Question 5

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5. In this problem you are asked to make a connection between the Hilbert space of photon states and the real space.

a) Construct a one photon state which at some moment of time is localized at a point $r = r_0$. (hint - make use of one photon states with defined momentum). Note that you actually have a freedom of defining what "localized" means here. Consider using for this purpose expectation values of electric or magnetic fields or the EM energy density. After making your choice consider how does this state develop in time? How would you make it move in a certain direction? A localized state can be defined as a state for which $\langle O\rangle = \delta(t-t_0)\,\delta(\mathbf{r}-\mathbf{r_0})$ where O can be either the intensity of the electric or the magnetic fields $\mathbf{E}$ and $\mathbf{B}$, or $H_{rad} = \frac{\epsilon_0}{2}\int d^3r\left(E^2+c^2B^2\right)$. To do the calculation, we'll put the photon in a box. A single photon state can be written (in the number basis) as $a^\dagger_{\mathbf{k}}|0\rangle$. We can also write the single photon state as $|\psi\rangle$ in a general basis (where we ignore polarization as it isn't relevant here)

$$|\psi\rangle = \sum_{\mathbf{k}}\phi_{\mathbf{k}}\left(a^\dagger_{\mathbf{k}}|0\rangle\right)$$
$$\text{s.t. } \sum_{\mathbf{k}}|\phi_{\mathbf{k}}|^2 = 1$$

where $a^\dagger_{\mathbf{k}}, a^\dagger_{\mathbf{k}}$ are the creation and annihilation operators creating/killing a single photon (field excitation) in mode $\mathbf{k}$ with energy $E_{|\mathbf{k}|} = \hbar c|\mathbf{k}|$. Like with the electron, we'd expect a localized state to require many "wavelengths" (or generally, many modes). This is okay since 1 photon is 1 excitation in the Fock basis. In another basis, it can be represented as anything else.

The most convenient expectation value to calculate is that of the intensity of the electric field $\mathbf{E}$ given by

$$\mathbf{E}(\mathbf{r}) = i\sum_{\mathbf{k}}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2V\epsilon_0}}\left(a_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}} - a^\dagger_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{r}}\right).$$

What we are actually looking for are coefficients $\phi(\mathbf{k})$, such that the photon is localized i.e. $\langle\psi|\frac{\mathbf{E}^2(\mathbf{r})}{V}|\psi\rangle \propto \delta(\mathbf{r}-\mathbf{r}')$

$$\langle\psi|\frac{I}{V}|\psi\rangle = \langle\psi|\frac{\mathbf{E}^2(\mathbf{r})}{V}|\psi\rangle = \langle\psi|\frac{\mathbf{E}(\mathbf{r})\cdot\mathbf{E}(\mathbf{r})}{V}|\psi\rangle = \ldots\text{(killed double operators)}$$
$$\ldots = -\frac{\hbar}{2V^2\epsilon_0}\sum_{\mathbf{k},\mathbf{k}',\mathbf{p},\mathbf{p}'}\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}\langle\psi| - a^\dagger_{\mathbf{k}}a_{\mathbf{k}'}e^{-i\mathbf{r}\cdot\left(\mathbf{k}-\mathbf{k}'\right)} - a_{\mathbf{k}}a^\dagger_{\mathbf{k}'}e^{i\mathbf{r}\cdot\left(\mathbf{k}-\mathbf{k}'\right)}|\psi\rangle = \ldots\text{(state expansion)}$$
$$\ldots = \frac{\hbar\omega}{2V^2\epsilon_0}\sum_{\mathbf{k},\mathbf{k}',\mathbf{p},\mathbf{p}'}\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}\phi^*_{\mathbf{p}'}\phi_{\mathbf{p}}\langle 0|a_{\mathbf{p}'}\left(a^\dagger_{\mathbf{k}}a_{\mathbf{k}'}e^{-i\mathbf{r}\cdot\left(\mathbf{k}-\mathbf{k}'\right)} + a_{\mathbf{k}}a^\dagger_{\mathbf{k}'}e^{i\mathbf{r}\cdot\left(\mathbf{k}-\mathbf{k}'\right)}\right)a^\dagger_{\mathbf{p}}|0\rangle = \ldots$$

we can now use the commutation relations

$$\left[a_{\mathbf{k}'}, a^\dagger_{\mathbf{k}}\right] = \delta(\mathbf{k}-\mathbf{k}')$$
$$\left[a_{\mathbf{k}'}, a_{\mathbf{k}}\right] = \left[a^\dagger_{\mathbf{k}'}, a^\dagger_{\mathbf{k}}\right] = 0$$

and get

$$... = \frac{\hbar}{2V^2\epsilon_0}\sum_{\mathbf{k},\mathbf{k'},\mathbf{p},\mathbf{p'}}\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k'}}}\phi^*_{\mathbf{p'}}\phi_{\mathbf{p}}\left(\underbrace{\langle 0|\, a_{\mathbf{p'}}a^\dagger_{\mathbf{k}}a_{\mathbf{k'}}a^\dagger_{\mathbf{p}}\,|0\rangle}_{\delta(\mathbf{k'}-\mathbf{p})\delta(\mathbf{k}-\mathbf{p'})} e^{-i\mathbf{r}\cdot(\mathbf{k}-\mathbf{k'})} + \underbrace{\langle 0|\, a_{\mathbf{p'}}a_{\mathbf{k}}a^\dagger_{\mathbf{k'}}a^\dagger_{\mathbf{p}}\,|0\rangle}_{\delta(\mathbf{p}-\mathbf{k})\delta(\mathbf{p'}-\mathbf{k'})+\delta(\mathbf{p}-\mathbf{p'})\delta(\mathbf{k}-\mathbf{k'})} e^{i\mathbf{r}\cdot(\mathbf{k}-\mathbf{k'})}\right) =$$

$$= \frac{\hbar}{2V^2\epsilon_0}\left(\sum_{\mathbf{k},\mathbf{k'},\mathbf{p},\mathbf{p'}}\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k'}}}\phi^*_{\mathbf{p'}}\phi_{\mathbf{p}}\delta(\mathbf{k'}-\mathbf{p})\,\delta(\mathbf{k}-\mathbf{p'})\,e^{-i\mathbf{r}\cdot(\mathbf{k}-\mathbf{k'})} + \right.$$

$$\left. + \sum_{\mathbf{k},\mathbf{k'},\mathbf{p},\mathbf{p'}}\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k'}}}\phi^*_{\mathbf{p'}}\phi_{\mathbf{p}}\left(\delta(\mathbf{p}-\mathbf{k})\,\delta(\mathbf{p'}-\mathbf{k'}) + \delta(\mathbf{p}-\mathbf{p'})\,\delta(\mathbf{k}-\mathbf{k'})\right)e^{i\mathbf{r}\cdot(\mathbf{k}-\mathbf{k'})}\right) =$$

$$= \frac{\hbar}{2V\epsilon_0}\left(\sum_{\mathbf{k},\mathbf{k'}}\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k'}}}\phi^*_{\mathbf{k}}\phi_{\mathbf{k'}}e^{-i\mathbf{r}\cdot(\mathbf{k}-\mathbf{k'})} + \phi^*_{\mathbf{k'}}\phi_{\mathbf{k}}e^{i\mathbf{r}\cdot(\mathbf{k}-\mathbf{k'})} + \underbrace{|\phi_{\mathbf{k}}|^2}_{*}\right) = ...$$

Disregarding the infinite energy term $(*)$, we see that for the choice $\phi(\mathbf{r},\mathbf{k}) = \frac{e^{i\mathbf{r}\cdot\mathbf{k}}}{\sqrt{\omega_{\mathbf{k}}}}$ we get

$$... = \frac{\hbar}{2V\epsilon_0}\left(\sum_{\mathbf{k},\mathbf{k'}} e^{i\mathbf{r'}\cdot(\mathbf{k}-\mathbf{k'})}e^{-i\mathbf{r}\cdot(\mathbf{k}-\mathbf{k'})} + e^{-i\mathbf{r'}\cdot(\mathbf{k}-\mathbf{k'})}e^{i\mathbf{r}\cdot(\mathbf{k}-\mathbf{k'})}\right) =$$

$$= \frac{\hbar}{2V\epsilon_0}\left(\sum_{\mathbf{k},\mathbf{k'}} 2e^{i(\mathbf{k}-\mathbf{k'})\cdot(\mathbf{r'}-\mathbf{r})}\right) \underset{\mathbf{k}+\mathbf{k'}\triangleq\mathbf{p'}}{\overset{\mathbf{k}-\mathbf{k'}\triangleq\mathbf{p}}{\Rightarrow}} \frac{\hbar}{\epsilon_0}\left(\underbrace{\int\frac{d^3p'}{\sqrt{V}}}_{=1 \text{ for finite V then take limit}} \int\frac{d^3p}{\sqrt{V}}e^{i\mathbf{p}\cdot(\mathbf{r'}-\mathbf{r})}\right) \underset{\text{up to a factor}_{V^\alpha}}{=} \frac{\hbar}{\epsilon_0}\delta(\mathbf{r'}-\mathbf{r})\,.$$

We've shown that for coefficients $\phi(\mathbf{r},\mathbf{k}) = \frac{e^{i\mathbf{r}\cdot\mathbf{k}}}{\sqrt{\omega_{\mathbf{k}}}}$, we obtain a localized photon wavefunction which is what we wanted to show. The time evolution of the state we found (in the Heisenberg picture) is given by the time evolution of the creation operator $a^\dagger_{\mathbf{k}}(t) = e^{-i\omega_{\mathbf{k}}t}a^\dagger_{\mathbf{k}}(0)$ where the dispersion relations are $\omega_{\mathbf{k}} = c\mathbf{k}$ thus

$$\psi(\mathbf{r},t) = \int \frac{e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)}}{\sqrt{V\omega_{\mathbf{k}}}}a^\dagger_{\mathbf{k}}\,|0\rangle\, d^3k.$$

In order for this wavefunction to move in a certain direction, it can't be totally localized (i.e. can't have $\psi \propto \delta(\mathbf{r}-\mathbf{r'})$). The reason is that "moving in a certain direction" means gaining more phase in one direction then another which can be accomplished by choosing the vector components of $|\mathbf{k}| = \sqrt{k_x^2+k_y^2+k_z^2}$ (for a localized state $k_i \in (-\infty,\infty)\,;\; i = x,y,z$) to be different from each other but still have their squares add up to $|\mathbf{k}|^2$. If a state is totally localized in space, constructing it would require all values of $k_i \in (-\infty,\infty)$and we wouldn't have the freedom of making the above choice. Considering a nearly localized state and an asymmetric choice of the spatial distribution of $\mathbf{k}$ such that, for example, $k_i \in (-a_i,a_i)\,;\; i = x,y$ while $k_z \in (-\infty,\infty)$, we can now write

$$\frac{\omega_{|\mathbf{k}|}}{v_i} = |\mathbf{k}| = |k_z|\sqrt{\frac{k_x^2}{k_z^2}+\frac{k_y^2}{k_z^2}+1} \overset{\text{where}_{k_z\gg k_{x,y}}}{\approx} |k_z|\left(\frac{k_x^2}{2k_z^2}+\frac{k_y^2}{2k_z^2}+1\right) = \frac{k_x^2}{2k_z}+\frac{k_y^2}{2k_z}+|k_z|$$

and see that the group velocity (which determines the direction of propagation of a wave) is

$$v_{group} \equiv \sum\frac{\partial\omega_{|\mathbf{k}|}}{\partial k_i}\hat{e}_i = \left[v_x\frac{k_x}{k_z}\hat{e}_x + v_y\frac{k_y}{k_z}\hat{e}_y + v_z\left(-\frac{k_x^2}{2k_z^2}-\frac{k_y^2}{2k_z^2}+\text{sign}_{k_z}\right)\hat{e}_z\right] \overset{\frac{k_i}{k_z}\ll 1}{\approx} v_z$$

which is a wave propagating in direction $\hat{e}_z$ at velocity v_z.

b) What is a quantum mechanical description of a radio wave with a given wavelength? An ideal monochromatic radio wave (made of a single wavelength) would be described by occupying a single mode of the field, i.e. a state like $|\psi\rangle = \left(a^\dagger_{\mathbf{k}}\right)^n|0\rangle$ where the power n indicates there could be many excitations (photons) with this wavelength (in this mode). Since $|\mathbf{k}|$ is well defined, the photons created wouldn't be localized (in space) but rather, smeared all over space.

c) Of a pulse of WiFi waves representing one bit, i.e. approximately a rectangular pulse of small duration. No real need for solving the problem but for your curiosity you may estimate this duration based on say 100

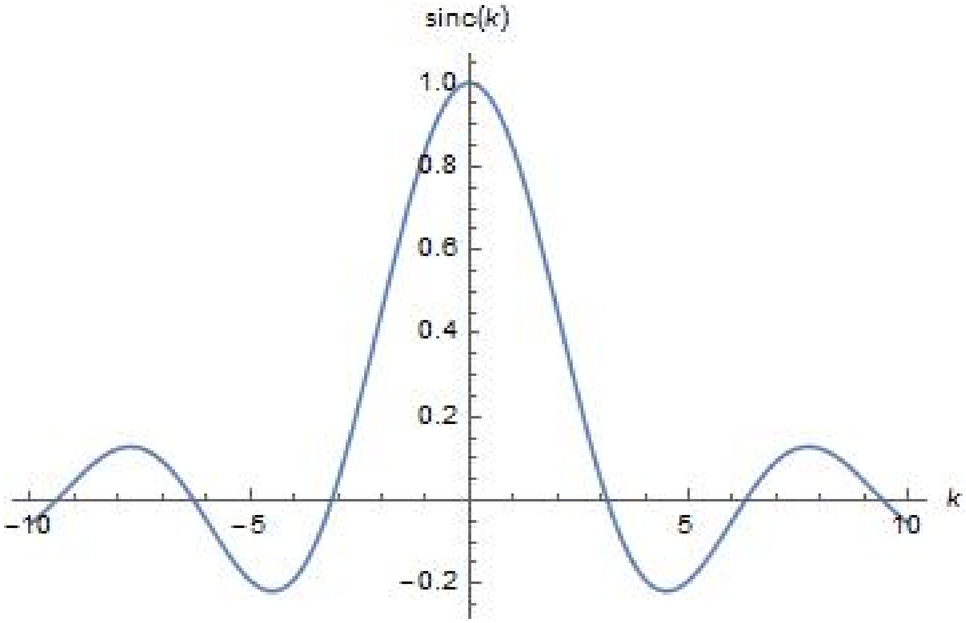

Figure 1: Sinc

Mbps data transmission speed. We know a finite wave packet can't be constructed using a single plane wave (i.e. a single wavelength) and would require using all of the different different wavelengths. Here, we'll use time-energy Fourier pair (contrary to the last section where we thought about space-momentum relations) to show to construct such a state.

The pulse is said to be rectangular and have a small time duration $\triangle t$. We'll first give an approximate solution for the limit $\triangle t \ll \frac{1}{\triangle \omega_{\mathbf{k}}}$ (where $\triangle \omega_{\mathbf{k}} \equiv \omega_{\mathbf{k}_{max}} - \omega_{\mathbf{k}_{min}}$ is related to the range of frequencies we'll take to build the wavefunction) and then discuss how this should be done in the case of the rectangular pulse.

In the limit $\triangle t \ll \frac{1}{\triangle \omega_{\mathbf{k}}}$, the pulse is approximately a delta function in t and $\triangle \omega_{\mathbf{k}}$ is very large (we'll approximate $\triangle \omega_{\mathbf{k}} \to \infty$). We can choose the expansion coefficients $\phi_{\mathbf{k}}(\mathbf{r}=0,t) = \frac{e^{i\omega_{\mathbf{k}}t}}{\sqrt{V}}$ to construct the state ψ. Since the dispersion relations are linear, we have $\omega_{\mathbf{k}} = \mathbf{c}\cdot\mathbf{k} \to \triangle \omega_{\mathbf{k}} = \mathbf{c}\cdot\triangle\mathbf{k}$ giving

$$\psi(r=0,t) = \int_0^\infty \frac{1}{c}\frac{e^{i\omega_{\mathbf{k}}t}}{\sqrt{V}}\left(a_{\mathbf{k}}^{\dagger n}|0\rangle\right)d\omega_{\mathbf{k}} \propto \int_{-\infty}^{\infty} \frac{e^{i\mathbf{c}\cdot\mathbf{k}t}}{\sqrt{V}}\left(a_{\mathbf{k}}^{\dagger n}|0\rangle\right)d^3k$$

which, when used to measure the intensity of the electric of magnetic fields (as above), will give a delta function in time.

Not using the above approximation, we cannot say that $\triangle\mathbf{k} \to \infty$ but instead we'll say $\triangle\mathbf{k} \approx 2\mathbf{k_0}$. To account for the reduction of the integration domain, we'll multiply the integral by a sinc function (see figure 1) with an average width of $2\mathbf{k_0}$ leaving us with

$$\psi(r=0,t) \propto \int_{-\infty}^{\infty} \frac{e^{i\mathbf{c}\cdot\mathbf{k}t}}{\sqrt{V}}\left(a_{\mathbf{k}}^{\dagger n}|0\rangle\right)\mathrm{sinc}\left(2\mathbf{k}_0\right)d^3k.$$

Now, when measuring the intensity of the electric of magnetic fields, we'll get a Fourier transform of a sinc function. This will result in a rect function (see figure 2) in time which is defined by

$$\mathrm{rect} = \begin{cases} 1 & \text{for } |t-t'| < \triangle t \\ 0 & \text{else} \end{cases} \quad \text{for } t,t' \geq 0$$

which is what we wanted to show.

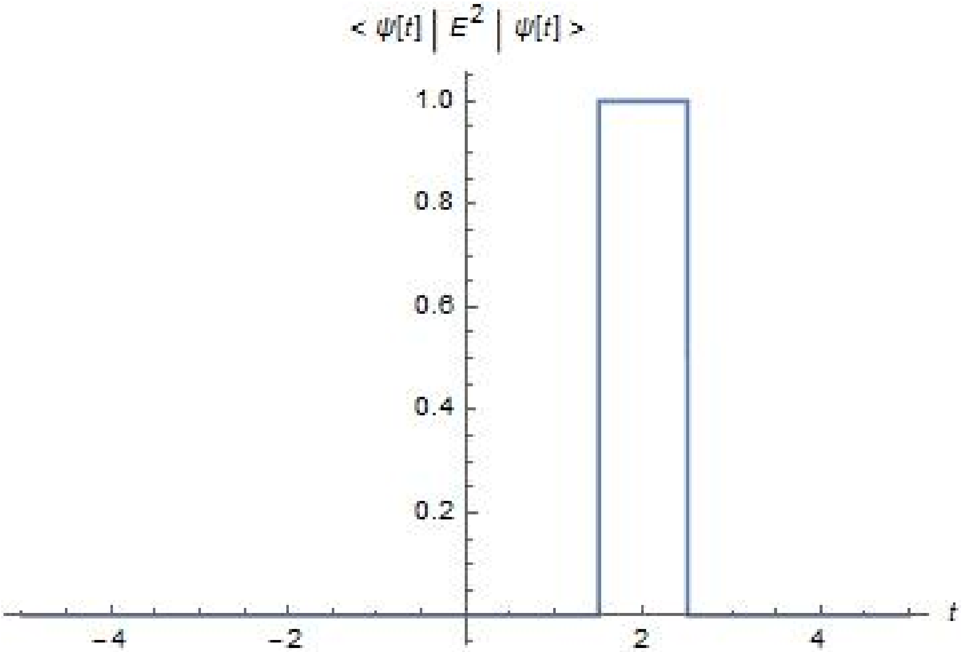

Figure 2: rect