1

$$\hat{U} = \hat{T} e^{-i/\hbar \int_{t_i}^{t_f} H \, dt}$$

$$\hat{U}^\dagger = e^{i/\hbar \int_{t_i}^{t_f} H^\dagger dt} \hat{T}^\dagger$$

$$U^\dagger U = e^{i/\hbar \int_{t_i}^{t_f} H^\dagger dt'} \underbrace{\hat{T}^\dagger \hat{T}}_{\mathbb{1}} e^{-i/\hbar \int_{t_i}^{t_f} H \, dt}$$

$$= \exp\left\{ i/\hbar \left[\overbrace{\int_{t_i}^{t_f} \underbrace{H^\dagger}_{H \text{ hermitian}} dt'}^{*} - \overbrace{\int_{t_i}^{t_f} H \, dt}^{**} \right] \right\}$$

Here the two integrals are over t and t'.

In general we can't do $\int_{t_i}^{t_f} (H - H) \, dt \, dt' = 0$.

But what we can do is do sums in the same Δt instead of the integral:

$$= \exp\left\{ i/\hbar \lim_{N \to \infty} \left[\left(H(t_i)\underbrace{(t_1 - t_i)}_{\varepsilon = \frac{t_f - t_i}{N}} + H(t_1)(t_2 - t_1) + \dots H(t_f)(t_f - t_N) \right) \leftarrow * \right.\right.$$
$$\left.\left. - \left(H(t_i)(t_1 - t_i) + H(t_1)(t_2 - t_1) + \dots H(t_f)(t_f - t_N) \right) \right] \right\} \leftarrow **$$

Now the two integrals cancel and we get $e^0 = 1$

then U is unitary.

(2)

$\delta^2 S$ disapears because $U'' = 0$.

$\Rightarrow \Lambda_{EF} = \Lambda_{FP}$

We have the freedom to choose $\vec{E} = E_x \hat{x}$

and free motion in $\hat{y}, \hat{z}$.

$$\Rightarrow K_{\hat{q}} = \sqrt{\frac{m}{2\pi i\hbar(t_f - t_i)}}\, e^{(i/\hbar)\left(\frac{m(q_f - q_i)^2}{2(t_f - t_i)}\right)} \qquad \hat{q} = \hat{y}, \hat{z}$$

For the $\hat{x}$ direction we just have to find S_{cl}.

$$S_{cl} = \int dt\ \tfrac{1}{2} m\dot{x}^2 + qEX \qquad X(t) = ?$$

$$m\ddot{x} = -qE \Rightarrow \quad X(t) = X_i + \dot{x}(t_i)(t - t_i) - \frac{qE}{2m}(t - t_i)^2$$

at x_f: $\quad x_f = x_i + \dot{x}(t_i)(t_f - t_i) - \frac{qE}{2m}(t - t_i)^2$

$$\frac{x_f - x_i}{t_f - t_i} + \frac{qE}{2m}(t - t_i) = \dot{x}(t_i)$$

$$\dot{x}(t) = \dot{x}(t_i) - \frac{qE}{m}(t - t_i) = \frac{x_f - x_i}{t_f - t_i} - \frac{qE}{2m}(t - t_i)$$

$$\Rightarrow S_{cl} = \int dt\ \frac{1}{2} m\left[\overbrace{\frac{x_f - x_i}{t_f - t_i}}^{\bar{v}} - \frac{qE}{2m}(t - t_i)\right]^2 + qE\left(x_i + \dot{x}(t_i)(t_f - t_i) - \frac{qE}{2m}(t - t_i)^2\right)$$

$$\frac{1}{2}m\bar{v}^2 + \frac{(qE)^2}{8m}(t - t_i)^2 - \frac{\bar{v}qE}{8}(t - t_i) + qEx_f + \frac{(qE)^2}{2m}(t - t_i)(t_f - t_i) - \frac{(qE)^2}{2m}(t - t_i)^2$$

$\Rightarrow$ S_{cl} = 3rd degree polynomial of $(t - t_i)$

$$\Rightarrow K_{tot} = K_x K_y K_z = \left(\frac{m}{2\pi i\hbar(t_f - t_i)}\right)^{3/2} \exp\left[\frac{i}{\hbar}\frac{m(y_f - y_i)^2}{2(t_f - t_i)} + \frac{i}{\hbar}\frac{m(z_f - z_i)^2}{2(t_f - t_i)} + \frac{i}{\hbar} S_{cl}\right]$$

(3)

$$\eta_0(t) = t - t_i$$

$$\eta(t) = \left|\frac{dx_{cl}(t)}{d\dot{x}_{cl}(t_i)}\right|$$

$$x_{cl}(t) = A\cos\omega t + B\sin\omega t$$

$$\dot{x}_{cl}(t) = -A\omega\sin\omega t + B\omega\cos\omega t$$

$t_i = 0 \Rightarrow A = x_i$ $\qquad$ $\dot{x}(t_i) = v \Rightarrow B = \frac{v}{\omega} = \frac{\dot{x}(t_i)}{\omega}$

$$\Rightarrow x(t) = x_i\cos\omega t + \frac{\dot{x}(t_i)}{\omega}\sin\omega t$$

$$\Rightarrow \eta(t) = \frac{\sin\omega t}{\omega} \qquad \sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega T}}$$

$$\Rightarrow F_{h.o} = \sqrt{\frac{m}{2\pi i(t_f - t_i)}}\sqrt{\frac{(t-t_i)\,\omega}{\sin\omega t}}$$

(at t_f we get the exact same answer)

And for K we just need $e^{iS_{cl}/\hbar}$ where:

$$S_{cl} = \frac{m\omega}{\sin\omega T}\left((x_i^2 + x_f^2)\cos\omega T - 2x_ix_f\right)$$

(4) $\frac{\partial S}{\partial x_f} = \frac{m\omega}{\sin\omega T}\left(2x_f\cos\omega T - 2x_i\right)$

$$\rightarrow \partial_{x_i} = \frac{-2m\omega}{\sin\omega T} \Rightarrow |A| = \frac{1}{\sqrt{2\pi\hbar}}\sqrt{\frac{2m\omega}{\sin\omega T}}$$

(5) we have $\frac{1}{\sqrt{i}} = e^{-i\frac{\pi}{2}}$ difference, so $\nu = 1$.