91

Quantum Mechanics - Problem Set 2

Gilad Weiss - 201170511 - Giladwei@gmail.com

November 30, 2014

1 Question 1

a)

First gauge:

$$\begin{aligned} A_x &= 0 \\ A_y &= Bx \\ A_z &= 0 \end{aligned}$$

Second gauge:

$$\begin{aligned} A_x &= -\frac{B}{2}y \\ A_y &= \frac{B}{2}x \\ A_z &= 0 \end{aligned}$$

Because there is no electric field and A doesn't depend on t we get:

$$\nabla A_0 = -E - \frac{1}{c}\frac{\partial A}{\partial t} = 0$$

So we'll use for both gauges $A_0 = 0$.
We'll use the fact that $H = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 + eA_0$.

Finding the eigenstates for the first gauge:

$$\begin{aligned} H &= \frac{1}{2m}\left(p - \frac{e}{c}Bx\hat{y}\right)^2 \\ &= \frac{p_x^2}{2m} + \frac{1}{2m}\left(p_y - \frac{eB}{c}x\right)^2 + \frac{p_z^2}{2m} \end{aligned}$$

✓

$$\psi(x,y,z) = C\phi(x)\exp(ik_y y)\exp(ik_z z)$$
$$H\psi = -\frac{\hbar^2}{2m}\phi''\frac{\psi}{\phi} + \frac{1}{2m}\left(k_y^2\hbar^2\psi - \hbar\frac{2eBx}{c}k_y\psi + \frac{e^2B^2}{c^2}x^2\psi\right) + \frac{\hbar^2}{2m}k_z^2\psi = E\psi$$
$$\phi'' - \frac{1}{\hbar^2}\left(k_y^2\hbar^2 - \hbar\frac{2eBx}{c}k_y + \frac{e^2B^2}{c^2}x^2 + \hbar^2k_z^2 - 2mE\right)\phi = 0$$
$$\phi'' - \frac{1}{\hbar^2}\left(\left(\frac{eB}{c}x - k_y\hbar\right)^2 + \hbar^2k_z^2 - 2mE\right)\phi = 0$$
$$\phi'' + \frac{2m}{\hbar^2}\left(-\frac{e^2B^2}{2mc^2}\left(x - \frac{ck_y\hbar}{eB}\right)^2 + E - \frac{\hbar^2k_z^2}{2m}\right)\phi = 0$$

By noticing and denoting that $x_0 = \frac{ck_y\hbar}{eB}$, $\epsilon = E - \frac{\hbar^2 k_z^2}{2m}$ and $\omega_c = \frac{eB}{mc}$ we get

$$\phi'' + \frac{2m}{\hbar^2}\left(\epsilon - \frac{m\omega_c^2}{2}\left(x - x_0\right)^2\right)\phi = 0$$

Which is the equation of a harmonic oscillator centered around x_0, so we get that:

$$\phi_n\left(x\right) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega_c\left(x - x_0\right)^2}{2\hbar}\right)H_n\left(\sqrt{\frac{m\omega_c}{\hbar}}\left(x - x_0\right)\right)$$

$$E_{n,k_z} = \frac{\hbar^2 k_z^2}{2m} + \hbar\omega_c\left(n + \frac{1}{2}\right)$$

We'll show that those eigenfunctions diagonalize x_0:

$$\begin{aligned}
x_0\psi &= \left(-i\frac{\hbar c}{eB}\frac{\partial}{\partial y}\right)\psi = -i\frac{\hbar c}{eB}\frac{\partial}{\partial y}\left(C\phi\left(x\right)\exp\left(ik_y y\right)\exp\left(ik_z z\right)\right)\\
&= -i\frac{\hbar c}{eB}C\phi\left(x\right)\exp\left(ik_z z\right)\frac{\partial}{\partial y}\left(\exp\left(ik_y y\right)\right)\\
&= -i\frac{\hbar c}{eB}ik_y C\phi\left(x\right)\exp\left(ik_z z\right)\exp\left(ik_y y\right)\\
&= \frac{\hbar c}{eB}k_y\psi
\end{aligned}$$

So we indeed got that the eigenfunctions diagonalize x_0. Furthermore because x_0 and y_0 do not commute, y_0 is completely unknown.

Finding the current density for the first gauge:

$$\begin{aligned}
j &= \frac{e}{2m}\psi^*\left(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}\right)\psi + c.c.\\
&= \frac{e}{2m}\psi^*\left(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}\right)\left(\phi\left(x\right)\exp\left(ik_y y\right)\exp\left(ik_z z\right)\right) + c.c.\\
&= \frac{e}{2m}\psi^*\left(-i\hbar\left(\hat{x}\frac{\phi'\left(x\right)}{\phi\left(x\right)}\psi + \hat{y}ik_y\psi + \hat{z}ik_z\psi\right) - \frac{eB}{c}x\hat{y}\psi\right) + c.c.\\
&= \frac{e}{2m}\psi^*\psi\left(-i\hbar\hat{x}\frac{\phi'\left(x\right)}{\phi\left(x\right)} + \hat{y}\hbar k_y + \hat{z}\hbar k_z - \frac{eB}{c}x\hat{y}\right) + c.c.\\
&= \frac{e}{m}\phi^2\left(x\right)\left(\hat{y}\hbar k_y + \hat{z}\hbar k_z - \frac{eB}{c}x\hat{y}\right)\\
&= \frac{e}{m}\left(\frac{eB}{c}x_0 - \frac{eB}{c}x\right)\phi^2\left(x\right)\hat{y} + \frac{e\hbar k_z}{m}\phi^2\left(x\right)\hat{z}\\
&= \frac{e^2 B}{mc}\left(x_0 - x\right)\phi^2\left(x\right)\hat{y} + \frac{e\hbar k_z}{m}\phi^2\left(x\right)\hat{z}
\end{aligned}$$

So we get that:

$$\begin{aligned}
j_x &= 0\\
j_y &= e\omega_c\left(x_0 - x\right)\phi^2\left(x\right)\\
j_z &= \frac{e}{m}\hbar k_z\phi^2\left(x\right)
\end{aligned}$$

In the x direction we've got that because we can know x_0 which is the x part of the guiding center, there is no current density.

In the y direction, again because we can know x_0 we have a current density that is 0 at x_0, positive to one side of x_0 and negative to the other side of x_0.

That is because the motion of the particle is analogues to a circular motion, and once x_0 is known we can see that the superposition of all possible circles (a column of circles centered at x_0 is a motion going up in one side of x_0 and down in the other.

In the z direction we have a current density centered around x_0 which is what we would expect in a free particle in z that's wave function is mainly around x_0

Finding the total current for the first gauge:

It's easy to see that $I_x = 0$ (because $j_x = 0$).

Because $\phi^2(x)$ is symmetric around $x - x_0$ while $e\omega_c(x_0 - x)$ is antisymmetric, once we integrate j_y we also find that:

$$I_y = \int j_y dA = 2\frac{e}{c}B\int_{-\infty}^{\infty}(x_0 - x)\phi^2(x)\,dxdz = 0$$

So we got that even though we have a current density, the total current is zero because of the ant symmetry around x_0.

✓

Finding the eigenstates for the second gauge:

For the second gauge we'll use the cylindrical coordinates:

$$\begin{aligned}
H &= \frac{1}{2m}\left(-i\hbar\nabla + \frac{eB}{2c}(y\hat{x} - x\hat{y})\right)^2 \\
&= \frac{1}{2m}\left(-i\hbar\left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{\partial}{\partial z}\hat{z}\right) + \frac{eB}{2c}r\left(\sin\theta\left(\cos\theta\hat{r} - \sin\theta\hat{\theta}\right) - \cos\theta\left(\sin\theta\hat{r} + \cos\theta\hat{\theta}\right)\right)\right)^2 \\
&= \frac{1}{2m}\left(-i\hbar\left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{\partial}{\partial z}\hat{z}\right) - \frac{eB}{2c}r\hat{\theta}\right)^2 \\
&= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2} + \frac{1}{2mr^2}\left(i\hbar\frac{\partial}{\partial\theta} + \frac{eB}{2c}r^2\right)^2
\end{aligned}$$

✓

Like in the previous time, we'll guess:

$$\psi(r,\theta,z) = \phi(r)\exp(ik_\theta\theta)\exp(ik_z z)$$

$$\begin{aligned}
H\psi &= \frac{\hbar^2}{2m}k_z^2\psi - \frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial r^2}\frac{\psi}{\phi} + \frac{1}{2mr^2}\left(i\hbar\frac{\partial}{\partial\theta} + \frac{eB}{2c}r^2\right)\left(-k_\theta\hbar\psi + \frac{eB}{2c}r^2\psi\right) \\
&= \frac{\hbar^2}{2m}k_z^2\psi - \frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial r^2}\frac{\psi}{\phi} + \frac{1}{2mr^2}\left(-k_\theta\hbar + \frac{eB}{2c}r^2\right)^2\psi \\
&= E\psi
\end{aligned}$$

So we get:

$$\frac{\hbar^2}{2m}k_z^2\phi - \frac{\hbar^2}{2m}\phi'' + \frac{1}{2mr^2}\left(-k_\theta\hbar + \frac{eB}{2c}r^2\right)^2\phi = E\phi$$ ✓

$$\phi'' + \frac{2m}{\hbar^2}\left[E - \frac{e^2B^2}{8c^2mr^2}\left(r^2 - \frac{2ck_\theta\hbar}{eB}\right)^2 - \frac{\hbar^2}{2m}k_z^2\right]\phi = 0$$

The exact solution of ϕ is complex and based on the Laguerre polynomials so I won't write it explicitly (we also don't need it because it doesn't depend on θ or on z), but I can find the energy:

$$E = \frac{1}{2m}\left(\left(\frac{eB}{2c}\right)^2 r^2 + \hbar^2\frac{k_\theta^2}{r^2} + \hbar^2 k_z^2 + \frac{\hbar}{m\omega_c}k_\theta\right)$$

This is not the energy.

Next we'll show that this wave function diagonalize $x_0^2 + y_0^2$:

$$\begin{aligned}
x_0^2 + y_0^2 &= \left(x + \frac{p_y - \frac{e}{c}A_y}{m\omega_c}\right)^2 + \left(y - \frac{p_x - \frac{e}{c}A_x}{m\omega_c}\right)^2 \\
&= \frac{1}{4}(x^2 + y^2) + \frac{1}{m\omega_c}(xp_y - yp_x) + \left(\frac{1}{m\omega_c}\right)^2(p_x^2 + p_y^2) \\
&= \frac{1}{4}r^2 + \frac{1}{m\omega_c}L_z + \left(\frac{1}{m\omega_c}\right)^2\left(p_r^2 + \frac{L_z^2}{r^2}\right)
\end{aligned}$$

✓

$$
\begin{aligned}
\left(x_0^2+y_0^2\right)\psi &= \left[\frac{1}{4}r^2+\frac{1}{m\omega_c}L_z+\left(\frac{1}{m\omega_c}\right)^2\left(p_r^2+\frac{L_z^2}{r^2}\right)\right]\psi \\
&= \exp\left(ik_\theta\theta\right)\exp\left(ik_z z\right)\left[\frac{1}{4}r^2+\left(\frac{1}{m\omega_c}\right)^2\frac{\hbar^2k_\theta^2}{r^2}+\frac{\hbar}{m\omega_c}k_\theta+\left(\frac{1}{m\omega_c}\right)^2p_r^2+\right]\phi\left(r\right) \\
&= \exp\left(ik_\theta\theta\right)\exp\left(ik_z z\right)\left[\frac{2}{m\omega_c^2}E-\left(\frac{\hbar k_z}{m\omega_c}\right)^2\right]\phi\left(r\right) \\
&= \left[\frac{2E}{m\omega_c^2}-\left(\frac{\hbar k_z}{m\omega_c}\right)^2\right]\psi
\end{aligned}
$$

You forgot the k_θ part.

So we indeed got that the eigenfunctions diagonalize $x_0^2+y_0^2$. Furthermore we can see that θ is completely unknown.

Finding the current density for the second gauge:

$$
\begin{aligned}
j &= \frac{e}{2m}\psi^*\left(-i\hbar\vec{\nabla}-\frac{e}{c}\vec{A}\right)\psi+c.c. \\
&= \frac{e}{2m}\psi^*\left(-i\hbar\left(\frac{\partial}{\partial r}\hat{r}+\frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta}+\frac{\partial}{\partial z}\hat{z}\right)-\frac{e}{c}r\hat{\theta}\right)\psi+c.c. \\
&= \frac{e}{2m}\psi^*\psi\left(-i\hbar\left(\frac{1}{\phi}\frac{\partial\phi}{\partial r}\hat{r}+\frac{ik_\theta}{r}\hat{\theta}+ik_z\hat{z}\right)-\frac{e}{c}r\hat{\theta}\right)+c.c. \\
&= \frac{e}{2m}\left|\phi\left(r\right)\right|^2\left(-i\hbar\frac{1}{\phi}\frac{\partial\phi}{\partial r}\hat{r}+\hbar\frac{k_\theta}{r}\hat{\theta}+\hbar k_z\hat{z}-\frac{e}{c}r\hat{\theta}\right)+c.c. \\
&= \frac{e}{2m}\left|\phi\left(r\right)\right|^2\left(2\hbar\frac{k_\theta}{r}\hat{\theta}+2\hbar k_z\hat{z}-2\frac{e}{c}r\hat{\theta}\right) \\
&= \frac{e}{m}\left|\phi\left(r\right)\right|^2\left(\left(\hbar\frac{k_\theta}{r}-\frac{B}{2}\frac{e}{c}r\right)\hat{\theta}+\hbar k_z\hat{z}\right)
\end{aligned}
$$

So we get that:

$$
\begin{aligned}
j_r &= 0 \\
j_\theta &= \frac{e}{m}\left(\frac{\hbar k_\theta}{r}-\frac{B}{2}\frac{er}{c}\right)\left|\phi\left(r\right)\right|^2 \\
j_z &= \frac{e}{m}\hbar k_z\left|\phi\left(r\right)\right|^2
\end{aligned}
$$

Finding the total current for the second gauge:

It's easy to see that $I_r=0$ (because $j_r=0$).

It makes more sense to integrate $\int \vec{j}\, d\vec{r}=0$

$$
I_\theta=\int j_\theta dA=\int_0^\infty\frac{e}{m}\left(\frac{\hbar k_\theta}{r}-\frac{er}{c}\right)\left|\phi\left(r\right)\right|^2 rdr
$$

Calculating this using Mathematica we get:

$$
I_\theta=\int_0^\infty\frac{e}{m}\left(\hbar k_\theta-\frac{e}{c}r^2\right)\left|\phi\left(r\right)\right|^2dr=0
$$

So we get no current in the θ direction as well.

The relation of the two gauges:

We know that the connection between the two gauges can be written as:

$$
\psi'\left(r,\theta,z\right)=\exp\left(-i\frac{e}{\hbar c}\chi\right)\psi\left(x,y,z\right)
$$

Where $\psi(x,y,z)$ is the solution in the first gauge, $\psi'(r,\theta,z)$ is the solution in the second gauge and χ is the transform between the two gauges.

In order to calculate χ we'll use the fact that:

$$A' = A + \nabla\chi$$

$$\nabla\chi = A' - A = -\frac{1}{2}B\begin{pmatrix} y \\ x \\ 0 \end{pmatrix}$$

$$\chi = -\frac{1}{2}Bxy$$

So we've got that:

$$\psi'(r,\theta,z) = \exp\left(i\frac{eB}{2\hbar c}xy\right)\psi(x,y,z)$$

b)

We have:

$$\begin{aligned}
A_{1,x} &= 0 \\
A_{1,y} &= Bx \\
A_{1,z} &= 0 \\
A_{2,x} &= -By \\
A_{2,y} &= 0 \\
A_{2,z} &= 0
\end{aligned}$$

We'll find the χ that transform those two gauges and satisfies:

$$\begin{aligned}
A_2 &= A_1 + \nabla\chi \\
\nabla\chi &= A_2 - A_1 = -B(y,x,0) \\
\chi &= -Bxy \\
S &= \exp\left(-i\frac{e}{\hbar c}\chi\right) = \exp\left(i\frac{eB}{\hbar c}xy\right)
\end{aligned}$$

By denoting C_1 and C_2 such that $H_i = \frac{1}{2m}C_i^2$ We'll show that $H_2 = SH_1S^\dagger$

$$\begin{aligned}
C_2S\varphi &= \left(-i\hbar\nabla - \frac{e}{c}By\hat{x}\right)\exp\left(i\frac{eB}{\hbar c}xy\right)\varphi \\
&= \left(-i\hbar\left(\exp\left(i\frac{eB}{\hbar c}xy\right)\nabla\varphi + \varphi\nabla\exp\left(i\frac{eB}{\hbar c}xy\right)\right) - \frac{e}{c}By\hat{x}\exp\left(i\frac{eB}{\hbar c}xy\right)\varphi\right) \\
&= \exp\left(i\frac{eB}{\hbar c}xy\right)\left(-i\hbar\left(\nabla + \left(i\frac{eB}{\hbar c}y\hat{x} + i\frac{eB}{\hbar c}x\hat{y}\right)\right) - \frac{e}{c}By\hat{x}\right)\varphi \\
&= \exp\left(i\frac{eB}{\hbar c}xy\right)\left(-i\hbar\nabla + \frac{eB}{c}x\hat{y}\right)\varphi \\
&= SC_1\varphi
\end{aligned}$$

$$\begin{aligned}
SH_1S^\dagger &= \frac{1}{2m}SC_1C_1S^\dagger = \frac{1}{2m}C_2SC_1S^\dagger \\
&= \frac{1}{2m}C_2C_2SS^\dagger = \frac{1}{2m}C_2^2SS^\dagger \\
&= H_2
\end{aligned}$$

Now we'll show that if $H_1\psi = E\psi$ is the eigenfunction of H_1 that we found in (a) than $S\psi$ is an eigenfunction of H_2 with the same E_ψ:

$$\begin{aligned}
H_2S\psi &= SH_1S^\dagger S\psi = SH_1\psi \\
&= SE_\psi\psi = E_\psi S\psi
\end{aligned}$$

So we showed that $S\psi$ is an eigenfunction of H_2 we're only left with showing that it is also a eigenfunction of x_0 in the second gauge:

$$\begin{aligned}
x_{0,(2)}S\psi &= \left(x - i\frac{\hbar c}{eB}\frac{\partial}{\partial y}\right)\left(\exp\left(i\frac{eB}{\hbar c}xy\right)\psi\right) \\
&= x\exp\left(i\frac{eB}{\hbar c}xy\right)\psi + i\frac{\hbar c}{eB}i\frac{eB}{\hbar c}x\exp\left(i\frac{eB}{\hbar c}xy\right)\psi + \exp\left(i\frac{eB}{\hbar c}xy\right)\frac{\hbar c}{eB}k_y\psi \\
&= x\exp\left(i\frac{eB}{\hbar c}xy\right)\psi - x\exp\left(i\frac{eB}{\hbar c}xy\right)\psi + \exp\left(i\frac{eB}{\hbar c}xy\right)\frac{\hbar c}{eB}k_y\psi \\
&= \frac{\hbar c}{mB}k_y S\psi
\end{aligned}$$

✓

Which means the new $S\psi$ is also an eigenfunction of the new x_0 as required.

c)

In answering we'll look at the gauge that diagonalize y_0 and gives us a free particle in x.

Adding $U(x)$ as described adds boundary conditions to the system that dictates that the wave function must be zero at $x = \pm\frac{L}{2}$. in the gauge we chose to look at the system the x part of the solution changes from $\exp(ik_x x)$ to $\cos\left(\frac{\pi(2k_X+1)}{L}x\right)$. ✗

The y part of the solution stays a harmonic oscillator, but will now depend on k_x so the energies will now be quantized in k_x as well.

Because the energies are a function of two quantum numbers, we'll still have degeneracy, we could have different combinations of the quantum numbers that will give the same energy, but this degeneracy will be much smaller than in the previous case.

This is not an eigenstate. Try it.

d)

Analyzing what happens (large L, small L, inequality for large L, decreasing L):

For the gauge:

$$\begin{aligned}
A_x &= 0 \\
A_y &= Bx \\
A_z &= 0
\end{aligned}$$

We can calculate the Hamiltonian:

$$H = \frac{p_x^2}{2m} + \frac{1}{2m}\left(p_y - \frac{eB}{c}x\right)^2 + \frac{p_z^2}{2m} + U(x)$$

✓

$$\psi(x,y,z) = \phi(x)\exp(ik_y y)\exp(ik_z z)$$

$$H\psi = -\frac{\hbar^2}{2m}\phi''\frac{\psi}{\phi} + \frac{1}{2m}\left(k_y^2\hbar^2\psi - \hbar\frac{2eBx}{c}k_y\psi + \frac{e^2B^2}{c^2}x^2\psi\right) + \frac{\hbar^2}{2m}k_z^2\psi + U(x)\psi = E\psi$$

$$\phi'' - \frac{1}{\hbar^2}\left(k_y^2\hbar^2 - \hbar\frac{2eBx}{c}k_y + \frac{e^2B^2}{c^2}x^2 + \hbar^2k_z^2 - 2m(E - U(x))\right)\phi = 0$$

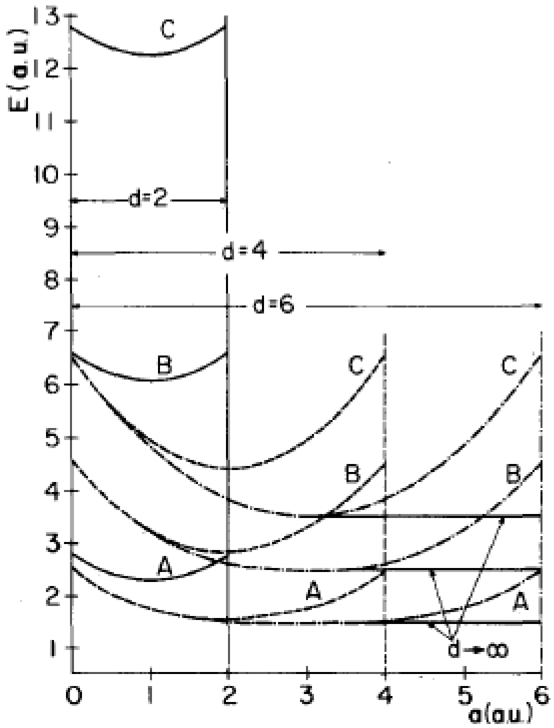
$$\phi'' - \frac{1}{\hbar^2}\left(\left(\frac{eB}{c}x - k_y\hbar\right)^2 + \hbar^2k_z^2 - 2m(E - U(x))\right)\phi = 0$$

$$\phi'' + \frac{2m}{\hbar^2}\left(-\frac{e^2B^2}{2mc^2}\left(x - \frac{ck_y\hbar}{eB}\right)^2 - U(x) + E - \frac{\hbar^2k_z^2}{2m}\right)\phi = 0$$

$$\phi'' + \frac{2m}{\hbar^2}\left(\epsilon - U(x) - \frac{m\omega_c^2}{2}(x - x_0)^2\right)\phi = 0$$

The problem we've got is of an harmonic oscillator in an infinite potential well, and as can be seen in [1] this will cause the energy levels to increase and to become further from each other, and the increase will be stronger the closer the energies are to the wall.

This can be understood by the following figure that shows how three energy levels (A, B and C) change as L gets smaller.



give a reference.
Define what "a" is, what "d" is.

Fig. 2. Ground state (A), and first two excited states (B) and (C) for different wall separations (d) as a function of relative position (a) of the oscillator center from one of walls (see text).

Some qualitative talk:

In the case of a large L there will be an effect only close to the wall, the limit of a large L is obviously related to the width of the oscillator, and from the equation we can see that this limit is:

$$L \gg \sqrt{\frac{\hbar}{m\omega_c}} = \sqrt{\frac{\hbar c}{eB}}$$

✓

In the case of a small L all energies will get a significant increase, and the lower L gets the higher the shift (and the different between two energies) gets.

Analyzing the current:

We know that the current is

$$j_i \propto \frac{\partial E}{\partial A_i}$$

For large L, the potential only effects the energy levels near the walls and then we can see:

$$\begin{aligned} j_y &\propto \frac{\partial E}{\partial A_y} = \frac{\partial E}{B\partial x} = \frac{\partial}{B\partial x}\left(\hbar\omega_c\left(n+\frac{1}{2}\right)+U\left(x\right)\right) \\ &= \frac{1}{B}\frac{\partial U}{\partial x}\left(x\right) \propto \pm\frac{1}{B}\delta\left(\frac{L}{2}\pm x\right) \end{aligned}$$

where did this come from? (it's wrong).

So we get that we'll only get a current density near the edges of the well and in different directions.

e)

First we'll try to solve for an electric field $\vec{D} = D\hat{x}$ without $U\left(x\right)$, we'll use the gauge:

$$
\begin{aligned}
A_x &= 0 \\
A_y &= Bx \\
A_z &= 0 \\
A_0 &= -Dx
\end{aligned}
$$

$$
\begin{aligned}
H &= \frac{1}{2m}\left(p - \frac{e}{c}Bx\hat{y}\right)^2 - eDx \\
&= \frac{p_x^2}{2m} + \frac{1}{2m}\left(p_y - \frac{eB}{c}x\right)^2 - eDx + \frac{p_z^2}{2m}
\end{aligned}
$$

$$\psi\left(x,y,z\right) = \phi\left(x\right)\exp\left(ik_y y\right)\exp\left(ik_z z\right)$$

$$H\psi = -\frac{\hbar^2}{2m}\phi''\frac{\psi}{\phi} + \frac{1}{2m}\left(k_y^2\hbar^2\psi - \hbar\frac{2eBx}{c}k_y\psi + \frac{e^2B^2}{c^2}x^2\psi\right) - eDx\psi + \frac{\hbar^2}{2m}k_z^2\psi = E\psi$$

$$\phi'' - \frac{1}{\hbar^2}\left(\frac{e^2B^2}{c^2}x^2 - 2\left(\hbar\frac{eB}{c}k_y + meD\right)x + k_y^2\hbar^2 + \hbar^2k_z^2 - 2mE\right)\phi = 0$$

$$\phi'' = \frac{1}{\hbar^2}\left(\left(\frac{eB}{c}x - \left(\hbar k_y + \frac{mcD}{B}\right)\right)^2 - \left(\hbar k_y + \frac{mcD}{B}\right)^2 + k_y^2\hbar^2 + \hbar^2k_z^2 - 2mE\right)\phi$$

$$\phi'' = \frac{1}{\hbar^2}\left(\left(\frac{eB}{c}x - \left(\hbar k_y + \frac{mcD}{B}\right)\right)^2 - 2\hbar k_y\frac{mcD}{B} - \frac{m^2c^2D^2}{B^2} + \hbar^2k_z^2 - 2mE\right)\phi$$

$$\phi'' = \frac{1}{\hbar^2}\left(\frac{e^2B^2}{c^2}\left(x - \left(\frac{c}{eB}\hbar k_y + \frac{mc^2D}{eB^2}\right)\right)^2 - 2\hbar k_y\frac{mcD}{B} - \frac{m^2c^2D^2}{B^2} + \hbar^2k_z^2 - 2mE\right)\phi$$

Defining $x_0 = \frac{ck_y\hbar}{eB} + \frac{mc^2D}{eB^2}$, $\epsilon = E - \frac{\hbar^2k_z^2}{2m} + \hbar k_y\frac{cD}{B} + \frac{mc^2D^2}{2B^2}$ and $\omega_c = \frac{eB}{mc}$:

$$\phi'' + \frac{2m}{\hbar^2}\left(\epsilon - \frac{m\omega_c^2}{2}\left(x - x_0\right)^2\right)\phi = 0$$

So again we got an harmonic oscillator with a shifted x_0 and an energy that depends on x_0:

$$
\begin{aligned}
ex_0D - \frac{mc^2D^2}{B^2} &= \frac{ck_y\hbar D}{B} \\
E_{n,x_0,k_z} &= \hbar\omega_c\left(n + \frac{1}{2}\right) + \frac{\hbar^2k_z^2}{2m} - \hbar k_y\frac{cD}{B} - \frac{mc^2D^2}{2B^2} \\
E_{n,x_0,k_z} &= \hbar\omega_c\left(n + \frac{1}{2}\right) + \frac{\hbar^2k_z^2}{2m} - ex_0D + \frac{mc^2D^2}{2B^2}
\end{aligned}
$$

So once we add an electric field we've got a constant shift in the energies and a shift that depends on x_0, that shift removes the degeneracy because now the energy depends on x_0 (we expected that to happen because now y_0 and H do not commute).

Combining the electric field and the potential we can see that the Hamiltonian will now cause the following equation:

$$\phi'' + \frac{2m}{\hbar^2}\left(\epsilon + U\left(x\right) - \frac{m\omega_c^2}{2}\left(x - x_0\right)^2\right)\phi = 0$$

And as we saw in the (d) the potential will only add to the energies near the walls and increase their difference, which means what we've got without the potential well will still be true qualitatively.

Next we'll calculate the current:

The edge states are important

$$
\begin{aligned}
j_y &\propto \cancel{\frac{\partial E}{\partial A_y} = \frac{\partial E}{B\partial x}} \\
&= \frac{\partial}{B\partial x}\left(\hbar\omega_c\left(n + \frac{1}{2}\right) + \frac{\hbar^2k_z^2}{2m} - exD + \frac{mc^2D^2}{2B^2} + U\left(x\right)\right) \\
&\propto -eD \pm \alpha\delta\left(\frac{L}{2} \pm x\right)
\end{aligned}
$$

You are mixing x and x_0. The states are <u>not</u> diagonal in x. but in x_0. $\frac{\partial E}{\partial x}$ is meaningless.

Which means we'll get the same current we got in (d) plus a constant value in the y direction, (which means the total current in the system won't be zero even away from the walls).

f)

Our findings above mostly fit to the classical case of cyclotron orbits:

In the classical case the particle moves in circles with a frequency of ω_c, similar to what we've found.

In the classical case the only current is in the case of an electric field or a potential well (like in the case of a metal where the only currents are on the edges) similar to what we've found.

The only fact that I can't explain is the fact that we've got a current density in the y direction when we've applied an electric field in the x direction, but this might be because of something similar to the hall effect.

2 Question 2

a)

$$\vec{B}(r,\theta,z) = \begin{cases} B\hat{z} & r < R \\ 0 & r \geq R \end{cases}$$

We'll look for the simple gauges inside the cylinder:

$$B = \nabla \times A = \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right)\hat{r} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial (rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right)\hat{z}$$

$$A = \frac{Br}{2}\hat{\theta} \quad or \quad A = -Br\theta\hat{r}$$

Note that this choice leads to $H(\theta + 2\pi) \neq H(\theta)$.

Because we need to choose A such that a integral just around 0 will give as zero while a integral around the cylinder will give us the flux, we'll choose:

$$A = \begin{cases} \frac{Br}{2}\hat{\theta} & r < R \\ \frac{BR^2}{2r}\hat{\theta} & r \geq R \end{cases}$$

✓

Next we'll solve Schrodinger equation:

$$H = \frac{p_r^2}{2m} + \frac{1}{2m}\left(p_\theta - \frac{e}{c}A_\theta(r)\right)^2 + \frac{p_z^2}{2m}$$

$$\psi(r,\theta,z) = \phi(r)\exp(ik_\theta\theta)\exp(ik_z z)$$

$$H\psi = \frac{1}{2m}p_r^2\psi + \frac{1}{2m}\left(\frac{\hbar}{r}k_\theta - \frac{e}{c}A_\theta(r)\right)^2\psi + \frac{\hbar^2}{2m}k_z^2\psi = E\psi$$

✓

$$\frac{p_r^2\phi(r)}{\phi(r)} + \left(\frac{\hbar k_\theta}{r} - \frac{e}{c}A_\theta(r)\right)^2 + \hbar^2k_z^2 - 2mE = 0$$

$$p_r^2\phi(r) + 2m\left(E - \frac{\hbar^2k_z^2}{2m} - \frac{1}{2m}\left(\frac{\hbar k_\theta}{r} - \frac{e}{c}A_\theta(r)\right)^2\right)\phi(r) = 0$$

Which can be written as a particle in a potential ($\epsilon = E - \frac{\hbar^2k_z^2}{2m}$ is the energy in the $x-y$ plane) :

$$\left[\frac{p_r^2}{2m} + \frac{\hbar^2k_z^2}{2m} + \frac{1}{2m}\left(\frac{\hbar k_\theta}{r} - \frac{e}{c}A_\theta(r)\right)^2\right]\phi(r) = E\phi(r)$$

$$\left[\frac{p_r^2}{2m} + V(r)\right]\phi(r) = \epsilon\phi(r)$$

$$V(r) = \frac{1}{2m}\left(\frac{\hbar k_\theta}{r} - \frac{e}{c}A_\theta(r)\right)^2$$

We can see that $V(r) \geq 0$ and that $V(r \to \infty) = 0$.

Because the energy must be greater than the minimal potential than we get that we must have a positive energy, and therefore $\epsilon > V(\infty)$ which means the particle can get to $r \to \infty$ which means the field can not bind the particle. ✓

Because $V(r \to \infty) = 0$ and the particle is unbound we can see that for $r \to \infty$ we get the free particle wave function:

$$H = \frac{p_r^2}{2m} + \frac{1}{2m}\left(p_\theta - \frac{e}{c}A_\theta(r)\right)^2 + \frac{p_z^2}{2m} \to \frac{p_r^2}{2m} + \frac{p_\theta^2}{2m} + \frac{p_z^2}{2m} = \frac{p^2}{2m} \times$$

For large r, p_θ and A behave the same.

Which means the energy spectrum is continues.

Another way to see the spectrum is continues is to compare the degrees of freedom to the constrains:

We have 2 second order equations, which means we have 4 degrees of freedom.

Constrains:

Continuously at R: $\phi(R^+) = \phi(R^-)$ and $\frac{\partial \phi}{\partial r}(R^+) = \frac{\partial \phi}{\partial r}(R^-)$ gives us 2 constrains

$r \to \infty$: Because we know the wave function acts as a free particle in $r \to \infty$ we can write $\psi(r \to \infty) = A\exp(ik_r r) + B\exp(-ik_r r)$. Now if we'll demand that ψ and its derivative will be zero at infinity than we'll get $A = B = 0$ which means the wave function will be 0 everywhere (Picard theorem) so we have no constrains in $r \to \infty$.

Normalization: Because we have a free particle at infinity we know we don't normalize the wave function, which means we don't get another constraint from here.

To conclude, we have 4 degrees of freedom and 2 constrains, which means that that we don't have a discretization in the energy spectrum.

I'm not convinced... $\psi(r \to \infty) = A(r)e^{ikr} + B(r)e^{-ikr}$ would be a better guess. where $A(r \to \infty) = A$, etc. I'm not sure the rest of your argument holds in this case.

b)

When $R \to \infty$ we only have one equation for r over all space, meaning we lost 2 degrees of freedom, and now we only have 2.

Constrains:

Continuously at R: When $R \to \infty$ those constrains disappear because we only have one equation.

As $V(r \to \infty) = \infty$ we get back the constrains at $r \to \infty$ because we must have $\phi(r \to \infty) = 0$ and $\frac{\partial \phi}{\partial r}(r \to \infty) = 0$. Additionally, we get back the constraint coming from the normalization (a state that gets to 0 at infinity needs to be normalized).

Finally we've got 3 constraints on 2 degrees of freedom, meaning we must discretize the energy spectrum.

Another way to see that is using the fact that $V(r \to \infty) = \infty$ to see that the state is not bounded and than according to [2] we know that the energy spectrum must be discrete.

You solved the $R \to \infty$ problem in question 1. Here you were asked about the evolution for large, finite R.

c)

Once we add a flux Φ to $x = y = 0$ we need to add to A a term that will cause every integral around $x = y = 0$ to increase by Φ for every loop but won't change $B = \nabla \times A' = \nabla \times A$ so it'll have to be $\Delta(A' - A) = \nabla\xi$:

$$\oint_{around\,zero} (A' - A)\,dr = \oint_{around\,zero} \nabla\xi = \xi_f - \xi_i = n\Phi$$

$$\xi = \frac{\Phi}{2\pi}\theta$$

$$A' - A = \nabla\xi = \frac{1}{r}\frac{\Phi}{2\pi}\hat{\theta}$$

$$A' = \begin{cases} \left(\frac{\Phi}{2\pi}\frac{1}{r} + \frac{B}{2}r\right)\hat{\theta} & r < R \\ \left(\frac{\Phi}{2\pi} + \frac{BR^2}{2}\right)\frac{1}{r}\hat{\theta} & r \geq R \end{cases}$$ ✓

Now that we have the new gauge, we can look at the new Hamiltonian and potential (noting $\Phi_0 = \frac{2\pi\hbar c}{e}$):

$$\begin{aligned} H &= \frac{p_r^2}{2m} + \frac{1}{2m}\left(p_\theta - \frac{e}{c}A'_\theta\right)^2 + \frac{p_z^2}{2m} \\ &= \frac{p_r^2}{2m} + \frac{1}{2m}\left(\frac{L_z}{r} - \frac{e}{c}\left(A_\theta + \frac{1}{r}\frac{\Phi}{2\pi}\right)\right)^2 + \frac{p_z^2}{2m} \\ &= \frac{p_r^2}{2m} + \frac{1}{2m}\left(\frac{1}{r}\left(L_z - \frac{e\Phi}{2\pi c}\right) - \frac{e}{c}A_\theta\right)^2 + \frac{p_z^2}{2m} \\ &= \frac{p_r^2}{2m} + \frac{1}{2m}\left(\frac{1}{r}\left(L_z - \hbar\frac{\Phi}{\Phi_0}\right) - \frac{e}{c}A_\theta\right)^2 + \frac{p_z^2}{2m} \end{aligned}$$ ✓

$$
\begin{aligned}
V(r) &= \frac{1}{2m}\left(\frac{\hbar}{r}k_\theta - \frac{e}{c}A'\right)^2 \\
&= \frac{1}{2m}\left(\frac{\hbar}{r}\left(k_\theta - \frac{e\Phi}{2\pi\hbar c}\right) - \frac{e}{c}A\right)^2 \\
&= \frac{1}{2m}\left(\frac{\hbar}{r}\left(k_\theta - \frac{\Phi}{\Phi_0}\right) - \frac{e}{c}A\right)^2
\end{aligned}
$$

From any of those two equations we can see that we have the change $L_z \to L_z - \hbar\frac{\Phi}{\Phi_0}$ and $k_\theta \to k_\theta - \frac{\Phi}{\Phi_0}$ (this is the same change because the eigenstates of L_z are without flux are $\hbar k_\theta$) so we can see that we'll only get a change in the form of a shift $k_\theta \to k_\theta - \frac{\Phi}{\Phi_0}$, which will add a $\exp\left(i\frac{\Phi}{\Phi_0}\theta\right)$ term to the wave function and give a shift to the eigenvalues.

Because k_θ must be an integer (we know that because $\psi(\theta + 2\pi) = \psi(\theta)\exp(2\pi i k_\theta) = \psi(\theta)$) we can see that for $\Phi = n\Phi_0$ (where n is an integer) we'll get the exact same solution and energies.

The radial wavefunctions also change, of course.

d)

Before we had the θ dependency $\psi_\theta = \exp\left(i\left(k_\theta - \frac{\Phi}{\Phi_0}\right)\theta\right)$ where k_θ was an integer, but now we don't have periodic boundary conditions, but our conditions are that $\psi_\theta(0) = \psi_\theta(2\pi - \Delta) = 0$ the first condition turns the exponent part into a sin while the second condition gives us:

$$
\begin{aligned}
\psi_\theta &= \sin\left(\left(k_\theta - \frac{\Phi}{\Phi_0}\right)\theta\right) \\
\psi_\theta(2\pi - \Delta) &= \sin\left(\left(k_\theta - \frac{\Phi}{\Phi_0}\right)(2\pi - \Delta)\right) = 0 \\
k_\theta - \frac{\Phi}{\Phi_0} &= \frac{\pi n}{2\pi - \Delta} \\
\psi_\theta &= \sin\left(\frac{\pi n}{2\pi - \Delta}\theta\right)
\end{aligned}
$$

This is not an eigenstate of π_θ:

$\left(-i\partial_\theta - \frac{e}{c}A(r)\right)^2 \sin(\alpha\theta) \neq \lambda(r)\sin(\alpha\theta)$

Which means we've lost the dependence in Φ. That fact means we'll also won't have the A-B effect anymore.

References

[1] J. L. Marin and S. A. Cruz, On the harmonic oscillator inside an infinite potential well

[2] C. Cohen-Tannoudji, Quantum Mechanics, Complement M_{III}, pages 351-358