QUESTION # 14

FIND EIGENFUNCTIONS AND EIGENVALUES OF $\psi_{op}(\vec{r})$

WE CAN EXPAND THE OPERATOR ON A BASIS

$$\psi_{op}(\vec{r}) = \sum_i \phi_i(\vec{r})\,\hat{a}_i \quad ; \quad \int \phi_i \phi_j^* \, dr = \delta_{ij}$$

$$[\psi(\vec{r}), \psi^+(\vec{r}\,')] = \sum_{ij} \phi_i(\vec{r})\,\phi_j^*(\vec{r}\,')\,[a_i, a_j^+] \stackrel{!}{=} \delta(\vec{r}-\vec{r}\,')$$

$$\Rightarrow \int dr'\,\delta(\vec{r}-\vec{r}\,')\,\phi_k(\vec{r}\,') = \int dr' \sum_{ij} \phi_i(\vec{r})\,\phi_j^*(\vec{r}\,')\,\phi_k(\vec{r}\,')\,[a_i, a_j^+] =$$

$$\Rightarrow \phi_k(\vec{r}) = \sum_i \phi_i(\vec{r})\,[a_i, a_k^+] \quad \Rightarrow \quad [a_i, a_k^+] = \delta_{ik}$$

WITH THIS ALGEBRA WE KNOW THE EIGENSTATES

$$a_i |\alpha_i\rangle = \alpha_i |\alpha_i\rangle$$ coherent state

$$|\alpha_i\rangle = e^{-\frac{1}{2}|\alpha_i|^2} \sum_{n=0}^{\infty} \frac{\alpha_i^n}{\sqrt{n!}} |n\rangle$$ number state

SO, THE EIGENSTATES OF $\psi_{op}(\vec{r})$ ARE

$$|\{\alpha_i\}\rangle = \bigotimes_i |\alpha_i\rangle$$

$$\psi_{op}(\vec{r})\,|\{\alpha_i\}\rangle = \left(\sum_i \phi_i(\vec{r})\,\alpha_i\right)|\{\alpha_i\}\rangle$$

WE ~~CAN DO~~ CANNOT DO THE SAME FOR $\psi_{op}^+(\vec{r})$, SINCE ~~THE~~ a^+ ~~$|\alpha_i\rangle$ ARE ALSO EIGENSTATES OF a^+.~~ HAS NO EIGENSTATES.

FOR THE FERMIONIC FIELD OPERATOR WE CAN DO THE SAME IF WE IMPOSE ANTI-COMMUTATION RELATIONS INSTEAD OF COMMUTATION.

$$\langle N_{op} \rangle = \langle \{\alpha_i\}| \psi^+_{op}(\vec{r})\, \psi_{op}(\vec{r})\, |\{\alpha_i\}\rangle =$$

$$= \langle \{\alpha_i\}| \sum_{ij} \int d^3r\, \phi_i^*(\vec{r})\, \phi_j(\vec{r})\, a_i^+ a_j\, | \{\alpha_i\}\rangle =$$

$$= \sum_{ij} \alpha_i^* \alpha_j\, \delta_{ij} = \sum_i |\alpha_i|^2$$

$$H_{op} \supset V_{int} = \int dr\, dr'\, \psi^+(\vec{r})\psi(\vec{r})\, V(r-r')\, \psi^+(r')\psi(r') =$$

$$= \int dr\, dr' \sum_{\substack{ij\\ke}} \phi_i^*(r)\phi_j(r)\phi_k^*(r')\phi_e(r')\, a_i^+ a_j\, a_k^+ a_e\, V_{ijke}$$

$$\langle V_{int} \rangle = \sum_{ij} |\alpha_i|^2 |\alpha_j|^2\, V_{ij}$$

IS V IS DIAGONALIZED IN OUR BASIS OF OPERATORS $\{a_i\}$.

$$H_{op} = H_0 + V_{int}$$

$$H_0 = \int d^3r\, \psi^+(\vec{r}) \overbrace{\left(-\frac{\hbar^2}{2m}\nabla^2 + U(\vec{r})\right)}^{h_0(r)} \psi(\vec{r})$$

ASSUME NOW THAT THE COEFFICIENT ϕ_i ARE EIGENFUNCTIONS OF h_0: $h_0\, \phi_i(r) = \varepsilon_i\, \phi_i(r)$

$$\Rightarrow\ H_0 = \sum_i \varepsilon_i\, a_i^+ a_i$$

$$V_{int} = \int d^3r\, d^3r'\, \psi^+(r)\psi(r)\, V(r-r')\, \psi^+(r')\psi(r') =$$

$$= \int d^3r\, d^3r' \sum_{\substack{ij\\ke}} \phi_i^*(r)\phi_j(r)\phi_k^*(r')\phi_e(r')\, V(r-r')\, a_i^+ a_j\, a_k^+ a_e$$

SINCE IT HAS TO BE A TWO BODY INTERACTION, ONLY TWO KIND OF BOSONS AT THE TIME CAN BE INVOLVED

$$\Rightarrow V_{ijke} = V_{ijke}\,\delta_{ij}\,\delta_{ke} \equiv V_{ik}$$

$$\Rightarrow V_{int} = \int d^3r\, d^3r' \sum_{ik} \phi_i^*(r)\,\phi_i(r)\,\phi_k^*(r')\,\phi_k(r')\,V_{ik}(r-r')$$

$$\cdot\, a_i^+ a_i\; a_k^+ a_k$$

$$\Rightarrow \langle V_{int} \rangle = \langle \{\alpha_i\} | V_{int} | \{\alpha_i\} \rangle =$$

$$= \int d^3r\, d^3r' \sum_{ik} \phi_i^*(r)\,\phi_i(r)\,\phi_k^*(r')\,\phi_k(r')\,V_{ik}(r-r')\cdot$$

$$|\alpha_i|^2\,|\alpha_k|^2$$

NOW, THE FIRST QUANTIZED INTERACTION IS

$$\int d^3r\, d^3r' \sum_{\substack{ij\\ ke}} \phi_i^*(r)\,\phi_j(r)\,\phi_k^*(r')\,\phi_e(r')\,V(r-r') =$$

$$= \langle \phi_i\,\phi_k | V | \phi_j\,\phi_e \rangle \equiv \tilde{V}_{ik,je}$$

SINCE IT HAS TO BE A TWO BODY INTERACTION, THE ONLY TERMS POSSIBLE ARE DIAGONAL IN THE SENSE

$$\tilde{V}_{ik,je} = V_{ik,je}\,\delta_{ji}\,\delta_{ke}$$

$$\Rightarrow \quad V_{int} = \sum_{\substack{ij\\ke}} \tilde{V}_{ik,je} \; a_i^+ a_j a_k^+ a_e =$$

$$= \sum_{\substack{ikj\\ke}} V_{ik,je} \, \delta_{ij} \, \delta_{ke} \; a_i^+ a_j a_k^+ a_e =$$

$$= \sum_{ik} V_{ik,ik} \; a_i^+ a_i a_k^+ a_k$$

$$\Rightarrow \quad \langle V_{int} \rangle = \langle \{\alpha_i\} | V_{int} | \{\alpha_i\} \rangle =$$

$$= \sum_{ik} V_{ik,ik} \, |\alpha_i|^2 \, |\alpha_k|^2$$