Q.M. II
Exercise 2

Alon Ner Gaon

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Question 1

Use parity qualities and calculate the following for the harmonic osilator:

$$\langle n|x|n\rangle \tag{1}$$
$$\langle n|p^2|n+1\rangle \tag{2}$$
$$\langle n|xpx|n\rangle \tag{3}$$

Solution:
Firstly, the eign states of the harmonic osilator $|n\rangle$ have a well defined parity.
Hence in (1) we have x wich is an anti-simetrical operator sandwiched by the same parity so

$$\langle n|x|n\rangle = 0.$$

In (2) we have a simetrical oparator p^2 sandwiched by an inverse parity, so again we have:

$$\langle n|p^2|n+1\rangle = 0.$$

In (3) we again have an anti-simetrical operator sandwiched by the same parity so

$$\langle n|xpx|n\rangle = 0.$$

Question 2.1

Given the space inversion operator Π :

$$\Pi : \vec{r} \to -\vec{r}.$$

Show how the spherical harmonics transmute under the space inversion:

$$\Pi Y_l^m(\theta,\phi) = ?$$

Solution:
The space inversion in spherical coordinates:

$$\begin{aligned}
&r \to r \\
&\theta \to \pi - \theta && (\cos\theta \to -\cos\theta) \\
&\phi \to \phi + \pi && \left(e^{im\phi} \to (-1)^m e^{im\phi}\right).
\end{aligned}$$

So after the space inversion the spherical harmonics become:

$$Y_l^m \propto (-1)^m P_l^m(-\cos\theta) e^{im\phi}.$$

And the parity of the assosiated legandre polynomials

$$P_l^m(-x) = (-1)^{l+m} P_l^m(x).$$

Finally we can conclude:

$$\Pi Y_l^m = (-1)^l Y_l^m.$$

So the parity of $|lm\rangle$ is defined only by the quantom number l, which means **$|lm\rangle$'s parity is degenarate $(2l+1)$ times**.

Question 2.2

Use the parity qualities of $|nlm\rangle$ to determine:

$$\langle nlm|x|nlm\rangle \qquad \langle nlm|\vec{r}|n,l+2,m\rangle \quad \langle nlm|r^2|n,l+1,m\rangle.$$

Solution:
As derived in 2.1: $|lm\rangle$'s parity is defined solely on l.
x is asimetrical, which means $\vec{r}$ is asimetrical, and r^2 is simetrical. Hence all of the above are 0.

Question 3

Let T be an hermitian generator of an arbitrary transformation, and:

$$[H,T] = 0.$$

Show that H is diagonal in the eignbase of T.

Solution:
We'll define the eignkets of T:

$$T|t\rangle = t|t\rangle.$$

And examine:

$$\begin{aligned}\langle t'|\,[H,T]\,|t\rangle &= \langle t'|HT - TH|t\rangle\\ &= \langle t'|HT|t\rangle - \langle t'|TH|t\rangle\\ &= \langle t'|Ht|t\rangle - \langle t'|t'H|t\rangle\\ 0 &= (t-t')\,\langle t'|H|t\rangle\end{aligned}$$

So for $t \neq t'$, $\langle t'|H|t\rangle$ must be equal to 0. And for $t' = t$, $\langle t'|H|t\rangle$ can take any value. □

Question 4

Show how the following operators transform under Space Inversion Π and under Time Reversal Θ:

$$\boldsymbol{S} \cdot \boldsymbol{p} \tag{4}$$
$$\boldsymbol{S} \cdot \boldsymbol{r} \tag{5}$$
$$\boldsymbol{S} \cdot \boldsymbol{L} \tag{6}$$
$$\boldsymbol{S} \cdot \boldsymbol{S}. \tag{7}$$

Solution:

Firstly we'll note that under Π:

$$\begin{aligned} r &\longrightarrow -r \\ p &\longrightarrow -p \\ S &\longrightarrow S \\ L &\longrightarrow L. \end{aligned}$$

And under Θ :

$$\begin{aligned} r &\longrightarrow r \\ p &\longrightarrow -p \\ S &\longrightarrow -S \\ L &\longrightarrow -L. \end{aligned}$$

$$\begin{aligned}
(4) \longrightarrow \Pi\boldsymbol{S} \cdot \boldsymbol{p}\Pi^{-1} &= -\boldsymbol{S} \cdot \boldsymbol{p} \\
(5) \longrightarrow \Pi\boldsymbol{S} \cdot \boldsymbol{r}\Pi^{-1} &= -\boldsymbol{S} \cdot \boldsymbol{r} \\
(6) \longrightarrow \Pi\boldsymbol{S} \cdot \boldsymbol{L}\Pi^{-1} &= \boldsymbol{S} \cdot \boldsymbol{L} \\
(7) \longrightarrow \Pi\boldsymbol{S} \cdot \boldsymbol{S}\Pi^{-1} &= \boldsymbol{S} \cdot \boldsymbol{S} \\
(4) \longrightarrow \Theta\boldsymbol{S} \cdot \boldsymbol{p}\Theta^{-1} &= \boldsymbol{S} \cdot \boldsymbol{p} \\
(5) \longrightarrow \Theta\boldsymbol{S} \cdot \boldsymbol{r}\Theta^{-1} &= -\boldsymbol{S} \cdot \boldsymbol{r} \\
(6) \longrightarrow \Theta\boldsymbol{S} \cdot \boldsymbol{L}\Theta^{-1} &= \boldsymbol{S} \cdot \boldsymbol{L} \\
(7) \longrightarrow \Theta\boldsymbol{S} \cdot \boldsymbol{S}\Theta^{-1} &= \boldsymbol{S} \cdot \boldsymbol{S}.
\end{aligned}$$

Question 5

Given a spinless undegenerate system which upholds:

$$[H, \Theta] = 0.$$

Prove that it is posible to choose the eignkets of the system to be real in the Position-Space.

Solution:
The undegeneracy means:

$$H|\psi\rangle = E_n|\psi\rangle.$$

Where E_n are uniqe.
Let us reverse the time:

$$\Theta H|\psi\rangle = E_n\Theta|\psi\rangle.$$

And use the commutation relation:

$$H\Theta|\psi\rangle = E_n\Theta|\psi\rangle.$$

Firstly we'll note that $|\psi\rangle = \Theta|\psi\rangle$ for E_n are uniqe.
Now let us recall that in the Position-Space the eignkets and the eignkets after time reversal are $\langle x|\psi\rangle$ and $\langle x|\psi\rangle^*$ respectivly. Hence:

$$\langle x|\psi\rangle = \langle x|\psi\rangle^*.$$

Question 6

Partical with spin $s = \frac{1}{2}$ is under the influence of a magnetic field:

$$\boldsymbol{B}(t) = B_\perp\left[\cos(\omega t)\,\hat{\boldsymbol{x}} + \sin(\omega t)\,\hat{\boldsymbol{y}}\right] + B_0\hat{\boldsymbol{z}}.$$

Parts 1+2

Write the hemiltonian in the base which diagonalize S_z using the pauli matrices.

Solution:

$$\begin{aligned}
H &= -\vec{M}\cdot\vec{B} \\
&= \frac{e}{2mc}\vec{S}\cdot\vec{B} \\
&= \frac{e\hbar}{2mc}\frac{\vec{S}}{\hbar}\cdot\vec{B} \\
&\approx \frac{1}{2}g\mu_B\vec{\sigma}\cdot\vec{B} \\
&= \frac{1}{2}g\mu_B\left[B_\perp\sigma_x\cos(\omega t) + B_\perp\sigma_y\sin(\omega t) + B_0\sigma_z\right] = \begin{pmatrix} B_0 & B_\perp e^{-i\omega t} \\ B_\perp e^{i\omega t} & B_0 \end{pmatrix} \\
&= \frac{1}{4}g\mu_B\left[B_\perp\sigma_x\left(e^{i\omega t} + e^{-i\omega t}\right) - iB_\perp\sigma_y\left(e^{i\omega t} - e^{-i\omega t}\right) + B_0\sigma_z\right] \\
&= \frac{1}{4}g\mu_B\left[B_\perp e^{i\omega t}\underbrace{\left(\sigma_x - i\sigma_y\right)}_{\propto S_-} + B_\perp e^{-i\omega t}\underbrace{\left(\sigma_x + i\sigma_y\right)}_{\propto S_+} + B_0\sigma_z\right].
\end{aligned}$$

Part 3

Let U be a unitary transformation $|\bar{\psi}\rangle = U|\psi\rangle$.
Show that $\mathcal{H}$ must transform as such:

$$\bar{\mathcal{H}} = U\mathcal{H}U^{\dagger} + i\hbar\frac{\partial U}{\partial t}U^{\dagger}.$$

To uphold the schrodiger equation:

$$\bar{\mathcal{H}}|\bar{\psi}\rangle = i\hbar\frac{\partial}{\partial t}|\bar{\psi}\rangle.$$

Solution:
By transforming the RHS:

$$\begin{aligned}
i\hbar\frac{\partial}{\partial t}|\bar{\psi}\rangle = i\hbar\frac{\partial}{\partial t}U|\psi\rangle &= i\hbar\left(\frac{\partial U}{\partial t}|\psi\rangle + U\frac{\partial|\psi\rangle}{\partial t}\right) \\
&= i\hbar\left(\frac{\partial U}{\partial t}U^{\dagger}U|\psi\rangle + U\left(\frac{1}{i\hbar}\mathcal{H}|\psi\rangle\right)\right) \\
&= i\hbar\left(\frac{\partial U}{\partial t}U^{\dagger}U|\psi\rangle + \frac{1}{i\hbar}U\left(\mathcal{H}U^{\dagger}U|\psi\rangle\right)\right) \\
&= \underbrace{\left(i\hbar\frac{\partial U}{\partial t}U^{\dagger} + U\mathcal{H}U^{\dagger}\right)}_{\bar{\mathcal{H}}}\underbrace{U|\psi\rangle}_{|\bar{\psi}\rangle}
\end{aligned}$$

Part 4

Let U the transformation be:

$$U = e^{i\frac{S_z}{\hbar}\Omega_0 t}.$$

Write U in the diagonalizing base of S_z and find $\bar{\mathcal{H}}$.
What Ω_0 must be so $\bar{\mathcal{H}}$ will be time independent?

Solution:
U is comprised solely from S_z hence:

$$U = \begin{pmatrix} e^{i\frac{\Omega_0}{2}t} & 0 \\ 0 & e^{-i\frac{\Omega_0}{2}t} \end{pmatrix}.$$

And due to the previeus part:

$$\begin{aligned}
\bar{\mathcal{H}} &\propto \begin{pmatrix} e^{i\frac{\Omega_0}{2}t} & 0 \\ 0 & e^{-i\frac{\Omega_0}{2}t} \end{pmatrix}\begin{pmatrix} B_0 & B_\perp e^{-i\omega t} \\ B_\perp e^{i\omega t} & B_0 \end{pmatrix}\begin{pmatrix} e^{-i\frac{\Omega_0}{2}t} & 0 \\ 0 & e^{i\frac{\Omega_0}{2}t} \end{pmatrix} + \begin{pmatrix} -\frac{\Omega_0}{2}\hbar & 0 \\ 0 & \frac{\Omega_0}{2}\hbar \end{pmatrix} \\
&= \begin{pmatrix} B_0 - \frac{\Omega_0}{2}\hbar & B_\perp e^{i(\Omega_0-\omega)t} \\ B_\perp e^{i(\omega-\Omega_0)t} & -B_0 + \frac{\Omega_0}{2}\hbar \end{pmatrix}.
\end{aligned}$$

So for $\bar{\mathcal{H}}$ to be time independent $\Omega_0 = \omega$.

From here on out I wo'nt show all the calculations, writing everything in LaTeX is too much for me right now.

Part 5

Find the eigenkets and eigen energies of $\bar{\mathcal{H}}$

Solution:
Because U is just a transformation of our choise we'll pick $\Omega_0 = \omega$ such that $\bar{\mathcal{H}}$ will be time independent:

$$\bar{\mathcal{H}} = \begin{pmatrix} B_0 - \frac{\Omega_0}{2}\hbar & B_\perp \\ B_\perp & -B_0 + \frac{\Omega_0}{2}\hbar \end{pmatrix}.$$

We'll define a modified magnetic field $B' = B_0 - \frac{\Omega_0}{2}\hbar$ and find the eigen-energies:

$$E_\pm = \pm\sqrt{B'^2 + B_\perp^2}.$$

And define:

$$E \equiv |E_\pm| = \sqrt{B'^2 + B_\perp^2}.$$

And the normelized eigenkets:

$$|+\rangle = \frac{1}{\sqrt{2E\left(E + B'\right)}}\begin{pmatrix} B' + E \\ B_\perp \end{pmatrix}$$
$$|-\rangle = \frac{1}{\sqrt{2E\left(E - B'\right)}}\begin{pmatrix} B' - E \\ B_\perp \end{pmatrix}.$$

Part 6

In $t = 0$ the state of the system $|\psi(0)\rangle = |\uparrow\rangle$.
What's the probability to find the system in the state $|\downarrow\rangle$ after time t?

Solution:

$$\begin{aligned}
|\psi(0)\rangle = |\uparrow\rangle &= \langle +|\uparrow\rangle|+\rangle + \langle -|\uparrow\rangle|-\rangle \\
&= \langle\uparrow|+\rangle|+\rangle + \langle\uparrow|-\rangle|-\rangle \\
&\Downarrow \\
|\psi(t)\rangle &= \langle\uparrow|+\rangle e^{-i\frac{E}{\hbar}t}|+\rangle + \langle\uparrow|-\rangle e^{i\frac{E}{\hbar}t}|-\rangle \\
\langle\downarrow|\psi(t)\rangle &= \langle\uparrow|+\rangle\langle\downarrow|+\rangle e^{-i\frac{E}{\hbar}t} + \langle\uparrow|-\rangle\langle\downarrow|-\rangle e^{i\frac{E}{\hbar}t} \\
&= \frac{B_\perp}{2E}e^{-i\frac{E}{\hbar}t} - \frac{B_\perp}{2E}e^{-i\frac{E}{\hbar}t} \\
P_t(|\downarrow\rangle) = |\langle\downarrow|\psi(t)\rangle|^2 &= \boxed{\frac{B_\perp^2}{4E^2}\sin^2\left(\frac{E}{\hbar}t\right)}.
\end{aligned}$$