Q.5 - A localized one photon state

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The most general one photon state can be written as

$$|\psi\rangle = \sum_{\alpha=1}^{2} \int d^3k \, \phi_\alpha(k) \, a^\dagger_{\bar{k},\alpha} |0\rangle \, ; \; \sum_\alpha \int d^3k \, |\phi_\alpha(k)|^2 = 1$$

To simplify I'll ignore polarization (has no effect in this problem).
I choose to define localization of the photon as localization of the electric field expectation value. Note that since neither $E, B, \mathcal{H}$ commute, defining localization of either of them will give different results.
Using the formula from class

$$E \propto i \int d^3k \sqrt{|k|} \left(a_{\bar{k}} e^{i\bar{k}\cdot\bar{r}} - a^\dagger_{\bar{k}} e^{-i\bar{k}\cdot\bar{r}} \right)$$

It is easy to see $\langle E \rangle = 0$ - odd number of creation operators. So the interesting value will be $\left\langle |E|^2 \right\rangle$

$$\left\langle |E|^2 \right\rangle \propto - \left\langle \int d^3k \, d^3k' \sqrt{|k|\,|k'|} \left(a_{\bar{k}} e^{i\bar{k}\cdot\bar{r}} - a^\dagger_{\bar{k}} e^{-i\bar{k}\cdot\bar{r}} \right) \left(a_{\bar{k}'} e^{i\bar{k}'\cdot\bar{r}} - a^\dagger_{\bar{k}'} e^{-i\bar{k}'\cdot\bar{r}} \right) \right\rangle$$

Since $|\psi\rangle$ is a one photon state $\langle a^2_{\bar{k}} \rangle = \left\langle a^{\dagger 2}_{\bar{k}} \right\rangle = 0, \; \forall \bar{k}$.

$$= \left\langle \int d^3k \int d^3k' \sqrt{|k|\,|k'|} \left(a_{\bar{k}} a^\dagger_{\bar{k}'} e^{i\left(\bar{k}-\bar{k}'\right)\cdot\bar{r}} + a^\dagger_{\bar{k}} a_{\bar{k}'} e^{-i\left(\bar{k}-\bar{k}'\right)\cdot\bar{r}} \right) \right\rangle$$

$$= \int d^3l \int dl' \int d^3k \int d^3k' \sqrt{|k|\,|k'|} \phi^*\left(\bar{l}\right) \phi\left(\bar{l}'\right) \left\langle 0 \left| a_{\bar{l}} \left(a_{\bar{k}} a^\dagger_{\bar{k}'} e^{i\left(\bar{k}-\bar{k}'\right)\cdot\bar{r}} + a^\dagger_{\bar{k}} a_{\bar{k}'} e^{-i\left(\bar{k}-\bar{k}'\right)\cdot\bar{r}} \right) a^\dagger_{\bar{l}'} \right| 0 \right\rangle$$

Using $\left\langle 0 \left| a_k a^\dagger_l \right| 0 \right\rangle = \delta(k-l)$, $\left\langle 0 \left| a^\dagger_k a_l \right| 0 \right\rangle = 0$ we get

$$= \int d^3l \int d^2l' \int d^3k \int d^3k' \sqrt{|k|\,|k'|} \phi^*\left(\bar{l}\right) \phi\left(\bar{l}'\right) \left\{ \begin{array}{c} e^{i\left(\bar{k}-\bar{k}'\right)\cdot\bar{r}} \left(\delta\left(\bar{k}-\bar{k}'\right) \delta\left(\bar{l}-\bar{l}'\right) + \delta\left(\bar{k}-\bar{l}'\right) \delta\left(\bar{k}'-\bar{l}\right) \right) \\ + e^{-i\left(\bar{k}-\bar{k}'\right)\cdot\bar{r}} \delta\left(\bar{l}'-\bar{k}'\right) \delta\left(\bar{k}-\bar{l}\right) \end{array} \right\}$$

$$= \int d^3l \int d^3k \, |k| \left|\phi\left(\bar{l}\right)\right|^2 + \int d^3l \int d^3k \sqrt{|k|\,|l|} \phi^*\left(\bar{l}\right) \phi\left(\bar{k}\right) e^{i\left(\bar{k}-\bar{l}\right)\cdot\bar{r}} + \int d^3l \int d^3k \sqrt{|k|\,|l|} \phi^*\left(\bar{k}\right) \phi\left(\bar{l}\right) e^{-i\left(\bar{k}-\bar{l}\right)\cdot\bar{r}}$$

$$= \int d^3l \int d^3k \, |k| \left|\phi\left(\bar{l}\right)\right|^2 + 2 \int d^3l \int d^3k \sqrt{|k|\,|l|} \phi^*\left(\bar{l}\right) \phi\left(\bar{k}\right) e^{i\left(\bar{k}-\bar{l}\right)\cdot\bar{r}}$$

The first term is related to the vacuum energy (and has no spatial dependance) so can be ignored. We want to get from the second term $\delta(r - r_0)$. First do so for $r_0 = 0$. Guess $\phi\left(\bar{k}\right) \propto \frac{1}{\sqrt{|k|}} e^{-\sigma^2 k^2}$.
Normalization factor

$$1 = \int d^3k \, |\phi(k)|^2 = \mathcal{N}^2 \int d^3k \, \frac{1}{|k|} e^{-2\sigma^2 k^2} = 4\pi \mathcal{N}^2 \int_0^\infty dk \, k e^{-2\sigma^2 k^2} = -\frac{\pi \mathcal{N}^2}{\sigma^2} \int_0^\infty dk \, \frac{\partial}{\partial k} \left(e^{-2\sigma^2 k^2} \right)$$

$$1 = \frac{\pi \mathcal{N}^2}{\sigma^2} \Rightarrow \mathcal{N} = \frac{\sigma}{\sqrt{\pi}}$$

Now insert our guess to the second term

$$\left\langle |E(r)|^2 \right\rangle \propto E_0 + \frac{\sigma^2}{\pi} \int d^3 l e^{-\sigma^2 l^2} e^{-i\bar{l}\cdot\bar{r}} \int d^3 k e^{-\sigma^2 k^2} e^{i\bar{k}\cdot\bar{r}}$$

$$\int d^3 k\, e^{-\sigma^2 k^2} e^{i\bar{k}\cdot\bar{r}} = \prod_i \int dk_i e^{-\sigma^2 k_i^2} e^{ik_i r_i} = e\left\{-\frac{r^2}{4\sigma^2}\right\} \prod_i \int dk_i e\left\{-\sigma^2 \left(k_i - i\frac{r_i}{2\sigma^2}\right)^2\right\} = \frac{\pi^{3/2}}{\sigma^3} e\left\{-\frac{r^2}{4\sigma^2}\right\}$$

$$\int d^3 l\, e^{-\sigma^2 l^2} e^{-i\bar{l}\cdot\bar{r}} = \frac{\pi^{3/2}}{\sigma^3} e\left\{-\frac{r^2}{4\sigma^2}\right\}$$

So finally we get

$$\left\langle |E(r)|^2 \right\rangle - E_0 \propto \frac{1}{\sigma^4} e\left\{-\frac{r^2}{2\sigma^2}\right\} \underbrace{\longrightarrow}_{\sigma\to 0} \delta(r)$$

By choosing ϕ as we did we got a localized electric field.
It is easy now to develop our state in time using our knowledge of the time evolution of the creation operator $a_k^\dagger(t) = a_k^\dagger(0)\, e^{-i\omega t}$, $\omega = c\,|k|$.

$$|\psi(t)\rangle = \int d^3 k\, \phi(k)\, a_k^\dagger(0)\, e^{-i\omega t}\, |0\rangle$$

As for the relevant electric field, we expect it to isotropically expand because it was vary well localized $\Rightarrow$ the momentum is unknown.
In order to understand the evolution of this expectation value in time we can consider the problem in the Heisenberg picture. As an operator E^2 satisfies the maxwell equations $\Longrightarrow$ satisfies wave equation. If the initail condition was a delta fucntion, the evolution would be as an isotropical spherical wave.
We can always add a phase to ϕ. To make the field move we can choose

$$\phi(\bar{k}) \to e^{i\bar{k}\cdot\bar{r}_0} \phi(\bar{k}) \Rightarrow \left\langle |E(r)|^2 \right\rangle - E_0 \propto e\left\{-\frac{(\bar{r}-\bar{r}_0)^2}{4\sigma^2}\right\}$$