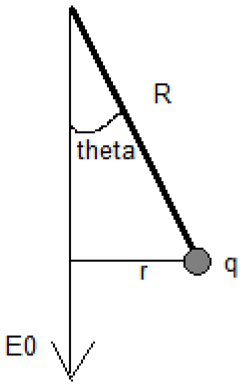
QM1 - Test - Question 7

February 8, 2015

a.

The situation: constant electric field $E = E_0\hat{z}$, pendulum with rod in lenght R, φ is the angle of rotation around $\hat{z}$ and θ is the angle in the (x,y) plane. There is no gravity in the system.



The degrees of freedom of the system: θ, φ where φ is the angle of rotation in the (x,y) plane and θ is the angle between the pendulum and $\hat{z}$. Therefore we expect the system to be described by two quantum numbers.

Note: In the reference he wrote that the hamiltonian we will get commutes with L_z - and claimed that there is conservation of L_z - I'm not sure that there is a point in saying this.

b. +c.

Note: This solution combines both bullets - you can first write all the expressions without doing the long wavelenght approximation and claim that this is the general expression.

The full hamiltonian:

$$\mathcal{H} = \frac{\left(\overline{P} - \frac{e}{c}\overline{A}\right)^2}{2m} - |q|\,|E_0|\,R\cos\theta = \frac{\overline{P}^2 - \frac{e}{2c}\left(\overline{PA} + \overline{AP}\right) + \left(\frac{e}{c}\right)^2\overline{A}^2}{2m} - |q|\,|E_0|\,R\cos\theta$$

We neglect $\overline{A}^2$ and divide the hamiltonian to the interacting and non intearcting parts:

$$\mathcal{H} = \underbrace{\frac{\overline{P}^2}{2m} - |q|\,|E_0|\,R\cos\theta}_{\mathcal{H}_0} \underbrace{- \frac{e}{2c}\left(\overline{PA} + \overline{AP}\right)}_{\mathcal{H}_{int}}$$

The non interacting part:

$$\mathcal{H}_0 = \frac{\overline{P}^2}{2m} - |q|\,|E_0|\,R\cos\theta = \frac{\overline{P}^2}{2m} - |q|\,|E_0|\,R\sqrt{1-\left(\frac{r}{R}\right)^2}$$

$$R\sqrt{1-\left(\frac{r}{R}\right)^2} \simeq_{\text{for small r}} R\left(1 - \frac{1}{2}\left(\frac{r}{R}\right)^2\right) = R - \frac{1}{2}\frac{r^2}{R} = R - \frac{1}{2}\frac{x^2+y^2}{R}$$

Therefore the non interacting part is

$$\mathcal{H}_0 = \frac{P_x^2 + P_y^2 + P_z^2}{2m} - R\,|q|\,|E_0| + \frac{m}{2}\frac{|q|\,|E_0|}{mR}\left(x^2+y^2\right)$$

We get two harmonic oscillators with frequency $\omega = \sqrt{\frac{|q||E_0|}{mR}}$- in x,y and a constant. (Also a free particle in z- we assumed small oscillations and therefore it means that P_z is small and we are going to neglect it)

Denoting the eigenstates $|n_x, n_y\rangle$ with energies:

$$E_{n_x,n_y} = \omega\left(n_x + n_y + 1\right) - R\,|q|\,|E_0|$$

Now we deal with the interaction part:

$$\mathcal{H}_{int} = -\frac{e}{2c}\left(\overline{PA} + \overline{AP}\right) = -\sum_{k\alpha}\left(\frac{\hbar}{\omega_k\Omega}\right)^{1/2} j_{-k}\lambda_{k\alpha}\left(a_{k\alpha} + a^{\dagger}_{-k\alpha}\right)$$

The general form of the hamiltonian:

$$H = H_{matter} + \underbrace{\sum_{k\alpha}\hbar\omega_k a^{\dagger}_{k\alpha}a_{k\alpha}}_{H_{radiation}} + U_{int} = H_{matter} + \sum_{k\alpha}\hbar\omega_k a^{\dagger}_{k\alpha}a_{k\alpha} - \sum_{k\alpha}\left(\frac{\hbar}{\omega_k\Omega}\right)^{1/2} j_{-k}\lambda_{k\alpha}\left(a_{k\alpha} + a^{\dagger}_{-k\alpha}\right)$$

In our case: $\mathcal{H}_0 = H_{matter}$. We start from some excited state of $\mathcal{H}_0$ - n_x or n_y (or both) are not zero, and no photons in the radiation part. We will finish with a state which is in a lower energy level in the matter part, and contains one photon in the radiation part.

The fermi golden rule:

$$\Gamma = \frac{2\pi}{\hbar}\left|\langle f|U_{int}|i\rangle\right|^2 \delta\left(E_f - E_i\right)$$

Here: $|i\rangle = |n_x, n_y\rangle \otimes \Pi_{k,\alpha}|N_{k,\alpha}\rangle$ where $\Pi_{k,\alpha}|N_{k,\alpha}\rangle$ is a state in which there are $N_{k,\alpha}$ photons in each energy ω_k and polarization α.

$|f\rangle = \left|n'_x, n'_y\right\rangle \otimes \Pi_{k\neq k'\ \&\ \alpha\neq\alpha'}|N_{k,\alpha}\rangle\,|N_{k',\alpha'} + 1\rangle$ - this is a state in which for a specific frequency k' there is one additional photon.

The condition on the energies:

$$\delta\left(E_f - E_i\right) = \delta\left(\omega\left(n'_x + n'_y - n_x - n_y\right) + \omega_k\right)$$

We are going to take the long wavelenght approximation which will not allow us to jump between states which have a difference of more than 1, the condition on the energies will give us: $n'_x + n'_y - n_x - n_y = 1$ therefore $\omega = \omega_k$

$$\begin{aligned}
& \left\langle n'_x n'_y; \Pi_{k\neq k'\ \&\ \alpha\neq\alpha'}N_{k,\alpha}, N_{k',\alpha'}\right| - \sum_{k\alpha}\left(\frac{\hbar}{\omega_k\Omega}\right)^{1/2}\overline{j_{-k}}\cdot\overline{\lambda_{k\alpha}}\left(a_{k\alpha} + a^{\dagger}_{-k\alpha}\right)\left|n_x n_y; \Pi_k N_k\right\rangle \\
(*) \quad = & \left\langle n'_x n'_y; \Pi_{\&\ \alpha\neq\alpha'}N_{k,\alpha}, N_{k',\alpha'}\right| - \left(\frac{\hbar}{\omega_{-k'}\Omega}\right)^{1/2}\overline{j_{k'}}\cdot\overline{\lambda_{-k'\alpha}}a^{\dagger}_{k'\alpha}\left|n_x n_y; \Pi_k N_k\right\rangle \\
= & \left\langle n'_x n'_y; \Pi_{\&\ \alpha\neq\alpha'}N_{k,\alpha}, N_{k',\alpha'}\right| - \left(\frac{\hbar}{\omega_{-k'}\Omega}\right)^{1/2}\overline{j_{k'}}\cdot\overline{\lambda_{-k'\alpha}}\sqrt{N_{k'}+1}\left|n_x n_y; \Pi_k N_k\right\rangle \\
= & -\sqrt{N_{k'}+1}\left(\frac{\hbar}{\omega_{-k'}\Omega}\right)^{1/2}\overline{\lambda_{-k'\alpha}}\cdot\langle n'_x n'_y|\overline{j_{k'}}|n_x n_y\rangle \underbrace{\langle\Pi_{\&\ \alpha\neq\alpha'}N_{k,\alpha}, N_{k',\alpha'}|\Pi_k N_k\rangle}_{=1}
\end{aligned}$$

$(*)$ - the only braket which will not give zero is the one for which $k = -k'$ - therefore we removed the sum on all the others.

The current:

General expression:

$$\overline{j}\left(r\right) = \frac{q_a}{2}\sum_a \frac{\hat{\overline{p}}_a}{m_a}\delta\left(r - \hat{\overline{r_0}}\right) + \delta\left(r - \hat{\overline{r_0}}\right)\frac{\hat{\overline{p}}_a}{m_a}$$

Here we have only one particle, therefore there is no sum:

$$j\left(r\right) = \frac{q}{2}\left[\frac{\hat{\overline{P}}}{m}\delta\left(r - \hat{\overline{r_0}}\right) + \delta\left(r - \hat{\overline{r_0}}\right)\frac{\hat{\overline{P}}}{m}\right]$$

We need j_k: we work in the long wavelength approximation, meaning $ka \ll 1$ when a is the characteristic lenght of the system. (Here $r \simeq a$ - this is the characteristic lenght) - therefore we take the 1st order in the exponent:

$$j_k = \int j_r e^{i\overline{k}\cdot\overline{r}} d^3r \simeq \int j_r d^3r$$

Here we get:

$$j_k = \int \frac{q}{2}\left(\frac{\hat{\overline{P}}}{m}\delta\left(r - \hat{\overline{r_0}}\right) + \delta\left(r - \hat{\overline{r_0}}\right)\frac{\hat{\overline{P}}}{m}\right) d^3r = q\frac{\overline{P}}{m}$$

Calculating only the braket, using:

$$P_x = i\sqrt{\frac{m\omega}{2}}\left(a_x^\dagger - a_x\right)\ ,\ P_y = i\sqrt{\frac{m\omega}{2}}\left(a_y^\dagger - a_y\right)$$

$$\begin{aligned}\langle n_x' n_y' | \overline{j_{k'}} | n_x n_y\rangle &= \frac{q}{m}\langle n_x' n_y' | P_x\hat{x} + P_y\hat{y} | n_x n_y\rangle = \frac{q}{m} i\sqrt{\frac{m\omega}{2}}\langle n_x' n_y' | \left(a_x^\dagger - a_x\right)\hat{x} + \left(a_y^\dagger - a_y\right)\hat{y} | n_x n_y\rangle \\ &= -qi\sqrt{\frac{\omega}{2m}}\langle n_x' n_y' | a_x\hat{x} + a_y\hat{y} | n_x n_y\rangle = -qi\sqrt{\frac{\omega}{2m}}\left(\sqrt{n_x}\delta_{n_y' n_y}\delta_{n_x'(n_x-1)}\hat{x} + \sqrt{n_y}\delta_{n_x' n_x}\delta_{n_y'(n_y-1)}\hat{y}\right)\end{aligned}$$

We get only transitions between adjacent states (only differenc of 1 between n_x, n_x' and n_y, n_y' - this is because we assumed the long wavelength approx.)

We need to calculate the dot product without the prefactors:

$$\overline{\lambda_{-k'\alpha}} \cdot \left(\sqrt{n_x}\delta_{n_y' n_y}\delta_{n_x'(n_x-1)}\hat{x} + \sqrt{n_y}\delta_{n_x' n_x}\delta_{n_y'(n_y-1)}\hat{y}\right)$$

The total expression: (Assuming, without loss of generality, from now, that we are in the state $n_x' = n_x - 1$ and $n_y = n_y'$)

$$\Gamma = \frac{2\pi}{\hbar}q^2\frac{\omega}{2m}\left(N_{k'}+1\right)\frac{\hbar}{\omega_{-k'}\Omega}\left|\overline{\lambda_{-k'\alpha}} \cdot \sqrt{n_x}\hat{x}\right|^2 = q^2\frac{\omega}{2m}\left(N_{k'}+1\right)\frac{2\pi}{\omega_{-k'}\Omega}\left|\left(\overline{\lambda_{-k'\alpha}}\right)_x\right|^2 n_x$$

This is the rate of transition.

Writing explicitly for a general polarization vector:

$$\begin{aligned}\hat{k} &= (sin\theta_k cos\varphi_k, sin\theta_k sin\varphi_k, cos\theta_k) \\ \overline{\lambda_{k,1}} &= (cos\theta_k cos\varphi_k, cos\theta_k sin\varphi_k, -sin\theta_k) \\ \overline{\lambda_{k,2}} &= (-sin\varphi_k, cos\varphi_k, 0)\end{aligned}$$

Choosing, like we did, $|i\rangle = |n_x, n_y\rangle$and $|f\rangle = |n_x - 1, n_y\rangle$ and then $\overline{\lambda_{-k'\alpha}} \cdot \sqrt{n_x}\hat{x} = cos\theta_k cos\varphi_k\sqrt{n_x}$

Then the rate for each polarization is:

$$\begin{aligned}\Gamma|_{\lambda_{k,1}} &= q^2\frac{\omega}{2m}\left(N_{k'}+1\right)\frac{2\pi}{\omega_{-k'}\Omega}cos^2\theta_k cos^2\varphi_k n_x \\ \Gamma|_{\lambda_{k,2}} &= q^2\frac{\omega}{2m}\left(N_{k'}+1\right)\frac{2\pi}{\omega_{-k'}\Omega}sin^2\varphi_k n_x\end{aligned}$$

For a general polarization:

$$\hat{\lambda} = cos\beta\hat{\lambda_{k,1}} + e^{i\alpha}sin\beta\hat{\lambda_{k,2}}$$

Then the rate is

$$\Gamma|_{\hat{\lambda}} = q^2\frac{\omega}{2m}\left(N_{k'}+1\right)\frac{2\pi}{\omega_{-k'}\Omega}n_x\left|cos\beta cos\theta_k cos\varphi_k - e^{i\alpha}sin\beta sin\varphi_k, cos\varphi_k\right|^2$$

d.

Note: wherever i wrote ω_k here it should be $\omega_{-k'}$.

The angular distribution: ($d\gamma$ is the solid angle)

$$\begin{aligned}
dN &= \sum_{\overline{k}\in d\Omega} \Gamma = \int_{k\in d\Omega} \Gamma \frac{\Omega d^3k}{(2\pi)^3} = \frac{\Omega d\gamma}{(2\pi)^3} \int \Gamma k^2 dk \\
&= \frac{\Omega d\gamma}{(2\pi)^3} \int q^2 \frac{\omega}{2m} (N_{k'}+1) \frac{2\pi}{\omega_{-k'}\Omega} \left| \left(\overline{\lambda_{-k'\alpha}}\right)_x \right|^2 n_x k^2 \delta (E_f - E_i)\, dk \\
&= \frac{d\gamma}{(2\pi)^2} \int q^2 \frac{\omega}{2m} (N_{k'}+1) \frac{1}{\omega_{-k'}} \left| \left(\overline{\lambda_{-k'\alpha}}\right)_x \right|^2 n_x \delta (\omega_k - \omega) \frac{\omega_k^2}{c^2} \frac{d\omega_k}{c} \\
&= \frac{d\gamma}{(2\pi)^2} q^2 \frac{}{2m} (N_{k'}+1) \left| \left(\overline{\lambda_{-k'\alpha}}\right)_x \right|^2 n_x \frac{\omega_k^2}{c^3}
\end{aligned}$$

Note: in the reference from last year he got ^{3} instead of .

If we cannot distinguish between polarizations: (Shimon told him to write the expression explicitly)

$$\frac{dN}{d\gamma}|_{tot} = \frac{dN}{d\gamma}|_{\overline{\lambda}_{k,1}} + \frac{dN}{d\gamma}|_{\overline{\lambda}_{k,2}}$$

Note: we already did the whole thing for general n_x, n_y - he probably wanted us to start with $n_x = n_y = 1$ or something.