電腦視覺與應用
Computer Vision and Applications

Lecture-06-1 Two-views geometry

Tzung-Han Lin
National Taiwan University of Science and Technology
Graduate Institute of Color and Illumination Technology
e-mail: thl@mail.ntust.edu.tw

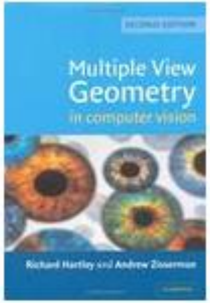
TAIWAN TECH National Taiwan University of Science and Technology

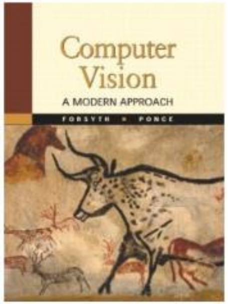
CIT 色彩與照明科技研究所 Graduate Institute of Color and Illumination Technology

Two-views geometry

幾何學

- Description for fundamental matrix, **F**, and essential matrix, **E**
- Lecture Reference at:
  - Multiple View Geometry in Computer Vision, Chapter 9, 11
  - Computer Vision A Modern Approach, Chapter 10.

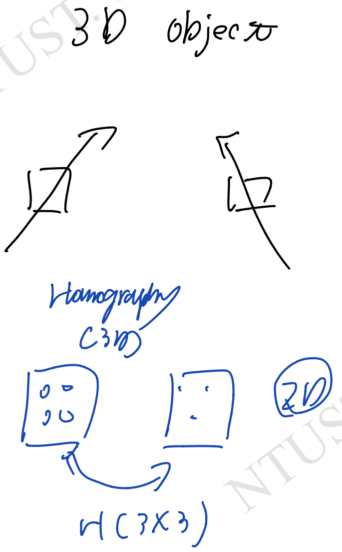




Two-views geometry – Outline

- epipole
- epipolar line
- epipolar plane

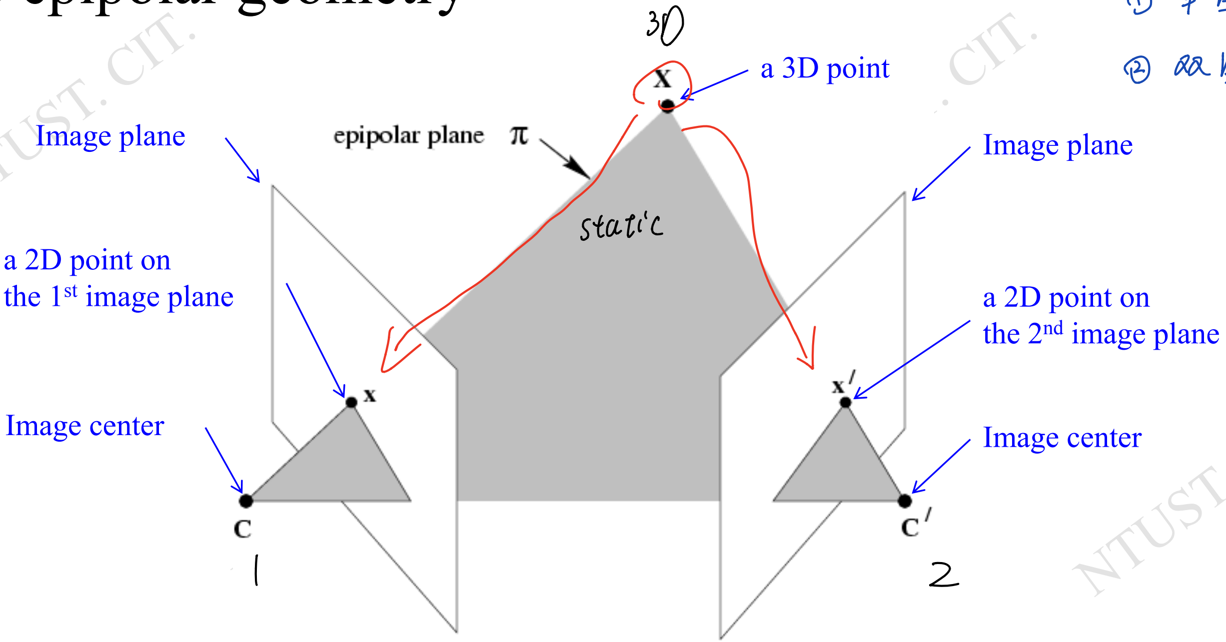
- Fundamental matrix **F**
- Essential matrix **E** → special case of **F**
- Computation for Fundamental Matrix **F**



The epipolar geometry

① 單目

② 双目



C, **C'**, **x**, **x'** and **X** are coplanar

Note:

Two images may have different intrinsic parameter, be taken at either the same or different periods.

The epipolar geometry

- In case of given **c**, **c'** (says **e**, **e'** as well) and **x**?
  → define an epipolar line for **x**

3X

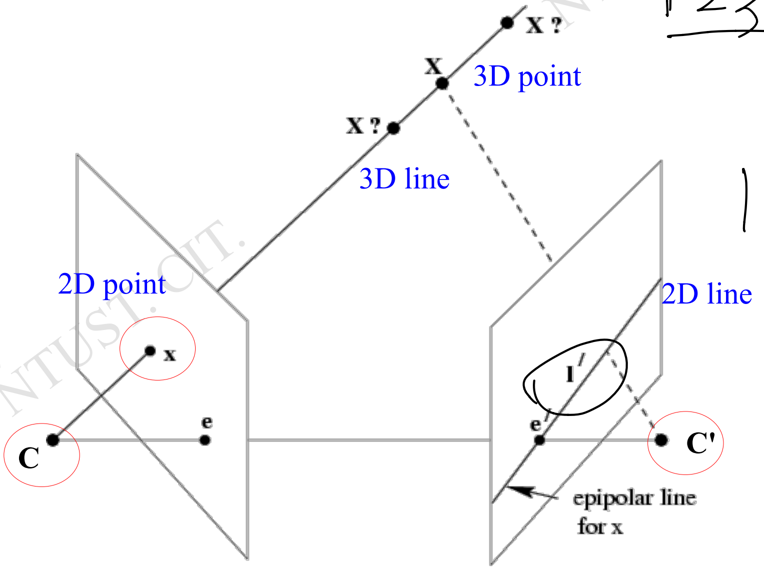
1X2

3X2

= 22

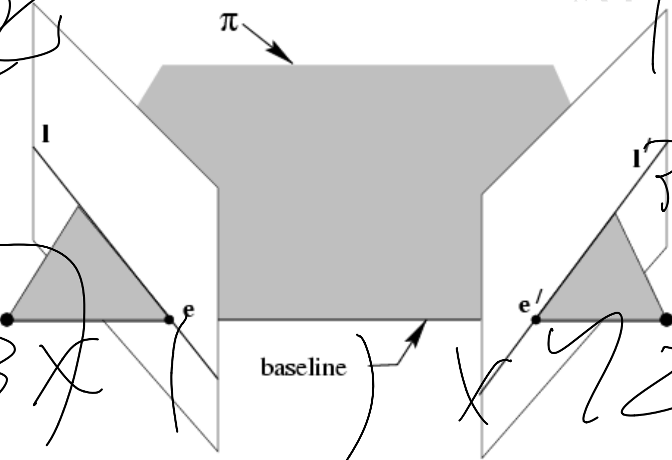
1X3

3X1



The epipolar geometry

3×720



3×1 3×3 3×1

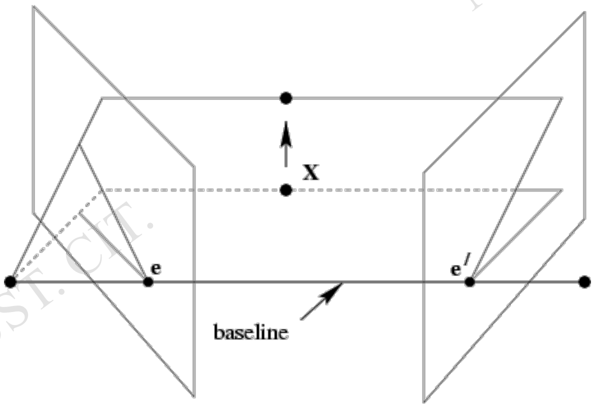
(3×3×3×1)

×720 3×1

All points on **π** project on **l** and **l'**

The epipolar geometry

- Family of planes π and lines **l** and **l'** intersection in **e** and **e'**



The epipolar geometry

- Summary for definition

epipoles **e**, **e**'

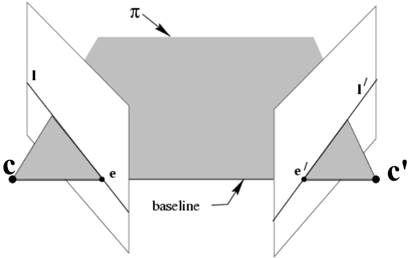
= intersection of baseline with image plane

= projection of projection center in other image

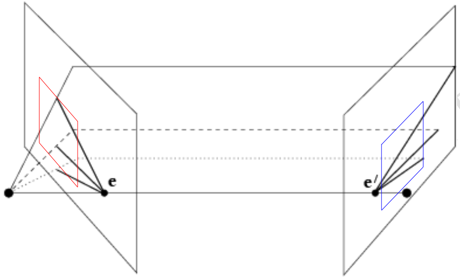
= vanishing point of camera motion direction

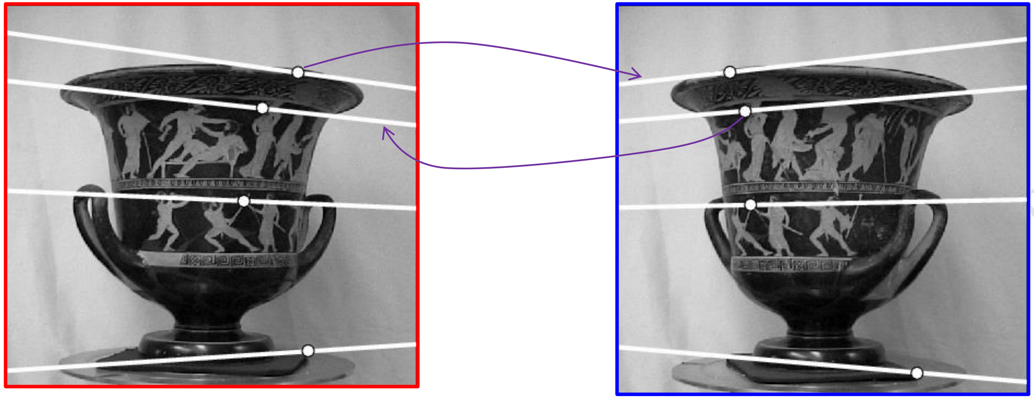
epipolar plane = plane containing baseline

epipolar line = intersection of epipolar plane with image

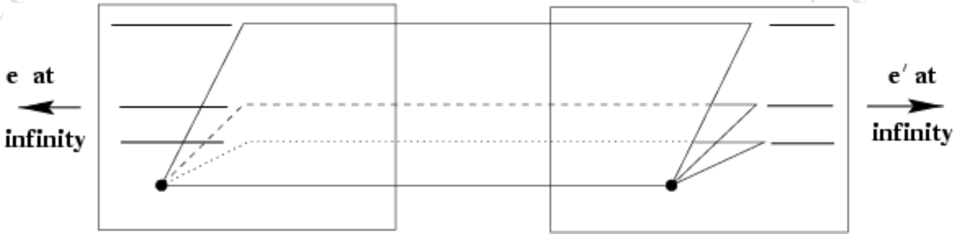


Example: converged stereo-camera

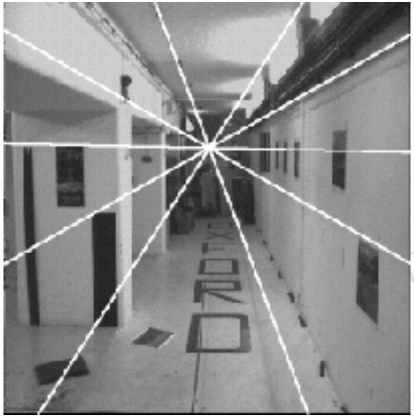


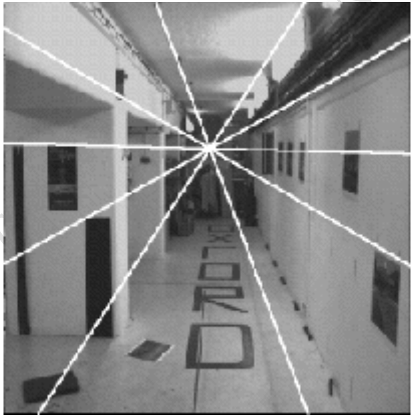


Example: motion parallel with image plane

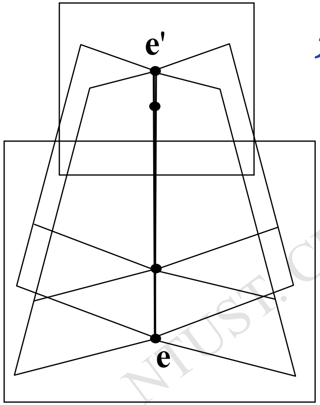


Example: forward motion





2 - View



2 nd

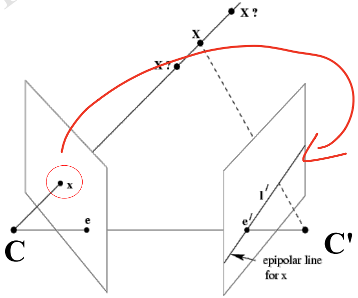
1 st

Fundamental matrix F

- Algebraic representation of epipolar geometry

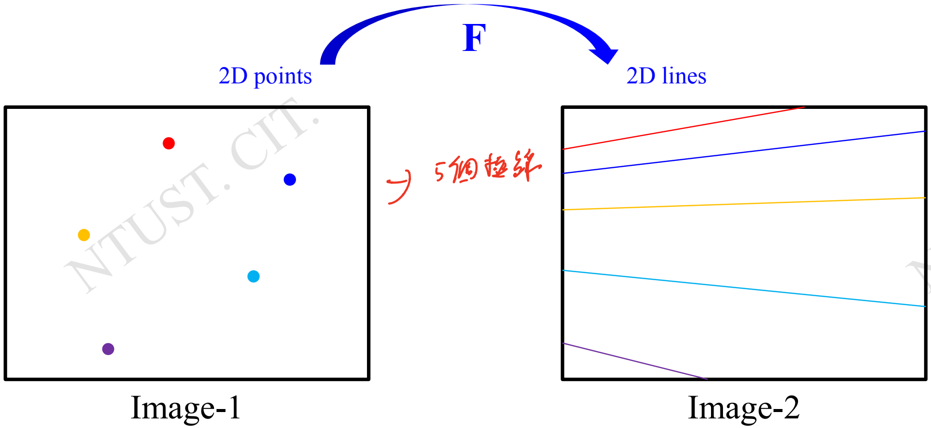
$$\mathbf{x} \mapsto \mathbf{l}'$$

- We will see that mapping is (singular) correlation (i.e. projective mapping from points to lines) represented by the fundamental matrix **F**.
- 註：已知兩張照片的F轉換，可從一張影像的特徵點，可以預估這些特徵點會落在另一張影像上的線上(一個點產生一條線)



Fundamental matrix F

- Points(or features) in image-1 are mapped into lines in image-2 by applying a 3x3 matrix **F**.
- Note!!! all points in image-1 are NOT necessary to be co-planar in 3D space. (different from 3x3 homography)



Fundamental matrix F

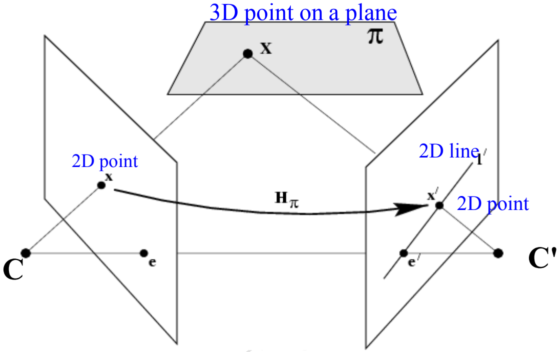
- How to determine **F** from two images?(from textbook Hartley04)
  - Using known 3x3 Homography (3D points on one plane) and the epipole
  - Algebraic method
  - Using known correspondences (feature matching between two images)→most popular method in practice

Determine fundamental matrix **F**

- from 3x3 homography
- from algebraic derivation
- from correspondence from two-views

Determine fundamental matrix F

1) from 3x3 homography



Note! notation

vector

real (實數,純量)

$$\mathbf{e}' = [e_1' \quad e_2' \quad e_3']^T$$

matrix (for calculation purpose)

real (實數,純量)

$$[\mathbf{e}']_\times = \begin{bmatrix} 0 & -e_3' & e_2' \\ e_3' & 0 & -e_1' \\ -e_2' & e_1' & 0 \end{bmatrix}$$

$$\mathbf{x}' = \mathbf{H}_\pi \mathbf{x}$$

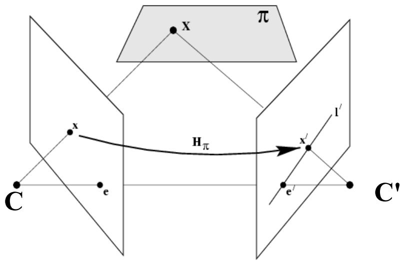
$$\underline{\mathbf{l}' = \mathbf{e}' \times \mathbf{x}' = \mathbf{e}' \times (\mathbf{H}_\pi \mathbf{x})} = ([\mathbf{e}']_\times \mathbf{H}_\pi)\mathbf{x} = \mathbf{F}\mathbf{x}$$

epipole (vector)

epipole (matrix)

Determine fundamental matrix F

1) from 3x3 homography—cont.



$$\mathbf{x}' = \mathbf{H}_\pi \mathbf{x}$$ →(3x3) homography mapping
2D points mapping to 2D points

$$\mathbf{l}' = \mathbf{F}\mathbf{x}$$ →2-Dimensional Mapping

Determine fundamental matrix F

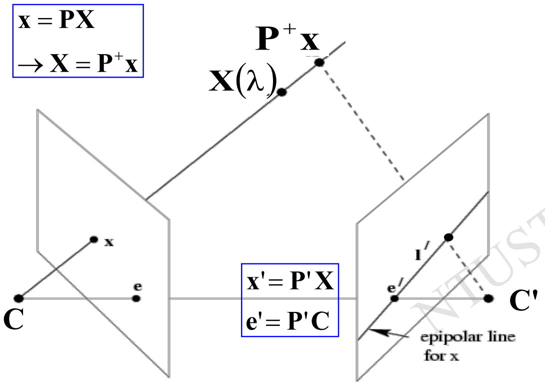
2) from algebraic derivation

$$\left(\mathbf{P}^{+}\mathbf{P}=\mathbf{I}\right)$$

$$\mathbf{X}(\lambda)=\mathbf{P}^{+}\mathbf{x}+\lambda\mathbf{C}$$

$$\mathbf{l}'=\mathbf{P}'\mathbf{C}\times\mathbf{P}'\mathbf{P}^{+}\mathbf{x}$$

$$\mathbf{F}=[\mathbf{e}']_{\times}\mathbf{P}'\mathbf{P}^{+}$$



This method is the same formula with the previous method, by replace $\mathbf{P}'\mathbf{P}^{+}$ with $\mathbf{H}_{\pi}$

Determine fundamental matrix F

3) from correspondence from two-views

- correspondence condition
- The fundamental matrix satisfies the condition that for any pair of corresponding points x↔x' in the two images

$l' = Fx$

(one point on one line could be written as:)

$$\mathbf{x}^T\mathbf{l} = 0 = \mathbf{l}^T\mathbf{x}$$

3x3

So, the governing equation will be

2nd at 1st

$$\boxed{\mathbf{x}'^T\mathbf{F}\mathbf{x} = 0}$$

line at 2nd

$x = \begin{bmatrix} x_1 \\ 3 \\ 1 \end{bmatrix}$

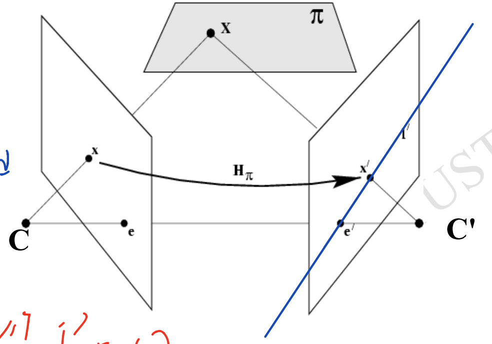
3x1

↳ 3x1 ⇒ line ⇒ epi

Since we have the following equation in image-2.

$$\because \mathbf{x}'^T\mathbf{l}' = 0$$

2nd 1st



$x'^T l' = 0$

Determine fundamental matrix F

3) from correspondence from two-views—cont.

- So called Weak calibration (for determining **F** in two views)

General form:

2D (1st)

2D (2nd)

$$\mathbf{x}'^{\mathrm{T}}\,\mathbf{F}\mathbf{x} = 0$$

⇒ line eq

8 DoF

9 unknown

Written in matrix form:

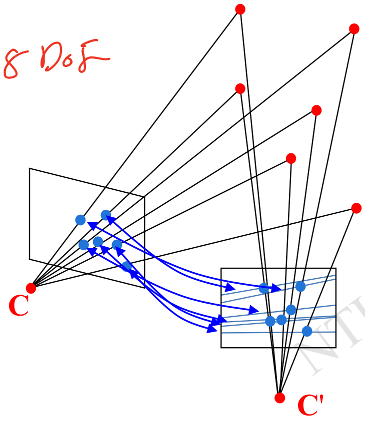
$$[u' \quad v' \quad 1]\begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix}\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = 0$$

2nd

1st

8 unknown

1

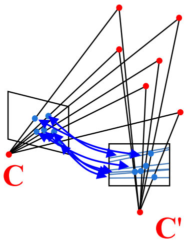


Determine fundamental matrix **F**

3) from correspondence from two-views—cont.

- Weak calibration

$$[u' \quad v' \quad 1]\begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix}\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = 0$$



$$\rightarrow \; F_{11}u'u + F_{12}u'v + F_{13}u' + F_{21}uv' + F_{22}vv' + F_{23}v' + F_{31}u + F_{32}v + \underset{\substack{\| \\ 1}}{F_{33}} = 0$$

$\hookrightarrow x'^T F x = 0$

Since **F** has 9-1 DOF, let F_{33}=1 and solving **F**.

In matrix form→

$$\boxed{[u'u \quad u'v \quad u' \quad uv' \quad vv' \quad v' \quad u \quad v]\begin{bmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \end{bmatrix} = -1}$$

8 unknowns, and one correspondence gives one constraint. It needs at least 8 correspondences.

Determine fundamental matrix F

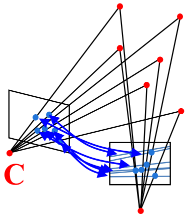
3) from correspondence from two-views—cont.

- Weak calibration

pixel

$$\mathbf{x}'^{T}\mathbf{F}\mathbf{x}=0$$

8 unknow



$$\begin{bmatrix} u_1'u_1 & u_1'v_1 & u_1' & u_1v_1' & v_1v_1' & v_1' & u_1 & v_1 \\ u_2'u_2 & u_2'v_2 & u_2' & u_2v_2' & v_2v_2' & v_2' & u_2 & v_2 \\ u_3'u_3 & u_3'v_3 & u_3' & u_3v_3' & v_3v_3' & v_3' & u_3 & v_3 \\ u_4'u_4 & u_4'v_4 & u_4' & u_4v_4' & v_4v_4' & v_4' & u_4 & v_4 \\ u_5'u_5 & u_5'v_5 & u_5' & u_5v_5' & v_5v_5' & v_5' & u_5 & v_5 \\ u_6'u_6 & u_6'v_6 & u_6' & u_6v_6' & v_6v_6' & v_6' & u_6 & v_6 \\ u_7'u_7 & u_7'v_7 & u_7' & u_7v_7' & v_7v_7' & v_7' & u_7 & v_7 \\ u_8'u_8 & u_8'v_8 & u_8' & u_8v_8' & v_8v_8' & v_8' & u_8 & v_8 \end{bmatrix} \begin{bmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$$

Solve **F** by taking an inverse operation to the above equation.
If you get more than 8 correspondences, least-square method or SVD may be used.
NOTE!! Since the order in every column is very different. The 1st column has values around 10^4~10^6, but the 3th column is 10^2~10^3. Without normalized, Least-Square-Method may yield poor results.

Property of the fundamental matrix F

$(AB)^T = B^T A^T$

- F is the unique 3x3 rank 2 matrix that satisfies $\mathbf{x}'^T\mathbf{F}\mathbf{x}=0$ for all $\mathbf{x}\leftrightarrow\mathbf{x}'$
  - Transpose: if **F** is fundamental matrix for the pair of cameras (**P**, **P'**), then $\mathbf{F}^T$ is fundamental matrix for (**P'**, **P**)
  - Epipolar lines: $\mathbf{l}'=\mathbf{F}\mathbf{x}$ & $\mathbf{l}=\mathbf{F}^T\mathbf{x}'$
  - Epipoles: on all epipolar lines, thus $\mathbf{e}'^T\mathbf{F}\mathbf{x}=0, \forall\mathbf{x} \Rightarrow \mathbf{e}'^T\mathbf{F}=0$, similarly $\mathbf{F}\mathbf{e}=0$
  - **F** has 7 DOF , i.e. 3x3-1(homogeneous)-1(rank2)
  - **F** is a correlation, projective mapping from a point **x** to a line $\mathbf{l}'=\mathbf{F}\mathbf{x}$ (not a proper correlation, i.e. not invertible)

$l \approx \underbrace{x'^T F}_{l} \overbrace{F x}^{l'} = 0$

$l = x'^T F$

$l^T = [x'^T F]^T = F^T x'$

Determine fundamental matrix **F**

$F \Rightarrow x'^T F x = 0$

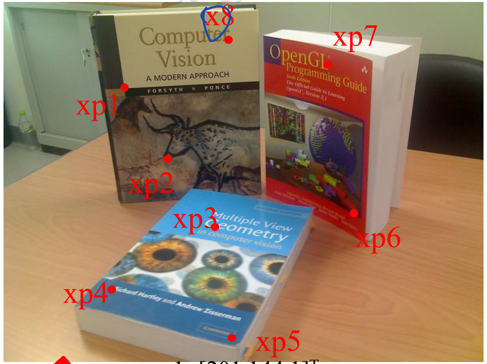
↓

pixel

- correspondence from two-views—example

3D

3D



3 點 退化

在直線上第3

點，並沒有貢獻

$x1=[227,212,1]^T$
$x2=[275,322,1]^T$
$x3=[449,370,1]^T$
$x4=[525,481,1]^T$
$x5=[699,432,1]^T$
$x6=[535,298,1]^T$
$x7=[498,118,1]^T$
$x8=[339,106,1]^T$

F?

Note! This is **NOT** 2D point to point mapping

$xp1=[201,144,1]^T$
$xp2=[275,261,1]^T$
$xp3=[349,369,1]^T$
$xp4=[182,479,1]^T$
$xp5=[380,562,1]^T$
$xp6=[584,351,1]^T$
$xp7=[542,108,1]^T$
$xp8=[373,64,1]^T$

Determine fundamental matrix F

- correspondence from two-views—example

$$\begin{bmatrix} u_1'u_1 & u_1'v_1 & u_1' & u_1v_1' & v_1v_1' & v_1' & u_1 & v_1 \\ u_2'u_2 & u_2'v_2 & u_2' & u_2v_2' & v_2v_2' & v_2' & u_2 & v_2 \\ u_3'u_3 & u_3'v_3 & u_3' & u_3v_3' & v_3v_3' & v_3' & u_3 & v_3 \\ u_4'u_4 & u_4'v_4 & u_4' & u_4v_4' & v_4v_4' & v_4' & u_4 & v_4 \\ u_5'u_5 & u_5'v_5 & u_5' & u_5v_5' & v_5v_5' & v_5' & u_5 & v_5 \\ u_6'u_6 & u_6'v_6 & u_6' & u_6v_6' & v_6v_6' & v_6' & u_6 & v_6 \\ u_7'u_7 & u_7'v_7 & u_7' & u_7v_7' & v_7v_7' & v_7' & u_7 & v_7 \\ u_8'u_8 & u_8'v_8 & u_8' & u_8v_8' & v_8v_8' & v_8' & u_8 & v_8 \end{bmatrix} \begin{bmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$$

8x8

x1=[227,212,1]^T
x2=[275,322,1]^T
x3=[449,370,1]^T
x4=[525,481,1]^T
x5=[699,432,1]^T
x6=[535,298,1]^T
x7=[498,118,1]^T
x8=[339,106,1]^T

xp1=[201,144,1]^T
xp2=[275,261,1]^T
xp3=[349,369,1]^T
xp4=[182,479,1]^T
xp5=[380,562,1]^T
xp6=[584,351,1]^T
xp7=[542,108,1]^T
xp8=[373,64,1]^T

```
A=[
xp1(1)*x1(1) xp1(1)*x1(2) xp1(1) x1(1)*xp1(2) x1(2)*xp1(2) xp1(2) x1(1) x1(2);
xp2(1)*x2(1) xp2(1)*x2(2) xp2(1) x2(1)*xp2(2) x2(2)*xp2(2) xp2(2) x2(1) x2(2);
xp3(1)*x3(1) xp3(1)*x3(2) xp3(1) x3(1)*xp3(2) x3(2)*xp3(2) xp3(2) x3(1) x3(2);
xp4(1)*x4(1) xp4(1)*x4(2) xp4(1) x4(1)*xp4(2) x4(2)*xp4(2) xp4(2) x4(1) x4(2);
xp5(1)*x5(1) xp5(1)*x5(2) xp5(1) x5(1)*xp5(2) x5(2)*xp5(2) xp5(2) x5(1) x5(2);
xp6(1)*x6(1) xp6(1)*x6(2) xp6(1) x6(1)*xp6(2) x6(2)*xp6(2) xp6(2) x6(1) x6(2);
xp7(1)*x7(1) xp7(1)*x7(2) xp7(1) x7(1)*xp7(2) x7(2)*xp7(2) xp7(2) x7(1) x7(2);
xp8(1)*x8(1) xp8(1)*x8(2) xp8(1) x8(1)*xp8(2) x8(2)*xp8(2) xp8(2) x8(1) x8(2)]
```

```
F =

    0.0000   -0.0000   -0.0007
   -0.0000    0.0000    0.0105
   -0.0011   -0.0093    1.0000
```

Determine fundamental matrix F

- correspondence from two-views—example, cont.

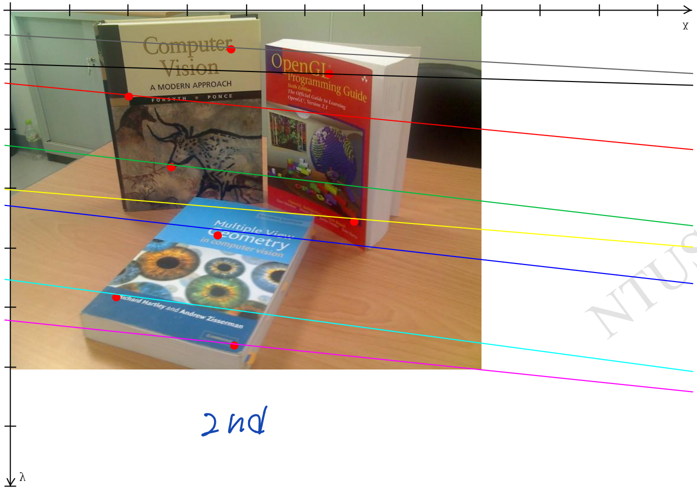
$$\mathbf{l'} = \mathbf{F}\mathbf{x}$$

F =

$$\begin{matrix} 0.0000 & -0.0000 & -0.0007 \\ -0.0000 & 0.0000 & 0.0105 \\ -0.0011 & -0.0093 & 1.0000 \end{matrix}$$

lp1 =	lp2 =	lp3 =	lp4 =	lp5 =	lp6 =	lp7 =	lp8 =
-0.0009	-0.0012	-0.0010	-0.0012	-0.0007	-0.0007	-0.0002	-0.0005
0.0098	0.0101	0.0089	0.0089	0.0073	0.0078	0.0071	0.0083
-1.2267	-2.3044	-2.9449	-4.0631	-3.8003	-2.3701	-0.6527	-0.3641

1st



2nd

Determine fundamental matrix F

- correspondence from two-views—example, cont.

$$\mathbf{l} = \mathbf{F}^T \mathbf{x}'$$

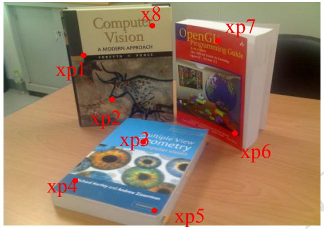
```
>> F'

ans =

  0.0000  -0.0000  -0.0011
 -0.0000   0.0000  -0.0093
 -0.0007   0.0105   1.0000
```

l1 =	l2 =	l3 =	l4 =	l5 =	l6 =	l7 =	l8 =
-0.0019	-0.0027	-0.0035	-0.0046	-0.0050	-0.0029	-0.0011	-0.0010
-0.0091	-0.0086	-0.0082	-0.0072	-0.0073	-0.0090	-0.0102	-0.0100
2.3599	3.5309	4.6076	5.8837	6.6067	4.2450	1.7302	1.3944

H
1st
H^{-1}
2nd

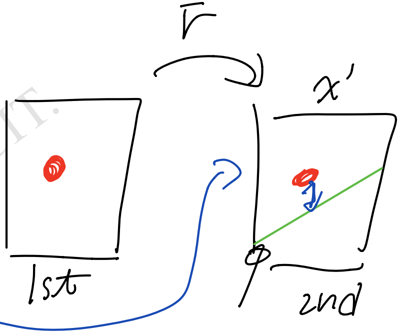
F
F^T

F^T



Determine fundamental matrix $\mathbf{F}$

- correspondence from two-views—example, cont.
  - Evaluation for error, for example check $\mathbf{l'}^T\mathbf{x'}$ or $\mathbf{l}^T\mathbf{x}$ or $\mathbf{x'}^T\mathbf{F}\mathbf{x}$



Fx

	$\mathbf{l'}^T\mathbf{x'}$	$\mathbf{l}^T\mathbf{x}$	$\mathbf{x'}^T\mathbf{F}\mathbf{x}$
1)	-2.4425e-015	-2.6645e-015	-2.6645e-015
2)	4.5741e-014	-3.5527e-015	-3.5527e-015
3)	-1.2390e-013	-4.4409e-015	-4.4409e-015
4)	-5.3291e-015	-5.3291e-015	-5.3291e-015
5)	-3.5527e-015	-3.5527e-015	-3.5527e-015
6)	-3.5527e-015	-3.5527e-015	-3.5527e-015
7)	-8.8818e-016	-8.8818e-016	-8.8818e-016
8)	-1.6653e-015	-1.5543e-015	-1.5543e-015
	2nd dist	1st dist	

$\Rightarrow x'Fx = 0$
φ

≤ 1 pixel

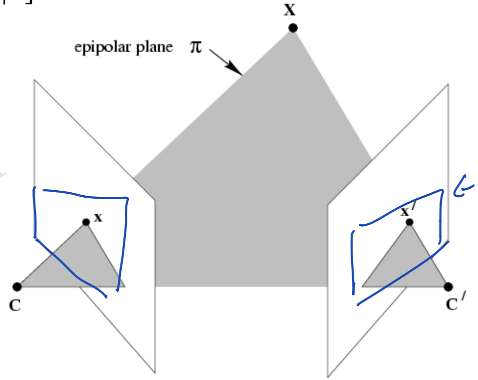
0.05

Essential matrix E

- The essential matrix is the specialization of the fundamental matrix to the case of normalized image coordinate. Historically, the essential matrix was introduced before the fundamental matrix, and the fundamental matrix may be thought of as the generalization of the essential matrix in which the assumption of calibrated cameras is removed.

3D

$$\mathbf{x} = \mathbf{P}\mathbf{X} = \mathbf{K}[\mathbf{R} \mid \mathbf{t}]\mathbf{X} \qquad \mathbf{x}' = \mathbf{P}'\mathbf{X} = \mathbf{K}'[\mathbf{R}' \mid \mathbf{t}']\mathbf{X}$$



F

← mm

$x'^T E x = 0$

physical unit : mm , inch

Essential matrix $\mathbf{E}$

Intrnic

- Normalized coordinates:
- Consider the image without $\mathbf{K}$ effect.

2D points on an image

2D points on a normalized image

$$\mathbf{x} = \mathbf{P}\mathbf{X} = \mathbf{K}[\mathbf{R} \mid \mathbf{t}]\mathbf{X} \quad \Rightarrow \quad \hat{\mathbf{x}} = \mathbf{K}^{-1}\mathbf{x} = \mathbf{K}^{-1}\mathbf{P}\mathbf{X} = [\mathbf{R} \mid \mathbf{t}]\mathbf{X}$$

$$\mathbf{x}' = \mathbf{P}'\mathbf{X} = \mathbf{K}'[\mathbf{R}' \mid \mathbf{t}']\mathbf{X} \quad \Rightarrow \quad \hat{\mathbf{x}}' = \mathbf{K}'^{-1}\mathbf{x}' = \mathbf{K}'^{-1}\mathbf{P}'\mathbf{X} = [\mathbf{R}' \mid \mathbf{t}']\mathbf{X}$$

$\mathbf{P} = \mathbf{K}[\mathbf{R} \mid \mathbf{t}]$ → general camera matrix

$\mathbf{K}^{-1}\mathbf{P} = [\mathbf{R} \mid \mathbf{t}]$ → normalized camera matrix

$$\boxed{\hat{\mathbf{x}}'^{T}\mathbf{E}\hat{\mathbf{x}} = 0}$$

similar to fundamental matrix format

→ $x'Fx = 0$

→ pixel

mm

$$\left.\begin{aligned} &\because \mathbf{x}'^{T}\mathbf{F}\mathbf{x} = 0 \\ &\hat{\mathbf{x}} = \mathbf{K}^{-1}\mathbf{x} \\ &\hat{\mathbf{x}}' = \mathbf{K}'^{-1}\mathbf{x}' \end{aligned}\right\} (\mathbf{K}'\hat{\mathbf{x}}')^{T}\mathbf{F}(\mathbf{K}\hat{\mathbf{x}}) = 0 \Rightarrow \hat{\mathbf{x}}'^{T}(\mathbf{K}'^{T}\mathbf{F}\mathbf{K})\hat{\mathbf{x}} = 0 \Rightarrow \underline{\mathbf{E} = \mathbf{K}'^{T}\mathbf{F}\mathbf{K}}$$

Essential matrix $\mathbf{E}$

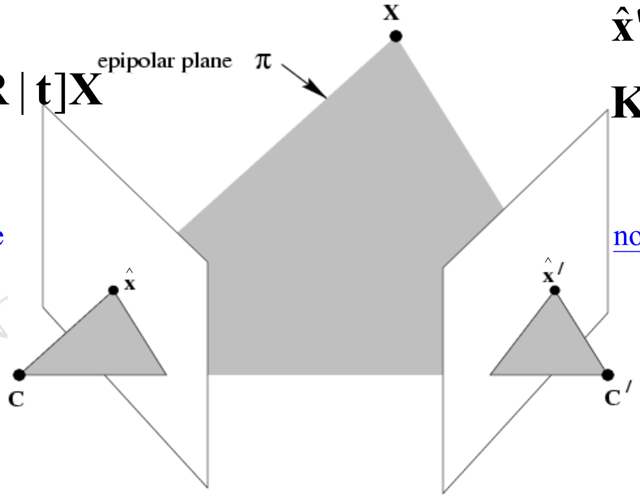
- Normalized coordinates:
- Consider the image without $\mathbf{K}$ effect.

$$\hat{\mathbf{x}} = \mathbf{K}^{-1}\mathbf{x} = \mathbf{K}^{-1}\mathbf{P}\mathbf{X} = [\mathbf{R}\,|\,\mathbf{t}]\mathbf{X}$$

normalized image

$$\hat{\mathbf{x}}' = \mathbf{K}'^{-1}\mathbf{x}' = \mathbf{K}'^{-1}\mathbf{P}'\mathbf{X} = [\mathbf{R}'\,|\,\mathbf{t}']\mathbf{X}$$

normalized image



Essential matrix E –example

- Continue the previous example:

F =

0.0000	-0.0000	-0.0007
-0.0000	0.0000	0.0105
-0.0011	-0.0093	1.0000

Assume we have intrinsic parameter of image 1&2:

K=[
857.249077 0.000000 402.813609
0.000000 866.660878 250.492920
0.000000 0.000000 1.000000]

Essential matrix can be determined by $\mathbf{E} = \mathbf{K}'^{T}\,\mathbf{F}\mathbf{K}$

Intrinsic parameter of **image-2**

Intrinsic parameter of **image-1**

Fundamental matrix from image-1 to image-2

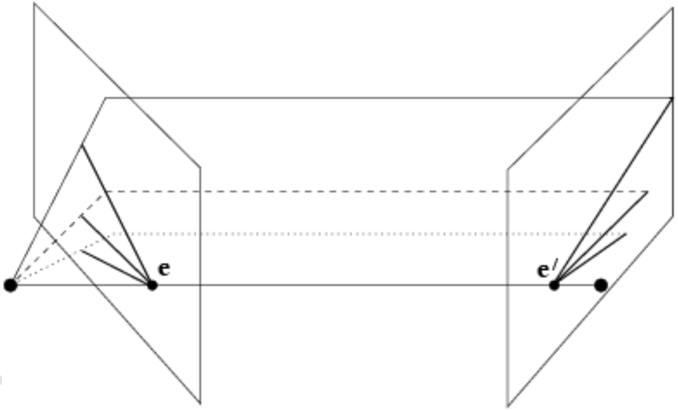
```
>> E=K'*F*K
```

E =

1.2509	-2.0515	-0.6399
-5.9498	4.1481	7.4831
-2.0853	-7.8357	0.0815

Epipolar geometry

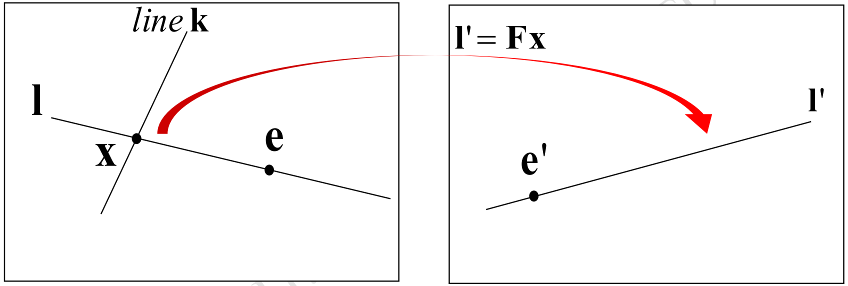
- Review:



$$\mathbf{l} = \mathbf{F}^{\mathrm{T}}\mathbf{x}' \qquad \mathbf{l}' = \mathbf{F}\mathbf{x}$$

Epipolar geometry-epipolar line homography

- **l**, **l'** epipolar lines in left and right images.



Suppose **l** and **l'** are corresponding epipolar lines, and **k** is ANY "line" NOT passing through the epipole **e**, then **l** and **l'** are related by

$$\mathbf{l}' = \mathbf{F}[\mathbf{k}]_\times \mathbf{l} \quad \rightarrow \quad \because \mathbf{k}^T\mathbf{e} \neq 0, \quad \mathbf{e}^T\mathbf{e} \neq 0$$

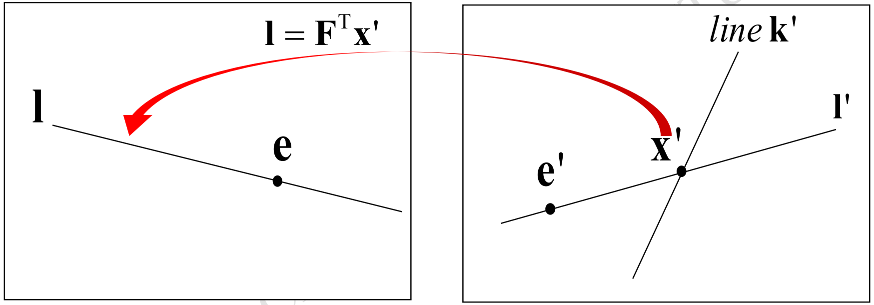
$$\boxed{\mathbf{l}' = \mathbf{F}[\mathbf{e}]_\times \mathbf{l}}$$

→Note: **e,** here, means the LINE for convenience not passing through the epipole **e**. So, we say **e** is one choice of line **k**.

Hartley04, sec.9.2.5

Epipolar geometry-epipolar line homography

- $\mathbf{l}$, $\mathbf{l}'$ epipolar lines in left and right images.

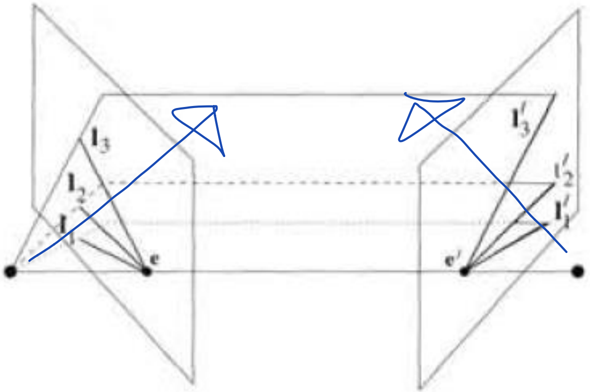


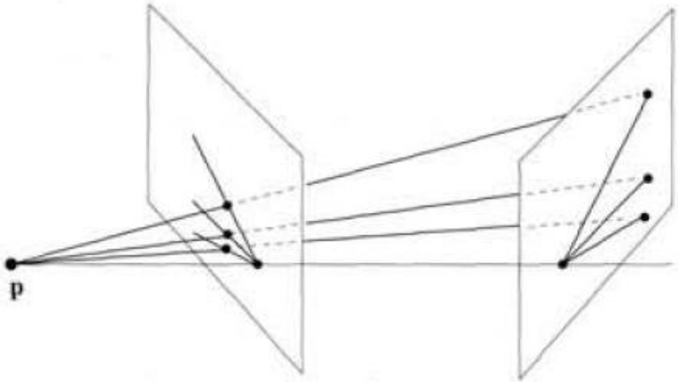
$$\boxed{\mathbf{l}' = \mathbf{F}\underbrace{[\mathbf{e}]_\times}\mathbf{l}} \qquad \boxed{\mathbf{l} = \mathbf{F}^T\underbrace{[\mathbf{e}']_\times}\mathbf{l}'}$$

A homography between two lines

Hartley04, sec.9.2.5

Epipolar geometry-epipolar line homography

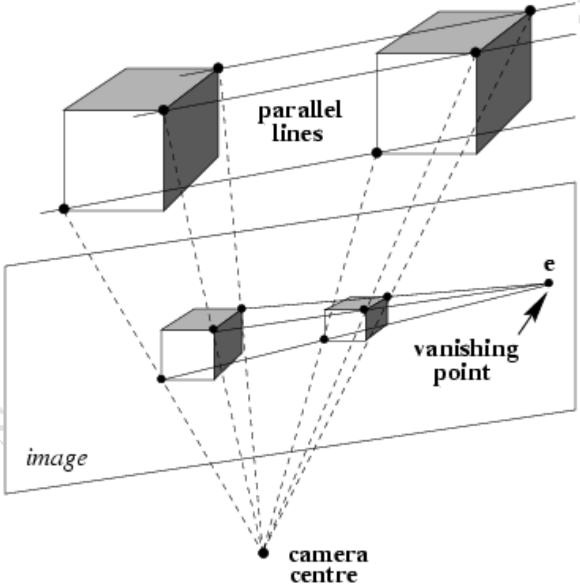




p is any point on the baseline

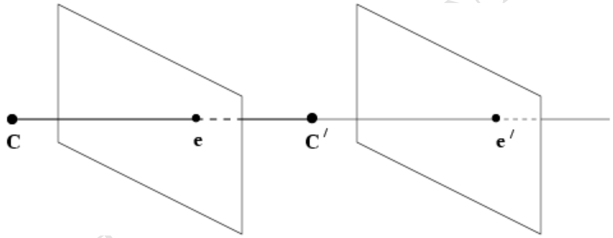
Fundamental matrix for pure translation

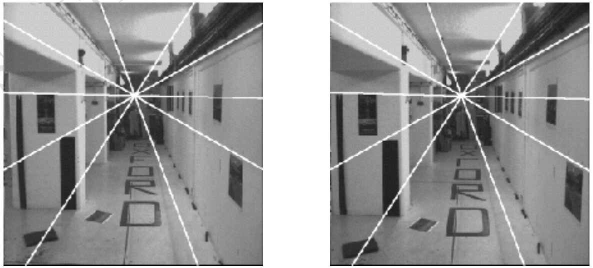
- Side direction



Fundamental matrix for pure translation

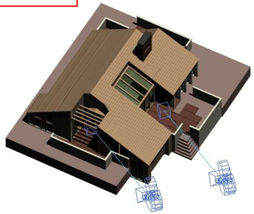
- Forward and backward





Fundamental matrix—example1

$x1=[375,219,1]^T$	$xp1=[108,219,1]^T$
$x2=[405,263,1]^T$	$xp2=[100,263,1]^T$
$x3=[433,560,1]^T$	$xp3=[123,559,1]^T$
$x4=[630,66,1]^T$	$xp4=[370,65,1]^T$
$x5=[678,96,1]^T$	$xp5=[452,96,1]^T$
$x6=[698,323,1]^T$	$xp6=[448,324,1]^T$
$x7=[696,367,1]^T$	$xp7=[403,367,1]^T$
$x8=[741,421,1]^T$	$xp8=[458,421,1]^T$

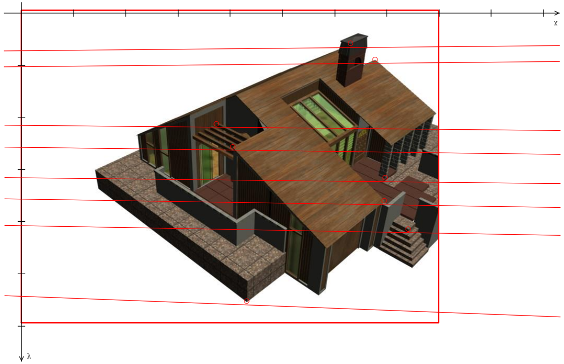
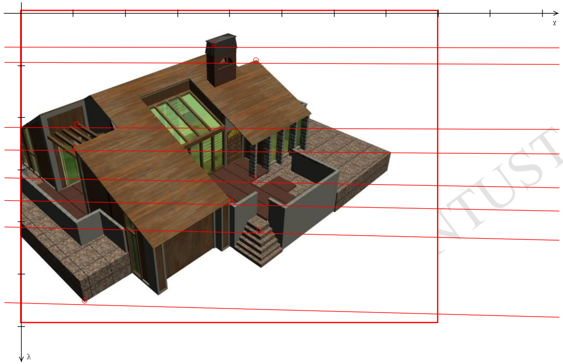


Fundamental matrix—example1, cont.

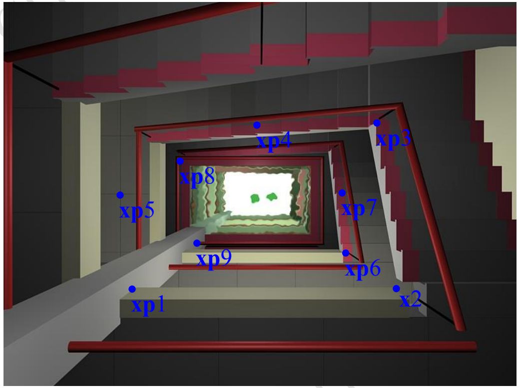
$(10^3)^2 \Rightarrow 10^6$

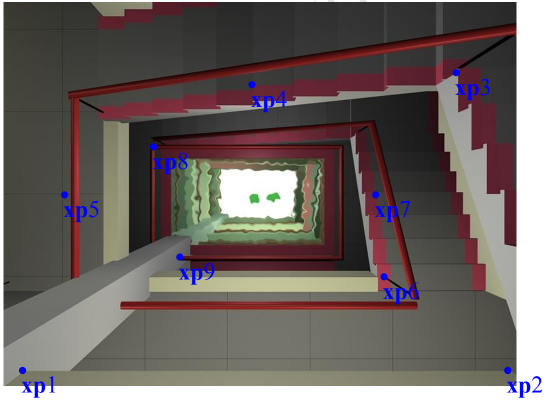
```
A=[
xp1(1)*x1(1) xp1(1)*x1(2) xp1(1) xp1(2)*x1(1) xp1(2)*x1(2) xp1(2) x1(1) x1(2) ;
xp2(1)*x2(1) xp2(1)*x2(2) xp2(1) xp2(2)*x2(1) xp2(2)*x2(2) xp2(2) x2(1) x2(2) ;
xp3(1)*x3(1) xp3(1)*x3(2) xp3(1) xp3(2)*x3(1) xp3(2)*x3(2) xp3(2) x3(1) x3(2) ;
xp4(1)*x4(1) xp4(1)*x4(2) xp4(1) xp4(2)*x4(1) xp4(2)*x4(2) xp4(2) x4(1) x4(2) ;
xp5(1)*x5(1) xp5(1)*x5(2) xp5(1) xp5(2)*x5(1) xp5(2)*x5(2) xp5(2) x5(1) x5(2) ;
xp6(1)*x6(1) xp6(1)*x6(2) xp6(1) xp6(2)*x6(1) xp6(2)*x6(2) xp6(2) x6(1) x6(2) ;
xp7(1)*x7(1) xp7(1)*x7(2) xp7(1) xp7(2)*x7(1) xp7(2)*x7(2) xp7(2) x7(1) x7(2) ;
xp8(1)*x8(1) xp8(1)*x8(2) xp8(1) xp8(2)*x8(1) xp8(2)*x8(2) xp8(2) x8(1) x8(2) ];
d=[-1 -1 -1 -1 -1 -1 -1 -1]';
f=inv(A)*d;
F=[f(1:3)';f(4:6)';f(7:8)' 1]
```

$\Rightarrow [\quad]^{-1}$

Fundamental matrix—example2

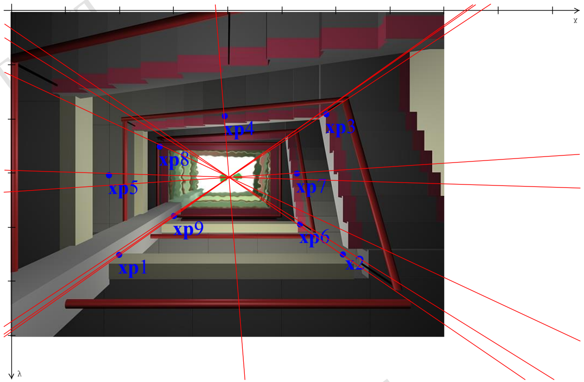


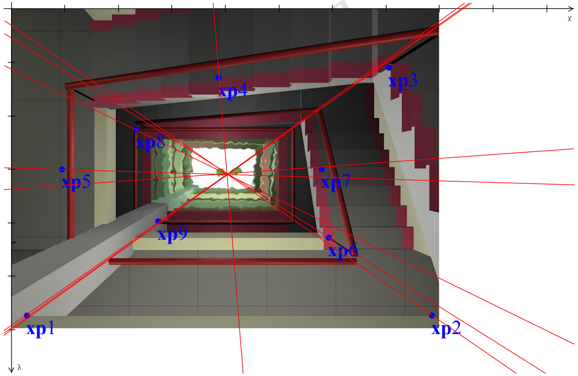


$x1=[207,446,1]^T$
$x2=[605,446,1]^T$
$x3=[586,182,1]^T$
$x4=[393,191,1]^T$
$x5=[182,301,1]^T$
$x6=[535,390,1]^T$
$x7=[532,299,1]^T$
$x8=[274,246,1]^T$
$x9=[303,377,1]^T$

$xp1=[34,577,1]^T$
$xp2=[790,577,1]^T$
$xp3=[705,105,1]^T$
$xp4=[389,131,1]^T$
$xp5=[92,300,1]^T$
$xp6=[592,428,1]^T$
$xp7=[581,297,1]^T$
$xp8=[236,230,1]^T$
$xp9=[275,401,1]^T$

Fundamental matrix—example2, cont.





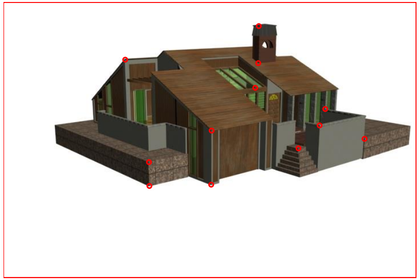
Solve it by OpenCV
F=[-0.000001 0.000648 -0.199148
-0.000636 -0.000002 0.256521
0.197009 -0.260813 1.000000]

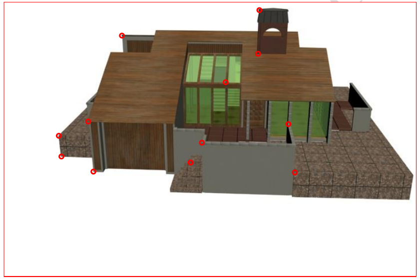
$\mathbf{l}_1^T=$	-0.1700	-0.2399	142.2416
$\mathbf{l}_2^T=$	-0.1708	0.2500	-8.3143
⋮	0.1295	0.1958	-112.4646
	0.1133	-0.0090	-42.8643
	0.0061	-0.2018	59.6347
	-0.0758	0.1219	-7.1046
	0.0075	0.1151	-38.5183
	0.0505	-0.1083	13.0009
$\mathbf{l}_9^T=$	-0.0583	-0.0834	49.0992

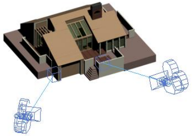
$\mathbf{l'}_1^T=$	0.0897	0.1240	-74.5417
$\mathbf{l'}_2^T=$	0.0893	-0.1292	3.8678
⋮	-0.0818	-0.1165	68.9793
	-0.0758	0.0062	28.6093
	-0.0043	0.1402	-41.6491
	0.0530	-0.0845	4.6827
	-0.0059	-0.0824	27.8257
	-0.0400	0.0818	-9.1795
$\mathbf{l'}_9^T=$	0.0448	0.0631	-37.6328

Fundamental matrix—example3

pinhole

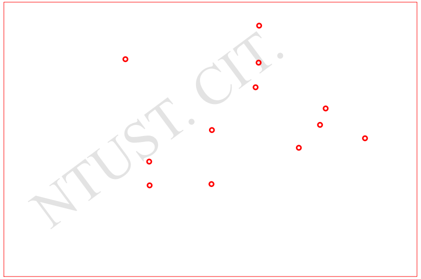


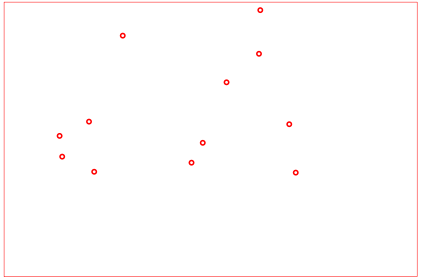




$x'Fx = 0$

x1=[211,99,1]^T
x2=[252,278,1]^T
x3=[253,320,1]^T
x4=[362,318,1]^T
x5=[362,223,1]^T
x6=[514,255,1]^T
x7=[630,238,1]^T
x8=[551,214,1]^T
x9=[561,185,1]^T
x10=[437,148,1]^T
x11=[444,105,1]^T
x12=[445,39,1]^T



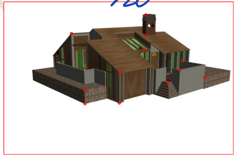


xp1=[205,57,1]^T
xp2=[95,233,1]^T
xp3=[99,270,1]^T
xp4=[156,296,1]^T
xp5=[146,210,1]^T
xp6=[325,279,1]^T
xp7=[507,296,1]^T
xp8=[345,246,1]^T
xp9=[496,211,1]^T
xp10=[386,140,1]^T
xp11=[442,90,1]^T
xp12=[445,12,1]^T

Fundamental matrix—example3, cont.

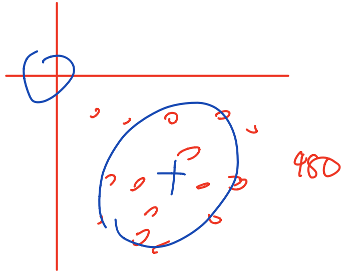
320

480



x

normalized



$x1=[211,99,1]^T$
$x2=[252,278,1]^T$
$x3=[253,320,1]^T$
$x4=[362,318,1]^T$
$x5=[362,223,1]^T$
$x6=[514,255,1]^T$
$x7=[630,238,1]^T$
$x8=[551,214,1]^T$
$x9=[561,185,1]^T$
$x10=[437,148,1]^T$
$x11=[444,105,1]^T$
$x12=[445,39,1]^T$

	A	B	C	D	E	F	G	H	I	J
1	x						length		nx	
2	211	99		-207.5	-102.8333		231.5836		-2.03549	-1.00875
3	252	278		-166.5	76.16667		183.0945		-1.63329	0.747163
4	253	320		-165.5	118.1667		203.3559		-1.62348	1.159165
5	362	318		-56.5	116.1667		129.178		-0.55424	1.139545
6	362	223		-56.5	21.16667		60.33471		-0.55424	0.207636
7	514	255		95.5	53.16667		109.3021		0.936814	0.521542
8	630	238		211.5	36.16667		214.57		2.074725	0.35478
9	551	214		132.5	12.16667		133.0574		1.299769	0.11935
10	561	185		142.5	-16.83333		143.4908		1.397864	-0.16513
11	437	148		18.5	-53.83333		56.92344		0.181477	-0.52808
12	444	105		25.5	-96.83333		100.1346		0.250144	-0.94989
13	445	39		26.5	-162.8333		164.9756		0.259954	-1.59733
14	Average									
15	418.5	201.83333		0	3.27E-13		144.1667			

Scale

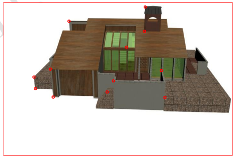
$$\mathbf{T} = \begin{bmatrix} \frac{\sqrt{2}}{144.1667} & 0 & 0 \\ 0 & \frac{\sqrt{2}}{144.1667} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -418.5 \\ 0 & 1 & -201.8333 \\ 0 & 0 & 1 \end{bmatrix}$$

T =

$$\begin{matrix} 0.0098 & 0 & -4.1053 \\ 0 & 0.0098 & -1.9799 \\ 0 & 0 & 1.0000 \end{matrix}$$

Fundamental matrix—example3, cont.

reduce computer loading



normalized

$xp1=[205,57,1]^T$
$xp2=[95,233,1]^T$
$xp3=[99,270,1]^T$
$xp4=[156,296,1]^T$
$xp5=[146,210,1]^T$
$xp6=[325,279,1]^T$
$xp7=[507,296,1]^T$
$xp8=[345,246,1]^T$
$xp9=[496,211,1]^T$
$xp10=[386,140,1]^T$
$xp11=[442,90,1]^T$
$xp12=[445,12,1]^T$

	A	B	C	D	E	F	G	H	I	J
1	xp						length		nxp	
2	205	57		-98.9167	-138		169.789596		-0.83388	-1.16336
3	95	233		-208.917	38		212.344469		-1.76119	0.320345
4	99	270		-204.917	75		218.210541		-1.72747	0.63226
5	156	296		-147.917	101		179.109855		-1.24696	0.851443
6	146	210		-157.917	15		158.627468		-1.33126	0.126452
7	325	279		21.08333	84		86.6054672		0.177735	0.708131
8	507	296		203.0833	101		226.812346		1.712019	0.851443
9	345	246		41.08333	51		65.4892379		0.346338	0.429937
10	496	211		192.0833	16		192.748559		1.619287	0.134882
11	386	140		82.08333	-55		98.8062428		0.691973	-0.46366
12	442	90		138.0833	-105		173.470479		1.16406	-0.88516
13	445	12		141.0833	-183		231.070351		1.189351	-1.54271
14	Average									
15	303.9167	195		6.44E-13	0		167.757051			

$$\mathbf{T}'=\begin{bmatrix} \frac{\sqrt{2}}{167.757051} & 0 & 0 \\ 0 & \frac{\sqrt{2}}{167.757051} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -303.9167 \\ 0 & 1 & -195 \\ 0 & 0 & 1 \end{bmatrix}$$

TP =

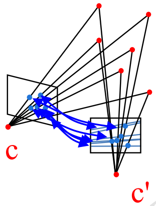
0.0084	0	-2.5621
0	0.0084	-1.6439
0	0	1.0000

Fundamental matrix—example3, cont.

- In this example, we have 12 correspondences (over-determine than 8), solve it by SVD.

$$\mathbf{x}'^{T}\mathbf{F}\mathbf{x}=0 \rightarrow [u' \quad v' \quad 1]\begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix}\begin{bmatrix} u \\ v \\ 1 \end{bmatrix}=0$$

$$\rightarrow F_{11}u'u+F_{12}u'v+F_{13}u'+F_{21}uv'+F_{22}vv'+F_{23}v'+F_{31}u+F_{32}v+F_{33}=0$$



$$[u'u \quad u'v \quad u' \quad uv' \quad vv' \quad v' \quad u \quad v \quad 1]\begin{bmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{bmatrix}=0$$

3X3 (Full)

Full 3x3

Fundamental matrix—example3, cont.

- NOTE! Data is normalized! So far, we are determining $\hat{\mathbf{F}}$

```
A=[
nxp1(1)*nx1(1) nxp1(1)*nx1(2) nxp1(1) nxp1(2)*nx1(1) nxp1(2)*nx1(2) nxp1(2) nx1(1) nx1(2) 1;
nxp2(1)*nx2(1) nxp2(1)*nx2(2) nxp2(1) nxp2(2)*nx2(1) nxp2(2)*nx2(2) nxp2(2) nx2(1) nx2(2) 1;
nxp3(1)*nx3(1) nxp3(1)*nx3(2) nxp3(1) nxp3(2)*nx3(1) nxp3(2)*nx3(2) nxp3(2) nx3(1) nx3(2) 1;
nxp4(1)*nx4(1) nxp4(1)*nx4(2) nxp4(1) nxp4(2)*nx4(1) nxp4(2)*nx4(2) nxp4(2) nx4(1) nx4(2) 1;
nxp5(1)*nx5(1) nxp5(1)*nx5(2) nxp5(1) nxp5(2)*nx5(1) nxp5(2)*nx5(2) nxp5(2) nx5(1) nx5(2) 1;
nxp6(1)*nx6(1) nxp6(1)*nx6(2) nxp6(1) nxp6(2)*nx6(1) nxp6(2)*nx6(2) nxp6(2) nx6(1) nx6(2) 1;
nxp7(1)*nx7(1) nxp7(1)*nx7(2) nxp7(1) nxp7(2)*nx7(1) nxp7(2)*nx7(2) nxp7(2) nx7(1) nx7(2) 1;
nxp8(1)*nx8(1) nxp8(1)*nx8(2) nxp8(1) nxp8(2)*nx8(1) nxp8(2)*nx8(2) nxp8(2) nx8(1) nx8(2) 1;
nxp9(1)*nx9(1) nxp9(1)*nx9(2) nxp9(1) nxp9(2)*nx9(1) nxp9(2)*nx9(2) nxp9(2) nx9(1) nx9(2) 1;
nxp10(1)*nx10(1) nxp10(1)*nx10(2) nxp10(1) nxp10(2)*nx10(1) nxp10(2)*nx10(2) nxp10(2) nx10(1) nx10(2) 1;
nxp11(1)*nx11(1) nxp11(1)*nx11(2) nxp11(1) nxp11(2)*nx11(1) nxp11(2)*nx11(2) nxp11(2) nx11(1) nx11(2) 1;
nxp12(1)*nx12(1) nxp12(1)*nx12(2) nxp12(1) nxp12(2)*nx12(1) nxp12(2)*nx12(2) nxp12(2) nx12(1) nx12(2) 1]

[U,S,V]=svd(A)
```

```
>> Fh=[V(1:3,9)';V(4:6,9)';V(7:9,9)']

Fh =
```

$$\hat{\mathbf{F}} = \begin{matrix} 0.0058 & 0.0290 & -0.0303 \\ 0.0353 & -0.0197 & -0.7377 \\ 0.1644 & 0.6520 & 0.0134 \end{matrix}$$

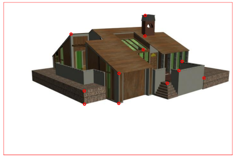
Denormalized for **F**

```
>> F=TP'*Fh*T

F =
```

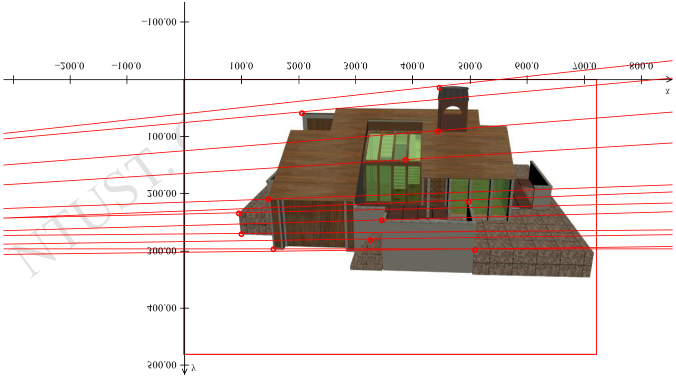
$$\mathbf{F} = \mathbf{T'}^{\mathrm{T}} \hat{\mathbf{F}} \mathbf{T} = \begin{matrix} 0.0000 & 0.0000 & -0.0009 \\ 0.0000 & -0.0000 & -0.0071 \\ 0.0009 & 0.0060 & -0.2791 \end{matrix}$$

Fundamental matrix—example3, cont.

- Find epipolar lines in the 2nd image for points of 1st image



>> F*x1	>> F*x2	>> F*x3	>> F*x4	>> F*x5	>> F*x6	>> F*x7	>> F*x8	>> F*x9	>> F*x10	>> F*x11	>> F*x12
ans =	ans =	ans =	ans =	ans =	ans =	ans =	ans =	ans =	ans =	ans =	ans =
-0.0006	-0.0002	-0.0001	-0.0000	-0.0002	-0.0001	-0.0001	-0.0002	-0.0002	-0.0004	-0.0005	-0.0006
-0.0067	-0.0068	-0.0069	-0.0066	-0.0064	-0.0060	-0.0057	-0.0059	-0.0058	-0.0061	-0.0060	-0.0059
0.5025	1.6102	1.8624	1.9482	1.3798	1.7077	1.7100	1.4955	1.3310	0.9984	0.7474	0.3534



12 点

12 line (2nd)

Fundamental matrix—example3, cont.

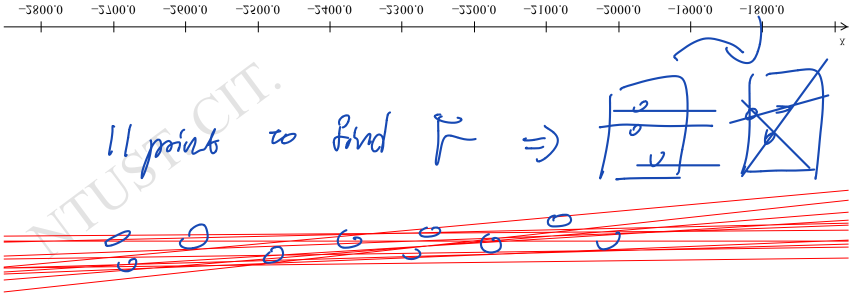
12-28 unknow

- Estimate the error, by $\mathbf{x}'^{T}\mathbf{F}\mathbf{x}$

>> xp1'*F*x1	>> ~~xp2'*F*x2~~	>> xp3'*F*x3	>> xp4'*F*x4	>> xp5'*F*x5	>> xp6'*F*x6	>> xp7'*F*x7	>> xp8'*F*x8	>> xp9'*F*x9	>> xp10'*F*x10	>> xp11'*F*x11	>> xp12'*F*x12
ans =	ans =	ans =	ans =	ans =	ans =	ans =	ans =	ans =	ans =	ans =	ans =
-5.3137e-004	~~0.0045~~	-0.0042	0.0020	-0.0020	-5.4760e-004	4.2649e-004	-2.6209e-004	-5.8426e-004	0.0023	-0.0018	7.5518e-004

Correspondences 2 has large error than others, let remove them then re-calculate (see next slide)

11 point to find F =>



[ ]
3X3

Fundamental matrix—example3, cont.

```
A=[
nxp1(1)*nx1(1) nxp1(1)*nx1(2) nxp1(1) nxp1(2)*nx1(1) nxp1(2)*nx1(2) nxp1(2) nx1(1) nx1(2) 1;
nxp3(1)*nx3(1) nxp3(1)*nx3(2) nxp3(1) nxp3(2)*nx3(1) nxp3(2)*nx3(2) nxp3(2) nx3(1) nx3(2) 1;
nxp4(1)*nx4(1) nxp4(1)*nx4(2) nxp4(1) nxp4(2)*nx4(1) nxp4(2)*nx4(2) nxp4(2) nx4(1) nx4(2) 1;
nxp5(1)*nx5(1) nxp5(1)*nx5(2) nxp5(1) nxp5(2)*nx5(1) nxp5(2)*nx5(2) nxp5(2) nx5(1) nx5(2) 1;
nxp6(1)*nx6(1) nxp6(1)*nx6(2) nxp6(1) nxp6(2)*nx6(1) nxp6(2)*nx6(2) nxp6(2) nx6(1) nx6(2) 1;
nxp7(1)*nx7(1) nxp7(1)*nx7(2) nxp7(1) nxp7(2)*nx7(1) nxp7(2)*nx7(2) nxp7(2) nx7(1) nx7(2) 1;
nxp8(1)*nx8(1) nxp8(1)*nx8(2) nxp8(1) nxp8(2)*nx8(1) nxp8(2)*nx8(2) nxp8(2) nx8(1) nx8(2) 1;
nxp9(1)*nx9(1) nxp9(1)*nx9(2) nxp9(1) nxp9(2)*nx9(1) nxp9(2)*nx9(2) nxp9(2) nx9(1) nx9(2) 1;
nxp10(1)*nx10(1) nxp10(1)*nx10(2) nxp10(1) nxp10(2)*nx10(1) nxp10(2)*nx10(2) nxp10(2) nx10(1) nx10(2) 1;
nxp11(1)*nx11(1) nxp11(1)*nx11(2) nxp11(1) nxp11(2)*nx11(1) nxp11(2)*nx11(2) nxp11(2) nx11(1) nx11(2) 1;
nxp12(1)*nx12(1) nxp12(1)*nx12(2) nxp12(1) nxp12(2)*nx12(1) nxp12(2)*nx12(2) nxp12(2) nx12(1) nx12(2) 1]
```

```
[U,S,V]=svd(A)
Fh=[V(1:3,9)';V(4:6,9)';V(7:9,9)']
F=TP'*Fh*T
```

>> xp1'*F*x1	>> xp3'*F*x3	>> xp4'*F*x4	>> xp5'*F*x5	>> xp6'*F*x6	>> xp7'*F*x7	>> xp8'*F*x8	>> xp9'*F*x9	>> xp10'*F*x10	>> xp11'*F*x11	>> xp12'*F*x12
ans =	ans =	ans =	ans =	ans =	ans =	ans =	ans =	ans =	ans =	ans =
-9.0214e-005	9.6499e-004	-0.0019	2.3422e-004	0.0025	-4.8966e-004	-7.3015e-004	6.9723e-005	-0.0015	8.8603e-004	5.1497e-005

Fundamental matrix—example3, cont.

```
A=[
nxp1(1)*nx1(1) nxp1(1)*nx1(2) nxp1(1) nxp1(2)*nx1(1) nxp1(2)*nx1(2) nxp1(2) nx1(1) nx1(2) 1;
nxp3(1)*nx3(1) nxp3(1)*nx3(2) nxp3(1) nxp3(2)*nx3(1) nxp3(2)*nx3(2) nxp3(2) nx3(1) nx3(2) 1;
nxp4(1)*nx4(1) nxp4(1)*nx4(2) nxp4(1) nxp4(2)*nx4(1) nxp4(2)*nx4(2) nxp4(2) nx4(1) nx4(2) 1;
nxp5(1)*nx5(1) nxp5(1)*nx5(2) nxp5(1) nxp5(2)*nx5(1) nxp5(2)*nx5(2) nxp5(2) nx5(1) nx5(2) 1;
nxp7(1)*nx7(1) nxp7(1)*nx7(2) nxp7(1) nxp7(2)*nx7(1) nxp7(2)*nx7(2) nxp7(2) nx7(1) nx7(2) 1;
nxp8(1)*nx8(1) nxp8(1)*nx8(2) nxp8(1) nxp8(2)*nx8(1) nxp8(2)*nx8(2) nxp8(2) nx8(1) nx8(2) 1;
nxp9(1)*nx9(1) nxp9(1)*nx9(2) nxp9(1) nxp9(2)*nx9(1) nxp9(2)*nx9(2) nxp9(2) nx9(1) nx9(2) 1;
nxp10(1)*nx10(1) nxp10(1)*nx10(2) nxp10(1) nxp10(2)*nx10(1) nxp10(2)*nx10(2) nxp10(2) nx10(1) nx10(2) 1;
nxp11(1)*nx11(1) nxp11(1)*nx11(2) nxp11(1) nxp11(2)*nx11(1) nxp11(2)*nx11(2) nxp11(2) nx11(1) nx11(2) 1;
nxp12(1)*nx12(1) nxp12(1)*nx12(2) nxp12(1) nxp12(2)*nx12(1) nxp12(2)*nx12(2) nxp12(2) nx12(1) nx12(2) 1]
```

```
[U,S,V]=svd(A)
Fh=[V(1:3,9)';V(4:6,9)';V(7:9,9)']
F=TP'*Fh*T
```

```
>> xp1'*F*x1

ans =

 -4.4279e-005
```

```
>> xp3'*F*x3

ans =

 -1.5922e-004
```

```
>> xp4'*F*x4

ans =

  1.1958e-004
```

```
>> xp5'*F*x5

ans =

  2.4719e-004
```

```
>> xp7'*F*x7

ans =

  1.0935e-004
```

```
>> xp8'*F*x8

ans =

 -2.8128e-004
```

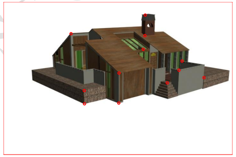
```
>> xp9'*F*x9

ans =

 -1.6777e-005
```

```
>> xp10'*F*x10

ans =

 -1.5967e-004
```

```
>> xp11'*F*x11

ans =

  2.4016e-004
```

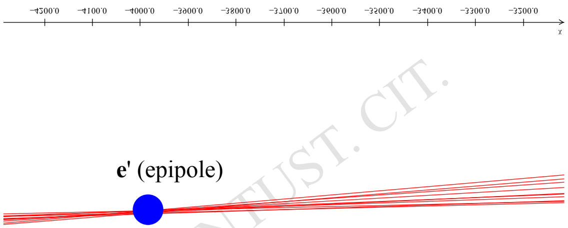
```
>> xp12'*F*x12

ans =

 -5.5050e-005
```

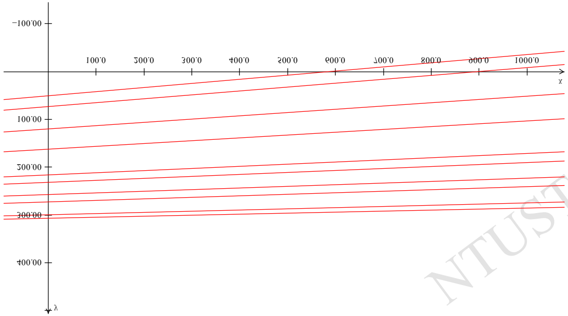
Fundamental matrix—example3, cont.



Find epipolar lines in the 2nd image for points of 1st image (10 correspondences)

>> F*x1	>> F*x3	>> F*x4	>> F*x5	>> F*x7	>> F*x8	>> F*x9	>> F*x10	>> F*x11	>> F*x12
ans =	ans =	ans =	ans =	ans =	ans =	ans =	ans =	ans =	ans =
0.0006	0.0002	0.0002	0.0003	0.0001	0.0002	0.0002	0.0004	0.0004	0.0005
0.0067	0.0071	0.0067	0.0065	0.0056	0.0058	0.0057	0.0060	0.0059	0.0058
-0.4978	-1.9467	-2.0165	-1.4073	-1.7067	-1.4929	-1.3145	-0.9833	-0.7129	-0.2904





Fundamental matrix—example3, cont.

```
A=[
nxp1(1)*nx1(1) nxp1(1)*nx1(2) nxp1(1) nxp1(2)*nx1(1) nxp1(2)*nx1(2) nxp1(2) nx1(1) nx1(2) 1;
nxp3(1)*nx3(1) nxp3(1)*nx3(2) nxp3(1) nxp3(2)*nx3(1) nxp3(2)*nx3(2) nxp3(2) nx3(1) nx3(2) 1;
nxp4(1)*nx4(1) nxp4(1)*nx4(2) nxp4(1) nxp4(2)*nx4(1) nxp4(2)*nx4(2) nxp4(2) nx4(1) nx4(2) 1;
nxp5(1)*nx5(1) nxp5(1)*nx5(2) nxp5(1) nxp5(2)*nx5(1) nxp5(2)*nx5(2) nxp5(2) nx5(1) nx5(2) 1;
nxp7(1)*nx7(1) nxp7(1)*nx7(2) nxp7(1) nxp7(2)*nx7(1) nxp7(2)*nx7(2) nxp7(2) nx7(1) nx7(2) 1;
nxp9(1)*nx9(1) nxp9(1)*nx9(2) nxp9(1) nxp9(2)*nx9(1) nxp9(2)*nx9(2) nxp9(2) nx9(1) nx9(2) 1;
nxp10(1)*nx10(1) nxp10(1)*nx10(2) nxp10(1) nxp10(2)*nx10(1) nxp10(2)*nx10(2) nxp10(2) nx10(1) nx10(2) 1;
nxp11(1)*nx11(1) nxp11(1)*nx11(2) nxp11(1) nxp11(2)*nx11(1) nxp11(2)*nx11(2) nxp11(2) nx11(1) nx11(2) 1;
nxp12(1)*nx12(1) nxp12(1)*nx12(2) nxp12(1) nxp12(2)*nx12(1) nxp12(2)*nx12(2) nxp12(2) nx12(1) nx12(2) 1]
```

```
[U,S,V]=svd(A)
Fh=[V(1:3,9)';V(4:6,9)';V(7:9,9)']
F=TP'*Fh*T
```

```
>> xp1'*F*x1
ans =
-1.2016e-005
```

```
>> xp3'*F*x3
ans =
1.7811e-005
```

```
>> xp4'*F*x4
ans =
-2.4047e-005
```

```
>> xp5'*F*x5
ans =
1.6801e-006
```

```
>> xp7'*F*x7
ans =
2.8201e-005
```

```
>> xp9'*F*x9
ans =
-5.3142e-005
```

```
>> xp10'*F*x10
ans =
3.7243e-005
```

```
>> xp11'*F*x11
ans =
3.0845e-006
```

```
>> xp12'*F*x12
ans =
1.1849e-006
```

Fundamental matrix—example3, cont.

ERROR Comparison: residual error (for equation or line)

```
A=[
nxp1(1)*nx1(1) nxp1(1)*nx1(2) nxp1(1) nxp1(2)*nx1(1) nxp1(2)*nx1(2) nxp1(2) nx1(1) nx1(2) 1;
nxp2(1)*nx2(1) nxp2(1)*nx2(2) nxp2(1) nxp2(2)*nx2(1) nxp2(2)*nx2(2) nxp2(2) nx2(1) nx2(2) 1;
nxp3(1)*nx3(1) nxp3(1)*nx3(2) nxp3(1) nxp3(2)*nx3(1) nxp3(2)*nx3(2) nxp3(2) nx3(1) nx3(2) 1;
nxp4(1)*nx4(1) nxp4(1)*nx4(2) nxp4(1) nxp4(2)*nx4(1) nxp4(2)*nx4(2) nxp4(2) nx4(1) nx4(2) 1;
nxp5(1)*nx5(1) nxp5(1)*nx5(2) nxp5(1) nxp5(2)*nx5(1) nxp5(2)*nx5(2) nxp5(2) nx5(1) nx5(2) 1;
nxp6(1)*nx6(1) nxp6(1)*nx6(2) nxp6(1) nxp6(2)*nx6(1) nxp6(2)*nx6(2) nxp6(2) nx6(1) nx6(2) 1;
nxp7(1)*nx7(1) nxp7(1)*nx7(2) nxp7(1) nxp7(2)*nx7(1) nxp7(2)*nx7(2) nxp7(2) nx7(1) nx7(2) 1;
nxp8(1)*nx8(1) nxp8(1)*nx8(2) nxp8(1) nxp8(2)*nx8(1) nxp8(2)*nx8(2) nxp8(2) nx8(1) nx8(2) 1;
nxp9(1)*nx9(1) nxp9(1)*nx9(2) nxp9(1) nxp9(2)*nx9(1) nxp9(2)*nx9(2) nxp9(2) nx9(1) nx9(2) 1;
nxp10(1)*nx10(1) nxp10(1)*nx10(2) nxp10(1) nxp10(2)*nx10(1) nxp10(2)*nx10(2) nxp10(2) nx10(1) nx10(2) 1;
nxp11(1)*nx11(1) nxp11(1)*nx11(2) nxp11(1) nxp11(2)*nx11(1) nxp11(2)*nx11(2) nxp11(2) nx11(1) nx11(2) 1;
nxp12(1)*nx12(1) nxp12(1)*nx12(2) nxp12(1) nxp12(2)*nx12(1) nxp12(2)*nx12(2) nxp12(2) nx12(1) nx12(2) 1]
```

```
[U,S,V]=svd(A)
Fh=[V(1:3,9)';V(4:6,9)';V(7:9,9)']
F=TP'*Fh*T

lp1=F*x1;lp1=lp1./sqrt(lp1(1)^2+lp1(2)^2);
……..
```

Error estimation function

Euclidean distance

xp1'*lp1	xp2'*lp2	xp3'*lp3	xp4'*lp4	xp5'*lp5	xp6'*lp6	xp7'*lp7	xp8'*lp8	xp9'*lp9	xp10'*lp10	xp11'*lp11	xp12'*lp12
ans =	ans =	ans =	ans =	ans =	ans =	ans =	ans =	ans =	ans =	ans =	ans =
-0.0795	0.6574	-0.6152	0.2976	-0.3051	-0.0908	0.0753	-0.0448	-0.1011	0.3848	-0.3072	0.1278

$x'^T Fx$

>> xp1'*F*x1	>> xp2'*F*x2	>> xp3'*F*x3	>> xp4'*F*x4	>> xp5'*F*x5	>> xp6'*F*x6	>> xp7'*F*x7	>> xp8'*F*x8	>> xp9'*F*x9	>> xp10'*F*x10	>> xp11'*F*x11	>> xp12'*F*x12
ans =	ans =	ans =	ans =	ans =	ans =	ans =	ans =	ans =	ans =	ans =	ans =
-5.3137e-004	0.0045	-0.0042	0.0020	-0.0020	-5.4760e-004	4.2649e-004	-2.6209e-004	-5.8426e-004	0.0023	-0.0018	7.5518e-004

Fundamental matrix computation

- Short summary
  - Why error occurs ?
  - Since the real camera is NOT the perfect pin-hole camera model
    →undistort images or avoid lens distortion in practice.
  - The image has physical limits in resolution and capacity.
    →earn more budget? Subpixel? Interpolation / super-resolution
  - Numerical issue. Currently, in computer vision field, less textbooks will teach you the Analytical Solution. Iteration error, round-off error, truncated error,
  - Matching error (correspondences) & measurement uncertainty

Fundamental matrix (enforcing singularity)

Rank 3

SVD

$$\mathbf{F} = \mathbf{USV}^{\mathrm{T}} = \mathbf{U}\begin{bmatrix} r & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & \boxed{t} \end{bmatrix}\mathbf{V}^{\mathrm{T}} \Rightarrow \mathbf{F'} = \mathbf{U}\boxed{\mathbf{S'}}\mathbf{V}^{\mathrm{T}} = \mathbf{U}\begin{bmatrix} r & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & \boxed{0} \end{bmatrix}\mathbf{V}^{\mathrm{T}}$$

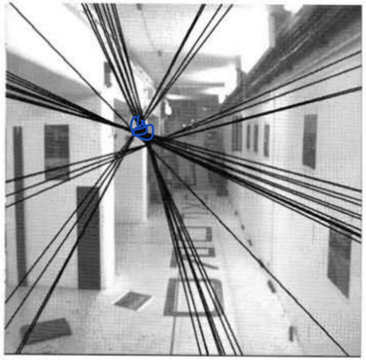
Rank 2

?

$U \cdot S \cdot V$

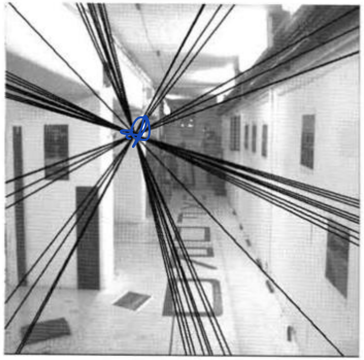
$[\ ]_{3 \times 3}$

The effect of a non-singular fundamental matrix



a

A singular fundamental matrix



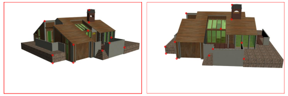
b

1st → F

det

Fundamental matrix (enforcing singularity)

- Example



Recall the previous example again.

Determine the normalized fundamental matrix $\hat{\mathbf{F}}$
Then, enforcing the singularity for $\hat{\mathbf{F}}$ to have $\hat{\mathbf{F}}'$
Denoramalized by $\hat{\mathbf{F}}'$ instead of $\hat{\mathbf{F}}$.

Eigen

$F = U \cdot S \cdot V^T$

Eigen vector

$\begin{bmatrix} D & & \\ & D & \\ & & 0 \end{bmatrix} = 0$

```
//SAMPLE CODE in MATLAB
A=[nxp1(1)*nx1(1) nxp1(1)*nx1(2) nxp1(1) nxp1(2)*nx1(1) nxp1(2)*nx1(2) nxp1(2) nx1(1) nx1(2) 1;
.......
nxp12(1)*nx12(1) nxp12(1)*nx12(2) nxp12(1) nxp12(2)*nx12(1) nxp12(2)*nx12(2) nxp12(2) nx12(1) nx12(2) 1]

[U,S,V]=svd(A)
f=[V(1:3,9)';V(4:6,9)';V(7:9,9)']

[Uf,Sf,Vf]=svd(f)

Sf(3,3)=0;

FP=Uf*Sf*Vf'
F=TP'*FP*T
```

$$\Rightarrow \begin{bmatrix} r & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & t \end{bmatrix} = \begin{matrix} 0.7413 & 0 & 0 \\ 0 & 0.6712 & 0 \\ 0 & 0 & \boxed{0.0031} \end{matrix}$$

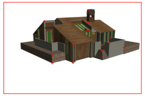
replace it by 0.0

$$\mathbf{F} = \mathbf{T}'^{T}\, \hat{\mathbf{F}}'\mathbf{T} = \begin{matrix} 0.0000 & 0.0000 & -0.0010 \\ 0.0000 & -0.0000 & -0.0071 \\ 0.0008 & 0.0060 & -0.2523 \end{matrix}$$

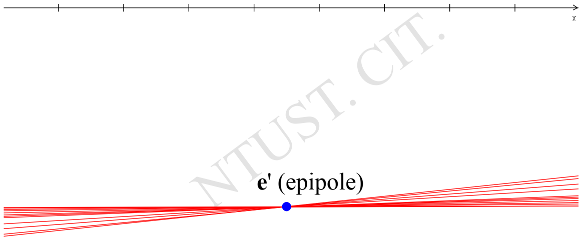
(let's redraw epipolar lines for image2, next slide)

Fundamental matrix (enforcing singularity)

- Example—cont.




>> F*x1	>> F*x2	>> F*x3	>> F*x4	>> F*x5	>> F*x6	>> F*x7	>> F*x8	>> F*x9	>> F*x10	>> F*x11	>> F*x12
ans =	ans =	ans =	ans =	ans =	ans =	ans =	ans =	ans =	ans =	ans =	ans =
-0.0006	-0.0002	-0.0001	-0.0000	-0.0002	-0.0001	-0.0000	-0.0001	-0.0002	-0.0004	-0.0005	-0.0006
-0.0067	-0.0068	-0.0069	-0.0066	-0.0064	-0.0060	-0.0057	-0.0059	-0.0058	-0.0061	-0.0060	-0.0059
0.5155	1.6236	1.8765	1.9542	1.3840	1.7012	1.6946	1.4855	1.3197	0.9956	0.7433	0.3480



Auto Fundamental matrix algorithm

Objective Compute the fundamental matrix between two images.

Algorithm

(i) **Interest points:** Compute interest points in each image.

≥ 8 points $\Rightarrow$ Avoid Degemented

(ii) **Putative correspondences:** Compute a set of interest point matches based on proximity and similarity of their intensity neighbourhood.

(iii) **RANSAC robust estimation:** Repeat for N samples, where N is determined adaptively as in algorithm 4.5(p121):

(a) Select a random sample of 7 correspondences and compute the fundamental matrix F as described in section 11.1.2. There will be one or three real solutions.

(b) Calculate the distance $d_\perp$ for each putative correspondence.

(c) Compute the number of inliers consistent with F by the number of correspondences for which $d_\perp < t$ pixels.

(d) If there are three real solutions for F the number of inliers is computed for each solution, and the solution with most inliers retained.

Choose the F with the largest number of inliers. In the case of ties choose the solution that has the lowest standard deviation of inliers.

(iv) **Non-linear estimation:** re-estimate F from all correspondences classified as inliers by minimizing a cost function, e.g. (11.6), using the Levenberg–Marquardt algorithm of section A6.2(p600).

(v) **Guided matching:** Further interest point correspondences are now determined using the estimated F to define a search strip about the epipolar line.

The last two steps can be iterated until the number of correspondences is stable.

① ≥ 8 points (set) degenerated

② 100% 100% 50% 10% RANSC

90% $\Rightarrow$ F $\rightarrow$ F

big 1 big 2

F small $\Rightarrow$ F big $\Rightarrow$ $U \cdot S \cdot V^T$

$U \cdot [\ ] \cdot V^T$

Enforce singularity Fundamental matrix algorithm

Objective

Find the fundamental matrix $\mathtt{F}$ that minimizes the algebraic error $\|\mathtt{A}\mathbf{f}\|$ subject to $\|\mathbf{f}\| = 1$ and $\det \mathtt{F} = 0$.

Algorithm

(i) Find a first approximation $\mathtt{F}_0$ for the fundamental matrix using the normalized 8-point algorithm 11.1. Then find the right null-vector $\mathbf{e}_0$ of $\mathtt{F}_0$.

(ii) Starting with the estimate $\mathbf{e}_i = \mathbf{e}_0$ for the epipole, compute the matrix $\mathtt{E}_i$ according to (11.4), then find the vector $\mathbf{f}_i = \mathtt{E}_i\mathbf{m}_i$ that minimizes $\|\mathtt{A}\mathbf{f}_i\|$ subject to $\|\mathbf{f}_i\| = 1$. This is done using algorithm A5.6(*p*595).

(iii) Compute the algebraic error $\epsilon_i = \mathtt{A}\mathbf{f}_i$. Since $\mathbf{f}_i$ and hence ϵ_i is defined only up to sign, correct the sign of ϵ_i (multiplying by minus 1 if necessary) so that $\mathbf{e}_i^\mathsf{T}\mathbf{e}_{i-1} > 0$ for $i > 0$. This is done to ensure that ϵ_i varies smoothly as a function of $\mathbf{e}_i$.

(iv) The previous two steps define a mapping $\mathbb{R}^3 \rightarrow \mathbb{R}^9$ mapping $\mathbf{e}_i \mapsto \epsilon_i$. Now use the Levenberg–Marquardt algorithm (section A6.2(*p*600)) to vary $\mathbf{e}_i$ iteratively so as to minimize $\|\epsilon_i\|$.

(v) Upon convergence, $\mathbf{f}_i$ represents the desired fundamental matrix.

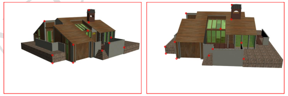
Fundamental matrix (enforcing singularity)

Of course, openCV provides findFundamentalMat function

- Parameters
  - points1 – Array of N points from the first image. The point coordinates should be floating-
  - point (single or double precision).
  - points2 – Array of the second image points of the same size and format as points1 .
  - method – Method for computing a fundamental matrix.
    - – CV_FM_7POINT for a 7-point algorithm. $N = 7$
    - – CV_FM_8POINT for an 8-point algorithm. $N \geq 8$
    - – CV_FM_RANSAC for the RANSAC algorithm. $N \geq 8$
    - – CV_FM_LMEDS for the LMedS algorithm. $N \geq 8$

Fundamental matrix using openCV

- findFundamentalMat



(RANSAC)
F=
-0.000219 -0.000913 0.292220
0.000103 -0.000245 0.737529
-0.142952 -0.450960 1.000000

(LMEDS, Levenberg-Marquardt)
F=
-0.000219 -0.000913 0.292220
0.000103 -0.000245 0.737529
-0.142952 -0.450960 1.000000

$$\mathbf{F} = \mathbf{USV}^{T}$$

1.331905 0.000000 0.000000
0.000000 0.281374 0.000000
0.000000 0.000000 0.000000

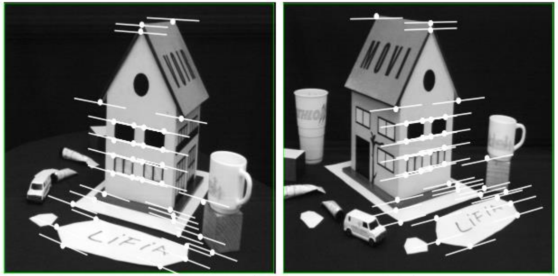
$$\mathbf{e}' = \mathbf{l}_1' \times \mathbf{l}_2' = [\mathbf{F}\mathbf{x}_1] \times [\mathbf{F}\mathbf{x}_2] = \begin{matrix} -552.206970 \\ 217.436905 \\ 1.000000 \end{matrix}$$

$$\mathbf{e} = \mathbf{l}_1 \times \mathbf{l}_2 = [\mathbf{F}^{T}\mathbf{x}_1'] \times [\mathbf{F}^{T}\mathbf{x}_2'] = \begin{matrix} -4089.085693 \\ 1298.432373 \\ 1.000000 \end{matrix}$$

Comparison of different methods

(a)

(b)



	Linear Least Squares	[Hartley, 1995]	[Luong *et al.*, 1993]
Av. Dist. 1	2.33 pixels	0.92 pixel	0.86 pixel
Av. Dist. 2	2.18 pixels	0.85 pixel	0.80 pixel

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