Intro. to Solid state
Exercise 2

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Question 1

Given that:

$$\bar{J} = n_e q_e \bar{v}_e + n_h q_h \bar{v}_h.$$

What is the electrical conductivity σ under a DC field?

Solution:

Drude equation of motion:

$$\frac{d}{dt}\bar{p} = -\frac{\bar{p}}{t} + q\bar{E}.$$

So for every charge carier:

$$\bar{p}_i = q_i \tau \bar{E}.$$

And if we multiply by the correct units:

$$n_i q_i \bar{v}_i = \frac{n_i q_i^2 \tau}{m_i}\bar{E}.$$

We get exactly the equation for the flow $\bar{J}$:

$$\bar{J} = \frac{n_i q_i^2 \tau}{m_i}\bar{E}.$$

And if we add up for the two charge cariers:

$$\bar{J} = \left(\frac{n_e q_e^2 \tau_e}{m_e} + \frac{n_h q_h^2 \tau_h}{m_h}\right)\bar{E}.$$

We get:

$$\sigma = \frac{n_e q_e^2 \tau_e}{m_e} + \frac{n_h q_h^2 \tau_h}{m_h}.$$

As expected if there's no interaction between the electrons and the holes the total conductivity is the sum of the two independet conductivities.

Question 2

Show that:

$$\bar{J} = \frac{e\tau}{m} k_B T \nabla n.$$

Solution:

$$J_x = -\frac{e}{2}\left[n\left(x - v_x\tau\right)v_x - n\left(x + v_x\tau\right)v_x\right]$$
$$J_x \approx -ev_x^2\tau\frac{d}{dx}n = -e\tau\frac{1}{3}\langle v^2\rangle\frac{d}{dx}n = -e\tau\frac{2}{3m}\langle\epsilon\rangle\frac{d}{dx}n$$
.

Every particle has $\frac{3}{2}k_B T$ from the Equipartition Theorem:

$$\bar{J} - \frac{e\tau}{m} k_B T \bar{\nabla} n.$$

Question 3.1

Given:

$$\bar{H} = H\hat{z}$$
$$\bar{E} = \bar{E}\left(\omega\right)e^{-i\omega t}.$$

And:

$$E_y = iE_x \qquad E_z = 0.$$

Show that:

$$\tau\frac{\mathrm{d}}{\mathrm{d}t}\bar{J}\left(t\right) = -\bar{J}\left(t\right) + \sigma_D\bar{E}\left(t\right) - \omega_c\tau\left(\bar{J}\left(t\right)\times\hat{z}\right).$$

Solution:

Drude equation of motion of a particle in an EM field:

$$\frac{d}{dt}\bar{p} = -\frac{\bar{p}}{\tau} - \frac{e}{c}\left[\bar{E} + \bar{v}\times\bar{H}\right].$$

And with units of the flow $\bar{J}$:

$$\frac{n\tau e}{m}\frac{d}{dt}\bar{p} = -ne\frac{\bar{p}}{\tau} - \frac{ne^2\tau}{m}\left[\bar{E} + \frac{1}{c}\bar{v}\times\bar{H}\right]$$
$$\Rightarrow -\tau\frac{\mathrm{d}}{\mathrm{d}t}\bar{J} = \bar{J} - \sigma_D\bar{E} - \underbrace{\frac{ne^2\tau}{mc}\bar{v}\times\bar{H}}_{\omega_c\tau\left(\bar{J}\times\hat{z}\right)}$$
$$\Rightarrow \tau\frac{\mathrm{d}}{\mathrm{d}t}\bar{J} = -\bar{J} + \sigma_D\bar{E} - \omega_c\tau\left(\bar{J}\times\hat{z}\right). \qquad (1)$$

Question 3.2

With a solution of the form:

$$\vec{J}(t) = \vec{J}(\omega) e^{-i\omega t}.$$

Show that:

$$\vec{J} = \frac{\sigma_D}{1 - i(\omega - \omega_c)\tau} \vec{E}.$$

Solution:
Placing the solution in (1) we obtain:

$$-\tau i\omega \vec{J}(\omega) = -\vec{J}(\omega) + \sigma_D \vec{E}(\omega) - \omega_c \tau \left(\vec{J}(\omega) \times \hat{z}\right).$$

$$\Rightarrow \begin{cases} -\tau i\omega J_x = -J_x + \sigma_D E_x - \omega_c \tau J_y & (2) \\ -\tau i\omega J_y = -J_y - i\sigma_D E_x + \omega_c \tau J_x & (3) \end{cases}$$

Now because the metal is isotropic the field can only induce a global phase and scaling on the flow i.e.:

$$\vec{J} \propto \sigma_D e^{-i\Phi} \vec{E}.$$

Which means that the flow upholds the same relation:

$$J_y = iJ_x \qquad J_z = 0.$$

This could also be seen by the sum $i(2) + (3)$.
Placing in (2) we obtain:

$$\begin{aligned} -\tau i\omega J_x &= -J_x + \sigma_D E_x - i\omega_c \tau J_x \\ J_x &= \frac{\sigma_D}{1 - i\tau(\omega - \omega_c)} E_x. \end{aligned}$$

And because the flow and the field share the same relation regarding their vector components we obtaioned what we were after.

Question 4

Show that Maxwell's equations:

$$\bar{\nabla} \cdot \vec{E} = \bar{\nabla} \cdot \vec{H} = 0 \tag{4}$$

$$\bar{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\vec{H} \tag{5}$$

$$\bar{\nabla} \times \vec{H} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial}{\partial t}\vec{E} \tag{6}$$

Under a constant field:

$$\vec{H} = H\hat{z} \tag{7}$$

Have a solution of the form:

$$E_x = E_0 e^{i(kz - \omega t)} \quad E_y = iE_x \quad E_z = 0.$$

If the following relations hold:

$$
\begin{aligned}
k^2c^2 &= \epsilon\omega^2 \\
\epsilon &= 1 - \frac{\omega_p^2}{\omega}\frac{1}{\omega - \omega_c + \frac{i}{\tau}} \\
\text{And } \omega_p^2 &\equiv \frac{4\pi n e^2}{m}
\end{aligned}
$$

Solution:
If we take the curl of ():

$$\bar{\nabla}\times\bar{\nabla}\times\vec{E} = \bar{\nabla}\times -\frac{1}{c}\frac{\partial}{\partial t}\vec{H}.$$

And substitue (6):

$$
\begin{aligned}
\bar{\nabla}\times\bar{\nabla}\times\vec{E} &= -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial}{\partial t}\vec{E}\right) \\
\bar{\nabla}\left(\bar{\nabla}\cdot\vec{E}\right) - \nabla^2\vec{E} &= -\frac{4\pi}{c^2}\frac{\partial}{\partial t}\vec{J} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} \\
\nabla^2\vec{E} &= \frac{4\pi}{c^2}\frac{\partial}{\partial t}\sigma(\omega)\vec{E} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} \\
-k^2 &= -i\omega\frac{4\pi}{c^2}\sigma(\omega) - \frac{1}{c^2}\omega^2 \\
k^2c^2 &= i\omega 4\pi\sigma(\omega) + \omega^2 \\
k^2c^2 &\stackrel{!}{=} \omega^2\underbrace{\left(\frac{i4\pi}{\omega}\frac{\sigma_D}{1 - i\tau(\omega - \omega_c)} + 1\right)}_{\epsilon(\omega)}
\end{aligned}
$$

$$\Rightarrow \epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega}\frac{1}{\omega - \omega_c + \frac{i}{\tau}}.$$

Under the assumption $\omega \ll \omega_c$:

$$\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega}\frac{1}{-\omega_c + \frac{i}{\tau}}.$$

And under the assumption $1 \ll \omega_c\tau$:

$$\epsilon(\omega) = 1 + \frac{\omega_p^2}{\omega\omega_c}\tau.$$

And under the assumption $\omega_c \ll \omega_p$:

$$
\begin{aligned}
\epsilon(\omega) &= \frac{4\pi\sigma_D}{\omega\omega_c} \\
\Rightarrow \omega^2\epsilon(\omega) &= \omega\frac{4\pi\sigma_D}{\omega_c} = k^2c^2 \\
\omega &= \frac{\omega_c k^2c^2}{4\pi\sigma_D}.
\end{aligned}
$$