1. Find a choice of angles $\theta, \theta', \phi, \phi'$ that violates the CHSH-Bell inequality assuming we start from the Bell state

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}\left(|H\rangle_1 |H\rangle_2 - |V\rangle_1 |V\rangle_2\right).$$

Hint: First, rewrite $|\Phi^-\rangle$ with respect to a basis $\{|\theta\rangle, |\theta^\perp\rangle\}$ for particle 1 and a basis $\{|\phi\rangle, |\phi^\perp\rangle\}$ for particle 2, and then show that $C(\theta,\phi) = \cos(2\theta + 2\phi)$ for $|\Phi^-\rangle$. Next, find angles such that

$$|C(\theta,\phi) + C(\theta,\phi') + C(\theta',\phi) - C(\theta',\phi')| > 2,$$

in violation of the CHSH-Bell inequality. Considerations for a correct choice of angles become more transparent if you first show that

$$C(\theta,\phi) + C(\theta,\phi') + C(\theta',\phi) - C(\theta',\phi') = 2\cos(2\theta + \phi + \phi')\cos(\phi' - \phi) + 2\sin(2\theta' + \phi + \phi')\sin(\phi' - \phi).$$

$$|H\rangle = \cos\theta\,|\theta\rangle - \sin\theta\,|\theta^\perp\rangle$$
$$|V\rangle = \sin\theta\,|\theta\rangle + \cos\theta\,|\theta^\perp\rangle$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}\Big[\left(\cos\theta|\theta\rangle - \sin\theta|\theta^\perp\rangle\right)\left(\cos\varphi|\varphi\rangle - \sin\varphi|\varphi^\perp\rangle\right) - \left(\sin\theta|\theta\rangle + \cos\theta|\theta^\perp\rangle\right)\left(\sin\varphi|\varphi\rangle + \cos\varphi|\varphi^\perp\rangle\right)\Big]$$

$$= \frac{1}{\sqrt{2}}\Big[\left(\cos\theta\cos\varphi - \sin\theta\sin\varphi\right)|\theta\rangle|\varphi\rangle - \left(\sin\theta\cos\varphi + \cos\theta\sin\varphi\right)|\theta^\perp\rangle|\varphi\rangle - \left(\cos\theta\sin\varphi + \sin\theta\cos\varphi\right)|\theta\rangle|\varphi^\perp\rangle + \left(\sin\theta\sin\varphi - \cos\theta\cos\varphi\right)|\theta^\perp\rangle|\varphi^\perp\rangle\Big)$$

ופונק' ההתאמה היא סכום ההסתברויות:

$$C(\theta,\varphi) = P(\theta,\varphi) + P(\theta^\perp,\varphi^\perp) - P(\theta^\perp,\varphi) - P(\theta,\varphi^\perp)$$

$$= \frac{1}{2}\Big[\left(\cos\theta\cos\varphi - \sin\theta\sin\varphi\right)^2 - \left(\sin\theta\cos\varphi + \cos\theta\sin\varphi\right)^2 - \left(\cos\theta\sin\varphi + \sin\theta\cos\varphi\right)^2 + \left(\sin\theta\sin\varphi - \cos\theta\cos\varphi\right)^2\Big]$$

$\sin(\theta \pm \varphi) = \sin\theta\cos\varphi \pm \cos\theta\sin\varphi$

$\cos(\theta \pm \varphi) = \cos\theta\cos\varphi \mp \sin\theta\sin\varphi$

$$= \cos^2(\theta+\varphi) - \sin^2(\theta+\varphi)^2$$

$$= \cos(2\theta + 2\varphi)$$

$C(\theta,\phi)+C(\theta,\phi')+C(\theta',\phi)-C(\theta',\phi') = \cos(2\theta+2\varphi)+\cos(2\theta+2\varphi')$

$\cos\theta-\cos\varphi=-2\sin\left(\frac{\theta+\varphi}{2}\right)\sin\left(\frac{\theta-\varphi}{2}\right)$

$\cos\theta+\cos\varphi=2\cos\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right)$

$+c(2\theta'+2\varphi)-\cos(2\theta'+2\varphi')$

$$=2\cos(2\theta+\varphi+\varphi')\cos(\varphi-\varphi')$$

$$-2\sin(2\theta'+\varphi+\varphi')\sin(\varphi-\varphi')$$

נב: $\varphi'=0$

$$2\left[\cos(2\theta+\varphi)\cos(\varphi)-\sin(2\theta'+\varphi)\sin(\varphi)\right]$$

נב: $\varphi=-\frac{\pi}{4}$

$$\sqrt{2}\left(\cos\left(2\theta-\frac{\pi}{4}\right)+\sin\left(2\theta'-\frac{\pi}{4}\right)\right)$$

$2\theta-\frac{\pi}{4}=0$ $\qquad$ $\theta'-\frac{\pi}{8}=\frac{\pi}{4}$

$\theta=\frac{\pi}{8}$ $\qquad$ $\theta'=\frac{3}{8}\pi$

לומר בכור: $\varphi'=0,\ \varphi=-\frac{\pi}{4},\ \theta=\frac{\pi}{8},\ \theta'=\frac{3\pi}{8}$

נקל $\sqrt{2}\,2>2$

2. Recall the definitions of the unitary transformations representing the beam splitter and the Kerr component in the optical setup that realizes the C-NOT gate:

$$\hat{U}_{\text{BS1}} = \exp\left[i\frac{\pi}{4}\left(\hat{a}_t^\dagger\hat{b}_t + \hat{b}_t^\dagger\hat{a}_t\right)\right] = \hat{U}_{\text{BS2}}^\dagger,$$
$$\hat{U}_{\text{Kerr}}(\eta) = \exp\left[i\eta\hat{a}_c^\dagger\hat{a}_c\hat{b}_t^\dagger\hat{b}_t\right].$$

(a) Show that

$$\hat{U}_{\text{BS1}}^\dagger\hat{U}_{\text{Kerr}}(\eta)\hat{U}_{\text{BS1}} = \exp\left[i\frac{\eta}{2}\hat{a}_c^\dagger\hat{a}_c\left(\hat{a}_t^\dagger\hat{a}_t + \hat{b}_t^\dagger\hat{b}_t\right)\right]\exp\left[\frac{\eta}{2}\hat{a}_c^\dagger\hat{a}_c\left(\hat{a}_t^\dagger\hat{b}_t - \hat{b}_t^\dagger\hat{a}_t\right)\right].$$

Hint: Expand $\hat{U}_{\text{Kerr}}$ into a power series and use the result of Question 3 in Homework 1.

$$U_{BS1}^\dagger U_{Kerr} U_{BS1} =$$

$$= e^{-i\frac{\pi}{4}(a_t b_t^\dagger + b_t a_t^\dagger)} \sum_{n=0}^{\infty} \frac{(i\eta a_c^\dagger a_c b_t^\dagger b_t)^n}{n!} e^{i\frac{\pi}{4}(a_t^\dagger b_t + b_t^\dagger a_t)}$$

הקומוטציות משפיעות על b_t כך:

$$b_t \to \frac{1}{\sqrt{2}}(a_t - i b_t)$$
$$\Downarrow$$
$$b_t^\dagger b_t \to \frac{1}{\sqrt{2}}(a_t^\dagger + i b_t^\dagger)(a_t - i b_t)$$

$$b_t^\dagger b_t = \frac{1}{2}\left(a_t^\dagger a_t + b_t^\dagger b_t + i(b_t^\dagger a_t - a_t^\dagger b_t)\right)$$

לכן נקבל:

$$= \sum_{n=0}^{\infty} \frac{\left[i\frac{\eta}{2} a_c^\dagger a_c\left(a_t^\dagger a_t + b_t^\dagger b_t + i(b_t^\dagger a_t - a_t^\dagger b_t)\right)\right]^n}{n!}$$

לפי תרגיל 8

$$= \exp\left[i\frac{\eta}{2} a_c^\dagger a_c\left(a_t^\dagger a_t + b_t^\dagger b_t\right) + \frac{\eta}{2} a_c^\dagger a_c\left(a_t^\dagger b_t - b_t^\dagger a_t\right)\right]$$

(b) Show that for any state of the target qubit (including any superposition of $|0\rangle_t$ and $|1\rangle_t$), the above is equivalent to

$$\hat{U}_{\mathrm{BS1}}^{\dagger}\hat{U}_{\mathrm{Kerr}}(\eta)\,\hat{U}_{\mathrm{BS1}} = \exp\left[i\frac{\eta}{2}\hat{a}_c^{\dagger}\hat{a}_c\right]\exp\left[\frac{\eta}{2}\hat{a}_c^{\dagger}\hat{a}_c\left(\hat{a}_t^{\dagger}\hat{b}_t - \hat{b}_t^{\dagger}\hat{a}_t\right)\right].$$

$$|1\rangle_t = |1\rangle_a |0\rangle_b$$

$$|0\rangle_t = |0\rangle_a |1\rangle_b$$

אחרי הפעלת $a_t^{\dagger}a_t + b_t^{\dagger}b_t \equiv U$ נקבל:

$$U|1\rangle = |1\rangle_a |0\rangle_b = |1\rangle$$

ואותו דבר לגבי: $U|0\rangle = |0\rangle$

לכן האופרטור U מתנהג פשוט 1 לכל

מצב ולכן גם על כל ס"פ:

$$U(\alpha|1\rangle + \beta|0\rangle) = \alpha|1\rangle + \beta|0\rangle$$

לכן ניתן פשוט להחליף את האופרטור ב-1

וקיבלנו מה שרצינו.

(a) Show that for two independent photonic modes, generated by creation operators $\hat{a}^\dagger$ and $\hat{b}^\dagger$, the following identity holds:

$$\exp\left[\frac{\pi}{2}\left(\hat{a}^\dagger\hat{b}-\hat{b}^\dagger\hat{a}\right)\right]|1\rangle_a|1\rangle_b=-|1\rangle_a|1\rangle_b\,.$$

$$\sum_{n=0}^{\infty}\frac{\left[\frac{\pi}{2}(a^\dagger b-b^\dagger a)\right]^n}{n!}|1\rangle_a|1\rangle_b \qquad a^\dagger|1\rangle=\sqrt{2}|2\rangle$$

$$(a^\dagger b-b^\dagger a)|1\rangle|1\rangle=\sqrt{2}\left(|2\rangle|0\rangle-|0\rangle|2\rangle\right)$$

אחרי עוד הפעלה:

$$-2|1\rangle|1\rangle-2|1\rangle|1\rangle=-4|1\rangle|1\rangle$$

אחרי עוד הפעלה נקבל:

$$4\sqrt{2}\left(|0\rangle|2\rangle-|2\rangle|0\rangle\right)$$

אחרי עוד הפעלה:

$$4\cdot 2\left(|1\rangle|1\rangle+|1\rangle|1\rangle\right)$$

$$=16|1\rangle|1\rangle$$

אומר כל 2 הפעלות שם מכפילים ב: -4

לכן סכום $n=2k$:

$$\sum_{k=0}^{\infty}\frac{(\frac{\pi}{2})^{2k}}{(2k)!}(-4)^k|1\rangle|1\rangle=\sum_{n=0}^{\infty}\frac{(\pi i)^n}{n!}|1\rangle|1\rangle=-|1\rangle|1\rangle$$

$$(-4)^k=(-1)^k4^k=(2i)^k$$

ב) אם $a_c=1 \iff |1\rangle_c$ תהפוך את הביטים האחרים