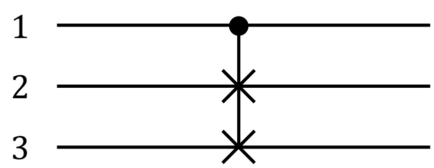
Fundamentals of Quantum Technology
Homework Sheet 2

1. Find a choice of angles $\theta, \theta', \phi, \phi'$ that violates the CHSH-Bell inequality assuming we start from the Bell state
$$\left|\Phi^-\right\rangle = \frac{1}{\sqrt{2}}\left(|H\rangle_1 |H\rangle_2 - |V\rangle_1 |V\rangle_2\right).$$
Hint: First, rewrite $|\Phi^-\rangle$ with respect to a basis $\left\{|\theta\rangle, \left|\theta^\perp\right\rangle\right\}$ for particle 1 and a basis $\left\{|\phi\rangle, \left|\phi^\perp\right\rangle\right\}$ for particle 2, and then show that $C(\theta,\phi) = \cos(2\theta + 2\phi)$ for $|\Phi^-\rangle$. Next, find angles such that
$$|C(\theta,\phi) + C(\theta,\phi') + C(\theta',\phi) - C(\theta',\phi')| > 2,$$
in violation of the CHSH-Bell inequality. Considerations for a correct choice of angles become more transparent if you first show that
$$\begin{aligned} C(\theta,\phi) + C(\theta,\phi') + C(\theta',\phi) - C(\theta',\phi') = {} & 2\cos(2\theta + \phi + \phi')\cos(\phi' - \phi) \\ & + 2\sin(2\theta' + \phi + \phi')\sin(\phi' - \phi). \end{aligned}$$

2. Recall the definitions of the unitary transformations representing the beam splitter and the Kerr component in the optical setup that realizes the C-NOT gate:
$$\begin{aligned} \hat{U}_{\mathrm{BS1}} &= \exp\left[i\frac{\pi}{4}\left(\hat{a}_t^\dagger \hat{b}_t + \hat{b}_t^\dagger \hat{a}_t\right)\right] = \hat{U}_{\mathrm{BS2}}^\dagger, \\ \hat{U}_{\mathrm{Kerr}}(\eta) &= \exp\left[i\eta \hat{a}_c^\dagger \hat{a}_c \hat{b}_t^\dagger \hat{b}_t\right]. \end{aligned}$$
   (a) Show that
$$\hat{U}_{\mathrm{BS1}}^\dagger \hat{U}_{\mathrm{Kerr}}(\eta)\hat{U}_{\mathrm{BS1}} = \exp\left[i\frac{\eta}{2}\hat{a}_c^\dagger \hat{a}_c\left(\hat{a}_t^\dagger \hat{a}_t + \hat{b}_t^\dagger \hat{b}_t\right)\right]\exp\left[\frac{\eta}{2}\hat{a}_c^\dagger \hat{a}_c\left(\hat{a}_t^\dagger \hat{b}_t - \hat{b}_t^\dagger \hat{a}_t\right)\right].$$
   Hint: Expand $\hat{U}_{\mathrm{Kerr}}$ into a power series and use the result of Question 3 in Homework 1.

   (b) Show that for any state of the target qubit (including any superposition of $|0\rangle_t$ and $|1\rangle_t$), the above is equivalent to
$$\hat{U}_{\mathrm{BS1}}^\dagger \hat{U}_{\mathrm{Kerr}}(\eta)\hat{U}_{\mathrm{BS1}} = \exp\left[i\frac{\eta}{2}\hat{a}_c^\dagger \hat{a}_c\right]\exp\left[\frac{\eta}{2}\hat{a}_c^\dagger \hat{a}_c\left(\hat{a}_t^\dagger \hat{b}_t - \hat{b}_t^\dagger \hat{a}_t\right)\right].$$

3. The Fredkin gate is an elementary 3-qubit gate where qubit 1 serves as a control qubit and qubits 2 and 3 serve as target qubits. It is drawn schematically as



and is a *controlled-swap* gate, meaning that it does not change the state of the qubits if the control qubit is in the state $|0\rangle_1$, while if it is in the state $|1\rangle_1$ then it performs the operation

$$|1\rangle_1 |x\rangle_2 |y\rangle_3 \longrightarrow |1\rangle_1 |y\rangle_2 |x\rangle_3 .$$

(a) Show that for two independent photonic modes, generated by creation operators $\hat{a}^\dagger$ and $\hat{b}^\dagger$, the following identity holds:

$$\exp\left[\frac{\pi}{2}\left(\hat{a}^\dagger\hat{b} - \hat{b}^\dagger\hat{a}\right)\right] |1\rangle_a |1\rangle_b = -|1\rangle_a |1\rangle_b .$$

(b) Design an optical realization of the Fredkin gate using 50:50 beam splitters, Kerr components and phase shifters (if necessary).
Hint: Note that the C-NOT gate we constructed in class simply swaps the states of the two photonic modes describing a single target qubit. Use a similar idea to swap the states of two target qubits. Use the result of Item (a) to address the case of a swap between two modes occupied by a single photon each.