Fundamentals of Quantum Technology
Homework Sheet 6

1. In class, we have defined the non-normalizable “phase eigenstates” $|\phi\rangle = \sum_{n=0}^{\infty} e^{in\phi}|n\rangle$ for a single-mode field.

   (a) Show that the phase states satisfy the following completeness relation:

   $$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi\, |\phi\rangle\langle\phi| = \hat{I}.$$

   (b) We have defined the phase distribution of a state $\hat{\rho}$ as $\mathcal{P}(\phi) = \frac{1}{2\pi}\langle\phi|\hat{\rho}|\phi\rangle$. Show that $\int_0^{2\pi} \mathrm{d}\phi \mathcal{P}(\phi) = 1$ for any state $\hat{\rho}$.
   Hint: Express $\hat{\rho}$ as an ensemble average.

2. Calculate the phase distribution $\mathcal{P}(\phi)$ for the following states:

   (a) The pure state $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\theta}|1\rangle\right)$.

   (b) The mixed state $\hat{\rho} = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$.

3. Recall that the density operator for a single-mode thermal field with frequency ω is given by

   $$\begin{aligned}\hat{\rho}_{\mathrm{Th}} &= \frac{1}{Z}\sum_{n=0}^{\infty} \exp(-\beta E_n)|n\rangle\langle n| \\ &= \frac{1}{1+\bar{n}}\sum_{n=0}^{\infty}\left(\frac{\bar{n}}{1+\bar{n}}\right)^n |n\rangle\langle n|,\end{aligned}$$

   where β is the inverse temperature and $\bar{n} = (\exp(\beta\hbar\omega) - 1)^{-1}$ is the average photon number.

   (a) Show the the variance of the photon number within the thermal state is given by

   $$\left\langle \hat{n}^2 \right\rangle - \langle n\rangle^2 = \bar{n}^2 + \bar{n}.$$

   (b) Show that the phase distribution for the thermal state is given by $\mathcal{P}(\phi) = 1/2\pi$.

4. Suppose that a single-mode field is initially in a coherent state, $|\psi(0)\rangle = |\alpha\rangle$, and we let it evolve with the Hamiltonian given by $\hat{H} = \hbar\omega\left(\hat{n} + \frac{1}{2}\right)$. Show that

   $$|\psi(t)\rangle = e^{-i\omega t/2}\left|\alpha e^{-i\omega t}\right\rangle.$$