$$\partial_t^2 \phi = v^2 \partial_x^2 \phi - \frac{\mu^2}{\alpha} \sin \alpha \phi$$

$$\partial_t^2 \phi - v^2 \partial_x^2 \phi + \frac{\mu^2}{\alpha} \sin \alpha \phi = 0$$

The EOM of a field is generally:

$$\partial_t^2 \phi - \partial_x^2 \phi + F(\phi) = 0$$

where $F$ is the generalized force the field expiriences.

$$F(\phi) = \frac{\partial V(\phi)}{\partial \phi}$$

$$\Rightarrow \quad V(\phi) = \int F(\phi)\, d\phi + C$$

in the sG field $V(\phi) = \frac{\mu^2}{\alpha^2}(1 - \cos \alpha \phi)$

where $C$ is taken to be $\frac{\mu^2}{\alpha^2}$ such that

$$V(\phi = 0) = 0$$

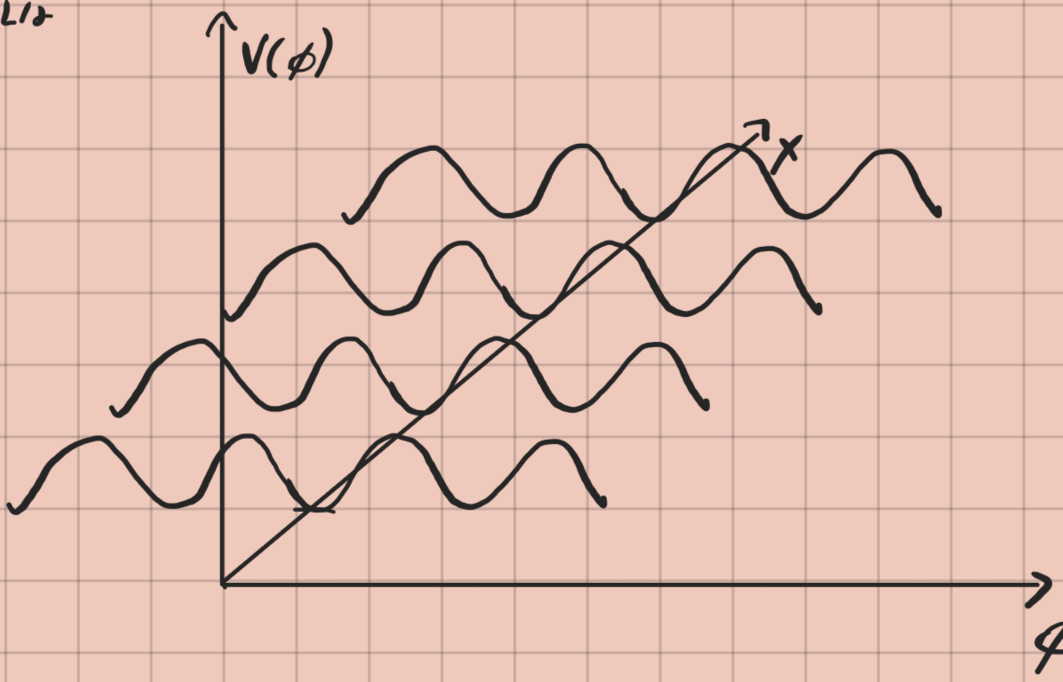
general Lagrangian density for 1+1 field:

$$\mathcal{L} = \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\partial_x \phi)^2 - V(\phi)$$

$$\mathcal{H} = \pi \frac{\partial \phi}{\partial t} - \mathcal{L} \qquad \left(\pi \equiv \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} = \partial_t \phi\right)$$

$$\mathcal{H} = \pi^2 - \frac{1}{2}\pi^2 + \frac{1}{2}(\partial_x \phi)^2 + V(\phi)$$

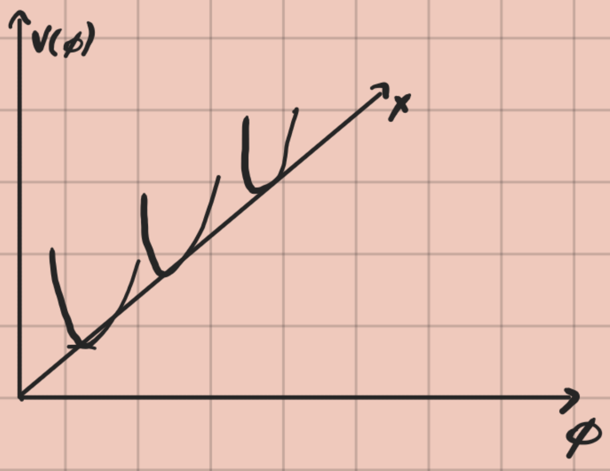
$$\Rightarrow H = \int_{-L/2}^{L/2} dx \left[\frac{1}{2}(\partial_t \phi)^2 + \frac{v^2}{2}(\partial_x \phi)^2 + \frac{\mu^2}{\alpha^2}(1-\cos\alpha\phi)\right]$$



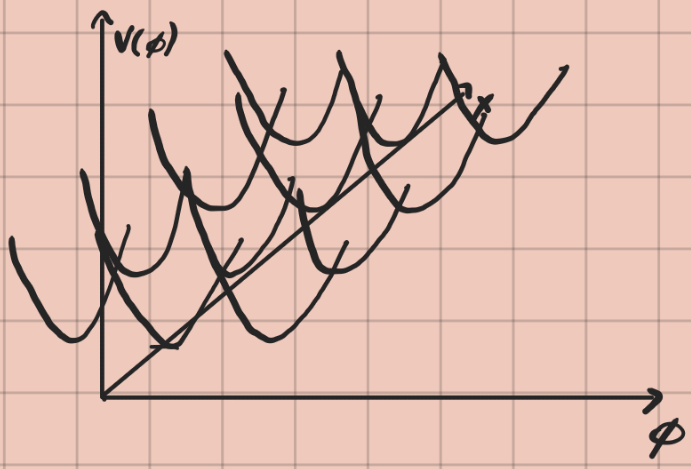

For the compact case (near $\phi=0$) it is similar to the harmonic oscillator $\left(1-\cos\phi \sim \frac{\phi^2}{2} - \frac{x^4}{24} \cdots\right)$.

For the non-compact case it is similar to Bloch electrons in a periodic lattice.

If we take just the harmonic term in $V(\phi)$:




compact




non-compact

$$H = \int_{-L/2}^{L/2} dx \left[ \frac{1}{2}(\partial_t \phi)^2 + \frac{v^2}{2}(\partial_x \phi)^2 + \frac{\mu^2}{\alpha^2}(1 - \cos \alpha\phi) \right]$$

second order

$\cancel{\pi^2} + \cancel{\delta\pi^2} + 2\pi\delta\pi$

$(\partial_x \phi + \partial_x \delta\phi)^2 = \cancel{(\partial_x \phi)^2} + \cancel{(\partial_x \delta\phi)^2} + 2\partial_x\phi\, \partial_x \delta\phi$

$$\delta H = \int_{-L/2}^{L/2} dx \left[ \frac{1}{2}(\pi + \delta\pi)^2 + \frac{v^2}{2}(\partial_x(\phi + \delta\phi))^2 + \frac{\mu^2}{\alpha^2}(\cancel{1} - \cos(\alpha\phi + \alpha\delta\phi)) \right.$$

$$\left. - \cancel{\frac{1}{2}\pi^2} - \cancel{\frac{v^2}{2}(\partial_x\phi)^2} - \frac{\mu^2}{\alpha^2}(\cancel{1} - \cos\alpha\phi) \right]$$

integration by parts

$$= \int_{-L/2}^{L/2} \pi\delta\pi - v^2 \partial_x^2 \phi\, \delta\phi + \frac{\mu^2}{\alpha^2}(\cos\alpha\phi - \cos(\alpha\phi + \alpha\delta\phi))$$

$$\cos\alpha\phi = 1 - \frac{(\alpha\phi)^2}{2} + \frac{(\alpha\phi)^4}{4!} - \frac{(\alpha\phi)^6}{6!} + \ldots$$

$(\alpha\phi)^2 + (\alpha\delta\phi)^2 + 2\alpha^2\phi\delta\phi \qquad 4\alpha^4\phi^3\delta\phi \qquad 6\alpha^6\phi^5\delta\phi$

$$-\cos(\alpha\phi + \alpha\delta\phi) = -1 + \frac{(\alpha\phi + \alpha\delta\phi)^2}{2} - \frac{(\alpha\phi + \alpha\delta\phi)^4}{4!} + \frac{(\alpha\phi + \alpha\delta\phi)^6}{6!}$$

up to first order in $\delta\phi$:

$$\cos\alpha\phi - \cos(\alpha\phi + \alpha\delta\phi) = \alpha^2\phi\,\delta\phi - \frac{\alpha^4\phi^3}{3!}\delta\phi + \frac{\alpha^6\phi^5}{5!}\delta\phi - \dots$$

$$= \alpha \sin\alpha\phi\,\delta\phi$$

$$\Rightarrow \delta H = \int_{-L/2}^{L/2} dx \left[ \pi\,\delta\pi - v^2\partial_x^2\phi\,\delta\phi + \frac{\mu^2}{\alpha}\sin(\alpha\phi)\,\delta\phi \right]$$

$$\Rightarrow \quad \frac{\delta H}{\delta \pi} = \pi$$

$$\frac{\delta H}{\delta \phi} = -v^2\frac{\partial^2\phi}{\partial x^2} + \frac{\mu^2}{\alpha}\sin\alpha\phi$$

$$\Rightarrow \quad \frac{\partial\phi}{\partial t} = \pi$$

$$\frac{\partial\pi}{\partial t} = -v^2\frac{\partial^2\phi}{\partial x^2} + \frac{\mu^2}{\alpha}\sin\alpha\phi \quad (*)$$

$\pi$ is the momenta, so we can think about $(*)$ like Newton's equation, and the right hand side of the eq. is the force. And $F = -\nabla U$ so $V(\phi)$ does include $\partial_x^2\phi$. Which means that the term that couple different

$\phi_{xx}$ is the second derivative. This term in the EOM makes that the curvature of $\phi$ causes motion in $\phi$. It is a restoring force (the sign) so we can add it to our plot:


## Quantization

$\phi(x,t) \to \hat{\phi}(x,t) \qquad \pi(x,t) = \hat{\pi} = -i\hbar\frac{\delta}{\delta\phi}$

$$[\hat{\phi}(x), \hat{\pi}(y)] = i\hbar\,\delta(x-y)$$

$$[\hat{\phi}(x), \hat{\phi}(y)] = [\hat{\pi}(x), \hat{\pi}(y)] = 0$$

$$\Rightarrow \hat{H} = \int_{-L/2}^{L/2} dx \left[-\frac{\hbar^2}{2}\frac{\delta^2}{\delta\phi^2} + \frac{v^2}{2}(\partial_x\hat{\phi})^2 + \frac{\mu^2}{\alpha^2}(1-\cos\alpha\hat{\phi})\right]$$

The schrödinger eq: $i\hbar \partial_t \Psi[\phi] = H \Psi[\phi]$

where $H\Psi[\phi] = \int_{-L/2}^{L/2} dx \left[ -\frac{\hbar^2}{2}\frac{\delta^2}{\delta\phi^2} + \frac{v^2}{2}(\partial_x \phi)^2 + \frac{\mu^2}{\alpha^2}(1-\cos\alpha\phi) \right] \Psi[\phi]$

Interpatation

$\Psi[\phi]$ is a wave functional - which is the probability amplitude for the field to be in a configuration $\phi(x,t)$.

The schrödinger eq. describe the time evolution for $\Psi[\phi]$

To extract physics we can first linearize the problem.

first take small fluctuations around the vacum state ($\phi = 0$, $\phi = \text{const}$)

$$\phi = \phi_0 + \eta(x,t)$$

where, for simplicity, we'll take $\phi_0 = 0$ to be the vacume state.

and for small $\phi$, $\sin\alpha\phi = \alpha\phi$

$$\Rightarrow \quad \partial_t^2 \phi - v^2 \partial_x^2 \phi + \mu^2 \phi = 0$$

rescalling

$$\partial_t^2 \phi - \partial_x^2 \phi + m^2 \phi = 0 \qquad m^2 = \frac{\mu^2}{v^2}$$

$$\Box \eta + m^2 \eta = 0$$

This is the Klein-Gordon equation and its solutions are plane waves : $e^{i(kx-\omega_k t)}$

$$\Rightarrow \qquad \omega_k^2 = k^2 + m^2$$

Thus a general solution is a superposition:

$$\eta(x,t) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \Big[ \underbrace{A(k)\, e^{i(kx-\omega_k t)} + A^*(k)}_{\text{coefficiants depending on initial conditions}}\, e^{-i(kx-\omega_k t)} \Big]$$

now we can quantize and use the creation and annihilation operators:

$$\hat{\eta}(x,t) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega_k}} \left( \hat{a}_k e^{i(kx-\omega_k t)} + \hat{a}_k^\dagger e^{-i(kx-\omega_k t)} \right)$$

Now the energy is just the sum of energies of these fluctuations: $H = \int \frac{dk}{\sqrt{2\pi}} \omega_k a_k a_k^\dagger$

(if we want we can add the energy of the vacuume, cuz we defined $\phi = \phi_0 + \eta$, $\phi_0 = 0$)

This is still a set of harmonic oscillators, but with despertion relation $\omega_k = \sqrt{k^2 + m^2}$

This is different then photons were $\omega = ck$.

Here we have a mass-gap, i.e. there is a minimum energy $E_0 = E(k=0) \sim m$ to add an excitation to the system. The quasi-particle has a rest mass of $m = \frac{\mu}{v}$.

This is a different partiscle if we would have neglected the non-linear term all togather. We would have gotten a linear dispersion relation:

$$\omega \sim |k|$$

Which is a photon, or an acoustic phonon. With a linear dispersion relation you can add an excitation with $k$ arbitrary close to zero.

(2) $$P_K = \sum \hbar \bar{k}\, a^+ a$$

The momentum of a coherent state will be the expactation value: $\langle \hat{P}_k \rangle = \langle \alpha | \hat{P} | \alpha \rangle$

For a given $\alpha, k$: $\langle \hat{p} \rangle = \langle \alpha | \hbar k\, a^+ a | \alpha \rangle$

$$= \hbar k\, \alpha^* \alpha = \hbar k\, |\alpha|^2$$

For the energy it will be the same:

$$\langle \hat{H}_k \rangle = \hbar \omega_k (\langle a^+ a \rangle + \tfrac{1}{2}) = \hbar \omega_k (|\alpha|^2 + \tfrac{1}{2})$$

For a pure number state: $\hat{a}^+ \hat{a} = \hat{n}$

$$\langle n | \hat{p} | n \rangle = \hbar k\, n$$

$$\langle n | \hat{H}_k | n \rangle = \hbar \omega_k (n + \tfrac{1}{2})$$

The coherent state has a Possonian distribution over the number states, with avarage photon number of $\langle n \rangle = |\alpha|^2$. Thus on avarage its momentum is $\hbar k|\alpha|^2$, and energy $\hbar\omega_k(|\alpha|^2+\frac{1}{2})$.

In the coherent state there's uncertainty in photon number, according to the Poisson dist.: $\Delta n = |\alpha|$, and scales like $\frac{\Delta n}{\langle n\rangle} = \frac{1}{\langle n\rangle}$.

For "bright" light with high $\langle n\rangle$ the fluctuations in photon number are small. As $\langle n\rangle$ decreases we approach the quantum regime where fluctuations are large (compared to $\langle n\rangle$)

In the classical limit $\langle n \rangle \to \infty \Leftrightarrow |\alpha| \to \infty$

$\Rightarrow \quad P_k = \hbar k |\alpha|^2 , \quad H_k = \hbar \omega_k (|\alpha|^2 + \frac{1}{2}) \longrightarrow \infty$

But this coicides withe the classical plane wave in infinite space, which also have infinite energy.

By "saying" $|\alpha|, \langle n \rangle \to \infty$ what we really mean is that the quantum fluctuations in photon number relative to the photon number goes to zero.

$$\frac{\Delta n}{\langle n \rangle} \to 0$$

In the strong quantum limit fluctuations are large $\Delta n \sim \langle n \rangle$. Then the fields $\bar{E}, \bar{B}$ can not be represeted as waves but as disrrete quanta of energy, photons, that is dominated by quantum statistics. For the coherent state it is simply Poissonic, but for other states it can be bunching / anti-bunching or other strong correlations.

For the extreme case of $\alpha = 0$, the vacuum state, you have zero mean fields but non-zero energy $\frac{1}{2}\hbar\omega$ and quantum fluctuations.

This matches exactly to the usuall points of view on light. For high occupation number light behaves more "wave-like" (classical limit), and for low occupation numbers light behaves more "particle-like" (quantum limit).