Quantum Computation 101 for Physicists

Class exercise 4

Today we will focus on more Bell games, additional to what you saw in class. These games present some ideas that can later be used in real applications of quantum computation.

1 Question 1 (Mermin–Peres Magic Square game)

Consider a 3×3 square which we want to fill with entries ± 1 such that the product of any row is 1 and the product of any column is -1. This can be shown to be impossible. We can convert this impossibility into a non-local game in the following manner.

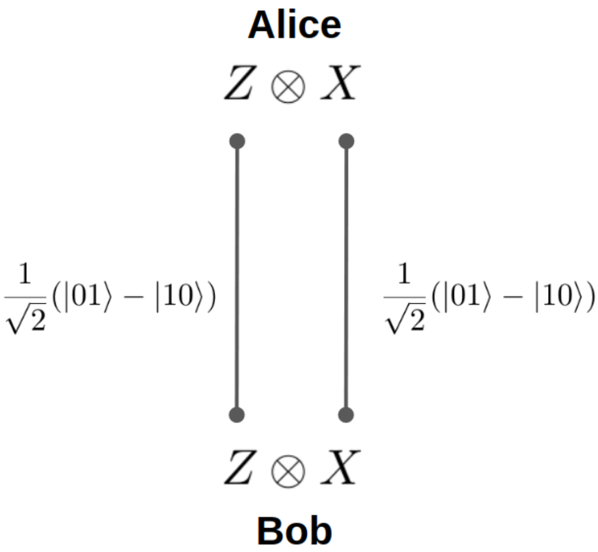
The game consists of two players, Alice and Bob. Alice and Bob are separated before the game begins. When the game starts, Alice and Bob get each a number between 1 and 3, which represents a row number for Alice and column number for Bob. The each return an array of three numbers in $\{\pm 1\}$, which represents the filling of the row or column they got. They win the game if the product of the numbers in the row Alice chose equals one, the product of the numbers of the column Bob chose equals -1, and the shared entry of Alice's row and Bob's column agree.

1. Show that this game cannot be won with probability $\geq 8/9$ if Alice and Bob only have classical correlations.
2. Show that we can find 9 2-qubit operators A_{ij} for $i, j \in \{1, 2, 3\}$ such that:
   - In each row and each column, the operators commute: $[A_{ij}, A_{i'j}] = [A_{ij}, A_{ij'}] = 0$.
   - The product of each row equals $\mathbb{I}$.
   - The product of each column equals $-\mathbb{I}$.
3. Use (b) to build a quantum solution to the game using 2 EPR pairs as entanglement.

1.1 Solution

1. Since there is no full solution to the magic square, Alice and Bob can only agree on the filling of 8 boxes. Say they agree on all boxes but box (i, j). Now, in 8 out of 9 cases, the row and column Alice and Bob get will not be exactly (i, j), this box will not be the shared box and Alice and Bob win the game. However, in one case the questions will be i for Alice and j for Bob.In this case there is no filling for this box on which both Alice and Bob agree. (There is no way to change the strategy in this case, since Alice does not know which column Bob got and vice versa).
2. We use outer product of Pauli matrices for the A operators:
$$A = \begin{pmatrix} X \otimes X & Y \otimes Y & Z \otimes Z \\ Y \otimes Z & Z \otimes X & X \otimes Y \\ Z \otimes Y & X \otimes Z & Y \otimes X \end{pmatrix}.$$
3. Since the operators in each row and each column commute, we can measure the three of them together. Each of our players has two qubits, so each of them measures the three operators in the row / column they got and fills the row / column accordingly. We start with the state
$$|B_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

Let's just look at the middle square to get persuaded that both Bob and Alice get the same result for their measurement: First, look at this drawing to make the process more clear:



Both Alice and Bob measure $\langle Z \rangle$ on their half of the first EPR-pair and $\langle X \rangle$ on their half of the second EPR-pair.

For the measurement of Z on each qubit in the EPR pair, we can read straight of the definition of the state, that Alice and Bob will neasure different values - the full state either collapses to $|01\rangle$ or to $|10\rangle$. Now, we just need to make sure that measuring X on each qubit will also give us different values for Alice and Bob. For this, we write the state in the eigenbasis of the operator X,

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle),$$
$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle),$$
$$|B_{11}\rangle = \frac{1}{2^{3/2}}((|+\rangle + |-\rangle)(|+\rangle - |-\rangle) - (|+\rangle - |-\rangle)(|+\rangle + |-\rangle)) = \frac{1}{\sqrt{2}}(|-+\rangle - |+-\rangle).$$

So we see that here, too, Alice and Bob will get different values. Multiplying the results of the Z and X measurements, Alice and Bob will get the same result for the middle box. Convince yourselves that the same trick will work for all squares!

2 Question 2 (Elizur-Vaidman's Bomb)

(Vaidman here is Lev Vaidman from TAU!)

In the model, we present a quantum bomb: a device that gets an input qubit and measures it. If the input qubit is in the state $|0\rangle$, nothing happens. If the input qubit is $|1\rangle$, the bomb explodes. Sometimes, the quantum bomb is malfunctioning and then we call it a dud - whatever the measurement's result is, nothing happens.

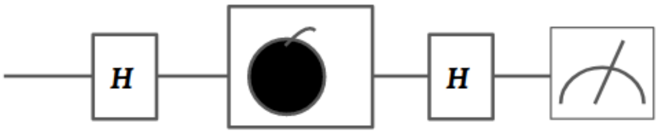
The problem is this - you are handed a quantum bomb, and you need to check whether it is a working bomb or a dud. But if you just check it, it will explode, and then the working bomb will be ruined.

Write a protocol that:

- If the bomb is a dud, the protocol's output is always 'dud'.
- If the bomb is a working bomb, with probability $\frac{1}{2}$ the bomb explodes, with probability $\frac{1}{4}$ the bomb is identified as a bomb but doesn't explode (success!) and with probability $\frac{1}{4}$ the bomb is identified as a dud. (In your homework, you will improve these statistics!)

2.1 Solution

For the solution, we use an important property of quantum systems - destructive interference. It is a useful method when designing quantum algorithms. we start with a qubit in the $|0\rangle$ state, and apply the following circuit:



Now, let's see what happens: If the bomb is a dud, then we basically just aplied the Hadamard gate twice and get $|0\rangle$ as an outcome. If the the bomb is a working bomb, then we have two options: Either the bomb measured the qubit in the $|1\rangle$ state and explodes - this happens with probability $\frac{1}{2}$. If we are luckier, the bomb measures the qubit in the state $|0\rangle$, and the second Hadamard gate will output the qubit in the state $|+\rangle$. Now, with probability $\frac{1}{2}$ we measure $|0\rangle$ and output a wrongful 'dud', but with probability $\frac{1}{2}$ we measure $|1\rangle$ which cannot happen for a dud, and in this case we know we have a working bomb.

3 Bell Basis

In class you saw the Bell basis for two qubits:

$$\begin{aligned}
|B_{00}\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\
|B_{01}\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \\
|B_{10}\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\
|B_{11}\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).
\end{aligned}$$

As Moshe mentioned, these are all maximally entangled. A result of that is that measuring one qubit, will instantly make the second qubit decay. The property is non-local - the position of both qubits in space does not change it. After performing the measurement on one qubit, we know the state of the other qubit with probability 1.

But this is not enough for being maximally entangled. Look at the following state:

$$|\psi_{10}\rangle = \frac{1}{\sqrt{10}}|00\rangle + \frac{3}{\sqrt{10}}|11\rangle.$$

Again, measuring one qubit will make the second one decay instantly, and once we measured one qubit, we know the state of the second qubit with probability 1. And yet, we claim that the state $|\psi_{10}\rangle$ above is less entangled than the states in the Bell basis. What is the difference?

One way to interpret the difference is in **information**. Before we measure one of the qubits, each qubit has a probability of $\frac{1}{10}$ to be $|0\rangle$ and probability $\frac{9}{10}$ to be $|1\rangle$. So, 9 out of 10 times, when measuring the first qubit we get $|1\rangle$ and then we know that the second qubit is also 1. But we didn't learn much by that - before we performed the measurement, we already knew that with very high probability the second qubit is at state $|1\rangle$. The information that one qubit has over the other is not as big as before.

A second way to see that $|\psi_{10}\rangle$ is less entangled than the states in the Bell basis is by recalling the definition you saw in class for entanglement: An entangled state cannot be written as a product state:

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle.$$

$|\psi_{10}\rangle$ Cannot be written as a product state. However, we can write it as:

$$|\psi_{10}\rangle = \frac{2}{\sqrt{10}}|1\rangle \otimes |1\rangle + \frac{1}{\sqrt{10}}\left(|00\rangle + |11\rangle\right).$$

So we see that it is closer to a product state - a product state with some correction. The more similar the coefficients are for both states, the more entanglement we have.