95

QM1 - Problem Set 6

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1

$$H=\sum_{\alpha}\varepsilon_{\alpha}a_{\alpha}^{\dagger}a_{\alpha}+\frac{1}{2}\sum_{\alpha\beta\gamma\delta}V_{\alpha\beta\gamma\delta}a_{\alpha}^{\dagger}a_{\beta}^{\dagger}a_{\delta}a_{\gamma}$$

(a)

$$\begin{aligned}
V_{\alpha\beta}^{(1)} &\equiv V_{\alpha\beta\alpha\beta}\\
V_{\alpha\beta}^{(2)} &\equiv V_{\alpha\beta\beta\alpha}\\
V_{\alpha\beta\gamma\delta} &= \delta_{\alpha\gamma}\delta_{\beta\delta}V_{\alpha\beta}^{(1)}+\delta_{\alpha\delta}\delta_{\beta\gamma}V_{\alpha\beta}^{(2)}\\
H &= \sum_{\alpha}\varepsilon_{\alpha}a_{\alpha}^{\dagger}a_{\alpha}+\frac{1}{2}\sum_{\alpha\beta}V_{\alpha\beta}^{(1)}a_{\alpha}^{\dagger}a_{\beta}^{\dagger}a_{\beta}a_{\alpha}-\frac{1}{2}\sum_{\alpha\beta}V_{\alpha\beta}^{(2)}a_{\alpha}^{\dagger}a_{\beta}^{\dagger}a_{\beta}a_{\alpha}\\
&= \sum_{\alpha}\varepsilon_{\alpha}a_{\alpha}^{\dagger}a_{\alpha}+\frac{1}{2}\sum_{\alpha\beta}\left(V_{\alpha\beta}^{(1)}-V_{\alpha\beta}^{(2)}\right)a_{\alpha}^{\dagger}a_{\beta}^{\dagger}a_{\beta}a_{\alpha}
\end{aligned}$$

(Noting that because we deal with fermions $V_{\alpha\alpha}^{(1)}=V_{\alpha\alpha}^{(2)}=0$)

So we see that for every a_i we have a $a_i^{\dagger}$, which means for every a_i we have a $U(1)$ symmetry and suggests that $N_i=a_i^{\dagger}a_i$ commutes with the Hamiltonian:

$$\begin{aligned}
[H,n_i] &= \sum_{\alpha}\varepsilon_{\alpha}\left[a_{\alpha}^{\dagger}a_{\alpha},N_i\right]+\frac{1}{2}\sum_{\alpha\beta}\left(V_{\alpha\beta}^{(1)}-V_{\alpha\beta}^{(2)}\right)\left[a_{\alpha}^{\dagger}a_{\beta}^{\dagger}a_{\beta}a_{\alpha},N_i\right]\\
&= 0
\end{aligned}$$

Which means that the number of particles is a conserved quantity, which means that our total symmetry is the adjoint where $U=\exp(iq_iN_i)$:

$$\begin{aligned}
H &\to \exp(iq_iN_i)\,H\exp(-iq_iN_i)\\
&= \left(1+iq_iN_i+O\left(n_i^2\right)\right)H\left(1-iq_iN_i+O\left(n_i^2\right)\right)\\
&= H+iq_iN_iH-iq_iHN_i+O\left(n_i^2\right)\\
&= H+iq_i\left[N_i,H\right]\\
&= H
\end{aligned}$$

So because the number of particles (n_i) in each state is conserved (a $U(1)\otimes U(1)\otimes\ldots$ symmetry) we know that our eigenvalues and eigenfunctions will be the states with a fixed number of particles (Fock states) and then we can calculate:

$$\begin{aligned}
H\left|\{n_i\}\right\rangle &= \left(\sum_{\alpha}\varepsilon_{\alpha}N_{\alpha}+\frac{1}{2}\sum_{\alpha\beta}\left(V_{\alpha\beta}^{(1)}-V_{\alpha\beta}^{(2)}\right)N_{\alpha}N_{\beta}\right)\left|\{n_i\}\right\rangle\\
&= \left(\sum_{\alpha}\varepsilon_{\alpha}n_{\alpha}+\frac{1}{2}\sum_{\alpha\beta}\left(V_{\alpha\beta}^{(1)}-V_{\alpha\beta}^{(2)}\right)n_{\alpha}n_{\beta}\right)\left|\{n_i\}\right\rangle
\end{aligned}$$

Which means $|\{n_i\}\rangle = \prod_i a_i^\dagger |0\rangle$ are indeed the eigenfunctions with the energies $E = \sum_i \varepsilon_i n_i + \frac{1}{2}\sum_{ij}\left(V_{ij}^{(1)} - V_{ij}^{(2)}\right) n_i n_j$.

b)

First we'll define our trial function:

$$\begin{aligned}
a_i^\dagger &= \sum_j C_{ij} b_j^\dagger \\
|\phi\rangle &= \prod_i b_i^\dagger |0\rangle \\
b_i^\dagger &= \sum_j D_{ij} a_j^\dagger = \sum_{jj'} D_{ij} C_{jj'} b_{j'}^\dagger \\
&\rightarrow \sum_j D_{ij} C_{jj'} = \delta_{ij'} \\
&\rightarrow b_i^\dagger = \sum_j C_{ij}^* a_j^\dagger
\end{aligned}$$

And then we'll calculate:

$$\begin{aligned}
a_\alpha^\dagger a_\alpha &= \sum_{j_1}\sum_{j_2} C_{\alpha j_1} C_{\alpha j_2}^* b_{j_1}^\dagger b_{j_2} \\
a_\alpha^\dagger a_\beta^\dagger a_\beta a_\alpha &= \sum_{j_1}\sum_{j_2}\sum_{j_3}\sum_{j_4} C_{\alpha j_1} C_{\beta j_2} C_{\beta j_3}^* C_{\alpha j_4}^* b_{j_1}^\dagger b_{j_2}^\dagger b_{j_3} b_{j_4} \\
a_\alpha^\dagger a_\beta^\dagger a_\alpha a_\beta &= \sum_{j_1}\sum_{j_2}\sum_{j_3}\sum_{j_4} C_{\alpha j_1} C_{\beta j_2} C_{\alpha j_3}^* C_{\beta j_4}^* b_{j_1}^\dagger b_{j_2}^\dagger b_{j_3} b_{j_4} \\
H &= \sum_{\alpha j_1 j_2} C_{\alpha j_1} C_{\alpha j_2}^* \varepsilon_\alpha b_{j_1}^\dagger b_{j_2} + \frac{1}{2}\sum_{\alpha\beta j_1 j_2 j_3 j_4} C_{\alpha j_1} C_{\beta j_2}\left(V_{\alpha\beta}^{(1)} C_{\beta j_3}^* C_{\alpha j_4}^* + V_{\alpha\beta}^{(2)} C_{\alpha j_3}^* C_{\beta j_4}^*\right) b_{j_1}^\dagger b_{j_2}^\dagger b_{j_3} b_{j_4} \\
\left\langle \phi \middle| b_{j_1}^\dagger b_{j_2} \middle| \phi \right\rangle &= \delta_{j_1 j_2} n_{j_1} \\
\left\langle \phi \middle| b_{j_1}^\dagger b_{j_2}^\dagger b_{j_3} b_{j_4} \middle| \phi \right\rangle &= \left[\delta_{j_1 j_4}\delta_{j_2 j_3} - \delta_{j_1 j_3}\delta_{j_2 j_4}\right] n_{j_1} n_{j_2} \\
\langle \phi | H | \phi \rangle &= \sum_{\alpha j_1} C_{\alpha j_1} C_{\alpha j_1}^* \varepsilon_\alpha n_{j_1} + \\
&\quad + \frac{1}{2}\sum_{\alpha\beta j_1 j_2} C_{\alpha j_1} C_{\beta j_2}\left[\left(V_{\alpha\beta}^{(1)} C_{\beta j_2}^* C_{\alpha j_1}^* + V_{\alpha\beta}^{(2)} C_{\alpha j_2}^* C_{\beta j_1}^*\right) - \left(V_{\alpha\beta}^{(1)} C_{\beta j_1}^* C_{\alpha j_2}^* + V_{\alpha\beta}^{(2)} C_{\alpha j_1}^* C_{\beta j_2}^*\right)\right] n_{j_1} n_{j_2} \\
&= \sum_{\alpha j_1} C_{\alpha j_1} C_{\alpha j_1}^* \varepsilon_\alpha n_{j_1} + \frac{1}{2}\sum_{\alpha\beta j_1 j_2}\left[\left(V_{\alpha\beta}^{(1)} - V_{\alpha\beta}^{(2)}\right)\left(|C_{\alpha j_1}|^2 |C_{\beta j_2}|^2 - C_{\alpha j_1} C_{\beta j_2} C_{\alpha j_2}^* C_{\beta j_1}^*\right)\right] n_{j_1} n_{j_2} \\
\langle \phi | \phi \rangle &= \sum_{ij} C_{ij} C_{ij}^*
\end{aligned}$$

Now we'll use Hartree-Fock by looking for the minimum by deriving with respect to C_{ij}, we want to calculate $\frac{d}{dC_{ij}}\left(\langle \phi | H | \phi \rangle - E\langle \phi | \phi \rangle\right) \equiv D_1 - D_2 = 0$:

$$
\begin{aligned}
D_1 &= \frac{d}{dC_{ij}} \langle \phi \,|\, H \,|\, \phi \rangle \\
&= C_{ij}^* \varepsilon_i n_j + \frac{1}{2} \sum_{\alpha j_2} \left[\left(V_{i\alpha}^{(1)} - V_{i\alpha}^{(2)} \right) \left(C_{\alpha j_2} C_{ij}^* C_{\alpha j_2}^* - C_{\alpha j_2} C_{ij_2}^* C_{\alpha j}^* \right) + \left(V_{\alpha i}^{(1)} - V_{\alpha i}^{(2)} \right) \left(C_{\alpha j_2} C_{\alpha j_2}^* C_{ij}^* - C_{\alpha j_2} C_{\alpha j}^* C_{ij_2}^* \right) \right] n_j n_{j_2} \\
&= C_{ij}^* \varepsilon_i n_j + \frac{1}{2} \sum_{\alpha j_2} \left[\left(V_{i\alpha}^{(1)} - V_{i\alpha}^{(2)} + V_{\alpha i}^{(1)} - V_{\alpha i}^{(2)} \right) C_{\alpha j_2} \left(C_{ij}^* C_{\alpha j_2}^* - C_{ij_2}^* C_{\alpha j}^* \right) \right] n_j n_{j_2} \\
D_2 &= \frac{d}{dC_{ij}} \left(\sum_{i'j'} E_{i'} C_{i'j'} C_{i'j'}^* \right) = E_i C_{ij}^* \\
E_i C_{ij}^* &= C_{ij}^* \varepsilon_i n_j + \frac{1}{2} \sum_{\alpha j_2} \left[\left(V_{i\alpha}^{(1)} - V_{i\alpha}^{(2)} + V_{\alpha i}^{(1)} - V_{\alpha i}^{(2)} \right) C_{\alpha j_2} \left(C_{ij}^* C_{\alpha j_2}^* - C_{ij_2}^* C_{\alpha j}^* \right) \right] n_j n_{j_2}
\end{aligned}
$$

To solve this Hartree-Fock equation, we'll guess that $C_{ij} = \delta_{ij}$ and then:

$$
\begin{aligned}
E_i \delta_{ij} &= \delta_{ij} \varepsilon_i n_i + \frac{1}{2} \sum_{\alpha j_2} \left[\left(V_{i\alpha}^{(1)} - V_{i\alpha}^{(2)} + V_{\alpha i}^{(1)} - V_{\alpha i}^{(2)} \right) \delta_{\alpha j_2} \delta_{ij} \delta_{\alpha j_2} \right] n_j n_{j_2} \\
&= \delta_{ij} \varepsilon_i n_i + \frac{1}{2} \sum_{\alpha} \left(V_{i\alpha}^{(1)} - V_{i\alpha}^{(2)} + V_{\alpha i}^{(1)} - V_{\alpha i}^{(2)} \right) \delta_{ij} n_i n_\alpha \\
E_i &= n_i \left[\varepsilon_i + \frac{1}{2} \sum_{\alpha} n_\alpha \left(V_{i\alpha}^{(1)} - V_{i\alpha}^{(2)} + V_{\alpha i}^{(1)} - V_{\alpha i}^{(2)} \right) \right]
\end{aligned}
$$

In order to find the energy we need to realize that what we've got is the energy of a particle in a single state $b_i^\dagger |0\rangle$, so we need to sum over all states:

$$
\begin{aligned}
E &= \sum_i E_i \\
&= \sum_i n_i \varepsilon_i + \frac{1}{4} \sum_{\alpha, i} n_i n_\alpha \left(V_{i\alpha}^{(1)} - V_{i\alpha}^{(2)} + V_{\alpha i}^{(1)} - V_{\alpha i}^{(2)} \right) \\
&= \sum_i n_i \varepsilon_i + \frac{1}{2} \sum_{\alpha, i} n_i n_\alpha \left(V_{i\alpha}^{(1)} - V_{i\alpha}^{(2)} \right)
\end{aligned}
$$

Note that we've divided and multiplied by 2 because of double summation and the fact that $\sum_{ij} V_{ij} = \sum_{ij} V_{ji}$.

So we reproduced the exact solutions from the previous section:

$$
\begin{aligned}
|\phi\rangle &= \prod_i b_i^\dagger |0\rangle = \prod_i \sum_j \delta_{ij} a_j^\dagger |0\rangle = \prod_i a_i^\dagger |0\rangle \\
E &= \sum_i \varepsilon_i n_i + \frac{1}{2} \sum_{i,j} n_i n_j \left(V_{ij}^{(1)} - V_{ij}^{(2)} \right)
\end{aligned}
$$

c)

First we'll calculate:

$$
\begin{aligned}
[H, a_i] &= \sum_\alpha \varepsilon_\alpha \left[a_\alpha^\dagger a_\alpha, a_i\right] + \frac{1}{2}\sum_{\alpha\beta} V_{\alpha\beta}^{(1)} \left[a_\alpha^\dagger a_\beta^\dagger a_\beta a_\alpha, a_i\right] + \frac{1}{2}\sum_{\alpha\beta} V_{\alpha\beta}^{(2)} \left[a_\alpha^\dagger a_\beta^\dagger a_\alpha a_\beta, a_i\right] \\
\left[H, a_i^\dagger\right] &= \sum_\alpha \varepsilon_\alpha \left[a_\alpha^\dagger a_\alpha, a_i^\dagger\right] + \frac{1}{2}\sum_{\alpha\beta} V_{\alpha\beta}^{(1)} \left[a_\alpha^\dagger a_\beta^\dagger a_\beta a_\alpha, a_i^\dagger\right] + \frac{1}{2}\sum_{\alpha\beta} V_{\alpha\beta}^{(2)} \left[a_\alpha^\dagger a_\beta^\dagger a_\alpha a_\beta, a_i^\dagger\right] \\
\left[a_\alpha^\dagger a_\alpha, a_i\right] &= a_\alpha^\dagger \left\{a_\alpha, a_i\right\} - \left\{a_\alpha^\dagger, a_i\right\} a_\alpha = -\delta_{\alpha i} a_\alpha \\
\left[a_\alpha^\dagger a_\alpha, a_i^\dagger\right] &= a_\alpha^\dagger \left\{a_\alpha, a_i^\dagger\right\} - \left\{a_\alpha^\dagger, a_i^\dagger\right\} a_\alpha = a_\alpha^\dagger \delta_{\alpha i} \\
\left[a_\alpha^\dagger a_\beta^\dagger a_\beta a_\alpha, a_i\right] &= a_\alpha^\dagger a_\beta^\dagger \left[a_\beta a_\alpha, a_i\right] + \left[a_\alpha^\dagger a_\beta^\dagger, a_i\right] a_\beta a_\alpha \\
&= \left(a_\alpha^\dagger \left\{a_\beta^\dagger, a_i\right\} - \left\{a_\alpha^\dagger, a_i\right\} a_\beta^\dagger\right) a_\beta a_\alpha \\
&= \left(a_\alpha^\dagger \delta_{\beta i} - \delta_{\alpha i} a_\beta^\dagger\right) a_\beta a_\alpha \\
&= a_\alpha^\dagger a_i a_\alpha \delta_{\beta i} - \delta_{\alpha i} a_\beta^\dagger a_\beta a_i \\
\left[a_\alpha^\dagger a_\beta^\dagger a_\beta a_\alpha, a_i^\dagger\right] &= a_\alpha^\dagger a_\beta^\dagger \left[a_\beta a_\alpha, a_i^\dagger\right] + \left[a_\alpha^\dagger a_\beta^\dagger, a_i^\dagger\right] a_\beta a_\alpha \\
&= a_\alpha^\dagger a_\beta^\dagger \left(a_\beta \left\{a_\alpha, a_i^\dagger\right\} - \left\{a_\beta, a_i^\dagger\right\} a_\alpha\right) \\
&= a_\alpha^\dagger a_\beta^\dagger \left(a_\beta \delta_{\alpha i} - \delta_{\beta i} a_\alpha\right) \\
&= a_i^\dagger a_\beta^\dagger a_\beta \delta_{\alpha i} - \delta_{\beta i} a_\alpha^\dagger a_i^\dagger a_\alpha \\
\left[a_\alpha^\dagger a_\beta^\dagger a_\alpha a_\beta, a_i\right] &= a_\alpha^\dagger a_\beta^\dagger \left[a_\alpha a_\beta, a_i\right] + \left[a_\alpha^\dagger a_\beta^\dagger, a_i\right] a_\alpha a_\beta \\
&= \left(a_\alpha^\dagger \left\{a_\beta^\dagger, a_i\right\} - \left\{a_\alpha^\dagger, a_i\right\} a_\beta^\dagger\right) a_\alpha a_\beta \\
&= \left(a_\alpha^\dagger \delta_{\beta i} - \delta_{\alpha i} a_\beta^\dagger\right) a_\alpha a_\beta \\
&= a_\alpha^\dagger a_\alpha a_i \delta_{\beta i} - \delta_{\alpha i} a_\beta^\dagger a_i a_\beta \\
\left[a_\alpha^\dagger a_\beta^\dagger a_\alpha a_\beta, a_i^\dagger\right] &= a_\alpha^\dagger a_\beta^\dagger \left[a_\alpha a_\beta, a_i^\dagger\right] + \left[a_\alpha^\dagger a_\beta^\dagger, a_i^\dagger\right] a_\alpha a_\beta \\
&= a_\alpha^\dagger a_\beta^\dagger \left(a_\alpha \left\{a_\beta, a_i^\dagger\right\} - \left\{a_\alpha, a_i^\dagger\right\} a_\beta\right) \\
&= a_\alpha^\dagger a_\beta^\dagger \left(a_\alpha \delta_{\beta i} - \delta_{\alpha i} a_\beta\right) \\
&= a_\alpha^\dagger a_i^\dagger a_\alpha \delta_{\beta i} - \delta_{\alpha i} a_i^\dagger a_\beta^\dagger a_\beta
\end{aligned}
$$

Next we can find:

$$
\begin{aligned}
[H,a_i] &= -\varepsilon_i a_i + \frac{1}{2}\left(\sum_\alpha V^{(1)}_{\alpha i} a^\dagger_\alpha a_i a_\alpha - \sum_\beta V^{(1)}_{i\beta} a^\dagger_\beta a_\beta a_i\right) + \frac{1}{2}\left(\sum_\alpha V^{(2)}_{\alpha i} a^\dagger_\alpha a_\alpha a_i - \sum_\beta V^{(2)}_{i\beta} a^\dagger_\beta a_i a_\beta\right) \\
&= -\varepsilon_i a_i + \frac{1}{2}\sum_\alpha \left(V^{(1)}_{\alpha i} a^\dagger_\alpha a_i a_\alpha - V^{(1)}_{i\alpha} a^\dagger_\alpha a_\alpha a_i + V^{(2)}_{\alpha i} a^\dagger_\alpha a_\alpha a_i - V^{(2)}_{i\alpha} a^\dagger_\alpha a_i a_\alpha\right) \\
&= -\varepsilon_i a_i + \frac{1}{2}\sum_\alpha \left(-V^{(1)}_{\alpha i} a^\dagger_\alpha a_\alpha - V^{(1)}_{i\alpha} a^\dagger_\alpha a_\alpha + V^{(2)}_{\alpha i} a^\dagger_\alpha a_\alpha + V^{(2)}_{i\alpha} a^\dagger_\alpha a_\alpha\right) a_i \\
&= -\left[\varepsilon_i + \frac{1}{2}\sum_\alpha \left(V^{(1)}_{\alpha i} + V^{(1)}_{i\alpha} - V^{(2)}_{\alpha i} - V^{(2)}_{i\alpha}\right) a^\dagger_\alpha a_\alpha\right] a_i \\
\left[H,a^\dagger_i\right] &= \varepsilon_i a^\dagger_i + \frac{1}{2}\left(\sum_\beta V^{(1)}_{i\beta} a^\dagger_i a^\dagger_\beta a_\beta - \sum_\alpha V^{(1)}_{\alpha i} a^\dagger_\alpha a^\dagger_i a_\alpha\right) + \frac{1}{2}\left(\sum_\alpha V^{(2)}_{\alpha i} a^\dagger_\alpha a^\dagger_i a_\alpha - \sum_\beta V^{(2)}_{i\beta} a^\dagger_i a^\dagger_\beta a_\beta\right) \\
&= \varepsilon_i a^\dagger_i + \frac{1}{2}\sum_\alpha \left(V^{(1)}_{i\alpha} a^\dagger_i a^\dagger_\alpha a_\alpha - V^{(1)}_{\alpha i} a^\dagger_\alpha a^\dagger_i a_\alpha + V^{(2)}_{\alpha i} a^\dagger_\alpha a^\dagger_i a_\alpha - V^{(2)}_{i\alpha} a^\dagger_i a^\dagger_\alpha a_\alpha\right) \\
&= \varepsilon_i a^\dagger_i + \frac{1}{2}\sum_\alpha \left(V^{(1)}_{i\alpha} a^\dagger_i a^\dagger_\alpha a_\alpha + V^{(1)}_{\alpha i} a^\dagger_i a^\dagger_\alpha a_\alpha - V^{(2)}_{\alpha i} a^\dagger_i a^\dagger_\alpha a_\alpha - V^{(2)}_{i\alpha} a^\dagger_i a^\dagger_\alpha a_\alpha\right) \\
&= a^\dagger_i \left[\varepsilon_i + \frac{1}{2}\sum_\alpha \left(V^{(1)}_{\alpha i} + V^{(1)}_{i\alpha} - V^{(2)}_{\alpha i} - V^{(2)}_{i\alpha}\right) a^\dagger_\alpha a_\alpha\right]
\end{aligned}
$$

Now we can look at the Heisenberg equations (noting that $a^\dagger_\alpha a_\alpha$ is conserved, and therefore we can replace it with n_α which is not time dependent):

$$
\begin{aligned}
B_i &= \varepsilon_i + \frac{1}{2}\sum_\alpha \left(V^{(1)}_{\alpha i} + V^{(1)}_{i\alpha} - V^{(2)}_{\alpha i} - V^{(2)}_{i\alpha}\right) n_\alpha \\
\frac{d}{dt} a_i &= \frac{i}{\hbar}[H,a_i] = -\frac{i}{\hbar} B_i a_i \\
a_i &= a_i(t=0)\exp\left(-\frac{i}{\hbar} B_i t\right) \\
\frac{d}{dt} a^\dagger_i &= \frac{i}{\hbar}\left[H,a^\dagger_i\right] = \frac{i}{\hbar} a^\dagger_i B_i \\
a^\dagger_i &= a^\dagger_i(t=0)\exp\left(\frac{i}{\hbar} B_i t\right)
\end{aligned}
$$

Next we'll try to find the spectrum. Assuming $|\psi\rangle$ is the eigenfunction with M particles ($H|\psi\rangle = E_M|\psi\rangle$):

$$
\begin{aligned}
H\left(a^\dagger|\psi\rangle\right) &= a^\dagger H|\psi\rangle + \left[H,a^\dagger\right]|\psi\rangle \\
&= a^\dagger E_M|\psi\rangle + B a^\dagger|\psi\rangle \\
&= (E_M + B)\left(a^\dagger|\psi\rangle\right) \\
H(a|\psi\rangle) &= aH|\psi\rangle + [H,a]|\psi\rangle \\
&= aE_M|\psi\rangle - Ba|\psi\rangle \\
&= (E_M - B)(a|\psi\rangle)
\end{aligned}
$$

So we've got that the wave functions for $M+1$ and $M-1$ particles are $a^\dagger|\psi\rangle$ and $a|\psi\rangle$, and the energies are $E_M \pm \Delta E$, where the energy difference from adding/removing a particle in a state i is:

$$
\Delta E_i = \varepsilon_i + \frac{1}{2}\sum_\alpha \left(V^{(1)}_{\alpha i} + V^{(1)}_{i\alpha} - V^{(2)}_{\alpha i} - V^{(2)}_{i\alpha}\right) n_\alpha
$$

From that we can get the same spectrum as before by summing only over the particles in the state (that were "added"):

$$
\begin{aligned}
|\psi\rangle &= \prod_{i=1}^{M} a_i^\dagger |0\rangle \\
E &= \sum_{i=1}^{M} \Delta E_i = \sum_{i=1}^{M} \left[\varepsilon_i + \frac{1}{2} \sum_{\alpha} \left(V_{\alpha i}^{(1)} - V_{\alpha i}^{(2)} \right) n_\alpha \right]
\end{aligned}
$$

2

$$
\begin{aligned}
H &= -\frac{\hbar^2}{2m} \nabla^2 + U(r) \\
\psi &= \sqrt{\rho} \exp(i\chi)
\end{aligned}
$$

First let us calculate:

$$
\begin{aligned}
H\psi &= -\frac{\hbar^2}{2m} \nabla^2 \left(\sqrt{\rho} \exp(i\chi) \right) + U\sqrt{\rho} \exp(i\chi) \\
&= -\frac{\hbar^2}{2m} \left[\left(\nabla^2 \sqrt{\rho} \right) \exp(i\chi) + 2 \left(\nabla \sqrt{\rho} \right) \left(\nabla \exp(i\chi) \right) + \sqrt{\rho} \left(\nabla^2 \exp(i\chi) \right) \right] + U\sqrt{\rho} \exp(i\chi) \\
&= -\frac{\hbar^2}{2m} \nabla \left(\frac{\nabla \rho}{2\sqrt{\rho}} \exp(i\chi) + i\sqrt{\rho} \exp(i\chi) \nabla \chi \right) + U\sqrt{\rho} \exp(i\chi) \\
&= \left[-\frac{\hbar^2}{2m} \left(\frac{\nabla^2 \rho}{2\sqrt{\rho}} - \frac{(\nabla \rho)^2}{4\sqrt{\rho^3}} + \frac{i}{\sqrt{\rho}} \nabla \rho \nabla \chi - \sqrt{\rho} (\nabla \chi)^2 + i\sqrt{\rho} \nabla^2 \chi \right) + U\sqrt{\rho} \right] \exp(i\chi) \\
i\hbar \frac{\partial \psi}{\partial t} &= i\hbar \left(\frac{1}{2\sqrt{\rho}} \frac{\partial \rho}{\partial t} + i\sqrt{\rho} \frac{\partial \chi}{\partial t} \right) \exp(i\chi)
\end{aligned}
$$

Now we can write the time-dependent Schrodinger equation:

$$
\begin{aligned}
0 &= H\psi - i\hbar \frac{\partial \psi}{\partial t} \\
0 &= -\frac{\hbar^2}{2m} \left(\frac{\nabla^2 \rho}{2\sqrt{\rho}} - \frac{(\nabla \rho)^2}{4\sqrt{\rho^3}} + \frac{i}{\sqrt{\rho}} \nabla \rho \nabla \chi - \sqrt{\rho} (\nabla \chi)^2 + i\sqrt{\rho} \nabla^2 \chi \right) + U\sqrt{\rho} - i\hbar \left(\frac{1}{2\sqrt{\rho}} \frac{\partial \rho}{\partial t} + i\sqrt{\rho} \frac{\partial \chi}{\partial t} \right) \\
&= \frac{\hbar^2}{2m} \left(-\frac{\nabla^2 \rho}{2\sqrt{\rho}} + \frac{(\nabla \rho)^2}{4\sqrt{\rho^3}} + \sqrt{\rho} (\nabla \chi)^2 + \frac{2m}{\hbar} \sqrt{\rho} \frac{\partial \chi}{\partial t} + \frac{2mU}{\hbar^2} \sqrt{\rho} \right) - i\frac{\hbar^2}{2m} \left(\frac{\nabla \rho \nabla \chi}{\sqrt{\rho}} + \sqrt{\rho} \nabla^2 \chi + \frac{m}{\hbar \sqrt{\rho}} \frac{\partial \rho}{\partial t} \right)
\end{aligned}
$$

Now using the fact that χ and ρ are real and defining $v = \nabla \chi$ we can get that:

$$
\begin{aligned}
0 &= -\frac{\nabla^2 \rho}{2\rho} + \frac{(\nabla \rho)^2}{4\rho^2} + v^2 + \frac{2m}{\hbar} \frac{\partial \chi}{\partial t} + \frac{2mU}{\hbar^2} \\
0 &= v \nabla \rho + \rho \nabla v + \frac{m}{\hbar} \frac{\partial \rho}{\partial t} = \frac{m}{\hbar} \frac{\partial \rho}{\partial t} + \nabla (\rho v)
\end{aligned}
$$

Defining $u = \frac{\hbar}{m} v$ we get that the second equation is:

$$
\frac{\partial \rho}{\partial t} + \nabla (\rho u) = 0
$$

Which is the continuity equation (and the first Euler equation).
Taking the ∇ of the first equation we get:

$$
\begin{aligned}
0 &= -\nabla \frac{\nabla^2 \rho}{2\rho} + \nabla \frac{(\nabla \rho)^2}{4\rho^2} + 2v \cdot \nabla v + \frac{2m}{\hbar} \frac{\partial v}{\partial t} + \frac{2m}{\hbar^2} \nabla U \\
0 &= \frac{\partial u}{\partial t} + (u \cdot \nabla) u + \left(\frac{\hbar}{2m} \right)^2 \underbrace{\nabla \left(\frac{(\nabla \rho)^2}{2\rho^2} - \frac{\nabla^2 \rho}{\rho} \right)}_{\equiv A} + \frac{1}{m} \nabla U(r)
\end{aligned}
$$

Trying to calculate A:

$$\begin{aligned}
A &= \partial_i\left(\frac{(\partial_j\rho)^2}{2\rho^2}-\frac{\partial_j\partial_j\rho}{\rho}\right)\\
&= \frac{(\partial_j\rho)\,\partial_i\partial_j\rho}{\rho^2}-\frac{(\partial_j\rho)^2\,\partial_i\rho}{\rho^3}-\frac{\partial_i\partial_j\partial_j\rho}{\rho}+\frac{(\partial_j\partial_j\rho)\,(\partial_i\rho)}{\rho^2}\\
&= \frac{1}{\rho}\left(\frac{(\partial_j\rho)\,\partial_i\partial_j\rho}{\rho}-\frac{(\partial_j\rho)^2\,\partial_i\rho}{\rho^2}-\partial_i\partial_j\partial_j\rho+\frac{(\partial_j\partial_j\rho)\,(\partial_i\rho)}{\rho}\right)
\end{aligned}$$

We can calculate and see that:

$$\partial_j\left(\frac{(\partial_i\rho)\,(\partial_j\rho)}{\rho}\right)=\left(\frac{(\partial_i\partial_j\rho)\,(\partial_j\rho)}{\rho}+\frac{(\partial_i\rho)\,(\partial_j\partial_j\rho)}{\rho}-\frac{(\partial_j\rho)\,(\partial_i\rho)\,(\partial_j\rho)}{\rho^2}\right)$$

Which is the term we had in A, and therefore:

$$\begin{aligned}
A\rho &= \partial_j\left(\frac{(\partial_i\rho)\,(\partial_j\rho)}{\rho}\right)-\partial_j\partial_i\partial_j\rho\\
&= \partial_j\left(\frac{(\partial_i\rho)\,(\partial_j\rho)}{\rho}-\partial_i\partial_j\rho\right)
\end{aligned}$$

Plugging that back into the equation we've got before we now have:

$$\begin{aligned}
\rho\frac{\partial u}{\partial t}+\rho\,(u\cdot\nabla)\,u+\left(\frac{\hbar}{2m}\right)^2\nabla\left(\frac{(\partial_i\rho)\,(\partial_j\rho)}{\rho}-\partial_i\partial_j\rho\right)+\frac{\rho}{m}\nabla U\,(r) &= 0\\
\rho\left(\frac{\partial u}{\partial t}+(u\cdot\nabla)\,u\right)+\left(\frac{\hbar}{2m}\right)^2\nabla\left(\frac{(\partial_i\rho)\,(\partial_j\rho)}{\rho}-\partial_i\partial_j\rho\right)+\frac{\rho}{m}\nabla U\,(r) &= 0
\end{aligned}$$

Which is indeed the Navier–Stokes equations, and the stress tensor is:

$$\tau=\left(\frac{\hbar}{2m}\right)^2\left(\frac{(\partial_i\rho)\,(\partial_j\rho)}{\rho}-\partial_i\partial_j\rho\right)$$

3

a)

We can write the equations (note that I've already divided the wave-function so now its normalization is $\int d^3r\,|\psi|^2=N$):

$$\begin{aligned}
i\hbar\frac{\partial\psi}{\partial t} &= \left(-\frac{\hbar^2\nabla^2}{2m}+V_0\,|\psi|^2\right)\psi=H_0\psi+H_V\psi\\
\psi &= \sqrt{\rho}\exp\,(i\chi)\\
i\hbar\frac{\partial\psi}{\partial t} &= i\hbar\left(\frac{1}{2\sqrt{\rho}}\frac{\partial\rho}{\partial t}+i\sqrt{\rho}\frac{\partial\chi}{\partial t}\right)\exp\,(i\chi)\\
H_0\psi &= -\frac{\hbar^2}{2m}\left(\frac{\nabla^2\rho}{2\sqrt{\rho}}-\frac{(\nabla\rho)^2}{4\sqrt{\rho^3}}+\frac{i}{\sqrt{\rho}}\nabla\rho\nabla\chi-\sqrt{\rho}\,(\nabla\chi)^2+i\sqrt{\rho}\nabla^2\chi\right)\exp\,(i\chi)\\
H_V\psi &= V_0\sqrt{\rho^3}\exp\,(i\chi)
\end{aligned}$$

Plugging that in and remembering that ρ and χ are real we get:

$$\begin{aligned}
\frac{2m}{\hbar}\frac{\partial\chi}{\partial t} &= \frac{\nabla^2\rho}{2\rho}-\frac{(\nabla\rho)^2}{4\rho^2}-(\nabla\chi)^2-\frac{2m\rho}{\hbar^2}V_0\\
\frac{m}{\hbar\rho}\frac{\partial\rho}{\partial t} &= -\frac{\nabla\rho}{\rho}\nabla\chi-\nabla^2\chi
\end{aligned}$$

Looking for a stationary solution ($\nabla\rho = 0 = \nabla\chi$) we get:

This means "uniform"

$$\begin{aligned}
\frac{\partial \chi}{\partial t} &= -\frac{\rho}{\hbar} V_0 \\
\frac{m}{\hbar\rho}\frac{\partial \rho}{\partial t} &= 0 \\
\rho &= \rho_0 \\
\chi &= \chi_0 - \frac{\rho_0}{\hbar} V_0 t \\
\psi_0 &= \sqrt{\rho_0}\exp\left(i\left(\chi_0 - \frac{\rho_0 V_0}{\hbar} t\right)\right)
\end{aligned}$$

Note that from normalization we can also get $\rho_0 = \frac{N}{\Omega}$.
Expanding ρ and χ around those solutions we get:

$$\begin{aligned}
\rho &= \rho_0 + \delta\rho \\
\chi &= \chi_0 - \frac{\rho_0}{\hbar} V_0 t + \delta\chi \\
\frac{\partial}{\partial t}\delta\chi &= \frac{\hbar}{2m}\frac{\nabla^2\delta\rho}{2(\rho_0+\delta\rho)} - \frac{\hbar}{2m}\frac{(\nabla\delta\rho)^2}{4(\rho_0+\delta\rho)^2} - (\nabla\delta\chi)^2 - \frac{\delta\rho}{\hbar}V_0 \\
\frac{\partial}{\partial t}\delta\rho &= -\frac{\hbar}{m}\nabla\delta\rho\nabla\delta\chi - \frac{\hbar}{m}(\rho_0+\delta\rho)\nabla^2\delta\chi
\end{aligned}$$

Dumping orders of δ^2 we get:

$$\begin{aligned}
\frac{\partial}{\partial t}\delta\chi &= \frac{\hbar}{4m\rho_0}\nabla^2\delta\rho - \frac{V_0}{\hbar}\delta\rho \\
\frac{\partial}{\partial t}\delta\rho &= -\frac{\hbar\rho_0}{m}\nabla^2\delta\chi \\
\frac{\partial^2}{\partial t^2}\delta\chi &= -\frac{\hbar^2}{4m^2}\nabla^4\delta\chi + \frac{\rho_0 V_0}{m}\nabla^2\delta\chi \\
\frac{\partial^2}{\partial t^2}\delta\rho &= -\frac{\hbar^2}{4m^2}\nabla^4\delta\rho + \frac{\rho_0 V_0}{m}\nabla^2\delta\rho
\end{aligned}$$

We can see that both equations are identical, so we'll solve for $f \in \{\delta\chi, \delta\rho\}$. Because the equation looks similar to the wave equation we'll try plane waves, and then:

$$\begin{aligned}
\frac{\partial^2}{\partial t^2} f &= -\omega^2 f \\
\nabla^2 f &= -k^2 f \\
\nabla^4 f &= k^4 f \\
-\omega^2 &= -\frac{\hbar^2}{4m^2}k^4 - \frac{\rho_0 V_0}{m}k^2 \\
\omega^2 &= \frac{\hbar^2}{4m^2}k^4 + \frac{\rho_0 V_0}{m}k^2
\end{aligned}$$

Now looking at $\frac{\partial}{\partial t}\delta\rho = -\frac{\hbar\rho_0}{m}\nabla^2\delta\chi$ and remembering both are real we can get:

$$\begin{aligned}
\delta\rho &= A_\rho \sin(\omega t - kr + \varphi_\rho) \\
\delta\chi &= A_\chi \sin(\omega t - kr + \varphi_\chi) \\
\omega A_\rho \cos(\omega t - kr + \varphi_\rho) &= \frac{\hbar\rho_0}{m}k^2 A_\chi \sin(\omega t - kr + \varphi_\chi)
\end{aligned}$$

So we need to have a phase difference of $\frac{\pi}{2}$ between $\delta\chi$ and $\delta\rho$ and we can write:

$$\begin{aligned}\delta\rho &= A\sin\left(\omega t - kr + \varphi\right)\\ \delta\chi &= \frac{m\omega}{\hbar\rho_0 k^2}A\cos\left(\omega t - kr + \varphi\right)\end{aligned}$$

Next we will consider the limit of $k \to 0$:

$$\begin{aligned}\omega^2\left(k \to 0\right) &\to \frac{\rho_0 V_0}{m}k^2\\ \omega &\to \sqrt{\frac{\rho_0 V_0}{m}}k\end{aligned}$$

Which mean we get a linear dispersion relation in this limit, and therefore we get that the energies goes as $\hbar\omega \to 0$ as well.

We can also see that:

$$A_\rho = \frac{\hbar\rho_0 k^2}{m\omega}A_\chi = \sqrt{\frac{\rho_0}{mV_0}}\hbar k A_\chi$$

So indeed we've got that in this limit the amplitude A_ρ goes linearly with k and therefore $\rho \to 0$.

We've got that $\rho \to 0$ and $E \to 0$ which means we have small excitations that correspond to the long wave length oscillations of the phase χ.

What we have obtained is the Goldstone modes, by looking at a certain location of the system we see an ordered system, while looking at the whole system we can see that the phase changes slowly, this should have been because of a symmetry breaking, but we couldn't understand how to see that the gauge symmetry was broken in that case. There's no gauge symmetry in this section.

b)

Adding the coulomb interaction means to add a $e\int d^3r'\left(\frac{|\psi|^2}{|r-r'|} - \frac{N}{\Omega}\right)|\psi\rangle$ term. If again we'll try $\psi = \sqrt{\rho}\exp\left(i\chi\right)$ we can get the similar equations to before:

$$\begin{aligned}\frac{\partial\rho}{\partial t} &= -\frac{\hbar}{m}\left(\left(\nabla\rho\right)\cdot\left(\nabla\chi\right) + \rho\left(\nabla^2\chi\right)\right)\\ \frac{\partial\chi}{\partial t} &= \frac{\hbar}{2m}\left(-\frac{\left(\nabla\rho\right)^2}{4\rho^2} + \frac{\nabla^2\rho}{2\rho} - \left(\nabla\chi\right)^2\right) - \frac{V_0\rho}{\hbar} - \frac{e}{\hbar}\int d^3r'\left(\frac{|\psi|^2}{|r-r'|} - \frac{N}{\Omega}\right)\end{aligned}$$

As we did before we can get that the uniform solution is:

$$\begin{aligned}\frac{\partial\chi}{\partial t} &= -\frac{\rho}{\hbar}V_0\\ \frac{m}{\hbar\rho}\frac{\partial\rho}{\partial t} &= 0\\ \rho &= \rho_0\\ \chi &= \chi_0 - \frac{\rho_0}{\hbar}V_0 t\\ \psi_0 &= \sqrt{\rho_0}\exp\left(i\left(\chi_0 - \frac{\rho_0 V_0}{\hbar}t\right)\right)\end{aligned}$$

As we did before we'll look for small variations:

$$\begin{aligned}\rho &= \rho_0 + \delta\rho\\ \chi &= \chi_0 - \frac{\rho_0}{\hbar}V_0 t + \delta\chi\end{aligned}$$

As we did before we can get the following equations (after dumping orders of δ^2 and above):

$$
\begin{aligned}
\frac{\partial}{\partial t}\delta\chi &= \frac{\hbar}{4m\rho_0}\nabla^2\delta\rho - \frac{V_0}{\hbar}\delta\rho + \frac{e}{\hbar}\int d^3r'\frac{\delta\rho}{|r-r'|} \\
\frac{\partial}{\partial t}\delta\rho &= -\frac{\hbar\rho_0}{m}\nabla^2\delta\chi \\
\frac{\partial^2}{\partial t^2}\delta\rho &= -\frac{\hbar^2}{4m^2}\nabla^4\delta\rho + \frac{\rho_0 V_0}{m}\nabla^2\delta\rho - \frac{e\rho_0}{m}\nabla^2\int d^3r'\frac{\delta\rho}{|r-r'|} \\
&= -\frac{\hbar^2}{4m^2}\nabla^4\delta\rho + \frac{\rho_0 V_0}{m}\nabla^2\delta\rho - \frac{e\rho_0}{m}4\pi\delta\rho
\end{aligned}
$$

As we did before we can guess plane wave (because we only have even powers of ∇) and then:

$$
\begin{aligned}
-\omega^2 &= -\frac{\hbar^2}{4m^2}k^4 - \frac{\rho_0 V_0}{m}k^2 - \frac{e\rho_0}{m}4\pi \\
\omega &= \sqrt{\frac{\hbar^2}{4m^2}k^4 + \frac{\rho_0 V_0}{m}k^2 + 4\pi\frac{e\rho_0}{m}}
\end{aligned}
$$

We can see that this time in the limit of $k \to 0$ we get $\omega \to \sqrt{\frac{\rho_0 V_0}{m}k^2 + 4\pi\frac{e\rho_0}{m}}$ which we can see that doesn't go to 0 but a constant.

Comparing the energy $E = \hbar\omega = \sqrt{\frac{\hbar^2\rho_0 V_0}{m}k^2 + 4\pi\frac{\hbar^2 e\rho_0}{m}}$ to the relativistic energy $E = \sqrt{p^2c^2 + m^2c^4}$ we see that the coulomb potential added for us the mass term. This is the so called Higgs mechanism in which the gauge field "swallows" the Goldstone mode and became massive.

4

$$
H = \frac{1}{2}\left(p^2 + \omega^2 q^2\right) + \lambda q\sigma_x + \varepsilon\sigma_z
$$

a)

The fast part of the Hamiltonian is $\lambda q\sigma_x + \varepsilon\sigma_z$, we want to solve it treating q as a parameter:

$$
H_{fast} = \lambda q\sigma_x + \varepsilon\sigma_z = \sqrt{(\lambda q)^2 + \varepsilon^2}\left(\frac{\lambda q}{\sqrt{(\lambda q)^2 + \varepsilon^2}}\sigma_x + \frac{\varepsilon}{\sqrt{(\lambda q)^2 + \varepsilon^2}}\sigma_z\right)
$$

And then we know the solution (by defining: $\sin(\theta) = \frac{\lambda q}{\sqrt{(\lambda q)^2+\varepsilon^2}}$):

$$
\begin{aligned}
|\psi_1\rangle &= \sin\frac{\theta}{2}|\uparrow\rangle - \cos\frac{\theta}{2}|\downarrow\rangle = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \end{pmatrix} \\
|\psi_2\rangle &= \cos\frac{\theta}{2}|\uparrow\rangle + \sin\frac{\theta}{2}|\uparrow\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}
\end{aligned}
$$

Where their energies are $\pm\sqrt{(\lambda q)^2 + \varepsilon^2}$ and we know that the solution for the slow part is the harmonic oscillator, so we get that the "potential surfaces" are:

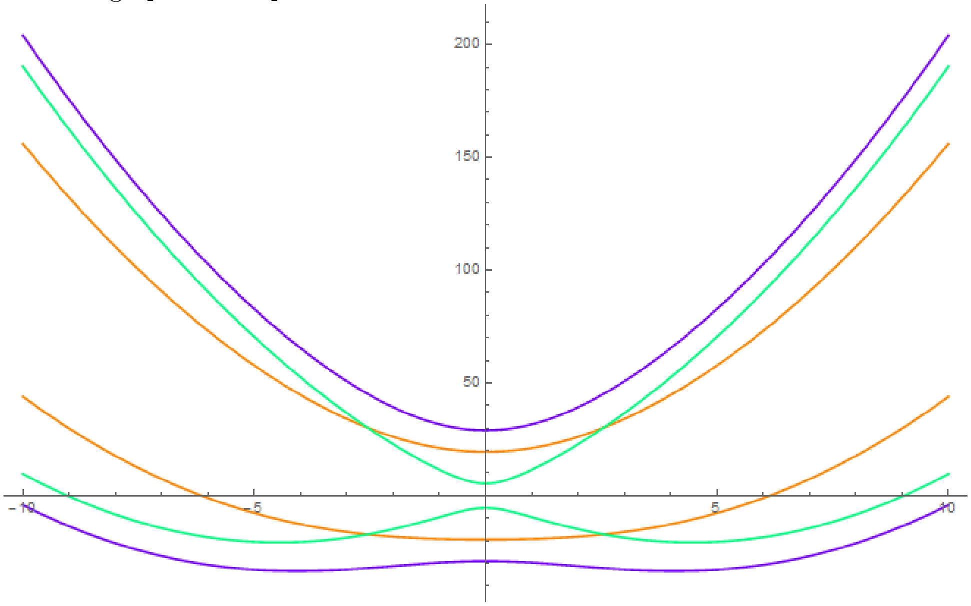
$$
U = \frac{1}{2}\omega^2 q^2 \pm \sqrt{(\lambda q)^2 + \varepsilon^2}
$$

In order to see if we have a Berry phase we need to calculate $\Im\left(\int_c \left\langle\psi\left|\frac{\partial}{\partial q}\right|\psi\right\rangle\right)$:

$$
\left\langle\psi_1\left|\frac{\partial}{\partial q}\right|\psi_1\right\rangle = \begin{pmatrix} \sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2}\frac{\partial\theta}{\partial q}\cos\frac{\theta}{2} \\ \frac{1}{2}\frac{\partial\theta}{\partial q}\sin\frac{\theta}{2} \end{pmatrix} \in Real
$$

And for $\left\langle\psi_2\left|\frac{\partial}{\partial q}\right|\psi_2\right\rangle$ we'll also get a real number which means they are both zero and therefore there will not be a Berry phase.

Here are graphs of the potential curves for some different values:



Now we'll do some semi-classical approximations:

For large $q \gg \frac{\varepsilon}{\lambda}$ we can get:

q is an observable, not a parameter

$$U \approx \frac{1}{2}\omega^2 q^2 \pm \lambda q = \frac{1}{2}\omega^2 \left(q \pm \frac{\lambda}{\omega^2}\right)^2 - \frac{\lambda^2}{2\omega^2}$$

-2

Which means we get two shifted harmonic oscillators with the same energies around $\pm\frac{\lambda}{\omega^2}$ and the energies are $E = \hbar\omega\left(N + \frac{1}{2}\right) - \frac{\lambda^2}{2\omega^2}$

For small $q \ll \frac{\varepsilon}{\lambda}$ we can get:

There are two cases:

$\varepsilon < \lambda^2/\omega^2$ $\varepsilon > \lambda^2/\omega^2$

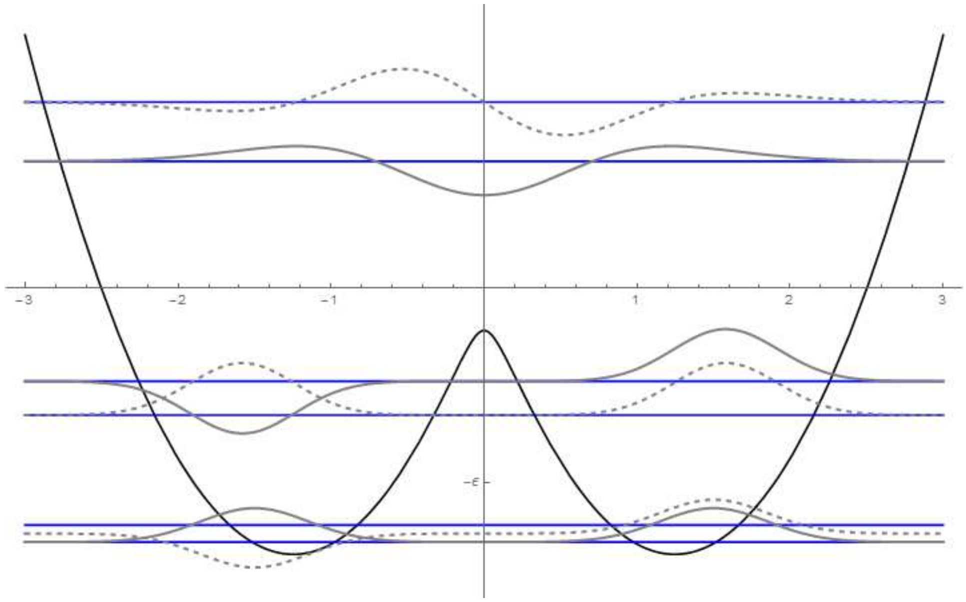
-2

$$\begin{aligned} U &\approx \frac{1}{2}\omega^2 q^2 \pm \varepsilon\left[1 + \frac{1}{2}\left(\frac{\lambda q}{\varepsilon}\right)^2\right] \\ &= \frac{1}{2}\left(\omega^2 \pm \frac{\lambda^2}{\varepsilon}\right) q^2 \pm \varepsilon \end{aligned}$$

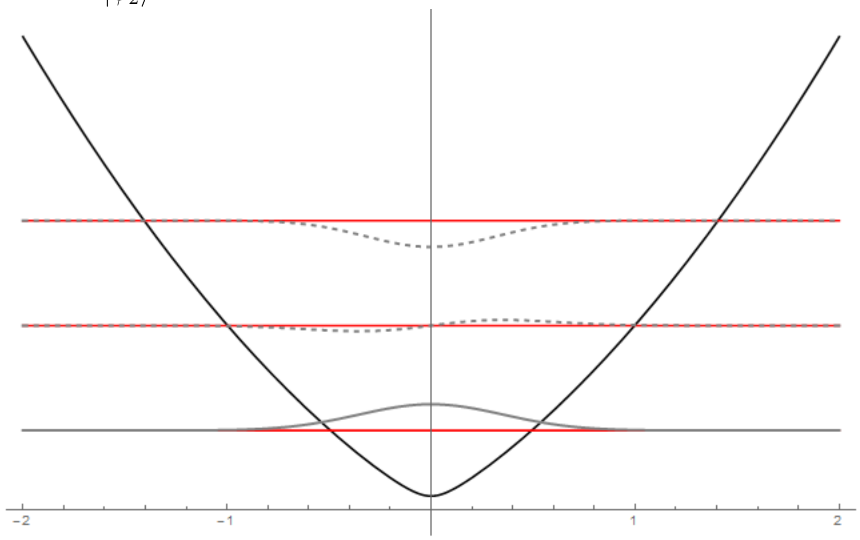
Which means we have two harmonic oscillator with different energies and centered around the same point ($q = 0$) and the energies are:

$$\begin{aligned} E_+ &= \hbar\sqrt{\omega^2 + \frac{\lambda^2}{\varepsilon}}\left(N + \frac{1}{2}\right) + \varepsilon \\ E_- &= \hbar\sqrt{\omega^2 - \frac{\lambda^2}{\varepsilon}}\left(N + \frac{1}{2}\right) - \varepsilon \end{aligned}$$

The energy levels and the wave functions of $|\psi_1\rangle$:



And for $|\psi_2\rangle$:



For using the Born-Oppenheimer approximation our criteria is that the energy levels are far away which means that our criteria is:

$$\varepsilon \gg \hbar\sqrt{\omega^2 \pm \frac{\lambda^2}{\epsilon}}$$

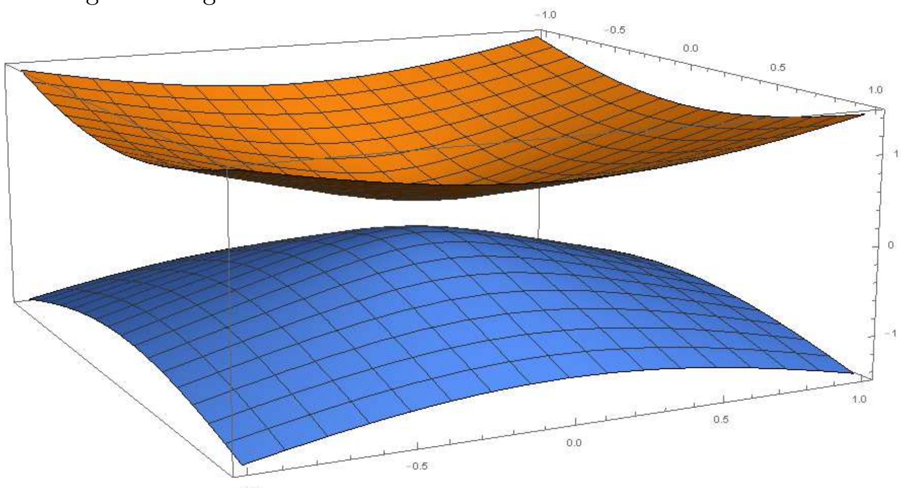
b)

$$\begin{aligned} H_{fast} &= \lambda q\sigma_x + \lambda p\sigma_y + \varepsilon\sigma_z \\ &= \sqrt{(\lambda q)^2 + (\lambda p)^2 + \varepsilon^2}\left(\frac{\lambda q}{\sqrt{(\lambda q)^2 + (\lambda p)^2 + \varepsilon^2}}\sigma_x + \frac{\lambda p}{\sqrt{(\lambda q)^2 + (\lambda p)^2 + \varepsilon^2}}\sigma_y + \frac{\varepsilon}{\sqrt{(\lambda q)^2 + (\lambda p)^2 + \varepsilon^2}}\sigma_z\right) \end{aligned}$$

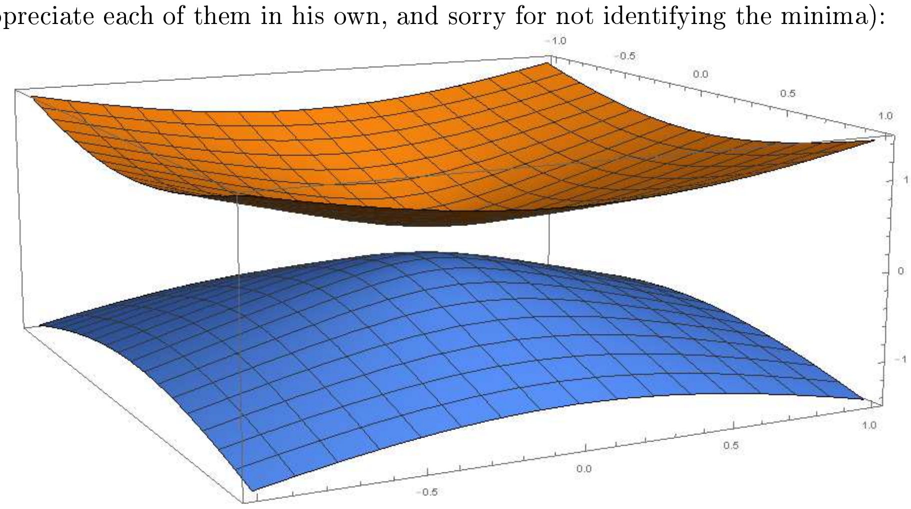
And then we know the solution:

$$\begin{aligned}
|\psi_1\rangle &= \begin{pmatrix} \sin\frac{\theta}{2} \\ -\exp(i\phi)\cos\frac{\theta}{2} \end{pmatrix} \\
|\psi_2\rangle &= \begin{pmatrix} \cos\frac{\theta}{2} \\ \exp(i\phi)\sin\frac{\theta}{2} \end{pmatrix} \\
E_{1,2} &= \pm\sqrt{(\lambda q)^2 + (\lambda p)^2 + \varepsilon^2} \\
\cos(\theta) &= \frac{\varepsilon}{\sqrt{(\lambda q)^2 + (\lambda p)^2 + \varepsilon^2}} \\
\tan(\phi) &= \frac{p}{q}
\end{aligned}$$

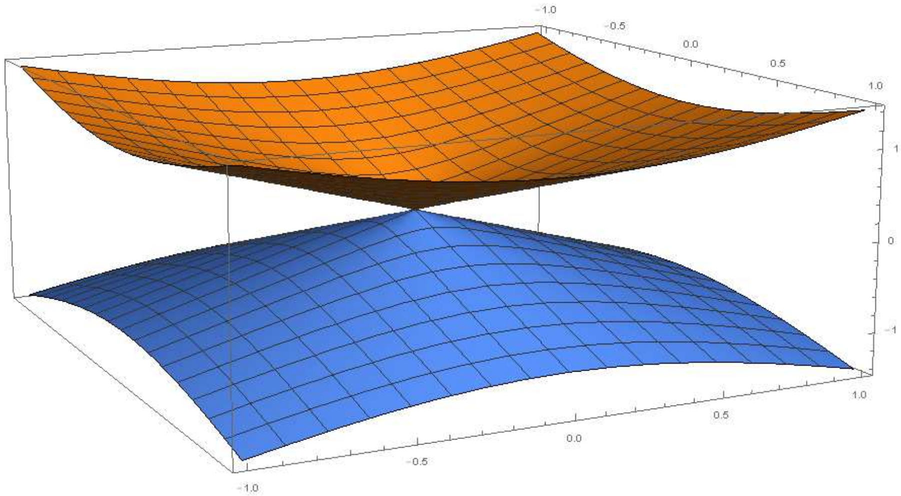
Plotting the energies for $\varepsilon > 0$:



Plotting the energies for $\varepsilon < 0$ (note that the difference here would have come from the difference between $(-\varepsilon)^2$ and ε^2, because I couldn't prove that there is a difference you might find that those graphs look a bit the same, still please appreciate each of them in his own, and sorry for not identifying the minima):



Plotting the energies for $\varepsilon = 0$:



And in here we can see that in that case we have an energy crossing when $p = q = 0$.
Next we'll calculate A_p and A_q for $|\psi_1\rangle$ (note that I dump all real parts):

$$\begin{aligned}
A_q &= -\Im\left(\left\langle \psi_1 \left| \frac{\partial}{\partial q} \right| \psi_1 \right\rangle\right) \\
&= -\Im\left(\begin{pmatrix} \sin\frac{\theta}{2} & -\exp(-i\phi)\cos\frac{\theta}{2} \end{pmatrix} \frac{\partial}{\partial q}\begin{pmatrix} \sin\frac{\theta}{2} \\ -\exp(i\phi)\cos\frac{\theta}{2} \end{pmatrix}\right) \\
&= -\Im\left(\begin{pmatrix} \sin\frac{\theta}{2} & -\exp(-i\phi)\cos\frac{\theta}{2} \end{pmatrix} \left[\begin{pmatrix} \frac{1}{2}\cos\frac{\theta}{2} \\ \frac{1}{2}\exp(i\phi)\sin\frac{\theta}{2} \end{pmatrix}\frac{\partial\theta}{\partial q} + \begin{pmatrix} \sin\frac{\theta}{2} \\ -i\exp(i\phi)\cos\frac{\theta}{2} \end{pmatrix}\frac{\partial\phi}{\partial q}\right]\right) \\
&= -\Im\left(i\cos^2\frac{\theta}{2}\frac{\partial\phi}{\partial q}\right) = \frac{p}{p^2+q^2}\cos^2\frac{\theta}{2} \\
&= \frac{p}{p^2+q^2}\left(\frac{\varepsilon}{2\sqrt{(\lambda q)^2+(\lambda p)^2+\varepsilon^2}} + \frac{1}{2}\right) \\
A_p &= -\Im\left(\left\langle \psi_1 \left| \frac{\partial}{\partial p} \right| \psi_1 \right\rangle\right) = -\cos^2\frac{\theta}{2}\frac{\partial\phi}{\partial p} \\
&= -\left(\frac{\varepsilon}{2\sqrt{(\lambda q)^2+(\lambda p)^2+\varepsilon^2}} + \frac{1}{2}\right)\frac{q}{p^2+q^2}
\end{aligned}$$

So we've got that:

$$\vec{A}(p,q) = \left[\frac{\varepsilon}{2\sqrt{(\lambda q)^2+(\lambda p)^2+\varepsilon^2}} + \frac{1}{2}\right]\frac{1}{p^2+q^2}(-q,p)$$

[Note that if we write it in "cylindrical coordinates" we get $\vec{A} = -\left(1 + \frac{\varepsilon}{\sqrt{(\lambda r)^2+\varepsilon^2}}\right)\frac{\hat{\phi}}{2r}$]
Next we'll calculate $F_{pq} = \partial_q A_p - \partial_p A_q$:

$$\begin{aligned}
F_{pq} &= \partial_q A_p - \partial_p A_q \\
&= (\nabla \times A)_z \\
&= -\frac{1}{2r}\left(\partial_r\left(rA_\phi\right) - \partial_\phi A_r\right) \\
&= \frac{1}{2r}\partial_r\left(1 + \frac{\varepsilon}{\sqrt{(\lambda r)^2 + \varepsilon^2}}\right) \\
&= -\frac{1}{2r}\frac{\varepsilon\lambda^2 r}{\sqrt{\left((\lambda r)^2 + \varepsilon^2\right)^3}} \\
&= -\frac{\varepsilon\lambda^2}{2\sqrt{(\lambda r)^2 + \varepsilon^2}^3} \\
&= -\frac{\varepsilon\lambda^2}{2\sqrt{(\lambda p)^2 + (\lambda q)^2 + \varepsilon^2}^3} \checkmark = -\frac{\varepsilon\lambda^2}{4d^3}
\end{aligned}$$

Comparing that to $B = \left(\frac{3d(m\cdot d)}{d^5} - \frac{m}{d^3}\right)$ we can see that the "source" of the of this "magnetic field" is $\frac{\varepsilon\lambda^2}{2}$ (the first term is zero because the vectors are perpendicular).

Next we'll calculate the Chern number:

$$\begin{aligned}
C &= \frac{1}{2\pi}\int dpdqF_{pq} \\
&= \frac{1}{2\pi}\int rdrd\phi F_{pq} \\
&= -\frac{1}{2\pi}\int \frac{\varepsilon\lambda^2 r}{2\sqrt{(\lambda r)^2 + \varepsilon^2}^3}drd\phi \\
&= -\frac{\varepsilon\lambda^2}{2}\int \frac{r}{\sqrt{(\lambda r)^2 + \varepsilon^2}^3}dr \\
&= -\frac{\varepsilon\lambda^2}{2}\frac{-1}{\varepsilon\lambda^2} \\
&= \frac{1}{2}\cdot \mathrm{sgn}(\varepsilon) \quad \checkmark\!\!\times \quad -1
\end{aligned}$$

So we've got that the Chern number is $\frac{1}{2}$, and as we saw in class the Berry phase for one loop is (using Stocks) $\int_c A\cdot\phi = 2\pi C = \pi$, so we've got that for every loop we get a Berry phase of π.

This result agrees with the fact that we have a degeneracy in the case of $\varepsilon = 0$ which creates the "source" we have seen in the last bullet.

And that's it, please enjoy :)