QM-1 Selected Problems for the Exam Feb 2015

The exam will be consist of two problems composed of questions in the following list.

You are advised to first form an understanding of how each problem below should be solved (the flowchart of the solution). Then work out the most important details (at least such which are not obvious to you) paying attention to subtle points.

1. a. Find the density matrix of a spin 1 state "pointing" along $\phi = 0$ and arbitrary θ direction. First write it for a pure state. Then repeat for a (mixed) state having probability w to have the spin along this direction. Calculate the entropy of this state. In the spin 1/2 the value $w = 1/2$ would be special. In what? Are there similar values of w here? What are the conditions to get a completely mixed state?

   b. Consider two othonormal states $|\psi_1>$ and $|\psi_2>$. In the density matrix $\rho = w_1|\psi_1><\psi_1| + w_2|\psi_2><\psi_2|$ what is the probability to find a system in the state $|\psi_1>$? What would be your answer if $|\psi_1>$ and $|\psi_2>$ were not orthogonal?

   c. Consider a linear combination $|\psi>= a|\psi_1> +b|\psi_2>$ and compare it with the density matrix ρ above with $w_1 = |a|^2$ and $w_2 = |b|^2$. Do they describe the same state? What are the similarities? The differences? Explain in as many details as you can.

   c. Think roughly of "coherence" as sensitivity to phases. In this understanding which are the above two states? Are they totally coherent? Are they totally incoherent? Argument your answer.

   d. In a general $N \times N$ density matrix one often calls the off diagonal matrix elements ρ_{ij} $(i \neq j)$ by names "coherences". Does this mean that when all coherences vanish one has an incoherent state? When in your understanding a density matrix describes a totally incoherent state?

   e. Give arguments for why one needs a bilocal $\rho(x, x')$ and can not get away with just $\rho(x, x)$.

2. Consider an electron moving in a two dimensional (x,y) plane and placed in a uniform magnetic field B which is directed perpendicu-

lar to the plane. In addition a potential $U(x)$ (which depends only on x) is acting on the electron.

a) What are the symmetries of this problem? What do they imply about possible degeneracies of the energy eigenstates? In the absence of $U(x)$ one could plot energy levels vs a continuum conserved quantum number. Will this be true with non zero $U(x)$? What will change in comparison with the original Landau levels?

b) Consider $U(x)$ in the form of a step

$$U(x) = 0 \quad \text{for} \quad x < 0\,; \quad U(x) = U_0 \quad \text{for} \quad x > 0$$

with a constant U_0. Use the symmetry considerations and reduce the Schrodinger equation to a one dimensional form. IDENTIFY THE POTENTIAL OF THIS PROBLEM and on what it depends and plot the most representative cases. On this basis using simple arguments and neglecting tunneling effects plot the qualitative behavior of the modified Landau levels in this case for the limit when U_0 is much larger than $\hbar\omega_c$. How would your considerations change in the opposite limit?

c) Try to qualitatively predict what will the tunneling change in this plot. This will correct possible subtle mistakes you have made at level crossings in the previous paragraph. How do you understand this in view of our discussion in class of the general level crossing problems?

d) Plot qualitatively how the classical orbits of various types will look like in this problem. On this basis try to predict the current distribution in different states of the modified Landau levels. Give examples of calculations supporting your predictions.

e) Assume you have an electron in one of the states far left of $x = 0$ and U_0 is much larger than $\hbar\omega_c$. How would you make the electron move towards the barrier at $x = 0$ and then over it to the far right. Hint: think of applying a uniform electric field to this electron and discuss what will it do working in the time dependent gauge and using adiabatic arguments (consult me or the tutor if not clear what these arguments are).

f) Assume (as it happens in realistic cases) you have electrons (as fermions) occupying all the states below chemical potential which is in the middle between the lowest Landau level and the next as they are to the far left of the barrier. Neglect the electron-electron interaction. Apply what you did in the previous paragraph to this distribution. This should lead to a Hall effect. Why?

g)(For extra credit) Using simple arguments calculate the Hall conductivity

3. In this problem you are asked to discuss quantum states of a two dimensional electron in two dimensional "trap" subject to a perpendicular magnetic field.

   Start with pp. 2) and 3) of the homework no. 2. Add to the well $U(x)$ also a well $V(y)$ in the y direction, i.e have the electron totally confined in a finite area.

   a) Assume that $U(x)$ and $V(y)$ are infinite potential wells. To have some physical intuition start by discussing (qualitatively) what are possible classical solutions in the resulting rectangular well $U(x) + V(y)$ and what you should expect "quantum mechanics will do to them". Start by easy cases of the relation between the parameters of the trap and magnetic field. Then try to go to harder cases.

   b) Now switch to quantum mech. Use your understanding (from your homework and tutorial) of the solutions in the absence of $V(y)$ to find what will be the effect of adding $V(y)$. Conduct this discussion by considering mixing the "unperturbed" states by the added $V(y)$. Your task will be easier if you assume that the magnetic field is very strong. How do your results fit into your classical intuition.

   c) Now discuss an exactly solvable example of this problem by taking

$$U(x) + V(y) = \frac{m}{2}(\omega_x^2 x^2 + \omega_y^2 y^2)$$

   What are your expectations of the solution on the basis of classical mechanics.

   d) Now to quantum mech. Start with $\omega_x = \omega_y$. Such a potential is symmetric. What is the corresponding conserved quantity (in the presence of the magnetic field)? Choose appropriate gauge so that this symmetry can be easily exploited and variables separated. What are the qualitative solutions of the problem after the variable separation. Find a simple way of conducting this discussion. How do these considerations fit your classical intuition?

e) Assume AB flux along the z-axis added at the origin. How will it change you results?

f) (For X credit). Find exact solutions of the above problem. First do it for the symmetric case $\omega_x = \omega_y$. Then for XX credit - for the general $\omega_x \neq \omega_y$ case.

4. Consider a classical field $\phi(x)$ in one space dim. which is described by the Hamiltonian

$$H = \int_{-\infty}^{\infty} dx \left[\frac{1}{2}\pi^2(x) + \frac{1}{2}(\partial_x\phi(x))^2 + U(\phi)\right]$$

where $\pi(x)$ are the canonical momenta, $U(\phi) = \frac{1}{2}m^2\phi^2(x) + \frac{1}{4}\lambda\phi^4(x)$ and m and λ are constants. In answering the following questions consider first the $\lambda = 0$ limit. Then for $\lambda \neq 0$ consider separately positive and negative values of m^2.

a) What is the equation of motion for ϕ? What are its simple (space independent) static solutions? How are they related to the stationary points of $U(\phi)$? What is the lowest energy solution? How many such solutions are there? Represent your answers graphically on the plane (ϕ, x) and view the potential $U(\phi)$ as plotted on the third axis perpendicular to this plane.

b) Expand around stable classical minima of the theory retaining only quadratic terms. Are there linear terms? Explain your answer? Now use your result to find eigenmodes of the small harmonic vibrations around these minima choosing periodic boundary conditions. Looking at the eigenmodes quantum mechanically what kind of particles do they represent, what is their energy-momentum relation, mass, etc? (Hint : energies and momenta are related to the Hamiltonian and the field momentum operators)

Note : the non linear terms (i.e. terms higher than quadratic in the expansion of H around stable classical minima) are typically discussed using perturbation theory. If written in terms of operators a_k and $a_k^\dagger$ of creation and annihilation of the particles defined in the quadratic part these terms describe various scattering processes of these particles.

c)For extra credit: Discuss qualitatively how does the wave functional of the ground state (the "vacuum") look for such a problem.

5. In this problem you are asked to make a connection between the Hilbert space of photon states and the real space.

   a) Construct a one photon state which at some moment of time is localized at a point $\mathbf{r} = \mathbf{r}_0$. (hint - make use of one photon states with defined momentum). Note that you actually have a freedom of defining what "localized" means here. Consider using for this purpose expectation values of electric or magnetic fields or the EM energy density. After making your choice consider how does this state develop in time? How would you make it move in a certain direction?

   b) What is a quantum mechanical description of a radio wave with a given wavelength?

   c) Of a pulse of WiFi waves representing one bit, i.e. approximately a rectangular pulse of small duration. No real need for solving the problem but for your curiosity you may estimate this duration based on say 100 Mbps data transmission speed.

6. a) Discuss the deexcitation by photon emission of a spherical rotor in its first excited state. What is the angular distribution of the emitted photon? What is its polarization? What is the total deexcitation rate? Formulate and answer these questions in a proper way. How will your answers change if all you know is that the initial state of the rotor had the energy $\hbar^2/I$?

   Note : The Hamiltonian of the spherical rotor is $H = \mathbf{L}^2/2I$ where I is the moment of inertia of the rotor. Its interaction with EM field is $V = -\mathbf{d} \cdot \mathbf{E}$ where $\mathbf{E}(\mathbf{r})$ is the electric field and $\mathbf{d}$ is the vector of the dipole moment which reflects the (fixed) distribution of charges inside the rotor. It has a constant length d and let us assume it is directed along the rotor axis.

7. Consider a charged pendulum (far from the Earth) in a static electric field $\mathbf{E}_0$. You can view charged pendulum as charge q with mass m

attached to a massless rigid rod of fixed length.

Discuss the emission of radiation by such a pendulum from an excited state.

- First understand the QM description of the degrees of freedom of the pendulum. Don't solve yet just understand what you need to solve in order to find the eigenstates and what is their form and qualitative behavior.
- Consider and write the general expression for the probability of emission in terms of the states of the pendulum which is valid irrespective of the parameters of the pendulum.
- Then consider suitable limits to simplify the calculations (long wavelength, small amplitude oscillations of the pendulum, etc).
- What is the angular distribution of the emitted radiation? Does it depend on the polarization? What would you tell experimentalist to expect if his photon detector does not distinguish between polarizations? Note - your answers depend strongly on the choice of the initial and final states. Start with simplest case doing all possible reasonable approximations. (for X credit) Discuss other pairs of states.

8. Consider the following simplified model of spinless fermions placed in a potential well and interacting via a very long range interaction

$$H = \sum_{i,j=1}^{\infty} h_{ij} a_i^+ a_j + \frac{V_0}{2} (\hat{N} - N_0)^2 \, .$$

Here h_{ij} is a single particle hamiltonian of the well. It is a hermitian matrix with known eigenvalues ε_n and eigenfunctions $\psi_n(i)$. The operator $\hat{N}$ is $\hat{N} = \sum_{j=1}^{\infty} a_j^+ a_j$ and N_0 is a fixed integer.

a) Explain how the above Hamiltonian can possibly be an approximation to a 2nd quantized Hamiltonians you saw in lectures. Find the ground state of the model for a fixed number M of electrons.

b) Derive and solve the Heisenberg equations of motion for the operators a_i and a_i^+. Use the solutions to find the excitation energies and the wave functions of systems with respectively M+1 and M-1 particles relative to energy of the M particle system.

c) Repeat for particle-hole operators $a_k^+ a_i$ and find the excitations of the M particle system.

d) What are the Hartree-Fock equations for this problem? Compare their solutions with the exact results found above. In particular pay discuss the relation between the Hartree-Fock energies as appear on the RHS of the equations and the various exact excitations energies found above.

9. a)Find eigenfunctions and eigenvalues of the boson operator $\psi_{op}(\mathbf{r})$. Can you do the same for $\psi_{op}^+(\mathbf{r})$? For the fermionic field operator?

   b) Return back to the case of the eigenstate of the bosonic $\psi_{op}(\mathbf{r})$ and calculate the average of H_{op} (with the two body interaction $V(\mathbf{r}-\mathbf{r}'))$ in this state. What is the average of N_{op}?

   Help : to facilitate the understanding of how to deal with the first question write momentarily $\psi_{op}(\mathbf{r})$ in terms of an expansion in some complete set. But then reformulate your answer for the "unexpanded" $\psi_{op}(\mathbf{r})$, i.e. give it with no relation to the set into which you have expanded. (it was anyways arbitrary, wasn't it?).

10. Describe differences and similarities between second quantization of the harmonic oscillator and its description in terms of a and a^+ operators, i.e. $a = p/(\sqrt{2m\hbar\omega}) - i(\sqrt{m\omega/2\hbar})x$, etc. Provide explanations and give examples.

11. Consider the following Hamiltonian

$$H = \int \psi^+(x)\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)\psi(x)\;dx + \int [\psi^+(x)+\psi(x)]V(x-y)\mu_z(y)\;dxdy +$$

$$+\int \mu_x(y)\;dy$$

where $\psi(x)$ and $\psi^+(x)$ are boson operators,

$$\mu_{x,z}(y) = \frac{1}{2}\sum_{\alpha,\beta=1}^{2} \xi_\alpha^+(y)\sigma_{\alpha\beta}^{x,z}\xi_\beta(y)\;,$$

$\xi_\alpha(y)$ and $\xi^+_\beta(y)$ are fermion operators, $\alpha,\ \beta = 1, 2$, $\sigma^z_{\alpha\beta}$ – is the Pauli matrix and $V(x-y) = \lambda\delta(x-y)$.

a) Use variational approximation with factorized wave function

$$|\Psi >= |\Psi_{\text{bosons}} > |\Psi_{\text{fermions}} >$$

for a ground state of H with one fermion and arbitrary number of bosons. (Hint: a) use your experience with a similarly looking homework problem b) good notation will take you far c) you may find it a bit easier to do some of the calculations in momentum representation) What is the meaning of the infinite energy term which you obtain in your solution?

b) Repeat a) for several fermions.

12. Consider the Hamiltonian

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2M} + \frac{k(x^2+y^2)}{2} + \alpha xy \ \ , \ \ |\alpha| < k$$

and assume that $M \gg m$. Find the energy levels and the corresponding wavefunctions using the Born-Oppenheimer approximation.

This problem has an exact solution. Find it and compare with the BO result.

13. Consider the relation of the Bogolibov spectrum (linear in k for small k) and superfluidity (i.e. the absence of a friction with walls of a moving superfluid when its velocity is smaller then some critical value). For this purpose model the superfluid by mean field bose condensate described by the Gross Pitaevsky equation and rather than considering a moving superfluid discuss the uniform motion of a small body (a small ball) through the condensate. Model the interaction of the moving body with the superfluid by an external potential $U(\mathbf{r}, t) = U(\mathbf{r} - \mathbf{r}_0 - \mathrm{v}t)$ where v is the velocity of the body and $U(\mathbf{r})$ tends to zero at large $|\mathbf{r}|$.

a) Consider the time dependent GP equation in the presence of $U(\mathbf{r}, t)$. Rewrite it in terms of ρ and χ in the polar representation $\psi = \sqrt{\rho}\exp(\chi)$.

b) Start with the condensate solution in the absence of U. Assume that the effect of $U(\mathbf{r}, t)$ produces small deviations $\delta\rho(\mathbf{r}, t)$ and $\delta\chi(\mathbf{r}, t)$

from the ρ_0 and χ_0 of the condensate. Write (linearized) equations for these small deviations.

c) Solve these equations using the Fourier transforms making the following simplifications - work in the long wave length limit i.e. keeping the lowest non trivial order in k (you can later include the neglected terms) and take for $U(\mathbf{r}, t)$ the simplest delta function shape $U(\mathbf{r}) = U_0\delta(\mathbf{r} - \mathbf{r}_0 - \mathrm{v}t)$.

Start by solving for the static body with $\mathrm{v} = 0$. Then consider what happens with this solution as the body moves. Show that for small velocities the moving body does not create a wake, i.e. does not leave behind ("emits")waves carrying energy. Pay attention to the analogy with Lorenz contraction of the EM field of a uniformly moving electric charge.

d) (For extra credit) Show that there is a critical velocity and find what happens above it. Pay attention to the analogy with the so called Cherenkov radiation or a supersonic boom of an airplane

(cf. `http://www.physics.upenn.edu/balloon/cerenkov_radiation.html`, etc)

14. An asymmetric double well changes in time in such a way that its lower minimum goes up while its higher minimum goes down so that their relative heights interchange (will be drawn on a blackboard in class at the beginning of the exam).

    Assume that the change is very slow. What will happen with a particle which was originally at the bottom of the lowest well. Describe the time development of the system. Do it qualitatively but with formulae, diagrams, etc, so that if needed all could be calculated. What quantitatively "very slow" means in this case? What will happen if the change is fast?