A two-level system coupled to a harmonic bath: A model for dissipation

Here we solve the dynamics of a two-level system (TLS) with a frequency ω_0 that is weakly coupled to a bath of harmonic oscillators with frequencies Ω_λ. The Hamiltonian is given by

$$\hat{\mathcal{H}} = \frac{\hbar\omega_0}{2}\hat{\sigma}_z + \hbar\sum_\lambda \Omega_\lambda \hat{B}^\dagger_\lambda \hat{B}_\lambda + \hbar\sum_\lambda \left[g_\lambda \hat{\sigma}_- \hat{B}^\dagger_\lambda + g^*_\lambda \hat{B}_\lambda \hat{\sigma}_+\right], \tag{1}$$

where the $\hat{B}_\lambda$ and the $\hat{B}^\dagger_\lambda$ are the annihilation and creation operators of the oscillators, respectively, and g_λ are coupling constants which will be assumed to be small. These operators satisfy the following commutation relations:

$$\begin{aligned}\left[\hat{B}_\lambda, \hat{B}^\dagger_{\lambda'}\right] &= \delta_{\lambda\lambda'}, \\ \left[\hat{B}_\lambda, \hat{B}_{\lambda'}\right] &= \left[\hat{B}^\dagger_\lambda, \hat{B}^\dagger_{\lambda'}\right] = 0,\end{aligned} \tag{2}$$

meaning that the different oscillators in the bath are independent from one another.

In the **Schrödinger picture**, the operators that act on the space of the TLS states are given by

$$\begin{aligned}\hat{\sigma}_z &= |e\rangle\langle e| - |g\rangle\langle g| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{[Schrödinger]} \\ \hat{\sigma}_+ &= |e\rangle\langle g| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{[Schrödinger]} \\ \hat{\sigma}_- &= |g\rangle\langle e| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad \text{[Schrödinger]}\end{aligned} \tag{3}$$

We will use the commutation relations of these operators as well, and they are given by

$$\begin{aligned}\left[\hat{\sigma}_z, \hat{\sigma}_\pm\right] &= \pm 2\hat{\sigma}_\pm, \\ \left[\hat{\sigma}_+, \hat{\sigma}_-\right] &= \hat{\sigma}_z.\end{aligned} \tag{4}$$

Importantly, any TLS operator commutes with any bath operator (as they represent independent objects).

We may generally write the density matrix of the TLS as

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & 1-\rho_{11} \end{pmatrix}, \tag{5}$$

and we can furthermore express ρ_{11} and ρ_{21} in terms of expectation values of the TLS operators, through the general relation $\left\langle \hat{O} \right\rangle = \mathrm{Tr}\left(\hat{O}\rho\right)$:

$$\begin{aligned}\langle\hat{\sigma}_z\rangle &= \mathrm{Tr}\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & 1-\rho_{11} \end{pmatrix}\right] = \mathrm{Tr}\left[\begin{pmatrix} \rho_{11} & \rho_{12} \\ -\rho_{21} & \rho_{11}-1 \end{pmatrix}\right] = 2\rho_{11} - 1, \\ \langle\hat{\sigma}_+\rangle &= \mathrm{Tr}\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & 1-\rho_{11} \end{pmatrix}\right] = \mathrm{Tr}\left[\begin{pmatrix} \rho_{21} & 1-\rho_{11} \\ 0 & 0 \end{pmatrix}\right] = \rho_{21}, \\ \langle\hat{\sigma}_-\rangle &= \mathrm{Tr}\left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & 1-\rho_{11} \end{pmatrix}\right] = \mathrm{Tr}\left[\begin{pmatrix} 0 & 0 \\ \rho_{11} & \rho_{12} \end{pmatrix}\right] = \rho_{12}.\end{aligned} \tag{6}$$

So we may rewrite the density matrix in the Schrödinger picture as

$$\rho=\begin{pmatrix} \frac{1+\langle\hat{\sigma}_z\rangle}{2} & \langle\hat{\sigma}_-\rangle \\ \langle\hat{\sigma}_+\rangle & \frac{1-\langle\hat{\sigma}_z\rangle}{2} \end{pmatrix}. \quad \text{[Schrödinger]} \tag{7}$$

This is essential because in what follows we will work in the **Heisenberg picture**, where the state of the system remains fixed and only the operators evolve in time; but because expectation values such as $\langle\hat{\sigma}_z\rangle$ and $\langle\hat{\sigma}_\pm\rangle$ are independent of the picture in which we choose to work, once we derive these expectation values in the Heisenberg picture we can plug them into the expression for ρ in the Schrödinger picture to obtain its dynamics.

Heisenberg picture evolution of operators

Within the Heisenberg picture, any operator $\hat{O}$[1] generally evolves according to the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{O}=\frac{i}{\hbar}\left[\hat{\mathcal{H}},\hat{O}\right]. \tag{8}$$

Another property of the Heisenberg picture, which will be important in our derivation, is that the Heisenberg picture conserves commutation relations between operators as long as they are estimated at an equal time. That is, if $\left[\hat{O}_1,\hat{O}_2\right]=\hat{O}_3$ in the Schrödinger picture, then also $\left[\hat{O}_1\left(t\right),\hat{O}_2\left(t\right)\right]=\hat{O}_3\left(t\right)$ in the Heisenberg picture. This in particular implies that, *at equal times*, a bath operator and a TLS operator always commute; for example, at any time t

$$\left[\hat{B}_\lambda\left(t\right),\hat{\sigma}_\pm\left(t\right)\right]=0. \tag{9}$$

Note the requirement that the same t appears in both operators; this need not (and will not) be the case if we substitute different times into the operators. In general, $\left[\hat{B}_\lambda\left(t_1\right),\hat{\sigma}_\pm\left(t_2\right)\right]\neq 0$ for $t_1\neq t_2$.

The time-evolution of the bath operators is therefore governed by the equation

$$\begin{aligned}\dot{\hat{B}}_\lambda&=\frac{i}{\hbar}\left[\hat{\mathcal{H}},\hat{B}_\lambda\right]=i\sum_{\lambda'}\Omega_{\lambda'}\left[\hat{B}^\dagger_{\lambda'}\hat{B}_{\lambda'},\hat{B}_\lambda\right]+i\sum_{\lambda'}g_{\lambda'}\hat{\sigma}_-\left[\hat{B}^\dagger_{\lambda'},\hat{B}_\lambda\right]\\&=\sum_{\lambda'}\left(-i\Omega_\lambda\hat{B}_\lambda-ig_\lambda\hat{\sigma}_-\right)\delta_{\lambda\lambda'}\\&=-i\Omega_\lambda\hat{B}_\lambda-ig_\lambda\hat{\sigma}_-.\end{aligned} \tag{10}$$

The integration of this equation simply yields

$$\begin{aligned}\hat{B}_\lambda\left(t\right)&=\hat{B}_\lambda\left(0\right)e^{-i\Omega_\lambda t}-ig_\lambda\int_0^t\mathrm{d}\tau\hat{\sigma}_-\left(\tau\right)e^{-i\Omega_\lambda(t-\tau)},\\ \hat{B}^\dagger_\lambda\left(t\right)&=\hat{B}^\dagger_\lambda\left(0\right)e^{i\Omega_\lambda t}+ig^*_\lambda\int_0^t\mathrm{d}\tau\hat{\sigma}_+\left(\tau\right)e^{i\Omega_\lambda(t-\tau)},\end{aligned} \tag{11}$$

as one can check by taking the derivative and obtaining the original equation of motion. The TLS ladder operators evolve according to

$$\dot{\hat{\sigma}}_-=\frac{i}{\hbar}\left[\hat{\mathcal{H}},\hat{\sigma}_-\right]=\frac{i\omega_0}{2}\left[\hat{\sigma}_z,\hat{\sigma}_-\right]+i\sum_\lambda g^*_\lambda\hat{B}_\lambda\left[\hat{\sigma}_+,\hat{\sigma}_-\right]=-i\omega_0\hat{\sigma}_-+i\sum_\lambda g^*_\lambda\hat{B}_\lambda\hat{\sigma}_z, \tag{12}$$

[1] This is true for any operator apart from the density operator ρ, which represents the state and therefore remains *fixed* in the Heisenberg picture.

yielding

$$\hat{\sigma}_-(t) = \hat{\sigma}_-(0)\, e^{-i\omega_0 t} + i\sum_\lambda g_\lambda^* \int_0^t \mathrm{d}\tau \hat{B}_\lambda(\tau)\, \hat{\sigma}_z(\tau)\, e^{-i\omega_0(t-\tau)},$$
$$\hat{\sigma}_+(t) = \hat{\sigma}_+(0)\, e^{i\omega_0 t} - i\sum_\lambda g_\lambda \int_0^t \mathrm{d}\tau \hat{\sigma}_z(\tau)\, \hat{B}_\lambda^\dagger(\tau)\, e^{i\omega_0(t-\tau)}. \tag{13}$$

So far these are *exact* expressions for the ladder operators and the bath operators. To derive an expression for $\hat{\sigma}_z(t)$ we must employ approximations, by keeping only terms that are at most second-order in the coupling constants g_λ.

Time evolution of $\hat{\sigma}_z$

The equation of motion for $\hat{\sigma}_z$ is given by

$$\dot{\hat{\sigma}}_z = \frac{i}{\hbar}\left[\hat{\mathcal{H}}, \hat{\sigma}_z\right] = i\sum_\lambda \left\{ g_\lambda [\hat{\sigma}_-, \hat{\sigma}_z]\, \hat{B}_\lambda^\dagger + g_\lambda^* \hat{B}_\lambda [\hat{\sigma}_+, \hat{\sigma}_z] \right\} = 2i\sum_\lambda \left\{ g_\lambda \hat{\sigma}_- \hat{B}_\lambda^\dagger - g_\lambda^* \hat{B}_\lambda \hat{\sigma}_+ \right\}. \tag{14}$$

We substitute the expressions that we derived for the bath and ladder operators, keeping terms up to second order in the coupling constants g_λ:

$$\begin{aligned}\dot{\hat{\sigma}}_z \approx\ & 2i\sum_\lambda \left\{ g_\lambda \hat{\sigma}_-(0)\, \hat{B}_\lambda^\dagger(0)\, e^{i(\Omega_\lambda - \omega_0)t} - g_\lambda^* \hat{B}_\lambda(0)\, \hat{\sigma}_+(0)\, e^{-i(\Omega_\lambda - \omega_0)t} \right\} \\ & - 2\sum_{\lambda,\lambda'} g_\lambda g_{\lambda'}^* \int_0^t \mathrm{d}\tau \hat{\sigma}_z(\tau)\, e^{-i\omega_0(t-\tau)} \hat{B}_{\lambda'}(\tau)\, \hat{B}_\lambda^\dagger(0)\, e^{i\Omega_\lambda t} \\ & - 2\sum_{\lambda,\lambda'} g_\lambda^* g_{\lambda'} \int_0^t \mathrm{d}\tau \hat{B}_\lambda(0)\, e^{-i\Omega_\lambda t} \hat{B}_{\lambda'}^\dagger(\tau)\, \hat{\sigma}_z(\tau)\, e^{i\omega_0(t-\tau)} \\ & - 2\sum_\lambda |g_\lambda|^2 \int_0^t \mathrm{d}\tau \left\{ \hat{\sigma}_-(0)\, e^{-i\omega_0 t} \hat{\sigma}_+(\tau)\, e^{i\Omega_\lambda(t-\tau)} + \hat{\sigma}_-(\tau)\, e^{-i\Omega_\lambda(t-\tau)} \hat{\sigma}_+(0)\, e^{i\omega_0 t} \right\}. \end{aligned} \tag{15}$$

The next assumption that we employ is that correlations between early-time dynamics and late-time dynamics decay very rapidly; in other words, the system "has a weak memory", and we can take the lower limit of the integrals to $-\infty$ because this will negligibly affect the result. This is called the **Markov approximation**, and it yields

$$\begin{aligned}\dot{\hat{\sigma}}_z \approx\ & 2i\sum_\lambda \left\{ g_\lambda \hat{\sigma}_-(0)\, \hat{B}_\lambda^\dagger(0)\, e^{i(\Omega_\lambda - \omega_0)t} - g_\lambda^* \hat{B}_\lambda(0)\, \hat{\sigma}_+(0)\, e^{-i(\Omega_\lambda - \omega_0)t} \right\} \\ & - 2\sum_{\lambda,\lambda'} g_\lambda g_{\lambda'}^* \int_{-\infty}^t \mathrm{d}\tau \hat{\sigma}_z(\tau)\, e^{-i\omega_0(t-\tau)} \hat{B}_{\lambda'}(\tau)\, \hat{B}_\lambda^\dagger(0)\, e^{i\Omega_\lambda t} \\ & - 2\sum_{\lambda,\lambda'} g_\lambda^* g_{\lambda'} \int_{-\infty}^t \mathrm{d}\tau \hat{B}_\lambda(0)\, e^{-i\Omega_\lambda t} \hat{B}_{\lambda'}^\dagger(\tau)\, \hat{\sigma}_z(\tau)\, e^{i\omega_0(t-\tau)} \\ & - 2\sum_\lambda |g_\lambda|^2 \int_{-\infty}^t \mathrm{d}\tau \left\{ \hat{\sigma}_-(0)\, e^{-i\omega_0 t} \hat{\sigma}_+(\tau)\, e^{i\Omega_\lambda(t-\tau)} + \hat{\sigma}_-(\tau)\, e^{-i\Omega_\lambda(t-\tau)} \hat{\sigma}_+(0)\, e^{i\omega_0 t} \right\}. \end{aligned} \tag{16}$$

Next, inside each integral we bring all operators within a product to an equal time τ; for example, in the third row we write $\hat{B}_\lambda(0) \approx \hat{B}_\lambda(\tau) e^{i\Omega_\lambda \tau}$, which is a valid approximation because the integral is already multiplied by two coupling constants, so that any additional term in the expansion of $\hat{B}_\lambda(\tau)$ will take us beyond our second-order approximation (see Eq. (11)). So now we have

$$\begin{aligned}\dot{\hat{\sigma}}_z &\approx 2i\sum_\lambda \left\{ g_\lambda \hat{\sigma}_-(0)\, \hat{B}_\lambda^\dagger(0)\, e^{i(\Omega_\lambda-\omega_0)t} - g_\lambda^* \hat{B}_\lambda(0)\, \hat{\sigma}_+(0)\, e^{-i(\Omega_\lambda-\omega_0)t} \right\} \\ &- 2\sum_{\lambda,\lambda'} g_\lambda g_{\lambda'}^* \int_{-\infty}^{t} \mathrm{d}\tau \hat{\sigma}_z(\tau)\, e^{-i\omega_0(t-\tau)} \hat{B}_{\lambda'}(\tau)\, \hat{B}_\lambda^\dagger(\tau)\, e^{i\Omega_\lambda(t-\tau)} \\ &- 2\sum_{\lambda,\lambda'} g_\lambda^* g_{\lambda'} \int_{-\infty}^{t} \mathrm{d}\tau e^{-i\Omega_\lambda(t-\tau)} \hat{B}_\lambda(\tau)\, \hat{B}_{\lambda'}^\dagger(\tau)\, \hat{\sigma}_z(\tau)\, e^{i\omega_0(t-\tau)} \\ &- 2\sum_\lambda |g_\lambda|^2 \int_{-\infty}^{t} \mathrm{d}\tau \left\{ \hat{\sigma}_-(\tau)\, e^{-i\omega_0(t-\tau)} \hat{\sigma}_+(\tau)\, e^{i\Omega_\lambda(t-\tau)} + \hat{\sigma}_-(\tau)\, e^{-i\Omega_\lambda(t-\tau)} \hat{\sigma}_+(\tau)\, e^{i\omega_0(t-\tau)} \right\}. \end{aligned} \tag{17}$$

This step was done because we now want to average over the bath degrees-of-freedom, leaving only the TLS degrees-of-freedom unfixed. We exploit the fact that at any time τ, our bath (by its definition as a thermal bath) is at thermal equilibrium, so that

$$\left\langle \hat{B}_\lambda(\tau)\, \hat{B}_{\lambda'}^\dagger(\tau) \right\rangle = \delta_{\lambda\lambda'} \left[1 + f_{\mathrm{BE}}(\Omega_\lambda)\right], \tag{18}$$

where

$$f_{\mathrm{BE}}(\omega) = \frac{1}{e^{\beta\hbar\omega} - 1} \tag{19}$$

is the Bose-Einstein distribution, and β is the inverse temperature. So in the second and third rows of the equation for $\dot{\hat{\sigma}}_z$ we replace $\hat{B}_\lambda(\tau)\, \hat{B}_{\lambda'}^\dagger(\tau)$ by this factor. The first row in this equation, on the other hand, is dropped completely when this average is taken, as the terms appearing in it contain only a single raising or lowering operator of a bath oscillator[2]. We are therefore left with

$$\begin{aligned}\dot{\hat{\sigma}}_z &\approx -2\sum_\lambda |g_\lambda|^2 \left(1 + f_{\mathrm{BE}}(\Omega_\lambda)\right) \int_{-\infty}^{t} \mathrm{d}\tau \hat{\sigma}_z(\tau) \left[e^{i(\Omega_\lambda-\omega_0)(t-\tau)} + e^{-i(\Omega_\lambda-\omega_0)(t-\tau)} \right] \\ &- 2\sum_\lambda |g_\lambda|^2 \int_{-\infty}^{t} \mathrm{d}\tau \hat{\sigma}_-(\tau)\, \hat{\sigma}_+(\tau) \left[e^{i(\Omega_\lambda-\omega_0)(t-\tau)} + e^{-i(\Omega_\lambda-\omega_0)(t-\tau)} \right]. \end{aligned} \tag{20}$$

[2] More precisely, each bath oscillator λ is in thermal equilibrium with respect to an inverse temperature β, such that its density operator is given by (recall Question 3 in Homework 3)

$$\rho_\lambda = \frac{1}{Z} \sum_{n=0}^{\infty} e^{-\hbar\Omega_\lambda(n+1/2)} \left|n\right\rangle \left\langle n\right|,$$

where Z is the partition function and $|n\rangle$ are the energy levels of the oscillator. Taking the thermal average of $\hat{B}_\lambda$, for example, is equivalent to computing $\mathrm{Tr}\left(\rho_\lambda \hat{B}_\lambda\right)$, but then

$$\begin{aligned}\mathrm{Tr}\left(\rho_\lambda \hat{B}_\lambda\right) &= \frac{1}{Z} \sum_{n=0}^{\infty} e^{-\hbar\Omega_\lambda(n+1/2)} \left\langle n\right| \hat{B}_\lambda \left|n\right\rangle \\ &= \frac{1}{Z} \sum_{n=0}^{\infty} e^{-\hbar\Omega_\lambda(n+1/2)} \left\langle n\right| \sqrt{n} \left|n-1\right\rangle = 0, \end{aligned}$$

because $\langle n \,|\, n-1 \rangle = 0$.

Because all the terms are already of second order in g_λ, in the first row we will approximate

$$\hat{\sigma}_z(\tau) \approx \hat{\sigma}_z(t), \tag{21}$$

and in the second row

$$\hat{\sigma}_-(\tau)\,\hat{\sigma}_+(\tau) \approx \hat{\sigma}_-(t)\,\hat{\sigma}_+(t) = \frac{\mathbb{I} - \hat{\sigma}_z(t)}{2}; \tag{22}$$

both approximations are valid to zeroth order in g_λ (as can be seen from Eqs. (13) and (14)). This leads to

$$\dot{\hat{\sigma}}_z \approx -\sum_\lambda |g_\lambda|^2 \left\{ \int_{-\infty}^{t} \mathrm{d}\tau \left[e^{i(\Omega_\lambda-\omega_0)(t-\tau)} + e^{-i(\Omega_\lambda-\omega_0)(t-\tau)} \right] \right\} \left[\mathbb{I} + \left(1 + 2f_{\mathrm{BE}}(\Omega_\lambda)\right) \hat{\sigma}_z \right]. \tag{23}$$

Changing variables in the integral ($\tau' = t - \tau$ for the first exponent and $\tau' = \tau - t$ for the second), we get

$$\begin{aligned} \dot{\hat{\sigma}}_z &\approx -\sum_\lambda |g_\lambda|^2 \left\{ \int_{-\infty}^{\infty} \mathrm{d}\tau' e^{i(\Omega_\lambda-\omega_0)\tau'} \right\} \left[\mathbb{I} + \left(1 + 2f_{\mathrm{BE}}(\Omega_\lambda)\right) \hat{\sigma}_z \right] \\ &= -\int \mathrm{d}\Omega \varrho(\Omega) |g(\Omega)|^2 \left\{ \int_{-\infty}^{\infty} \mathrm{d}\tau' e^{i(\Omega-\omega_0)\tau'} \right\} \left[\mathbb{I} + \left(1 + 2f_{\mathrm{BE}}(\Omega)\right) \hat{\sigma}_z \right], \end{aligned} \tag{24}$$

where ϱ is the density of states of bath frequencies (we replaced the discrete spectrum with a continuous one). We now use the indentity

$$\int_{-\infty}^{\infty} \mathrm{d}\tau' e^{i(\Omega-\omega_0)\tau'} = 2\pi\delta(\Omega - \omega_0), \tag{25}$$

yielding

$$\begin{aligned} \dot{\hat{\sigma}}_z &\approx -2\pi \int \mathrm{d}\Omega \varrho(\Omega) |g(\Omega)|^2 \delta(\Omega - \omega_0) \left[\mathbb{I} + \left(1 + 2f_{\mathrm{BE}}(\Omega)\right) \hat{\sigma}_z \right] \\ &= -2\pi |g(\omega_0)|^2 \varrho(\omega_0) \left[\mathbb{I} + \left(1 + 2f_{\mathrm{BE}}(\omega_0)\right) \hat{\sigma}_z \right] \\ &= -2\pi |g(\omega_0)|^2 \varrho(\omega_0) \left[\mathbb{I} + \coth\left(\frac{\beta\hbar\omega_0}{2}\right) \hat{\sigma}_z \right]. \end{aligned} \tag{26}$$

In total, the equation of motion amounts to

$$\dot{\hat{\sigma}}_z \approx -2\kappa \left[\tanh\left(\frac{\beta\hbar\omega_0}{2}\right) \mathbb{I} + \hat{\sigma}_z \right], \tag{27}$$

where we have defined the decay rate

$$\kappa \equiv \pi |g(\omega_0)|^2 \varrho(\omega_0) \coth\left(\frac{\beta\hbar\omega_0}{2}\right). \tag{28}$$

The solution to this equation is simple:

$$\boxed{\hat{\sigma}_z(t) = e^{-2\kappa t} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \left(1 - e^{-2\kappa t}\right) \tanh\left(\frac{\beta\hbar\omega_0}{2}\right) \mathbb{I}. \quad \text{[Heisenberg]}} \tag{29}$$

What is then the expectation value $\langle\hat{\sigma}_z\rangle$? In the Heisenberg picture, the density operator remains fixed in time,

$$\rho = \begin{pmatrix} \rho_{11}(0) & \rho_{12}(0) \\ \rho_{21}(0) & 1-\rho_{11}(0) \end{pmatrix}. \quad \text{[Heisenberg]} \tag{30}$$

Let us calculate $\langle\sigma_z\rangle$:

$$\langle\hat{\sigma}_z\rangle = \mathrm{Tr}\left(\rho\hat{\sigma}_z(t)\right) = \left(2\rho_{11}(0)-1\right)e^{-2\kappa t} + \left(e^{-2\kappa t}-1\right)\tanh\left(\frac{\beta\hbar\omega_0}{2}\right), \tag{31}$$

and in particular

$$\lim_{t\to\infty}\langle\hat{\sigma}_z\rangle = -\tanh\left(\frac{\beta\hbar\omega_0}{2}\right) = \frac{e^{-\beta\hbar\omega_0}-1}{e^{-\beta\hbar\omega_0}+1}, \tag{32}$$

which is the thermal equilibrium result. So in the long time limit, the TLS relaxes to a thermal state with respect to the inverse temperature β (*regardless* of the initial state of the TLS).

In the Schrödinger picture, the corresponding dynamics of the diagonal terms in ρ is given by

$$\begin{aligned}\rho_{11}(t) = \frac{1+\langle\hat{\sigma}_z\rangle}{2} &= \rho_{11}(0)e^{-2\kappa t} + \left(1-e^{-2\kappa t}\right)\cdot\frac{1-\tanh\left(\frac{\beta\hbar\omega_0}{2}\right)}{2} \\ &= \rho_{11}(0)e^{-2\kappa t} + \left(1-e^{-2\kappa t}\right)\cdot\frac{1}{e^{\beta\hbar\omega_0}+1},\end{aligned} \tag{33}$$

and

$$\rho_{22}(t) = 1-\rho_{11}(t) = \rho_{22}(0)e^{-2\kappa t} + \left(1-e^{-2\kappa t}\right)\cdot\frac{e^{\beta\hbar\omega_0}}{e^{\beta\hbar\omega_0}+1}. \tag{34}$$

Time evolution of $\hat{\sigma}_\pm$

Back to the Heisenberg picture, we now solve the time evolution of the ladder operators more explicitly by limiting ourselves, as before, to terms which are second-order in the coupling constants. Namely, we solve the equation

$$\dot{\hat{\sigma}}_- = -i\omega_0\hat{\sigma}_- + i\sum_\lambda g_\lambda^*\hat{B}_\lambda\hat{\sigma}_z \tag{35}$$

by substituting

$$\begin{aligned}\hat{B}_\lambda(t) &= \hat{B}_\lambda(0)e^{-i\Omega_\lambda t} - ig_\lambda\int_0^t \mathrm{d}\tau\hat{\sigma}_-(\tau)e^{-i\Omega_\lambda(t-\tau)}, \\ \hat{\sigma}_z(t) &= \hat{\sigma}_z(0) + 2i\sum_\lambda\int_0^t \mathrm{d}\tau\left\{g_\lambda\hat{\sigma}_-(\tau)\hat{B}_\lambda^\dagger(\tau) - g_\lambda^*\hat{B}_\lambda(\tau)\hat{\sigma}_+(\tau)\right\}.\end{aligned} \tag{36}$$

This yields

$$\begin{aligned}\dot{\hat{\sigma}}_- \approx{}& -i\omega_0\hat{\sigma}_- + i\sum_\lambda g_\lambda^*\hat{B}_\lambda(0)\hat{\sigma}_z(0)e^{-i\Omega_\lambda t} + \sum_\lambda|g_\lambda|^2\int_0^t \mathrm{d}\tau\hat{\sigma}_-(\tau)e^{-i\Omega_\lambda(t-\tau)}\hat{\sigma}_z(0) \\ &-2\sum_{\lambda,\lambda'}g_\lambda^*\int_0^t \mathrm{d}\tau\hat{B}_\lambda(0)e^{-i\Omega_\lambda t}\left\{g_{\lambda'}\hat{\sigma}_-(\tau)\hat{B}_{\lambda'}^\dagger(\tau) - g_{\lambda'}^*\hat{B}_{\lambda'}(\tau)\hat{\sigma}_+(\tau)\right\}.\end{aligned} \tag{37}$$

Now, in the integral in the first row we use $\hat{\sigma}_-(\tau) \approx \hat{\sigma}_-(0)\, e^{-i\omega_0\tau}$ and then use $\sigma_-(0)\,\sigma_z(0) = \sigma_-(0)$, and in the second row we approximate as before $\hat{B}_\lambda(0) \approx \hat{B}_\lambda(\tau)\, e^{i\Omega_\lambda\tau}$; we also employ the Markov approximation by taking the lower limits of the integrals to $-\infty$, obtaining

$$\begin{aligned}
\dot{\hat{\sigma}}_- \approx & -i\omega_0\hat{\sigma}_- + i\sum_\lambda g_\lambda^*\hat{B}_\lambda(0)\,\hat{\sigma}_z(0)\, e^{-i\Omega_\lambda t} + \sum_\lambda |g_\lambda|^2 \int_{-\infty}^{t} \mathrm{d}\tau e^{-i\Omega_\lambda(t-\tau)} e^{-i\omega_0\tau}\hat{\sigma}_-(0) \\
& -2\sum_{\lambda,\lambda'} g_\lambda^* \int_{-\infty}^{t} \mathrm{d}\tau e^{-i\Omega_\lambda(t-\tau)} \left\{ g_{\lambda'}\hat{\sigma}_-(\tau)\,\hat{B}_\lambda(\tau)\,\hat{B}_{\lambda'}^\dagger(\tau) - g_{\lambda'}^*\hat{B}_\lambda(\tau)\,\hat{B}_{\lambda'}(\tau)\,\hat{\sigma}_+(\tau) \right\}. \qquad (38)
\end{aligned}$$

Averaging the bath operators yields

$$\begin{aligned}
\dot{\hat{\sigma}}_- & \approx -i\omega_0\hat{\sigma}_- - \sum_\lambda |g_\lambda|^2 \left(1 + 2f_{\mathrm{BE}}(\Omega_\lambda)\right) \int_{-\infty}^{t} \mathrm{d}\tau e^{-i\Omega_\lambda(t-\tau)}\hat{\sigma}_-(\tau) \\
& \approx -i\omega_0\hat{\sigma}_- - \left\{ \sum_\lambda |g_\lambda|^2 \left(1 + 2f_{\mathrm{BE}}(\Omega_\lambda)\right) \int_{-\infty}^{t} \mathrm{d}\tau e^{i(\omega_0-\Omega_\lambda)(t-\tau)} \right\} \hat{\sigma}_-(t) \\
& = -i\omega_0\hat{\sigma}_- - \left\{ \int \mathrm{d}\Omega \varrho(\Omega)\,|g(\Omega)|^2 \left(1 + 2f_{\mathrm{BE}}(\Omega)\right) \int_0^{\infty} \mathrm{d}\tau' e^{i(\omega_0-\Omega)\tau'} \right\} \hat{\sigma}_-(t) \\
& = -i\omega_0\hat{\sigma}_- - \left\{ \int \mathrm{d}\Omega \left[\pi\delta(\omega_0-\Omega) + i\mathcal{P}\frac{1}{\omega_0-\Omega} \right] \varrho(\Omega)\,|g(\Omega)|^2 \left(1 + 2f_{\mathrm{BE}}(\Omega)\right) \right\} \hat{\sigma}_-(t). \qquad (39)
\end{aligned}$$

In our last step we used the identity

$$\int_0^{\infty} \mathrm{d}\tau' e^{i\omega\tau'} = \pi\delta(\omega) + i\mathcal{P}\frac{1}{\omega}, \qquad (40)$$

where $\mathcal{P}$ denotes Cauchy's principal value

We may now define a renormalized frequency, featuring the **Lamb shift**,

$$\widetilde{\omega}_0 \equiv \omega_0 + \mathcal{P}\int \mathrm{d}\Omega \varrho(\Omega) \frac{|g(\Omega)|^2}{\omega_0-\Omega} \coth\left(\frac{\beta\hbar\Omega}{2}\right), \qquad (41)$$

so that we end up with the equation

$$\dot{\hat{\sigma}}_- = -\left(\kappa + i\widetilde{\omega}_0\right)\hat{\sigma}_-, \qquad (42)$$

where κ was defined in Eq. (28). The solution is again simple:

$$\boxed{\hat{\sigma}_-(t) = e^{-\left(\kappa+i\widetilde{\omega}_0\right)t}\hat{\sigma}_-(0) = e^{-\left(\kappa+i\widetilde{\omega}_0\right)t} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad \text{[Heisenberg]}} \qquad (43)$$

For the raising operator we get

$$\boxed{\hat{\sigma}_+(t) = \hat{\sigma}_-(t)^\dagger = e^{-\left(\kappa-i\widetilde{\omega}_0\right)t} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad \text{[Heisenberg]}} \qquad (44)$$

Note that these results imply that the ladder operators oscillate with a modified frequency $\widetilde{\omega}_0$ rather than with ω_0 (the difference is of second-order in the coupling constants); we may take this modification (the Lamb shift) into account in the dynamics of $\hat{\sigma}_z$, replacing ω_0 with $\widetilde{\omega}_0$ such that

$$\boxed{\hat{\sigma}_z(t) = e^{-2\kappa t} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \left(1 - e^{-2\kappa t}\right) \tanh\left(\frac{\beta\hbar\widetilde{\omega}_0}{2}\right) \mathbb{I}. \quad \text{[Heisenberg]}} \qquad (45)$$

Full density operator dynamics (Schrödinger picture)

From the correspondence of expectation values, we conclude that within the Schrödinger picture the terms of density operator exhibit the following dynamics:

$$\begin{aligned}
\rho_{11}\left(t\right) &= \rho_{11}\left(0\right)e^{-2\kappa t} + \left(1 - e^{-2\kappa t}\right)\cdot\frac{1}{e^{\beta\hbar\widetilde{\omega_0}}+1},\\
\rho_{22}\left(t\right) &= \rho_{22}\left(0\right)e^{-2\kappa t} + \left(1 - e^{-2\kappa t}\right)\cdot\frac{e^{\beta\hbar\widetilde{\omega_0}}}{e^{\beta\hbar\widetilde{\omega_0}}+1},\\
\rho_{12}\left(t\right) &= \rho_{12}\left(0\right)e^{-\left(\kappa+i\widetilde{\omega_0}\right)t},\\
\rho_{21}\left(t\right) &= \rho_{21}\left(0\right)e^{-\left(\kappa-i\widetilde{\omega_0}\right)t}.
\end{aligned} \tag{46}$$