Quantum Computation 101 for Physicists

Home exercise 6

1. (a) Show that if the number of solutions is $t = N/4$, then Grover's algorithm always finds a solution with probability 1 after just one query.

   (b) find all values of t for which we can find a solution with probability 1 (using any number of Grover iterations, as long as once we are done, we are guaranteed to measure a solution).

2. In most cases, we do not know the exact number of solutions. In this question you will show that even without knowing the exact number of solutions, the number of required Grover iterations is still $O(\sqrt{N})$. Assume we have an unknown number of solutions k. Assume k is small enough such that the optimal number of iterations is of order $\sqrt{N}$.

   (a) Show that if we define $k' = k/2$ and perform the optimal number for k' solutions rather than k, we can still measure a solution with probability close to $1/2$.

   (b) Denote by j the number of necessary repetitions of the algorithm if we perform $k/2$ iterations, such that we can find a solution with a very high probability. Note that j does not depend on N, and if we require the probability of success to be $1 - \epsilon$, then $j = O(\log_2(1/\epsilon))$, which is a very weak dependence.

   Show that the following protocol finds a solution with high probability in $O(\sqrt{N})$ Grover iterations:

   - define $k' = 1$
   - while a solution was not found, do:

     – repeat j times: perform k', measure, and test if the result is a solution.

     – Assign $k' = 2k'$.