QMII
Exercise 3

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Question 1

Ginen the transformation $U(\lambda) = e^{-i\lambda\frac{G}{\hbar}}$ and the generator $G = \frac{1}{2}(xp + px)$.

Part 1

Show how an infinitesimal transformation $U(\epsilon)$ transforms the position and momentum operators $x,\ p$.

Solution:

$$U(\epsilon) \approx 1 - i\epsilon\frac{G}{\hbar} = 1 - i\frac{\epsilon}{2\hbar}(xp + px).$$

$$\begin{aligned}
UxU^{\dagger} &\approx \left[1 - i\frac{\epsilon}{2\hbar}(xp + px)\right] x \left[1 - i\frac{\epsilon}{2\hbar}(xp + px)\right] \\
&= x - i\frac{\epsilon}{2\hbar}(xp + px)x + i\frac{\epsilon}{2\hbar}x(xp + px) \\
&= x + i\frac{\epsilon}{2\hbar}[x, xp + px] \\
&= x + i\frac{\epsilon}{2\hbar}[x, 2xp - i\hbar] \\
&= x + i\frac{\epsilon}{\hbar}[x, xp] \\
&= x + i\frac{\epsilon}{\hbar}x[x, p] \\
&= x + i\frac{\epsilon}{\hbar}xi\hbar \\
&= \boxed{(1 - \epsilon)x}.
\end{aligned}$$

$$
\begin{aligned}
UpU^\dagger &\approx \left[1 - i\frac{\epsilon}{2\hbar}(xp+px)\right] p \left[1 - i\frac{\epsilon}{2\hbar}(xp+px)\right] \\
&= x - i\frac{\epsilon}{2\hbar}(xp+px)\,p + i\frac{\epsilon}{2\hbar}p\,(xp+px) \\
&= x + i\frac{\epsilon}{2\hbar}[p, xp+px] \\
&= x + i\frac{\epsilon}{2\hbar}[p, 2xp - i\hbar] \\
&= x + i\frac{\epsilon}{\hbar}[p, xp] \\
&= x + i\frac{\epsilon}{\hbar}p\,[p, x] \\
&= x - i\frac{\epsilon}{\hbar}pi\hbar \\
&= \boxed{(1+\epsilon)\,p}.
\end{aligned}
$$

Part 2

Given the results of the previous part, show how a finite transformation transforms x, p.

Solution:
Dividing λ into N equal sections and letting $N \longrightarrow \infty$ we obtain:

$$
\begin{aligned}
x' &= \left(1 - \frac{\lambda}{N}\right)^N x \overset{(N\to\infty)}{=} e^{-\lambda}x \\
p' &= \left(1 + \frac{\lambda}{N}\right)^N p \overset{(N\to\infty)}{=} e^{\lambda}p.
\end{aligned}
$$

Part 3

Show what the transformation U does to $|x\rangle$.

Solution:
Firstly we'll note that U and x do not commute so we'll have to find these two expressions:

$$
Ux|x\rangle \quad ; \quad xU|x\rangle.
$$

$$
\begin{gathered}
Ux|x_0\rangle = Ux_0|x_0\rangle = x_0U|x_0\rangle \\
UxU^\dagger U|x_0\rangle = e^{-\lambda}xU|x_0\rangle \\
\Rightarrow Ux|x_0\rangle = x_0U|x_0\rangle = e^{-\lambda}xU|x_0\rangle \\
\Rightarrow xU|x_0\rangle = e^{\lambda}x_0U|x_0\rangle
\end{gathered}
$$

Which means that $U|x_0\rangle$ is an eigenket of x with eigenvalue $e^{\lambda}x_0$:

$$
U|x_0\rangle \propto |e^{\lambda}x_0\rangle.
$$

Now we only need to see if U pulls out a constant:

$$\delta\left(x-x'\right)=\langle x'|x\rangle=\langle x'|U^{\dagger}U|x\rangle=|A|^2\langle e^{\lambda}x'|e^{\lambda}x\rangle=|A|^2\delta\left(e^{\lambda}\left(x-x'\right)\right)=\frac{|A|^2}{e^{\lambda}}\delta\left(x-x'\right)$$
$$\Rightarrow\frac{|A|^2}{e^{\lambda}}=1$$
$$\Rightarrow A=e^{\frac{\lambda}{2}}$$
$$\Rightarrow\boxed{U|x\rangle=e^{\frac{\lambda}{2}}|e^{\lambda}x\rangle}$$

Question 2

Show that:

$$R_{z'}\left(\gamma\right)R_{y'}\left(\beta\right)R_{z}\left(\alpha\right)=R_{z}\left(\alpha\right)R_{y}\left(\beta\right)R_{z}\left(\gamma\right).$$

Solution:
Firstly we'll note that y' after the z rotation is:

$$R_{y'}\left(\beta\right)=R_{z}\left(\alpha\right)R_{y}\left(\beta\right)R_{z}^{-1}\left(\alpha\right).$$

And that z' after the y rotation is:

$$R_{z'}\left(\gamma\right)=R_{y'}\left(\beta\right)R_{z}\left(\gamma\right)R_{y'}^{-1}\left(\beta\right).$$

So:

$$R_{z'}\left(\gamma\right)R_{y'}\left(\beta\right)R_{z}\left(\alpha\right)=R_{y'}\left(\beta\right)R_{z}\left(\gamma\right)\cancel{R_{y'}^{-1}\left(\beta\right)R_{y'}\left(\beta\right)}R_{z}\left(\alpha\right).$$

If we raplace now $R_{y'}\left(\beta\right)$ andremember that rotations around the same axis commute:

$$R_{z'}\left(\gamma\right)R_{y'}\left(\beta\right)R_{z}\left(\alpha\right)=R_{z}\left(\alpha\right)R_{y}\left(\beta\right)\cancel{R_{z}^{-1}\left(\alpha\right)R_{z}\left(\alpha\right)}R_{z}\left(\gamma\right)=R_{z}\left(\alpha\right)R_{y}\left(\beta\right)R_{z}\left(\gamma\right)\checkmark.$$

Question 3

Given the folloing rotation in $j=\frac{1}{2}$:

$$\mathcal{D}\left(\hat{\boldsymbol{n}},\theta\right)=\frac{i}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}.$$

Find $\hat{\boldsymbol{n}}$ and θ.

Solution:

$$\mathcal{D}=\frac{i}{\sqrt{2}}\left(\sigma_x+\sigma_z\right)=i\boldsymbol{\sigma}\cdot\left(\frac{\hat{\boldsymbol{x}}+\hat{\boldsymbol{z}}}{\sqrt{2}}\right).$$
$$\Rightarrow\begin{cases}\cos\frac{\theta}{2}=0\\\sin\frac{\theta}{2}=-1\end{cases}\quad\Rightarrow\boxed{\hat{\boldsymbol{n}}=\frac{\hat{\boldsymbol{x}}+\hat{\boldsymbol{z}}}{\sqrt{2}},\ \theta=3\pi}.$$

Question 4

Use the identity for $j = 1$:

$$e^{-i\theta J_y/\hbar} = 1 - i\sin\theta\frac{J_y}{\hbar} + (\cos\theta - 1)\left(\frac{J_y}{\hbar}\right)^2.$$

To find the elements of $d^{(j=1)}_{m'm}$.

Solution:
Firstly we'll note that in $j = 1$:

$$\frac{L_y}{\hbar} = \frac{1}{i\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \Rightarrow \left(\frac{L_y}{\hbar}\right)^2 = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}.$$

Hence:

$$e^{-i\theta J_y/\hbar} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - i\sin\theta\frac{1}{i\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} + (\cos\theta - 1)\begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}.$$

$$\Rightarrow \boxed{d^{(j=1)}_{m'm} = \begin{pmatrix} \frac{1+\cos\theta}{2} & -\frac{\sin\theta}{\sqrt{2}} & \frac{1-\cos\theta}{2} \\ \frac{\sin\theta}{\sqrt{2}} & \cos\theta & -\frac{\sin\theta}{\sqrt{2}} \\ \frac{1-\cos\theta}{2} & \frac{\sin\theta}{\sqrt{2}} & \frac{1+\cos\theta}{2} \end{pmatrix}}.$$

Question 5

Given a state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

a) Find $|\psi'\rangle$ after a rotation of $\frac{\pi}{2}$ around the $\hat{z}$ axis.

b) Find $|\psi'\rangle$ after a rotation of $\frac{\pi}{2}$ around the $\hat{y}$ axis.

Solution:

a)

$$R_z\left(\frac{\pi}{2}\right) = e^{-i\frac{\pi}{2}J_z/\hbar} = \begin{pmatrix} e^{-i\frac{\pi}{2}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -e^{-i\frac{\pi}{2}} \end{pmatrix}.$$

$$\Rightarrow |\psi'\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\frac{\pi}{2}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -e^{-i\frac{\pi}{2}} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \boxed{\frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}}.$$

b)

$$R_y\left(\frac{\pi}{2}\right) = e^{-i\frac{\pi}{2}J_y/\hbar} = d^{j=1}_{m'm}\left(\theta = \frac{\pi}{2}\right) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}.$$

$$\Rightarrow |\psi'\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \boxed{\frac{1}{2}\begin{pmatrix} \frac{1}{\sqrt{2}} - 1 \\ 1 \\ \frac{1}{\sqrt{2}} + 1 \end{pmatrix}}.$$

Question 6

The dynamics of a particle with $j = 1$ is given by the hemiltonian:

$$\mathcal{H} = \epsilon \begin{pmatrix} 2 & \frac{1-i}{2} & 0 \\ \frac{1+i}{2} & 2 & \frac{1-i}{2} \\ 0 & \frac{1+i}{2} & 2 \end{pmatrix}.$$

a) Write $\mathcal{H}$ as a sum of the elements of $\boldsymbol{J}$ and J^2 i.e.:

$$\mathcal{H} = aJ^2 + b\boldsymbol{J} \cdot \hat{\boldsymbol{a}}.$$

b) Is $\mathcal{H}$ symmetrical under an arbitrary rotation?

c) With wich angle θ and around wich axis $\hat{\boldsymbol{n}}$, $\hat{\boldsymbol{a}}$ must be turned such that it will point in $\hat{\boldsymbol{z}}$?

d) What are the euler angles α, β, γ that corrospond to the rotation from the previous section?

e) Find the rotation matrix $\mathcal{D}(\hat{\boldsymbol{n}}, \theta)$.

f) Show that the rotation matrix diagonalize $\mathcal{H}$.

<u>Solution</u>:

a)

$$\mathcal{H} = \frac{\epsilon}{\hbar^2} J^2 + \frac{\epsilon}{\hbar} \boldsymbol{J} \cdot \left(\frac{\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}}{\sqrt{2}} \right).$$

$$\Rightarrow E_{lm} = \epsilon \left(j \left(j+1 \right) + m \right) = \left(2+m \right) \epsilon = \begin{cases} 3\epsilon & m = 1 \\ 2\epsilon & m = 0 \\ \epsilon & m = -1 \end{cases}.$$

b) No, the system is only symmetrical to rotations around the $\hat{\boldsymbol{a}} = \frac{\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}}{\sqrt{2}}$ axis.

c) Around $\hat{\boldsymbol{n}} = \frac{\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}}}{\sqrt{2}}$ with $\theta = \frac{\pi}{2}$.

d)

$$\gamma = -\frac{\pi}{4} \qquad \beta = -\frac{\pi}{2} \qquad \alpha = 0.$$

e) With $d^{(j=1)}_{m'm}$ wich we already calculated:

$$\begin{aligned} \mathcal{D} &= e^{i\frac{\pi}{2} J_y/\hbar} e^{i\frac{\pi}{4} J_z/\hbar} \\ &= \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} e^{i\frac{\pi}{4}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\frac{\pi}{4}} \end{pmatrix} \\ &= \boxed{\frac{1}{2} \begin{pmatrix} e^{i\frac{\pi}{4}} & \sqrt{2} & e^{-i\frac{\pi}{4}} \\ -\sqrt{2}e^{i\frac{\pi}{4}} & 0 & \sqrt{2}e^{-i\frac{\pi}{4}} \\ e^{i\frac{\pi}{4}} & -\sqrt{2} & e^{-i\frac{\pi}{4}} \end{pmatrix}}. \end{aligned}$$

f)

$$\mathcal{DHD}^\dagger = 2\epsilon\mathbb{I} + \frac{\epsilon}{4}\begin{pmatrix} e^{i\frac{\pi}{4}} & \sqrt{2} & e^{-i\frac{\pi}{4}} \\ -\sqrt{2}e^{i\frac{\pi}{4}} & 0 & \sqrt{2}e^{-i\frac{\pi}{4}} \\ e^{i\frac{\pi}{4}} & -\sqrt{2} & e^{-i\frac{\pi}{4}} \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{\sqrt{2}}e^{-i\frac{\pi}{4}} & 0 \\ \frac{1}{\sqrt{2}}e^{i\frac{\pi}{4}} & 0 & \frac{1}{\sqrt{2}}e^{-i\frac{\pi}{4}} \\ 0 & \frac{1}{\sqrt{2}}e^{i\frac{\pi}{4}} & 0 \end{pmatrix}\begin{pmatrix} e^{-i\frac{\pi}{4}} & -\sqrt{2}e^{-i\frac{\pi}{4}} & e^{-i\frac{\pi}{4}} \\ \sqrt{2} & 0 & -\sqrt{2} \\ e^{i\frac{\pi}{4}} & \sqrt{2}e^{i\frac{\pi}{4}} & e^{i\frac{\pi}{4}} \end{pmatrix}$$

$$= 2\epsilon\mathbb{I} + \frac{\epsilon}{4}\begin{pmatrix} e^{i\frac{\pi}{4}} & \sqrt{2} & e^{-i\frac{\pi}{4}} \\ -\sqrt{2}e^{i\frac{\pi}{4}} & 0 & \sqrt{2}e^{-i\frac{\pi}{4}} \\ e^{i\frac{\pi}{4}} & -\sqrt{2} & e^{-i\frac{\pi}{4}} \end{pmatrix}\begin{pmatrix} e^{-i\frac{\pi}{4}} & 0 & -e^{-i\frac{\pi}{4}} \\ \sqrt{2} & 0 & \sqrt{2} \\ e^{i\frac{\pi}{4}} & 0 & -e^{i\frac{\pi}{4}} \end{pmatrix}$$

$$= 2\epsilon\mathbb{I} + \frac{\epsilon}{4}\begin{pmatrix} 4 & & \\ & 0 & \\ & & -4 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 3\epsilon & & \\ & 2\epsilon & \\ & & \epsilon \end{pmatrix}} \checkmark .$$