Introduction to Particles and Nuclear Physics - Home Exercise 5 - Solution

Question 1

1. We begin from the simplified expression for the cross-section:

$$\sigma = \frac{S}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2)^2}} \int |M|^2 (2\pi)^4 \delta^4(p_1 + p_2 + \sum_{j=3}^{N} p_j) \prod_{k=3}^{N} \frac{d^3\mathbf{p}_k}{2\sqrt{\mathbf{p}_k^2 + m_k^2}} \frac{1}{(2\pi)^3}$$

And we can plug into this our result derived in class for the lab frame:

$$\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2)^2} = m_2 |\mathbf{p}_1|$$

And again write out the expression, after taking out the various numerical factors from the integral and using $\mathbf{p}_2 = 0$ since we're in the lab frame:

$$\sigma = \frac{S}{64\pi^2 |\mathbf{p}_1| m_2} \int |M|^2 \frac{\delta^3(\mathbf{p}_1 - \mathbf{p}_3 - \mathbf{p}_4)\delta(E_1 + m_2 - p_3^0 - p_4^0)\ d^3\mathbf{p}_3\ d^3\mathbf{p}_4}{\sqrt{\mathbf{p}_3^2 + m_3^2}\sqrt{\mathbf{p}_4^2 + m_4^2}}$$

Now we can perform the spatial integral over $d^3\mathbf{p}_4$ using the δ^3 function, sending $\mathbf{p}_4 \to \mathbf{p}_1 - \mathbf{p}_3$ and writing explicitly the p^0 components:

$$\sigma = \frac{S}{64\pi^2 |\mathbf{p}_1| m_2} \int |M|^2 \frac{\delta(E_1 + m_2 - \sqrt{\mathbf{p}_3^2 + m_3^2} - \sqrt{(\mathbf{p}_1 - \mathbf{p}_3)^2 + m_4^2})\ d^3\mathbf{p}_3}{\sqrt{\mathbf{p}_3^2 + m_3^2}\sqrt{(\mathbf{p}_1 - \mathbf{p}_3)^2 + m_4^2}}$$

Writing $d^3\mathbf{p}_3 = |\mathbf{p}_3|^2 d|\mathbf{p}_3| d\Omega$ and $(\mathbf{p}_1 - \mathbf{p}_3)^2 = |\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 - 2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta$ (with θ the scattering angle of particle 3):

$$\frac{d\sigma}{d\Omega} = \frac{S}{64\pi^2 |\mathbf{p}_1| m_2} \int |M|^2 \frac{\delta(E_1 + m_2 - \sqrt{|\mathbf{p}_3|^2 + m_3^2} - \sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 - 2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta + m_4^2})|\mathbf{p}_3|}{\sqrt{|\mathbf{p}_3|^2 + m_3^2}\sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 - 2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta + m_4^2}}$$

To solve this integral we can define a new variable as:

$$z = \sqrt{|\mathbf{p}_3|^2 + m_3^2} - \sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 - 2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta + m_4^2})$$

$$\Rightarrow \quad \frac{dz}{d|\mathbf{p}_3|} = \frac{|\mathbf{p}_3| z - |\mathbf{p}_1|\cos\theta\sqrt{|\mathbf{p}_3|^2 + m_3^2}}{\sqrt{|\mathbf{p}_3|^2 + m_3^2}\sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 - 2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta + m_4^2}}$$

$$\Rightarrow \quad \frac{dz}{||\mathbf{p}_3| z - |\mathbf{p}_1|\cos\theta\sqrt{\mathbf{p}_3^2 + m_3^2}|} = \frac{d|\mathbf{p}_3|}{\sqrt{|\mathbf{p}_3|^2 + m_3^2}\sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 - 2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta + m_4^2}}$$

Which we can plug into the integral (with $|\mathbf{p}_3|$ now implicitly a function of z):

$$\frac{d\sigma}{d\Omega} = \frac{S}{64\pi^2 |\mathbf{p}_1| m_2} \int |M|^2 \frac{\delta(E_1 + m_2 - z)|\mathbf{p}_3|^2 dz}{||\mathbf{p}_3| z - |\mathbf{p}_1|\cos\theta\sqrt{|\mathbf{p}_3|^2 + m_3^2}|}$$

Recalling that $\sqrt{|\mathbf{p}_3|^2 + m_3^2} = E_3$ we arrive at the expression for the differential cross section:

$$\frac{d\sigma}{d\Omega} = \frac{1}{(8\pi)^2}\frac{S|M|^2}{|\mathbf{p}_1|m_2}\frac{|\mathbf{p}_3|^2}{||\mathbf{p}_3|(E_1+m_2) - |\mathbf{p}_1|E_3\cos\theta|}$$

2. Now we can take the limit of $m_3 = m_1 = 0$ and $m_2 = m_4 = m$, so $|\mathbf{p}_1| = E_1$ and $|\mathbf{p}_3| = E_3$ and the expression simplifies to:

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{1}{(8\pi)^2}\frac{S|M|^2}{E_1 m}\frac{E_3^2}{|E_3(E_1+m) - E_1E_3\cos\theta|}\\ &= \frac{1}{(8\pi)^2}\frac{S|M|^2}{E_1 m}\frac{E_3}{|E_1(1-\cos\theta)+m|}\end{aligned}$$

To further simplify things, we can use some kinematic calculations:

$$\begin{aligned}& p_1 - p_3 = p_2 - p_4\\ \Rightarrow\quad & p_1^2 + p_3^2 - 2p_1\cdot p_3 = p_2^2 + p_4^2 - 2p_2\cdot p_4\\ \Rightarrow\quad & E_1E_3(1-\cos\theta) = p_2\cdot p_4 - m^2 = E_2E_4 - \mathbf{p_2}\cdot\mathbf{p_4} - m^2 = E_4 m - m^2\end{aligned}$$

Where in the last equality we used the fact that $\mathbf{p_2} = 0$ and $E_2 = m$ since we're in the lab frame. From energy conservation we can replace $(E_4 = E_1 + E_2 - E_3 = E_1 - E_3 + m)$ and so:

$$E_1E_3(1-\cos\theta) = m(E_4 - m) = m(E_1 - E_3 + m - m) = m(E_1 - E_3)$$

Which gives us the following identity (after grouping on one side all the terms multiplied by E_3:

$$\frac{1}{E_1(1-\cos\theta)+m} = \frac{E_3}{mE_1}$$

Plugging this in we arrive at the result:

$$\frac{d\sigma}{d\Omega} = \mathcal{S}\left(\frac{E_3}{8\pi m E_1}\right)^2|\mathcal{M}|^2$$

Question 2

From our definition of a particle lifetime, τ, we understand that at $t = \tau$ the probability that the particle decayed is e^{-1}.
Therefore we can use the average travel distance of the π^+:

$$\begin{aligned}d =& \gamma\beta c\tau = \frac{p}{m}c\tau\\ \rightarrow\quad p =& m\frac{d}{c\tau} = 140\ MeV\frac{50\cdot 10^3 m}{3\cdot 10^8\frac{m}{sec}\times 2.6\cdot 10^-8 sec} \approx 900\ GeV\end{aligned}$$

Question 3