Fundamentals of Quantum Technology
Week 5: Dissipation in a Two-Level System

1 A two-level system coupled to a harmonic bath

In the lectures you discussed the dynamics of a two-level system (TLS) that is weakly coupled to a large bath of harmonic oscillators. The Hamiltonian that describes the full system is given by

$$\hat{\mathcal{H}} = \frac{\hbar\omega_0}{2}\hat{\sigma}_z + \hbar\sum_\lambda \Omega_\lambda \hat{B}_\lambda^\dagger \hat{B}_\lambda + \hbar\sum_\lambda \left[g_\lambda \hat{\sigma}_- \hat{B}_\lambda^\dagger + g_\lambda^* \hat{B}_\lambda \hat{\sigma}_+\right],$$

where $\hbar\omega_0 = E_e - E_g$ as usual, Ω_λ are the frequencies of the harmonic oscillators comprising the bath, $\hat{B}_\lambda$ and $\hat{B}_\lambda^\dagger$ are their annihilation and creation operators, and g_λ are the coupling constants. The strategy we employ is to solve for the dynamics of the TLS operators $\hat{\sigma}_z$ and $\hat{\sigma}_\pm$ within the Heisenberg picture, and then immediately derive the dynamics of ρ (the density matrix of the TLS) in the Schrödinger picture, through the relation

$$\rho = \begin{pmatrix} \frac{1+\langle\hat{\sigma}_z\rangle}{2} & \langle\hat{\sigma}_-\rangle \\ \langle\hat{\sigma}_+\rangle & \frac{1-\langle\hat{\sigma}_z\rangle}{2} \end{pmatrix}. \quad \text{[Schrödinger]}$$

Recall that expectation values are independent of the picture in which we choose to work.

In the Heisenberg picture, equations of motion for the operators are given by the Heisenberg equation

$$\dot{\hat{O}} = \frac{i}{\hbar}\left[\hat{\mathcal{H}}, \hat{O}\right].$$

We use the commutation relations

$$\begin{aligned} &\left[\hat{B}_\lambda, \hat{B}_{\lambda'}^\dagger\right] = \delta_{\lambda\lambda'}, \quad \left[\hat{B}_\lambda, \hat{B}_{\lambda'}\right] = \left[\hat{B}_\lambda^\dagger, \hat{B}_{\lambda'}^\dagger\right] = 0, \\ &\left[\hat{\sigma}_z, \hat{\sigma}_\pm\right] = \pm 2\hat{\sigma}_\pm, \quad \left[\hat{\sigma}_+, \hat{\sigma}_-\right] = \hat{\sigma}_z, \end{aligned}$$

to obtain the following equations of motion:

$$\begin{aligned} \dot{\hat{B}}_\lambda &= -i\Omega_\lambda \hat{B}_\lambda - i g_\lambda \hat{\sigma}_-, \\ \dot{\hat{\sigma}}_- &= -i\omega_0 \hat{\sigma}_- + i\sum_\lambda g_\lambda^* \hat{B}_\lambda \hat{\sigma}_z, \\ \dot{\hat{\sigma}}_z &= 2i\sum_\lambda \left\{g_\lambda \hat{\sigma}_- \hat{B}_\lambda^\dagger - g_\lambda^* \hat{B}_\lambda \hat{\sigma}_+\right\}. \end{aligned}$$

In the lecture you wrote exact solutions (which were not, however, closed expressions) for the first two equations and their conjugates, substituted these solutions into the equation for $\dot{\hat{\sigma}}_z$, and then kept terms only up to second order in the coupling constants g_λ. This yielded the approximate solution

$$\hat{\sigma}_z(t) = e^{-2\kappa t}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \left(1 - e^{-2\kappa t}\right)\tanh\left(\frac{\beta\hbar\omega_0}{2}\right)\mathbb{I},$$

with the decay rate κ being defined as

$$\kappa \equiv \pi \left|g\left(\omega_0\right)\right|^2 \varrho\left(\omega_0\right) \coth\left(\frac{\beta\hbar\omega_0}{2}\right).$$

We will now present the solution for $\hat{\sigma}_{\pm}(t)$, using a similar analysis. We can write the exact solutions

$$\begin{aligned}
\hat{B}_{\lambda}(t) &= \hat{B}_{\lambda}(0)\, e^{-i\Omega_{\lambda}t} - i g_{\lambda}\int_0^t \mathrm{d}\tau \hat{\sigma}_{-}(\tau)\, e^{-i\Omega_{\lambda}(t-\tau)}, \\
\hat{\sigma}_z(t) &= \hat{\sigma}_z(0) + 2i\sum_{\lambda}\int_0^t \mathrm{d}\tau \left\{ g_{\lambda}\hat{\sigma}_{-}(\tau)\, \hat{B}_{\lambda}^{\dagger}(\tau) - g_{\lambda}^{*}\hat{B}_{\lambda}(\tau)\, \hat{\sigma}_{+}(\tau)\right\},
\end{aligned}$$

and substitute them into the equation of motion for $\hat{\sigma}_{-}$, keeping terms up to second order in the coupling constants,

$$\begin{aligned}
\dot{\hat{\sigma}}_{-} \approx{}& -i\omega_0\hat{\sigma}_{-} + i\sum_{\lambda} g_{\lambda}^{*}\hat{B}_{\lambda}(0)\, \hat{\sigma}_z(0)\, e^{-i\Omega_{\lambda}t} + \sum_{\lambda}\left|g_{\lambda}\right|^2 \int_0^t \mathrm{d}\tau \hat{\sigma}_{-}(\tau)\, e^{-i\Omega_{\lambda}(t-\tau)}\hat{\sigma}_z(0) \\
&- 2\sum_{\lambda,\lambda'} g_{\lambda}^{*}\int_0^t \mathrm{d}\tau \hat{B}_{\lambda}(0)\, e^{-i\Omega_{\lambda}t}\left\{ g_{\lambda'}\hat{\sigma}_{-}(\tau)\, \hat{B}_{\lambda'}^{\dagger}(\tau) - g_{\lambda'}^{*}\hat{B}_{\lambda'}(\tau)\, \hat{\sigma}_{+}(\tau)\right\}.
\end{aligned}$$

What we will want to do now is to bring operators within each integral to equal times (for reasons that will become clear shortly). Since both integrals already appear as second-order terms in the equation, we can use the zeroth-order approximations

$$\begin{aligned}
\hat{\sigma}_{-}(\tau) &\approx \hat{\sigma}_{-}(0)\, e^{-i\omega_0\tau}, \\
\hat{B}_{\lambda}(\tau) &\approx \hat{B}_{\lambda}(0)\, e^{-i\Omega_{\lambda}\tau}.
\end{aligned}$$

In the first integral, this yields

$$\begin{aligned}
\hat{\sigma}_{-}(\tau)\, e^{-i\Omega_{\lambda}(t-\tau)}\hat{\sigma}_z(0) &\approx e^{-i\omega_0\tau}e^{-i\Omega_{\lambda}(t-\tau)}\hat{\sigma}_{-}(0)\, \hat{\sigma}_z(0) \\
&= e^{-i\omega_0\tau}e^{-i\Omega_{\lambda}(t-\tau)}\hat{\sigma}_{-}(0) \\
&\approx e^{-i\Omega_{\lambda}(t-\tau)}\hat{\sigma}_{-}(\tau),
\end{aligned}$$

where in the transition between the first row and the second row we exploited our knowledge of the forms of Pauli operators at $t = 0$. In the second integral we may write $\hat{B}_{\lambda}(0) \approx \hat{B}_{\lambda}(\tau)\, e^{i\Omega_{\lambda}\tau}$. We also take the lower limit of each integral to $-\infty$; this is called the **Markov approximation**, and it implies that the dynamics of the bath is very fast such that the system is "memory-less", and its behavior is affected only negligibly by events in the distant past.

All in all,

$$\begin{aligned}
\dot{\hat{\sigma}}_{-} \approx{}& -i\omega_0\hat{\sigma}_{-} + i\sum_{\lambda} g_{\lambda}^{*}\hat{B}_{\lambda}(0)\, \hat{\sigma}_z(0)\, e^{-i\Omega_{\lambda}t} + \sum_{\lambda}\left|g_{\lambda}\right|^2 \int_{-\infty}^t \mathrm{d}\tau e^{-i\Omega_{\lambda}(t-\tau)}\hat{\sigma}_{-}(\tau) \\
&- 2\sum_{\lambda,\lambda'} g_{\lambda}^{*}\int_{-\infty}^t \mathrm{d}\tau e^{-i\Omega_{\lambda}(t-\tau)}\left\{ g_{\lambda'}\hat{\sigma}_{-}(\tau)\, \hat{B}_{\lambda}(\tau)\, \hat{B}_{\lambda'}^{\dagger}(\tau) - g_{\lambda'}^{*}\hat{B}_{\lambda}(\tau)\, \hat{B}_{\lambda'}(\tau)\, \hat{\sigma}_{+}(\tau)\right\}.
\end{aligned}$$

We now average over bath operators, recalling that the bath is assumed to be at thermal equilibrium in any given moment in time, which specifically implies that

$$\left\langle \hat{B}_\lambda (t) \right\rangle = 0,$$

and

$$\left\langle \hat{B}_\lambda (t) \hat{B}^\dagger_{\lambda'} (t) \right\rangle = \left\langle \delta_{\lambda\lambda'} + \hat{B}^\dagger_{\lambda'} (t) \hat{B}_\lambda (t) \right\rangle = \delta_{\lambda\lambda'} \left(1 + f_{\mathrm{BE}} (\Omega_\lambda)\right),$$

where we used the Bose-Einstein distribution function

$$f_{\mathrm{BE}} (\omega) = \frac{1}{e^{\beta\hbar\omega} - 1}.$$

This averaging leaves us only with the degrees-of-freedom of the TLS appearing in the equation, such that

$$\begin{aligned}
\dot{\hat{\sigma}}_- &\approx -i\omega_0 \hat{\sigma}_- (t) - \sum_\lambda |g_\lambda|^2 \left(1 + 2 f_{\mathrm{BE}} (\Omega_\lambda)\right) \int\limits_{-\infty}^{t} \mathrm{d}\tau e^{-i\Omega_\lambda (t-\tau)} \hat{\sigma}_- (\tau) \\
&\approx - \left\{ i\omega_0 + \sum_\lambda |g_\lambda|^2 \coth\left(\frac{\beta\hbar\Omega_\lambda}{2}\right) \int\limits_{-\infty}^{t} \mathrm{d}\tau e^{i(\omega_0 - \Omega_\lambda)(t-\tau)} \right\} \hat{\sigma}_- (t) \\
&= - \left\{ i\omega_0 + \int \mathrm{d}\Omega \varrho (\Omega) |g (\Omega)|^2 \coth\left(\frac{\beta\hbar\Omega}{2}\right) \int\limits_{-\infty}^{t} \mathrm{d}\tau e^{i(\omega_0 - \Omega)(t-\tau)} \right\} \hat{\sigma}_- (t) \\
&= - \left\{ i\omega_0 + \int \mathrm{d}\Omega \varrho (\Omega) |g (\Omega)|^2 \coth\left(\frac{\beta\hbar\Omega}{2}\right) \int\limits_{0}^{\infty} \mathrm{d}\tau e^{i(\omega_0 - \Omega)\tau} \right\} \hat{\sigma}_- (t).
\end{aligned}$$

Next, we use the identity

$$\int\limits_0^\infty \mathrm{d}\tau e^{i\omega\tau} = \pi\delta (\omega) + i\mathcal{P}\frac{1}{\omega},$$

where $\mathcal{P}$ is Cauchy's principal value. We obtain

$$\begin{aligned}
\dot{\hat{\sigma}}_- &\approx - \left\{ i\omega_0 + i\mathcal{P} \int \mathrm{d}\Omega \varrho (\Omega) \frac{|g (\Omega)|^2}{\omega_0 - \Omega} \coth\left(\frac{\beta\hbar\Omega}{2}\right) + \pi |g (\omega_0)|^2 \varrho (\omega_0) \coth\left(\frac{\beta\hbar\omega_0}{2}\right) \right\} \hat{\sigma}_- \\
&\equiv - \left(i\widetilde{\omega}_0 + \kappa\right) \hat{\sigma}_-,
\end{aligned}$$

where κ has the same definition as in the case of the solution for $\hat{\sigma}_z$ (the decay rate of the diagonal terms is twice the decay rate of the off-diagonal terms).

The solution of this equation is straightforward, and it also yields the solution for $\hat{\sigma}_+$ through conjugation:

$$\boxed{\hat{\sigma}_\pm (t) = e^{\pm i\widetilde{\omega}_0 t - i\kappa t} \hat{\sigma}_\pm (0).}$$

Note the shift of the oscillation frequency compared to ω_0:

$$\widetilde{\omega}_0 = \omega_0 + \mathcal{P} \int \mathrm{d}\Omega \varrho (\Omega) \frac{|g (\Omega)|^2}{\omega_0 - \Omega} \coth\left(\frac{\beta\hbar\Omega}{2}\right).$$

If we take this correction into account in the original equation of motion for $\dot{\hat{\sigma}}_\pm$, it will be reflected in the solution for $\hat{\sigma}_z$ as well:

$$\boxed{\hat{\sigma}_z (t) = e^{-2\kappa t} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \left(1 - e^{-2\kappa t}\right) \tanh\left(\frac{\beta\hbar\widetilde{\omega}_0}{2}\right) \mathbb{I}.}$$

This is called the **Lamb shift** – the energy levels of the TLS are shifted ("renormalized") due to the interaction with the environment. While it is an important physical effect that is evident in practical implementations, it is often ignored in calculations because it only slightly changes each energy level, and does not cause these levels to mix.

At home you will examine the consequences of a different type of coupling to the bath,

$$\hat{\mathcal{H}} = \frac{\hbar\omega_0}{2}\hat{\sigma}_z + \hbar\sum_\lambda \Omega_\lambda \hat{B}_\lambda^\dagger \hat{B}_\lambda + \hbar\hat{\sigma}_z \sum_\lambda \left[\mu_\lambda \hat{B}_\lambda^\dagger + \mu_\lambda^* \hat{B}_\lambda\right].$$

Here $\left[\hat{\mathcal{H}}, \hat{\sigma}_z\right] = 0$ and therefore there is no energy relaxation, but you will see that the interaction with the $\Omega_\lambda = 0$ oscillators still induces dephasing. Namely, you will show that in the Heisenberg picture

$$\dot{\hat{\sigma}}_z = 0, \quad \dot{\hat{\sigma}}_\pm = \left(\pm i\widetilde{\omega}_0 - \Gamma\right)\hat{\sigma}_\pm,$$

with $\widetilde{\omega}_0$ a renormalized frequency, as before. When both types of coupling are present, we arrive at the following equations for the expectation values of the TLS operators:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{\sigma}_z\rangle &= -2\kappa\left[\tanh\left(\frac{\beta\hbar\widetilde{\omega}_0}{2}\right) + \langle\hat{\sigma}_z\rangle\right] \equiv -\frac{1}{T_1}\left[\tanh\left(\frac{\beta\hbar\widetilde{\omega}_0}{2}\right) + \langle\hat{\sigma}_z\rangle\right], \\
\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{\sigma}_\pm\rangle &= \left(\pm i\omega_0' - \kappa - \Gamma\right)\langle\hat{\sigma}_\pm\rangle \equiv \left(\pm i\omega_0' - \frac{1}{2T_1} - \frac{1}{T_2^*}\right)\langle\hat{\sigma}_\pm\rangle \equiv \left(\pm i\omega_0' - \frac{1}{T_2}\right)\langle\hat{\sigma}_\pm\rangle,
\end{aligned}$$

where ω_0' is the shifted frequency due to both couplings. Typically, $T_2 < T_1$.

2 The open Rabi problem

At home you will derive similar expectation-value equations of motion for the semiclassical Rabi model. Combining the dynamics due to the Rabi driving field and the interaction with the environment, the total equations of motion (ignoring the Lamb shift and assuming $\mathcal{V}$ is real) are

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{\sigma}_z\rangle &= -\frac{i\mathcal{V}}{\hbar}\left[\langle\hat{\sigma}_+\rangle - \langle\hat{\sigma}_-\rangle\right] - \frac{1}{T_1}\left[\tanh\left(\frac{\beta\hbar\omega_0}{2}\right) + \langle\hat{\sigma}_z\rangle\right], \\
\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{\sigma}_\pm\rangle &= \mp\frac{i\mathcal{V}}{2\hbar}\langle\hat{\sigma}_z\rangle - \frac{1}{T_2}\langle\hat{\sigma}_\pm\rangle.
\end{aligned}$$

The two equations are coupled, and one can easily identify in each equation the part corresponding to coherent evolution and the part responsible for decoherence.

Exercise

Given the effects of decoherence, what elementary requirement is there regarding $\mathcal{V}$ such that single-qubit gates can still be applied?

Solution

The frequency of rotations of the Bloch sphere that are performed using Rabi pulses is $\mathcal{V}/\hbar$, meaning that they are performed over a typical time scale of $\sim \hbar/\mathcal{V}$. To avoid detrimental effects of decoherence, we must require

$$\frac{\hbar}{\mathcal{V}} \ll T_2 \longrightarrow \mathcal{V} \gg \frac{\hbar}{T_2}.$$