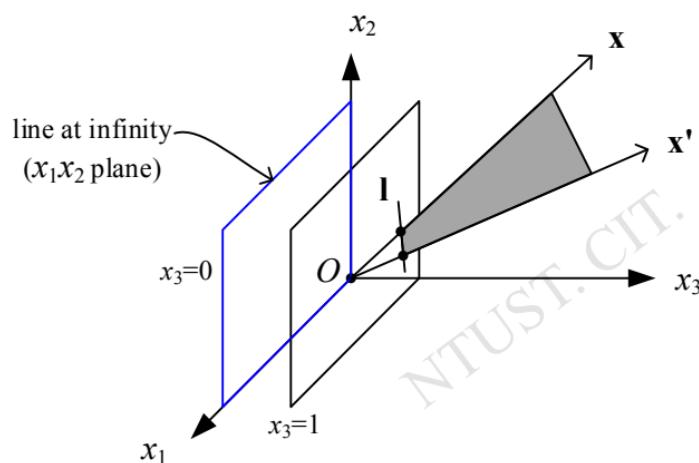
$$\mathbf{P}^2 = \mathbf{R}^2 \cup \mathbf{l}_\infty$$

Note that in  $\mathbf{P}^2$  there is no distinction  
between ideal points and others



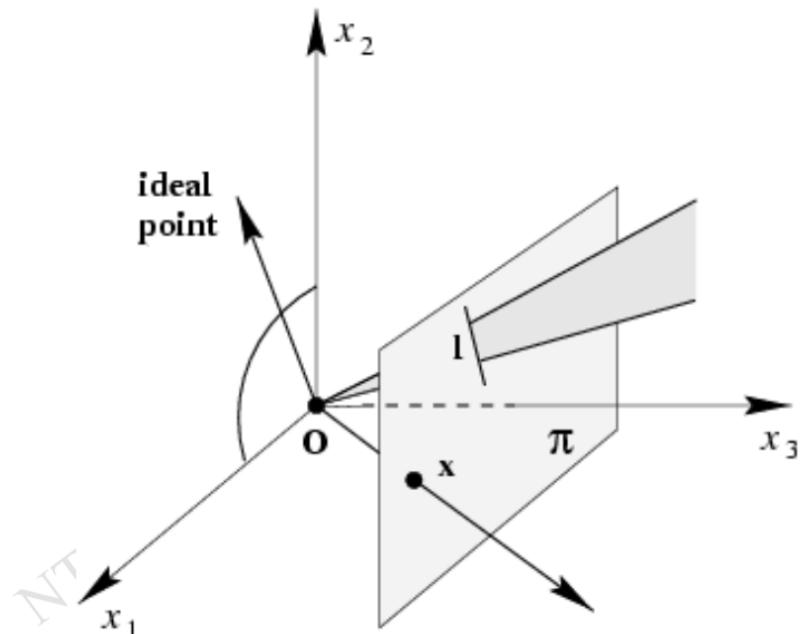
# Ideal points and the line at infinity

- Schematic of homogenous coordinates:





# A model for the projective plane

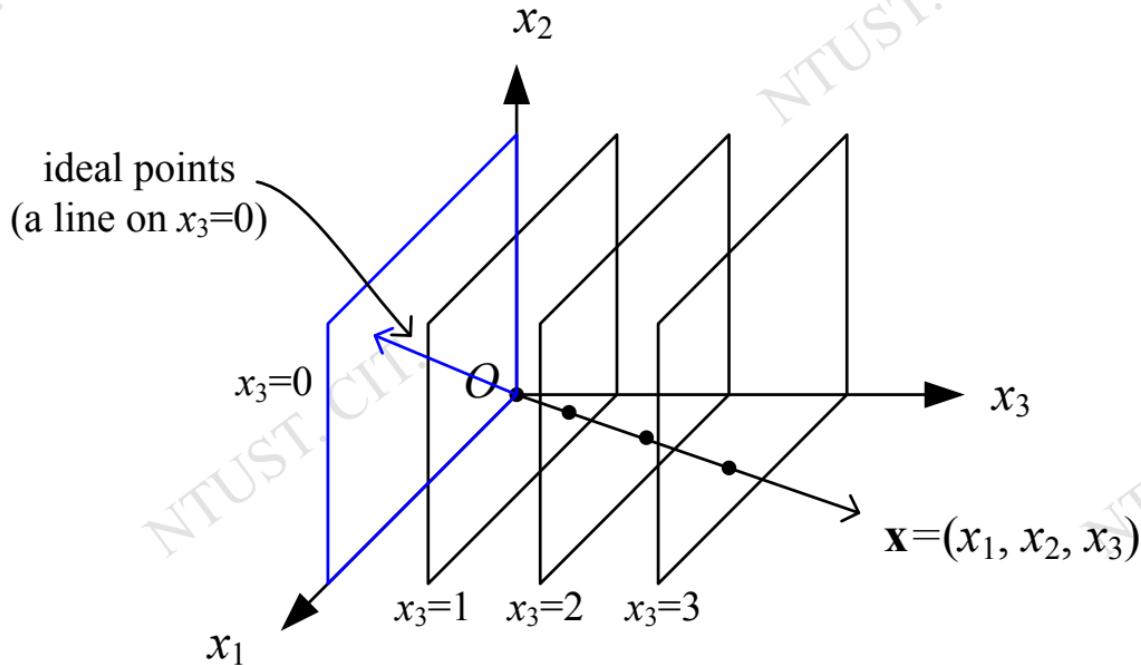


exactly one line through two points

exactly one point at intersection of two lines



# A model for the projective plane—cont.





# Duality of 2D lines and points

Duality principle:

- To any theorem of 2-dimensional projective geometry there corresponds a dual theorem, which may be derived by interchanging the role of points and lines in the original theorem

$$\mathbf{x}$$



$$\mathbf{l}$$

$$\mathbf{x}^T \mathbf{l} = 0$$



$$\mathbf{l}^T \mathbf{x} = 0$$

$$\mathbf{x} = \mathbf{l} \times \mathbf{l}'$$



$$\mathbf{l} = \mathbf{x} \times \mathbf{x}'$$



# Conics

- Curve described by 2<sup>nd</sup>-degree equation in the plane

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

- or homogenized  $x \mapsto \begin{smallmatrix} x_1 \\ x_3 \end{smallmatrix}, y \mapsto \begin{smallmatrix} x_2 \\ x_3 \end{smallmatrix}$

$$ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

- or in matrix form

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad \rightarrow \quad \mathbf{x}^\top \mathbf{C} \mathbf{x} = 0$$

5DOF:  $\{a:b:c:d:e:f\}$       with     $\mathbf{C} = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$

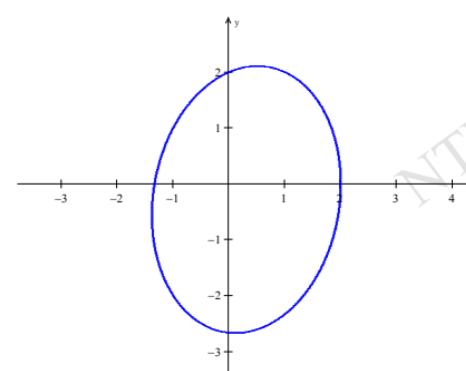
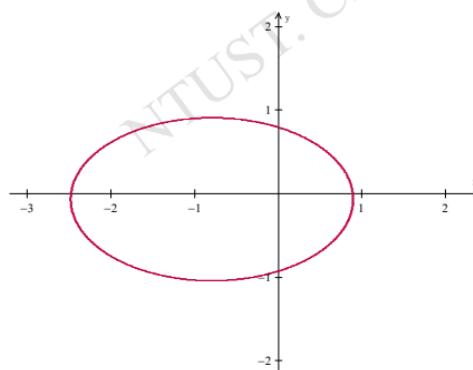


# Conics (example)

## ■ Example

$$\begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 4 \\ 0 & 15 & 1 \\ 4 & 1 & -11 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} 6 & -0.5 & 2 \\ -0.5 & 3 & 1 \\ 2 & 1 & -16 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$





# Conics

- Five points define a conic

For each point the conic passes through

$$ax_i^2 + bx_iy_i + cy_i^2 + dx_i + ey_i + f = 0$$

or in matrix form

$$(x_i^2, x_iy_i, y_i^2, x_i, y_i, f) \cdot \mathbf{c} = 0 \quad \mathbf{c} = (a, b, c, d, e, f)^T$$

stacking constraints yields

$$\begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{bmatrix} \mathbf{c} = 0$$

$$\begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$$



# Conics

- Five points define a conic, an example:

5 points determine a conic:

(-6,1.6733,1)

(-3,2.8636,1)

(0,3.1623,1)

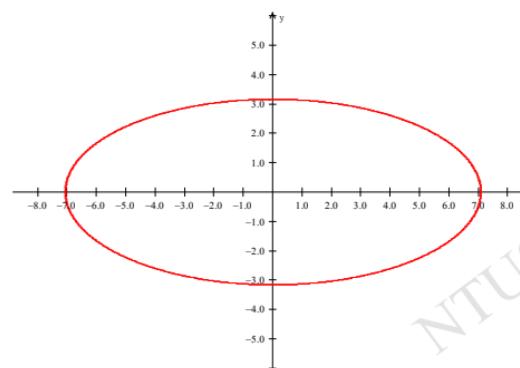
(3,2.8636,1)

(6,1.6733,1)

$$\begin{matrix} 36.0000 & -10.0399 & 2.8000 & -6.0000 & 1.6733 \\ 9.0000 & -8.5907 & 8.2000 & -3.0000 & 2.8636 \\ 0 & 0 & 10.0000 & 0 & 3.1623 \\ 9.0000 & 8.5907 & 8.2000 & 3.0000 & 2.8636 \\ 36.0000 & 10.0399 & 2.8000 & 6.0000 & 1.6733 \end{matrix} * \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$$

Solve it, then get  $\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} -0.02 \\ 0 \\ -0.1 \\ -0.000 \\ -0.000 \end{bmatrix} \rightarrow 0.2x^2 + y^2 - 10 = 0$

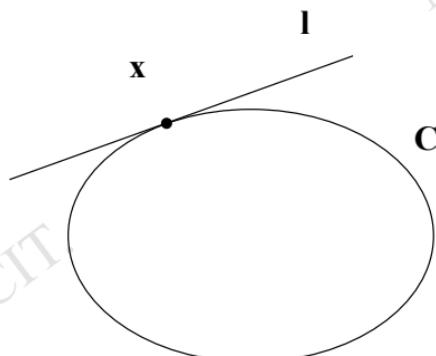
$$0.2x^2 + y^2 - 10 = 0$$





# Tangent lines to conics

- The line  $l$  tangent to  $C$  at point  $x$  on  $C$  is given by  $l = Cx$



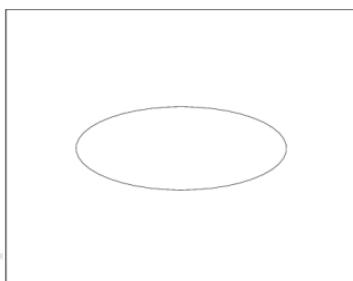
Since  $\mathbf{x}^T \mathbf{C} \mathbf{x} = 0 = \mathbf{x}^T \mathbf{l} = \mathbf{l}^T \mathbf{x}$



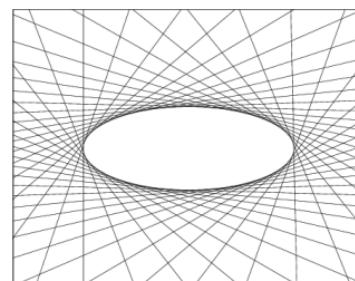
# Dual conics

A line tangent to the conic  $C$  satisfies  $\mathbf{l}^T \mathbf{C}^* \mathbf{l} = 0$

- In general ( $C$  full rank):  $\mathbf{C}^* = \mathbf{C}^{-1}$
- Dual conics = line conics = conic envelopes (包絡線)



$$\mathbf{x}^T \mathbf{C} \mathbf{x} = 0$$



$$\mathbf{l}^T \mathbf{C}^* \mathbf{l} = 0$$



# Dual conics

- A line tangent to the conic  $C$  satisfies

$$\mathbf{l}^T \mathbf{C}^* \mathbf{l} = 0 \rightarrow \mathbf{C}^* = \mathbf{C}^{-1}$$

- Proof:

Since  $\mathbf{x}^T \mathbf{C} \mathbf{x} = 0$

And line on the conic :  $\mathbf{l} = \mathbf{C} \mathbf{x} \rightarrow \mathbf{x} = \mathbf{C}^{-1} \mathbf{l}$  (says tangent points)

$$\mathbf{x}^T \mathbf{C} \mathbf{x} = 0$$

$$(\mathbf{C}^{-1} \mathbf{l})^T \mathbf{C} (\mathbf{C}^{-1} \mathbf{l}) = 0$$

$$\mathbf{l}^T \mathbf{C}^{-T} \mathbf{C} \mathbf{C}^{-1} \mathbf{l} = 0$$

$$\mathbf{l}^T \mathbf{C}^{-T} \mathbf{l} = 0$$

$$\mathbf{l}^T \mathbf{C}^* \mathbf{l} = 0$$

$$\therefore \mathbf{C}^* = \mathbf{C}^{-1}$$

(hint: since  $\mathbf{C}$  is symmetric)



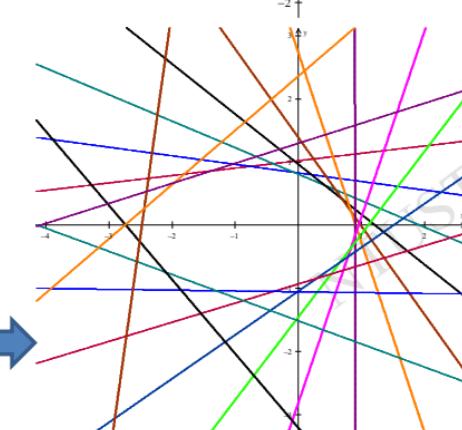
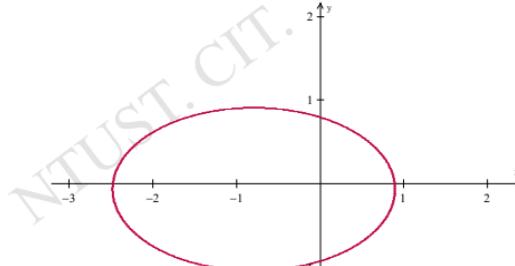
# Dual conics (example)

## ■ Example

$$\begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 4 \\ 0 & 15 & 1 \\ 4 & 1 & -11 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 5 & 0 & 4 \\ 0 & 15 & 1 \\ 4 & 1 & -11 \end{bmatrix}^{-1} = \begin{bmatrix} 0.1551 & -0.0037 & 0.0561 \\ -0.0037 & 0.0664 & 0.0047 \\ 0.0561 & 0.0047 & -0.0701 \end{bmatrix}$$

$$I^T \begin{bmatrix} 0.1551 & -0.0037 & 0.0561 \\ -0.0037 & 0.0664 & 0.0047 \\ 0.0561 & 0.0047 & -0.0701 \end{bmatrix} I = 0 \quad \rightarrow$$





# Degenerate conics

- A conic is degenerate if matrix C is not of full rank

e.g. two lines (rank 2)

$$\mathbf{C} = \mathbf{l}\mathbf{m}^T + \mathbf{m}\mathbf{l}^T$$

e.g. repeated line (rank 1)

$$\mathbf{C} = \mathbf{I}\mathbf{I}^T$$

Example:

$$\mathbf{x}^T \mathbf{C} \mathbf{x} = 0$$

$$\mathbf{x}^T (\mathbf{l}\mathbf{m}^T + \mathbf{m}\mathbf{l}^T) \mathbf{x} = (\mathbf{x}^T \mathbf{l})(\mathbf{m}^T \mathbf{x}) + (\mathbf{x}^T \mathbf{m})(\mathbf{l}^T \mathbf{x}) = 0$$

So, either  $\mathbf{x}^T \mathbf{l} = 0$ , or  $\mathbf{x}^T \mathbf{m} = 0 \rightarrow$  two lines

Example:

$$\mathbf{l} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

Degenerate line conics: 2 points (rank 2), double point (rank1)

Note that for degenerate conics

$$(\mathbf{C}^*)^* \neq \mathbf{C}$$

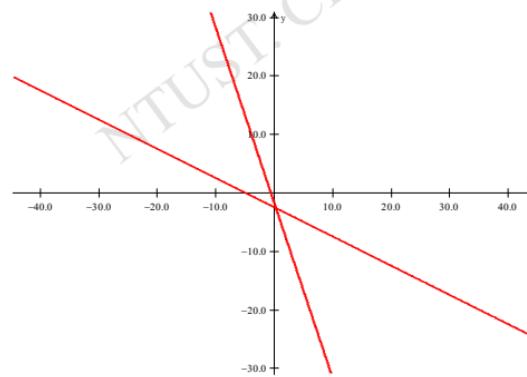


# Degenerate conics (example)

## ■ Example

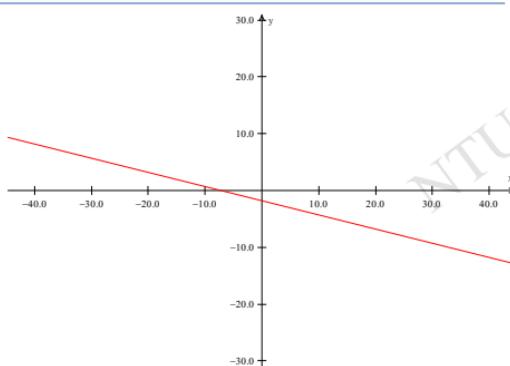
$$\mathbf{C} = \mathbf{l}\mathbf{m}^T + \mathbf{m}\mathbf{l}^T$$

$$\mathbf{l} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \mathbf{m} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 6 & 7 & 17 \\ 7 & 4 & 9 \\ 17 & 9 & 20 \end{bmatrix}$$



$$\mathbf{C} = \mathbf{l}\mathbf{l}^T$$

$$\mathbf{l} = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 4 & 7 \\ 4 & 16 & 28 \\ 7 & 28 & 49 \end{bmatrix}$$





# Projective transformations

- Definition:

A projectivity is an invertible mapping  $h$  from  $P^2$  to itself such that three points  $x_1, x_2, x_3$  lie on the same line if and only if  $h(x_1), h(x_2), h(x_3)$  do. → 在同一線上的3個點經過轉換仍然共線

- Theorem:

A mapping  $h: P^2 \rightarrow P^2$  is a projectivity if and only if there exists a non-singular  $3 \times 3$  matrix  $\mathbf{H}$  such that for any point in  $P^2$  represented by a vector  $\mathbf{x}$  it is true that  $h(\mathbf{x}) = \mathbf{Hx}$

- Definition: Projective transformation

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{or} \quad \mathbf{x}' = \mathbf{Hx}$$

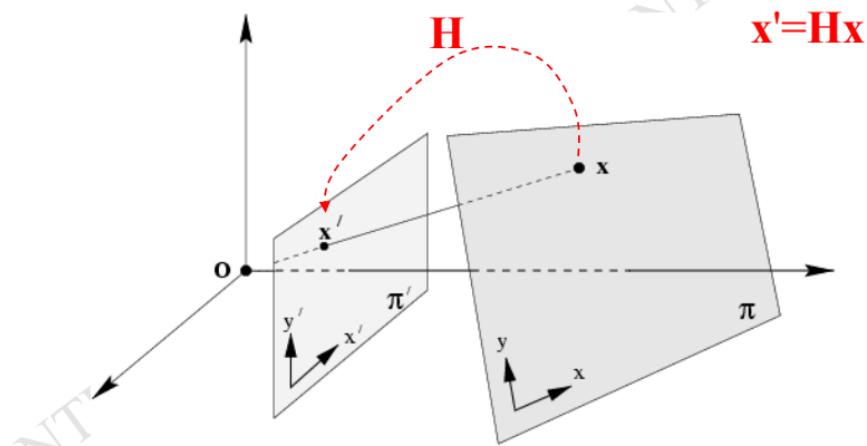
8DOF

- Projectivity = Collineation = Projective Transformation = Homography



# Application: mapping between planes

- Homography



*central projection* may be expressed by  $x' = Hx$



# Homography: to remove projective effect



select four points in a plane with known coordinates

$$x' = \frac{x'_1}{x'_3} = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}} \quad y' = \frac{x'_2}{x'_3} = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

$$\begin{aligned} x'(h_{31}x + h_{32}y + h_{33}) &= h_{11}x + h_{12}y + h_{13} \\ y'(h_{31}x + h_{32}y + h_{33}) &= h_{21}x + h_{22}y + h_{23} \end{aligned} \quad (\text{linear in } h_{ij})$$

(2 constraints/point, 8DOF  $\Rightarrow$  4 points needed)

Note! **NO calibration at all necessary**, better ways to compute



# Homography: to remove projective effect

- Rewrite equation

$$xh_{11} + yh_{12} + h_{13} - x'xh_{31} - x'yh_{32} - x'h_{33} = 0$$

$$xh_{21} + yh_{22} + h_{23} - y'xh_{31} - y'yh_{32} - y'h_{33} = 0$$

- Normalize hij with h33, (replace  $h_{ij}/h_{33}$  with  $h_{ij}$  temporarily)

$$xh_{11} + yh_{12} + h_{13} - x'xh_{31} - x'yh_{32} = x'$$

$$xh_{21} + yh_{22} + h_{23} - y'xh_{31} - y'yh_{32} = y'$$

- In matrix form:

$$\begin{bmatrix} x & y & 1 & 0 & 0 & 0 & -x'x & -x'y \\ 0 & 0 & 0 & x & y & 1 & -y'x & -y'y \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

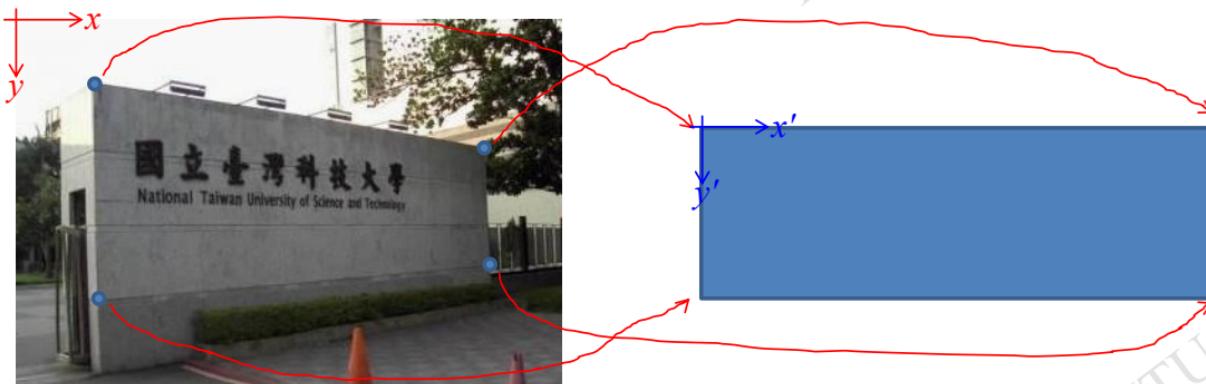
8 unknowns to be solved

If you have one set of correspondence, you get  $(x, y)$  &  $(x', y')$ . So, you need at least four correspondences for solving 8 unknowns. Note: here, one correspondence forms two equations (需要四組對應點)



# Homography: to remove projective effect

- For example: Take a picture, then remove the distortion. Someday...



A picture you took

The result you want

**Define your problem, first!!!**



# Homography: to remove projective effect

- For example, —cont.



Correspondence:

$(54,45) \rightarrow (0,0)$

$(58,196) \rightarrow (0,100)$

$(332,172) \rightarrow (400,100)$

$(329,91) \rightarrow (400,0)$

Then, find  $\mathbf{H}$

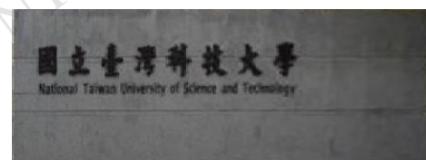
$$\begin{bmatrix} 54 & 45 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 54 & 45 & 1 & 0 & 0 \\ 58 & 196 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 58 & 196 & 1 & -5800 & -19600 \\ 332 & 172 & 1 & 0 & 0 & 0 & -132800 & -68800 \\ 0 & 0 & 0 & 332 & 172 & 1 & -33200 & -17200 \\ 329 & 91 & 1 & 0 & 0 & 0 & -131600 & -36400 \\ 0 & 0 & 0 & 329 & 91 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 100 \\ 400 \\ 100 \\ 400 \\ 0 \end{bmatrix}$$



# Homography: to remove projective effect

- For example, —cont.

$$\begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \end{bmatrix} = \begin{bmatrix} 54 & 45 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 54 & 45 & 1 & 0 & 0 \\ 58 & 196 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 58 & 196 & 1 & -5800 & -19600 \\ 332 & 172 & 1 & 0 & 0 & 0 & -132800 & -68800 \\ 0 & 0 & 0 & 332 & 172 & 1 & -33200 & -17200 \\ 329 & 91 & 1 & 0 & 0 & 0 & -131600 & -36400 \\ 0 & 0 & 0 & 329 & 91 & 1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 100 \\ 400 \\ 100 \\ 400 \\ 0 \end{bmatrix}$$



$$\begin{aligned}
 & \begin{array}{ccccccccccccc}
 -0.0030 & -0.0086 & -0.0007 & 0.0085 & -0.0008 & -0.0160 & 0.0045 & 0.0160 & 0 & 0.7210 \\
 -0.0065 & 0.0002 & 0.0066 & -0.0002 & 0.0000 & 0.0004 & -0.0001 & -0.0004 & 0 & -0.0191 \\
 1.4550 & 0.4537 & -0.2620 & -0.4511 & 0.0425 & 0.8424 & -0.2355 & -0.8450 & 0 & -38.0771 \\
 0.0003 & -0.0024 & -0.0003 & -0.0012 & 0.0006 & 0.0002 & -0.0006 & 0.0035 & * 100 & -0.1029 \\
 -0.0020 & -0.0072 & 0.0020 & 0.0073 & -0.0038 & -0.0012 & 0.0038 & 0.0011 & 400 & 0.6150 \\
 0.0735 & 1.4550 & -0.0730 & -0.2620 & 0.1364 & 0.0425 & -0.1368 & -0.2355 & 100 & -22.1199 \\
 0.0000 & -0.0000 & -0.0000 & 0.0000 & 0.0000 & -0.0000 & -0.0000 & 0.0000 & 400 & -0.0016 \\
 -0.0000 & 0.0000 & 0.0000 & -0.0000 & -0.0000 & 0.0000 & 0.0000 & -0.0000 & 0 & 0.0001
 \end{array} \\
 & = \rightarrow \mathbf{H} = \begin{bmatrix} 0.721 & -0.0191 & -38.0771 \\ -0.1029 & 0.6150 & -22.1199 \\ -0.0016 & 0.0001 & 1 \end{bmatrix}
 \end{aligned}$$



# Homography: to remove projective effect

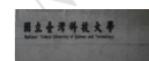
- For example, —cont.

	p1 =	p2 =	p3 =	p4 =
original points:	54	58	332	329
	45	196	172	91
	1	1	1	1



	>> H*p1	>> H*p2	>> H*p3	>> H*p4
x' = Hx	ans =	ans =	ans =	ans =
	0	-0.0000	198.0257	197.4097
	0.0000	92.4536	49.5064	0.0000
	0.9192	0.9245	0.4951	0.4935

	0	-0.0000	400.0000	400.0000
normalized	0.0000	100.0000	100.0000	0.0000
	1.0000	1.0000	1.0000	1.0000

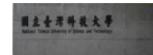


Desired points:  $(0,0,1)^T$   $(0,100,1)^T$   $(400,100,1)^T$   $(400, 0,1)^T$



# Homography: to remove projective effect

- For example, —cont. (inverse mapping)

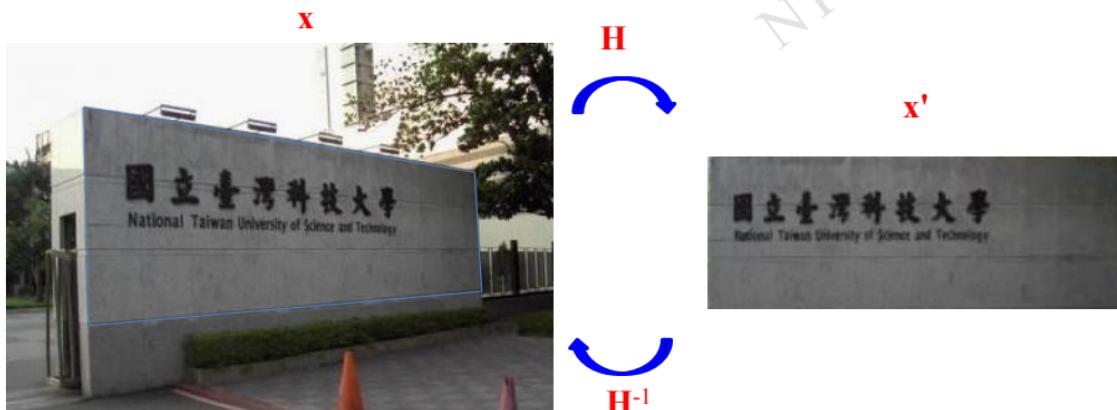
	pp1 =	pp2 =	pp3 =	pp4 =	
desired points:	0	0	400	400	
	0	100	100	0	
	1	1	1	1	
	>> inv(H)*pp1	>> inv(H)*pp2	>> inv(H)*pp3	>> inv(H)*pp4	
x=H <sup>-1</sup> x'	ans =	ans =	ans =	ans =	
	58.7480	62.7342	670.6200	666.6338	
	48.9567	211.9982	347.4296	184.3881	
	1.0879	1.0816	2.0199	2.0262	
normalized	54.0000	58.0000	332.0000	329.0000	
	45.0000	196.0000	172.0000	91.0000	
	1.0000	1.0000	1.0000	1.0000	

Original points:  $(54,45,1)^T$      $(58,196,1)^T$      $(332,172,1)^T$      $(329,91,1)^T$



# Homography: to remove projective effect

- For example, —cont. (inverse mapping)



Filling correct COLOR:

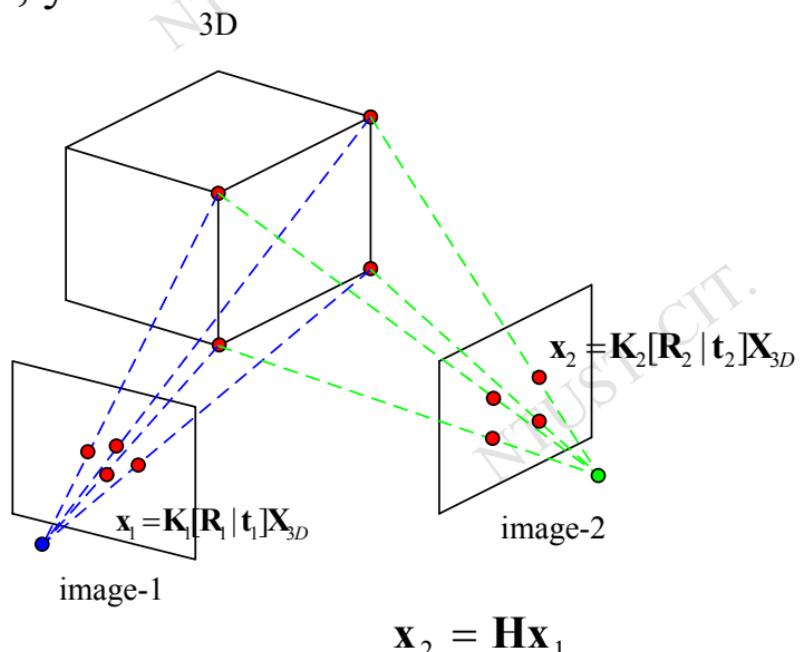
You are knowing to filling color in a “400x100” image. For each pixel, you need to calculate its color by applying  $H^{-1}$  to its coordinate.



# Homography: What is it?

If you have at least 4 corresponding points, a homography can dominate the transformation between two images. So, you do NOT need to determine  $\mathbf{K}[\mathbf{R}|\mathbf{t}]$  for 2 views

(Note: ONLY planar structure in 3D)





# Homography in OpenCV

- openCV provides various kinds of mapping operations in computer vision.

Correspondence:

$(54,45) \rightarrow (0,0)$   
 $(58,196) \rightarrow (0,100)$   
 $(332,172) \rightarrow (400,100)$   
 $(329,91) \rightarrow (400,0)$

Then, find H

Sample Code:

**Mat H = findHomography( xSet, xpSet,  
CV\_RANSAC );**

Source Points:

54.000000 45.000000  
58.000000 196.000000  
332.000000 172.000000  
329.000000 91.000000

Destination Points:

0.000000 0.000000  
0.000000 100.000000  
400.000000 100.000000  
400.000000 0.000000

Homography Matrix:

0.721049 -0.019101 -38.077095  
-0.102873 0.615001 -22.119894  
-0.001561 0.000077 1.000000



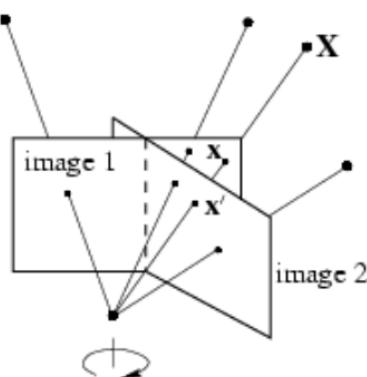
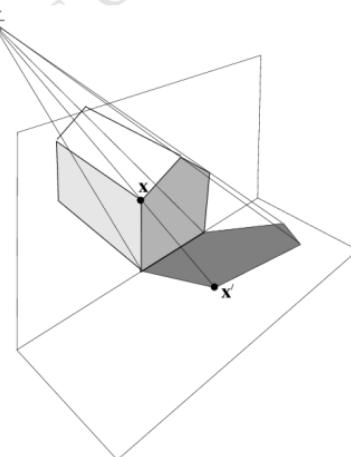
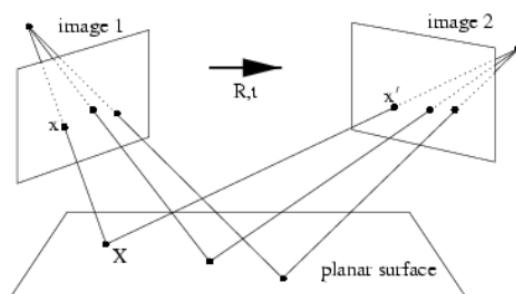
# Homography: example: develop a program

- Example





# Homography: more examples





# Transformation of lines and conics

Homography for specific shape

- For a point transformation

$$\mathbf{x}' = \mathbf{H} \mathbf{x}$$

- Transformation for lines

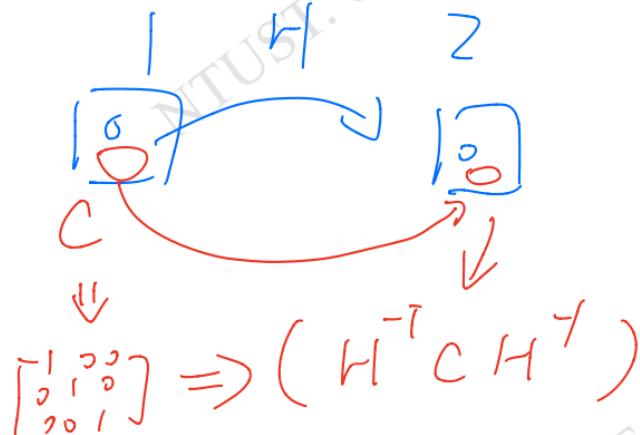
$$\mathbf{l}' = \mathbf{H}^{-T} \mathbf{l}$$

- Transformation for conics

$$\mathbf{C}' = \mathbf{H}^{-T} \mathbf{C} \mathbf{H}^{-1}$$

- Transformation for dual conics

$$\mathbf{C}'^* = \mathbf{H} \mathbf{C}^* \mathbf{H}^T$$


$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow (\mathbf{H}^{-T} \mathbf{C} \mathbf{H}^{-1})$$



# Transformation of lines and conics

- Transformation for lines  $\mathbf{l}' = \mathbf{H}^{-T}\mathbf{l}$  (proof)

If we have  $\mathbf{x}' = \mathbf{H}\mathbf{x}$

And  $\mathbf{x}'$  on line  $\mathbf{l}'$ , and  $\mathbf{x}$  on  $\mathbf{l}$ .

So, we have  $\mathbf{l}'^T \mathbf{x}' = 0$   $\mathbf{l}^T \mathbf{x} = 0$

Rewrite  $\mathbf{l}'^T \mathbf{x}' = 0 = \mathbf{l}^T \mathbf{H}^{-1} \mathbf{H}\mathbf{x}$

Then,  $\mathbf{l}'^T = \mathbf{l}^T \mathbf{H}^{-1} \Rightarrow \mathbf{l}' = (\mathbf{l}'^T)^T = (\mathbf{l}^T \mathbf{H}^{-1})^T = \mathbf{H}^{-T} \mathbf{l}$

Get:  $\mathbf{l}' = \mathbf{H}^{-T} \mathbf{l}$



# Transformation of lines and conics

- Transformation for conics  $\mathbf{C}' = \mathbf{H}^{-T} \mathbf{C} \mathbf{H}^{-1}$  (proof)

If we have  $\mathbf{x}' = \mathbf{H}\mathbf{x}$

And know a conic equation:  $\mathbf{x}^T \mathbf{C} \mathbf{x} = 0$

So, we have  $\mathbf{x} = \mathbf{H}^{-1}\mathbf{x}'$

Rewrite equation:  $\mathbf{x}^T \mathbf{C} \mathbf{x} = 0 \rightarrow (\mathbf{H}^{-1}\mathbf{x}')^T \mathbf{C} \mathbf{H}^{-1}\mathbf{x}' = 0$

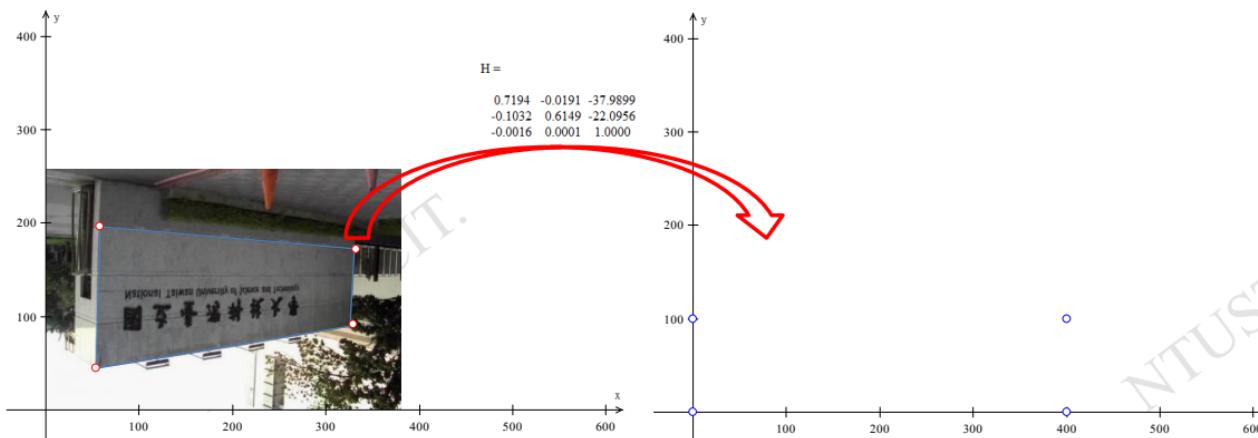
Then,  $\mathbf{x}'^T (\mathbf{H}^{-T} \mathbf{C} \mathbf{H}^{-1}) \mathbf{x}' = 0 = \mathbf{x}'^T \mathbf{C}' \mathbf{x}'$

Get:  $\mathbf{C}' = \mathbf{H}^{-T} \mathbf{C} \mathbf{H}^{-1}$



# Transformation of lines and conics

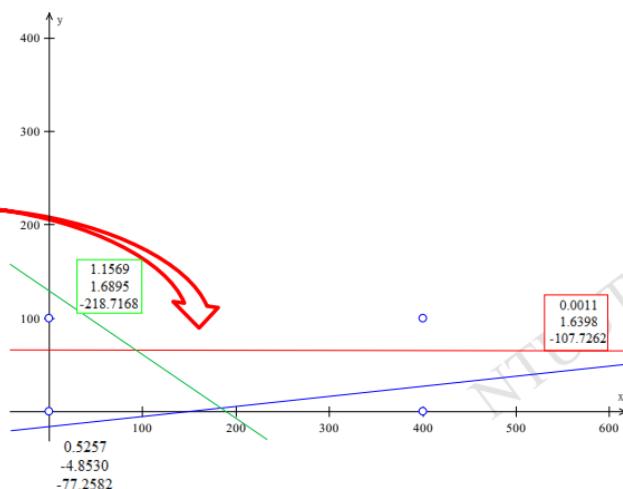
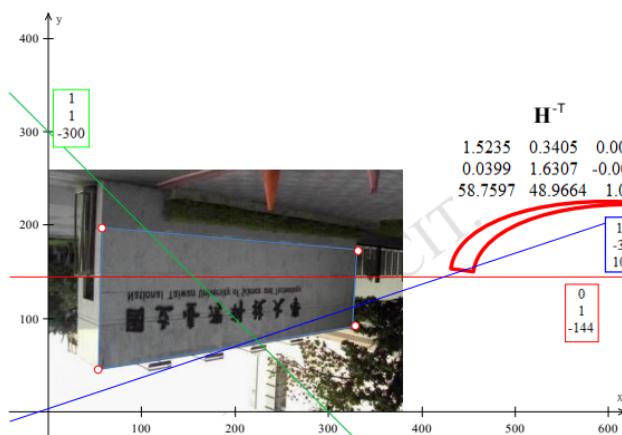
- Example: (the same with previous, but mirror for convenience)





# Transformation of lines and conics

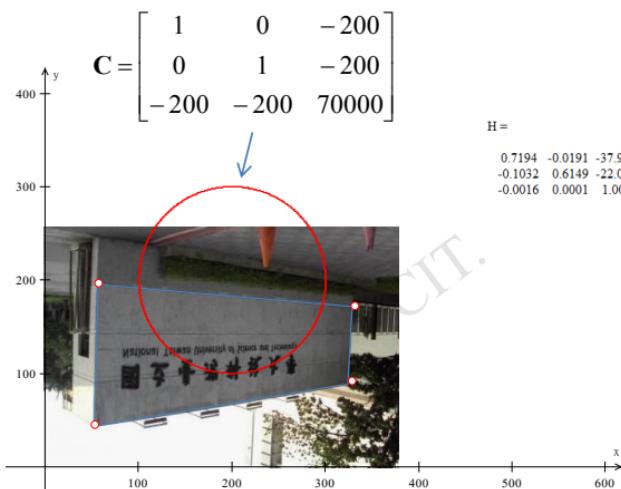
- Example: lines transformation



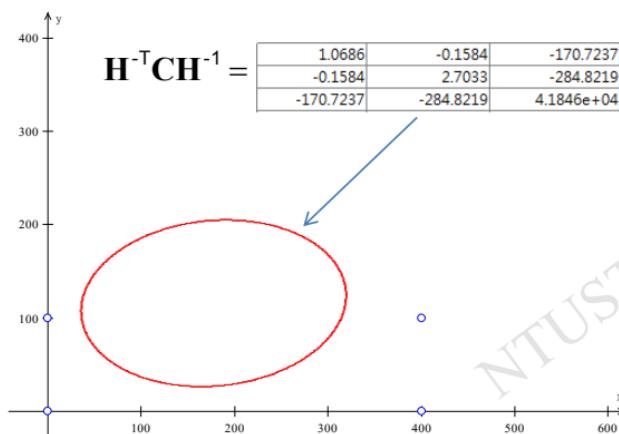


# Transformation of lines and conics

- Example: conics transformation  $\mathbf{C}' = \mathbf{H}^{-T}\mathbf{C}\mathbf{H}^{-1}$



$$\mathbf{H}^{-T}\mathbf{C}\mathbf{H}^{-1} = \begin{bmatrix} 1.0686 & -0.1584 & -170.7237 \\ -0.1584 & 2.7033 & -284.8219 \\ -170.7237 & -284.8219 & 4.1846e+04 \end{bmatrix}$$





# A hierarchy of transformations

- Projective linear group
  - Affine group (last row (0,0,1))
    - Euclidean group (upper left 2x2 orthogonal)
      - Oriented Euclidean group (upper left 2x2 det 1)

Alternative, characterize transformation in terms of elements or quantities that are preserved or *invariant*  
e.g. Euclidean transformations leave distances unchanged





# Four classic types of transformation

- Isometrics
- Similarities
- Affine mapping
- Projective mapping



# Four classic types of transformation—cont.

## ■ Class I: Isometries (iso=same, metric=measure)

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} \varepsilon \cos \theta & -\sin \theta & t_x \\ \varepsilon \sin \theta & \cos \theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \quad \varepsilon = \pm 1$$

orientation preserving:  $\varepsilon = 1$

orientation reversing:  $\varepsilon = -1$

$$x' = \mathbf{H}_E x = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} x \quad \mathbf{R}^\top \mathbf{R} = \mathbf{I}$$

3DOF (1 rotation, 2 translation)

special cases: pure rotation, pure translation

**Invariants:** length, angle, area      此類型轉換夾角 面積 長度不會改變!!!



# Four classic types of transformation—cont.

## ■ Class II: Similarities (isometry + scale)

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} s \cos \theta & -s \sin \theta & t_x \\ s \sin \theta & s \cos \theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$x' = \mathbf{H}_s x = \begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} x \quad \mathbf{R}^\top \mathbf{R} = \mathbf{I}$$

4DOF (1 scale, 1 rotation, 2 translation)

also known as *equi-form* (shape preserving)

*metric structure* = structure up to similarity (in literature)

**Invariants:** ratios of length, angle, ratios of areas,

parallel lines

此類型轉換使影像中的：  
長度比例  
夾角 面積比例 平行線不會改變!!!



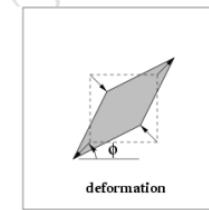
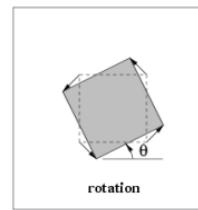
# Four classic types of transformation—cont.

## ■ Class III: Affine transformations

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\mathbf{x}' = \mathbf{H}_A \mathbf{x} = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \mathbf{x}$$

$$\mathbf{A} = \mathbf{R}(\theta)\mathbf{R}(-\phi)\mathbf{D}\mathbf{R}(\phi) \quad \mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$



6DOF (2 scale, 2 rotation, 2 translation)

non-isotropic scaling! (2DOF: scale ratio and orientation)

**Invariants:** parallel lines, ratios of parallel lengths,  
ratios of areas



# Four classic types of transformation—cont.

## ■ Class IV: Projective transformations

$$\mathbf{x}' = \mathbf{H}_P \mathbf{x} = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^\top & v \end{bmatrix} \mathbf{x} \quad \mathbf{v} = (v_1, v_2)^\top$$

8DOF (2 scale, 2 rotation, 2 translation, 2 line at infinity)

Action non-homogeneous over the plane

**Invariants:** cross-ratio of four points on a line (ratio of ratio)



# Four classic types of transformation—cont.

- Action of affinities and projectivities on line at infinity

$$\begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0}^T & v \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ 0 \end{pmatrix}$$

Line at infinity stays at infinity,  
but points move along line

$$\begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^T & v \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ v_1 x_1 + v_2 x_2 \end{pmatrix}$$

Line at infinity becomes finite,  
allows to observe vanishing points, horizon,



# Four classic types of transformation—cont.

- Decomposition of projective transformations

$$\mathbf{H} = \mathbf{H}_S \mathbf{H}_A \mathbf{H}_P = \begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{v}^T & v \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^T & v \end{bmatrix}$$

Similarity      Affine      Projective

decomposition unique (if chosen  $s>0$ )

$\mathbf{K}$  upper-triangular,  $\det \mathbf{K} = 1$

Example:

$$\mathbf{H} = \begin{bmatrix} 1.707 & 0.586 & 1.0 \\ 2.707 & 8.242 & 2.0 \\ 1.0 & 2.0 & 1.0 \end{bmatrix}$$

Step: 1. Determine  $\mathbf{v}^T$

Step: 2. Find  $\mathbf{K}$

Step: 3. then  $s$  and  $\mathbf{R}$

$$\mathbf{H} = \begin{bmatrix} 2\cos 45^\circ & -2\sin 45^\circ & 1.0 \\ 2\sin 45^\circ & 2\cos 45^\circ & 2.0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{Step 1}} \begin{bmatrix} 0.5 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{Step 2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$



# Four classic types of transformation—cont.

- Decomposition of projective transformations
  - Example

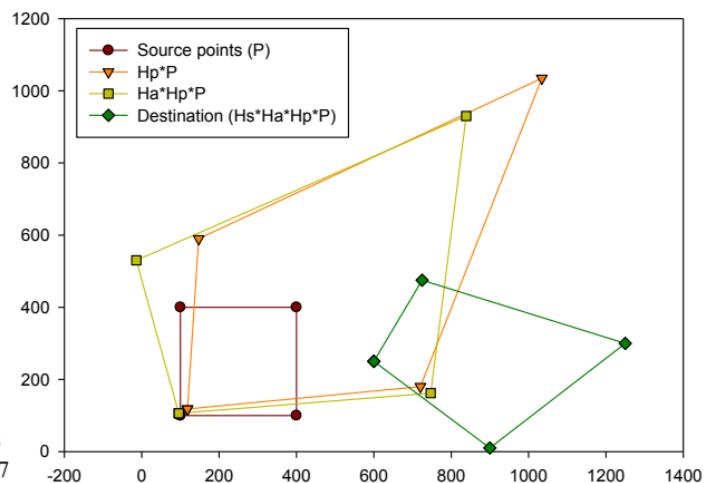
Source Points	Destination Points
(100,100)	(600,250)
(400,100)	(900,10)
(400,400)	(1250,300)
(100,400)	(725,475)

$H =$   
-0.027759 -0.054527 516.226013  
-0.687046 0.368143 243.555847  
-0.000972 -0.000562 1.000000

---

$H_p =$   
1.000000 0.000000 0.000000  
0.000000 1.000000 0.000000  
-0.000972 -0.000562 1.000000

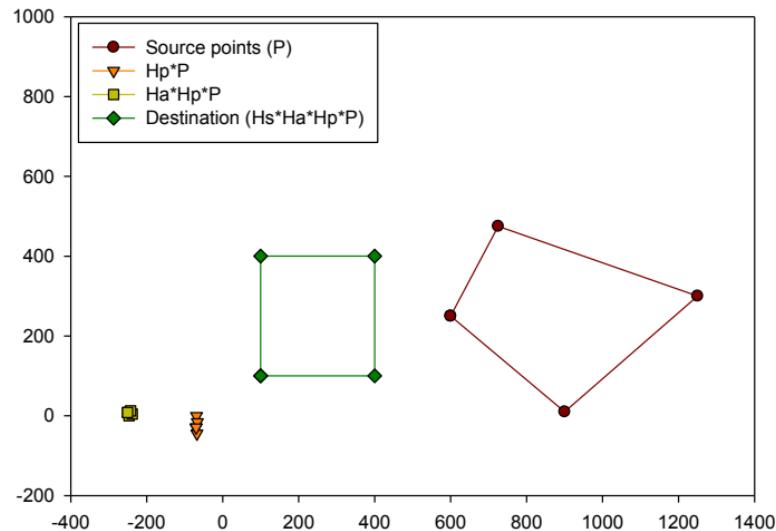
$H_a =$   
1.112456 -0.301632 0.000000  
-0.000000 0.898912 0.000000  
0.000000 0.000000 1.000000       $H_s =$   
0.425900 0.404881 516.226013  
-0.404881 0.425900 243.555847  
0.000000 0.000000 1.000000





# Four classic types of transformation—cont.

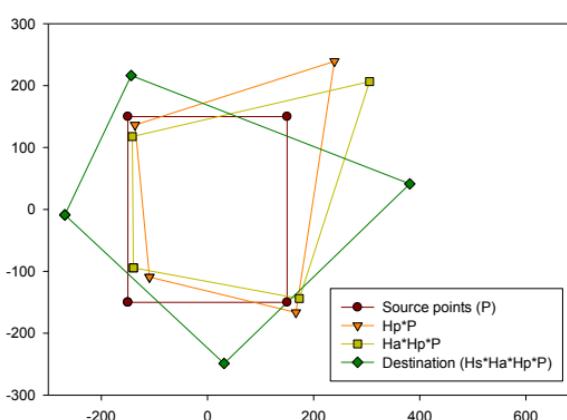
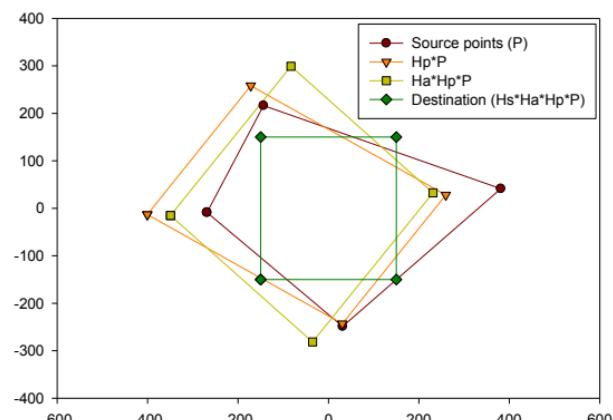
- Decomposition of projective transformations
  - (Inverse) Example





# Four classic types of transformation—cont.

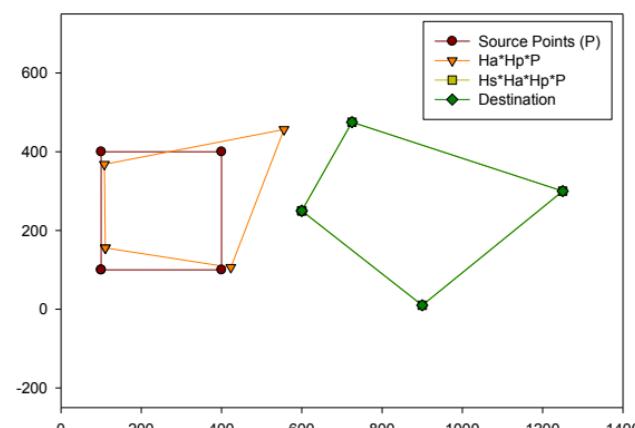
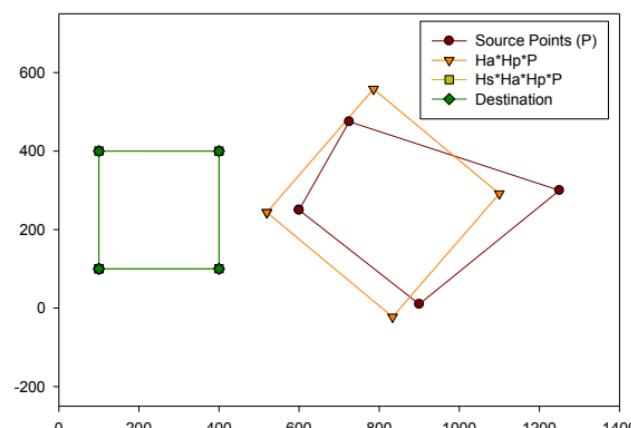
- Decomposition of projective transformations
  - Example—cont.





# Four classic types of transformation—cont.

- Decomposition of projective transformations
  - Example—cont.

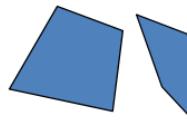




# Overview transformations

Projective  
8dof

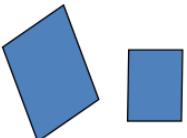
$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$



Concurrency, collinearity, order  
of contact (intersection, tangency,  
inflection, etc.), cross ratio

Affine  
6dof

$$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

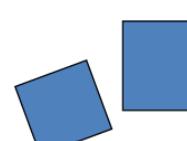


Parallelism, ratio of areas, ratio  
of lengths on parallel lines (e.g  
midpoints), linear combinations  
of vectors (centroids).

**The line at infinity  $l_\infty$**

Similarity  
4dof

$$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



Ratios of lengths, angles.

**The circular points I,J**

Euclidean  
3dof

$$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



lengths, areas.



# Number of invariants?

- The number of functional invariants is equal to, or greater than, the number of degrees of freedom of the configuration less the number of degrees of freedom of the transformation
- e.g. configuration of 4 points in general position has 8 dof (2/pt) and so 4 similarity, 2 affinity and zero projective invariants



# Short summary

- Points and lines

$$\mathbf{l}^T \mathbf{x} = 0 \quad \mathbf{x} = \mathbf{l} \times \mathbf{l}' \quad \mathbf{l} = \mathbf{x} \times \mathbf{x}' \quad \mathbf{l}_\infty = (0,0,1)^T$$

- Conics and dual conics

$$\mathbf{x}^T \mathbf{C} \mathbf{x} = 0 \quad \mathbf{l}^T \mathbf{C}^* \mathbf{l} = 0 \quad \mathbf{C}^* = \mathbf{C}^{-1} \quad \mathbf{l} = \mathbf{C} \mathbf{x}$$

- Projective transformations

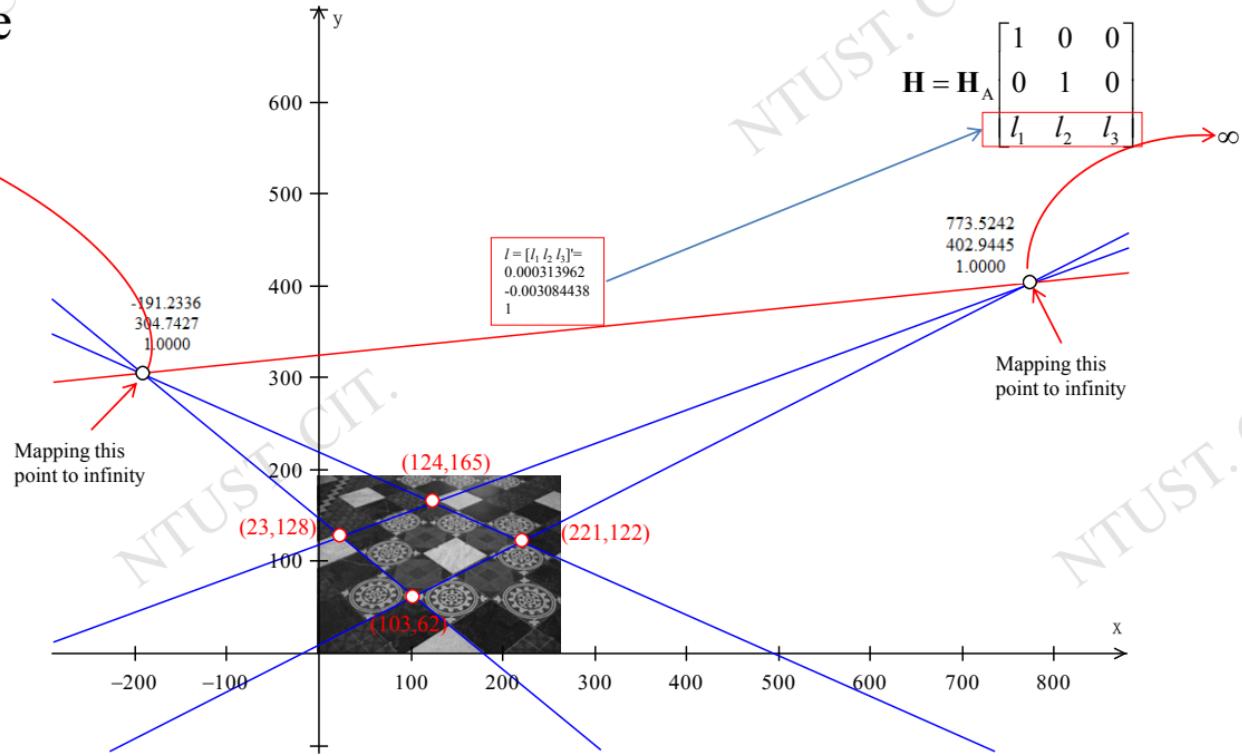
$$\mathbf{x}' = \mathbf{H} \mathbf{x} \quad \mathbf{l}' = \mathbf{H}^{-T} \mathbf{l}$$

$$\mathbf{C}' = \mathbf{H}^{-T} \mathbf{C} \mathbf{H}^{-1} \quad \mathbf{C}'^* = \mathbf{H} \mathbf{C}^* \mathbf{H}^T$$



# Affine rectification via the vanishing line

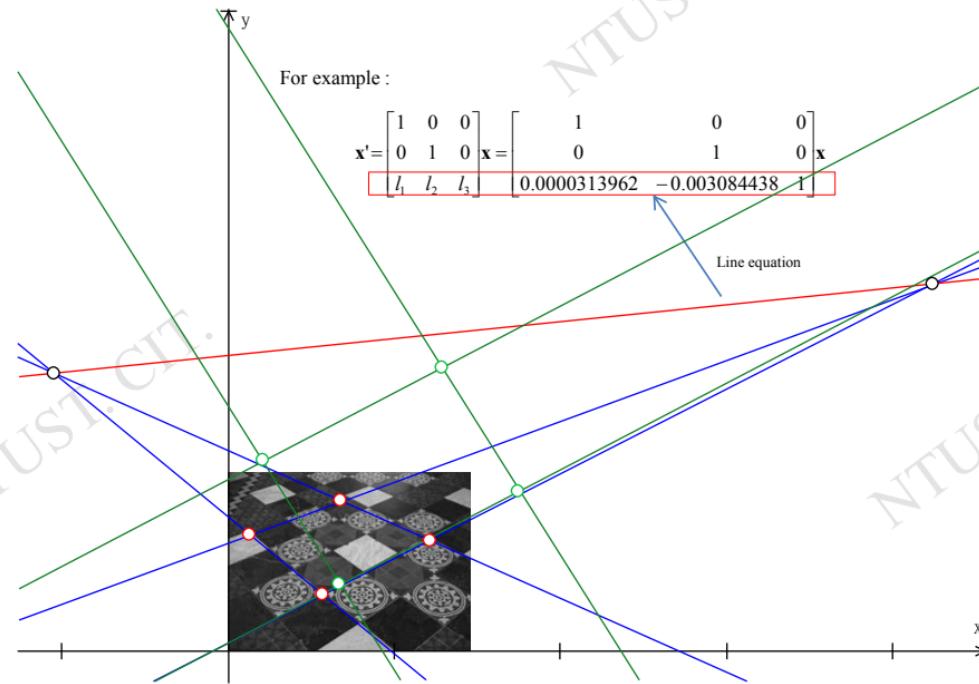
## ■ Example





# Affine rectification via the vanishing line

## ■ Example—cont.





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