Quantum Mechanics 1 - Exam Problems

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Problem 2

a) Without the potential the operators

$$x_0 = x + \frac{v_y}{\omega_c}, \qquad y_0 = y - \frac{v_x}{\omega_c}$$

commute with the Hamiltonian

$$H = \frac{m}{2}\left(v_x^2 + v_y^2\right)$$

where

$$v_i = \frac{1}{m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)_i$$

now that we introduced

$$U(x) = \frac{\alpha x^2}{2}$$

y_0 will no longer be a constant of motion because v_x does not commute with $U(x)$, the problem is no longer invariant for translations in x.

To find the Heisenberg equation of motion for y_0 are now

$$\begin{aligned}
-i\hbar\frac{\mathrm{d}y_0}{\mathrm{d}t} &= [H, y_0] \\
&= \left[\frac{\alpha x^2}{2}, y_0\right] \\
&= \frac{1}{m\omega_c}\left[\frac{\alpha x^2}{2}, -p_x\right] \\
&= -\frac{\alpha}{m\omega_c}x\,[x, p_x] = -i\hbar\frac{\alpha x}{m\omega_c}
\end{aligned}$$

so we get

$$\frac{\mathrm{d}y_0}{\mathrm{d}t} = \frac{\alpha}{m\omega_c}x$$

b) Let us solve the problem choosing a gauge

$$\mathbf{A} = Bx\hat{\mathbf{y}}$$

we get

$$H = \frac{p_x^2}{2m} + \frac{1}{2m}\left(p_y - \frac{eB}{c}x\right)^2 + \frac{\alpha x^2}{2}$$

plugging a solution

$$\psi = e^{ik_y y}\phi\left(x\right)$$

we get an equation

$$H\phi\left(x\right) = \left[\frac{p_x^2}{2m} + \frac{1}{2m}\left(\hbar k_y - \frac{eB}{c}x\right)^2 + \frac{\alpha x^2}{2}\right]\phi\left(x\right) = E\phi(x)$$

we denote

$$x_0 = \ell^2 k_y, \qquad \omega_c = \frac{eB}{mc}, \qquad \ell^2 = \frac{\hbar c}{eB}$$

and we get

$$\begin{aligned}
H &= \frac{p_x^2}{2m} + \frac{m\omega_c^2}{2}\left(x - x_0\right)^2 + \frac{\alpha x^2}{2} \\
&= \frac{p_x^2}{2m} + \frac{m\omega_c^2}{2}\left(x^2 - 2xx_0 + x_0^2\right) + \frac{\alpha x^2}{2} \\
&= \frac{p_x^2}{2m} + \frac{1}{2}\left(\left(\alpha + m\omega_c^2\right)x^2 - 2m\omega_c^2 x x_0 + m\omega_c^2 x_0^2\right) \\
&= \frac{p_x^2}{2m} + \frac{\alpha + m\omega_c^2}{2}\left(x - \frac{m\omega_c^2 x_0}{\alpha + m\omega_c^2}\right)^2 + \frac{m\omega_c^2 x_0^2}{2} - \frac{m^2\omega_c^4 x_0^2}{2\left(\alpha + m\omega_c^2\right)} \\
&= \frac{p_x^2}{2m} + \frac{\alpha + m\omega_c^2}{2}\left(x - \frac{m\omega_c^2 x_0}{\alpha + m\omega_c^2}\right)^2 + \frac{1}{2}\frac{\alpha m\omega_c^2 x_0^2}{\alpha + m\omega_c^2} \\
&\equiv \frac{p_x^2}{2m} + \frac{m\omega_\alpha^2}{2}\left(x - \frac{\omega_c^2 x_0}{\omega_\alpha^2}\right)^2 + \frac{1}{2}\frac{\alpha m\omega_c^2 x_0^2}{\alpha + m\omega_c^2}
\end{aligned}$$

where we defined

$$\omega_\alpha^2 = \frac{\alpha + m\omega_c^2}{m}$$

This is a shifted harmonic oscillator so the energy levels are now

$$E_n = \hbar\omega_\alpha\left(n + \frac{1}{2}\right) + \frac{\alpha}{2}\left(\frac{\omega_c^2}{\omega_\alpha^2}\right)x_0^2$$

we see that there is no longer a degeneracy of the levels in $x_0 = \ell^2 k_y$ as there is when there is no potential. The wave functions are now

$$\begin{aligned}
\psi_{x_0,n} &= e^{ix_0 y/\ell^2}\chi_n\left(x - \left(\frac{\omega_c^2}{\omega_\alpha^2}\right)x_0\right) \\
&\equiv e^{ix_0 y/\ell^2}\chi_n\left(x - x_\alpha\right)
\end{aligned}$$

where χ_n is the wave function of the n^{th} energy level of harmonic oscillator with frequency ω_α.

c) The current density is given by

$$\mathbf{j} = -\frac{e\hbar}{m}\mathrm{Im}\left(\psi\boldsymbol{\nabla}\psi^*\right) - \frac{e^2}{mc}\mathbf{A}\left|\psi\right|^2$$

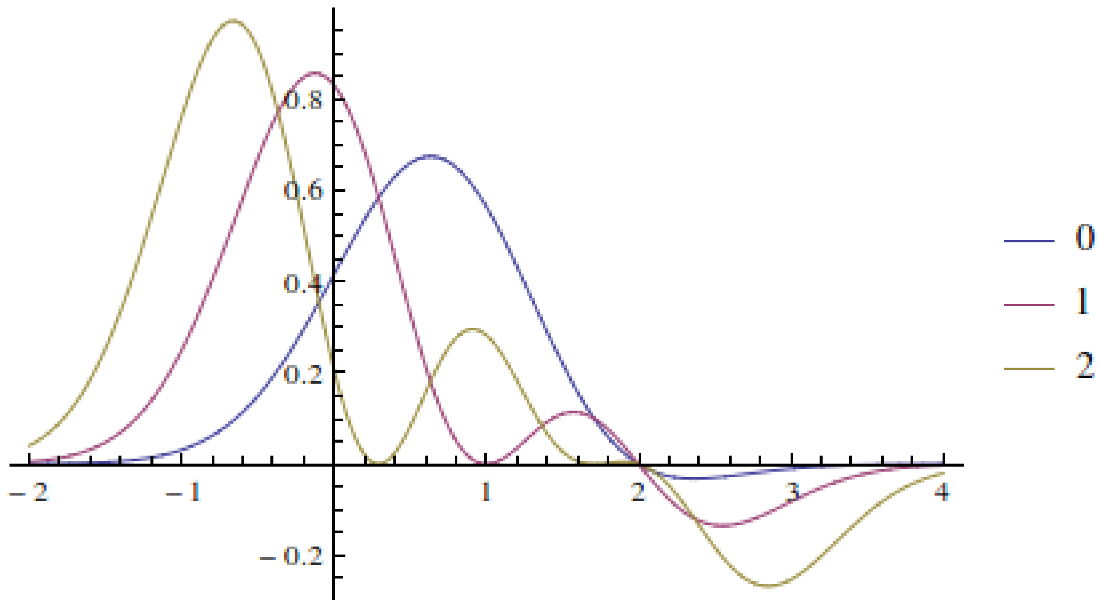
we have

$$j_x = -\frac{e\hbar}{m}Im\left(\chi_n\left(x - x_\alpha\right)\partial_x\chi_n^*\left(x - x_\alpha\right)\right) = 0$$

since χ_n is a real function.

$$\begin{aligned}
j_y &= -\frac{e\hbar}{m}\left|\chi_n\left(x - x_\alpha\right)\right|^2\mathrm{Im}\left(-i\frac{x_0}{\ell^2}\right) - \frac{e^2}{mc}Bx\left|\chi_n\left(x - x_\alpha\right)\right|^2 \\
&= \frac{e\hbar}{m\ell^2}\left(x_0 - x\right)\left|\chi_n\left(x - x_\alpha\right)\right|^2
\end{aligned}$$

to draw schematically the current we note that $x_\alpha < x_0$, for values $x_\alpha = 1$ and $x_0 = 2$ we get



The total current in the y direction is given by

$$I_{n,x_0} = \int j_y \mathrm{d}x = \frac{e\hbar x_0}{m\ell^2} - \int x \left|\chi_n\left(x - x_\alpha\right)\right|^2$$
$$= \frac{e\hbar\left(x_0 - x_\alpha\right)}{m\ell^2}$$

Problem 5 (Yuval, Ido, Ori)

a) The Hamiltonian of the problem is given by

$$H_0 = \frac{p^2}{2m} - \frac{e^2 Z}{r} - \frac{g\mu_B}{2} B_0 \sigma_z$$

in the presence of the radiation EM field we will have

$$H = H_0 + H_r + V_{int}$$

where

$$V_{int} = -\frac{1}{c}\int \mathbf{j}\cdot\mathbf{A}\mathrm{d}^3 r - \frac{g\mu_B}{2}\left(\nabla\times\mathbf{A}\right)\cdot\boldsymbol{\sigma}$$
$$= -\frac{1}{c}\int \mathbf{j}\cdot\mathbf{A}\mathrm{d}^3 r - i\frac{g\mu_B c}{2}\sum_{\mathbf{k},\alpha}\left(\frac{\hbar}{\omega_{\mathbf{k}}\Omega}\right)^{1/2} e^{i\mathbf{k}\cdot\mathbf{r}_{op}}\left[\left(\mathbf{k}\times\boldsymbol{\lambda}_\alpha\right)\cdot\boldsymbol{\sigma}\right]\left(a_{\mathbf{k},\alpha} + a^\dagger_{-\mathbf{k},\alpha}\right)$$

b) We treat V_{int} as a perturbation and neglect the interaction of the radiation field with the orbital part, we use the Fermi golden rule to calculate the rate of transitions

$$\Gamma = \frac{2\pi}{\hbar}\left|\langle f|V_{int}|i\rangle\right|^2\delta\left(\varepsilon_f - \varepsilon_i - \hbar\omega_{\mathbf{k}}\right)$$

where $\omega_{\mathbf{k}}$ is the frequency of the emitted photon.
plugging in

$$|i\rangle = |n_{\mathbf{k}} = 0, \sigma_z = 1, n\ell m\rangle$$
$$|f\rangle = |n_{\mathbf{k}} = 1, \sigma_z = -1, n\ell m\rangle$$

we get the rate for a specific polarization

$$\Gamma = \left(\frac{g\mu_B c}{2}\right)^2 \frac{2\pi}{\hbar}\left(\frac{\hbar}{\omega_{\mathbf{k}}\Omega}\right)\left|\left\langle n\ell m|e^{i\mathbf{k}\cdot\mathbf{r}}|n\ell m\right\rangle\right|^2 \times \left|\langle\downarrow|(\mathbf{k}\times\boldsymbol{\lambda}_\alpha)\cdot\boldsymbol{\sigma}|\uparrow\rangle\right|^2 \delta\left(g\mu_B B_0 - \hbar\omega_{\mathbf{k}}\right)$$

To discuss the actual measured quantity we need to talk about the rate of emission par unit solid angle γ

$$\begin{aligned}
\frac{\mathrm{d}N}{\mathrm{d}\gamma} &= \int \Gamma \frac{\Omega k^2 \mathrm{d}k}{(2\pi)^3} \\
&= \left(\frac{g\mu_B c}{2}\right)^2 \frac{\omega_{\mathbf{k}}}{4\pi^2 c^3 \hbar}\left|\left\langle n\ell m|e^{i\mathbf{k}\cdot\mathbf{r}}|n\ell m\right\rangle\right|^2 \left|\langle\downarrow|(\mathbf{k}\times\boldsymbol{\lambda}_\alpha)\cdot\boldsymbol{\sigma}|\uparrow\rangle\right|^2\Bigg|_{\omega_{\mathbf{k}}=\frac{g\mu_B B_0}{\hbar}} \\
&= \frac{(g\mu_B)^3 B_0}{16\pi^2\hbar^2 c}\left|\left\langle n\ell m|e^{i\mathbf{k}\cdot\mathbf{r}}|n\ell m\right\rangle\right|^2 \left|\langle\downarrow|(\mathbf{k}\times\boldsymbol{\lambda}_\alpha)\cdot\boldsymbol{\sigma}|\uparrow\rangle\right|^2
\end{aligned}$$

we assume that the energy of the emitted photon $g\mu_B B_0$ is much smaller than the gap between atomic levels , therefore $\frac{1}{k} = \frac{c\hbar}{g\mu_B B_0}$ is much larger than the length scale of the state $|n\ell m\rangle$, say the Bohr radius . Therefore we can approximate

$$\left\langle n\ell m|e^{i\mathbf{k}\cdot\mathbf{r}}|n\ell m\right\rangle \approx \langle n\ell m|n\ell m\rangle = 1$$

so we get

$$\frac{\mathrm{d}N}{\mathrm{d}\gamma} = \frac{(g\mu_B)^3 B_0}{16\pi^2\hbar c}\left|\langle\downarrow|(\mathbf{k}\times\boldsymbol{\lambda}_\alpha)\cdot\boldsymbol{\sigma}|\uparrow\rangle\right|^2$$

Now we concentrate on the matrix element

$$\left|\langle\downarrow|(\mathbf{k}\times\boldsymbol{\lambda}_\alpha)\cdot\boldsymbol{\sigma}|\uparrow\rangle\right|^2$$

only the components in the x, y directions contribute to this expression.
Denoting

$$\mathbf{k} = k\left(\sin\theta_{\mathbf{k}}\cos\phi_{\mathbf{k}}, \sin\theta_{\mathbf{k}}\sin\phi_{\mathbf{k}}, \cos\theta_{\mathbf{k}}\right)$$

we choose a polarization basis

$$\begin{aligned}
\lambda_{\mathbf{k},1} &= (\sin\phi_{\mathbf{k}}, -\cos\phi_{\mathbf{k}}, 0) \\
\lambda_{\mathbf{k},2} &= (\cos\theta_{\mathbf{k}}\cos\phi_{\mathbf{k}}, \cos\theta_{\mathbf{k}}\sin\phi_{\mathbf{k}}, -\sin\theta_{\mathbf{k}})
\end{aligned}$$

we get

$$\mathbf{k}\times\boldsymbol{\lambda}_1 = -\boldsymbol{\lambda}_2, \qquad \mathbf{k}\times\boldsymbol{\lambda}_2 = -\boldsymbol{\lambda}_1$$

so

$$\begin{aligned}
(\mathbf{k}\times\boldsymbol{\lambda}_1)\cdot(\sigma_x, \sigma_y, 0) &= -\boldsymbol{\lambda}_2\cdot(\sigma_x,\sigma_y,0) \\
&= -\begin{pmatrix} & \cos\theta_{\mathbf{k}}\cos\phi_{\mathbf{k}} - i\cos\theta_{\mathbf{k}}\sin\phi_{\mathbf{k}} \\ \cos\theta_{\mathbf{k}}\cos\phi_{\mathbf{k}} + i\cos\theta_{\mathbf{k}}\sin\phi_{\mathbf{k}} & \end{pmatrix} \\
&= -\cos\theta_{\mathbf{k}}\begin{pmatrix} 0 & e^{-i\phi_{\mathbf{k}}} \\ e^{i\phi_{\mathbf{k}}} & 0 \end{pmatrix} \\
(\mathbf{k}\times\boldsymbol{\lambda}_2)\cdot(\sigma_x, \sigma_y, 0) &= -\begin{pmatrix} 0 & \sin\phi_{\mathbf{k}} + i\cos\phi_{\mathbf{k}} \\ \sin\phi_{\mathbf{k}} - i\cos\phi_{\mathbf{k}} & 0 \end{pmatrix} \\
&= i\begin{pmatrix} 0 & e^{-i\phi_{\mathbf{k}}} \\ e^{i\phi_{\mathbf{k}}} & 0 \end{pmatrix}
\end{aligned}$$

and the matrix elements

$$\left|\langle\downarrow|(\mathbf{k}\times\boldsymbol{\lambda}_1)\cdot\boldsymbol{\sigma}|\uparrow\rangle\right|^2=\cos^2\theta_{\mathbf{k}}$$
$$\left|\langle\downarrow|(\mathbf{k}\times\boldsymbol{\lambda}_2)\cdot\boldsymbol{\sigma}|\uparrow\rangle\right|^2=1$$

If we look at photons in a specific polarization

$$\lambda_{\beta,\phi}=\cos\beta\lambda_1+e^{i\phi}\sin\beta\lambda_2$$

we need to sum the contributions from both polarizations

$$\left|\langle\downarrow|(\mathbf{k}\times\boldsymbol{\lambda}_{\beta,\phi})\cdot\boldsymbol{\sigma}|\uparrow\rangle\right|^2=\left|\cos\beta\cos\theta_{\mathbf{k}}+ie^{i\phi}\sin\beta\right|^2$$

so we get

$$\left.\frac{\mathrm{d}N}{\mathrm{d}\gamma}\right|_{\lambda_\beta}=\frac{(g\mu_B)^3B_0}{16\pi^2\hbar c}\left|\cos\beta\cos\theta_{\mathbf{k}}+ie^{i\phi}\sin\beta\right|^2$$

If the detector does not distinguish between polarizations we just sum over the emission rates for both polarizations

$$\frac{\mathrm{d}N}{\mathrm{d}\gamma}=\frac{(g\mu_B)^3B_0}{16\pi^2\hbar c}\left(1+\cos^2\theta_{\mathbf{k}}\right)$$

c) Now we have

$$|f'\rangle=|n_{\mathbf{k}}=0,\sigma_z=1,n\ell m\rangle$$
$$|i'\rangle=|n_{\mathbf{k}}=1,\sigma_z=-1,n\ell m\rangle$$

and the Fermi golden rule gives us

$$\Gamma=\frac{2\pi}{\hbar}\left|\langle f'|V_{int}|i'\rangle\right|^2\delta\left(-g\mu_BB_0+\hbar\omega_{\mathbf{k}}\right)$$

since

$$\left|\langle f'|V_{int}|i'\rangle\right|^2=\left|\langle i'|V_{int}|f'\rangle\right|^2=\left|\langle f|V_{int}|i\rangle\right|^2$$

the expression we have here is exactly the same as the one in the previous section, so the probability stays the same and so thus the dependence on the polarization of the photon.
From the previous section we can see that if the photon is polarized in the $\hat{\mathbf{z}}$ direction there will be no absorption. If the polarization is in the xy plane we will have maximal absorption.

Problem 6 (Ori)

The rate of de-excitation is given by

$$\Gamma=\frac{2\pi}{\hbar}\left|\langle\ell=0,m=0;n_{\mathbf{k}}=1|V|\ell=1,m;n_{\mathbf{k}}=0\rangle\right|^2\delta\left(\frac{\hbar^2}{I}-\hbar\omega_{\mathbf{k}}\right)$$

Plugging in

$$V=-\mathbf{d}\cdot\mathbf{E}=-i\mathbf{d}\cdot\sum_{\mathbf{k},\alpha}\sqrt{\frac{\hbar\omega}{\Omega}}e^{i\mathbf{k}\cdot\mathbf{r}}\boldsymbol{\lambda}_\alpha\left(a_{\mathbf{k}}-a^\dagger_{-\mathbf{k}}\right)$$

we get

$$\Gamma = \frac{2\pi\omega_{\mathbf{k}}}{\Omega}\left|\boldsymbol{\lambda}_\alpha \cdot \langle 0,0|\mathbf{d}|1,m\rangle\right|^2 \delta\left(\frac{\hbar^2}{I} - \hbar\omega_{\mathbf{k}}\right)$$

and the emission rate per unit solid angle

$$\begin{aligned}
\frac{\mathrm{d}N}{\mathrm{d}\gamma} &= \int \Gamma \frac{\Omega k^2 \mathrm{d}k}{(2\pi)^3} \\
&= \frac{1}{4\pi^2}\int \left|\boldsymbol{\lambda}_\alpha \cdot \langle 0,0|\mathbf{d}|1,m\rangle\right|^2 \delta\left(\frac{\hbar^2}{I} - \hbar\omega_{\mathbf{k}}\right)\omega_{\mathbf{k}}\frac{\omega^2\mathrm{d}\omega}{c^3} \\
&= \frac{1}{4\pi^2 c^3 \hbar}\left(\frac{\hbar}{I}\right)^3 \left|\boldsymbol{\lambda}_\alpha \cdot \langle 0,0|\mathbf{d}|1,m\rangle\right|^2
\end{aligned}$$

so we get

$$\frac{\mathrm{d}N}{\mathrm{d}\gamma} = \frac{1}{4\pi^2 c^3}\left(\frac{\hbar}{I}\right)^3 \left|\boldsymbol{\lambda}_\alpha \cdot \langle 0,0|\mathbf{d}|1,m\rangle\right|^2$$

We write

$$\begin{gathered}
d_x = d\tfrac{1}{2}\sqrt{\tfrac{8\pi}{3}}\left[y_{1,-1}(\theta,\phi) - y_{1,1}(\theta,\phi)\right] \\
d_y = d\tfrac{i}{2}\sqrt{\tfrac{8\pi}{3}}\left[y_{1,1}(\theta,\phi) + y_{1,1}(\theta,\phi)\right] \\
d_z = d\sqrt{\tfrac{8\pi}{3}}y_{1,0}(\theta,\phi)
\end{gathered}$$

It will be convenient to denote

$$\begin{aligned}
\mathbf{d} &= \sqrt{\frac{4\pi}{3}}d\left[\frac{1}{\sqrt{2}}(\hat{\mathbf{x}} + i\hat{\mathbf{y}})Y_{1,-1} + \hat{\mathbf{z}}Y_{10} + \frac{1}{\sqrt{2}}(\hat{\mathbf{x}} - i\hat{\mathbf{y}})Y_{1,1}\right] \\
&\equiv \sqrt{\frac{4\pi}{3}}d\left[\hat{\mathbf{e}}_1 Y_{1,-1} + \hat{\mathbf{e}}_0 Y_{10} + \hat{\mathbf{e}}_{-1}Y_{1,1}\right] \\
&= \sqrt{\frac{4\pi}{3}}d\sum_m \hat{\mathbf{e}}_m^* Y_{1m}
\end{aligned}$$

from this we get that

$$\langle 0,0|\mathbf{d}|1,m\rangle = \frac{d}{\sqrt{3}}\hat{\mathbf{e}}_m$$

therefore we get that the emission rate is

$$\frac{\mathrm{d}N}{\mathrm{d}\gamma} = \frac{d^2}{12\pi^2 c^3}\left(\frac{\hbar}{I}\right)^3 \left|\boldsymbol{\lambda}_{\mathbf{k},\alpha} \cdot \hat{\mathbf{e}}_m\right|^2$$

Denoting

$$\begin{gathered}
\hat{\mathbf{k}} = (\sin\theta_{\mathbf{k}}\cos\phi_{\mathbf{k}}, \sin\theta_{\mathbf{k}}\sin\phi_{\mathbf{k}}, \cos\theta_{\mathbf{k}}) \\
\boldsymbol{\lambda}_{\mathbf{k},1} = (\cos\theta_{\mathbf{k}}\cos\phi_{\mathbf{k}}, \cos\theta_{\mathbf{k}}\sin\phi_{\mathbf{k}}, -\sin\theta_{\mathbf{k}}) \\
\boldsymbol{\lambda}_{\mathbf{k},2} = (-\sin\phi_{\mathbf{k}}, \cos\phi_{\mathbf{k}}, 0)
\end{gathered}$$

For $m = 0$ we get

$$\begin{gathered}
\alpha = 1 \Rightarrow \lambda_{\mathbf{k},1} \cdot \hat{\mathbf{e}}_0 = -\sin\theta_{\mathbf{k}} \\
\alpha = 2 \Rightarrow \lambda_{\mathbf{k},2} \cdot \hat{\mathbf{e}}_0 = 0
\end{gathered}$$

so

$$\frac{\mathrm{d}N}{\mathrm{d}\gamma} = \frac{d^2}{12\pi^2 c^3}\left(\frac{\hbar}{I}\right)^3 \sin^2\theta_{\mathbf{k}}$$

For $m = \pm 1$ we have

$$\begin{aligned}
\lambda_{\mathbf{k},1}\cdot\hat{\mathbf{e}}_{\pm 1} &= \left[\cos\theta_{\mathbf{k}}\cos\phi_{\mathbf{k}} \pm i\cos\theta_{\mathbf{k}}\sin\phi_{\mathbf{k}}\right]/\sqrt{2} \\
&= \frac{e^{\pm i\phi_{\mathbf{k}}}}{\sqrt{2}}\cos\theta_{\mathbf{k}} \\
\lambda_{\mathbf{k},2}\cdot\hat{\mathbf{e}}_{\pm 1} &= \frac{-\sin\phi_{\mathbf{k}} \pm i\cos\phi_{\mathbf{k}}}{\sqrt{2}} \\
&= \pm\frac{i}{\sqrt{2}}\left(\cos\phi_{\mathbf{k}} \pm i\sin\phi_{\mathbf{k}}\right) \\
&= \pm\frac{i}{\sqrt{2}}e^{\pm i\phi_{\mathbf{k}}}
\end{aligned}$$

so the angular distribution is

$$\frac{\mathrm{d}N}{\mathrm{d}\gamma} = \frac{d^2}{24\pi^2 c^3}\left(\frac{\hbar}{I}\right)^3\left(1+\cos^2\theta_{\mathbf{k}}\right)$$

For $m = 0$ the polarization of the photon is

$$\boldsymbol{\lambda}_{\mathbf{k},m=0} = \frac{1}{\sqrt{\left|\boldsymbol{\lambda}_{\mathbf{k},1}\cdot\hat{\mathbf{e}}_0\right|^2+\left|\boldsymbol{\lambda}_{\mathbf{k},2}\cdot\hat{\mathbf{e}}_0\right|^2}}\left[\left(\boldsymbol{\lambda}_{\mathbf{k},1}\cdot\hat{\mathbf{e}}_0\right)\boldsymbol{\lambda}_{\mathbf{k},1} + \left(\boldsymbol{\lambda}_{\mathbf{k},2}\cdot\hat{\mathbf{e}}_0\right)\boldsymbol{\lambda}_{\mathbf{k},2}\right]$$

it will be some general linear polarization.
For $m = \pm 1$ we get

$$\begin{aligned}
\boldsymbol{\lambda}_{\mathbf{k},m=\pm 1} &= \frac{1}{\sqrt{\left|\boldsymbol{\lambda}_{\mathbf{k},1}\cdot\hat{\mathbf{e}}_{\pm 1}\right|^2+\left|\boldsymbol{\lambda}_{\mathbf{k},2}\cdot\hat{\mathbf{e}}_{\pm 1}\right|^2}}\left[\left(\boldsymbol{\lambda}_{\mathbf{k},1}\cdot\hat{\mathbf{e}}_{\pm 1}\right)\boldsymbol{\lambda}_{\mathbf{k},1} + \left(\boldsymbol{\lambda}_{\mathbf{k},2}\cdot\hat{\mathbf{e}}_{\pm 1}\right)\boldsymbol{\lambda}_{\mathbf{k},2}\right] \\
&= \frac{1}{\sqrt{1+\cos^2\theta_{\mathbf{k}}}}\left(\cos\theta_{\mathbf{k}}\boldsymbol{\lambda}_{\mathbf{k},1} \pm i\boldsymbol{\lambda}_{\mathbf{k},2}\right)
\end{aligned}$$

Problem 7

To understand how the quantum Hamiltonian looks like we need to figure out the classical Hamiltonian first, to deal with the constraint (hard rod) we start with the Lagrangian of the classical system

$$\mathcal{L} = \frac{mL^2}{2}\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) - qE_0 L\left(1-\cos\theta\right)$$

the canonical momenta

$$p_\theta = mL^2\dot{\theta} \Rightarrow \dot{\theta} = \tfrac{p_\theta}{mL^2}$$
$$p_\phi = mL^2\sin^2\theta\dot{\phi} \Rightarrow \dot{\phi} = \tfrac{p_\phi}{mL^2\sin^2\theta}$$

so the Hamiltonian is

$$H = \frac{1}{2mL^2}\left(p_\theta^2 + \frac{1}{\sin^2\theta}p_\phi^2\right) + qE_0 L\left(1-\cos\theta\right)$$

now we can quantize the system imposing canonical commutation relations.

We see that p_ϕ commutes with the Hamiltonian so the eigenstates of this system will be eigenstates of p_ϕ as well (those are basically eigenstates of L_z).
The interaction term between the EM radiation field and our particle is given by

$$V_{int} = \frac{q}{2m}\left[\mathbf{p}\cdot\mathbf{A}\left(\mathbf{r}\right)+\mathbf{A}\left(\mathbf{r}\right)\cdot\mathbf{p}\right]$$

where $\mathbf{p}$ is the linear momentum of the particle, so we need to find expressions for $\mathbf{r}_{op}, \mathbf{p}_{op}$ in terms of θ, ϕ classically we have

$$\mathbf{p} = p_\phi\hat{\boldsymbol{\phi}} + p_\theta\hat{\boldsymbol{\theta}}$$

since $\hat{\boldsymbol{\phi}}$ and $\hat{\boldsymbol{\theta}}$ are functions of θ, ϕ they do not commute with p_ϕ, p_θ respectively so when we quantize we will take

$$\mathbf{p} = \frac{1}{2}\left(\hat{\boldsymbol{\phi}}p_\phi + p_\phi\hat{\boldsymbol{\phi}} + \hat{\boldsymbol{\theta}}p_\theta + p_\theta\hat{\boldsymbol{\theta}}\right)$$

To compute the emission rate

$$\Gamma = \frac{2\pi}{\hbar}\left|\langle n_f, m_f; n_{\mathbf{k}}+1|V_{int}|n_i, m_i; n_{\mathbf{k}}\rangle\right|^2\delta\left(\Delta\varepsilon - \hbar\omega_{\mathbf{k}}\right)$$

we plug in the expression of $\mathbf{A}$

$$\mathbf{A} = \sum_{\mathbf{k}\alpha}\left(\frac{\hbar c^2}{\omega_{\mathbf{k}}\Omega}\right)^{1/2}\boldsymbol{\lambda}_\alpha e^{-i\mathbf{k}\cdot\mathbf{r}}\left(a_{-\mathbf{k}} + a^\dagger_{\mathbf{k}}\right)$$

so for a specific mode we get

$$\Gamma = \frac{2\pi q^2\left(n_{\mathbf{k}}+1\right)c}{4m^2\omega_{\mathbf{k}}\Omega}\left|\langle n_f, m_f|\mathbf{p}\cdot\boldsymbol{\lambda}_\alpha e^{-i\mathbf{k}\cdot\mathbf{r}} + e^{-i\mathbf{k}\cdot\mathbf{r}}\mathbf{p}\cdot\boldsymbol{\lambda}_\alpha|n_i, m_i\rangle\right|^2\delta\left(\Delta\varepsilon - \hbar\omega_{\mathbf{k}}\right)$$

and the rate per unit solid angle

$$\frac{\mathrm{d}N}{\mathrm{d}\gamma} = \frac{q^2\omega}{4\pi^2\hbar c^2}\left|\frac{1}{2m}\langle n_f, m_f|\mathbf{p}\cdot\boldsymbol{\lambda}_\alpha e^{-i\mathbf{k}\cdot\mathbf{r}} + e^{-i\mathbf{k}\cdot\mathbf{r}}\mathbf{p}\cdot\boldsymbol{\lambda}_\alpha|n_i, m_i\rangle\right|^2$$

In the long wave length approximation we will have

$$e^{-i\mathbf{k}\cdot\mathbf{r}} \approx 1$$

so the matrix element we need to calculate is

$$\begin{aligned}\left|\frac{1}{m}\langle n_f, m_f|\mathbf{p}\cdot\boldsymbol{\lambda}_\alpha|n_i, m_i\rangle\right|^2 &= \left|\frac{1}{i\hbar}\langle n_f, m_f|\left[\mathbf{r}, H_0\right]\cdot\boldsymbol{\lambda}_\alpha|n_i, m_i\rangle\right|^2 \\ &= \omega^2\left|\langle n_f, m_f|\mathbf{r}\cdot\boldsymbol{\lambda}_{\mathbf{k},\alpha}|n_i, m_i\rangle\right|^2\end{aligned}$$

and we need to calculate

$$\frac{\mathrm{d}N}{\mathrm{d}\gamma} = \frac{q^2\omega^3}{4\pi^2\hbar c^2}\left|\langle n_f, m_f|\mathbf{r}\cdot\boldsymbol{\lambda}_{\mathbf{k},\alpha}|n_i, m_i\rangle\right|^2$$

now we need to do some approximations to find the eigenstates of the pendulum, if we consider small oscillations we get

$$H \approx \frac{1}{2mL^2}\left(p_\theta^2 + \frac{1}{\theta^2}p_\phi^2\right) + \frac{qE_0L}{2}\theta^2$$

Problem 8 (Yuval,Ido,Ori)

$$H=\int\psi^{\dagger}(x)\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+U(x)\right)\psi(x)\,\mathrm{d}x+\int\psi^{\dagger}(x)\psi(x)V(x-y)\mu(y)\,\mathrm{d}x\mathrm{d}y$$
$$\mu(y)=\tfrac{1}{2}\sum_{\alpha,\beta=1}^{2}\xi_{\alpha}^{\dagger}(y)\sigma_{\alpha\beta}^{x}\xi_{\beta}(y)$$
$$U(x)=\frac{m\Omega^2x^2}{2},\qquad V(x-y)=\frac{m\omega^2(x-y)^2}{2},\qquad \Omega>\omega$$
$$h\equiv-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+U(x)$$

a) Let us first look at the interaction part

$$H_{int}=\frac{m\omega^2}{2}\int\psi^{\dagger}(x)\psi(x)(x-y)^2\left(\xi_1^{\dagger}(y)\xi_2(y)+\xi_2^{\dagger}(y)\xi_1(y)\right)\mathrm{d}x\mathrm{d}y$$

now define

$$\eta_{+}^{\dagger}(y)=\frac{1}{\sqrt{2}}\left(\xi_1^{\dagger}(y)+\xi_2^{\dagger}(y)\right)$$
$$\eta_{-}^{\dagger}(y)=\frac{1}{\sqrt{2}}\left(\xi_1^{\dagger}(y)-\xi_2^{\dagger}(y)\right)$$

so we have

$$\begin{aligned}
H_{int}\eta_{+}^{\dagger}(y')|0\rangle &= \frac{1}{\sqrt{2}}\int\mathrm{d}y\,(x-y)^2\left(\xi_1^{\dagger}(y)\xi_2(y)+\xi_2^{\dagger}(y)\xi_1(y)\right)\left(\xi_1^{\dagger}(y')+\xi_2^{\dagger}(y')\right)|0\rangle\\
&= \frac{1}{\sqrt{2}}\int\mathrm{d}y\,(x-y)^2\left(\xi_1^{\dagger}(y)\xi_2(y)\xi_1^{\dagger}(y')+\xi_2^{\dagger}(y)\xi_1(y)\xi_1^{\dagger}(y')+\xi_2^{\dagger}(y)\xi_1(y)\xi_2^{\dagger}(y')+\xi_1^{\dagger}(y)\xi_2(y)\xi_2^{\dagger}(y')\right)|0\rangle\\
&= \frac{1}{\sqrt{2}}\int\mathrm{d}y\,(x-y)^2\left(\xi_2^{\dagger}(y)\xi_1(y)\xi_1^{\dagger}(y')+\xi_1^{\dagger}(y)\xi_2(y)\xi_2^{\dagger}(y')\right)|0\rangle\\
&= \frac{1}{\sqrt{2}}\int\mathrm{d}y\,(x-y)^2\delta(y-y')\left(\xi_2^{\dagger}(y)+\xi_1^{\dagger}(y)\right)|0\rangle\\
&= (x-y')^2\eta_{+}^{\dagger}(y')|0\rangle\\
H_{int}\eta_{-}^{\dagger}(y')|0\rangle &= -(x-y')^2\eta_{-}^{\dagger}(y')|0\rangle
\end{aligned}$$

We try a product state of the form

$$|\psi_{\pm,y'}\rangle=|\phi_{\pm,y'}\rangle\otimes\eta_{\pm}^{\dagger}(y')|0\rangle$$

plugging this into the Hamiltonian we get an equation for $|\phi_{\pm,y'}\rangle$

$$H|\phi_{\pm,y'}\rangle=\int\psi^{\dagger}(x)h\psi(x)\,\mathrm{d}x|\phi_{\pm,y'}\rangle\pm\int\psi^{\dagger}(x)\psi(x)\frac{m\omega^2(x-y')^2}{2}\mathrm{d}x|\phi_{\pm,y'}\rangle$$

so we see our Hamiltonian is now

$$H_{\pm}=\int\psi^{\dagger}(x)\left(-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2+\frac{1}{2}m\left[\Omega^2x^2\pm\omega^2(x-y')^2\right]\right)\psi(x)\,\mathrm{d}x$$

we complete the square

$$\begin{aligned}
h' &= -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2+\frac{1}{2}m\left[\Omega^2x^2\pm\omega^2(x-y')^2\right]\\
&= -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2+\frac{m}{2}\left[\left(\Omega^2\pm\omega^2\right)x^2\mp2\omega^2xy'\pm\omega^2y'^2\right]\\
&= -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2+\frac{m}{2}\left(\Omega^2\pm\omega^2\right)\left(x\mp\frac{\omega^2y'}{(\Omega^2\pm\omega^2)}\right)^2\pm\frac{m\omega^2}{2}y'^2-\frac{m\omega^4y'^2}{2(\Omega^2\pm\omega^2)}\\
&= -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2+\frac{m}{2}\left(\Omega^2\pm\omega^2\right)\left(x\mp\frac{\omega^2y'}{(\Omega^2\pm\omega^2)}\right)^2\pm\frac{m}{2}\frac{\omega^2\Omega^2}{(\Omega^2\pm\omega^2)}y'^2\\
&\equiv -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2+\frac{m}{2}\left(\Omega^2\pm\omega^2\right)(x\mp x_0)^2\pm\frac{m\Omega^2}{2}x_0^2
\end{aligned}$$

we expand

$$\psi_{y',\pm}(x)=\sum_{k=0}^{\infty}b_k\chi_k(x\pm x_0)$$

and get

$$H=\sum_{k=0}^{\infty}\varepsilon_{k,\pm,y'}b^{\dagger}_{k,\pm,y'}b_{k,\pm,y'}$$

with

$$\varepsilon_{n,\pm,y'}=\hbar\sqrt{\Omega^2\pm\omega^2}\left(n+\frac{1}{2}\right)\pm\frac{m\Omega^2}{2}x_0^2$$

so the eigenstate will be

$$\Psi_{y',\pm,\{n_i\}}=\Pi_i\left(b^{\dagger}_{i,\pm,y'}\right)^{n_i}|0\rangle_B\otimes\eta^{\dagger}_{\pm}(y')|0\rangle_F$$

with energy

$$E_{y',\pm,\{n_i\}}=\sum_i n_i\varepsilon_{n,\pm,y'}$$

b) Now we take a fermion state

$$|\phi_F\rangle=\Pi_i\eta^{\dagger}_{\sigma_i}(y_i)|0\rangle,\qquad\sigma_i=\pm$$

so we get

$$\begin{aligned}H_{int}|\phi_F\rangle &= \int\mathrm{d}y\,(x-y)^2\left(\xi_1^{\dagger}(y)\,\xi_2(y)+\xi_2^{\dagger}(y)\,\xi_1(y)\right)\Pi_i\eta^{\dagger}_{\sigma_i}(y_i)|0\rangle\\ &= \int\mathrm{d}y\,(x-y)^2\left(\eta_+^{\dagger}(y)\,\eta_+(y)-\eta_-^{\dagger}(y)\,\eta_-(y)\right)\Pi_i\eta^{\dagger}_{\sigma_i}(y_i)|0\rangle\end{aligned}$$

one can show (in the exam show it) that

$$\left\{\eta_j(y),\eta_i^{\dagger}(y')\right\}=\delta(y-y')\,\delta_{ij}$$

$$\{\eta_j(y),\eta_i(y')\}=\left\{\eta_j^{\dagger}(y),\eta_i^{\dagger}(y')\right\}=0$$

so after commuting everything properly we get

$$H_{int}|\phi_F\rangle=\frac{m\omega^2}{2}\sum_i\sigma_i(x-y_i)^2|\phi_F\rangle$$

Plugging in a product wave function like in the previous section we get

$$\begin{aligned}H_B &= \int\psi^{\dagger}(x)\left(-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2+\frac{1}{2}m\left[\Omega^2x^2+\omega^2\sum_i\sigma_i(x-y_i)^2\right]\right)\psi(x)\,\mathrm{d}x\\ &\equiv \int\psi^{\dagger}(x)\,h'\psi(x)\,\mathrm{d}x\end{aligned}$$

we complete the square

$$\begin{aligned}h' &= -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2+\frac{1}{2}m\left[\Omega^2x^2+\omega^2\sum_i\sigma_i(x-y_i)^2\right]\\ &= -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2+\frac{1}{2}m\left[\left(\Omega^2+\omega^2\sum_i\sigma_i\right)x^2-2\omega^2\sum_i\sigma_ixy_i+\omega^2\sum_i\sigma_iy_i^2\right]\\ &= -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2+\frac{m}{2}\left(\Omega^2+\omega^2\sum_i\sigma_i\right)\left(x-\frac{\omega^2\sum_i\sigma_iy_i}{\Omega^2+\omega^2\sum_i\sigma_i}\right)^2+\frac{m}{2}\left[\omega^2\sum_i\sigma_iy_i^2-\frac{\omega^4\left(\sum_i\sigma_iy_i\right)^2}{\Omega^2+\omega^2\sum_i\sigma_i}\right]\\ &= -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2+\frac{m}{2}\left(\Omega^2+\omega^2\sum_i\sigma_i\right)(x-x_0)^2+\frac{m}{2}\left[\omega^2\sum_i\sigma_iy_i^2-\frac{\omega^4\left(\sum_i\sigma_iy_i\right)^2}{\Omega^2+\omega^2\sum_i\sigma_i}\right]\end{aligned}$$

expanding $\psi(x)$

$$\psi(x) = \sum_{n=0}^{\infty} b_n \chi_n (x - x_0)$$

we get the Hamiltonian

$$H_B = \sum \varepsilon_n b_n^\dagger b_n$$
$$\varepsilon_n = \hbar \left(\Omega^2 + \omega^2 \sum_i \sigma_i\right)^{1/2} \left(n + \tfrac{1}{2}\right) + \tfrac{m}{2} \left[\omega^2 \sum_i \sigma_i y_i^2 - \tfrac{\omega^4 \left(\sum_i \sigma_i y_i\right)^2}{\Omega^2 + \omega^2 \sum_i \sigma_i}\right]$$

and the general eigenstate is as in the previous section.

Problem 9

$$H = \sum_{i,j=1}^{\infty} h_{ij} a_i^\dagger a_j + \frac{V_0}{2} \left(\hat{N} - N_0\right)^2$$

a) We define the state

$$b_n^\dagger |0\rangle = \sum_{j=1}^{\infty} \psi_n (j) \, a_j^\dagger |0\rangle$$

this state satisfies

$$\begin{aligned}
\sum_{i,j=1}^{\infty} h_{ij} a_i^\dagger a_j b_n^\dagger |0\rangle &= \sum_{i,j,j'=1}^{\infty} a_i^\dagger h_{ij} \psi_n (j') \, a_j a_{j'}^\dagger |0\rangle \\
&= \sum_{i,j=1}^{\infty} a_i^\dagger h_{ij} \psi_n (j) \, |0\rangle \\
&= \varepsilon_n \sum_{i=1}^{\infty} \psi_n (i) \, a_i^\dagger |0\rangle \\
&= \varepsilon_n b_n^\dagger |0\rangle
\end{aligned}$$

Also

$$\begin{aligned}
\hat{N} b_n^\dagger |0\rangle &= \sum_{i,j=1}^{\infty} a_i^\dagger a_i \psi_n (j) \, a_j^\dagger |0\rangle \\
&= \sum_{i=1}^{\infty} \psi_n (i) \, a_i^\dagger |0\rangle \\
&= b_n^\dagger |0\rangle
\end{aligned}$$

So we can work in a new basis defining

$$b_n^\dagger = \sum_{j=1}^{\infty} \psi_n (j) \, a_j^\dagger, \qquad b_n = \sum_{j=1}^{\infty} \psi_n^* (j) \, a_j$$

we get

$$\begin{aligned}
b_{n'} b_n^\dagger &= \sum_{jj'} a_{j'} a_j^\dagger \psi_{n'}^* (j') \psi_n (j) \\
&= \sum_{jj'} \left(\delta_{jj'} - a_j^\dagger a_{j'} \right) \psi_{n'}^* (j') \psi_n (j) \\
&= \sum_{j} \psi_{n'}^* (j) \psi_n (j) - b_n^\dagger b_{n'} \\
&= \delta_{nn'} - b_n^\dagger b_{n'}
\end{aligned}$$

so

$$\left\{ b_{n'}, b_n^\dagger \right\} = \delta_{nn'}$$

similarly we show that

$$\left\{ b_n, b_{n'} \right\} = \left\{ b_n^\dagger, b_{n'}^\dagger \right\} = 0$$

so b_n's are indeed fermion operators.
In this basis the Hamiltonian is

$$H = \sum_{n=1}^{\infty} \varepsilon_n b_n^\dagger b_n + \frac{V_0}{2} \left(\sum_{n=1}^{\infty} b_n^\dagger b_n - N_0 \right)^2$$

The ground state is

$$|g\rangle = \left(\Pi_{i=1}^{M} b_i^\dagger \right) |0\rangle, \qquad E_0 = \sum_{i=1}^{M} \varepsilon_i$$

b) The Heisenberg equations

$$\begin{aligned}
-i\hbar \frac{\mathrm{d} a_k}{\mathrm{d} t} &= [H, a_k] \\
&= \sum_{i,j=1}^{\infty} h_{ij} \left[a_i^\dagger a_j, a_k \right] + \frac{V_0}{2} \left[\hat{N}^2 - 2 N_0 \hat{N}, a_k \right]
\end{aligned}$$

we use the identity

$$[AB, C] = A \{B, C\} - \{A, C\} B$$

and get

$$\begin{aligned}
\left[a_i^\dagger a_j, a_k \right] &= - \left\{ a_i^\dagger, a_k \right\} a_j = -\delta_{ik} a_j \\
\left[\hat{N}, a_k \right] &= \left[a_k^\dagger a_k, a_k \right] = -a_k \\
\left[\hat{N}^2, a_k \right] &= \hat{N} \left[\hat{N}, a_k \right] + \left[\hat{N}, a_k \right] \hat{N} = - \left(2\hat{N} + 1 \right) a_k
\end{aligned}$$

so we get

$$-i\hbar \frac{\mathrm{d} a_k}{\mathrm{d} t} = - \sum_{j=1}^{\infty} h_{kj} a_j - V_0 \left(\hat{N} - N_0 + \frac{1}{2} \right) a_k$$

Similarly for $a_k^\dagger$ we get

$$\begin{aligned}
\left[a_i^\dagger a_j, a_k^\dagger \right] &= a_i^\dagger \left\{ a_j, a_k^\dagger \right\} = \delta_{jk} a_i^\dagger \\
\left[\hat{N}, a_k^\dagger \right] &= \left[a_k^\dagger a_k, a_k^\dagger \right] = a_k^\dagger \\
\left[\hat{N}^2, a_k^\dagger \right] &= \hat{N} \left[\hat{N}, a_k^\dagger \right] + \left[\hat{N}, a_k^\dagger \right] \hat{N} = \left(2\hat{N} - 1 \right) a_k^\dagger
\end{aligned}$$

so we got

$$\begin{aligned} -i\hbar\frac{\mathrm{d}a_k}{\mathrm{d}t} &= -\sum_{j=1}^{\infty} h_{kj}a_j - V_0\left(\hat{N}-N_0+\tfrac{1}{2}\right)a_k \\ -i\hbar\frac{\mathrm{d}a_k^\dagger}{\mathrm{d}t} &= \sum_{i=1}^{\infty} a_i^\dagger h_{ik} + V_0\left(\hat{N}-N_0-\tfrac{1}{2}\right)a_k^\dagger \end{aligned}$$

To solve the first equation we do a basis transformation, we multiply it by $\psi_n^*(k)$ and sum over k

$$\begin{aligned} -i\hbar\frac{\mathrm{d}b_n}{\mathrm{d}t} &= -\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left(h_{jk}\psi_n(k)\right)^* a_j - V_0\left(\hat{N}-N_0+\frac{1}{2}\right)b_n \\ &= -\varepsilon_n\sum_{j=1}^{\infty}\psi_n^*(j)\,a_j - V_0\left(\hat{N}-N_0+\frac{1}{2}\right)b_n \\ &= \left[-\varepsilon_n - V_0\left(\hat{N}-N_0+\frac{1}{2}\right)\right]b_n \end{aligned}$$

so solution for this equation is

$$b_n(t) = \exp\left[-\frac{i}{\hbar}\left(\varepsilon_n + V_0\left(\hat{N}-N_0+\frac{1}{2}\right)\right)t\right]b_n(0)$$

similarly we get

$$-i\hbar\frac{\mathrm{d}b_n^\dagger}{\mathrm{d}t} = \left[\varepsilon_n + V_0\left(\hat{N}-N_0-\frac{1}{2}\right)\right]b_n^\dagger$$

so

$$b_n^\dagger(t) = \exp\left[\frac{i}{\hbar}\left(\varepsilon_n + V_0\left(\hat{N}-N_0-\frac{1}{2}\right)\right)t\right]b_n^\dagger(0)$$

to go back to the solution in terms of $a_k, a_k^\dagger$ we use

$$a_k = \sum_n \psi_n^*(k)\,b_n, \qquad a_k^\dagger = \sum_n \psi_n(k)\,b_n^\dagger$$

Now we use these solutions to find the excitation energies we have

$$\begin{gathered} b_k(t)\,|M\rangle = e^{\frac{i}{\hbar}Ht}b_k(0)\,e^{-\frac{i}{\hbar}Ht}|M\rangle \\ \exp\left[-\tfrac{i}{\hbar}\left(\varepsilon_k + V_0\left(\hat{N}-N_0+\tfrac{1}{2}\right)\right)t\right]b_k(0)\,|M\rangle = e^{\frac{i}{\hbar}E_{M-1}^{-k}}e^{-\frac{i}{\hbar}E_M}|M-1,-k\rangle \\ \exp\left[-\tfrac{i}{\hbar}\left(\varepsilon_k + V_0\left(M-1-N_0+\tfrac{1}{2}\right)\right)t\right]|M-1,-k\rangle = e^{\frac{i}{\hbar}E_{M-1}^{-k}}e^{-\frac{i}{\hbar}E_M}|M-1,-k\rangle \end{gathered}$$

comparing the exponents we get

$$E_M - E_{M-1}^{-k} = \varepsilon_k + V_0\left(M-N_0-\frac{1}{2}\right)$$

similar computation for the other case.

c) now we need to calculate the commutation relations for $a_k^\dagger a_n$

$$\begin{gathered} \left[a_i^\dagger a_j, a_k^\dagger a_n\right] = \left[a_i^\dagger a_j, a_k^\dagger\right]a_n + a_k^\dagger\left[a_i^\dagger a_j, a_n\right] = \delta_{jk}a_i^\dagger a_n - \delta_{in}a_k^\dagger a_j \\ \left[\hat{N}, a_k^\dagger a_n\right] = \left[\hat{N}, a_k^\dagger\right]a_n + a_k^\dagger\left[\hat{N}, a_n\right] = a_k^\dagger a_n - a_k^\dagger a_n = 0 \\ \left[\hat{N}^2, a_k^\dagger a_n\right] = 0 \end{gathered}$$

so we get

$$\begin{aligned}
-i\hbar\frac{\mathrm{d}\left(a_k^\dagger a_n\right)}{\mathrm{d}t} &= \sum_{i,j} h_{ij}\left(\delta_{jk}a_i^\dagger a_n - \delta_{in}a_k^\dagger a_j\right) \\
&= \left(\sum_i a_i^\dagger h_{ik}\right)a_n - a_k^\dagger\left(\sum_j h_{nj}a_j\right)
\end{aligned}$$

note that

$$\begin{aligned}
b_m^\dagger b_\ell &= \left(\sum_{j=1}^{\infty}\psi_m\left(j\right)a_j^\dagger\right)\left(\sum_{j'=1}^{\infty}\psi_\ell^*\left(j'\right)a_{j'}\right) \\
&= \sum_{jj'}\psi_m\left(j\right)\psi_\ell^*\left(j'\right)a_j^\dagger a_{j'}
\end{aligned}$$

so if we multiply the equation by $\psi_m\left(k\right)\psi_\ell^*\left(n\right)$ and sum over all indices we get

$$\begin{aligned}
-i\hbar\frac{\mathrm{d}\left(b_m^\dagger b_\ell\right)}{\mathrm{d}t} &= \sum_i a_i^\dagger h_{ik}\psi_m\left(k\right)\psi_\ell^*\left(n\right)a_n - \psi_m\left(k\right)a_k^\dagger\sum_j\psi_\ell^*\left(n\right)h_{nj}a_j \\
&= \varepsilon_m b_m^\dagger b_\ell - \varepsilon_\ell b_m^\dagger b_\ell = \left(\varepsilon_m-\varepsilon_\ell\right)b_m^\dagger b_\ell
\end{aligned}$$

so the solution is

$$b_m^\dagger b_\ell\left(t\right) = \exp\left(\frac{i}{\hbar}\left(\varepsilon_m-\varepsilon_\ell\right)\right)b_m^\dagger b_\ell\left(0\right)$$

from here we continue in the same fashion.

Problem 10 (Aviram, Tal, Idan, Nathan)

Upper signs for Bosons lower signs for Fermions.

a)

$$\begin{aligned}
H &= \sum_\alpha \varepsilon_\alpha a_\alpha^\dagger a_\alpha + \frac{1}{2}\sum_{\alpha\beta\gamma\delta}V_{\alpha\beta\gamma\delta}a_\alpha^\dagger a_\beta^\dagger a_\delta a_\gamma \\
&= \sum_\alpha \varepsilon_\alpha a_\alpha^\dagger a_\alpha + \frac{1}{2}\sum_{\alpha\beta}\left(V_{\alpha\beta\alpha\beta}\pm V_{\alpha\beta\beta\alpha}\right)a_\alpha^\dagger a_\beta^\dagger a_\alpha a_\beta \\
&= \sum_\alpha \varepsilon_\alpha a_\alpha^\dagger a_\alpha + \frac{1}{2}\sum_{\alpha\beta}\left(V_{\alpha\beta\alpha\beta}\pm V_{\alpha\beta\beta\alpha}\right)\left(\mp\delta_{\beta\alpha}a_\alpha^\dagger a_\beta \pm a_\alpha^\dagger a_\alpha a_\beta^\dagger a_\beta\right) \\
&= \sum_\alpha \varepsilon_\alpha N_\alpha + \frac{1}{2}\sum_{\alpha\beta}\left(V_{\alpha\beta\alpha\beta}\pm V_{\alpha\beta\beta\alpha}\right)\left(\mp\delta_{\beta\alpha}N_\alpha \pm N_\alpha N_\beta\right) \\
&= \sum_\alpha \varepsilon_\alpha N_\alpha + \frac{1}{2}\sum_{\alpha}\left(V_{\alpha\alpha\alpha\alpha}\pm V_{\alpha\alpha\alpha\alpha}\right)\left(\mp N_\alpha\right) + \frac{1}{2}\sum_{\alpha\beta}\left(V_{\alpha\beta\alpha\beta}\pm V_{\alpha\beta\beta\alpha}\right)\left(\pm N_\alpha N_\beta\right) \\
&= \sum_\alpha \varepsilon_\alpha N_\alpha + \sum_\alpha V_{\alpha\alpha\alpha\alpha}\left(N_\alpha-1\right)N_\alpha + \frac{1}{2}\sum_{\alpha\neq\beta}\left(V_{\alpha\beta\beta\alpha}\pm V_{\alpha\beta\alpha\beta}\right)\left(N_\alpha N_\beta\right)
\end{aligned}$$

As we can see the Hamiltonian depends only on N_j operators and they are the generators

$$e^{-i\alpha N_j}He^{i\alpha N_j} \approx \left(1-i\alpha N_j\right)H\left(1+i\alpha N_j\right) = H + i\alpha\left[H,N_j\right]$$

since obviously

$$[H, N_j] = 0$$

we get

$$e^{-i\alpha N_j} H e^{i\alpha N_j} = H$$

Our solution will be an outer product of number states

$$|\psi\rangle = |n_1\rangle \otimes |n_2\rangle ... = \bigotimes_i |n_i\rangle$$

$$H\left(\bigotimes_i |n_i\rangle\right) = \left[\sum_\alpha \varepsilon_\alpha n_\alpha + \sum_\alpha V_{\alpha\alpha\alpha\alpha}\left(n_\alpha - 1\right) n_\alpha + \frac{1}{2}\sum_{\alpha\neq\beta}\left(V_{\alpha\beta\alpha\beta} \pm V_{\alpha\beta\beta\alpha}\right)\left(\pm n_\alpha n_\beta\right)\right]\left(\bigotimes_i |n_i\rangle\right)$$

b) For deriving the HF equations it will be more convenient to stay with the expression

$$H = \sum_\alpha \varepsilon_\alpha a_\alpha^\dagger a_\alpha + \frac{1}{2}\sum_{\alpha\beta} a_\alpha^\dagger a_\beta^\dagger \left(V_{\alpha\beta\alpha\beta} \pm V_{\alpha\beta\beta\alpha}\right) a_\alpha a_\beta$$

$$|\psi_{trial}\rangle = \Pi_i |n_i\rangle = \Pi_{k=1}^N b_{i_k}^\dagger |0\rangle$$

where

$$b_i = \sum_j \phi_{ij} a_j, \qquad b_i^\dagger = \sum_j \phi_{ij}^* a_j^\dagger$$

and ϕ_{ij} is a unitary matrix, note that $b_i, b_i^\dagger$ are fermion operators and $\hat{N}|\psi_{trial}\rangle = N|\psi_{trial}\rangle$.
The inverse transformation

$$a_i = \sum_j \phi_{ji}^* b_j, \qquad a_i^\dagger = \sum \phi_{ji}^* b_j^\dagger$$

expressing the Hamiltonian in terms of b_i

$$\sum_\alpha \varepsilon_\alpha \left(\sum_j \phi_{j\alpha} b_j^\dagger\right)\left(\sum_k \phi_{k\alpha}^* b_k\right) = \sum_{\alpha,k,j} \varepsilon_\alpha \phi_{j\alpha}\phi_{k\alpha}^* b_j^\dagger b_k$$

$$\frac{1}{2}\sum_{\alpha\beta}\left(V_{\alpha\beta\alpha\beta} \pm V_{\alpha\beta\beta\alpha}\right) a_\alpha^\dagger a_\beta^\dagger a_\alpha a_\beta = \frac{1}{2}\sum_{\alpha\beta,j_i}\left(V_{\alpha\beta\alpha\beta} \pm V_{\alpha\beta\beta\alpha}\right)\phi_{j_1\alpha}\phi_{j_2\beta}\phi_{j_3\alpha}^*\phi_{j_4\beta}^* b_{j_1}^\dagger b_{j_2}^\dagger b_{j_3} b_{j_4}$$

so we have

$$H = \sum_\alpha \sum_{kj} \varepsilon_\alpha \phi_{j\alpha}\phi_{k\alpha}^* b_j^\dagger b_k + \frac{1}{2}\sum_{\alpha\beta}\sum_{ijkl}\left(V_{\alpha\beta\alpha\beta} \pm V_{\alpha\beta\beta\alpha}\right)\phi_{j_1\alpha}\phi_{j_2\beta}\phi_{j_3\alpha}^*\phi_{j_4\beta}^* b_i^\dagger b_j^\dagger b_k b_l$$

Fermions

To derive the HF equations we need to calculate $\langle\psi|H|\psi\rangle$,

$$\begin{aligned}
\left\langle \psi | b_j^\dagger b_k | \psi \right\rangle &= \left\langle 0 | b_{i_1}..b_{i_n} b_j^\dagger b_k b_{i_n}^\dagger .. b_{i_1}^\dagger | 0 \right\rangle \\
&= \left\langle 0 | b_{i_1}..b_{i_{n-1}}\left(\delta_{i_n j} - b_j^\dagger b_{i_n}\right) b_k b_{i_n}^\dagger .. b_{i_1}^\dagger | 0 \right\rangle \\
&= \delta_{i_n j}\left\langle 0 | b_{i_1}..b_{i_{n-1}} b_k b_{i_n}^\dagger .. b_{i_1}^\dagger | 0 \right\rangle + \left\langle 0 | b_{i_1}..b_{i_{n-1}} b_j^\dagger b_k b_{i_{n-1}}..b_{i_1} | 0 \right\rangle \\
&= .. = \sum_{l=1}^N \delta_{i_l j}\delta_{i_l k} = \delta_{jk} n_j
\end{aligned}$$

where $n_j = 1$ if the state j is occupied and 0 otherwise

Similarly (there is an easier way to see this by expecting the final result and explaining why it must be true)

$$
\begin{aligned}
\left\langle \psi | b_i^\dagger b_j^\dagger b_k b_l | \psi \right\rangle &= \left\langle 0 | b_{i_1} .. b_{i_n} b_i^\dagger b_j^\dagger b_k b_l b_{i_n}^\dagger .. b_{i_1}^\dagger | 0 \right\rangle \\
&= \delta_{i i_n} \left\langle 0 | b_{i_1} .. b_{i_{n-1}} b_j^\dagger b_k b_l b_{i_n}^\dagger .. b_{i_1}^\dagger | 0 \right\rangle - \left\langle 0 | b_{i_1} .. b_{i_{n-1}} b_i^\dagger b_{i_n} b_j^\dagger b_k b_l b_{i_n}^\dagger .. b_{i_1}^\dagger | 0 \right\rangle \\
&= \delta_{i i_n} \delta_{i_n l} \left\langle 0 | b_{i_1} .. b_{i_{n-1}} b_j^\dagger b_k b_{i_{n-1}}^\dagger .. b_{i_1}^\dagger | 0 \right\rangle - \delta_{i_n j} \left\langle 0 | b_{i_1} .. b_{i_{n-1}} b_i^\dagger b_k b_l b_{i_n}^\dagger .. b_{i_1}^\dagger | 0 \right\rangle \\
&\quad + \left\langle 0 | b_{i_1} .. b_{i_{n-1}} b_i^\dagger b_j^\dagger b_{i_n} b_k b_l b_{i_n}^\dagger .. b_{i_1}^\dagger | 0 \right\rangle \\
&= \delta_{i i_n} \delta_{i_n l} \sum_{l=1}^{N-1} \delta_{i_l j} \delta_{i_l k} - \delta_{i_n j} \delta_{i_n l} \left\langle 0 | b_{i_1} .. b_{i_{n-1}} b_i^\dagger b_k b_{i_{n-1}}^\dagger .. b_{i_1}^\dagger | 0 \right\rangle \\
&\quad + \delta_{i_n j} \delta_{i_n k} \left\langle 0 | b_{i_1} .. b_{i_{n-1}} b_i^\dagger b_l b_{i_{n-1}}^\dagger .. b_{i_1}^\dagger | 0 \right\rangle + \left\langle 0 | b_{i_1} .. b_{i_{n-1}} b_i^\dagger b_j^\dagger b_k b_l b_{i_{n-1}}^\dagger .. b_{i_1}^\dagger | 0 \right\rangle \\
&= \delta_{i i_n} \delta_{i_n l} \sum_{l=1}^{N-1} \delta_{i_l j} \delta_{i_l k} - \delta_{i_n j} \delta_{i_n l} \sum_{l=1}^{N-1} \delta_{i_l i} \delta_{i_l k} + \delta_{i_n j} \delta_{i_n k} \sum_{x=1}^{N-1} \delta_{i_x i} \delta_{i_x l} + \left\langle 0 | b_{i_1} .. b_{i_{n-1}} b_i^\dagger b_j^\dagger b_k b_l b_{i_{n-1}}^\dagger .. b_{i_1}^\dagger | 0 \right\rangle \\
&= \left(\delta_{i_n l} n_j + \delta_{i_n j} n_l \right) \delta_{il} \delta_{jk} - \delta_{i_n j} n_k \delta_{jl} \delta_{ik} + \left\langle 0 | b_{i_1} .. b_{i_{n-1}} b_i^\dagger b_j^\dagger b_k b_l b_{i_{n-1}}^\dagger .. b_{i_1}^\dagger | 0 \right\rangle \\
&= ... = n_j n_i \left(\delta_{il} \delta_{jk} - \delta_{jl} \delta_{ik} \right)
\end{aligned}
$$

so we get

$$
\begin{aligned}
\langle \psi | H | \psi \rangle &= \sum_\alpha \sum_{kj} \varepsilon_\alpha \phi_{j\alpha} \phi_{k\alpha}^* \delta_{jk} n_j + \frac{1}{2} \sum_{\alpha\beta} \sum_{ijkl} \left(V_{\alpha\beta\alpha\beta} - V_{\alpha\beta\beta\alpha} \right) \phi_{i\alpha} \phi_{j\beta} \phi_{k\alpha}^* \phi_{l\beta}^* n_j n_i \left(\delta_{il} \delta_{jk} - \delta_{jl} \delta_{ik} \right) \\
&= \sum_{\alpha,j} \varepsilon_\alpha \left| \phi_{j\alpha} \right|^2 n_j + \frac{1}{2} \sum_{\alpha\beta} \sum_{ij} \left(V_{\alpha\beta\alpha\beta} - V_{\alpha\beta\beta\alpha} \right) \phi_{i\alpha} \phi_{j\beta} \phi_{j\alpha}^* \phi_{i\beta}^* n_j n_i \\
&\quad - \frac{1}{2} \sum_{\alpha\beta} \sum_{ij} \left(V_{\alpha\beta\alpha\beta} - V_{\alpha\beta\beta\alpha} \right) \phi_{i\alpha} \phi_{j\beta} \phi_{i\alpha}^* \phi_{j\beta}^* n_j n_i \\
&= \sum_{\alpha,j} \varepsilon_\alpha \left| \phi_{j\alpha} \right|^2 n_j + \frac{1}{2} \sum_{\alpha\beta} \sum_{ij} \left(V_{\alpha\beta\alpha\beta} - V_{\alpha\beta\beta\alpha} \right) n_j n_i \left(\phi_{i\alpha} \phi_{j\beta} \phi_{j\alpha}^* \phi_{i\beta}^* - \phi_{i\alpha} \phi_{j\beta} \phi_{i\alpha}^* \phi_{j\beta}^* \right)
\end{aligned}
$$

now we need to diffrentiate with respect to ϕ_{nm}^* , assuming $n_n = 1$ we get

$$
\begin{aligned}
\frac{\delta \langle \psi | H | \psi \rangle}{\delta \phi_{nm}^*} &= \varepsilon_m \phi_{nm} + \frac{1}{2} \sum_{\alpha\beta} \sum_{ij} \left(V_{\alpha\beta\alpha\beta} - V_{\alpha\beta\beta\alpha} \right) n_j n_i \\
&\quad \times \left(\phi_{i\alpha} \phi_{j\beta} \phi_{j\alpha}^* \delta_{ni} \delta_{m\beta} + \phi_{i\alpha} \phi_{j\beta} \phi_{i\beta}^* \delta_{jn} \delta_{\alpha m} - \phi_{i\alpha} \phi_{j\beta} \phi_{i\alpha}^* \delta_{nj} \delta_{\beta m} - \phi_{i\alpha} \phi_{j\beta} \phi_{j\beta}^* \delta_{ni} \delta_{\alpha m} \right) \\
&= \varepsilon_m \phi_{nm} + \frac{1}{2} \sum_{\alpha j} n_j \left(V_{\alpha m\alpha m} - V_{\alpha mm\alpha} \right) \phi_{n\alpha} \phi_{jm} \phi_{j\alpha}^* + \frac{1}{2} \sum_{\beta i} n_i \left(V_{m\beta m\beta} - V_{m\beta\beta m} \right) \phi_{im} \phi_{n\beta} \phi_{i\beta}^* \\
&\quad - \frac{1}{2} \sum_{\alpha i} n_i \left(V_{\alpha m\alpha m} - V_{\alpha mm\alpha} \right) \left| \phi_{i\alpha} \right|^2 \phi_{nm} - \frac{1}{2} \sum_{\beta j} n_j \left(V_{m\beta m\beta} - V_{m\beta\beta m} \right) \left| \phi_{j\beta} \right|^2 \phi_{nm} \\
&= \varepsilon_m \phi_{nm} + \frac{1}{2} \sum_{\alpha j} \Big[\left(V_{\alpha m\alpha m} - V_{\alpha mm\alpha} \right) \phi_{n\alpha} \phi_{jm} \phi_{j\alpha}^* + \left(V_{m\alpha m\alpha} - V_{m\alpha\alpha m} \right) \phi_{jm} \phi_{n\alpha} \phi_{j\alpha}^* \\
&\quad - \left(V_{\alpha m\alpha m} - V_{\alpha mm\alpha} \right) \left| \phi_{j\alpha} \right|^2 \phi_{nm} - \left(V_{m\alpha m\alpha} - V_{\alpha mm\alpha} \right) \left| \phi_{j\alpha} \right|^2 \phi_{nm} \Big]
\end{aligned}
$$

and the equation is

$$
\begin{aligned}
E_n \phi_{nm} &= \varepsilon_m \phi_{nm} + \frac{1}{2} \sum_{\alpha j} \Big[\left(V_{\alpha m\alpha m} - V_{\alpha mm\alpha} \right) \phi_{n\alpha} \phi_{jm} \phi_{j\alpha}^* + \left(V_{m\alpha m\alpha} - V_{m\alpha\alpha m} \right) \phi_{jm} \phi_{n\alpha} \phi_{j\alpha}^* \\
&\quad - \left(V_{\alpha m\alpha m} - V_{\alpha mm\alpha} \right) \left| \phi_{j\alpha} \right|^2 \phi_{nm} - \left(V_{m\alpha m\alpha} - V_{\alpha mm\alpha} \right) \left| \phi_{j\alpha} \right|^2 \phi_{nm} \Big]
\end{aligned}
$$

now we plug a solution of the form

$$\phi_{nm} = \delta_{nm}$$

so

$$\begin{aligned}
E_n\delta_{nm} &= \varepsilon_n\delta_{nm} + \frac{1}{2}\sum_{\alpha j}\left[\left(V_{\alpha m\alpha m} - V_{\alpha mm\alpha}\right)\delta_{n\alpha}\delta_{jm}\delta_{j\alpha} + \left(V_{m\alpha m\alpha} - V_{m\alpha\alpha m}\right)\delta_{jm}\delta_{n\alpha}\delta_{j\alpha}\right. \\
&\quad \left. - \left(V_{\alpha m\alpha m} - V_{\alpha mm\alpha}\right)\left|\delta_{j\alpha}\right|^2\delta_{nm} - \left(V_{m\alpha m\alpha} - V_{\alpha mm\alpha}\right)\left|\delta_{j\alpha}\right|^2\delta_{nm}\right] \\
&= \varepsilon_n\delta_{nm} + \frac{1}{2}\sum_{j}\left(V_{jmjm} - V_{jmmj}\right)\delta_{nm} - \left(V_{mjmj} - V_{jmmj}\right)\delta_{nm} \\
&= \varepsilon_n\delta_{nm} + \frac{1}{2}\sum_{j}\left(V_{jmjm} - V_{mjmj}\right)\delta_{nm} \\
&= \varepsilon_n\delta_{nm} + \frac{1}{2}\sum_{j}\left(V_{jnjn} - V_{njnj}\right)\delta_{nm}
\end{aligned}$$

so we get

$$E_n = \varepsilon_n + \frac{1}{2}\sum_{j\in occ}\left(V_{jnjn} - V_{njnj}\right)$$

In this case the HF solution is exact.

Bosons

now we put all the particles in the same state

$$|\psi\rangle = \frac{1}{\sqrt{N!}}\left(b_g^{\dagger}\right)^N|0\rangle$$

$$\left\langle\psi|b_j^{\dagger}b_k|\psi\right\rangle = N\delta_{gk}\delta_{jk}$$

$$\left\langle\psi|b_i^{\dagger}b_j^{\dagger}b_kb_l|\psi\right\rangle = \left(N^2 - N\right)\delta_{gi}\delta_{ij}\delta_{jk}\delta_{kl}$$

repeat as for fermions...

Problem 11

$$H = \int\psi^{\dagger}(x)\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)\psi(x)\,\mathrm{d}x + \int\left[\psi^{\dagger}(x) + \psi(x)\right]\lambda\delta(x-y)\,\mu(y)\,\mathrm{d}x\mathrm{d}y$$

where

$$\mu(y) = \frac{1}{2}\left(\xi_1^{\dagger}\xi_1 - \xi_2^{\dagger}\xi_2\right)(y)$$

1) Work in momentum representation

$$\psi(x) = \tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}a(k)\,e^{ikx}\mathrm{d}k$$
$$\psi^{\dagger}(x) = \tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}a^{\dagger}(k)\,e^{-ikx}\mathrm{d}k$$

we get as usual

$$\left[a(k), a^{\dagger}(k')\right] = \delta(k-k')$$

plugging this in we get

$$H = \int\mathrm{d}k\frac{\hbar^2k^2}{2m}a^{\dagger}(k)\,a(k) + H_{int}$$

2) Plug in a state of the form

$$|\phi_B\rangle \otimes \xi_\alpha^\dagger (y') |0_F\rangle$$

so

$$H_{int}|\phi_B\rangle \otimes \xi_\alpha^\dagger (y') |0_F\rangle = \left(\psi^\dagger (y') + \psi (y')\right) \frac{\lambda}{2} (-1)^{\alpha+1} |\phi_B\rangle \otimes \xi_\alpha^\dagger (y') |0_F\rangle$$

3) Look for boson eigenfunctions of

$$\begin{aligned}
H_B &= \int \mathrm{d}k \frac{\hbar^2 k^2}{2m} a^\dagger (k) a (k) + \left(\psi^\dagger (y) + \psi (y)\right) \frac{\lambda}{2} (-1)^{\alpha+1} \\
&= \int \mathrm{d}k \frac{\hbar^2 k^2}{2m} \left(a^\dagger (k) a (k) + \frac{m}{\hbar^2 k^2} \frac{\lambda (-1)^{\alpha+1}}{\sqrt{2\pi}} \left(a^\dagger (k) e^{-iky} + a (k) e^{iky} \right) \right) \\
&= \int \mathrm{d}k \frac{\hbar^2 k^2}{2m} \left(a^\dagger (k) + \frac{m}{\hbar^2 k^2} \frac{\lambda (-1)^{\alpha+1}}{\sqrt{2\pi}} e^{iky} \right) \left(a (k) + \frac{m}{\hbar^2 k^2} \frac{\lambda (-1)^{\alpha+1}}{\sqrt{2\pi}} e^{-iky} \right) \\
&\quad - \int \mathrm{d}k \frac{1}{2\pi} \frac{m}{\hbar^2 k^2} \frac{\lambda^2}{2}
\end{aligned}$$

4) Define

$$A_{y,\alpha} (k) = a (k) + \tfrac{m}{\hbar^2 k^2} \tfrac{\lambda (-1)^{\alpha+1}}{\sqrt{2\pi}} e^{-iky}$$
$$A_{y,\alpha}^\dagger = a^\dagger (k) + \tfrac{m}{\hbar^2 k^2} \tfrac{\lambda (-1)^{\alpha+1}}{\sqrt{2\pi}} e^{iky}$$

same commutation relations .
Define vacuum

$$A_{y,\alpha} (k) |\Omega_{B,y,\alpha}\rangle = 0$$

so a boson eigenstate

$$\Pi_{i=1}^n \frac{1}{\sqrt{m_i!}} \left(A_{y,\alpha}^\dagger (k_i) \right)^{m_i} |\Omega_{B,y,\alpha}\rangle$$

this is an eigenstate of $H_B (y', \alpha)$ with

$$E = \sum_{i=1}^n m_i \frac{\hbar^2 k_i^2}{2m} - \frac{m\lambda^2}{4\pi\hbar^2} \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{k^2}$$

5) Divergence is for small k . The Hamiltonian describes point interactions which can be a relevant approximation in large energy scales but this does not make sense in small energies.

Problem 12 (Yuval, Ido, Idan, Ori)

Let us write

$$\psi (\mathbf{r}) = \sum_i \phi_i (\mathbf{r}) a_i$$

If we take an eigenstate of one of the a_i's

$$\psi (\mathbf{r}) |\alpha\rangle = \alpha \phi_i (\mathbf{r}) |\alpha\rangle$$

if we find an eigenstate of the operator a_i it will also be an eigenstate of $\psi (\mathbf{r})$, this is a coherent state

$$|\alpha; i\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \left(a_i^\dagger \right)^n |0\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha a_i^\dagger} |0\rangle$$

Now for a general function we write

$$\Phi(\mathbf{r}) = \sum_i \alpha_i \phi_i(\mathbf{r})$$

and define

$$|\Phi\rangle = \Pi_i \exp\left(-\frac{|\alpha_i|^2}{2} + \alpha_i a_i^\dagger\right)|0\rangle$$

so we get

$$\psi(\mathbf{r})|\Phi\rangle = \sum \alpha_i \phi_i(\mathbf{r})|\Phi\rangle = \Phi(\mathbf{r})|\Phi\rangle$$

We want to express this without the creation operators we note that

$$a_i^\dagger = \int \mathrm{d}\mathbf{r}\phi_i(\mathbf{r})\psi^\dagger(\mathbf{r})$$

so we get

$$\begin{aligned}
|\Phi\rangle &= \Pi_i \exp\left(-\frac{|\alpha_i|^2}{2} + \alpha_i \int \mathrm{d}\mathbf{r}\phi_i(\mathbf{r})\psi^\dagger(\mathbf{r})\right)|0\rangle \\
&= \exp\left(-\frac{\int|\Phi|^2}{2}\right)\exp\left(\int \sum_i \alpha_i\phi_i\psi^\dagger \mathrm{d}\mathbf{r}\right)|0\rangle \\
&= \exp\left(-\frac{\int|\Phi|^2}{2}\right)\exp\left(\int \Phi(\mathbf{r})\psi^\dagger(\mathbf{r})\,\mathrm{d}\mathbf{r}\right)|0\rangle
\end{aligned}$$

we can write this more explicitly

$$|\Phi\rangle = \exp\left(-\frac{\int|\Phi|^2}{2}\right)\sum_{n=0}^{\infty}\frac{1}{n!}\left(\int \Phi(\mathbf{r})\psi^\dagger(\mathbf{r})\,\mathrm{d}\mathbf{r}\right)^n|0\rangle$$

using the fact that

$$\begin{aligned}
\left[\psi(\mathbf{r}'),\left(\int \Phi(\mathbf{r})\psi^\dagger(\mathbf{r})\,\mathrm{d}\mathbf{r}\right)^n\right] &= n\left(\int \Phi(\mathbf{r})\psi^\dagger(\mathbf{r})\,\mathrm{d}\mathbf{r}\right)^{n-1}\left[\psi(\mathbf{r}'),\int \Phi(\mathbf{r})\psi^\dagger(\mathbf{r})\,\mathrm{d}\mathbf{r}\right] \\
&= n\Phi(\mathbf{r}')\left(\int \Phi(\mathbf{r})\psi^\dagger(\mathbf{r})\,\mathrm{d}\mathbf{r}\right)^{n-1}
\end{aligned}$$

we can see that

$$\begin{aligned}
\psi(\mathbf{r}')|\Phi\rangle &= \exp\left(-\frac{\int|\Phi|^2\mathrm{d}\mathbf{r}}{2}\right)\sum_{n=0}^{\infty}\frac{1}{n!}\left[\psi(\mathbf{r}'),\left(\int \Phi(\mathbf{r})\psi^\dagger(\mathbf{r})\,\mathrm{d}\mathbf{r}\right)^n\right]|0\rangle \\
&= \exp\left(-\frac{\int|\Phi|^2\mathrm{d}\mathbf{r}}{2}\right)\sum_{n=1}^{\infty}\frac{\Phi(\mathbf{r}')}{(n-1)!}\left(\int \Phi(\mathbf{r})\psi^\dagger(\mathbf{r})\,\mathrm{d}\mathbf{r}\right)^{n-1}|0\rangle \\
&= \Phi(\mathbf{r}')|\Phi\rangle
\end{aligned}$$

Let us show that we cannot find eigenstates of $\psi^\dagger(\mathbf{r})$.
Assume $|\Phi\rangle$ is an eigenstate of $\psi^\dagger(\mathbf{r})$ and write it in the following way

$$|\Phi\rangle = \sum_{n=0}^{\infty}\alpha_n|\Phi_n\rangle$$

where $|\Phi_n\rangle$ is a projection of $|\Phi\rangle$ on the subspace of the Fock space with n particles .

Now note that $\psi^{\dagger}(\mathbf{r})|\Phi_n\rangle$ is a state with $n+1$ particles so

$$\sum_{n=0}^{\infty}\alpha_n|\Phi_n\rangle \propto \psi^{\dagger}(\mathbf{r})|\Phi_n\rangle = \sum_{n=1}^{\infty}\alpha_n|\Phi'_n\rangle$$

from this we get

$$\alpha_0 = 0$$

by induction we get

$$\alpha_n = 0$$

for all n, so $|\Phi\rangle = 0$ and we got a contradiction.

For fermions we also can't find an eigenstate of $\psi(\mathbf{r})$ (other then the vacuum which trivially gives zero), indeed assume that $|\Phi\rangle$
is an eigenstate with $\Phi \neq 0$, so we get

$$\psi(\mathbf{r})|\Phi\rangle = \Phi|\Phi\rangle \Rightarrow \psi(\mathbf{r})(\psi(\mathbf{r})|\Phi\rangle) = \Phi^2|\Phi\rangle$$

on the other hand

$$\psi(\mathbf{r})\psi(\mathbf{r})|\Phi\rangle = -\psi(\mathbf{r})\psi(\mathbf{r})|\Phi\rangle \Rightarrow \psi(\mathbf{r})\psi(\mathbf{r})|\Phi\rangle = 0$$

and we get a contradiction.

Returning to the eigenstate of bosonic $\psi(\mathbf{r})$, we have

$$H_{op} = \int \psi^{\dagger}(\mathbf{r})\underbrace{\left(-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 + U(\mathbf{r})\right)}_{\equiv h}\psi(\mathbf{r})\,\mathrm{d}\mathbf{r} + \int \psi^{\dagger}(\mathbf{r})\psi^{\dagger}(\mathbf{r}')V(\mathbf{r}-\mathbf{r}')\psi(\mathbf{r}')\psi(\mathbf{r})\,\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{r}'$$

so we get

$$\begin{aligned}
\langle\Phi|H_{op}|\Phi\rangle &= \int \left\langle\Phi|\psi^{\dagger}(\mathbf{r})h\psi(\mathbf{r})|\Phi\right\rangle \mathrm{d}\mathbf{r} + \int \left\langle\Phi|\psi^{\dagger}(\mathbf{r})\psi^{\dagger}(\mathbf{r}')V(\mathbf{r}-\mathbf{r}')\psi(\mathbf{r}')\psi(\mathbf{r})|\Phi\right\rangle \mathrm{d}\mathbf{r}\mathrm{d}\mathbf{r}' \\
&= \int \langle\Phi|\Phi\rangle\,\Phi^*(\mathbf{r})h\Phi(\mathbf{r})\,\mathrm{d}\mathbf{r} + \int \langle\Phi|\Phi\rangle\,\Phi^*(\mathbf{r})\Phi^*(\mathbf{r}')V(\mathbf{r}-\mathbf{r}')\Phi(\mathbf{r}')\Phi(\mathbf{r})\,\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{r}' \\
&= \int \Phi^*(\mathbf{r})h\Phi(\mathbf{r})\,\mathrm{d}\mathbf{r} + \int \Phi^*(\mathbf{r})\Phi^*(\mathbf{r}')V(\mathbf{r}-\mathbf{r}')\Phi(\mathbf{r}')\Phi(\mathbf{r})\,\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{r}'
\end{aligned}$$

bThe number operator

$$N_{op} = \int \mathrm{d}\mathbf{r}\psi^{\dagger}(\mathbf{r})\psi(\mathbf{r})$$

so we get

$$\langle\Phi|N_{op}|\Phi\rangle = \int \mathrm{d}\mathbf{r}|\Phi|^2$$

Problem 15

$$\begin{gathered}
H = -\tfrac{\hbar^2}{2mR^2}\tfrac{\partial^2}{\partial\phi^2} - \tfrac{g\hbar}{2}\mathbf{B}\cdot\boldsymbol{\sigma} \\
\mathbf{B} = B_1\hat{\mathbf{z}} + B_2(-\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi)
\end{gathered}$$

a) Treating the spin as the fast coordinate we use the Born Oppenheimer approximation

$$\psi = \xi_n(\phi)\otimes(\alpha(\phi)|\uparrow\rangle + \beta(\phi)|\downarrow\rangle)$$

the Hamiltonian for the spin part

$$\begin{aligned}
-\frac{g\hbar}{2}\mathbf{B}\cdot\boldsymbol{\sigma} &= -\frac{g\hbar}{2}\begin{pmatrix} B_1 & B_2\left(-\sin\phi - i\cos\phi\right) \\ B_2\left(-\sin\phi + i\cos\phi\right) & B_1 \end{pmatrix} \\
&= -\frac{g\hbar}{2}\begin{pmatrix} B_1 & -iB_2e^{-i\phi} \\ iB_2e^{i\phi} & -B_1 \end{pmatrix}
\end{aligned}$$

so we need to diagonalize the matrix, the characteristic polynomial

$$\lambda^2 - B_1^2 - B_2^2 = 0$$
$$\lambda_\pm = \pm\sqrt{B_1^2 + B_2^2}$$

The eigenvectors

$$|\pm\rangle = \frac{1}{\sqrt{2\left(B_1^2 + B_2^2 \mp B_1\sqrt{B_1^2 + B_2^2}\right)}}\left(iB_2e^{-i\phi}|\uparrow\rangle + \left(B_1 \mp \sqrt{B_1^2 + B_2^2}\right)|\downarrow\rangle\right)$$

with energies

$$\varepsilon_\pm = \mp\frac{g\hbar}{2}\sqrt{B_1^2 + B_2^2}$$

so the B.O Hamiltonian

$$H_{BO} = -\frac{\hbar^2}{2m}\left(\frac{1}{R}\partial_\phi + iA_\pm\right)^2 + \varepsilon_\pm$$

where

$$\begin{aligned}
A_\pm &= \frac{1}{R}\left\langle \pm|\partial_\phi|\pm\right\rangle \\
&= \frac{1}{R}\left(\langle\pm|\right)\left(\frac{B_2e^{-i\phi}|\uparrow\rangle}{\sqrt{2\left(B_1^2 + B_2^2 \mp B_1\sqrt{B_1^2 + B_2^2}\right)}}\right) \\
&= \frac{i}{R}\frac{B_2^2}{2\left(B_1^2 + B_2^2 \mp B_1\sqrt{B_1^2 + B_2^2}\right)} \\
&\equiv i\frac{b_\pm}{R}
\end{aligned}$$

to find the eigenstates we guess

$$\xi = e^{im\phi}$$

where m must be integer and we get a condition on the energies and on m .
The Berry vector potential we got is

$$\mathbf{A}_\pm = i\frac{b_\pm}{R}\hat{\boldsymbol{\phi}}$$

it is not uniquely defined since the eigenstates $|\pm\rangle$ are defined up to a phase $e^{i\chi(\phi)}$, so if we take

$$|\pm\rangle \to e^{i\chi(\phi)}|\pm\rangle$$

we get

$$\mathbf{A}_\pm \to \mathbf{A}_\pm + i\boldsymbol{\nabla}_\phi\chi\left(\phi\right)$$