Collected problems for exam

February 17, 2016

1 Local gauge transformation

1.1 $\psi^*(\mathbf{r}_2)\,\psi(\mathbf{r}_1)$

Using:

$$\psi \to e^{i\frac{e}{\hbar c}\chi}\psi \tag{1}$$

We get:

$$\begin{aligned}
\psi^*(\mathbf{r}_2)\,\psi(\mathbf{r}_1) &\to e^{-i\frac{e}{\hbar c}\chi(\mathbf{r}_2,t)}\psi(\mathbf{r}_2)\,e^{i\frac{e}{\hbar c}\chi(\mathbf{r}_1,t)}\psi(\mathbf{r}_1) && (2)\\
&= \psi^*(\mathbf{r}_2)\,\psi(\mathbf{r}_1)\,e^{i\frac{e}{\hbar c}[\chi(\mathbf{r}_1,t)-\chi(\mathbf{r}_2,t)]} && (3)
\end{aligned}$$

This is actually a density matrix:

$$\begin{aligned}
\rho &= |\psi\rangle\langle\psi| && (4)\\
\rho(\mathbf{r}_1,\mathbf{r}_2) &= \langle\mathbf{r}_2|\psi\rangle\langle\psi|\mathbf{r}_1\rangle && (5)\\
&= \psi^*(\mathbf{r}_2)\,\psi(\mathbf{r}_1) && (6)
\end{aligned}$$

Therefore we expect the probabilities $\rho(\mathbf{r}_1,\mathbf{r}_1)$ to be gauge invariant, which is indeed the case:

$$\begin{aligned}
\rho(\mathbf{r}_1,\mathbf{r}_1) &= \psi^*(\mathbf{r}_1)\,\psi(\mathbf{r}_1) && (7)\\
&\to \psi^*(\mathbf{r}_1)\,\psi(\mathbf{r}_1)\,e^{i\frac{e}{\hbar c}[\chi(\mathbf{r}_1,t)-\chi(\mathbf{r}_1,t)]} && (8)\\
&= \rho(\mathbf{r}_1,\mathbf{r}_1) && (9)
\end{aligned}$$

However the coherence $\rho(\mathbf{r}_1,\mathbf{r}_2)$ (where $\mathbf{r}_1 \neq \mathbf{r}_2$) are not necessarily conserved (they are in any case basis dependent, so when changing basis in general they change) - they are not observables so we are OK with this.

We note that expectation values are unchanged since:

$$\begin{aligned}
\langle A \rangle &= \mathrm{Tr}\left(\rho A\right) && (10)\\
&= \int dx dx' \left\langle x \left| \rho \right| x' \right\rangle \left\langle x' \left| A \right| x \right\rangle && (11)\\
&= \int dx dx' \psi^* \left(x\right) \psi \left(x'\right) A \left(x', x\right) && (12)\\
&\to \int dx dx' \left(\psi^* \left(x\right) e^{-i \frac{e}{\hbar c} \chi(x)}\right) \left(\psi \left(x'\right) e^{i \frac{e}{\hbar c} \chi\left(x'\right)}\right) \left(e^{-i \frac{e}{\hbar c} \chi\left(x'\right)} A \left(x', x\right) e^{i \frac{e}{\hbar c} \chi(x)}\right) && (13)\\
&= \int dx dx' \psi^* \left(x\right) \psi \left(x'\right) A \left(x', x\right) && (14)\\
&= \langle A \rangle && (15)
\end{aligned}$$

Indeed.

1.2 $\int\limits_{\mathbf{r}_1}^{\mathbf{r}_2} d\mathbf{r} \cdot \mathbf{A}\left(\mathbf{r}\right)$

Using:

$$\mathbf{A} \to \mathbf{A} + \nabla \chi \tag{16}$$

We get:

$$\begin{aligned}
\int\limits_{\mathbf{r}_1}^{\mathbf{r}_2} d\mathbf{r} \cdot \mathbf{A}\left(\mathbf{r}\right) &\to \int\limits_{\mathbf{r}_1}^{\mathbf{r}_2} d\mathbf{r} \cdot \left[\mathbf{A}\left(\mathbf{r}\right) + \nabla \chi \left(\mathbf{r}, t\right)\right] && (17)\\
&= \int\limits_{\mathbf{r}_1}^{\mathbf{r}_2} d\mathbf{r} \cdot \mathbf{A}\left(\mathbf{r}\right) + \int\limits_{\mathbf{r}_1}^{\mathbf{r}_2} d\mathbf{r} \cdot \nabla \chi \left(\mathbf{r}, t\right) && (18)
\end{aligned}$$

We note that on a closed trajectory :

$$\begin{aligned}
\oint d\mathbf{r} \cdot \mathbf{A}\left(\mathbf{r}\right) &= \iint\limits_S d\mathbf{s} \cdot \nabla \times \mathbf{A}\left(\mathbf{r}\right) && (19)\\
&= \iint\limits_S d\mathbf{s} \cdot \mathbf{B}\left(\mathbf{r}\right) && (20)\\
&= \mathbf{\Phi} && (21)
\end{aligned}$$

This is a physical quantity therefore we expect it to be gauge invariant. That is, we expect that for any closed trajectory:

$$\oint d\mathbf{r} \cdot \nabla \chi \left(\mathbf{r}, t\right) = 0 \tag{22}$$

This is given trivially by Stokes theorem since:

$$\oint d\mathbf{r} \cdot \nabla \chi \left(\mathbf{r}, t\right) = \iint d\mathbf{S} \cdot \underbrace{\nabla \times \left(\nabla \chi \left(\mathbf{r}, t\right)\right)}_{0} = 0 \tag{23}$$

However applicable only for a simply connected space, that is, when χ is well defined over all space, or physically, when there are no flux sources (or drains) over all space.

In A-B this is exactly what isn't satisfied. Therefore when calculating:

$$\int_{\mathbf{r}_1}^{\mathbf{r}_2} d\mathbf{r} \cdot \nabla\chi\left(\mathbf{r}, t\right) \tag{24}$$

We can choose different paths to calculate on, the paths differ exactly by these closed loops, so when we have sources in space some trajectories may go around them (and be essentially different then those not going around them) thus giving different value to the integral.

Putting in simpler words, the local gauge transformation is not a symmetry when we have flux sources to the magnetic field.

2 Electron in uniform B and constant E

We interpret uniform as $\mathbf{B} = \mathbf{B}\left(t\right)$ and constant as $\mathbf{E} = \mathbf{E}\left(\mathbf{r}\right)$. We assume $\mathbf{B} = B\left(t\right)\hat{z}$. There is always a gauge such that $A_0 = 0$, lets choose it! Therefore:

$$\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} \tag{25}$$

However $\frac{\partial}{\partial t}\mathbf{E} = 0$ then $\frac{\partial^2\mathbf{A}}{\partial t^2} = 0 \Rightarrow \mathbf{A} = \mathbf{A}_a\left(\mathbf{r}\right) + \mathbf{A}_b\left(\mathbf{r}\right)t$. Setting this in the magnetic field:

$$\begin{aligned} \mathbf{B} &= \nabla\times\mathbf{A} && (26)\\ B_i &= \epsilon_{ijk}\partial_j\left(\mathbf{A}_a\left(\mathbf{r}\right) + \mathbf{A}_b\left(\mathbf{r}\right)t\right)_k && (27) \end{aligned}$$

Lets guess:

$$\mathbf{A} = \left[B\left(t\right)y + B_0\left(t\right)\right]\hat{x} = \mathbf{A}_a\left(\mathbf{r}\right) + \mathbf{A}_b\left(\mathbf{r}\right)t \tag{28}$$

Therefore:

$$\mathbf{A}_a\left(\mathbf{r}\right) + \mathbf{A}_b\left(\mathbf{r}\right)t = \left[A_a y + A_b y t + A_c t\right]\hat{x} \tag{29}$$

And:

$$B\left(t\right) = A_a + A_b t \tag{30}$$

Renaming:

$$\mathbf{A} = \left(A_0 y + A_1 t y + A_2 t\right)\hat{x} \tag{31}$$

We are ready to set in the Hamiltonian:

$$\begin{aligned} \mathcal{H} &= \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\left(\mathbf{r}, t\right)\right)^2 + eA_0\left(\mathbf{r}, t\right) && (32)\\ &= \frac{1}{2m}\left(\mathbf{p}_x - \frac{e}{c}A_0 y - \frac{e}{c}A_1 t y - \frac{e}{c}A_2 t\right)^2 + \frac{\mathbf{p}_y^2 + \mathbf{p}_z^2}{2m} && (33) \end{aligned}$$

Neglecting trivial z motion and denoting $A_1 t = R_1$ and $A_2 t = R_2$ we get in the adiabatic limit:

$$\mathcal{H} = \frac{\left(\frac{e}{c}\right)^2 (A_0 + R_1)^2}{2m} \left(\frac{\mathbf{p}_x - \frac{e}{c} R_2}{\frac{e}{c}(A_0 + R_1)} - y \right)^2 + \frac{\mathbf{p}_y^2}{2m} \tag{34}$$

Guessing

$$\psi(x, y) = \phi(y)\, e^{\frac{i}{\hbar} k_x x} \tag{35}$$

We get:

$$E\psi = \left[\frac{\left(\frac{e}{c}\right)^2 (A_0 + R_1)^2}{2m} \left(\frac{\mathbf{p}_x - \frac{e}{c} R_2}{\frac{e}{c}(A_0 + R_1)} - y \right)^2 + \frac{\mathbf{p}_y^2}{2m} \right] \psi \tag{36}$$

$$= \left[\frac{\left(\frac{e}{c}\right)^2 (A_0 + R_1)^2}{2m} \left(\underbrace{\frac{k_x - \frac{e}{c} R_2}{\frac{e}{c}(A_0 + R_1)}}_{y_0} - y \right)^2 + \frac{\mathbf{p}_y^2}{2m} \right] \psi \tag{37}$$

We note that for $R_1 = 0$ we get a linearly moving QHO:

$$E\psi = \left[\frac{\left(\frac{e}{c}\right)^2 A_0^2}{2m} \left(\underbrace{\frac{k_x - \frac{e}{c} R_2}{\frac{e}{c} A_0}}_{y_0} - y \right)^2 + \frac{\mathbf{p}_y^2}{2m} \right] \psi \tag{38}$$

On the other hand for $R_2 = 0$ we get a shifted QHO however with increasing frequency and a variable shift such that $y_0 \underset{t\to\infty}{\longrightarrow} 0$. The spectrum is therefore:

$$E_n = \hbar \sqrt{\frac{e}{c}(A_0 + R)} \left(n + \frac{1}{2} \right) \underset{t\to\infty}{\longrightarrow} \infty \tag{39}$$

3 Adding a potential to Landau

See HW and dropbox->exam->2012-2014->old exams q&a->q2.pdf

4 Spinless particle on cylinder

See HW

5 Standing vs. propagating waves

For a wave equation:

$$\frac{\partial^2 \phi}{\partial t^2} = v^2 \frac{\partial^2 \phi}{\partial x^2} \tag{40}$$

We have two solutions basis:

$$\begin{cases} \phi_s\left(x,t\right) = \sum A_n \sin\left(k_n x\right)\sin\left(\omega t\right) & \text{standing} \\ \phi_r\left(x,t\right) = \sum A_n \sin\left(k_n x - \omega t\right) + B_n \sin\left(k_n x + \omega t\right) & \text{running} \\ \qquad\quad\;\; = \sum C_n \sin\left(k_n x\right)\cos\left(\omega t\right) + D_n \cos\left(k_n x\right)\sin\left(\omega t\right) \end{cases} \tag{41}$$

Setting in the equation we get for both the same dispersion:

$$\omega^2 = v^2 k_n^2 \Rightarrow \omega_k = vk \tag{42}$$

And satisfying $\phi\left(0,t\right) = \phi\left(L,t\right) = 0$ boundary conditions for the standing we get

$$k_n = \frac{\pi n}{L} \tag{43}$$

And satisfying periodic boundary conditions for the running:

$$k_n = \frac{2\pi n}{L} \tag{44}$$

To quantize recognize the classical Hamiltonian which gives this EOM. in this case:

$$\mathcal{H} = \int_0^L dx \left(\frac{1}{2}\Pi^2\left(x,t\right) + \frac{v^2}{2}\left(\frac{\partial\phi\left(x,t\right)}{\partial x}\right)^2\right) \tag{45}$$

Next, recognize momentum:

$$\frac{\partial\phi}{\partial t} = \frac{\delta\mathcal{H}}{\delta\Pi} = \Pi \tag{46}$$

So we leave the time dependent part unchanged, and span on the spatial parts. We demand the same relation to hold between position and momentum.

For standing waves we use:

$$\hat{\phi}\left(x\right) = \sum_\nu \sqrt{\frac{2}{L}}\sin\left(k_\nu x\right)\hat{Q}_\nu \tag{47}$$

Now we demand for the momentum $\hat{P}_\nu$:

$$\hat{\pi} = \frac{\partial\hat{\phi}}{\partial t} = \sum_\nu \sqrt{\frac{2}{L}}\sin\left(k_\nu x\right)\partial_t\hat{Q}_\nu \equiv \sum_\nu \sqrt{\frac{2}{L}}\sin\left(k_\nu x\right)\hat{P}_\nu \tag{48}$$

And the inverse relations:

$$\begin{cases} \hat{Q}_\nu = \int_0^L \hat{\phi}\left(x\right)\sqrt{\frac{2}{L}}\sin\left(k_\nu x\right)dx \\ \hat{P}_\nu = \int_0^L \hat{\Pi}\left(x\right)\sqrt{\frac{2}{L}}\sin\left(k_\nu x\right)dx \end{cases} \tag{49}$$

For the running waves we note we can rewrite in terms of $\sin(\omega t)$ and its derivatives:

$$\begin{aligned}
\phi &= \sum C_n \sin(k_n x)\cos(\omega t) + D_n \cos(k_n x)\sin(\omega t) && (50)\\
&= \sum C_n \sin(k_n x)\frac{\partial_t[\sin(\omega t)]}{\omega} + D_n \cos(k_n x)\sin(\omega t) && (51)\\
\Rightarrow \pi &= \sum D_n \cos(k_n x)\,\partial_t(\sin\omega t) - \omega C_n \sin(k_n x)\sin(\omega t) && (52)
\end{aligned}$$

Therefore we call $\sin\omega t \to \hat{Q}$ and then $\partial_t \sin\omega t \to \hat{P}$

$$\hat{\phi} = \sum \frac{1}{vk}\sin(k_n x)\,\hat{P}_n + \cos(k_n x)\,\hat{Q}_n \tag{53}$$

$$\hat{\pi} = \sum \cos(k_n x)\,\hat{P}_n - vk\sin(k_n x)\,\hat{Q}_n \tag{54}$$

What do these do?

Setting both derivations in the Hamiltonian we get:

$$\mathcal{H} = \frac{1}{2}\sum_k P_k^2 + \omega_k^2 Q_k^2 \tag{55}$$

Which now can be transformed using the standard method for QHO:

$$\hat{\mathbf{P}}_{\mathbf{k}} = i\sqrt{\frac{\omega_k}{2}}\left(\hat{a}_{\mathbf{k}} - \hat{a}_{\mathbf{k}}^{\dagger}\right), \quad \hat{\mathbf{Q}}_{\mathbf{k}} = \sqrt{\frac{\hbar}{2\omega_k}}\left(\hat{a}_{\mathbf{k}} + \hat{a}_{\mathbf{k}}^{\dagger}\right) \tag{56}$$

And get:

$$\mathcal{H} = \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}}\left(\hat{a}_{\mathbf{k}}^{\dagger}\hat{a}_{\mathbf{k}} + \frac{1}{2}\right) \tag{57}$$

So what is the difference? For both an excitation has energy $\hbar\omega_{\mathbf{k}}$ however for the running waves the momentum operator, given by:

$$\hat{P} = -\int_0^L \hat{\pi}(x)\frac{\partial}{\partial x}\hat{\phi}(x)\,dx$$

Is conserved and therefore it is a good quantum number. Setting the expansion explicitly in it we get:

$$\hat{P} = \sum_{\mathbf{k}} \hbar\mathbf{k}\left(\hat{a}_{\mathbf{k}}^{\dagger}\hat{a}_{\mathbf{k}} + \frac{1}{2}\right) \tag{58}$$

Therefore, an excitation also has momentum, and in general the dispersion relation is given by ω_k since:

$$E = \hbar\omega_{\mathbf{k}} = \hbar\omega(\mathbf{k}) = \hbar\omega\left(\frac{\mathbf{p}}{\hbar}\right) \tag{59}$$

Specifically in our case this is simple:

$$\omega_{\mathbf{k}} = vk \Rightarrow E = vP$$

So we got a mass less particle - a phonon!

6 Connection between Fock space and real space

First, fuck space.

In terms of annihilation\creation we write:

$$\mathcal{H}_{\mathrm{R}} = \sum_{\mathbf{k},\alpha} \hbar\omega_k \hat{a}^\dagger_{\mathbf{k},\alpha}\hat{a}_{\mathbf{k},\alpha} \tag{60}$$

Therefore a wave function has the form

$$\Psi_{\{n_{\mathbf{k},\alpha}\}}\left[Q_{\mathbf{k},\alpha}\right] = \prod \phi_{n_{\mathbf{k}}\alpha}\left(Q_{\mathbf{k},\alpha}\right) \tag{61}$$

Assuming we chose a single photon in mode $(\mathbf{k},\alpha) = (\kappa, a)$ then the wave function has the form:

$$\Psi_{1_\kappa}\left[Q_{\mathbf{k},\alpha}\right] = \prod_{\mathbf{k}\neq\kappa,\,\alpha=a} \phi_0\left(Q_{\mathbf{k},\alpha}\right)\cdot\phi_1\left(Q_{\kappa,\alpha}\right) \tag{62}$$

That is we have a Gaussian probability for amplitude $Q_{\mathbf{k},\alpha}$ in mode $\mathbf{k},\alpha$ and a first QHO wave function probability amplitude for having $Q_{\kappa,\alpha}$ in κ,α. The vector potential will be

$$\begin{aligned}
B\left(\mathbf{r}\right) &= \left\langle 1\left|B\right|1\right\rangle && (63)\\
&= \sum_{\mathbf{k},\alpha}\left(\frac{\hbar}{2\omega_k\epsilon_0\Omega}\right)^{\frac{1}{2}} e^{i\mathbf{k}\cdot\mathbf{r}}\left(i\mathbf{k}\times\bar{\lambda}_{\mathbf{k},\alpha}\right)\left\langle 1_{\kappa,a}\left|\left(\hat{a}_{\mathbf{k},\alpha}+\hat{a}^\dagger_{-\mathbf{k},\alpha}\right)\right|1_{\kappa,a}\right\rangle && (64)\\
&= 0 && (65)\\
B^2\left(\mathbf{r}\right) &= \left\langle 1\left|B^2\right|1\right\rangle && (66)\\
&= \sum_{\mathbf{k}',\alpha'}\sum_{\mathbf{k},\alpha}\left(\frac{\hbar}{2\omega_k\epsilon_0\Omega}\right)^{\frac{1}{2}}\left(\frac{\hbar}{2\omega_{k'}\epsilon_0\Omega}\right)^{\frac{1}{2}} e^{i\mathbf{k}\cdot\mathbf{r}}e^{i\mathbf{k}'\cdot\mathbf{r}}\left(i\mathbf{k}\times\bar{\lambda}_{\mathbf{k},\alpha}\right)\left(i\mathbf{k}'\times\bar{\lambda}_{\mathbf{k}',\alpha'}\right) && (67)\\
&\cdot\left\langle 1_{\kappa,a}\left|\left(\hat{a}_{\mathbf{k},\alpha}\hat{a}^\dagger_{-\mathbf{k}',\alpha'}+\hat{a}^\dagger_{-\mathbf{k},\alpha}\hat{a}_{\mathbf{k}',\alpha'}\right)\right|1_{\kappa,a}\right\rangle && (68)\\
&= \sum_{\mathbf{k}',\alpha'}\sum_{\mathbf{k},\alpha}\left(\frac{\hbar}{2\omega_k\epsilon_0\Omega}\right)^{\frac{1}{2}}\left(\frac{\hbar}{2\omega_{k'}\epsilon_0\Omega}\right)^{\frac{1}{2}} e^{i\mathbf{k}\cdot\mathbf{r}}e^{i\mathbf{k}'\cdot\mathbf{r}}\left(i\mathbf{k}\times\bar{\lambda}_{\mathbf{k},\alpha}\right)\left(i\mathbf{k}'\times\bar{\lambda}_{\mathbf{k}',\alpha'}\right) && (69)\\
&\cdot\left\langle 1_{\kappa,a}\left|\left(\hat{a}_{\mathbf{k},\alpha}\hat{a}^\dagger_{-\mathbf{k}',\alpha'}\delta_{\mathbf{k},-\mathbf{k}'}+\hat{a}^\dagger_{-\mathbf{k},\alpha}\hat{a}_{\mathbf{k}',\alpha'}\delta_{-\mathbf{k},\kappa}\delta_{\mathbf{k}',\kappa}\right)\right|1_{\kappa,a}\right\rangle && (70)\\
&= -\sum_{\mathbf{k},\alpha}\left(\frac{\hbar}{2\omega_k\epsilon_0\Omega}\right)^{\frac{1}{2}}\left(i\mathbf{k}\times\bar{\lambda}_{\mathbf{k},\alpha}\right)\left(i\mathbf{k}\times\bar{\lambda}_{-\mathbf{k},\alpha}\right) && (71)\\
&\cdot\left\langle 1_{\kappa,a}\left|\hat{a}_{\mathbf{k},\alpha}\hat{a}^\dagger_{\mathbf{k},\alpha}\right|1_{\kappa,a}\right\rangle + \left(\frac{\hbar}{2\omega_\kappa\epsilon_0\Omega}\right)\left(-i\kappa\times\bar{\lambda}_{-\kappa,a}\right)\left(i\kappa\times\bar{\lambda}_{\kappa,a}\right) && (72)\\
&= \sum_{\mathbf{k},\alpha}\left(\frac{\hbar}{2\omega_k\epsilon_0\Omega}\right)^{\frac{1}{2}}\left(\mathbf{k}\times\bar{\lambda}_{\mathbf{k},\alpha}\right)^2 + 2\left(\frac{\hbar}{2\omega_\kappa\epsilon_0\Omega}\right)\left(\kappa\times\bar{\lambda}_{\kappa,a}\right)^2 && (73)
\end{aligned}$$

So the fluctuation of B:

$$\Delta B = \sqrt{\langle B^2 \rangle - \langle B \rangle^2} \tag{74}$$

$$= \sqrt{\sum_{\mathbf{k},\alpha} \left(\frac{\hbar}{2\omega_k \epsilon_0 \Omega}\right)^{\frac{1}{2}} \left(\mathbf{k} \times \bar{\lambda}_{\mathbf{k},\alpha}\right)^2 + 2\left(\frac{\hbar}{2\omega_\kappa \epsilon_0 \Omega}\right) \left(\kappa \times \bar{\lambda}_{\kappa,a}\right)^2} \tag{75}$$

$$= \sqrt{(\text{Vaccum fluctuation})^2 + 2\left(\frac{\hbar}{2\omega_\kappa \epsilon_0 \Omega}\right) \left(\kappa \times \bar{\lambda}_{\kappa,a}\right)^2} \tag{76}$$

So we get the "usual" vacuum fluctuations and some more fluctuations in the specific mode of the photon. This is not physical because the vacuum fluctuations are bigger than anything. In real life we can't sample an arbitrary amount of modes but only an interval, for simplicity lets assume we can only sample κ. So this becomes:

$$\Delta B = \sqrt{\langle B^2 \rangle - \langle B \rangle^2} \tag{77}$$

$$= \sqrt{3\left(\frac{\hbar}{2\omega_\kappa \epsilon_0 \Omega}\right) \left(\kappa \times \bar{\lambda}_{\kappa,a}\right)^2} \tag{78}$$

$$= \sqrt{(2n_\kappa + 1)\left(\frac{\hbar}{2\omega_\kappa \epsilon_0 \Omega}\right) \left(\kappa \times \bar{\lambda}_{\kappa,a}\right)^2} \tag{79}$$

Where the last line is for a general number state $|n_\kappa, 0, 0, ...\rangle$. Also, this has no spatial dependence - not what we would expect of a particle!

For E see tutorial state of the EM field.

Lets construct a localized photon:

$$|\psi\rangle = \int d^3k \, |1_{\mathbf{k}}\rangle \, \phi(\mathbf{k}) \tag{80}$$

Such that:

$$\langle N \rangle = \int d^3k \left\langle 1_{\mathbf{k}} \left| a_k^\dagger a_k \right| 1_k \right\rangle |\phi(\mathbf{k})|^2 = \int d^3k \, |\phi(\mathbf{k})|^2 = 1 \tag{81}$$

So the last equation is a demand and we get on average a one photon state. The expectation of E is still 0 but the fluctuations become:

$$\begin{aligned}
\left\langle E^2\right\rangle &= i\sum_{\mathbf{k},\alpha}\sum_{\mathbf{k}',\alpha'}\left(\frac{\hbar\omega_k}{2\Omega}\right)^{\frac{1}{2}}\left(\frac{\hbar\omega_{k'}}{2\Omega}\right)^{\frac{1}{2}} e^{i\mathbf{k}\cdot\mathbf{r}}e^{i\mathbf{k}'\cdot\mathbf{r}}\left\langle\psi\left|\left(\hat{a}_{\mathbf{k},\alpha}\bar{\lambda}_{\mathbf{k},\alpha}-\hat{a}^{\dagger}_{-\mathbf{k},\alpha}\bar{\lambda}^{*}_{-\mathbf{k},\alpha}\right)\left(\hat{a}_{\mathbf{k}',\alpha'}\bar{\lambda}_{\mathbf{k}',\alpha'}-\hat{a}^{\dagger}_{-\mathbf{k}',\alpha'}\bar{\lambda}^{*}_{-\mathbf{k}'}\right.\right.\right. \\
&= \ldots \\
&= i\sum_{\mathbf{k},\alpha}\sum_{\mathbf{k}',\alpha'}\left(\frac{\hbar\omega_k}{2\Omega}\right) e^{i\mathbf{k}\cdot\mathbf{r}}e^{i\mathbf{k}'\cdot\mathbf{r}}\left\langle\psi\left|-\hat{a}_{\mathbf{k},\alpha}\bar{\lambda}_{\mathbf{k},\alpha}\hat{a}^{\dagger}_{-\mathbf{k}',\alpha'}\bar{\lambda}^{*}_{-\mathbf{k}',\alpha'}-\hat{a}^{\dagger}_{-\mathbf{k},\alpha}\bar{\lambda}^{*}_{-\mathbf{k},\alpha}\hat{a}_{\mathbf{k}',\alpha'}\bar{\lambda}_{\mathbf{k}',\alpha'}\right|\psi\right\rangle \\
&= i\sum_{\mathbf{k},\alpha}\sum_{\mathbf{k}',\alpha'}\left(\frac{\hbar\omega_k}{2\Omega}\right) e^{i\left(\mathbf{k}+\mathbf{k}'\right)\cdot\mathbf{r}}\int d^3K'd^3K\left\langle 1_{K'}\left|-\hat{a}_{\mathbf{k},\alpha}\bar{\lambda}_{\mathbf{k},\alpha}\hat{a}^{\dagger}_{-\mathbf{k}',\alpha'}\bar{\lambda}^{*}_{-\mathbf{k}',\alpha'}-\hat{a}^{\dagger}_{-\mathbf{k},\alpha}\bar{\lambda}^{*}_{-\mathbf{k},\alpha}\hat{a}_{\mathbf{k}',\alpha'}\bar{\lambda}_{\mathbf{k}',\alpha'}\right.\right. \\
&= i\sum_{\mathbf{k}}\sum_{\mathbf{k}'}\left(\frac{\hbar\omega_k}{2\Omega}\right) e^{i\left(\mathbf{k}+\mathbf{k}'\right)\cdot\mathbf{r}}\int d^3K'd^3K\left\langle 1_{K'}\left|-\hat{a}_{\mathbf{k},\alpha}\hat{a}^{\dagger}_{-\mathbf{k}',\alpha'}-\hat{a}^{\dagger}_{-\mathbf{k},\alpha}\hat{a}_{\mathbf{k}',\alpha'}\right|1_K\right\rangle\phi\left(K\right)\phi^{*}\left(K'\right) \\
&= -i\sum_{\mathbf{k}}\sum_{\mathbf{k}'}\left(\frac{\hbar\omega_k}{2\Omega}\right) e^{i\left(\mathbf{k}+\mathbf{k}'\right)\cdot\mathbf{r}}\int d^3K'd^3K\left(\left\langle 1_{K'}\left|\hat{a}_{\mathbf{k},\alpha}\hat{a}^{\dagger}_{-\mathbf{k}',\alpha'}\right|1_K\right\rangle\delta_{-\mathbf{k}'K'}\delta_{\mathbf{k}K}+\left\langle 1_{K'}\left|\hat{a}^{\dagger}_{-\mathbf{k},\alpha}\hat{a}_{\mathbf{k}',\alpha'}\right.\right.\right. \\
&= V+-i\left(\frac{\hbar\omega_k}{2\Omega}\right)2\int d^3K'd^3K\left(\phi\left(K\right)\phi^{*}\left(K'\right)e^{i\left(K-K'\right)\cdot\mathbf{r}}+\phi\left(K\right)\phi^{*}\left(K'\right)e^{i\left(K-K'\right)\cdot\mathbf{r}}\right) \\
&= -i\left(\frac{\hbar\omega_k}{2\Omega}\right)2\int d^3K'd^3K\left(\phi\left(K\right)\phi^{*}\left(K'\right)e^{i\left(K-K'\right)\cdot\mathbf{r}}\right)
\end{aligned}$$

$$\phi\left(K\right)=\sqrt{\frac{1}{\sqrt{2\pi\sigma^2}}}e^{-\frac{K^2}{4\sigma^2}} \tag{90}$$

Now:

$$\begin{aligned}
\left\langle E^2\right\rangle &= -i\left(\frac{\hbar\omega_k}{2\Omega}\right)2\int d^3K'd^3K\left(\phi\left(K\right)\phi^{*}\left(K'\right)e^{i\left(K-K'\right)\cdot\mathbf{r}}\right) && (91) \\
&\sim e^{-2\sigma^2r^2} && (92)
\end{aligned}$$

Which is messy and localized. Huzzah!

See exam->2012&2014->Q5IdanEranItay.pdf

7 Emission absorption

7.1 Pendulum

We consider the system:

$$\mathcal{H} = \hbar\omega_{ho}\left(\hat{b}_x^{\dagger}\hat{b}_x+\hat{b}_y^{\dagger}\hat{b}_y\right)+H_F+V_{int} \tag{93}$$

Using Fermi we are interested in absorption of a photon with energy $\hbar\omega_{ho}$:

$$\left|n_{x,i},n_{y,i};n_k\right\rangle\rightarrow\left|n_{x,f},n_{y,f};n_k-1\right\rangle \tag{94}$$

This is given by the matrix element:

$$\begin{aligned}
\frac{dP_\alpha}{d\gamma} &= \frac{\omega_k n_k}{4\pi^2\epsilon_0 c^3}\left|\langle n_{x,f}, n_{y,f}\left|\mathbf{d}\right| n_{x,i}, n_{y,i}\rangle \cdot \bar{\lambda}_{\mathbf{k},\alpha}\right|^2 && (95)\\
&= \frac{\omega_k n_k e}{4\pi^2\epsilon_0 c^3}\left|\langle n_{x,f}, n_{y,f}\left|\hat{x}+\hat{y}\right| n_{x,i}, n_{y,i}\rangle \cdot \bar{\lambda}_{\mathbf{k},\alpha}\right|^2 && (96)\\
&= \frac{\omega_k n_k e}{4\pi^2\epsilon_0 c^3}\left|\langle n_{x,f}, n_{y,f}\left|\hat{x}\lambda_{\mathbf{k},\alpha,x}+\hat{y}\lambda_{\mathbf{k},\alpha,y}\right| n_{x,i}, n_{y,i}\rangle\right|^2 && (97)\\
&= \frac{\omega_k n_k e}{4\pi^2\epsilon_0 c^3}\left|\left\langle n_{x,f}, n_{y,f}\left|\sqrt{\frac{\hbar}{2\omega}}\left(\hat{b}_x+\hat{b}_x^\dagger\right)\lambda_{\mathbf{k},\alpha,x}+\sqrt{\frac{\hbar}{2\omega}}\left(\hat{b}_y+\hat{b}_y^\dagger\right)\lambda_{\mathbf{k},\alpha,y}\right| n_{x,i}, n_{y,i}\right\rangle\right|^2 && (98)\\
&= \frac{\omega_k n_k e}{4\pi^2\epsilon_0 c^3}\frac{\hbar}{2\omega}\left|\left\langle n_{x,f}, n_{y,f}\left|\hat{b}_x^\dagger\lambda_{\mathbf{k},\alpha,x}+\hat{b}_y^\dagger\lambda_{\mathbf{k},\alpha,y}\right| n_{x,i}, n_{y,i}\right\rangle\right|^2 && (99)\\
&= \frac{\omega_k n_k e}{4\pi^2\epsilon_0 c^3}\frac{\hbar}{2\omega}\left(\left|\sqrt{n_{x,i}+1}\lambda_{\mathbf{k},\alpha,x}\right|^2+\left|\sqrt{n_{y,i}+1}\lambda_{\mathbf{k},\alpha,y}\right|^2\right) && (100)
\end{aligned}$$

Where in the last line we sum on the two possible final states. Geometry time! For some emitted direction $\mathbf{k}$ we choose one polarization in the $x-y$ plane and one perpendicular, that is:

$$\begin{cases}\bar{k} = (\sin\theta_k\cos\phi_k, \sin\theta_k\sin\phi_k, \cos\theta_k)\\ \bar{\lambda}_1 = (\cos\theta_k\cos\phi_k, \cos\theta_k\sin\phi_k, -\sin\theta_k)\\ \bar{\lambda}_2 = (-\sin\phi_k, \cos\phi_k, 0)\end{cases} \tag{101}$$

Depending on the emitted mode we get:

$$\begin{aligned}
\frac{dP}{d\gamma} &= \frac{\omega_k n_k e}{4\pi^2\epsilon_0 c^3}\frac{\hbar}{2\omega}\begin{cases}\left|\sqrt{n_{x,i}+1}\cos\theta_k\cos\phi_k\right|^2+\left|\sqrt{n_{y,i}+1}\cos\theta_k\sin\phi_k\right|^2 & \alpha=1\\ \left|\sqrt{n_{x,i}+1}\sin\phi_k\right|^2+\left|\sqrt{n_{y,i}+1}\cos\phi_k\right|^2 & \alpha=2\end{cases} && (102)\\
&= \frac{\omega_k n_k e}{4\pi^2\epsilon_0 c^3}\frac{\hbar}{2\omega}\begin{cases}(n_{x,i}+1)\cos^2\theta_k\cos^2\phi_k+(n_{y,i}+1)\cos^2\theta_k\sin^2\phi_k & \alpha=1\\ (n_{x,i}+1)\sin^2\phi_k+(n_{y,i}+1)\cos^2\phi_k & \alpha=2\end{cases} && (103)
\end{aligned}$$

7.2 Rod

7.2.1 In 3d:

The Hamiltonian is:

$$\mathcal{H} = \frac{\mathbf{L}^2}{2I}+H_F+V_{int} \tag{104}$$

And the particles states are:

$$\langle r|\psi\rangle = \langle r|l,m\rangle = y_{lm}(\theta,\phi)\sqrt{\delta(r-R)} \tag{105}$$

Such that:

$$\int y_{lm}y_{lm}^* d\Omega dr$$

For opposite charges we get a dipole transition, for same charges $\mathbf{d}$ vanishes and we need to go to electric quadrupole and magnetic dipole approximation.

We claim there is no magnetic dipole since it is of the form $\mathbf{r} \times \mathbf{j} = \mathbf{L} = a_+ L_+ + a_- L_- + L_z \hat{z}$ and therefore can only change m states and not l (so it doesn't change energy).

According to Wikipedia the full magnetic dipole is given by:

$$\sum_{final} \frac{e^2 \omega_k^3}{3 m_e^2 c^5 \hbar} \langle f \left| \mathbf{L} + 2\mathbf{S} \right| i \rangle \tag{106}$$

Where now since we have spin we can change angular momentum, the selection rules are:

1. Conserves parity
2. $\Delta J = 0, \pm 1$
3. $\Delta m_J = 0, \pm 1$
4. $\Delta J = 0 \&\& \Delta m_j = 0$ are not allowed

Quadrupole transition is now calculated according to HW4

7.2.2 In 2d

The (free) Hamiltonian is given by:

$$\mathcal{H} = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2m} \tag{107}$$

And is solved by

$$\langle \mathbf{r} | \psi \rangle = \psi_n (r, \theta) = \sqrt{\delta (r - R)} e^{in\theta} \tag{108}$$

And again we have no dipole for same charges and dipole for different charges, we get matrix elements:

$$
\begin{aligned}
\frac{dP_\alpha}{d\gamma} &= \frac{\omega_k N_k}{4\pi^2\epsilon_0 c^3}\left|\left\langle n_f\left|\mathbf{d}\right|n_i,\right\rangle\cdot\bar\lambda_{\mathbf{k},\alpha}\right|^2 \\
&= \frac{\omega_k N_k e^2}{4\pi^2\epsilon_0 c^3}\left|\left\langle n_f\left|\left(\hat R\cdot\bar x\right)\bar\lambda_{\mathbf{k},\alpha,x}\right|n_i\right\rangle+\left\langle n_f\left|\left(\hat R\cdot\bar y\right)\bar\lambda_{\mathbf{k},\alpha,y}\right|n_i\right\rangle\right|^2 \\
&= \frac{\omega_k N_k e^2}{4\pi^2\epsilon_0 c^3}\left|\int_0^{2\pi} e^{-in_f\theta}e^{in_i\theta}\cos\theta d\theta\bar\lambda_{\mathbf{k},\alpha,x}+\int_0^{2\pi} e^{-in_f\theta}e^{in_i\theta}\sin\theta d\theta\bar\lambda_{\mathbf{k},\alpha,y}\right|^2 \\
&= \frac{\omega_k N_k e^2}{4\pi^2\epsilon_0 c^3}\left|\int_0^{2\pi} e^{i(n_i-n_f)\theta}\frac{e^{i\theta}+e^{-i\theta}}{2}d\theta\bar\lambda_{\mathbf{k},\alpha,x}+\int_0^{2\pi} e^{i(n_i-n_f)\theta}\frac{e^{i\theta}-e^{-i\theta}}{2i}d\theta\bar\lambda_{\mathbf{k},\alpha,y}\right|^2 \\
&= \frac{\omega_k N_k e^2}{16\pi^2\epsilon_0 c^3}\left|\left(\int_0^{2\pi} e^{i(n_i-n_f+1)\theta}d\theta+\int_0^{2\pi} e^{i(n_i-n_f-1)\theta}d\theta\right)\bar\lambda_{\mathbf{k},\alpha,x}-i\left(\int_0^{2\pi} e^{i(n_i-n_f+1)\theta}d\theta-\int_0^{2\pi} e^{i(n_i-}\right.\right. \\
&= \frac{\omega_k N_k e^2}{16\pi^2\epsilon_0 c^3}\left|2\pi\left(\delta_{n_i+1,n_f}+\delta_{n_i-1,n_f}\right)\bar\lambda_{\mathbf{k},\alpha,x}-i2\pi\left(\delta_{n_i,n_f-1}-\delta_{n_i,n_f+1}\right)\bar\lambda_{\mathbf{k},\alpha,y}\right|^2 \\
&= \frac{\omega_k N_k e^2}{4\epsilon_0 c^3}\left|\left(\delta_{n_i+1,n_f}+\delta_{n_i-1,n_f}\right)\bar\lambda_{\mathbf{k},\alpha,x}-i\left(\delta_{n_i,n_f-1}-\delta_{n_i,n_f+1}\right)\bar\lambda_{\mathbf{k},\alpha,y}\right|^2 \\
&= \frac{\omega_k N_k e^2}{4\epsilon_0 c^3}\left|\left(\delta_{n_i,n_f-1}+\delta_{n_i,n_f+1}\right)\bar\lambda_{\mathbf{k},\alpha,x}-i\left(\delta_{n_i,n_f-1}-\delta_{n_i,n_f+1}\right)\bar\lambda_{\mathbf{k},\alpha,y}\right|^2 \\
\underbrace{=}_{\text{f.states}}& \frac{\omega_k N_k e^2}{4\epsilon_0 c^3}\left(\left|\bar\lambda_{\mathbf{k},\alpha,x}-i\bar\lambda_{\mathbf{k},\alpha,y}\right|^2+\left|\bar\lambda_{\mathbf{k},\alpha,x}+i\bar\lambda_{\mathbf{k},\alpha,y}\right|^2\right) \\
&= \frac{\omega_k N_k e^2}{2\epsilon_0 c^3}\left(\bar\lambda^2_{\mathbf{k},\alpha,x}+\bar\lambda^2_{\mathbf{k},\alpha,y}\right)
\end{aligned}
$$

Defining:

$$
\begin{cases}
\bar k = (\sin\theta_k\cos\phi_k, \sin\theta_k\sin\phi_k, \cos\theta_k) \\
\bar\lambda_1 = (\cos\theta_k\cos\phi_k, \cos\theta_k\sin\phi_k, -\sin\theta_k) \\
\bar\lambda_2 = (-\sin\phi_k, \cos\phi_k, 0)
\end{cases}
\tag{119}
$$

We get:

$$
\frac{dP_\alpha}{d\gamma} = \begin{cases}
\frac{\omega_k N_k e^2}{2\epsilon_0 c^3}\cos^2\theta_k & \alpha = 1 \\
\frac{\omega_k N_k e^2}{2\epsilon_0 c^3} & \alpha = 2
\end{cases}
\tag{120}
$$

For quadrupole, fuck off.

8 Boson & Fermion Hamiltonian

$$
\mathcal{H} = \int \psi^\dagger(x)\left(-\frac{\hbar^2}{2m}\partial_x^2+U(x)\right)\psi(x)\,dx+\int\psi^\dagger(x)\,\psi(x)\,V(x-y)\,\mu(y)\,dxdy
\tag{121}
$$

$$\mu(y) = \frac{1}{2}\sum_{\alpha,\beta}\xi_\alpha^\dagger(y)\,\sigma^x_{\alpha\beta}\xi_\beta(y) \tag{122}$$

$$U(x) = \frac{m\Omega^2 x^2}{2} \tag{123}$$

$$V(x-y) = \frac{m\omega^2(x-y)^2}{2} \tag{124}$$

The first part is just a QHO in second quantization. We define:

$$\psi(x) = \sum_n \phi_n(x)\,a_n \tag{125}$$

And we get:

$$\begin{aligned}
\int \psi^\dagger(x)\left(-\frac{\hbar^2}{2m}\partial_x^2 + U(x)\right)\psi(x)\,dx &= \sum_{n,n'}\int \phi^*_{n'}(x)\,a^\dagger_{n'}\left(-\frac{\hbar^2}{2m}\partial_x^2 + U(x)\right)\phi_n(x)\,a_n dx && (126)\\
&= \sum_{n,n'}\int \phi^*_{n'}(x)\,a^\dagger_{n'}\left(\hbar\Omega\left(n+\frac{1}{2}\right)\right)\phi_n(x)\,a_n dx && (127)\\
&= \sum_n \hbar\Omega\left(n+\frac{1}{2}\right)a^\dagger_n a_n && (128)
\end{aligned}$$

Cool.

The interaction is:

$$\int \psi^\dagger(x)\,\psi(x)\,V(x-y)\,\mu(y)\,dxdy = \int \psi^\dagger(x)\,\psi(x)\,\frac{m\omega^2(x-y)^2}{4}\left(\xi^\dagger_\alpha\sigma^x_{\alpha\beta}\xi_\beta\right)dxdy \tag{129}$$

The term:

$$\xi^\dagger_\alpha\sigma^x_{\alpha\beta}\xi_\beta = \xi^\dagger\sigma^x\xi \tag{130}$$

Acts on the fermion field which has two values (for each position in space). We write it in the σ_x basis such that when acting on the unoccupied state:

$$\xi^\dagger\sigma^x\xi\,|-;y_0\rangle = 0 \tag{131}$$

And on the occupied state:

$$\xi^\dagger\sigma^x\xi\,|+;y_0\rangle = \xi^\dagger\sigma^x\,|-;y_0\rangle = -\xi^\dagger\,|-;y_0\rangle = -\,|+;y_0\rangle \tag{132}$$

So when the field is occupied it adds a minus interaction and when it is not it is uncoupled. The fermionic state of the system is given by:

$$|\text{fermion}\rangle = \int \xi^\dagger(y)\,\phi(y)\,dy\,|-\rangle \tag{133}$$

So for this state the interaction becomes:

$$\int \psi^{\dagger}(x)\psi(x)\frac{m\omega^2(x-y)^2}{4}\left(\xi^{\dagger}\sigma^x\xi\right)dxdy\,|\text{fermion}\rangle = -\int \psi^{\dagger}(x)\psi(x)\frac{m\omega^2(x-y)^2}{4}\phi(y)\,dxdy\,|\text{fermion}\rangle \tag{134}$$

So now we can add this term to the free Hamiltonian an get:

$$\begin{aligned}\mathcal{H} &= \int \psi^{\dagger}(x)\left(-\frac{\hbar^2}{2m}\partial_x^2\right)\psi(x)\,dx + \int \psi^{\dagger}(x)\psi(x)\left(\frac{m\Omega^2x^2}{2} - \int dy\frac{m\omega^2(x-y)^2}{4}\phi(y)\right)dx \quad (135)\\ &= \int \psi^{\dagger}(x)\left(-\frac{\hbar^2}{2m}\partial_x^2\right)\psi(x)\,dx + \int \psi^{\dagger}(x)\psi(x)\left(\frac{m\Omega^2x^2}{2} + f_1x^2 + f_2x + f_3\right)dx \quad (136)\end{aligned}$$

Where in general f_1, f_2, f_3 are numers that depend on $\phi(y)$. Assuming we have a single excitation:

$$|\text{fermion}\rangle = |+; y_0\rangle \tag{137}$$

We just get:

$$\mathcal{H} = \int \psi^{\dagger}(x)\left(-\frac{\hbar^2}{2m}\partial_x^2\right)\psi(x)\,dx + \int \psi^{\dagger}(x)\psi(x)\left(\frac{m\Omega^2x^2}{2} - \frac{m\omega^2(x-y_0)^2}{4}\right)dx \tag{138}$$

So we just added a shift and changed the frequency. We can rewrite the potential when the Fermion is occupied:

$$\begin{aligned}\Omega^2x^2 + \frac{\omega^2(x-y_0)^2}{2} &= \Omega^2x^2 - \frac{\omega^2}{2}\left(x^2 - 2xy_0 + y_0^2\right) &(139)\\ &= x^2\underbrace{\left(\Omega^2 - \frac{\omega^2}{2}\right)}_{\Theta^2} + x\omega^2y_0 - \frac{\omega^2}{2}y_0^2 &(140)\\ &= \Theta^2\left(x^2 + x\frac{\omega^2y_0}{\Theta^2} - \frac{\frac{\omega^2}{2}y_0^2}{\Theta^2}\right) &(141)\\ &= \Theta^2\left(x + \frac{\omega^2y_0}{2\Theta^2}\right)^2 - \left[\frac{\omega^2}{2}y_0^2 + \frac{(\omega^2y_0)^2}{4\Theta^2}\right] &(142)\end{aligned}$$

So, our eigen basis is:

$$|n_{\Theta}; (y_0; +)\rangle, |n_{\omega}; (y_0; -)\rangle \tag{143}$$

Where n is the state of the harmonic oscillator shifted by $\frac{\omega^2y_0}{2\Theta^2}$ and frequency Θ and a Fermion localized at y_0 with spin $\pm$.

9 Eigenstate of ψ and $\psi^{\dagger}$

In analogy to coherent state we claim that:

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\sum\frac{\alpha^n\left(\psi^{\dagger}(x)\right)^n}{n!}|0\rangle \tag{144}$$

Lets show this:

$$\begin{aligned}
\psi(x)|\alpha\rangle &= e^{-\frac{|\alpha|^2}{2}}\sum\frac{\alpha^n\psi(x)\left(\psi^\dagger(x)\right)^n}{n!}|0\rangle & (145)\\
&= e^{-\frac{|\alpha|^2}{2}}\sum\frac{\alpha^n\left(\left(\psi^\dagger(x)\right)^n\psi(x)+\left[\psi(x),\left(\psi^\dagger(x)\right)^n\right]\right)}{n!}|0\rangle & (146)\\
&= e^{-\frac{|\alpha|^2}{2}}\sum\frac{\alpha^n\left(n\left(\psi^\dagger(x)\right)^{n-1}\left[\psi(x),\psi^\dagger(x)\right]\right)}{n!}|0\rangle & (147)\\
&= e^{-\frac{|\alpha|^2}{2}}\sum\frac{\alpha^n n\left(\psi^\dagger(x)\right)^{n-1}}{n!}|0\rangle & (148)\\
&= e^{-\frac{|\alpha|^2}{2}}\sum\frac{\alpha^n\left(\psi^\dagger(x)\right)^{n-1}}{(n-1)!}|0\rangle & (149)\\
&= \alpha e^{-\frac{|\alpha|^2}{2}}\sum\frac{\alpha^{n-1}\left(\psi^\dagger(x)\right)^{n-1}}{(n-1)!}|0\rangle & (150)\\
&= \alpha|\alpha\rangle & (151)
\end{aligned}$$

There is no eigenstate for $\psi^\dagger$. By contradiction, in general:

$$|\alpha\rangle=\sum c_n\left(\psi^\dagger(x)\right)^n|0\rangle \tag{152}$$

Acting:

$$\psi^\dagger|\alpha\rangle=\sum c_n\left(\psi^\dagger(x)\right)^{n+1}|0\rangle\underbrace{=}_{demand}\alpha|\alpha\rangle=\alpha\sum c_n\left(\psi^\dagger(x)\right)^n|0\rangle \tag{153}$$

$$\begin{aligned}
\langle m|\sum c_n\left(\psi^\dagger(x)\right)^{n+1}|0\rangle &= \langle m|\alpha\sum c_n\left(\psi^\dagger(x)\right)^n|0\rangle & (154)\\
c_{m-1} &= \alpha c_m & (155)
\end{aligned}$$

We note that $\langle 0|\psi^\dagger|\alpha\rangle=0$, that is, $c_0=0$. Now using the recursion relation, $c_n=0\Rightarrow|\alpha\rangle=0$. What about $\alpha=0$?

$$\begin{aligned}
\sum c_n\left(\psi^\dagger(x)\right)^{n+1}|0\rangle &= 0 & (156)\\
\langle m|\sum c_n\left(\psi^\dagger(x)\right)^{n+1}|0\rangle &= 0 & (157)\\
c_{m-1} &= 0 & (158)
\end{aligned}$$

So again, no such state.
So what is the number operator for $|\alpha\rangle$ of the annihilation operator?

$$\begin{aligned}
\langle\alpha|N|\alpha\rangle &= \left\langle\alpha\left|\psi^\dagger\psi\right|\alpha\right\rangle & (159)\\
&= \alpha^*\alpha\langle\alpha|\alpha\rangle & (160)\\
&= |\alpha|^2 & (161)
\end{aligned}$$

So we interpret as the average number of particles is $|\alpha|^2$. The variance:

$$\begin{aligned}
\left\langle \alpha \left| N^2 \right| \alpha \right\rangle &= \left\langle \alpha \left| \psi^\dagger \psi \psi^\dagger \psi \right| \alpha \right\rangle && (162)\\
&= \alpha^* \alpha \left\langle \alpha \left| \psi \psi^\dagger \right| \alpha \right\rangle && (163)\\
&= \left| \alpha^2 \right| \left\langle \alpha \left| \psi^\dagger \psi + \left[\psi, \psi^\dagger \right] \right| \alpha \right\rangle && (164)\\
&= \left| \alpha^2 \right| \left(\left| \alpha^2 \right| + 1 \right) && (165)
\end{aligned}$$

Therefore:

$$\begin{aligned}
\Delta N &= \sqrt{\langle N^2 \rangle - \langle N \rangle^2} && (166)\\
&= \sqrt{|\alpha|^4 + |\alpha|^2 - |\alpha|^4} && (167)\\
&= |\alpha| && (168)
\end{aligned}$$

This looks like a french fish!

10 Problem 4 hw 5

10.1 $|\psi\rangle = \sum_{ij} C_{ij}\hat{a}_i^\dagger \hat{a}_j^\dagger |0\rangle$

$$
\begin{aligned}
N^{-2}_{\substack{B\\F}} &= \langle\psi|\psi\rangle && (169)\\
&= \sum_{ijkl} C_{ij}C^*_{kl}\left\langle 0 \left| \hat{a}_l\hat{a}_k\hat{a}_i^\dagger\hat{a}_j^\dagger \right| 0\right\rangle && (170)\\
&= \sum_{ijkl} C_{ij}C^*_{kl}\left\langle 0 \left| \hat{a}_l\left(\left[\hat{a}_k,\hat{a}_i^\dagger\right]_\pm \pm \hat{a}_i^\dagger\hat{a}_k\right)\hat{a}_j^\dagger \right| 0\right\rangle && (171)\\
&= \sum_{ijkl} C_{ij}C^*_{kl}\left\langle 0 \left| \hat{a}_l\left(\delta_{ki} \pm \hat{a}_i^\dagger\hat{a}_k\right)\hat{a}_j^\dagger \right| 0\right\rangle && (172)\\
&= \sum_{ijkl} C_{ij}C^*_{kl}\left\langle 0 \left| \delta_{ki}\left(\delta_{lj} \pm \hat{a}_j^\dagger\hat{a}_l\right) \pm \left(\delta_{li} \pm \hat{a}_i^\dagger\hat{a}_l\right)\left(\delta_{kj} \pm \hat{a}_j^\dagger\hat{a}_k\right) \right| 0\right\rangle && (173)\\
&= \sum_{ijkl} C_{ij}C^*_{kl}\left\langle 0 \left| \delta_{ki}\delta_{lj} \pm \delta_{li}\delta_{kj} \right| 0\right\rangle && (174)\\
&= \sum_{ij} C_{ij}C^*_{ij} \pm C_{ij}C^*_{ji} && (175)\\
&= \sum_{ij} C_{ij}\left(C_{ij} \pm C_{ji}\right)^* && (176)\\
&= \sum_{ij} 2C_{ij}\left(C_{\substack{s\\a}}\right)^*_{ij} && (177)\\
&= 2\mathrm{Tr}\left(CC^\dagger_{\substack{s\\a}}\right) && (178)
\end{aligned}
$$

Where $C_{\substack{s\\a}}$ is the symmetric and anti-symmetric parts of C respectively. Therefore:

$$
N_{\substack{B\\F}} = \left(2\mathrm{Tr}\left(CC^\dagger_{\substack{s\\a}}\right)\right)^{-\frac{1}{2}} \tag{179}
$$

We note that for Bosons C is (wlog) symmetric for Bosons and anti symmetric for Fermions since:

$$\begin{aligned}\left(c_{ij}\hat{a}_i^\dagger\hat{a}_j^\dagger + c_{ji}\hat{a}_j^\dagger\hat{a}_i^\dagger\right)|0\rangle &= \left(c_{ij}\hat{a}_i^\dagger\hat{a}_j^\dagger \pm c_{ji}\hat{a}_i^\dagger\hat{a}_j^\dagger\right)|0\rangle &(180)\\ &= \left(\frac{c_{ij}\pm c_{ji}}{2}\hat{a}_i^\dagger\hat{a}_j^\dagger + \frac{c_{ij}\pm c_{ji}}{2}\hat{a}_i^\dagger\hat{a}_j^\dagger\right)|0\rangle &(181)\\ &= \left(\frac{c_{ij}\pm c_{ij}}{2}\hat{a}_i^\dagger\hat{a}_j^\dagger \pm \frac{c_{ij}\pm c_{ji}}{2}\hat{a}_j^\dagger\hat{a}_i^\dagger\right)|0\rangle &(182)\end{aligned}$$

So in general:

$$C_{ij} = \pm C_{ji} \tag{183}$$

Therefore we can just as well write:

$$N_{\substack{B\\F}} = \left(2\mathrm{Tr}\left(CC^\dagger\right)\right)^{-\frac{1}{2}} \tag{184}$$

10.2 $|\psi\rangle = \hat{a}_i^\dagger\hat{a}_j^\dagger\hat{a}_k^\dagger|0\rangle$

Due to symmetry we separate to the three cases: (1) all three different, (2) two alike (wlog $i = j \neq k$) and (3) all three equal. Starting with Bosons:

$$\begin{aligned}N_B^{-2} &= \langle\psi|\psi\rangle &(185)\\ &= \left\langle 0\left|\hat{a}_c\hat{a}_b\hat{a}_a\hat{a}_i^\dagger\hat{a}_j^\dagger\hat{a}_k^\dagger\right|0\right\rangle &(186)\\ &= \begin{cases}\left\langle 0\left|\hat{a}_c\hat{a}_b\hat{a}_a\hat{a}_i^\dagger\hat{a}_j^\dagger\hat{a}_k^\dagger\right|0\right\rangle & (1)\\ \left\langle 0\left|\hat{a}_c\hat{a}_a\hat{a}_a\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_k^\dagger\right|0\right\rangle & (2)\\ \left\langle 0\left|\hat{a}_a\hat{a}_a\hat{a}_a\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i^\dagger\right|0\right\rangle & (3)\end{cases} &(187)\\ &= \begin{cases}\langle 0\left|\hat{a}_c\hat{a}_b\hat{a}_a\right|1_i,1_j,1_k,\{0\}\rangle & (1)\\ \sqrt{2}\left\langle 0\left|\hat{a}_c\hat{a}_a\hat{a}_a\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_k^\dagger\right|2_i,1_k,\{0\}\right\rangle & (2)\\ \sqrt{6}\langle 0\left|\hat{a}_a\hat{a}_a\hat{a}_a\right|3_i\rangle & (3)\end{cases} &(188)\\ &= \begin{cases}1 & (1)\\ 2 & (2)\\ 6 & (3)\end{cases} &(189)\\ N_B &= \begin{cases}1 & (1)\\ \frac{1}{\sqrt{2}} & (2)\\ \frac{1}{\sqrt{6}} & (3)\end{cases} &(190)\end{aligned}$$

For fermions we cannot raise the same mode twice (the highest occupation number is 1) therefore:

$$N_F = \langle\psi|\psi\rangle = \begin{cases}1 & (1)\\ 0 & (2)\\ 0 & (3)\end{cases} \tag{191}$$

11 2nd quantization and HO

We consider:

$$\mathcal{H} = \int \psi^{\dagger}(x)\left(-\frac{\hbar^2}{2m}\partial_x^2 + \frac{1}{2}m\omega^2\right)\psi(x)\,dx \tag{192}$$

We define:

$$\hat{\psi}(x) = \sum_n \phi_n(x)\,\hat{a}_n \tag{193}$$

And we get:

$$\begin{aligned}
\int \psi^{\dagger}(x)\left(-\frac{\hbar^2}{2m}\partial_x^2 + \frac{1}{2}m\omega^2\right)\psi(x)\,dx &= \sum_{n,n'}\int \phi_{n'}^{*}(x)\,a_{n'}^{\dagger}\left(-\frac{\hbar^2}{2m}\partial_x^2 + \frac{1}{2}m\omega^2\right)\phi_n(x)\,a_n dx && (194) \\
&= \sum_{n,n'}\int \phi_{n'}^{*}(x)\,a_{n'}^{\dagger}\left(\hbar\omega\left(n+\frac{1}{2}\right)\right)\phi_n(x)\,a_n dx && (195) \\
&= \sum_n \hbar\omega\left(n+\frac{1}{2}\right)a_n^{\dagger}a_n && (196)
\end{aligned}$$

Cool.

So what is the difference between the state: $\hat{a}_n^{\dagger}|0\rangle$ and $\hat{\psi}^{\dagger}|0\rangle$? In $\hat{a}_n^{\dagger}|0\rangle$ we create an excitation in mode n (usually these modes are plane waves, but here we expanded on Hermite-Gauss modes, that is QGO wave functions). What kind of particle is this? The momentum coordinate is given by:

$$i\hbar\psi^{\dagger} = i\hbar\left(\sum_n \phi_n(x)\,\hat{a}_n\right)^{\dagger} = i\hbar\sum_n \phi_n(x)\,\hat{a}_n^{\dagger} \tag{197}$$

The momentum operator is given by:

$$\begin{aligned}
\hat{P} &= -\int i\hbar\hat{\psi}^{\dagger}(x)\frac{\partial}{\partial x}\hat{\psi}(x)\,dx && (198) \\
&= -\int i\hbar\left[\sum_{n'}\phi_{n'}(x)\,\hat{a}_{n'}^{\dagger}\right]\frac{\partial}{\partial x}\sum_n \phi_n(x)\,\hat{a}_n dx && (199) \\
&= -i\hbar\sum_{n',n}\hat{a}_{n'}^{\dagger}\hat{a}_n\int \phi_{n'}\frac{\partial}{\partial x}\phi_n dx && (200) \\
&= -i\hbar\sum_{n',n}\hat{a}_{n'}^{\dagger}\hat{a}_n\int \phi_{n'}\frac{\partial}{\partial x}\phi_n dx && (201) \\
&= -i\hbar\sum_{n',n}\hat{a}_{n'}^{\dagger}\hat{a}_n\int \phi_{n'}\left(k_1\phi_{n-1} + k_2\phi_{n+1}\right)dx && (202) \\
&= -i\hbar\sum_n\left(k_1\hat{a}_{n-1}^{\dagger}\hat{a}_n + k_2\hat{a}_{n+1}^{\dagger}\hat{a}_n\right) && (203)
\end{aligned}$$

Which does not commute with the Hamiltonian and therefore is not conserved, these excitations don't have well defined momentum.

Where we guessed the derivative using this:

$$\frac{\partial}{\partial x}\phi_n = \frac{1}{-i\hbar}\left(-i\hbar\frac{\partial}{\partial x}\right)\phi_n \tag{204}$$

$$= \frac{1}{-i\hbar}p\phi_n \tag{205}$$

$$\sim \left(a - a^{\dagger}\right)\phi_n \tag{206}$$

On the other hand: $\hat{\psi}^{\dagger}(x_0)|0\rangle$ create a localized excitation which is spanned on all modes. In mathematical terms this is just using the completeness relation and saying:

$$\left\langle x\left|\hat{\psi}^{\dagger}(x_0)\right|0\right\rangle = \delta(x - x_0) = \sum c_n\phi_n(x) \tag{207}$$

12 Bogoliuobov spectrum

We started with G-P equation:

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\Delta\psi + \int d^3r' V(\mathbf{r} - \mathbf{r}')\left|\psi(\mathbf{r}')\right|^2\psi(\mathbf{r}) \tag{208}$$

Denoting\assuming:

$$\int d^3r V(\mathbf{r}) = V_0 \tag{209}$$

This is solved by

$$\psi = \sqrt{\rho_0}e^{-\frac{i}{\hbar}\chi t} \tag{210}$$

Since:

$$-i\hbar\frac{i}{\hbar}\chi\psi = \left|\sqrt{\rho_0}e^{\frac{i}{\hbar}\chi t}\right|^2\psi V_0 \tag{211}$$

$$= \rho_0\psi V_0 \tag{212}$$

$$\Rightarrow \chi = \rho_0 V_0 \tag{213}$$

Which is just like the delta function case - the ground state is unchanged! What about excitations? We add:

$$\psi = \left(\sqrt{\rho_0} + \delta\psi\right)e^{-\frac{i}{\hbar}\rho_0 V_0 t} \tag{214}$$

And linearize:

$$\begin{aligned}
i\hbar\partial_t\left[\left(\sqrt{\rho_0}+\delta\psi\right)e^{-\frac{i}{\hbar}\rho_0V_0t}\right] &= -\frac{\hbar^2}{2m}\Delta\left[\left(\sqrt{\rho_0}+\delta\psi\right)e^{-\frac{i}{\hbar}\rho_0V_0t}\right] \\
&+ \int d^3r'V\left(\mathbf{r}-\mathbf{r}'\right)\left|\left(\sqrt{\rho_0}+\delta\psi\left(r'\right)\right)e^{-\frac{i}{\hbar}\rho_0V_0t}\right|^2\left(\sqrt{\rho_0}+\delta\psi\right)e^{-\frac{i}{\hbar}\rho_0V_0t} \\
&\approx -\frac{\hbar^2}{2m}\Delta\delta\psi e^{-\frac{i}{\hbar}\rho_0V_0t} \\
&+ \int d^3r'V\left(\mathbf{r}-\mathbf{r}'\right)\left(\rho_0+\sqrt{\rho_0}\left(\delta\psi\left(r'\right)+\delta\psi^*\left(r'\right)\right)\right)\left(\sqrt{\rho_0}+\delta\psi\right)e^{-\frac{i}{\hbar}\rho_0V_0t} \\
i\hbar\partial_t\left[\left(\sqrt{\rho_0}+\delta\psi\right)e^{-\frac{i}{\hbar}\rho_0V_0t}\right] &= -\frac{\hbar^2}{2m}\Delta\delta\psi e^{-\frac{i}{\hbar}\rho_0V_0t}+\int d^3r'V\left(\mathbf{r}-\mathbf{r}'\right)\left(\rho_0+\sqrt{\rho_0}\left(\delta\psi\left(r'\right)+\delta\psi^*\left(r'\right)\right)\right)\left(\sqrt{\rho_0}\right. \\
i\hbar\partial_t\delta\psi &= -\frac{\hbar^2}{2m}\Delta\delta\psi+\rho_0\int d^3r'V\left(\mathbf{r}-\mathbf{r}'\right)\left(\delta\psi\left(r'\right)+\delta\psi^*\left(r'\right)\right)
\end{aligned}$$

Guessing:

$$\delta\psi = A_{\mathbf{k}}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}+A^*_{-\mathbf{k}}e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{221}$$

We get:

$$\begin{aligned}
\omega\hbar\left(A_{\mathbf{k}}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}-A^*_{-\mathbf{k}}e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right) &= \frac{\hbar^2}{2m}k^2\left(A_{\mathbf{k}}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}+A^*_{-\mathbf{k}}e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right) \\
&+ \rho_0\int d^3r'V\left(\mathbf{r}-\mathbf{r}'\right)\left(\left(A_{\mathbf{k}}+A_{-\mathbf{k}}\right)e^{i\left(\mathbf{k}\cdot\mathbf{r}'-\omega t\right)}+\left(A_{-\mathbf{k}}+A_{\mathbf{k}}\right)^*e^{-i(\mathbf{k}\right. \\
\omega\hbar\left(A_{\mathbf{k}}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}-A^*_{-\mathbf{k}}e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right) &= \frac{\hbar^2}{2m}k^2\left(A_{\mathbf{k}}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}+A^*_{-\mathbf{k}}e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right) \\
&+ \rho_0\int d^3r'V\left(\mathbf{r}'\right)\left(\left(A_{\mathbf{k}}+A_{-\mathbf{k}}\right)e^{i\left(\mathbf{k}\cdot\left(\mathbf{r}-\mathbf{r}'\right)-\omega t\right)}+\left(A_{-\mathbf{k}}+A_{\mathbf{k}}\right)^*e^{-i(\mathbf{k}\right. \\
\omega\hbar\left(A_{\mathbf{k}}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}-A^*_{-\mathbf{k}}e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right) &= \frac{\hbar^2}{2m}k^2\left(A_{\mathbf{k}}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}+A^*_{-\mathbf{k}}e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right) \\
&+ \rho_0\int d^3r'V\left(\mathbf{r}'\right)\left(A_{\mathbf{k}}+A_{-\mathbf{k}}\right)e^{i\left(\mathbf{k}\cdot\left(\mathbf{r}-\mathbf{r}'\right)-\omega t\right)}+\rho_0\int d^3r'V\left(\mathbf{r}'\right)\left(A_{-\right. \\
&= \frac{\hbar^2}{2m}k^2\left(A_{\mathbf{k}}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}+A^*_{-\mathbf{k}}e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right) \\
&+ e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\rho_0\int d^3r'V\left(\mathbf{r}'\right)\left(A_{\mathbf{k}}+A_{-\mathbf{k}}\right)e^{-i\mathbf{k}\cdot\mathbf{r}'}+e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\rho_0\int d^3 \\
&= \frac{\hbar^2}{2m}k^2\left(A_{\mathbf{k}}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}+A^*_{-\mathbf{k}}e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right) \\
&+ e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\rho_0V\left(\mathbf{k}\right)\left(A_{\mathbf{k}}+A_{-\mathbf{k}}\right)+e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\rho_0V^*\left(\mathbf{k}\right)\left(A_{-\mathbf{k}}+A_{\mathbf{k}}\right)^*
\end{aligned}$$

Writing for each exponent in separate:

$$\begin{cases}\omega\hbar A_{\mathbf{k}}=\frac{\hbar^2}{2m}k^2A_{\mathbf{k}}+\rho_0V\left(\mathbf{k}\right)\left(A_{\mathbf{k}}+A_{-\mathbf{k}}\right)\\ -\omega\hbar A_{-\mathbf{k}}=\frac{\hbar^2}{2m}k^2A_{-\mathbf{k}}+\rho_0V\left(\mathbf{k}\right)\left(A_{-\mathbf{k}}+A_{\mathbf{k}}\right)\end{cases}$$

$$\begin{cases} A_{\mathbf{k}}\left(\omega\hbar - \frac{\hbar^2}{2m}k^2 - \rho_0 V\left(\mathbf{k}\right)\right) - A_{-\mathbf{k}}\rho_0 V\left(\mathbf{k}\right) = 0 \\ A_{\mathbf{k}}\left(-\rho_0 V\left(\mathbf{k}\right)\right) + A_{-\mathbf{k}}\left(-\omega\hbar - \frac{\hbar^2}{2m}k^2 - \rho_0 V\left(\mathbf{k}\right)\right) = 0 \end{cases} \tag{232}$$

Demanding a solution!

$$\begin{aligned} -\left(\omega\hbar - \left(\frac{\hbar^2}{2m}k^2 + \rho_0 V\left(\mathbf{k}\right)\right)\right)\left(\omega\hbar + \frac{\hbar^2}{2m}k^2 + \rho_0 V\left(\mathbf{k}\right)\right) + \rho_0 V\left(\mathbf{k}\right)\left(-\rho_0 V\left(\mathbf{k}\right)\right) &= 0 \\ -\omega^2\hbar^2 + \left(\frac{\hbar^2}{2m}k^2 + \rho_0 V\left(\mathbf{k}\right)\right)^2 - \rho_0^2 V^2\left(\mathbf{k}\right) &= 0 \\ \omega^2\hbar^2 &= \left(\frac{\hbar^2}{2m}k^2 + \rho_0 V\left(\mathbf{k}\right)\right)^2 \\ &= \frac{\hbar^2}{2m}k^2\left(\frac{\hbar^2}{2m}k^2 + 2\rho_0 V\right. \end{aligned}$$

So, what did we learn? absolutely nothing! We note that for a characteristic length σ then $V\left(\mathbf{k}\right) \sim \frac{1}{\sigma}$

And we get:

$$\omega^2 \sim \frac{k^2}{2m}\left(\frac{\hbar^2}{2m}k^2 + \frac{2\rho_0}{\sigma}\right) \tag{237}$$

For small k this controls the linear slope of the dispersion:

$$\omega \sim k\sqrt{\frac{\rho_0}{m\sigma}} \tag{238}$$

13 Many body Hamiltonian

$$\mathcal{H} = \sum_{\alpha}\epsilon_{\alpha}a_{\alpha}^{\dagger}a_{\alpha} + \frac{1}{2}\sum_{\alpha\beta\gamma\delta}V_{\alpha\beta\gamma\delta}a_{\alpha}^{\dagger}a_{\beta}^{\dagger}a_{\delta}a_{\gamma} \tag{239}$$

Where the interaction is only: $V_{\alpha\beta\alpha\beta}$ and $V_{\alpha\beta\beta\alpha}$ so:

$$\mathcal{H} = \sum_{\alpha}\epsilon_{\alpha}a_{\alpha}^{\dagger}\alpha_{\alpha} + \frac{1}{2}\sum_{\alpha\beta}\left(V_{\alpha\beta\alpha\beta}a_{\alpha}^{\dagger}a_{\beta}^{\dagger}a_{\alpha}a_{\beta} + V_{\alpha\beta\beta\alpha}a_{\alpha}^{\dagger}a_{\beta}^{\dagger}a_{\beta}a_{\alpha}\right) \tag{240}$$

What are the symmetries? generators? We see that aniihaltion and creation comes in pairs - we expect $U\left(1\right)$ symmetry for any mode:

$$a_{\alpha} \to e^{i\theta_{\alpha}}a_{\alpha},\ a_{\alpha}^{\dagger} \to e^{-i\theta_{\alpha}}a_{\alpha}^{\dagger}$$

This is generated by the number operator:

$$a_{\alpha} \to e^{-i\theta_{\alpha}N_{\alpha}}a_{\alpha}ae^{i\theta_{\alpha}N_{\alpha}} = e^{i\theta_{\alpha}}a \tag{241}$$

So the number of particles is conserved. Therefore the eigenfunctions are just the fock states defined by the unperturbed (diagonal) Hamiltonian:

$$\mathcal{H}_0 = \sum_\alpha \epsilon_\alpha a_\alpha^\dagger a_\alpha \tag{242}$$

Therefore:

$$|\{n_\alpha\}\rangle = \prod_\alpha \frac{\left(a_\alpha^\dagger\right)^{n_\alpha}}{\sqrt{n_\alpha!}} |0\rangle \tag{243}$$

The energy is:

$$\begin{aligned}\left[\sum_\alpha \epsilon_\alpha a_\alpha^\dagger a_\alpha + \frac{1}{2}\sum_{\alpha\beta}\left(V_{\alpha\beta\alpha\beta} a_\alpha^\dagger a_\beta^\dagger a_\alpha a_\beta + V_{\alpha\beta\beta\alpha} a_\alpha^\dagger a_\beta^\dagger a_\beta a_\alpha\right)\right]|\{n_\alpha\}\rangle &= \left[\sum_\alpha \epsilon_\alpha a_\alpha^\dagger a_\alpha + \frac{1}{2}\sum_{\alpha\beta}\left(\pm V_{\alpha\beta\alpha\beta} a_\alpha^\dagger\right.\right. \\ &= \left[\sum_\alpha n_\alpha \epsilon_\alpha + \frac{1}{2}\sum_{\alpha\beta}\left(V_{\alpha\beta\alpha\beta} \pm V_{\alpha\beta}\right.\right.\end{aligned}$$

Where + is for Bosons and − for fermions.

Hartree Fock: The ansatz is that the state is a product state of single particles:

$$|\psi\rangle = \hat{b}_{i_1}^\dagger \hat{b}_{i_2}^\dagger \cdot ... \cdot \hat{b}_{i_n}^\dagger |0\rangle \tag{246}$$

Where the $\hat{b}$'s are some arbitrary basis such that:

$$\hat{a}_i = \sum U_{ik}\hat{b}_k$$

Where $U^\dagger = U^{-1}$. Rewriting the free Hamiltonian we get:

$$\begin{aligned}\mathcal{H}_0 &= \sum_\alpha \epsilon_\alpha a_\alpha^\dagger a_\alpha &(247)\\ &= \sum_{\alpha k,k'} \epsilon_\alpha \left(U_{\alpha k'}\hat{b}_{k'}\right)^\dagger U_{\alpha k}\hat{b}_k &(248)\\ &= \sum_{k,k'}\left(\sum_\alpha \epsilon_\alpha U_{\alpha k} U_{k'\alpha}^*\right)\hat{b}_{k'}^\dagger \hat{b}_k &(249)\\ &= \sum_{k,k'} \omega_{kk'}\hat{b}_{k'}^\dagger \hat{b}_k &(250)\end{aligned}$$

For the intetraction term:

$$
\begin{aligned}
\frac{1}{2}\sum_{\alpha\beta}\left(V_{\alpha\beta\alpha\beta}a^\dagger_\alpha a^\dagger_\beta a_\alpha a_\beta + V_{\alpha\beta\beta\alpha}a^\dagger_\alpha a^\dagger_\beta a_\beta a_\alpha\right) &= \frac{1}{2}\sum_{\alpha\beta}\sum_{k,k'}\left(V_{\alpha\beta\alpha\beta}\left(U_{\alpha k}\hat{b}_k\right)^\dagger\left(U_{\beta k'}\hat{b}_{k'}\right)^\dagger\left(U_{\alpha k}\hat{b}_k\right)\left(U_{\beta k'}\hat{b}_{k'}\right.\right. && (25\\
&+ V_{\alpha\beta\beta\alpha}\left(U_{\alpha k}\hat{b}_k\right)^\dagger\left(U_{\beta k'}\hat{b}_{k'}\right)^\dagger\left(U_{\beta k'}\hat{b}_{k'}\right)\left(U_{\alpha k}\hat{b}_k\right)\Big) && (25\\
&= \frac{1}{2}\sum_{\alpha\beta}\sum_{k,k'}\Big(V_{\alpha\beta\alpha\beta}U^*_{k\alpha}U^*_{k'\beta}U_{\alpha k}U_{\beta k'}\hat{b}^\dagger_k\hat{b}^\dagger_{k'}\hat{b}_k\hat{b}_{k'} && (25\\
&+ V_{\alpha\beta\beta\alpha}U^*_{k\alpha}U^*_{k'\beta}U_{\beta k'}U_{\alpha k}\hat{b}^\dagger_k\hat{b}^\dagger_{k'}\hat{b}_{k'}\hat{b}_k\Big) && (25\\
&= \frac{1}{2}\sum_{k,k'}\left(V_{kk'}\hat{b}^\dagger_k\hat{b}^\dagger_{k'}\hat{b}_k\hat{b}_{k'} + W_{kk'}\hat{b}^\dagger_k\hat{b}^\dagger_{k'}\hat{b}_{k'}\hat{b}_k\right) && (25
\end{aligned}
$$

The free term overlap reads:

$$
\begin{aligned}
\left\langle\psi\left|\hat{\mathcal{H}}_0\right|\psi\right\rangle &= \sum_i \langle i\,|h|\,i\rangle\, n_i && (256)\\
&= \sum_i \omega_{ii}n_i, \qquad n_i = \begin{cases}1 & \text{occupied}\\ 0 & \text{not occupied}\end{cases}\sum_i \langle i\,|h|\,i\rangle\, n_i && (257)\\
&= \sum_{i,\alpha}\left(\epsilon_\alpha U_{\alpha i}U^*_{i\alpha}\right)n_i && (258)
\end{aligned}
$$

We use the identity:

$$
\begin{aligned}
[AB, C] &= ABC - CAB && (259)\\
&= ABC + ACB - ACB - CAB && (260)\\
&= A\{B, C\} - \{A, C\}B && (261)
\end{aligned}
$$

Heisenberg equations:

$$
\begin{aligned}
\partial_t \hat{a}_\theta &= \frac{i}{\hbar}\left[\mathcal{H}, \hat{a}_\theta\right] && (262)\\
&= \frac{i}{\hbar}\left[\sum_\alpha \epsilon_\alpha a_\alpha^\dagger a_\alpha + \frac{1}{2}\sum_{\alpha\beta\gamma\delta} V_{\alpha\beta\gamma\delta} a_\alpha^\dagger a_\beta^\dagger a_\delta a_\gamma, \hat{a}_\theta\right] && (263)\\
&= \frac{i}{\hbar}\sum_\alpha \epsilon_\alpha \left[a_\alpha^\dagger a_\alpha, \hat{a}_\theta\right] + \frac{i}{\hbar}\frac{1}{2}\sum_{\alpha\beta\gamma\delta} V_{\alpha\beta\gamma\delta}\left[a_\alpha^\dagger a_\beta^\dagger, \hat{a}_\theta\right] a_\delta a_\gamma + \frac{i}{\hbar}\frac{1}{2}\sum_{\alpha\beta\gamma\delta} V_{\alpha\beta\gamma\delta} a_\alpha^\dagger a_\beta^\dagger \left[a_\delta a_\gamma, \hat{a}_\theta\right] && (264)\\
&= \frac{i}{\hbar}\sum_\alpha \epsilon_\alpha \left(a_\alpha^\dagger \left\{a_\alpha, \hat{a}_\theta\right\} - \left\{a_\alpha^\dagger, \hat{a}_\theta\right\} a_\alpha\right) + \frac{i}{\hbar}\frac{1}{2}\sum_{\alpha\beta\gamma\delta} V_{\alpha\beta\gamma\delta}\left(a_\alpha^\dagger \left\{a_\beta^\dagger, \hat{a}_\theta\right\} - \left\{a_\alpha^\dagger, \hat{a}_\theta\right\} a_\beta^\dagger\right) a_\delta a_\gamma && (265)\\
&\quad + \frac{i}{\hbar}\frac{1}{2}\sum_{\alpha\beta\gamma\delta} V_{\alpha\beta\gamma\delta} a_\alpha^\dagger a_\beta^\dagger \left(a_\delta \left\{a_\gamma, \hat{a}_\theta\right\} - \left\{a_\delta, \hat{a}_\theta\right\} a_\gamma\right) && (266)\\
&= -\frac{i}{\hbar}\epsilon_\theta a_\theta + \frac{i}{\hbar}\frac{1}{2}\sum_\alpha \left(V_{\alpha\theta\theta\alpha} - V_{\alpha\theta\alpha\theta} - V_{\theta\alpha\theta\alpha} + V_{\theta\alpha\alpha\theta}\right) a_\alpha^\dagger a_\alpha a_\theta && (267)\\
&= -\frac{i}{\hbar}\left(\epsilon_\theta - \frac{1}{2}\sum_{\alpha\neq\theta} V_{\theta\alpha}\hat{N}_\alpha\right) a_\theta && (268)
\end{aligned}
$$

We note that:

$$\left[N_{\alpha\neq\theta}, a_\theta\right] = 0, \quad \left[\hat{N}, \mathcal{H}\right] = 0 \tag{269}$$

Therefore:

$$a_\theta(t) = e^{-\frac{i}{\hbar}\left(\epsilon_\theta - \frac{1}{2}\sum\limits_{\alpha\neq\theta} V_{\theta\alpha}\hat{N}_\alpha\right)t} a_\theta(0) \tag{270}$$

And:

$$a_\theta^\dagger(t) = e^{\frac{i}{\hbar}\left(\epsilon_\theta - \frac{1}{2}\sum\limits_{\alpha\neq\theta} V_{\theta\alpha}\hat{N}_\alpha\right)t} a_\theta^\dagger(0) \tag{271}$$

14 Double degenerate energy level

Assuming the Hamiltonian:

$$\mathcal{H} = \begin{pmatrix} E_0 & & & & \\ & E_0 & & & \\ & & E_1 & & \\ & & & E_2 & \\ & & & & \ddots \end{pmatrix} \tag{272}$$

With eigenstates:

$$|0,1\rangle, |0,2\rangle, |1\rangle, |2\rangle, \ldots |n\rangle \tag{273}$$

We can compare the pure state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0,1\rangle + |0,2\rangle) \tag{274}$$

With the density operator:

$$\begin{aligned} |\psi\rangle\langle\psi| &= \frac{1}{2}(|0,1\rangle\langle 0,1| + |0,2\rangle\langle 0,2| + |0,1\rangle\langle 0,2| + |0,2\rangle\langle 0,1|) && (275)\\ &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} && (276) \end{aligned}$$

To the mixed state:

$$\rho = \frac{1}{2}|0,1\rangle\langle 0,1| + \frac{1}{2}|0,2\rangle\langle 0,2| = \frac{1}{2}\mathbf{1} \tag{277}$$

What is the probability to measure E_0? In both 1! What is the resulting state after the measurement? For the pure state, nothing changes. For the fully mixed, nothing changes.

15 Various density matrices

1. Pure state:
$$|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle \Rightarrow \rho = |\psi\rangle\langle\psi| \tag{278}$$
2. Fully mixed:
$$\rho = \frac{1}{2}\mathbf{1} \tag{279}$$
3. Thermal state:
$$\begin{aligned} \rho &= \sum_n e^{-\beta E_n}|n\rangle\langle n| && (280)\\ &= \frac{e^{-\beta E_\uparrow}|\uparrow\rangle\langle\uparrow| + e^{-\beta E_\downarrow}|\downarrow\rangle\langle\downarrow|}{e^{-\beta E_\uparrow} + e^{-\beta E_\downarrow}} && (281)\\ &= \frac{|\uparrow\rangle\langle\uparrow| + e^{-\beta\Delta}|\downarrow\rangle\langle\downarrow|}{1 + e^{-\beta\Delta}} && (282) \end{aligned}$$

16 Discuss coherence, provide understanding

Coherence is the off diagonal elements in the density matrix.

$$\rho = \sum p_n|\psi_n\rangle\langle\psi_n|, \qquad \sum p_n = 1 \tag{283}$$

For a two level system:

$$\rho^\dagger = \rho \Rightarrow \rho = w_1|\xi_1\rangle\langle\xi_1| + w_2|\xi_2\rangle\langle\xi_2| \tag{284}$$

This tells us the "correct" measuring basis. A non-zero coherence between two states means that you cannot decompose your ensemble into a merely statistical mixture of sub-ensembles such that no sub-ensemble contains both basis states.

Lets show this - assuming we have a quantum state $|+\rangle = \frac{|\uparrow\rangle+|\downarrow\rangle}{\sqrt{2}}$ so the density matrix is:

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{285}$$

So the coherences are $\frac{1}{2}$. However the "classical" analog is just probability $\frac{1}{2}$ of $|\uparrow\rangle$ or $|\downarrow\rangle$ which is:

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{286}$$

So the coherence are reminiscent of the quantum nature of our state. We can say that this is a quantum property because the results are correlated:

$$\langle \psi_1 \left| \rho \right| \psi_2 \rangle \neq 0 \tag{287}$$

17 Double well in time

The lower well goes up and the higher goes down. To simplify we write:

$$\mathcal{H} = \begin{pmatrix} -\alpha t & \Delta \\ \Delta & \alpha t \end{pmatrix} \tag{288}$$

We can diagonalize this in any moment:

$$(-\alpha t - E)(\alpha t - E) - \Delta^2 = 0 \tag{289}$$

$$E^2 - \left(\Delta^2 + \alpha t^2\right) = 0 \tag{290}$$

$$E_\pm = \pm\sqrt{\Delta^2 + \alpha t^2} \tag{291}$$

So if the transition is adiabatic we get avoided crossing. The eigenvectors are given by:

$$\frac{1}{\sqrt{1+\left(\frac{\Delta}{\sqrt{\Delta^2+\alpha^2t^2}+\alpha t}\right)^2}}\begin{pmatrix} 1 \\ -\frac{\Delta}{\sqrt{\Delta^2+\alpha^2t^2}+\alpha t} \end{pmatrix} \leftrightarrow E_- \tag{292}$$

$$\frac{1}{\sqrt{1+\left(\frac{\sqrt{\Delta^2+\alpha^2t^2}-\alpha t}{\Delta}\right)^2}}\begin{pmatrix} \frac{\sqrt{\Delta^2+\alpha^2t^2}-\alpha t}{\Delta} \\ 1 \end{pmatrix} \leftrightarrow E_+ \tag{293}$$

We note that:

$$\lim_{t\to\infty} |-\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lim_{t\to\infty} |+\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{294}$$

So they flip! Just for fun, we note that for $t = 0$ we get:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \quad \leftrightarrow \quad E_- \tag{295}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \quad \leftrightarrow \quad E_+ \tag{296}$$

Cool.

18 Born-Oppenheimer

$$\mathcal{H} = \frac{p_x^2}{2m} + \frac{p_y^2}{2M} + \frac{k\left(x^2+y^2\right)}{2} + \alpha xy, \quad |\alpha| \lll k, \; M \gg m \tag{297}$$

So we rewrite:

$$\mathcal{H} = \mathcal{H}_x\left(y\right) + \mathcal{H}_y$$

So for the fast co.s we get a sfhited HO:

$$\begin{aligned}
\mathcal{H}_x &= \frac{p_x^2}{2m} + \frac{kx^2}{2} + \alpha xy & (298)\\
&= \frac{p_x^2}{2m} + \frac{k}{2}\left(x^2 + \frac{2\alpha xy}{k}\right) & (299)\\
&= \frac{p_x^2}{2m} + \frac{m}{2}\left(\frac{k}{m}\right)\left(x + \frac{\alpha y}{k}\right)^2 - \frac{k}{2}\left(\frac{\alpha y}{k}\right)^2 & (300)
\end{aligned}$$

I know this! The spectrum is:

$$\epsilon_n = \hbar\omega\left(n + \frac{1}{2}\right) - \frac{k}{2}\left(\frac{\alpha y}{k}\right)^2, \quad \omega = \sqrt{\frac{k}{m}} \tag{301}$$

And the eigenstates are:

$$|n; y\rangle = \phi_n\left(x; y\right) = \phi_n\left(x + x_0\right), \quad x_0 = \frac{\alpha y}{k} \tag{302}$$

And ϕ_n is the n'th QHO eigenstate. Berry phase is given by:

$$\begin{aligned}
\mathcal{A} &= \langle \phi_n \left|\nabla_y\right| \phi_n \rangle\\
&= \left\langle \phi_n\left(x + \frac{\alpha y}{k}\right)\left|\partial_y\right| \phi_n\left(x + \frac{\alpha y}{k}\right)\right\rangle\\
&= \left\langle \phi_n\left(x + \frac{\alpha y}{k}\right)\right| k_1\phi_{n-1}\left(x + \frac{\alpha y}{k}\right) + k_2\phi_{n+1}\left(x + \frac{\alpha y}{k}\right)\Big\rangle\\
&= 0
\end{aligned}$$

No Berry phase!

So setting in B-O equation we get:

$$\begin{aligned}
\left[-\frac{\hbar^2}{2m}\left(\nabla - i\bar{A}_m(Q)\right)^2 + V(Q) + \epsilon_m(Q)\right]\xi_m(Q) &= E\xi_m(Q) &(303)\\
\left[-\frac{\hbar^2}{2m}\partial_y^2 + \frac{ky^2}{2} + \epsilon_m(y)\right]\xi_m(y) &= E\xi_m(y) &(304)\\
\left[-\frac{\hbar^2}{2M}\partial_y^2 + \frac{ky^2}{2} + \hbar\omega\left(m+\frac{1}{2}\right) - \frac{k}{2}\left(\frac{\alpha y}{k}\right)^2\right]\xi_m(y) &= E\xi_m(y) &(305)\\
\left[-\frac{\hbar^2}{2M}\partial_y^2 + \frac{kM}{2M}\left(1-\frac{\alpha^2}{k^2}\right)y^2\right]\xi_m(y) + \hbar\omega\left(m+\frac{1}{2}\right)\xi_m(y) &= E\xi_m(y) &(306)
\end{aligned}$$

We get:

$$E_{n;m} = \hbar\Omega\left(n+\frac{1}{2}\right) + \hbar\omega\left(m+\frac{1}{2}\right),\ \Omega = \sqrt{\frac{k}{M}\left(1-\frac{\alpha^2}{k^2}\right)} \tag{307}$$

$$\xi_{n;m}(y) = \xi_n(y) = \phi_{n,\Omega}(y) \tag{308}$$

And:

$$\begin{aligned}
\psi_{n,n}(x,y) &= \xi_n(y)\,\phi_{m,\omega}(x;y) &(309)\\
&= \phi_{n,\Omega}(y)\,\phi_{m,\omega}\left(x+\frac{\alpha y}{k}\right) &(310)
\end{aligned}$$

And the energy:

$$E_{n,m} = \hbar\Omega\left(n+\frac{1}{2}\right) + \hbar\omega\left(m+\frac{1}{2}\right) \tag{311}$$

Comparing to the formal solution:

$$\mathcal{H} = \frac{1}{2}\begin{pmatrix}p_x\\p_y\end{pmatrix}^T\begin{pmatrix}\frac{1}{m} & \\ & \frac{1}{M}\end{pmatrix}\begin{pmatrix}p_x\\p_y\end{pmatrix} + \frac{1}{2}\begin{pmatrix}x\\y\end{pmatrix}^T\begin{pmatrix}k & \alpha\\ \alpha & k\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} \tag{312}$$

$$p'_x = p_x\sqrt{\frac{M}{m}}, \qquad x' = x\sqrt{\frac{m}{M}} \tag{313}$$

And we get:

$$\begin{aligned}
\mathcal{H} &= \frac{1}{2}\begin{pmatrix}\sqrt{\frac{m}{M}}p'_x\\p_y\end{pmatrix}^T\begin{pmatrix}\frac{1}{m} & \\ & \frac{1}{M}\end{pmatrix}\begin{pmatrix}\sqrt{\frac{m}{M}}p'_x\\p_y\end{pmatrix} + \frac{1}{2}\begin{pmatrix}\sqrt{\frac{M}{m}}x'\\y\end{pmatrix}^T\begin{pmatrix}k & \alpha\\ \alpha & k\end{pmatrix}\begin{pmatrix}\sqrt{\frac{M}{m}}x'\\y\end{pmatrix} &(314)\\
&= \frac{1}{2M}\begin{pmatrix}p'_x\\p_y\end{pmatrix}^T\begin{pmatrix}p'_x\\p_y\end{pmatrix} + \frac{M}{2}\begin{pmatrix}x'\\y\end{pmatrix}^T\begin{pmatrix}\frac{k}{m} & \sqrt{\frac{1}{Mm}}\alpha\\ \sqrt{\frac{1}{Mm}}\alpha & \frac{k}{M}\end{pmatrix}\begin{pmatrix}x'\\y\end{pmatrix} &(315)
\end{aligned}$$

Diagonialize! The eigenvalues are given by:

$$\left(\frac{k}{m}-\omega^2\right)\left(\frac{k}{M}-\omega^2\right)-\frac{1}{Mm}\alpha^2 = 0 \tag{316}$$

$$\omega^2 = \frac{k\left(m+M\right)\pm\sqrt{k^2\left(m-M\right)^2+4\alpha^2 mM}}{2mM} \tag{317}$$

$$\omega = \sqrt{\frac{k\left(\mu+1\right)\pm\sqrt{k^2\left(1-2\mu\right)+4\alpha^2\mu}}{2m}} \tag{318}$$

$$\approx \begin{cases}\sqrt{\frac{k}{m}} & fast\\ \sqrt{\frac{k}{M}}\sqrt{1-\frac{\alpha^2}{k^2}} & slow\end{cases} \tag{319}$$

So the energies are correct in the sense that the exact spectrum is:

$$E_{nm} = \hbar\left[\omega_{slow}\left(n+\frac{1}{2}\right)+\omega_{fast}\left(m+\frac{1}{2}\right)\right] = E_{n;m} \tag{320}$$

19 Berry

$$\mathcal{H} = \lambda\left(q\sigma_x + p\sigma_y\right)+\epsilon\sigma_z \tag{321}$$

Discuss solutions as a function of q,p. The "correct" basis for this is spin in the direction of $\mathbf{n}=(\lambda q,\lambda p,\epsilon)$. In this basis the Hamiltonian is just:

$$\mathcal{H} = \mathbf{n}\cdot\bar{\sigma}_n = n\begin{pmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{pmatrix}\cdot\bar{\sigma} \tag{322}$$

Solved by:

$$\left|+\right\rangle = \begin{pmatrix}\cos\frac{\theta}{2}\\ e^{i\phi}\sin\frac{\theta}{2}\end{pmatrix},\ \left|-\right\rangle = \begin{pmatrix}-\sin\frac{\theta}{2}\\ e^{i\phi}\cos\frac{\theta}{2}\end{pmatrix},\qquad \mathcal{H}\left|\pm\right\rangle = \pm n\left|\pm\right\rangle \tag{323}$$

The energies are just $\pm n$. Therefore:

$$E_{\pm n} = \pm\sqrt{\lambda^2 q^2+\lambda^2 p^2+\epsilon^2} \tag{324}$$

We note that if $\epsilon=0$ then at $(p,q)=(0,0)$ we have a degeneracy point which acts as a source for Berry phase. Any trajectory that doesn't circle this point doesn't accumulate a geometric phase. So we just assume a circular trajectory around the origin in (p,q) space.;

Lets calcaulte $\bar{\mathbb{A}}_\pm$:

$$q = n\cos\phi,\ p = n\sin\phi \tag{325}$$

$$\bar{\mathbb{A}}_\pm\left(\bar{R}\right) = -\Im\left\langle\pm\left|\partial_p\hat{p}+\partial_q\hat{q}\right|\pm\right\rangle \tag{326}$$

Therefore:

$$\left(\partial_\phi \frac{\partial\phi}{\partial p}\right)|+\rangle = \left(\frac{\partial\phi}{\partial p}\partial_\phi\right)\begin{pmatrix}\cos\frac{\theta}{2}\\ e^{i\phi}\sin\frac{\theta}{2}\end{pmatrix} = \frac{\partial\phi}{\partial p}\begin{pmatrix}0\\ ie^{i\phi}\sin\frac{\theta}{2}\end{pmatrix} \tag{327}$$

$$\left(\partial_\phi \frac{\partial\phi}{\partial p}\right)|-\rangle = \left(\frac{\partial\phi}{\partial p}\partial_\phi\right)\begin{pmatrix}-\sin\frac{\theta}{2}\\ e^{i\phi}\cos\frac{\theta}{2}\end{pmatrix} = \frac{\partial\phi}{\partial p}\begin{pmatrix}0\\ ie^{i\phi}\cos\frac{\theta}{2}\end{pmatrix} \tag{328}$$

$$\left(\partial_\phi \frac{\partial\phi}{\partial q}\right)|-\rangle = \left(\frac{\partial\phi}{\partial q}\partial_\phi\right)\begin{pmatrix}-\sin\frac{\theta}{2}\\ e^{i\phi}\cos\frac{\theta}{2}\end{pmatrix} = \frac{\partial\phi}{\partial q}\begin{pmatrix}0\\ ie^{i\phi}\cos\frac{\theta}{2}\end{pmatrix} \tag{329}$$

$$\left(\partial_\phi \frac{\partial\phi}{\partial q}\right)|+\rangle = \left(\frac{\partial\phi}{\partial q}\partial_\phi\right)\begin{pmatrix}\cos\frac{\theta}{2}\\ e^{i\phi}\sin\frac{\theta}{2}\end{pmatrix} = \frac{\partial\phi}{\partial q}\begin{pmatrix}0\\ ie^{i\phi}\sin\frac{\theta}{2}\end{pmatrix} \tag{330}$$

Therefore:

$$\bar{\mathbb{A}}_+\left(\bar{R}\right) = -\Im\left[\begin{pmatrix}\cos\frac{\theta}{2}\\ e^{-i\phi}\sin\frac{\theta}{2}\end{pmatrix}^T\left(\frac{\partial\phi}{\partial p}\begin{pmatrix}0\\ ie^{i\phi}\sin\frac{\theta}{2}\end{pmatrix}\hat{p} + \frac{\partial\phi}{\partial q}\begin{pmatrix}0\\ ie^{i\phi}\sin\frac{\theta}{2}\end{pmatrix}\hat{q}\right)\right] \tag{331}$$

$$= -\Im\left[\begin{pmatrix}\cos\frac{\theta}{2}\\ e^{i\phi}\sin\frac{\theta}{2}\end{pmatrix}^T\left(\begin{pmatrix}0\\ \frac{\partial\phi}{\partial p}ie^{i\phi}\sin\frac{\theta}{2}\end{pmatrix}\hat{p} + \begin{pmatrix}0\\ \frac{\partial\phi}{\partial q}ie^{i\phi}\sin\frac{\theta}{2}\end{pmatrix}\hat{q}\right)\right] \tag{332}$$

$$= -\left(\frac{\partial\phi}{\partial p}\sin^2\frac{\theta}{2}\hat{p} + \sin^2\frac{\theta}{2}\frac{\partial\phi}{\partial q}\hat{q}\right) \tag{333}$$

Now:

$$\tan\phi = \frac{p}{q} \Rightarrow \phi = \arctan\frac{p}{q} \tag{334}$$

$$\frac{\partial\phi}{\partial p} = \frac{q}{p^2+q^2},\ \frac{\partial\phi}{\partial q} = \frac{-p}{p^2+q^2} \tag{335}$$

Therefore:

$$\bar{\mathbb{A}}_+\left(\bar{R}\right) = -\frac{q\hat{p}-p\hat{q}}{p^2+q^2}\sin^2\frac{\theta}{2} \tag{336}$$

The chern number is 1, why? fuck you that's why! The rotor:
In general, if $\hat{n} = \hat{z}$ then:

$$\langle + |\sigma_x| - \rangle = (1,0)\begin{pmatrix} & 1\\ 1 & \end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = 1 \tag{337}$$

$$\langle + |\sigma_y| - \rangle = (1,0)\begin{pmatrix} & -i\\ i & \end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = -i \tag{338}$$

In this case the berry phase is given by:

$$\mathbf{V}_+\left(\bar{R}\right) = \lambda^2\Im\left[\frac{\langle + |\sigma_x\hat{q}+\sigma_y\hat{p}| - \rangle \times \langle - |\sigma_x\hat{q}+\sigma_y\hat{p}| + \rangle}{4n^2}\right] \tag{339}$$

$$\mathbf{V}_+\left(\bar{R}\right) = \lambda^2\Im\left[\frac{(\hat{q}-i\hat{p})\times(\hat{q}+i\hat{p})}{4n^2}\right] \tag{340}$$

$$= \lambda^2\frac{\mathbf{n}}{2n^3} \tag{341}$$

$$\mathbf{V}_-\left(\bar{R}\right) = -\lambda^2\frac{\mathbf{n}}{2n^3} \tag{342}$$

Since this is a vector quantity, it transforms like a vector. So in fact this is general and not only for $\hat{z}$.

20 Scattering of a spherical potential well

The potential of a spherical well is given by:

$$V(\mathbf{r}) = -V_0 \Theta(r_0 - r), \quad r_0 > 0 \tag{343}$$

Then using Born approximation we get:

$$\begin{aligned}
f(\theta) &= -\frac{m}{2\pi\hbar^2} \int_0^{2\pi} d\phi \int_0^{\infty} r^2 dr V(r) \int_0^{\pi} \sin\theta' d\theta' e^{-iqr\cos\theta'} & (344) \\
&= -\frac{2m}{\hbar^2} \int_0^{\infty} r dr u(r) \frac{\sin qr}{q} & (345) \\
&= V_0 \frac{2m}{q\hbar^2} \int_0^{r_0} dr \cdot r \sin qr & (346) \\
&= -V_0 \frac{2m}{q\hbar^2} \partial_q \int_0^{r_0} dr \cos qr & (347) \\
&= -V_0 \frac{2m}{q\hbar^2} \partial_q \left[\frac{1}{q} \sin(qr_0) \right] & (348) \\
&= V_0 \frac{2m}{\hbar^2} \left[\frac{\sin(qr_0)}{q^3} - \frac{r_0}{q^2} \cos(qr_0) \right] & (349)
\end{aligned}$$

The cross section:

$$\frac{d\sigma}{d\Omega} = V_0^2 \frac{4m^2}{\hbar^4} \left[\frac{\sin(qr_0)}{q^3} - \frac{r_0}{q^2} \cos(qr_0) \right]^2 \tag{350}$$

How about a delta potential?!

$$V(\mathbf{r}) = -V_0 \frac{\delta(r)}{4\pi r^2} \tag{351}$$

Then using Born approximation we get:

$$
\begin{aligned}
f(\theta) &= -\frac{m}{2\pi\hbar^2}\int_0^{2\pi} d\phi \int_0^{\infty} r^2 dr V(r) \int_0^{\pi} \sin\theta' d\theta' e^{-iqr\cos\theta'} && (352)\\
&= V_0 \frac{2m}{4\pi\hbar^2}\int_0^{\infty} dr\delta(r)\frac{\sin qr}{rq} && (353)\\
&= V_0 \frac{2m}{4\pi\hbar^2} && (354)
\end{aligned}
$$

The cross section:

$$
\frac{d\sigma}{d\Omega} = \left[V_0 \frac{m}{2\pi\hbar^2}\right]^2 \tag{355}
$$