דף נוסחאות

תנע זוויתי:

$$[J_i, J_j] = i\hbar\epsilon_{ijk}J_k \qquad J_z\left|jm\right\rangle = \hbar m\left|jm\right\rangle$$
$$\left[J^2, \mathbf{J}\right] = 0 \qquad J^2\left|jm\right\rangle = \hbar^2 j\left(j+1\right)\left|j,m\right\rangle$$
$$J_\pm = J_x \pm iJ_y \qquad [J_z, J_\pm] = \pm\hbar J_\pm$$
$$[J_+, J_-] = 2\hbar J_z \qquad J_\pm\left|jm\right\rangle = \hbar\sqrt{j\left(j+1\right) - m\left(m\pm1\right)}\left|j,m\pm1\right\rangle$$

ספין $\frac{1}{2}$:

$$\mathbf{S} = \frac{\hbar}{2}\boldsymbol{\sigma}: \qquad \sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad \sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \qquad \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

בור פוטנציאל אינסופי:

$$\left\langle x|n\right\rangle = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right), \qquad E_n = \frac{\pi^2\hbar^2}{2mL^2}n^2$$

אוסצילטור הרמוני:

$$\left\langle x|n\right\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)e^{-m\omega x^2/2\hbar}, \qquad E_n = \hbar\omega\left(n+\frac{1}{2}\right)$$

$$H_0\left(\xi\right) = 1, \qquad H_1\left(\xi\right) = 2\xi, \qquad H_2\left(\xi\right) = 4\xi^2 - 2, \qquad H_3\left(\xi\right) = 8\xi^3 - 12\xi$$

אופרטורי העלאה והורדה:

$$\begin{cases} a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{i}{m\omega}p\right) \\ a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{i}{m\omega}p\right)\end{cases} \qquad \begin{cases} a\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle \\ a^\dagger\left|n\right\rangle = \sqrt{n+1}\left|n+1\right\rangle\end{cases} \qquad \begin{cases} x = \sqrt{\frac{\hbar}{2m\omega}}\left(a^\dagger + a\right) \\ p = i\sqrt{\frac{m\omega\hbar}{2}}\left(a^\dagger - a\right)\end{cases}$$

אטומי מימן ודמוי-מימן:

$$\left\langle \mathbf{r}|n\ell m\right\rangle = R_{n\ell}\left(r\right)Y_m^{(\ell)}\left(\theta,\varphi\right), \qquad E_n = -\frac{e^2}{2a_0}\frac{Z^2}{n^2} = -\frac{Z^2}{n^2}13.6\,\mathrm{eV}$$

$$R_{10}\left(r\right) = 2\left(\frac{Z}{a_0}\right)^{3/2}e^{-Zr/a_0} \qquad R_{30}\left(r\right) = 2\left(\frac{Z}{3a_0}\right)^{3/2}\left(1 - \frac{2Zr}{3a_0} + \frac{2\left(Zr\right)^2}{27a_0^2}\right)e^{-Zr/3a_0}$$

$$R_{20}\left(r\right) = 2\left(\frac{Z}{2a_0}\right)^{3/2}\left(1 - \frac{Zr}{2a_0}\right)e^{-Zr/2a_0} \qquad R_{31}\left(r\right) = \frac{4\sqrt{2}}{3}\left(\frac{Z}{3a_0}\right)^{3/2}\frac{Zr}{a_0}\left(1 - \frac{Zr}{6a_0}\right)e^{-Zr/3a_0}$$

$$R_{21}\left(r\right) = \frac{1}{\sqrt{3}}\left(\frac{Z}{2a_0}\right)^{3/2}\frac{Zr}{a_0}e^{-Zr/2a_0} \qquad R_{32}\left(r\right) = \frac{2\sqrt{2}}{27\sqrt{5}}\left(\frac{Z}{3a_0}\right)^{3/2}\left(\frac{Zr}{a_0}\right)^2e^{-Zr/3a_0}$$

הזזות וסיבובים:

$$\mathcal{T}\left(\mathbf{a}\right)=e^{-i\mathbf{a}\cdot\mathbf{p}/\hbar},\qquad\mathcal{D}\left(\hat{\mathbf{n}},\theta\right)=e^{-i\theta\mathbf{J}\cdot\hat{\mathbf{n}}/\hbar}$$

זוויות אוילר:

$$\mathcal{D}\left(\hat{\mathbf{n}},\theta\right)=e^{-i\alpha J_z/\hbar}e^{-i\beta J_y/\hbar}e^{-i\gamma J_z/\hbar},\qquad\mathcal{D}_{m',m}^{(j)}=e^{-i(m'\alpha+m\gamma)}d_{m',m}^{(j)}\left(\beta\right)$$

אופרטור וקטורי:

$$V_i'=R_{ij}^{-1}V_j,\quad\left[J_i,V_j\right]=i\hbar\epsilon_{ijk}V_k,\qquad V_0^{(1)}=V_z,\quad V_{\pm}^{(1)}=\mp\frac{1}{\sqrt{2}}\left(V_x\pm iV_y\right)$$

טנזורים כדוריים:

$$T_q^{(k)\prime}=\sum_{q'=-k}^{k}T_{q'}^{(k)}\mathcal{D}_{q'q}^{(k)},\qquad\begin{cases}\left[J_z,T_q^{(k)}\right]=\hbar qT_q^{(k)}\\ \left[J_{\pm},T_q^{(k)}\right]=\hbar\sqrt{k\left(k+1\right)-q\left(q\pm1\right)}T_{q\pm1}^{(k)}\end{cases}$$

משפט ויגנר-אקרט:

$$\langle\alpha';j',m'|T_q^{(k)}|\alpha;j,m\rangle=\langle j,k;m,q|j',m'\rangle\left\langle\alpha'j'\middle\|T^{(k)}\middle\|\alpha j\right\rangle$$

תורת הפרעות בלתי-תלויה בזמן (ללא ניוון):

$$\Delta E_n^{(1)}=\left\langle n^{(0)}\middle|V\middle|n^{(0)}\right\rangle,\quad\Delta E_n^{(2)}=\sum_{n\neq m}\frac{\left|\left\langle m^{(0)}\middle|V\middle|n^{(0)}\right\rangle\right|^2}{E_n^{(0)}-E_m^{(0)}},\quad\left|n^{(1)}\right\rangle=\sum_{n\neq m}\frac{\left\langle m^{(0)}\middle|V\middle|n^{(0)}\right\rangle}{E_n^{(0)}-E_m^{(0)}}\left|m^{(0)}\right\rangle$$

קירוב WKB:

$$\psi_{\pm}\left(x\right)\approx\frac{1}{\sqrt{p\left(x\right)}}\exp\left[\pm\frac{i}{\hbar}\int^{x}p\left(x'\right)\mathrm{d}x'\right],\qquad p\left(x\right)=\sqrt{2m\left[E-V\left(x\right)\right]}$$

$$T\approx e^{-2\gamma},\qquad\gamma=\frac{1}{\hbar}\int_{x_1}^{x_2}\left|p\left(x\right)\right|\mathrm{d}x$$

תורת הפרעות תלויה בזמן:

$$i\hbar\frac{\partial}{\partial t}c_m=\sum_n\langle m|V\left(t\right)|n\rangle e^{i\omega_{mn}t}c_n\left(t\right),\qquad P_{n\to m}=\left|\frac{1}{i\hbar}\int_0^t\langle m|V\left(\tau\right)|n\rangle e^{i\omega_{mn}\tau}\mathrm{d}\tau\right|^2$$

כלל הזהב של פרמי:

$$V=\mathcal{V}e^{i\omega t}+\text{h.c.}:\qquad\Gamma_{i\to f}=\sum_{|f\rangle}\frac{2\pi}{\hbar}\left|\langle f|\mathcal{V}|i\rangle\right|^2\delta\left(E_f-E_i\mp\hbar\omega\right)$$

קצב מעברים בקירוב הדיפול:

$$\Gamma_{i\to f}^{\text{dip.}}=\sum_{\lambda=1,2}\sum_{\mathbf{k}}\frac{2\pi}{\hbar}\left(\frac{e\omega A_0}{c}\right)^2\left|\langle f|\boldsymbol{\epsilon}^{(\lambda)}\cdot\mathbf{r}|i\rangle\right|^2\delta\left(E_f-E_i\mp\hbar\omega\right)$$

$$A_0^{\text{abs.}}=\sqrt{\frac{2\pi\hbar c}{k}\frac{N}{V}},\qquad A_0^{\text{emis.}}=\sqrt{\frac{2\pi\hbar c}{k}\frac{N+1}{V}}$$

קירוב בורן מסדר 1:

$$f_{\text{Born}}^{(1)}\left(\mathbf{k}_f,\mathbf{k}_i\right)=-\frac{m}{2\pi\hbar^2}\left(2\pi\right)^3\,\langle\mathbf{k}_f|V|\mathbf{k}_i\rangle=-\frac{m}{2\pi\hbar^2}\tilde{V}\left(\mathbf{k}_f-\mathbf{k}_i\right)$$

תנע מועבר בפיזור אלסטי:

$$\mathbf{q}=\mathbf{k}_f-\mathbf{k}_i,\qquad q=2k\sin\frac{\theta}{2},\qquad \mathrm{d}\left(\cos\theta\right)=\frac{q\,\mathrm{d}q}{k^2}$$

גלים חלקיים/הסחות פאזה:

$$f\left(\theta\right)=\sum_{\ell=0}^{\infty}\left(2\ell+1\right)f_\ell\left(k\right)P_\ell\left(\cos\theta\right):\qquad f_\ell\left(k\right)=\frac{S_\ell\left(k\right)-1}{2ik}=\frac{e^{i\delta_\ell}}{k}\sin\left(\delta_\ell\right),\quad S_\ell\left(k\right)=e^{2i\delta_\ell}$$

חתך פעולה כולל והמשפט האופטי:

$$\sigma_{\text{tot}}=\frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}\left(2\ell+1\right)\sin^2\left(\delta_\ell\right)=\frac{4\pi}{k}\,\mathrm{Im}\{f\left(\theta=0\right)\}$$

אינטגרלים של אקספוננטים:

$$\int_0^\infty x^n e^{-x}\,\mathrm{d}x=n!,\qquad \int_{-\infty}^{\infty}e^{-\alpha x^2}\,\mathrm{d}x=\sqrt{\frac{\pi}{\alpha}}$$

$$\int_{-\infty}^{\infty}x^2e^{-\alpha x^2}\,\mathrm{d}x=\frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}},\qquad \int_{-\infty}^{\infty}x^4e^{-\alpha x^2}\,\mathrm{d}x=\frac{3}{4\alpha^2}\sqrt{\frac{\pi}{\alpha}}$$

טרנספורם פורייה:

$$\tilde{V}\left(\mathbf{q}\right)=\int V\left(\mathbf{r}\right)e^{-i\mathbf{q}\cdot\mathbf{r}}\,\mathrm{d}^3r\,,\qquad V\left(\mathbf{r}\right)=\frac{1}{\left(2\pi\right)^3}\int\tilde{V}\left(\mathbf{q}\right)e^{i\mathbf{q}\cdot\mathbf{r}}\,\mathrm{d}^3q$$

זהויות טריגונומטריות:

$$\sin\left(2x\right)=2\sin x\cos x,\qquad \cos\left(2x\right)=\cos^2x-\sin^2x,\qquad \tan\left(x+y\right)=\frac{\tan x+\tan y}{1-\tan x\tan y}$$

נוסחת BCH:

$$e^ABe^{-A}=B+[A,B]+\frac{1}{2!}[A,[A,B]]+\frac{1}{3!}[A,[A,[A,B]]]+\dots$$

45. Clebsch-Gordan Coefficients, Spherical Harmonics, and d Functions

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2	Coefficients		
m_1	m_2			
.	.			

1/2×1/2

		1
		+1
+1/2	+1/2	1

		1	0
		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

		1
		−1
−1/2	−1/2	1

1×1/2

		3/2
		+3/2
+1	+1/2	1

		3/2	1/2
		+1/2	+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

		3/2	1/2
		−1/2	−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

		3/2
		−3/2
−1	−1/2	1

2×1

		3
		+3
+2	+1	1

		3	2
		+2	+2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

		3	2	1
		+1	+1	+1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

		3	2	1
		0	0	0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

		3	2	1
		−1	−1	−1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

		3	2
		−2	−2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

		3
		−3
−2	−1	1

1×1

		2
		+2
+1	+1	1

		2	1
		+1	+1
+1	0	1/2	1/2
0	+1	1/2	−1/2

		2	1	0
		0	0	0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

		2	1
		−1	−1
0	−1	1/2	1/2
−1	0	1/2	−1/2

		2
		−2
−1	−1	1

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$d^{\,\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

3/2×1

		5/2
		+5/2
+3/2	+1	1

		5/2	3/2
		+3/2	+3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

		5/2	3/2	1/2
		+1/2	+1/2	+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

		5/2	3/2	1/2
		−1/2	−1/2	−1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

		5/2	3/2
		−3/2	−3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

		5/2
		−5/2
−3/2	−1	1

2×1/2

		5/2
		+5/2
+2	+1/2	1

		5/2	3/2
		+3/2	+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

		5/2	3/2
		+1/2	+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

		5/2	3/2
		−1/2	−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

		5/2	3/2
		−3/2	−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

		5/2
		−5/2
−2	−1/2	1

3/2×1/2

		2
		+2
+3/2	+1/2	1

		2	1
		+1	+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

		2	1
		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

		2	1
		−1	−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

		2
		−2
−3/2	−1/2	1

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^{\,j}_{m',m} = (-1)^{m-m'} d^{\,j}_{m,m'} = d^{\,j}_{-m,-m'}$$

$$d^{\,1}_{0,0} = \cos\theta \qquad d^{\,1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{\,1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2}$$

$$d^{\,1}_{1,1} = \frac{1+\cos\theta}{2} \qquad d^{\,1}_{1,0} = -\frac{\sin\theta}{\sqrt{2}} \qquad d^{\,1}_{1,-1} = \frac{1-\cos\theta}{2}$$

3/2×3/2

		3
		+3
+3/2	+3/2	1

		3	2
		+2	+2
+3/2	+1/2	1/2	1/2
+1/2	+3/2	1/2	−1/2

		3	2	1
		+1	+1	+1
+3/2	−1/2	1/5	1/2	3/10
+1/2	+1/2	3/5	0	−2/5
−1/2	+3/2	1/5	−1/2	3/10

		3	2	1	0
		0	0	0	0
+3/2	−3/2	1/20	1/4	9/20	1/4
+1/2	−1/2	9/20	1/4	−1/20	−1/4
−1/2	+1/2	9/20	−1/4	−1/20	1/4
−3/2	+3/2	1/20	−1/4	9/20	−1/4

		3	2	1
		−1	−1	−1
+1/2	−3/2	1/5	1/2	3/10
−1/2	−1/2	3/5	0	−2/5
−3/2	+1/2	1/5	−1/2	3/10

		3	2
		−2	−2
−1/2	−3/2	1/2	1/2
−3/2	−1/2	1/2	−1/2

		3
		−3
−3/2	−3/2	1

2×3/2

		7/2
		+7/2
+2	+3/2	1

		7/2	5/2
		+5/2	+5/2
+2	+1/2	3/7	4/7
+1	+3/2	4/7	−3/7

		7/2	5/2	3/2
		+3/2	+3/2	+3/2
+2	−1/2	1/7	16/35	2/5
+1	+1/2	4/7	1/35	−2/5
0	+3/2	2/7	−18/35	1/5

		7/2	5/2	3/2	1/2
		+1/2	+1/2	+1/2	+1/2
+2	−3/2	1/35	6/35	2/5	2/5
+1	−1/2	12/35	5/14	0	−3/10
0	+1/2	18/35	−3/35	−1/5	1/5
−1	+3/2	4/35	−27/70	2/5	−1/10

		7/2	5/2	3/2	1/2
		−1/2	−1/2	−1/2	−1/2
+1	−3/2	4/35	27/70	2/5	1/10
0	−1/2	18/35	3/35	−1/5	−1/5
−1	+1/2	12/35	−5/14	0	3/10
−2	+3/2	1/35	−6/35	2/5	−2/5

		7/2	5/2	3/2
		−3/2	−3/2	−3/2
0	−3/2	2/7	18/35	1/5
−1	−1/2	4/7	−1/35	−2/5
−2	+1/2	1/7	−16/35	2/5

		7/2	5/2
		−5/2	−5/2
−1	−3/2	4/7	3/7
−2	−1/2	3/7	−4/7

		7/2
		−7/2
−2	−3/2	1

2×2

		4
		+4
+2	+2	1

		4	3
		+3	+3
+2	+1	1/2	1/2
+1	+2	1/2	−1/2

		4	3	2
		+2	+2	+2
+2	0	3/14	1/2	2/7
+1	+1	4/7	0	−3/7
0	+2	3/14	−1/2	2/7

		4	3	2	1
		+1	+1	+1	+1
+2	−1	1/14	3/10	3/7	1/5
+1	0	3/7	1/5	−1/14	−3/10
0	+1	3/7	−1/5	−1/14	3/10
−1	+2	1/14	−3/10	3/7	−1/5

		4	3	2	1	0
		0	0	0	0	0
+2	−2	1/70	1/10	2/7	2/5	1/5
+1	−1	8/35	2/5	1/14	−1/10	−1/5
0	0	18/35	0	−2/7	0	1/5
−1	+1	8/35	−2/5	1/14	1/10	−1/5
−2	+2	1/70	−1/10	2/7	−2/5	1/5

		4	3	2	1
		−1	−1	−1	−1
+1	−2	1/14	3/10	3/7	1/5
0	−1	3/7	1/5	−1/14	−3/10
−1	0	3/7	−1/5	−1/14	3/10
−2	+1	1/14	−3/10	3/7	−1/5

		4	3	2
		−2	−2	−2
0	−2	3/14	1/2	2/7
−1	−1	4/7	0	−3/7
−2	0	3/14	−1/2	2/7

		4	3
		−3	−3
−1	−2	1/2	1/2
−2	−1	1/2	−1/2

		4
		−4
−2	−2	1

$$d^{\,3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,1/2} = -\sqrt{3}\,\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-1/2} = \sqrt{3}\,\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^{\,2}_{2,2} = \Big(\frac{1+\cos\theta}{2}\Big)^2$$

$$d^{\,2}_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^{\,2}_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,-2} = \Big(\frac{1-\cos\theta}{2}\Big)^2$$

$$d^{\,2}_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^{\,2}_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\,\cos\theta$$

$$d^{\,2}_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^{\,2}_{0,0} = \Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

Figure 45.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974).