Quantized Electromagnetic Field

Selected chapter of lecture notes on Quantum Mechanics

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Contents

1 Introduction

In this Chapter we will show how the quantum mechanical description of one or several particles is extended to the quantization of electromagnetic field. In contrast to particles which are described by the coordinates of their positions $\mathbf{r}_a$, $a = 1, ..., N$ (N- the number of particles) the electromagnetic field is described by the configuration of the electric and magnetic fields $\mathbf{E}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$. In order to learn how such extended systems are treated in quantum mechanics we shall start with a much simpler system - that of a one dimensional string.

2 Simple System First - Quantum Mechanics of a Guitar String

2.1 Classical string

We consider a string depicted in Fig. 1

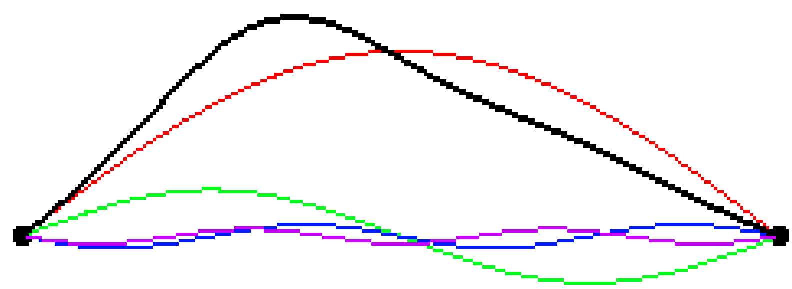

Figure 1: Configurations of a guitar string. Denoting the abscissa of the figure (the equilibrium position of the string) by x and the ordinate (the string deviations from the equilibrium) by ϕ the string configurations are described by a function $\phi(x)$.

Classically its general configuration is conveniently described by a function $\phi(x)$ which determines the deviations ϕ of the string from the equilibrium position $\phi = 0$ at every point of the axis x. For simplicity we shall assume that the ends of the string are fixed at $x = 0$ and $x = L$

$$\phi(0) = 0 \ \ , \ \ \phi(L) = 0 \ \ . \tag{1}$$

In the following section we will extend our discussion to a more relevant example of a string with periodic boundary conditions - the so called closed string.

We will assume that classically the string is described by a simple linear wave equation

$$\frac{\partial^2 \phi(x,t)}{\partial t^2} = v^2 \frac{\partial^2 \phi(x,t)}{\partial x^2} \tag{2}$$

where v has dimensionality of velocity and is actually the phase (as well as group) velocity of the waves

$$\phi(x,t) = A \sin[kx \pm \omega(k)t + \alpha] \ .$$

These are solutions of the wave equation as can be easily verified by direct substitution. Here A and α are arbitrary constant amplitude and phase and the dispersion relation is

$$\omega(k) = vk \ . \tag{3}$$

It will be very useful to view the function $\phi(x)$ as a collection of the coordinates describing the "position" of the string. To emphasize this one might think of $\phi(x)$ as a set $\{\phi_x\}$ indicating that x is actually an index numbering different coordinates. To make it even more precise the x variable can be discretized and $\phi(x)$ reduced to $N+1$ variables as follows

$$\phi_x \equiv \phi(x = n\Delta x) \ \ , \ \ \Delta x = L/N \ \ , \ \ n = 0, 1, ..., N$$

Formally one should at the end let $N \to \infty$, $\Delta x \to 0$ but in practice it is enough to have Δx much smaller than the smallest wave length Λ of the waves which one intends to consider. The physical reasons behind the cutoff Λ may actually be the requirement that it is much larger than the microscopic length scales related to say the distances between the constituents of which the string is built.

2.2 Quantum description of the string

2.2.1 The wave functional

Our goal is to quantize the classical string as described above. We shall use the the straightforward generalization of the canonical quantization procedure for system with few degrees of freedom like one or several particles. This means that instead of having a definite $\phi(x)$ describing the string configuration we must assume that for each $\phi(x)$ there is a probability amplitude $\Psi[\phi(x)]$ which contains all the (quantum) information about the string. In particular $|\Psi[\phi(x)]|^2$ gives the probability density to find a particular configuration $\phi(x)$.

Mathematically $\Psi[\phi(x)]$ represents a correspondence between the set of all functions $\phi(x)$ subject to the conditions Eq. (1) and a set of complex numbers Ψ. Such a correspondence is called a functional. Examples of functionals should be familiar to the reader already from classical mechanics where the classical action $S[q(t)]$ is a functional of the trajectories $q(t)$.

The functional $\Psi[\phi(x)]$ is called the wave functional. Using the approximate discretized form ϕ_x of the functions $\phi(x)$ the functional $\Psi[\phi(x)]$ can actually be viewed as a function of $N-1$ variables $\{\phi_{x=n\Delta x}\}$. The variables ϕ_0 and $\phi_{x=N\Delta x}$ are fixed to 0 to comply with Eq. (1).

Classically string dynamics is described by the time dependence $\phi(x,t)$ as governed by the equation (2). Quantum mechanical time evolution should be described by the time dependence of the wave functional $\Psi[\phi(x), t]$. What governs this time evolution? Continuing the analogy with the few degrees of freedom system this should be the Schrödinger equation

$$i\hbar \frac{\partial \Psi[\phi(x), t]}{\partial t} = H_{\text{op}} \Psi[\phi(x), t] \tag{4}$$

with H_{op} the Hamiltonian operator of the string. We will now determine this operator and will learn how to find the operators of the string observables following the standard route.

2.2.2 The string Hamiltonian

We will start by finding the classical Hamiltonian function of the string. For this we shall rewrite the string equation (2) in the Hamiltonian form. It is useful to note that this equation represents a set of coupled Newton equations for the string coordinates ϕ_x. This can be seen by rewriting it in a discretized form

$$\frac{d^2\phi_x}{dt^2} = \frac{v^2}{\Delta x^2} \left(\phi_{x+\Delta x} - 2\phi_x + \phi_{x-\Delta x}\right) \tag{5}$$

where we used the discretized form of the second derivative

$$\frac{\partial^2 \phi(x)}{\partial x^2} \to \frac{1}{\Delta x} \left(\frac{\phi_{x+\Delta x} - \phi_x}{\Delta x} - \frac{\phi_x - \phi_{x-\Delta x}}{\Delta x} \right)$$

We will rewrite the second order in time wave (Newton) equation of the string as a pair of first order equations

$$\frac{\partial \phi(x,t)}{\partial t} = \pi(x,t) \quad , \quad \frac{\partial \pi(x,t)}{\partial t} = v^2 \frac{\partial^2 \phi(x,t)}{\partial x^2} \tag{6}$$

where (as usual) the first equation is actually the definition of the momenta. As a next step let us introduce the following functional

$$H[\pi(x), \phi(x)] = \int_0^L dx \left[\frac{1}{2}\pi^2(x) + \frac{v^2}{2} \left(\frac{\partial \phi(x)}{\partial x} \right)^2 \right] \tag{7}$$

Using it we can write the pair (6) as

$$\frac{\partial \phi(x,t)}{\partial t} = \frac{\delta H[\pi(x,t), \phi(x,t)]}{\delta \pi(x,t)} \quad , \quad \frac{\partial \pi(x,t)}{\partial t} = -\frac{\delta H[\pi(x,t), \phi(x,t)]}{\delta \phi(x,t)} \quad . \tag{8}$$

Here the notation $\delta/\delta\pi(x,t)$ and $\delta/\delta\phi(x,t)$ stands for the variational derivative (see below) with respect to $\pi(x,t)$ and $\phi(x,t)$ respectively. We now show that the above equations are indeed equivalent to the pair (6) recalling in passing how the variational derivatives are defined and calculated.

We let the functions $\pi(x)$ and $\phi(x)$ in the functional (7) to have infinitesimal variations $\delta\pi(x)$ and $\delta\phi(x)$. The corresponding variation δH due to this is

$$\begin{aligned}\delta H &\equiv H[\pi(x)+\delta\pi(x),\phi(x)+\delta\phi(x)] - H[\pi(x),\phi(x)] = \\ &= \int_0^L dx \left[\pi(x)\delta\pi(x) + v^2 \frac{\partial\phi(x)}{\partial x}\frac{\partial\delta\phi(x)}{\partial x}\right] \;+\; \text{higher order terms} \\ &= \int_0^L dx \left[\pi(x)\delta\pi(x) - v^2 \frac{\partial^2\phi(x)}{\partial x^2}\delta\phi(x)\right] \;+\; \text{higher order terms}\end{aligned}$$

where we used integration by parts in the second term.

In analogy with the relation of the differential dF of a function of many variables $F(q_1, q_2, ..., q_N)$ and its partial derivatives

$$dF = \sum_{n=1}^{N} \frac{\partial F}{\partial q_n} dq_n$$

the functional derivatives of $H[\pi(x,t),\phi(x,t)]$ are by definition the functions which multiply $\delta\pi(x,t)$ and $\delta\phi(x,t)$ respectively in the expression for the variation δH,

$$\frac{\delta H[\pi(x),\phi(x)]}{\delta\pi(x)} = \pi(x) \quad , \frac{\delta H[\pi(x),\phi(x)]}{\delta\phi(x)} = -v^2\frac{\partial^2\phi(x)}{\partial x^2} \tag{9}$$

Inserting these relations into Eq. (8) we see that they indeed reproduce Eq. (6).

Equations (8) have the Hamiltonian form with $H[\pi(x),\phi(x)]$ as the Hamiltonian and $\pi(x)$, $\phi(x)$ as the momenta and coordinates. It should perhaps be more clear if for a moment we use the notation π_x and ϕ_x instead of $\pi(x)$ and $\phi(x)$ treating x as a label. The equations (8) in these notations are

$$\frac{\partial\phi_x(t)}{\partial t} = \frac{\partial H[\pi_x(t),\phi_x(t)]}{\partial\pi_x(t)} \quad , \quad \frac{\partial\pi_x(t)}{\partial t} = -\frac{\delta H[\pi_x(t),\phi_x(t)]}{\partial\phi_x(t)} \quad .$$

2.2.3 Basic quantum operators for the string

We shall now proceed to define the quantum mechanical operator $H_{\rm op}$. We will do this by first determining what are the operators corresponding to $\pi(x)$ and $\phi(x)$ and then replacing with them the latter in the expression (7) for the classical Hamiltonian.

Since in our formulation $\phi(x)$ are the coordinates of the string the corresponding operator $\phi_{\rm op}(x)$ should be just the operator of multiplication by $\phi(x)$, i.e. its action on an arbitrary wave functional is

$$\phi_{\rm op}(x)\Psi[\phi(x')] = \phi(x)\Psi[\phi(x')] \tag{10}$$

To avoid confusion we use different arguments of ϕ's in the operator and in Ψ. This would perhaps be easier to understand if (again momentarily) we shall switch to the notation $\phi_{x'}$ instead of $\phi(x')$. Then the functional $\Psi[\phi(x')]$ is just a function $\Psi(\{\phi_{x'}\})$ of the set of all $\phi_{x'}$ variables. The action of the operator $\hat{\phi}_x$, i.e. the operator of the $x-$th component the coordinates of the string is just a multiplication by ϕ_x with this particular x. Note that in order to avoid the double subscript we here used $\hat{\phi}$ to denote the operator.

In the same way we can determine the operator corresponding to the momentum $\pi(x)$. In the "simplified" notations it should be $\hat{\pi}_x = -i\hbar\partial/\partial\phi_x$ which means that in terms of the functional derivatives it is

$$\pi_{\rm op}(x)\Psi[\phi(x')] = -i\hbar\frac{\delta}{\delta\phi(x)}\Psi[\phi(x')] \tag{11}$$

We note that the commutator of the basic operators is

$$[\phi_{\rm op}(x), \pi_{\rm op}(y)] = i\hbar\delta(x-y) \tag{12}$$

This can verified by acting with the commutator on an arbitrary wave functional

$$[\phi_{\rm op}(x), \pi_{\rm op}(y)]\Psi[\phi(x')] = -i\hbar\left(\phi(x)\frac{\delta}{\delta\phi(y)}\Psi[\phi(x')] - \frac{\delta}{\delta\phi(y)}\phi(x)\Psi[\phi(x')]\right) =$$

$$= -i\hbar\left(\phi(x)\frac{\delta}{\delta\phi(y)}\Psi[\phi(x')] - \frac{\delta\phi(x)}{\delta\phi(y)}\Psi[\phi(x')] - \phi(x)\frac{\delta}{\delta\phi(y)}\Psi[\phi(x')]\right) =$$
$$= i\hbar\delta(x-y)\Psi[\phi(x')]$$

Substituting the operators $\phi_{\rm op}(x)$ and $\pi_{\rm op}(x)$ in the Hamiltonian (7) we obtain

$$H_{\rm op} = \int_0^L dx\left[\frac{1}{2}\pi_{\rm op}^2(x) + \frac{v^2}{2}\left(\frac{\partial\phi_{\rm op}(x)}{\partial x}\right)^2\right] \tag{13}$$

Armed with this explicit form of the Hamiltonian operator of our system we can proceed to solve the Schrödinger equation (4). Since the Hamiltonian is time independent it will be sufficient to solve the stationary equation

$$H_{\rm op}\Psi = E\Psi \tag{14}$$

Knowing all its solutions will allow to find the most general solution of (4).

The string Hamiltonian operator (7) may look formidable but is actually quite simple because of its quadratic dependence on the coordinates and momenta. This of course is a direct consequence of the linearity of the string equation (2). As is seen from the discretized form (5) such equations describe coupled harmonic oscillators. The standard way of solving such problems is to make a transformation to normal modes.

2.3 Reminder - normal modes of vibrations

Let us recall how the transformation to normal modes is done in the general context represented by the set of N coupled equations

$$m_l \ddot{q}_l = -\sum_{n=1}^{N} k_{ln} q_n \quad , l = 1, ..., N \tag{15}$$

with masses m_l and N by N symmetric matrix of elastic constants $k_{ln} = k_{nl}$. For simplicity we shall assume in the following that all the masses are equal $m_1 = ... = m_N = m$. The Hamiltonian of this problem is the standard sum of the kinetic and potential energies

$$H = \frac{1}{2m} \sum_{l=1}^{N} p_l^2 + \frac{1}{2} \sum_{l,n=1}^{N} k_{ln} q_l q_n \tag{16}$$

Let us try the following solution of the equations (15)

$$q_l = Re(C_l e^{i\omega t}), \quad l = 1, ..., N$$

where Re stands for real part and C_l's are constants. This form assumes that all the degrees of freedom vibrate with the same frequency. Inserting this into the equations (15) with all $m_i = m$. we obtain

$$\sum_{n=1}^{N} (k_{ln} - m\omega^2 \delta_{ln}) C_n = 0 \tag{17}$$

To have a non trivial solution one must demand that

$$\det(k_{ln} - m\omega^2 \delta_{ln}) = 0 \tag{18}$$

which shows that ω^2 is an eigenvalue of the matrix k_{ln}/m which in turns means that in general one will have N such solutions which will have $\omega^2 > 0$ provided k_{ln} is positive definite.

Let us denote by ω_ν and $\{C_n^\nu\}$ the set of N solutions of Eq.(17). The symmetry of k_{ij} assures orthogonality of the eigenvectors $\{C_n^\nu\}$'s with different eigenvalues ω_ν. For a degenerate case i.e. if some $\omega_\mu = \omega_\nu$ one has a freedom to choose $\{C_n^\mu\}$ and $\{C_n^\nu\}$ to ensure that orthogonality holds also in this case. We also note that since the equations (17) are homogeneous at least one of the components in a given vector $\{C_n^\nu\}$ is arbitrary and can be used to set normalization of $\{C_n^\nu\}$'s to unity. We thus have orthonormality

$$\sum_{n=1}^{N} C_n^\mu C_n^\nu = \delta_{\mu\nu}$$

The N vectors $\{C_n^\nu\}$ each with N components form an $N \times N$ matrix. The orthonormality conditions (together with completeness which we do not discuss) mean that this matrix is orthogonal (unitary for complex C_n's). Let us use it to make the transformation to new coordinates

$$q_n = \sum_{\nu=1}^{N} C_n^\nu Q_\nu \tag{19}$$

Inserting this in Eqs. (15) (with $m_i = m$) and using Eqs. (17) one obtains

$$\sum_{\nu=1}^{N} [m C_l^\nu \ddot{Q}_\nu + \sum_{n=1}^{N} k_{ln} C_n^\nu Q_\nu] = \sum_{\nu=1}^{N} m C_l^\nu [\ddot{Q}_\nu + \omega_\nu^2 Q_\nu] = 0 \tag{20}$$

Due to orthogonality of C's one finds that the equations for Q_ν's are decoupled. Indeed multiplying by C_l^μ and summing over l one obtains

$$\ddot{Q}_\mu + \omega_\mu^2 Q_\mu = 0 \ \ , \ \ \mu = 1, ..., N \ . \tag{21}$$

The transformation (19) from the original coordinates q_l to the new Q_ν is called the transformation to normal modes and Q_ν – the normal mode coordinates.

How does the Hamiltonian look in the new coordinates? To answer this we need to add to (19) also the transformation to the corresponding normal modes momenta. Since in our case $p_l = m\dot{q}_l$ it is clear that p's transform like q's

$$p_n = \sum_{\nu=1}^{N} C_n^\nu P_\nu \tag{22}$$

Inserting this and (19) into the Hamiltonian (16) we obtain using the orthonormality of C_n^μ's and equations (17)

$$H = \frac{1}{2m}\sum_{\mu=1}^{\infty}[P_\mu^2 + \omega_\mu^2 Q_\mu^2] \tag{23}$$

In normal mode variables the Hamiltonian is just a collection of independent oscillators.

2.4 String as a collection of decoupled oscillators

2.4.1 Normal modes of the guitar string

We will now use the technique described in the previous subsection to transform the string Hamiltonian to a collection of independent oscillators. We are looking for the analog of the transformation (19) from the string coordinates $\phi(x)$ to the normal modes coordinates. Since x here plays the role of the index n in q_n the analog of the matrix C_n^ν of the transformation should be functions of x defined on the interval $0 \le x \le L$. We denote the set of these functions by $u_n(x)$ and write

$$\phi(x) = \sum_\nu u_\nu(x) Q_\nu \ \ , \ \ u_\nu(0) = u_\nu(L) = 0 \tag{24}$$

where we indicated that $u_\nu(x)$ should vanish at the end points of the string to assure the boundary conditions (1).

By comparing the string equation (2) and the coupled oscillators equation (15) we see that the role of the coupling matrix k_{ij} is played by $-v^2\partial^2/\partial x^2$ so the functions $u_\nu(x)$ must satisfy (cf., Eq. (17))

$$-v^2\frac{\partial^2}{\partial x^2}u(x) = \omega^2 u(x) \ \ , \ \ u(0) = u(L) = 0 \tag{25}$$

which has orthonormal eigenfunction solutions

$$u_\nu(x) = \sqrt{\frac{2}{L}}\sin k_\nu x \ \ , \quad k_\nu = \frac{\pi\nu}{L} \ \ , \ \ \nu = 1, 2, ... \tag{26}$$

with eigenvalues

$$\omega_\nu = v k_\nu \tag{27}$$

We have fixed the coefficients in $u_\nu(x)$'s so that these functions are normalized.

To conclude - the functions (26) represent the configurations of the string normal modes in which all the points of the string oscillate with the same frequency ω_ν which depends on the wave number k of the mode, Eq. (27). We remind that in general the relation $\omega = \omega(k)$ of the frequency upon the wave vector is called the dispersion relation. It is the most important characteristic of linear waves.

2.4.2 Quantum mechanics of string normal modes

Using $u_\nu(x)$'s we can transform the Hamiltonian operator (13) of the string to a sum of independent oscillators. We view the operators $\phi_{\rm op}(x)$ and $\pi_{\rm op}(x)$ as functions of x and expand

$$\phi_{\rm op}(x) = \sum_{\nu=1}^{\infty} \sqrt{\frac{2}{L}} \sin k_\nu x \, \hat{Q}_\nu \quad , \quad \pi_{\rm op}(x) = \sum_{\nu=1}^{\infty} \sqrt{\frac{2}{L}} \sin k_\nu x \, \hat{P}_\nu \tag{28}$$

Since we expand operator valued functions the coefficients $\hat{Q}_\nu$ and $\hat{P}_\nu$ here are operators which we denoted by hats above to avoid double subscripts.

There are important relations which these operators must satisfy in order to preserve the canonical commutation relations (12) between $\phi_{\rm op}(x)$ and $\pi_{\rm op}(x)$. These operators must themselves be canonical, i.e. they must obey

$$[\hat{Q}_\mu, \hat{P}_\nu] = i\hbar\delta_{\mu\nu} \quad , \quad [\hat{Q}_\mu, \hat{Q}_\nu] = [\hat{P}_\mu, \hat{P}_\nu] = 0 \tag{29}$$

This can be verified in one of the two ways. We can insert the expansions (28) in $[\phi_{\rm op}(x), \pi_{\rm op}(y)]$. Using the first commutator above and the completeness relation $\sum_\nu u_\nu(x)u_\nu(y) = \delta(x-y)$ we will obtain that (12) is indeed satisfied. The other two commutators simply assure that $\phi_{\rm op}(x)$ and $\pi_{\rm op}(x)$ commute at different points. The other way is to "invert" (28)

$$\hat{Q}_\nu = \int_0^L \phi_{\rm op}(x) \sqrt{\frac{2}{L}} \sin k_\nu x \, dx \quad , \quad \hat{P}_\nu = \int_0^L \pi_{\rm op}(x) \sqrt{\frac{2}{L}} \sin k_\nu x \, dx \tag{30}$$

and calculate the needed commutators. Incidentally the above relations also demonstrate how the operators $\hat{Q}_\nu$ and $\hat{P}_\nu$ should act on wave functionals $\Psi[\phi(x)]$.

The commutation relations (29) mean that $\hat{Q}_\mu$ and $\hat{P}_\mu$ are respectively coordinate and momentum operators of the normal modes of the string. As can be seen from (28) classically they are coordinates and momenta representing the amplitudes and their velocities of all the harmonic standing waves which the string can support.

Inserting the above expansions in (13) and using the orthonormality property of the set (26) we obtain

$$H_{\text{op}} = \frac{1}{2}\sum_{\nu=1}^{\infty}(\hat{P}_\nu^2 + \omega_\nu^2\hat{Q}_\nu^2) \tag{31}$$

The Hamiltonian operator is reduced to a sum of terms each representing harmonic oscillator with unit mass and frequency ω_ν. It is important to observe that the underlying waves on the elastic string can only be seen in the dependence of ω of the oscillators on the corresponding wave vectors encoded in the dispersion relation (27).

Classically the simple form (31) of the Hamiltonian in terms of the normal modes' dynamical variables suggests to switch the string description from $\phi(x,t), \pi(x,t)$ to the set $\{Q_\nu(t), P_\nu(t)\}$. Quantum mechanically we note that the relation (10) and the first of (30) implies that the operators $\hat{Q}_\nu$ are simple multiplication operators

$$\hat{Q}_\nu\Psi[\phi(x)] = Q_\nu\Psi[\phi(x)]$$

Following the commutation relations (29) the canonically conjugate operators $\hat{P}_\nu$ can be taken as

$$\hat{P}_\nu\Psi[\phi(x)] = -i\hbar\frac{\partial}{\partial Q_\nu}\Psi[\phi(x)]$$

This suggests to switch to the description in which wave functionals $\Psi[\phi(x)]$ are viewed as functions of (formally infinite number of) the variables Q_ν

$$\Psi[\phi(x)] \to \Psi(\{Q_\nu\})$$

We now note that the terms in the sum representing H_{op}, Eq. (31) commute between themselves on account of the last pair of commutators in Eq. (29). This means that the eigenfunctions of H_{op} are products of the eigenfunctions of all individual terms in the sum and the corresponding eigenvalues are sums of individual eigenvalues

$$\begin{aligned} E_{\{N_\nu\}} &= \sum_{\nu=1}^{\infty}\hbar\omega_\nu(N_\nu + \frac{1}{2}) \;=\; E_{\text{ground state}} + \sum_{\nu=1}^{\infty}\hbar\omega_\nu N_\nu \;,\; N_\nu = 0, 1, 2, ... \\ E_{\text{ground state}} &= \sum_{\nu=1}^{\infty}\frac{\hbar\omega_\nu}{2} \quad , \quad \Psi_{\{N_\nu\}}(\{Q_\nu\}) = \prod_{\nu=1}^{\infty}\psi_{N_\nu}(\beta_\nu Q_\nu) \;,\; \beta_\nu = \sqrt{\omega_\nu/\hbar} \end{aligned} \tag{32}$$

The eigenfunctions ψ_N are the well known harmonic oscillator eigenfunctions

$$\psi_N(y) = \frac{1}{\sqrt{2^N N!}}\left(\frac{\omega}{\pi\hbar}\right)^{1/4} e^{-y^2/2}H_N(y) \tag{33}$$

where $H_N(y)$ denotes N-th order Hermite polynomial. In Fig. 2 graphs of several of these functions are shown.

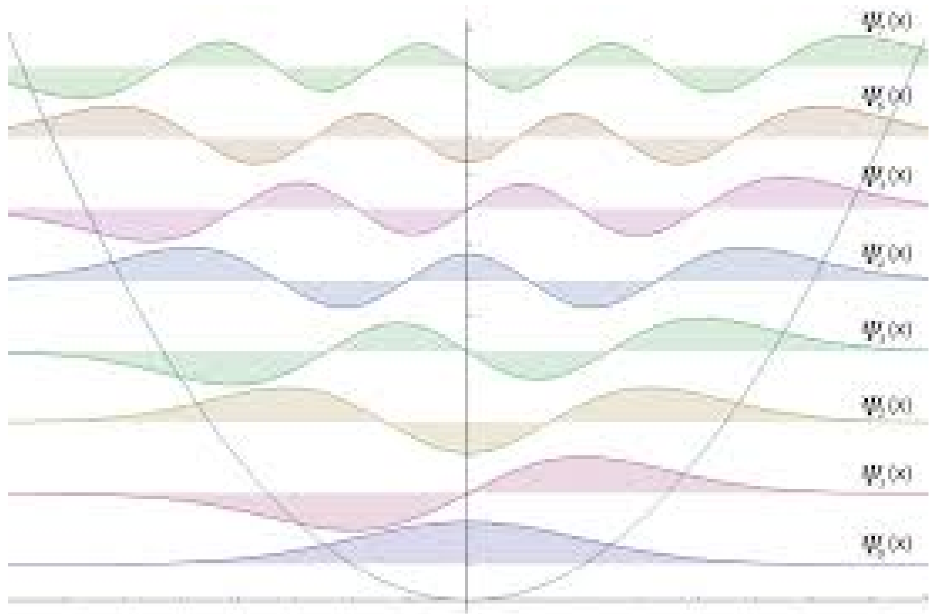

Figure 2: Energy levels and corresponding wave functions of harmonic oscillator. The energy levels are "equidistant", separated by equal energy intervals

The energies (32) exhibit the most important result of our discussion of string quantization - that it can be viewed as a collection of independent quanta with energies $\hbar\omega_\nu = \hbar v k_\nu$. This is a consequence of two general features - string is a linear dynamical system and therefore a collection of normal modes oscillators while the quantum energy levels of oscillators are "equidistant", i.e. separated by equal energy intervals $\hbar\omega$.

In the next Section we will consider a closed string which can support not only normal modes in a form of standing waves like the present fixed end string but also traveling waves. We will show that the corresponding energy quanta of such modes carry mechanical momentum and could therefore be considered as particles.

2.4.3 String ground state. The Casimir effect

The ground state energy in (32) is formally a sum of infinitely many "zero point motion" terms. This is an "ultraviolet" infinity related to the formal possibility to have waves with $k_\nu \to \infty$, i.e. vanishingly small wavelengths $\lambda_\nu = 2\pi/k_\nu$. In practice of course the simple description given by Eq. (2) ceases to be valid at atomic scales and should be replaced by a more elaborate model. As a (much more practical) alternative one could introduce a cutoff k_{cutoff} for "allowed" normal modes in the model and limit the validity of the model (2) to scales $\gg \lambda_{cutoff} = 2\pi/k_{cutoff}$.

Denoting by ν_c the largest integer ν corresponding to the $k_{cutoff} = \pi\nu_c/L$ the ground state wavefunction is a product of ν_c Gaussians

$$\begin{aligned} \Psi_{\text{ground state}}(\{Q_\nu\}) &= \prod_{\nu=1}^{\nu_c} \psi_0(\beta_\nu Q_\nu) = \\ &= \prod_{\nu=1}^{\nu_c} \left(\frac{\omega_\nu}{\pi\hbar}\right)^{1/4} \exp\left(-\sum_{\nu=1}^{\nu_c} \omega_\nu Q_\nu^2/2\hbar\right) \end{aligned} \tag{34}$$

which express the "zero point" fluctuations of the quantum string which is not at rest even in its lowest energy state.

As we will see in the forthcoming sections the ground state of the EM field is expected to exhibit similar zero point fluctuations of the fields in its ground state which is the vacuum of the theory. Is it possible to observe these vacuum fluctuations? In a 1948 famous paper Ref. [1] Casimir proposed a way to do this using what has become known as a Casimir effect. We will now explain its principle idea in the simple example of the guitar string ground state.

Let us consider what will happen with the quantum guitar string if we "fret" it, i.e. press with an imaginary finger hard at some position $x = d$ so that the string will not vibrate at this point, Fig. 3. Obviously this changes the normal modes of the string by excluding the modes which do not vanish at $x = d$. This means that the ground state energy density will change. In fact the new normal modes will consist of two families with $\omega'_\nu = vk_\nu = v\pi\nu/d$ and $\omega''_\nu = v\pi\nu/(L-d)$. The corresponding ground state energy will correspondingly consist of two parts

$$E_0(d) = \sum_{\nu=1}^{\nu_c} \frac{\hbar v \pi}{2} \left[\frac{\nu}{d} + \frac{\nu}{L-d}\right] \tag{35}$$

It is clear that for $d = L/2$ both parts are equal while for $d < L/2$ ($d > L/2$) the first term, i.e. the energy of the narrower (wider) part is smaller (larger) than the second term.

The finite cutoff frequency $\sim \nu_c$ in the above expression "regularizes" the (ultraviolet) divergence of the sum $\sum_\nu \nu$. To eliminate ν_c from the final result one must "renormalize" it which can be done, cf., Ref. [2], by calculating $E_0(d)$ relative to the symmetric configuration at $d = L/2$ with the result[1]

$$\Delta E_0(d) \equiv E_0(d) - E_0(L/2) = -\frac{\pi\hbar v}{24}\left(\frac{1}{d} + \frac{1}{L-d} - \frac{4}{L}\right)$$

It is seen that $\Delta E_0(d)$ is symmetric with respect to $d \to L - d$ and decreases monotonically as $d \to 0$ and $d \to L$ as

$$\Delta E_0(d)|_{d \ll L/2} \to -\frac{\pi\hbar v}{24}\frac{1}{d} \quad , \quad \Delta E_0(d)|_{L-d \ll L/2} \to -\frac{\pi\hbar v}{24}\frac{1}{L-d} \tag{36}$$

The dependence on d means that the function $E_0(d)$ can be considered as the potential energy of the separation point of the string at $x = d$ and that there is a force

$$F(d) = -\frac{\partial E_0(d)}{\partial d}$$

acting on what causes the separation between the two parts of the string (the imaginary fretting finger). This force "tries to drive" the separation towards the end points of the string. A simple physical intuition behind this force is the imbalance of the ground state fluctuations radiation pressure on both sides of the separation point $x = d$ when $d \neq L/2$. One must be aware however that things are more delicate as the sign of the force depends on the type of boundary conditions assumed at $x = d$. For details cf., Ref. [2].

The force $F(d)$ is called the Casimir force and its appearance is a manifestation of the Casimir effect. We will return to this effect below in the context of the vacuum fluctuations of the quantized EM field.

---

[1] The common way of calculating is to use a "soft" cutoff, i.e. to replace e.g. $\sum_{\nu=1}^{\nu_c} \nu$ by $\sum_{\nu=1}^{\infty} \nu e^{-\nu/\nu_c}$, calculate the last sum for $\nu_c \to \infty$ using

$$\sum_{\nu=1}^{\infty} \nu e^{-\alpha\nu} = -\frac{\partial}{\partial\alpha}\sum_{\nu=1}^{\infty} e^{-\alpha\nu} = \frac{\partial}{\partial\alpha}\frac{1}{1-e^{\alpha}} \underset{\alpha\to 0}{\longrightarrow} \frac{1}{\alpha^2} - \frac{1}{12} + ...$$

with $\alpha = 1/\nu_c$

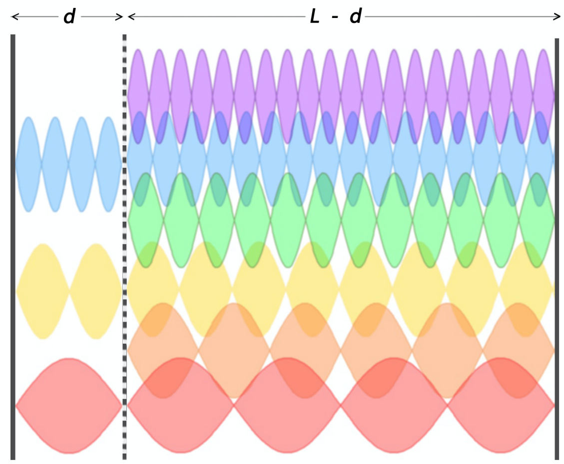

Figure 3: Guitar string fretted at $x = d$ and its (schematically drawn) normal modes

3 Quantization of Traveling Waves. Closed String

The normal modes of a string were described in the previous section are standing waves as can be most clearly seen by considering the expansions (28). Classically $\hat{Q}_\nu$'s and $\hat{P}_\nu$ are functions $Q_\nu(t), P_\nu(t)$ each depending harmonically on time with frequency ω_ν. Thus each term in (28) is a standing wave $\sim Q_\nu(t)\sin k_\nu x$ and $\sim P_\nu(t)\sin k_\nu x$.

3.1 Expansion in traveling waves

3.1.1 Periodic boundary conditions

The standing wave solution of the equations (25) defining the normal modes is a consequence of the fixed ends boundary conditions (1) for the guitar string. These were reflected in the boundary conditions $u_\nu(0) = u_\nu(L) = 0$ in the normal modes equations Eqs. (25). In this section we will explore a more interesting and practical situation when the normal modes are traveling waves. This is realized if one assumes periodic boundary conditions for a string, i.e. for every x

$$\phi(x,t) = \phi(x+L,t) \tag{37}$$

This effectively means that such string does not have ends, i.e. it is closed and equivalent to a circle. Note in passing that differentiating the periodicity condition with respect to x shows that also the derivatives $\partial\phi/\partial x$, $\partial^2\phi/\partial x^2$, etc of ϕ are periodic. According to the string equation (2) this means that so are the time derivatives.

The periodic boundary conditions for $\phi(x)$ are translated into conditions

$$u_\nu(x) = u_\nu(x+L)$$

replacing the fixed ends conditions in the normal modes equation (25). The solutions are now coming as an infinite set of degenerate pairs each with the same frequency

$$\sqrt{\frac{2}{L}}\sin k_\nu x \quad , \quad \sqrt{\frac{2}{L}}\cos k_\nu x \quad , \quad k_\nu = \frac{2\pi\nu}{L} \quad , \quad \omega_\nu = v k_\nu \quad , \quad \nu = 1, 2, ... \tag{38}$$

This degeneracy is "compensated" by the the values of k_ν being at twice larger intervals $\Delta k = 2\pi/L$ than in the standing wave solutions (26) with $\Delta k = \pi/L$. In real space this means that the normal modes of the closed string have integer number $L/\lambda_\nu = \nu$ of the wavelengths $\lambda_\nu = 2\pi/k_\nu = L/\nu$ over the string length L rather than integer number of half wavelengths $\lambda_\nu = 2L/\nu$ as it was in the fixed ends string case.

Solutions belonging to different frequencies are automatically orthogonal and we chose them to be orthogonal also within each degenerate pair. Here is a helpful integral

$$\int_0^L \sin k_\nu x \cos k_\mu x dx = \int_0^L \frac{1}{2}\left[\sin(k_\nu + k_\mu)x + \sin(k_\nu - k_\mu)x\right] dx =$$
$$= -\frac{1}{2}\left[\frac{1}{k_\nu + k_\mu}\cos(k_\nu + k_\mu)x\Big|_0^L + \frac{1}{k_\nu - k_\mu}\cos(k_\nu - k_\mu)x\Big|_0^L\right] = 0$$

We also normalize them as in the fixed ends case.

As always with degeneracies the choice above is of course not unique. Another useful possibility is

$$\frac{1}{\sqrt{L}}e^{ik_\nu x} \quad , \quad k_\nu = \frac{2\pi\nu}{L} \quad , \quad \omega_\nu = v|k_\nu| \quad , \quad \nu = \pm 1, \pm 2, ... \tag{39}$$

again with degenerate in frequency orthogonal pairs. We will see the results of such a choice below, cf., Eq.(64). In the following sections we will use the freedom in specifying the degeneracy of normal modes in a closed string to find the traveling waves expansion.

Let us also note that a non vibrating constant solution $u_0(x) = \text{const}$ exists with $\omega = 0$. This means that the string configuration in this mode is constant independent of x and has linear time dependence $\phi(x,t) = at + b$. It describes a uniformly moving string and plays an important role in the so called super string theory. It will not be of interest to us and will not be included in our considerations.

3.1.2 Trying the simplest expansion

The most natural expansion using the above degenerate modes would be

$$\begin{aligned}\phi(x,t) &= \sqrt{\frac{2}{L}}\sum_{\nu=1}^{\infty}\left[Q_{1,\nu}(t)\sin k_\nu x + Q_{2,\nu}(t)\cos k_\nu x\right]\\ \pi(x,t) &= \sqrt{\frac{2}{L}}\sum_{\nu=1}^{\infty}\left[P_{1,\nu}(t)\sin k_\nu x + P_{2,\nu}(t)\cos k_\nu x\right]\end{aligned} \tag{40}$$

with two independent sets of amplitudes $\{Q_{1,\nu}(t), P_{1,\nu}(t)\}$ and $\{Q_{2,\nu}(t), P_{2,\nu}(t)\}$ for the two degenerate modes. Inserting this in the string equation and projecting each mode leads to the decoupled equations

$$\ddot{Q}_{i,\nu} + \omega_\nu^2 Q_{i,\nu} = 0 \quad ; \quad i = 1,2 \quad ; \quad \nu = 1,2,... \tag{41}$$

and the corresponding Hamiltonian

$$H = \sum_{i=1,2}\sum_{\nu=1}^{\infty} H_{i,\nu} \quad , \quad H_{i,\nu} = \frac{1}{2}\left(P_{i,\nu}^2 + \omega_\nu Q_{i,\nu}^2\right)$$

The general solution

$$\begin{aligned}Q_{i,\nu}(t) &= Q_{i,\nu}(0)\cos\omega_\nu t + P_{i,\nu}(0)/\omega_\nu \sin\omega_\nu t\\ P_{i,\nu}(t) &= -\omega_\nu Q_\nu(0)\sin\omega_\nu t + P_\nu(0)\cos\omega_\nu t\end{aligned} \tag{42}$$

with arbitrary initial conditions $Q_{i,\nu}(0)$ and $P_{i,\nu}(0) = \dot{Q}_{i,\nu}(0)$ shows however that the expansion above will be in terms of standing or traveling waves depending on these conditions.

3.1.3 Transforming to new normal modes variables

To obtain an expansion in traveling waves let us use the degeneracy of the two modes at every k_ν and do the following transformation

$$\begin{aligned}Q_{1,\nu} &= \frac{1}{\sqrt{2}}(Q_k - Q_{-k}) \quad , \quad P_{1,\nu} = \frac{1}{\sqrt{2}}(P_k - P_{-k})\\ Q_{2,\nu} &= \frac{1}{\omega_k\sqrt{2}}(P_k + P_{-k}) \quad , \quad P_{2,\nu} = -\frac{\omega_k}{\sqrt{2}}(Q_k + Q_{-k})\end{aligned} \tag{43}$$

to new variables $Q_{\pm k}, P_{\pm k}$. Note that to simplify notations we dropped the subscript ν in the right hand side and denoted accordingly $\omega_k = v|k|$. We note that the above transformation mixes coordinates and momenta. In the Appendix we show that this transformation is canonical.

Substituting (43) in the expansion (40) we obtain

$$\begin{aligned} \phi(x,t) &= \sqrt{\frac{1}{L}} \sum_k \left[Q_k(t) \sin kx + \frac{P_k(t)}{v|k|} \cos kx \right] \\ \pi(x,t) &= \sqrt{\frac{1}{L}} \sum_k \left[P_k(t) \sin kx - v|k| Q_k(t) \cos kx \right] \ , \ k = \frac{2\pi\nu}{L} \ , \ \nu = \pm 1, \pm 2, ... \end{aligned} \tag{44}$$

where we have combined together the sums over Q_k, P_k and Q_{-k}, P_{-k} modes by extending the sums to include the negative values of k. We show in Appendix that this is the desired expansion in traveling waves - waves with positive and negative k's moving in opposite directions. The hamiltonian in the new variables has the sum of independent oscillators form

$$H = \frac{1}{2} \sum_k \left(P_k^2 + \omega_k^2 Q_k^2 \right) \tag{45}$$

with the sum extending over both positive and negarive k's. The key point to note is that compared to (40) this is achieved in the expansion (44) by making the amplitudes of the second degenerate mode not independent but proportional to the canonical conjugate of the amplitudes of the first mode and extending the sum to the negative k's.

3.1.4 Inverting the transformation

We remark that inverting (44) requires some care. The simplest is to recall that $\sqrt{2/L} \sin kx$ and $\sqrt{2/L} \cos kx$ form orthonormal set for positive $k > 0$ and use the expansion (40) together with the relations (43). This means that it is the following combinations of Q_k and P_k which are simple projections

$$\begin{aligned}
\frac{1}{\sqrt{2}}(Q_k - Q_{-k}) &= \sqrt{\frac{2}{L}}\int_0^L dx\ \phi(x)\sin kx \\
\frac{1}{\sqrt{2}}(P_k - P_{-k}) &= \sqrt{\frac{2}{L}}\int_0^L dx\ \pi(x)\sin kx \\
\frac{1}{\omega_k\sqrt{2}}(P_k + P_{-k}) &= \sqrt{\frac{2}{L}}\int_0^L dx\ \phi(x)\cos kx \\
-\frac{\omega_k}{\sqrt{2}}(Q_k + Q_{-k}) &= \sqrt{\frac{2}{L}}\int_0^L dx\ \pi(x)\cos kx
\end{aligned} \tag{46}$$

from which the expressions for each of the Q_k and P_k follow by a simple calculation.

$$\begin{aligned}
Q_k &= \frac{1}{\sqrt{L}}\int_0^L dx[\phi(x)\sin kx - \frac{1}{\omega_k}\pi(x)\cos kx] \\
P_k &= \frac{1}{\sqrt{L}}\int_0^L dx[\pi(x)\sin kx + \omega_k\phi(x)\cos kx]
\end{aligned} \tag{47}$$

3.1.5 The physics of the new variables

Let us write the sums in each of the terms in the expansions (44) as single functions. For this purpose let us transform

$$Q_k = C_k\cos\alpha_k \quad , \quad P_k = -\omega_k C_k \sin\alpha_k \quad , \quad C_k = \frac{1}{\omega_k}\sqrt{P_k^2 + \omega_k^2 Q_k^2} \tag{48}$$

This gives

$$\begin{aligned}
Q_k \sin kx + \frac{P_k}{\omega_k}\cos kx &= C_k \sin(kx - \alpha_k) \\
P_k \sin kx - \omega_k Q_k \cos kx &= -\omega_k C_k \cos(kx - \alpha_k)
\end{aligned} \tag{49}$$

Since the amplitude C_k is proportional to the square root of the energy of the mode it is a constant of the motion for the mode time oscillations. It is not difficult to show that the phase $\alpha_k(t)$ is just

$$\alpha_k(t) = \omega_k t + \alpha_k(0)$$

Indeed writing

$$C_k = \sqrt{2I_k/\omega_k}$$

one finds that Eq.(48) is essentially a canonical transformation from P_k, Q_k to the action-angle variables I_k, α_k for a harmonic oscillator, cf., Ref.[3], with

$$\begin{aligned} I_k &= \frac{1}{2\omega_k}(P_k^2 + \omega_k^2 Q_k^2) \quad , \quad \tan\alpha_k = -P_k/\omega_k Q_k \\ \dot{I}_k &= 0 \quad , \quad \dot{\alpha}_k = \omega_k \end{aligned} \tag{50}$$

3.1.6 Mechanical momentum of the string

Apart of the Hamiltonian an important quantity describing the physics of a string is its mechanical momentum (please do not confuse this $\mathbb{P}$ with the canonical P_k's)

$$\mathbb{P} = -\int_0^L \pi(x,t)\frac{\partial}{\partial x}\phi(x,t)dx \tag{51}$$

It is conserved by the string equations of motion (2) as can be seen from the following calculation. Defining the density of P

$$\mathcal{P}(x,t) = -\pi(x,t)\frac{\partial}{\partial x}\phi(x,t)$$

we have

$$\begin{aligned} \frac{\partial\mathcal{P}(x,t)}{\partial t} &= -\frac{\partial\pi(x,t)}{\partial t}\frac{\partial\phi(x,t)}{\partial x} - \pi(x,t)\frac{\partial^2\phi(x,t)}{\partial t\partial x} \\ &= -v^2\frac{\partial^2\phi(x,t)}{\partial x^2}\frac{\partial\phi(x,t)}{\partial x} - \pi(x,t)\frac{\partial\pi(x,t)}{\partial x} \\ &= -\frac{1}{2}\frac{\partial}{\partial x}\left[v^2\left(\frac{\partial\phi(x,t)}{\partial x}\right)^2 + \pi^2(x,t)\right] \end{aligned}$$

This is one dimensional analogue of the continuity equation which connects the time derivative of $\mathcal{P}(x,t)$ and the space derivative of the density of the Hamiltonian

$$\mathcal{H}(x,t) = \frac{1}{2}\left[\pi^2(x,t) + v^2\left(\frac{\partial\phi(x,t)}{\partial x}\right)^2\right]$$

Integrating this and using the boundary conditions (must use periodic not fixed ends) we obtain the conservation law

$$\frac{\partial\mathbb{P}}{\partial t} = \frac{\partial}{\partial t}\int_0^L \mathcal{P}(x,t)dx = 0 \tag{52}$$

3.2 Quantum mechanics of the traveling waves

3.2.1 The basic operators. String Hamiltonian and momentum

As in the case of the fixed ends string the closed string is quantized by introducing wave functionals $\Psi[\phi(x), t]$ for the string coordinates and the operators $\phi_{op}(x) = \phi(x)$ and $\pi_{op}(x) = -i\hbar\delta/\delta\phi(x)$. The Hamiltonian operator is the same given by Eq.(13) since the string equations are the same. What is different are the boundary conditions which led to modified (degenerate) normal modes and the corresponding expansions (44).

Using these expansion for the operators

$$
\begin{aligned}
\phi_{op}(x) &= \sqrt{\frac{1}{L}}\sum_k \left[\sin kx\ \hat{Q}_k + \frac{1}{v|k|}\cos kx\ \hat{P}_k\right] \\
\pi_{op}(x) &= \sqrt{\frac{1}{L}}\sum_k \left[\sin kx\ \hat{P}_k - v|k|\cos kx\ \hat{Q}_k\right]
\end{aligned}
\tag{53}
$$

we are led to the canonical commutators for the traveling waves amplitudes

$$
[\hat{Q}_k, \hat{P}_{k'}] = i\hbar\delta_{kk'} \quad , \quad [\hat{Q}_k, \hat{Q}_{k'}] = [\hat{P}_k, \hat{P}_{k'}] = 0 \tag{54}
$$

These of course follow from the basic commutators (12) and the expressions (47). The Hamiltonian operator in terms of $\hat{Q}_k$'s and $\hat{P}_k$'s has the same form of decoupled oscillators

$$
H_{op} = \frac{1}{2}\sum_k \left[\hat{P}_k^2 + \omega_k^2 \hat{Q}_k^2\right] \tag{55}
$$

Quantum mechanically the string momentum $\mathbb{P}$, Eq. (51), becomes an operator

$$
P_{\rm op} = -\frac{1}{2}\int_0^L dx \left\{\pi_{\rm op}(x)\frac{\partial}{\partial x}\phi_{\rm op}(x)dx + \left[\frac{\partial}{\partial x}\phi_{\rm op}(x)\right]\pi_{\rm op}(x)\right\} \tag{56}
$$

As usual with products of non commuting operators, here $\pi_{\rm op}(x)$ and $\phi_{\rm op}(x)$, one must use a symmetrized expression.

The operator $P_{\rm op}$ is the generator of translations

$$
\phi_{\rm op}(x) \ \rightarrow\ \phi_{\rm op}(x+a) \quad , \quad \pi_{\rm op}(x) \ \rightarrow\ \pi_{\rm op}(x+a)
$$

Indeed using the basic commutators (12) one can easily verify that

$$
[P_{\rm op}, \phi_{\rm op}(x)] = -i\hbar\frac{\partial}{\partial x}\phi_{\rm op}(x) \quad , \quad [P_{\rm op}, \pi_{\rm op}(x)] = -i\hbar\frac{\partial}{\partial x}\pi_{\rm op}(x) \tag{57}
$$

as it should be for the generator of translations.

Inserting the expansions (53) in the momentum $P_{\rm op}$ we obtain

$$
\begin{aligned}
P_{\rm op} &= -\frac{1}{L}\sum_{kk'}\int_0^L \left[\sin kx\ \hat{P}_k - v|k|\cos kx\ \hat{Q}_k\right]\left[k'\cos k'x\ \hat{Q}_{k'} - \frac{k'}{v|k'|}\sin k'x\ \hat{P}_{k'}\right]dx \\
&= \frac{1}{2}\sum_k \left[\frac{k}{v|k|}\hat{P}_k^2 + vk|k|\hat{Q}_k^2\right] = \sum_k \frac{k}{\omega_k}\frac{1}{2}\left[\hat{P}_k^2 + \omega_k^2\hat{Q}_k^2\right]
\end{aligned}
$$

3.2.2 The eigenstates. Energies and momenta of traveling waves quanta

The traveling waves Hamiltonian (55) has the same decoupled normal modes oscillators form as the one for the standing waves (31) so formally its solutions have the same form as (32)

$$
\begin{aligned}
E_{\{N_k\}} &= E_{\text{ground state}} + \sum_k \hbar\omega_k N_k \quad , \quad N_k = 0, 1, 2, ... \\
\Psi_{\{N_k\}}(\{Q_k\}) &= \prod_k \psi_{N_k}(\beta_k Q_k) \quad , \quad \beta_k = \sqrt{\omega_k/\hbar}
\end{aligned}
\tag{58}
$$

with familiar harmonic oscillator eigenfunctions $\psi_N(\beta Q)$, Eq. (33). There are however important differences.

Since $\omega_k = v|k|$ the traveling waves energy quanta $\epsilon_k = \hbar\omega_k$ are doubly degenerate with respect to the sign (direction) of k. Even more profound is that these quanta also carry momentum. Indeed comparing the expression (58) for the string momentum P_{op} with the Hamiltonian H_{op} one observes that P_{op} has the same eigenfunctions (not surprising) with the eigenvalues[2]

$$
P_{\{N_k\}} = \sum_k \hbar k N_k \tag{59}
$$

Each quantum has "mechanical" momentum $p_k = \hbar k$. So the closed string can be considered as a collection of traveling waves "quasi"particles with energy momentum relation

$$
\epsilon_k \equiv \hbar\omega_k = \hbar v|k| = v|p_k| \quad \Rightarrow \quad \epsilon(p) = v|p| \tag{60}
$$

It is useful to pay attention that this result can be viewed (obtained by a shortcut) as a consequence of the three fundamental relations - two basic quantum mechanical

[2]The presence of $\pm\hbar k$ terms in this expression helps to cancel the 1/2 "zero point motion" term present in the expression for the energy.

relations - the Plank-Einstein $\epsilon = \hbar\omega$ and the de Broglie $p = \hbar k$ and the string dispersion relation $\omega = v|k|$. In a similar way we will find below that the quanta of the EM field will be particles (photons) with energy-momentum relation $\epsilon = \hbar\omega = \hbar ck = cp$ i.e. of massless relativistic particles. The quanta of the Schrödinger field will have $\epsilon = \hbar\omega = \hbar^2k^2/2m = p^2/2m$, i.e. the energy-momentum relation of non relativistic particles.

3.2.3 Transformation to creation and annihilation operators

In practice it is very convenient to introduce creation and annihilation operators in the standard way

$$\begin{aligned} \hat{Q}_k = \sqrt{\hbar/2\omega_k}\left(\hat{a}_k + \hat{a}_k^\dagger\right) \quad &, \quad \hat{P}_k = i\sqrt{\hbar\omega_k/2}\left(\hat{a}_k^\dagger - \hat{a}_k\right) \\ \hat{a}_k = \sqrt{1/2\hbar\omega_k}\left(i\hat{P}_k + \omega_k\,\hat{Q}_k\right) \quad &, \quad \hat{a}_k^\dagger = \sqrt{1/2\hbar\omega_k}\left(-i\hat{P}_k + \omega_k\,\hat{Q}_k\right) \\ [\hat{a}_k, \hat{a}_{k'}^\dagger] = \delta_{kk'} \quad &, \quad [\hat{a}_k, \hat{a}_{k'}] = 0 = [\hat{a}_k^\dagger, \hat{a}_{k'}^\dagger] \end{aligned} \tag{61}$$

Using these operators we can write the Hamiltonian and the momentum operator as

$$\begin{aligned} \hat{H}_{\rm op} &= E_0 + \sum_k \hbar\omega_k \hat{a}_k^\dagger \hat{a}_k \\ \hat{P}_{\rm op} &= \sum_k \hbar k \hat{a}_k^\dagger \hat{a}_k \end{aligned} \tag{62}$$

and their eigenstates

$$|\{N_k\} >= \prod_k |N_k >= \prod_k \frac{(\hat{a}_k^\dagger)^{N_k}}{(N_k!)^{1/2}}|0 > \tag{63}$$

Great advantage of using $\hat{a}$ and $\hat{a}^\dagger$ operators rather than $\hat{P}$ and $\hat{Q}$ is the simplicity of the "action" of these operators on the "number states", i.e the states with a fixed numbers of quasi particles in each normal mode. Schematically

$$\hat{a}|n >= \sqrt{n}|n-1 > \quad , \quad \hat{a}^\dagger|n >= \sqrt{n+1}|n+1 >$$

In detailed notation

$$\hat{a}_k|\{N_k\} >= \sqrt{N_k}|N_k - 1 > \prod_{k'\neq k} |\{N_{k'}\} >$$

$$\hat{a}_k^\dagger|\{N_k\}>=\sqrt{N_k+1}|N_k+1>\prod_{k'\neq k}|\{N_{k'}\}>$$

One says that the operators $\hat{a}_k^\dagger$ and $\hat{a}_k$ create and destroy quasi particles of energy $\hbar\omega$ and momentum $\hbar k$.

It is useful to express the field operators in terms of a and $a^\dagger$. Using (53) we obtain

$$\begin{aligned}\phi_{op}(x) &= -i\sum_k\sqrt{\frac{\hbar}{2\omega_k L}}\left[\hat{a}_k e^{ikx}-\hat{a}_k^\dagger e^{-ikx}\right] = -i\sum_k\sqrt{\frac{\hbar}{2\omega_k L}}\left[\hat{a}_k-\hat{a}_{-k}^\dagger\right]e^{ikx} \\ \pi_{op}(x) &= -\sum_k\sqrt{\frac{\hbar\omega_k}{2L}}\left[\hat{a}_k e^{ikx}+\hat{a}_k^\dagger e^{-ikx}\right] = -\sum_k\sqrt{\frac{\hbar\omega_k}{2L}}\left[\hat{a}_k+\hat{a}_{-k}^\dagger\right]e^{ikx}\end{aligned} \tag{64}$$

which are sums of terms which either create a quantum with momentum $\hbar k$ or annihilate one with the opposite momentum $-\hbar k$.

4 Quantization of the EM Field

The quantization of the electromagnetic field follows the same route as with the simple string above. The classical equations of the field are the Maxwell equations. We will cast them into Hamilton form and identify the Hamiltonian and the canonical coordinates and momenta of the field. We will then replace them by operators with canonical commutation relations acting on the appropriate wave functionals.

4.1 Hamilton form of the Maxwell equations

The Maxwell equations have the familiar form[3]

$$\begin{aligned}\nabla\cdot\mathbf{E}(\mathbf{r},t) &= \frac{\rho(\mathbf{r})}{\epsilon_0} \quad , \quad \nabla\cdot\mathbf{B}(\mathbf{r},t)=0 \\ \nabla\times\mathbf{E}(\mathbf{r},t) &= -\frac{\partial\mathbf{B}(\mathbf{r},t)}{\partial t} \quad , \quad \nabla\times\mathbf{B}(\mathbf{r},t)=\frac{1}{c^2}\frac{\partial\mathbf{E}(\mathbf{r},t)}{\partial t}+\frac{\mathbf{j}(\mathbf{r},t)}{c^2\epsilon_0}\end{aligned} \tag{65}$$

Here c is the light velocity and ϵ_0 is the vacuum permittivity constant $\epsilon_0 = 8.85\cdot 10^{-12}Fm^{-1}$ related to our choice of the SI units system.

The Maxwell equations describe the EM field configuration for a given distribution of the electric current $\mathbf{j}(\mathbf{r},t)$ and density $\rho(\mathbf{r},t)$ of electric charges. Assuming that

[3]In this Chapter we use the SI system of units.

we are dealing with a system of N charges and denoting by $\mathbf{r}_a(t)$, $\mathbf{v}_a(t), a = 1, ..., N$ their positions and velocities we have

$$\rho(\mathbf{r}, t) = \sum_{a=1}^{N} q_a \delta(\mathbf{r} - \mathbf{r}_a(t)) \quad , \quad \mathbf{j}(\mathbf{r}, t) = \sum_{a=1}^{N} q_a \mathbf{v}_a \delta(\mathbf{r} - \mathbf{r}_a(t)) \tag{66}$$

These expressions must be supplemented by the mechanical equations of motion for the charges as they move in the given $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$. These equations are just the Newton equations for the charges

$$m_a \frac{d\mathbf{v}_a}{dt} = q_a \mathbf{E}(\mathbf{r}_a, t) + q_a (\mathbf{v}_a \times \mathbf{B}(\mathbf{r}_a, t)) \quad , \quad \mathbf{v}_a = \frac{d\mathbf{r}_a}{dt} \quad , \quad a =, ..., N \tag{67}$$

The coupled equations (65), (66) and (67) provide the complete system which determines how the positions of the charges and their motion determine the EM field and how this field determines the motion of the charges. Our first goal will be to cast this system in the Hamiltonian form thereby determining its canonical variables and the Hamiltonian.

4.1.1 Vector potential. The $A_0 = 0$ gauge

We start by noting that the first pair of Maxwell equations in (65) does not involve time derivatives. They are in a sense constraints on the possible functional dependence of $\mathbf{E}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$. Both constraints are easy to resolve. The condition $\nabla \cdot \mathbf{B} = 0$ means that there are no magnetic charges in nature and that $\mathbf{B}$ can be represented as a curl of an arbitrary vector function

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{68}$$

which is conventionally called the vector potential. In the Chapter where we considered the motion in an external EM field we have seen that the quantum mechanical formulation was impossible without an explicit use of this function. Also presently we will find that the quantization of EM field can not avoid using $\mathbf{A}$.

The second pair of the Maxwell equations in (65) are dynamical equations. Inserting Eq. (68) in the first we obtain

$$\nabla \times (\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}) = 0 \quad \Rightarrow \quad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla A_0 \tag{69}$$

Here A_0 is (in the non relativistic parlance) the "scalar potential" which together with $\mathbf{A}$ completely determine the fields $\mathbf{E}$ and $\mathbf{B}$. The potentials $\mathbf{A}$ and A_0 are not uniquely defined. We can choose instead different function $A'_0(\mathbf{r}, t)$ and $\mathbf{A}'(\mathbf{r}, t)$ related to A_0 and $\mathbf{A}$ by the gauge transformation

$$A'_0 = A_0 - \frac{\partial \chi}{\partial t} \quad , \quad \mathbf{A}' = \mathbf{A} + \nabla \chi \tag{70}$$

with arbitrary function $\chi(\mathbf{r}, t)$. We shall use this freedom and take A_0 to be identically equal to zero and write

$$\frac{\partial \mathbf{A}}{\partial t} = -\mathbf{E} \tag{71}$$

This choice is called "working in the $A_0 = 0$ gauge". Importantly this choice does not exhaust the full gauge freedom. We can still add to $\mathbf{A}$ a gradient of a *time independent* function $\chi(\mathbf{r})$ without changing our $A_0 = 0$ assumption.

Inserting (68) in the last of the Maxwell equations (65) we obtain

$$\frac{\partial \mathbf{E}}{\partial t} = c^2 \nabla \times \nabla \times \mathbf{A} - \frac{\mathbf{j}}{\epsilon_0} \tag{72}$$

Using (71) this equation becomes

$$\frac{\partial^2 \mathbf{A}}{\partial t^2} = -c^2 \nabla \times \nabla \times \mathbf{A} + \frac{\mathbf{j}}{\epsilon_0} \tag{73}$$

Regarding the 2nd time derivative on the left as acceleration of $\mathbf{A}(\mathbf{r}, t)$ one can view the above equation (73) as coupled Newton equations for the degrees of freedom $\mathbf{A}(\mathbf{r})$. In this view at every point in space there are three such degrees of freedom (field coordinates) which can symbolically be represented as $A_{i,\mathbf{r}}$. The 3 dimensional vector index is $i = 1, ..., 3$ and $\mathbf{r}$ is running over all points in the 3 dimensional space in a way similar to x running over points of the x axis in the example of a string. The coupling between different $A_{i,\mathbf{r}}$ is via complicated combination of second order vector derivative $\nabla \times \nabla \times$ connecting different vector components of $\mathbf{A}(\mathbf{r})$ in neighboring points.

The last term in Eq. (73) is the "force" acting on the field coordinates by the matter. Ignoring this force for a moment (i.e. considering the EM field in an empty space region) we can view the coordinates $\mathbf{A}(\mathbf{r})$ as representing coupled oscillators. This is because the above equation without the last term is linear. Although complicated from the vector analysis point of view the derivatives combination $\nabla \times \nabla \times$ is a linear operation.

4.1.2 The Hamiltonian

Continuing with the "mechanical" interpretation of the EM field dynamics we notice that the pair of the 1st order equations (71) and (72) without the last term can be regarded together as Hamilton equations with the following field Hamiltonian

$$H_f = \frac{\epsilon_0}{2} \int d^3r \left[\mathbf{E}^2(\mathbf{r}) + c^2 (\nabla \times \mathbf{A}(\mathbf{r}))^2 \right] \tag{74}$$

and canonical variables $A_i(\mathbf{r})$ as coordinates and $-\epsilon_0 E_i(\mathbf{r})$ as momenta. We will verify this in a moment but first we note that perhaps the simplest way to guess the expression of the Hamiltonian is to notice that on account of Eq. (71) the first term in it has the form of the kinetic energy. One can determine how it changes with time by forming a scalar product of the left hand side of (72) with $\mathbf{E}$. Multiplying the first term on the right hand side (remember we still are ignoring the current term) with the equal quantity $-\partial \mathbf{A}/\partial t$ we can integrate both sides over $\mathbf{r}$. After simple manipulations[4] one can show that the change in time of the kinetic energy is equal to minus the change in time of the second term in H_f which has the meaning of the potential energy. This of course verifies that H_f is conserved, $dH_f/dt = 0$.

Returning to the Hamiltonian (74) we form its first variation

$$\begin{aligned}
\delta H_f &= \int d^3r \left[\epsilon_0 \mathbf{E} \cdot \delta \mathbf{E} + \epsilon_0 c^2 (\nabla \times \mathbf{A}) \cdot (\nabla \times \delta \mathbf{A}) \right] \\
&= \int d^3 \left[\epsilon_0 \mathbf{E} \cdot \delta \mathbf{E} + \epsilon_0 c^2 \epsilon_{ijk} \partial_j A_k \epsilon_{ilm} \partial_l \delta A_m \right] \\
&= \int d^3r \left[\epsilon_0 \mathbf{E} \cdot \delta \mathbf{E} - \epsilon_0 c^2 \delta A_m \epsilon_{ilm} \epsilon_{ijk} \partial_l \partial_j A_k \right] \\
&= \int d^3r \left[\epsilon_0 \mathbf{E} \cdot \delta \mathbf{E} + \epsilon_0 c^2 \delta \mathbf{A} \cdot \nabla \times \nabla \times \mathbf{A}) \right]
\end{aligned}$$

where we performed integration by parts in the second term. From this it follows

[4]Here are the details

$$\begin{aligned}
\int d^3r\, \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} = -c^2 \int d^3r \frac{\partial \mathbf{A}}{\partial t} \cdot \nabla \times \nabla \times \mathbf{A} &= -c^2 \int d^3r \partial_t A_i \epsilon_{ijk} \partial_j \epsilon_{klm} \partial_l A_m = \\
= (\text{integrate by parts}) = c^2 \int d^3r \epsilon_{ijk} \partial_j \partial_t A_i \epsilon_{klm} \partial_l A_m &= -c^2 \int d^3r \left(\nabla \times \frac{\partial \mathbf{A}}{\partial t} \right) \cdot (\nabla \times \mathbf{A}) \ .
\end{aligned}$$

Rewrite as $\frac{d}{dt} \frac{1}{2} \int d^3r\, \mathbf{E}^2 = -\frac{c^2}{2} \frac{d}{dt} \int d^3r (\nabla \times \mathbf{A})^2$

that

$$\frac{\delta H_f}{\delta(-\epsilon_0 \mathbf{E})} = -\mathbf{E} \quad , \quad \frac{\delta H_f}{\delta \mathbf{A}} = \epsilon_0 c^2 \nabla \times \nabla \times \mathbf{A} \tag{75}$$

showing that the Hamilton equations with this Hamiltonian and canonical variables $\mathbf{A}(\mathbf{r})$ and $-\epsilon_0 \mathbf{E}(\mathbf{r})$

$$\frac{\partial \mathbf{A}}{\partial t} = \frac{\delta H_f}{\delta(-\epsilon_0 \mathbf{E})} = -\mathbf{E} \quad , \quad \frac{\partial(-\epsilon_0 \mathbf{E})}{\partial t} = -\frac{\delta H_f}{\delta \mathbf{A}} = -\epsilon_0 c^2 \nabla \times \nabla \times \mathbf{A} \tag{76}$$

indeed coincide with Eqs. (71) and (72) without the current term.

Let us now show how to account for the last term in Eq. (72) and for the Newton equations (67) governing the dynamics of the matter. Here we are guided by our knowledge of the Hamiltonian of charges moving a given EM field (cf. chapter Motion in External EM Field). We simply add it to H_f above and obtain

$$H = \frac{\epsilon_0}{2} \int d^3 r \left[\mathbf{E}^2(\mathbf{r}) + c^2 (\nabla \times \mathbf{A}(\mathbf{r}))^2 \right] + \sum_{a=1}^{N} \frac{1}{2m_a} \left[\mathbf{p}_a - q_a \mathbf{A}(\mathbf{r}_a) \right]^2 \tag{77}$$

Using exactly the same calculation as in the chapter Motion in External EM Field we can show that the Hamilton equations

$$\frac{d\mathbf{r}_a}{dt} = \frac{\partial H}{\partial \mathbf{p}_a} \quad , \quad \frac{d\mathbf{p}_a}{dt} = -\frac{\partial H}{\partial \mathbf{r}_a} \tag{78}$$

are equivalent to the Newton equations (67).

What is left is to consider the first variation of the last term in Eq. (77) with respect to $\mathbf{A}(\mathbf{r})$. We obtain

$$-\sum_{a=1}^{N} \frac{q_a}{m_a} \left[\mathbf{p}_a - q_a \mathbf{A}(\mathbf{r}_a) \right] \cdot \delta \mathbf{A}(\mathbf{r}_a) = -\sum_{a=1}^{N} \frac{q_a}{m_a} \left[\mathbf{p}_a - q_a \mathbf{A}(\mathbf{r}_a) \right] \cdot \int d^3 r \delta(\mathbf{r} - \mathbf{r}_a) \delta \mathbf{A}(\mathbf{r})$$

$$= \; -\int d^3 r \sum_{a=1}^{N} \frac{q_a}{m_a} \left[\mathbf{p}_a - q_a \mathbf{A}(\mathbf{r}_a) \right] \delta(\mathbf{r} - \mathbf{r}_a) \cdot \delta \mathbf{A}(\mathbf{r}) = -\int d^3 r \; \mathbf{j}(\mathbf{r}) \cdot \delta \mathbf{A}(\mathbf{r})$$

where $\mathbf{j}(\mathbf{r})$ is the current as defined in Eq. (67) with

$$\mathbf{v}_a = \frac{1}{m_a} \left[\mathbf{p}_a - q_a \mathbf{A}(\mathbf{r}_a) \right]$$

With this result the second equation of (76) with H_f replaced by the full H, Eq. (77) now reads

$$\frac{\partial(-\epsilon_0 \mathbf{E})}{\partial t} = -\frac{\delta H}{\delta \mathbf{A}} = -\epsilon_0 c^2 \nabla \times \nabla \times \mathbf{A} + \mathbf{j} \tag{79}$$

reproducing the complete equation (72). We also note that since the second term in the full H does not depend on $\mathbf{E}$ the first equation in (76) remains unchanged when H_f is replaced in it by H.

4.2 Canonical quantization

Having established the canonical structure of the theory we can now quantize it. Attentive reader should have noticed that we have not yet accounted for the first equation in the set (65) expressing the Gauss law. We also seem to be missing from the Hamiltonian (77) the regular Coulomb interaction energy between the charges $\{q_a\}$. We will address these issues shortly but meanwhile let us proceed with the quantization.

Moving from classical to quantum description we recognise that the coordinate set of our system consists of the vector potential $\mathbf{A}(\mathbf{r})$ (i.e. 3 vector components in each point of the position space, i.e. $3 \times \infty^3$ variables) and 3N vectors $\{\mathbf{r}_a\}$ of the particles' positions. Accordingly we introduce the wave functional of the field $\mathbf{A}(\mathbf{r})$ which also depends (is a function of) the N particles' positions and time

$$\Psi = \Psi[\mathbf{A}(\mathbf{r}), \mathbf{r}_1, ..., \mathbf{r}_N, t] \tag{80}$$

This should be viewed as a correspondence between all field configurations and set of N particles' positions $\{\mathbf{r}_a\}$ and (in general) complex probability amplitudes which in general change with time.

The physical operators are constructed from the corresponding classical quantities by the canonical substitution

$$\begin{aligned} \mathbf{A}(\mathbf{r}) \to \mathbf{A}_{op}(\mathbf{r}) \;&=\; \mathbf{A}(\mathbf{r}) \;\;,\;\; \mathbf{E}(\mathbf{r}) \to \mathbf{E}_{op}(\mathbf{r}) = \frac{i\hbar}{\epsilon_0}\frac{\delta}{\delta \mathbf{A}(\mathbf{r})} \\ \mathbf{r}_a \;\to\; \hat{\mathbf{r}}_a \;&=\; \mathbf{r}_a \;\;,\;\; \mathbf{p}_a \;\to\; \hat{\mathbf{p}}_a = -i\hbar\nabla_a \end{aligned} \tag{81}$$

where we accounted for the fact that it is the combination $-\epsilon_0 \mathbf{E}(\mathbf{r})$ which is canonical to $\mathbf{A}(\mathbf{r})$ not just $\mathbf{E}(\mathbf{r})$. Using the equality

$$\frac{\delta A_j(\mathbf{r}')}{\delta A_i(\mathbf{r})} = \delta_{ij}\delta(\mathbf{r} - \mathbf{r}').$$

it follows that the field operators obey the commutation relations

$$\left[\hat{E}_i(\mathbf{r}), \hat{A}_j(\mathbf{r}')\right] = \frac{i\hbar}{\epsilon_0}\delta_{ij}\delta(\mathbf{r}-\mathbf{r}') \ , \ \left[\hat{A}_i(\mathbf{r}), \hat{A}_j(\mathbf{r}')\right] = \left[\hat{E}_i(\mathbf{r}), \hat{E}_j(\mathbf{r}')\right] = 0 \qquad (82)$$

The time dependence of the wave functional/function $\Psi[\mathbf{A}(\mathbf{r}), \mathbf{r}_1, ..., \mathbf{r}_N, t]$ is governed by the Schrödinger equation

$$i\hbar\frac{\partial\Psi(t)}{\partial t} = H_{op}\Psi(t) \qquad (83)$$

in which the Hamiltonian operator H_{op} is obtained by replacing in the classical expression (77) the fields $\mathbf{A}(\mathbf{r})$, $\mathbf{E}(\mathbf{r})$ and the particle variables $\{\mathbf{r}_a\}$, $\{\mathbf{p}_a\}$ by the corresponding operators (81),

$$H_{op} = \frac{\epsilon_0}{2}\int d^3r \left[\mathbf{E}^2_{op}(\mathbf{r}) + c^2(\nabla\times\mathbf{A}_{op}(\mathbf{r}))^2\right] + \sum_{a=1}^{N}\frac{1}{2m_a}\left[-i\hbar\nabla_a - q_a\mathbf{A}_{op}(\hat{\mathbf{r}}_a)\right]^2 \ . \quad (84)$$

The above Hamiltonian does not depend on time as we are dealing with a closed EM field + matter system. The energy is therefore conserved and quantum mechanically we can reduce in the standard way the solution of the above time dependent Schrödinger equation to solving the static equation

$$H_{op}\Psi = E\Psi \qquad (85)$$

This is a complicated equation for the coupled field-matter system. No exact solution is possible. We will discuss approximate solutions below. But before that we have to clarify several formal but very important issues which will allow us to somewhat simplify the problem.

4.3 Gauge invariance

4.3.1 The Gauss law

We will now show that the Hamiltonian formulation presented above neatly accounts for both the Gauss law and the Coulomb interaction between the charges. The key to this is to observe that the Hamiltonian is invariant under the gauge transformation

$$\begin{aligned} &\mathbf{A}(\mathbf{r},t) \to \mathbf{A}(\mathbf{r},t) + \nabla\chi(\mathbf{r}) \quad &, \quad &\mathbf{E}(\mathbf{r},t) \to \mathbf{E}(\mathbf{r},t) \\ &\mathbf{p}_a(t) \to \mathbf{p}_a(t) + q_a\nabla\chi(\mathbf{r}_a(t)) \quad &, \quad &\mathbf{r}_a(t) \to \mathbf{r}_a(t) \end{aligned} \qquad (86)$$

with an arbitrary **time independent** function $\chi(\mathbf{r})$. This is the residual gauge transformation we have briefly mentioned after Eq. (71).

The symmetry of H under (86) is the result of the way the vector potential $\mathbf{A}(\mathbf{r})$ enters it, i.e. only via the combinations $\nabla \times \mathbf{A}$ and $[\mathbf{p}_a - q_a\mathbf{A}(\mathbf{r}_a)]$. It is a local symmetry meaning that it is characterised by parameters $\chi(\mathbf{r})$ which depend on $\mathbf{r}$. Schematically there are ∞^3 parameters corresponding to the "number" of points in the 3D space of vectors $\mathbf{r}$. As we will show below the generators of this symmetry are

$$g_{op}(\mathbf{r}) = -\epsilon_0 \nabla \cdot \mathbf{E}_{op}(\mathbf{r}) + \rho_{op}(\mathbf{r}) \tag{87}$$

Their dependence on $\mathbf{r}$ means that there are ∞^3 generators corresponding to ∞^3 parameters $\chi(\mathbf{r})$ in (86).

Classically expressions corresponding to the generators of symmetries of the Hamiltonian are conserved by the Hamilton equations. Momentum and angular momentum are of course the classic examples of such conservations. Accordingly let us show that equations (71), (72) and (67) conserve the above expression for the generator $g_{op}(\mathbf{r}, t)$ when it is taken as classical and when $\mathbf{E}$ and ρ in it are allowed to evolve according to these equations. We have

$$\frac{\partial}{\partial t}[-\epsilon_0 \nabla \cdot \mathbf{E}(\mathbf{r}) + \rho(\mathbf{r})] = -\epsilon_0 \nabla \cdot \frac{\partial \mathbf{E}}{\partial t} + \frac{\partial \rho}{\partial t} = \nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0 \tag{88}$$

where we used Eq.(72) and the continuity equation for the charges.[5]

The vanishing of $\partial g(\mathbf{r})/\partial t$ means that local quantities $-\epsilon_0 \nabla \cdot \mathbf{E}(\mathbf{r}) + \rho(\mathbf{r})$ form a constant, time independent function of $\mathbf{r}$. It is natural to denote this function by $\rho_0(\mathbf{r})$

$$-\epsilon_0 \nabla \cdot \mathbf{E}(\mathbf{r}) + \rho(\mathbf{r}) = \rho_0(\mathbf{r})$$

and interpret it as a density of fixed static electric charges. We notice that these charges appear in addition to the density $\rho(\mathbf{r})$ of the dynamical charges q_a described by the equations (67). Under normal circumstances there are no such extra static charges. In fact their presence would violate such symmetries as translational, rotational, Lorenz. So one should assume that $\rho_0 = 0$. Using this in the above relation we recover the Gauss law.

[5] Continuity equation is a general relation between $\rho(\mathbf{r}, t)$ and $\mathbf{j}(\mathbf{r}, t)$ given by (66)

$$\begin{aligned} \frac{\partial \rho(\mathbf{r}, t)}{\partial t} &= \frac{\partial}{\partial t} \sum_{a=1}^{N} q_a \delta(\mathbf{r} - \mathbf{r}_a(t)) = \sum_{a=1}^{N} q_a \nabla_{\mathbf{r}_a} \delta(\mathbf{r} - \mathbf{r}_a(t)) \cdot \frac{d\mathbf{r}_a}{dt} = \\ &= (\text{using } \ \nabla_{\mathbf{r}_a} \delta(\mathbf{r} - \mathbf{r}_a) = -\nabla_{\mathbf{r}} \delta(\mathbf{r} - \mathbf{r}_a) \ ) = -\nabla_{\mathbf{r}} \cdot \mathbf{j}(\mathbf{r}, t) \end{aligned}$$

4.3.2 Quantum mechanics of the gauge transformation

Let us work out the quantum mechanics of the gauge transformation (86). What we want to show is that it is generated by the ∞^3 operators $g_{op}(\mathbf{r})$, Eq. (87), i.e. that the following relations hold

$$e^{-(i/\hbar)\int d^3r\chi(\mathbf{r})g_{op}(\mathbf{r})}\begin{pmatrix}\mathbf{A}_{op}(\mathbf{r}')\\ \hat{\mathbf{p}}_a\end{pmatrix}e^{(i/\hbar)\int d^3r\chi(\mathbf{r})g_{op}(\mathbf{r})}=\begin{pmatrix}\mathbf{A}_{op}(\mathbf{r}')+\nabla\chi(\mathbf{r}')\\ \hat{\mathbf{p}}_a+q_a\nabla\chi(\mathbf{r}_a)\end{pmatrix}\tag{89}$$

As usual it is enough to consider the infinitesimal $\chi(\mathbf{r})$ for which the left hand side reduces to

$$-\frac{i}{\hbar}\int d^3r\chi(\mathbf{r})\left[g_{op}(\mathbf{r}),\begin{pmatrix}\mathbf{A}_{op}(\mathbf{r}')\\ \hat{\mathbf{p}}_a\end{pmatrix}\right]=-\frac{i}{\hbar}\int d^3r\chi(\mathbf{r})\begin{pmatrix}-i\hbar\nabla_{\mathbf{r}}\delta(\mathbf{r}-\mathbf{r}')\\ -i\hbar q_a\nabla_{\mathbf{r}}\delta(\mathbf{r}-\mathbf{r}_a)\end{pmatrix}\tag{90}$$

Here we omitted the identity term and used

$$[g_{op}(\mathbf{r}),\mathbf{A}_{op}(\mathbf{r}')]=[-\epsilon_0\nabla\cdot\mathbf{E}_{op}(\mathbf{r}),\mathbf{A}_{op}(\mathbf{r}')]=-i\hbar\nabla_{\mathbf{r}}\delta(\mathbf{r}-\mathbf{r}')\tag{91}$$

and

$$[g_{op}(\mathbf{r}),\hat{\mathbf{p}}_a]=[\rho_{op}(\mathbf{r}),\hat{\mathbf{p}}_a]=i\hbar\nabla_{\mathbf{r}_a}\delta(\mathbf{r}-\mathbf{r}_a)=-i\hbar\nabla_{\mathbf{r}}\delta(\mathbf{r}-\mathbf{r}_a)\tag{92}$$

Now we do the integration by parts and use the delta function

$$-\frac{i}{\hbar}\int d^3r\chi(\mathbf{r})\begin{pmatrix}-i\hbar\nabla_{\mathbf{r}}\delta(\mathbf{r}-\mathbf{r}')\\ -i\hbar q_a\nabla_{\mathbf{r}}\delta(\mathbf{r}-\mathbf{r}_a)\end{pmatrix}=\int d^3r\nabla\chi(\mathbf{r})\begin{pmatrix}\delta(\mathbf{r}-\mathbf{r}')\\ q_a\delta(\mathbf{r}-\mathbf{r}_a)\end{pmatrix}=\begin{pmatrix}\nabla\chi(\mathbf{r}')\\ q_a\nabla\chi(\mathbf{r}_a)\end{pmatrix}\tag{93}$$

obtaining exactly what is needed to get the $\chi(\mathbf{r})$ dependent term in the right hand side of Eq.(89).

Thus ∞^3 operators $g_{op}(\mathbf{r})$ are indeed the generators of the gauge transformation. Since the Hamiltonian operator H_{op} is invariant under this transformation one must have that H_{op} commutes with $g_{op}(\mathbf{r})$

$$[H_{op},g_{op}(\mathbf{r})]=0\tag{94}$$

As in simpler quantum mechanical systems this means that H_{op} and $g_{op}(\mathbf{r})$ have common eigenfunctions. We write symbolically

$$H_{op}\Psi=E\Psi\quad,\quad g_{op}(\mathbf{r})\Psi=\rho_0(\mathbf{r})\Psi\tag{95}$$

where we denoted by $\rho_0(\mathbf{r})$ the eigenvalues of the ∞^3 operators $q_{op}(\mathbf{r})$. As in the classical case the meaning of $\rho_0(\mathbf{r})$ is the density of static ("background") electric charges. They are "background" because they are not a part of the dynamics. Just sit there as a part of initial conditions. Their presence would violate basic symmetries (translational, rotational, Lorenz) so the physics dictates that one must select only the eigenfunctions which belong in the "sector" of the system Hilbert space for which

$$g_{op}(\mathbf{r})\Psi = 0 \tag{96}$$

In other words - the *gauge invariant sector*.

4.3.3 Separating the longitudinal components of the fields

In the Hamiltonian formulation the electromagnetic field has $3 \times \infty^3$ degrees of freedom which in our formulation are described by the coordinates $A_i(\mathbf{r})$ and the corresponding momenta $-\epsilon_0 E_i(\mathbf{r})$, $i = 1, 2, 3$. Using the local gauge symmetry (86) one can eliminate one third of these degrees of freedom. For this reason let us represent the functions $\mathbf{A}(\mathbf{r})$ and $\mathbf{E}(\mathbf{r})$ as sum of the so called transverse and longitudinal components

$$\mathbf{A}(\mathbf{r}) = \mathbf{A}_T(\mathbf{r}) + \mathbf{A}_L(\mathbf{r}) \;\; , \;\; \mathbf{E}(\mathbf{r}) = \mathbf{E}_T(\mathbf{r}) + \mathbf{E}_L(\mathbf{r}) \tag{97}$$

where $\mathbf{A}_T$, $\mathbf{A}_L$, $\mathbf{E}_T$ and $\mathbf{E}_L$ satisfy

$$\nabla \cdot \mathbf{A}_T(\mathbf{r}) = \nabla \cdot \mathbf{E}_T(\mathbf{r}) = 0 \quad , \quad \nabla \times \mathbf{A}_L(\mathbf{r}) = \nabla \times \mathbf{E}_L(\mathbf{r}) = 0 \tag{98}$$

Such a representation is possible for any vector field. This can be shown (and the origin of the names longitudinal and transverse understood) using Fourier expansions. Let us take for example $\mathbf{A}(\mathbf{r})$ and expand

$$\mathbf{A}(\mathbf{r}) = \sum_{\mathbf{k}} \mathbf{A}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \tag{99}$$

For convenience in order to have discrete values of $\mathbf{k}$ we consider the fields in a large but finite volume. The precise boundary conditions are not important for this discussion.

The Fourier amplitudes $\mathbf{A}_{\mathbf{k}}$ are vectors. Their directions in principle bear no relation to the direction of the corresponding wave vectors $\mathbf{k}$. We can however represent each of them as a sum of two vectors which are parallel and perpendicular to "their" $\mathbf{k}$

$$\mathbf{A} = \mathbf{A}^{(L)}_{\mathbf{k}} + \mathbf{A}^{(T)}_{\mathbf{k}} \quad , \quad \text{with } \; \mathbf{A}^{(L)}_{\mathbf{k}} \times \mathbf{k} = 0 \;\; , \;\; \mathbf{A}^{(T)}_{\mathbf{k}} \cdot \mathbf{k} = 0$$

Using this we can write the Fourier expansion as a sum

$$\mathbf{A}(\mathbf{r}) = \sum_{\mathbf{k}} \mathbf{A}^{(L)}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} + \sum_{\mathbf{k}} \mathbf{A}^{(T)}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \tag{100}$$

Using

$$\nabla \cdot (\mathbf{a}e^{i\mathbf{k}\cdot\mathbf{r}}) = i\mathbf{k} \cdot \mathbf{a}e^{i\mathbf{k}\cdot\mathbf{r}} \quad , \quad \nabla \times (\mathbf{a}e^{i\mathbf{k}\cdot\mathbf{r}}) = i\mathbf{k} \times \mathbf{a}e^{i\mathbf{k}\cdot\mathbf{r}}$$

we see that the two terms in the Fourier expansion of $\mathbf{A}(\mathbf{r})$ are respectively $\mathbf{A}_L(\mathbf{r})$ and $\mathbf{A}_T(\mathbf{r})$ as appear in (97).

We note also that longitudinal components of the vector fields can be written as a gradient of a scalar function. Therefore we can write

$$\mathbf{A}(\mathbf{r}) = \mathbf{A}_T(\mathbf{r}) + \nabla\xi(\mathbf{r}) \quad , \quad \mathbf{E}(\mathbf{r}) = \mathbf{E}_T(\mathbf{r}) - \nabla\phi(\mathbf{r}) \tag{101}$$

where two scalar functions $\xi(\mathbf{r})$ and -$\phi(\mathbf{r})$ fully determine the longitudinal components $\mathbf{A}_L(\mathbf{r})$ and $\mathbf{E}_L(\mathbf{r})$ respectively. As will become clear in the next section $\phi(\mathbf{r})$ is the scalar electric potential so familiar from the Coulomb and other electrostatic problems.

4.3.4 Recovering the Coulomb interaction. Resulting Hamiltonian

Inserting the above expressions for $\mathbf{A}(\mathbf{r})$ and $\mathbf{E}(\mathbf{r})$ in the Hamiltonian (77) we obtain

$$H = \frac{\epsilon_0}{2}\int d^3r \left[\mathbf{E}_T^2(\mathbf{r}) + (\nabla\phi)^2 + c^2(\nabla\times\mathbf{A}_T(\mathbf{r}))^2\right] + \sum_{a=1}^{N}\frac{1}{2m_a}\left[\mathbf{p}_a - q_a\mathbf{A}_T(\mathbf{r}_a)\right]^2 \tag{102}$$

We have transformed $\mathbf{p}_a$ to $\mathbf{p}_a + q_a\nabla\xi(\mathbf{r}_a)$. The mixed term containing $\mathbf{E}_T\cdot\nabla\phi$ does not appear since it vanishes as can be seen after integrating it by parts

$$\int d^3r\ \mathbf{E}_T\cdot\nabla\phi = -\int d^3r\ (\nabla\cdot\mathbf{E}_T)\phi \ = \ 0$$

As a last step in transforming H we note that the Gauss law allows to express $\phi(\mathbf{r})$ in terms of $\rho(\mathbf{r})$

$$\nabla\cdot\mathbf{E}(\mathbf{r}) = -\nabla^2\phi(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\epsilon_0} \quad \Rightarrow \quad \phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\rho(\mathbf{r}') \tag{103}$$

We see that in this formulation the familiar scalar potential appears in the longitudinal component of the electric field $\mathbf{E}_L(\mathbf{r})$ and is completely determined by the density of the charge. This allows to express the term in H containing $(\nabla\phi)^2$ as

$$\frac{\epsilon_0}{2}\int d^3r(\nabla\phi)^2 = -\frac{\epsilon_0}{2}\int d^3r\phi\nabla^2\phi = \frac{1}{2}\int d^3r\phi(\mathbf{r})\rho(\mathbf{r}) = \frac{1}{8\pi\epsilon_0}\int d^3rd^3r'\frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

This is just the Coulomb interaction between the charges in $\rho(\mathbf{r})$ and can be written using the expression for $\rho(\mathbf{r})$ given in (67) as

$$\frac{\epsilon_0}{2}\int d^3r(\nabla\phi)^2 = \frac{1}{8\pi\epsilon_0}\sum_{a\neq b}^{N}\frac{q_a q_b}{|\mathbf{r}_a-\mathbf{r}_b|}+\sum_{a=1}^{N}\epsilon^a_{\text{self interaction}} \tag{104}$$

where $\epsilon^a_{\text{self interaction}}$ are constants which express the Coulomb self-energy of each particle. They diverge for point particles. We will not deal with this in details but assuming that particles have small but finite sizes (cutoffs) we will simply disregard this constant term.

To conclude, the Hamiltonian has the form

$$H = H_r + \sum_{a=1}^{N}\frac{1}{2m_a}\left[\mathbf{p}_a - q_a\mathbf{A}_T(\mathbf{r}_a)\right]^2 + V_{\text{Coul}} \tag{105}$$

where we defined the radiation and the Coulomb interaction parts of the Hamiltonian

$$\begin{aligned} H_r &= \frac{\epsilon_0}{2}\int d^3r\left[\mathbf{E}_T^2(\mathbf{r}) + c^2(\nabla\times\mathbf{A}_T(\mathbf{r}))^2\right] \qquad (106)\\ V_{\text{Coul}} &= \frac{1}{8\pi\epsilon_0}\sum_{a\neq b}^{N}\frac{q_a q_b}{|\mathbf{r}_a-\mathbf{r}_b|} \end{aligned}$$

It is often convenient to write this Hamiltonian as a sum of three parts

$$H = H_r + H_{\text{matter}} + U_{\text{radiation}-\text{matter interaction}} \tag{107}$$

where H_r is given by (106) and

$$\begin{aligned} H_{\text{matter}} &= \sum_{a=1}^{N}\frac{\mathbf{p}_a^2}{2m_a} + V_{\text{Coul}} \qquad (108)\\ U_{\text{radiation}-\text{matter interaction}} &= -\sum_{a=1}^{N}\frac{q_a}{m_a}\mathbf{p}_a\cdot\mathbf{A}_T(\mathbf{r}_a) + \qquad (109)\\ &\quad + \sum_{a=1}^{N}\frac{q_a^2}{2m_a}\mathbf{A}_T(\mathbf{r}_a)\cdot\mathbf{A}_T(\mathbf{r}_a) \end{aligned}$$

Note that when switching to operators there will be no operator ordering ambiguity in the term $\hat{\mathbf{p}}_a\cdot\mathbf{A}_T(\mathbf{r}_a)$ since the difference, i.e. the commutator

$$\hat{\mathbf{p}}_a\cdot\mathbf{A}_T(\mathbf{r}_a) - \mathbf{A}_T(\mathbf{r}_a)\cdot\hat{\mathbf{p}}_a = -i\hbar\nabla_a\cdot\mathbf{A}_T(\mathbf{r}_a) = 0$$

4.3.5 An aside - separating transverse and longitudinal Maxwell equations

Let us now examine how the Hamilton (Maxwell) equations (71) and (72) look in terms of the transverse fields $\mathbf{A}_T(\mathbf{r})$, $\mathbf{E}_T(\mathbf{r})$. One can see that each equation separates into two relating separately the transverse and longitudinal components

$$\begin{aligned} \frac{\partial \mathbf{A}_T}{\partial t} &= -\mathbf{E}_T \quad , \quad \frac{\partial \mathbf{A}_L}{\partial t} = -\mathbf{E}_L \\ \frac{\partial \mathbf{E}_T}{\partial t} &= c^2 \nabla \times \nabla \times \mathbf{A}_T - \frac{\mathbf{j}_T}{\epsilon_0} \quad , \quad \frac{\partial \mathbf{E}_L}{\partial t} = -\frac{\mathbf{j}_L}{\epsilon_0} \end{aligned} \tag{110}$$

The last equation is equivalent to the continuity equation for the current

$$\frac{\partial \rho(\mathbf{r}, t)}{\partial t} = -\frac{\nabla \cdot \mathbf{j}_L(\mathbf{r}, t)}{\epsilon_0} = -\frac{\nabla \cdot \mathbf{j}(\mathbf{r}, t)}{\epsilon_0} \tag{111}$$

as can be see by taking divergence of both parts and using the Gauss law for $\nabla \cdot \mathbf{E}_L$. The equation $\partial \mathbf{A}_L / \partial t = -\mathbf{E}_L$, shows how the longitudinal component of $\mathbf{A}$ which does not enter the Hamiltonian (i.e. is the "cyclic" coordinate) develops in time for a given $\mathbf{E}_L$ which in turn is determined by the Gauss law via the charge density. One can see this as analogous to say the motion of the angular coordinates in a spherically symmetric problem as determined by the (conserved) angular momentum.

5 Photons

5.1 Field oscillators

In the present and following sections we will disregard the radiation-matter interaction and will concentrate of the radiation part described by H_r. Since this Hamiltonian is quadratic we will continue as in the case of a string. We will impose periodic boundary conditions and will expand $\mathbf{E}_T(\mathbf{r})$ and $\mathbf{A}_T(\mathbf{r})$ in terms of traveling waves. As we will see H_r will become a sum of decoupled oscillators so the traveling waves are the normal modes of the radiation.

5.1.1 Expansion in traveling waves

Following the string example, cf., Eq.(44) and the footnote[6] below we expand the field canonical coordinates and momenta $\mathbf{A}(\mathbf{r})$ and $-\epsilon_0\mathbf{E}(\mathbf{r})$, cf. Eq. (76) in a large volume Ω

$$
\begin{aligned}
\mathbf{A}(\mathbf{r}) &= \frac{1}{\sqrt{\Omega\epsilon_0}}\sum_{\mathbf{k}}\left(\mathbf{q_k}\sin(\mathbf{k}\cdot\mathbf{r})+\frac{1}{ck}\mathbf{p_k}\cos(\mathbf{k}\cdot\mathbf{r})\right) \qquad (112)\\
\mathbf{E}(\mathbf{r}) &= -\frac{1}{\sqrt{\Omega\epsilon_0}}\sum_{\mathbf{k}}\left(\mathbf{p_k}\sin(\mathbf{k}\cdot\mathbf{r})-ck\,\mathbf{q_k}\cos(\mathbf{k}\cdot\mathbf{r})\right)
\end{aligned}
$$

with vector expansion coefficients $\mathbf{q_k}$ and $\mathbf{p_k}$. To make the above expansions more symmetric with respects to the appearance of ϵ_0 we changed our canonical variables to

$$\mathbf{A}(\mathbf{r})\to\mathbf{A}(\mathbf{r})/\sqrt{\epsilon_0}\ \ ,\ \ -\epsilon_0\mathbf{E}(\mathbf{r})\to-\sqrt{\epsilon_0}\,\mathbf{E}(\mathbf{r})$$

The periodic boundary conditions lead to discrete values of the wave vectors

$$\mathbf{k}=\left\{(n_x,n_y,n_x)\frac{2\pi}{\Omega^{1/3}}\right\}\ \ ,\ \ n_i=0,\pm1,\pm2\ldots \qquad (113)$$

where we assumed the volume to be a cube, i.e. have the same length, width and height each equal to $\Omega^{1/3}$.

The more conventional form of the expansion Eq. (112) found in the literature (cf., cf. Landau and Lifshitz, Classical Field Theory, Sec.52 or Ref. [7]) is written in terms of canonically transformed variables

$$\mathbf{q_k}\to-\frac{1}{\omega_k}\mathbf{p_k}\ \ ;\ \ \mathbf{p_k}\to\omega_k\mathbf{q_k} \qquad (114)$$

[6]For easy comparison we reproduce this expansion here

$$
\begin{aligned}
\phi(x,t) &= \sqrt{\frac{1}{L}}\sum_k\left[Q_k(t)\sin kx+\frac{P_k(t)}{v|k|}\cos kx\right]\\
\pi(x,t) &= \sqrt{\frac{1}{L}}\sum_k\left[P_k(t)\sin kx-v|k|Q_k(t)\cos kx\right]\ ,\ k=\frac{2\pi\nu}{L}\ ,\ \nu=\pm1,\pm2,...
\end{aligned}
$$

which for the transverse components of the fields results in

$$\begin{aligned} \mathbf{A}_T(\mathbf{r}) &= \frac{1}{\sqrt{\Omega\epsilon_0}} \sum_{\mathbf{k}} \left[\mathbf{Q}_{\mathbf{k}} \cos(\mathbf{k}\cdot\mathbf{r}) - \frac{1}{\omega_k} \mathbf{P}_{\mathbf{k}} \sin(\mathbf{k}\cdot\mathbf{r}) \right] \\ \mathbf{E}_T(\mathbf{r}) &= -\frac{1}{\sqrt{\Omega\epsilon_0}} \sum_{\mathbf{k}} \left[\mathbf{P}_{\mathbf{k}} \cos(\mathbf{k}\cdot\mathbf{r}) + \omega_k \mathbf{Q}_{\mathbf{k}} \sin(\mathbf{k}\cdot\mathbf{r}) \right] \end{aligned} \tag{115}$$

The transversality of $\mathbf{A}_T(\mathbf{r})$ and $\mathbf{E}_T(\mathbf{r})$ means that the vectors $\mathbf{Q}_{\mathbf{k}}$ and $\mathbf{P}_{\mathbf{k}}$ are always orthogonal to the corresponding $\mathbf{k}$,

$$\mathbf{k}\cdot\mathbf{Q}_{\mathbf{k}} = 0 \quad , \quad \mathbf{k}\cdot\mathbf{P}_{\mathbf{k}} = 0.$$

It is convenient to use a pair of fixed unit polarization vectors $\boldsymbol{\lambda}_{\mathbf{k}\alpha}$, $\alpha = 1, 2$ with

$$\boldsymbol{\lambda}_{\mathbf{k}1}\cdot\boldsymbol{\lambda}_{\mathbf{k}2} = 0 \quad , \quad \boldsymbol{\lambda}_{\mathbf{k}\alpha}\cdot\mathbf{k} = 0 \quad , \quad \alpha = 1, 2 \tag{116}$$

so we can write

$$\mathbf{Q}_{\mathbf{k}} = \sum_{\alpha=1,2} Q_{\mathbf{k}\alpha}\boldsymbol{\lambda}_{\mathbf{k}\alpha} \quad , \quad \mathbf{P}_{\mathbf{k}} = \sum_{\alpha=1,2} P_{\mathbf{k}\alpha}\boldsymbol{\lambda}_{\mathbf{k}\alpha} \tag{117}$$

We now insert expansions (115) and (117) into the Maxwell equations for the transverse component, i.e. into the 1st and 3rd equation of the set (110) in the absence of the current (recall we are discussing pure radiation). We obtain in the straightforward manner separate linear equations for $Q_{\mathbf{k}\alpha}$ and $P_{\mathbf{k}\alpha}$

$$\dot{Q}_{\mathbf{k}\alpha} = P_{\mathbf{k}\alpha} \quad , \quad \dot{P}_{\mathbf{k}\alpha} = -\omega_k^2 Q_{\mathbf{k}\alpha} \quad \text{with} \quad \omega_k = ck \tag{118}$$

One clearly sees that these are Hamilton equations of harmonic oscillators labeled by $\mathbf{k}\alpha$ each with the Hamiltonian

$$H_{\mathbf{k}\alpha} = \frac{1}{2}\left(P_{\mathbf{k}\alpha}^2 + \omega_k^2 Q_{\mathbf{k}\alpha}^2\right) \tag{119}$$

and $Q_{\mathbf{k}\alpha}$ and $P_{\mathbf{k}\alpha}$ being the generalized coordinates and momenta. One could also obtain this by inserting expansions (115) and (117) into H_r to find

$$H_r = \sum_{\mathbf{k}\alpha} H_{\mathbf{k}\alpha} = \frac{1}{2}\sum_{\mathbf{k}\alpha}\left(P_{\mathbf{k}\alpha}^2 + \omega_k^2 Q_{\mathbf{k}\alpha}^2\right) \quad \text{with} \quad \omega_k = ck \tag{120}$$

It is seen that indeed we represent H_r as a sum of decoupled oscillators with frequencies given by the well know dispersion relation of the EM waves. There are two oscillators with different polarizations for each $\mathbf{k}$. Since ω_k depends on the magnitude of k all the oscillators with $|\mathbf{k}| = k$ have the same frequency.

5.1.2 Field wave functions and eigenstates. Photons appear

We now turn to the quantum mechanics of the radiation. It is the easiest to do this in the decoupled eigenmodes of the field as encoded in the Hamiltonian (120). Instead of classical time dependent variables $Q_{\mathbf{k}\alpha}(t)$ and $P_{\mathbf{k}\alpha}(t)$ we consider wave function $\Psi(\{Q_{\mathbf{k}\alpha}\}, t)$ which contains all the quantum information. This is "extracted" by using operators for every physical quantity which are build of two sets of basic operators of the "coordinate" and "momentum".

$$\begin{gathered} Q_{\mathbf{k}\alpha} \;\; \rightarrow \;\; \hat{Q}_{\mathbf{k}\alpha} = Q_{\mathbf{k}\alpha} \quad ; \quad P_{\mathbf{k}\alpha} \;\; \rightarrow \;\; \hat{P}_{\mathbf{k}\alpha} = -i\hbar \frac{\partial}{\partial Q_{\mathbf{k}\alpha}} \\ \left[P_{\mathbf{k}\alpha}, P_{\mathbf{k}'\alpha'} \right] = \left[Q_{\mathbf{k}\alpha}, Q_{\mathbf{k}'\alpha'} \right] = 0 \quad , \left[Q_{\mathbf{k}\alpha}, P_{\mathbf{k}'\alpha'} \right] = i\hbar \delta_{\mathbf{k}\mathbf{k}'} \delta_{\alpha\alpha'} \end{gathered} \tag{121}$$

All in the usual way as in quantum mechanics of mechanical systems.

The time dependence of $\Psi(\{Q_{\mathbf{k}\alpha}\}, t)$ is governed by the Schrödinger equation

$$i\hbar \frac{\partial \Psi(\{Q_{\mathbf{k}\alpha}\}, t)}{\partial t} = \hat{H}_r \Psi(\{Q_{\mathbf{k}\alpha}\}, t) \tag{122}$$

with the Hamiltonian operator $\hat{H}_r$ obtained from Eq. (120) by replacing in it the coordinates and momenta with the corresponding operators

$$\hat{H}_r = \frac{1}{2} \sum_{\mathbf{k}\alpha} \left(\hat{P}^2_{\mathbf{k}\alpha} + \omega_k^2 \hat{Q}^2_{\mathbf{k}\alpha} \right) \tag{123}$$

The most important solutions of the Schrödinger equation are the stationary states which are the eigenstates of $\hat{H}_r$

$$\hat{H}_r \Psi(\{Q_{\mathbf{k}\alpha}\}) = \mathcal{E} \Psi(\{Q_{\mathbf{k}\alpha}\}) \tag{124}$$

Since $\hat{H}_r$ is a sum of independent terms each representing an oscillator the eigenvalues of $\hat{H}_r$ are sums of eigenenergies of independent oscillators

$$\begin{gathered} \mathcal{E}_{\{N_{\mathbf{k}\alpha}\}} = \sum_{\mathbf{k}\alpha} \hbar\omega_k \left(N_{\mathbf{k}\alpha} + \frac{1}{2} \right) \equiv \mathcal{E}_0 + \sum_{\mathbf{k}\alpha} \hbar\omega_k N_{\mathbf{k}\alpha} \\ \mathcal{E}_0 = \sum_{\mathbf{k}\alpha} \frac{\hbar\omega_k}{2} \quad , \quad N_{\mathbf{k}\alpha} = 0, 1, 2, 3, ... \end{gathered} \tag{125}$$

The corresponding eigenfunctions are products

$$\Psi_{\{N_{\mathbf{k}_\alpha}\}}(\{Q_{\mathbf{k}\alpha}\}) = \prod_{\mathbf{k}\alpha} \psi_{N_{\mathbf{k}_\alpha}}(Q_{\mathbf{k}\alpha}) \quad , \quad N_{\mathbf{k}\alpha} = 0, 1, 2, 3, ... \tag{126}$$

where $\psi_{N_{\mathbf{k}_\alpha}}(Q_{\mathbf{k}\alpha})$ are the standard eigenfunctions of a harmonic oscillator, cf., Eq.(33), with unit mass and frequency $\omega_k = ck$.

As in our discussion of the string quantization (and actually in the quantization of any linear dynamical system) we find that the EM field can be viewed as a collection of energy quanta

$$\epsilon_{\mathbf{k}\alpha} = \hbar\omega_k$$

in its normal modes. In the following sections we will show that these quanta have all the characteristics of particles. They carry momentum, angular momentum and spin and have energy-momentum relation of massless particles moving with the speed of light, cf., Eq.(141). These quantum particles are photons.

Focusing on the details we note that the EM modes are 3D vector waves. In our developments we have chosen them as plane waves with wave vectors $\mathbf{k}$ and polarisations $\boldsymbol{\lambda}_{\mathbf{k}\alpha}$. But let us note that the eigenfrequencies of these modes $\omega_k = ck$ and as a result the energies $\epsilon_{\mathbf{k}\alpha}$ of the quanta depend only on the magnitude of $\mathbf{k}$, i.e. on the wavelength and not on the direction of the vector $\mathbf{k}$ and the polarisation of the modes.

As with the ordinary matter particles with e.g. $\epsilon_{\mathbf{k}} = \hbar^2k^2/2m$ this means that there is a continuum degeneracy of the modes and therefore of the quantum mechanical states of the (free) photons. This degeneracy results in a freedom to change the basis states with a given energy ϵ from the vector plane wave (like we did above) to e.g. spherical (vector spherical!) or cylindrical (vector cylindrical!) etc waves. The quantum numbers $\mathbf{k}\alpha$ will then be replaced by appropriately changed ones like k, l, m replacing k_x, k_y, k_z in the scalar waves. A recent reference to the vector spherical waves is e.g. Ref.[4]. We will encounter them and provide a brief review in the Chapter Photon Interactions with Matter.

5.1.3 The wave function of the vacuum. The Casimir effect

The ground state of the EM field is the vacuum of the theory in the absence of matter and other quantum fields. Its wave function is the product of Gaussians familiar from our discussion of the guitar string, Eq. (34),

$$\Psi_{\{N_{\mathbf{k}_\alpha}=0\}}(\{Q_{\mathbf{k}\alpha}\}) = \prod_{\mathbf{k}\alpha} \psi_0(Q_{\mathbf{k}\alpha}) = \prod_{\mathbf{k}\alpha} \left(\frac{\omega_k}{\pi\hbar}\right)^{1/4} \exp\left(-\sum_{\mathbf{k}\alpha} \omega_k Q^2_{\mathbf{k}\alpha}/2\hbar\right) \tag{127}$$

It provides perhaps the simplest example of quantum vacuum fluctuations of field degrees of freedom in a quantum field theory.

Can one observe these fluctuations? In a ground breaking paper, Ref.[1], Casimir addressed this issue. He suggested that such fluctuations induce "attraction between two perfectly conducting plates". On such plates the parallel to the plates components of the electric field must vanish so that the field normal modes for which this doesn't happen will be excluded from the field degrees of freedom and consequently from the vacuum fluctuations. This is schematically illustrated in Fig.4. The density of the normal modes frequencies between the plates will be smaller than in the free space outside.

We have considered a one dimensional version of this effect in the context of the quantum guitar string, Sec.2.4.3. We have shown there that it leads to an attractive force between the (analogue of) the plates with the more narrow spacing than that of the other part of the string. The same happens in the realistic 3D case with quantized EM field. Casimir calculations predicted that an attractive force per unit area (pressure) at plate separation a is given by

$$P = -\frac{\hbar \pi^2 c}{240 a^4}$$

Note that the inverse quartic dependence on the distance is most unusual in physics. It is also worth mentioning that for certain special combinations of the plates materials, the Casimir force can be repulsive. The results obtained by Casimir were later extended to various geometries of the plates and his predictions were confirmed experimentally, cf., Ref.[6].

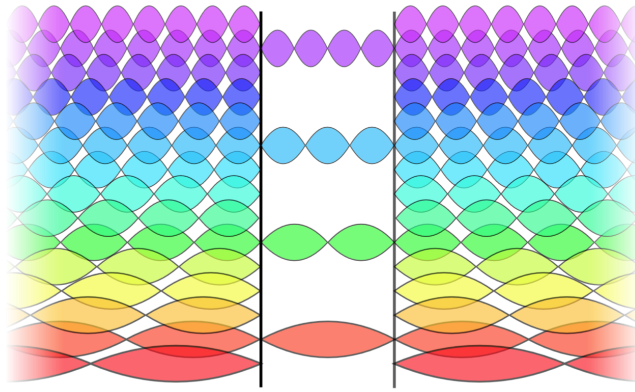

Figure 4: Schematic depiction of the normal modes of the field vibrations in the presence of two plates which enforce vanishing of the field at the plates positions (Fig.1 from Ref.[5]). This leads to the difference in the frequency densities of the field normal modes between the plates as compared to the outside free space. This difference depends on the distance between the plates and leads to the Casimir effect of plate attraction

5.1.4 Photon creation and annihilation operators. Field operators

In practice it is convenient to introduce creation and annihilation operators of photons in the standard way

$$\begin{gathered}\hat{Q}_{\mathbf{k}\alpha} = \sqrt{\hbar/2\omega_k}\left(\hat{a}^\dagger_{\mathbf{k}\alpha} + \hat{a}_{\mathbf{k}\alpha}\right) \quad , \quad \hat{P}_{\mathbf{k}\alpha} = i\sqrt{\hbar\omega_k/2}\left(\hat{a}^\dagger_{\mathbf{k}\alpha} - \hat{a}_{\mathbf{k}\alpha}\right) \\ \hat{a}_{\mathbf{k}\alpha} = \sqrt{1/2\hbar\omega_k}\left(i\hat{P}_{\mathbf{k}\alpha} + \omega_k\,\hat{Q}_{\mathbf{k}\alpha}\right) \quad , \quad \hat{a}^\dagger_{\mathbf{k}\alpha} = \sqrt{1/2\hbar\omega_k}\left(-i\hat{P}_{\mathbf{k}\alpha} + \omega_k\,\hat{Q}_{\mathbf{k}\alpha}\right) \\ [\hat{a}_{\mathbf{k}\alpha}, \hat{a}^\dagger_{\mathbf{k}'\alpha'}] = \delta_{\mathbf{k}\mathbf{k}'}\delta_{\alpha\alpha'} \quad , \quad [\hat{a}_{\mathbf{k}\alpha}, \hat{a}_{\mathbf{k}'\alpha'}] = 0 = [\hat{a}^\dagger_{\mathbf{k}\alpha}, \hat{a}^\dagger_{\mathbf{k}'\alpha'}]\end{gathered} \tag{128}$$

Using these operators we can write the Hamiltonian

$$\hat{H}_r = \mathcal{E}_0 + \sum_{\mathbf{k}\alpha} \hbar\omega_k \hat{a}^\dagger_{\mathbf{k}\alpha}\hat{a}_{\mathbf{k}\alpha} \tag{129}$$

and its eigenstates

$$|\{N_{\mathbf{k}\alpha}\} >= \prod_{\mathbf{k}\alpha} |N_{\mathbf{k}\alpha} >= \prod_{\mathbf{k}\alpha} \frac{(\hat{a}^\dagger_{\mathbf{k}\alpha})^{N_{\mathbf{k}\alpha}}}{(N_{\mathbf{k}\alpha}!)^{1/2}}|0> \tag{130}$$

Great advantage of using $\hat{a}$ and $\hat{a}^\dagger$ operators rather than $\hat{P}$ and $\hat{Q}$ in dealing with photons is the simplicity of the "action" of these operators on the "number states", i.e. the states with a fixed photon numbers in each normal mode. Schematically

$$\hat{a}|n>=\sqrt{n}|n-1> \quad , \quad \hat{a}^\dagger|n>=\sqrt{n+1}|n+1>$$

In "full glory"

$$\hat{a}_{\mathbf{k}\alpha}|\{N_{\mathbf{k}\alpha}\}>=\sqrt{N_{\mathbf{k}\alpha}}|N_{\mathbf{k}\alpha}-1>\prod_{\mathbf{k}'\neq\mathbf{k},\alpha'\neq\alpha}|\{N_{\mathbf{k}'\alpha'}\}>$$

$$a^\dagger_{\mathbf{k}\alpha}|\{N_{\mathbf{k}\alpha}\}>=\sqrt{N_{\mathbf{k}\alpha}+1}|N_{\mathbf{k}\alpha}+1>\prod_{\mathbf{k}'\neq\mathbf{k},\alpha'\neq\alpha}|\{N_{\mathbf{k}'\alpha'}\}>$$

One says that the operators $\hat{a}^\dagger_{\mathbf{k}\alpha}$ and $\hat{a}_{\mathbf{k}\alpha}$ create and destroy(annihilate) photons.

It is useful to express the operators of the fields (115) as well as $\hat{\mathbf{B}}(\mathbf{r})$ in terms of the creation and annihilation operators

$$\begin{aligned}
\hat{\mathbf{A}}_T(\mathbf{r}) &= \sum_{\mathbf{k}\alpha}\left(\frac{\hbar}{2\epsilon_0\omega_k\Omega}\right)^{1/2}\left[\hat{a}_{\mathbf{k}\alpha}\boldsymbol{\lambda}_{\mathbf{k}\alpha}e^{i\mathbf{k}\cdot\mathbf{r}}+\hat{a}^\dagger_{\mathbf{k}\alpha}\boldsymbol{\lambda}_{\mathbf{k}\alpha}e^{-i\mathbf{k}\cdot\mathbf{r}}\right]= \\
&= \sum_{\mathbf{k}\alpha}\left(\frac{\hbar}{2\epsilon_0\omega_k\Omega}\right)^{1/2}\boldsymbol{\lambda}_{\mathbf{k}\alpha}e^{i\mathbf{k}\cdot\mathbf{r}}(\hat{a}_{\mathbf{k}\alpha}+\hat{a}^\dagger_{-\mathbf{k}\alpha}) \\
\hat{\mathbf{E}}_T(\mathbf{r}) &= \sum_{\mathbf{k}\alpha} i\left(\frac{\hbar\omega_k}{2\epsilon_0\Omega}\right)^{1/2}\left[\hat{a}_{\mathbf{k}\alpha}\boldsymbol{\lambda}_{\mathbf{k}\alpha}e^{i\mathbf{k}\cdot\mathbf{r}}-\hat{a}^\dagger_{\mathbf{k}\alpha}\boldsymbol{\lambda}_{\mathbf{k}\alpha}e^{-i\mathbf{k}\cdot\mathbf{r}}\right] \\
\hat{\mathbf{B}}(\mathbf{r}) &= \sum_{\mathbf{k}\alpha} i\left(\frac{\hbar}{2\epsilon_0\omega_k\Omega}\right)^{1/2}\left[\hat{a}_{\mathbf{k}\alpha}(\mathbf{k}\times\boldsymbol{\lambda}_{\mathbf{k}\alpha})e^{i\mathbf{k}\cdot\mathbf{r}}-\hat{a}^\dagger_{\mathbf{k}\alpha}(\mathbf{k}\times\boldsymbol{\lambda}_{\mathbf{k}\alpha})e^{-i\mathbf{k}\cdot\mathbf{r}}\right]
\end{aligned} \tag{131}$$

It is important to note that once the longitudinal components of the fields $\mathbf{A}_{op}(\mathbf{r})$ and $\mathbf{E}_{op}(\mathbf{r})$ were separated the remaining transverse parts $\hat{\mathbf{A}}_T(\mathbf{r})$ and $\hat{\mathbf{E}}_T(\mathbf{r})$ do not obey the canonical commutations (82). Rather the delta function there gets replaced by the so called "transverse" delta function cf., Ref.[7], Ch.İII.A.1.

5.2 Photon momentum

In this subsection we will consider the operator of the momentum of the EM field $\mathbf{P}_{\text{field}}$. Using it we will be able to show that photons are not just "portions" of energy of the EM field but that they also carry a "corresponding" portion of its momentum. Moreover the relation between the energies and momenta of these portions are as of *massless* particles traveling with the speed of light.

5.2.1 Generators of translations in the matter-field system

Quantum mechanically it is probably the easiest to guess the expression of the momentum by recalling that it is the generator of infinitesimal translations. For the interacting system of particles (matter) and EM field described by the Hamiltonian (84) the operation of infinitesimal translation is the transformation

$$\begin{aligned} e^{(i/\hbar)\mathbf{a}\cdot\hat{\mathbf{P}}}\mathbf{A}_{op}(\mathbf{r})e^{(-i/\hbar)\mathbf{a}\cdot\hat{\mathbf{P}}} &= \mathbf{A}_{op}(\mathbf{r}+\mathbf{a}) \approx \mathbf{A}_{op}(\mathbf{r}) + (\mathbf{a}\cdot\nabla)\mathbf{A}_{op}(\mathbf{r}) \\ e^{(i/\hbar)\mathbf{a}\cdot\hat{\mathbf{P}}}\mathbf{E}_{op}(\mathbf{r})e^{(-i/\hbar)\mathbf{a}\cdot\hat{\mathbf{P}}} &= \mathbf{E}_{op}(\mathbf{r}+\mathbf{a}) \approx \mathbf{E}_{op}(\mathbf{r}) + (\mathbf{a}\cdot\nabla)\mathbf{E}_{op}(\mathbf{r}) \\ e^{(i/\hbar)\mathbf{a}\cdot\hat{\mathbf{P}}}\hat{\mathbf{r}}_a e^{(-i/\hbar)\mathbf{a}\cdot\hat{\mathbf{P}}} &= \hat{\mathbf{r}}_a + \mathbf{a}, \quad a = 1, ..., N \end{aligned} \tag{132}$$

Therefore the(vector) momentum operator $\hat{\mathbf{P}} = \{\hat{P}_x, \hat{P}_y, \hat{P}_z\}$ of the system should be such that for each of its component $\hat{P}_k$, the commutators hold

$$[\hat{P}_k, \hat{A}_j(\mathbf{r})] = -i\hbar\partial_k\hat{A}_j(\mathbf{r}) \quad , \quad [\hat{P}_k, \hat{E}_j(\mathbf{r})] = -i\hbar\partial_k E_j(\mathbf{r}) \tag{133}$$

$$[\hat{P}_k, r_{j,a}] = -i\hbar\frac{\partial r_{j,a}}{\partial r_{k,a}} = -i\hbar\delta_{kj} \tag{134}$$

It is actually very easy to guess what such $\hat{\mathbf{P}}$ should be

$$\hat{\mathbf{P}} = \hat{\mathbf{P}}_{\text{matter}} + \hat{\mathbf{P}}_{\text{field}} = \sum_{a=1}^{N}\hat{\mathbf{p}}_a + \frac{\epsilon_0}{2}\sum_{j=1}^{3}\int d^3r\left\{\hat{E}_j(\mathbf{r})\nabla\hat{A}_j(\mathbf{r}) + h.c.\right\} \tag{135}$$

where $\hat{p}_a = -i\hbar\nabla_a$, $a = 1, ...N$ and the "h.c." abbreviation stands for "hermitian conjugate".

Indeed the first term $\hat{\mathbf{P}}_{\text{matter}}$ has the required commutator with $\hat{\mathbf{r}}_a$ while commuting with $\mathbf{E}_{op}(\mathbf{r})$ and $\mathbf{A}_{op}(\mathbf{r})$ and the second term $\hat{\mathbf{P}}_{\text{field}}$ commutes with $\hat{\mathbf{r}}_a$ and satisfies

$$\begin{aligned} [\hat{P}_{\text{field},k}, \hat{A}_j(\mathbf{r})] &= \frac{\epsilon_0}{2}\sum_{n=1}^{3}\int d^3r'\left\{[\hat{E}_n(\mathbf{r}')\partial'_k\hat{A}_n(\mathbf{r}'), \hat{A}_j(\mathbf{r})] + ...\right\} = \\ &= -\epsilon_0\sum_{n=1}^{3}\int d^3r'\frac{i\hbar\delta_{nj}}{\epsilon_0}\delta(\mathbf{r}'-\mathbf{r}))\partial'_k\hat{A}_n(\mathbf{r}') = -i\hbar\partial_k\hat{A}_j(\mathbf{r}) \end{aligned}$$

$$
\begin{aligned}
[\hat{P}_{\text{field},k}, \hat{E}_j(\mathbf{r})] &= \frac{\epsilon_0}{2}\sum_{n=1}^{3}\int d^3r' \left\{[\hat{E}_n(\mathbf{r}')\partial'_k\hat{A}_n(\mathbf{r}'), \hat{E}_j(\mathbf{r})] + ...\right\} = \\
&= -\epsilon_0\sum_{n=1}^{3}\int d^3r'\hat{E}_n(\mathbf{r}'))\partial'_k\frac{-i\hbar\delta_{nj}}{\epsilon_0}\delta(\mathbf{r}'-\mathbf{r})) = -i\hbar\partial_k\hat{E}_j(\mathbf{r})
\end{aligned}
$$

where in the last line we used the integration by parts.

Let us recall the classical expression for the conserved momentum in the presence of the EM field. On the basis of the Maxwell and Newton equations Eqs.(65 - 67) one finds that

$$
\mathbf{P} = \sum_{a=1}^{N} m_a\mathbf{v}_a + \epsilon_0\int d^3r\,\mathbf{E}(\mathbf{r})\times\mathbf{B}(\mathbf{r}) \tag{136}
$$

is the conserved total momentum of the field-matter system

$$
\frac{d\mathbf{P}}{dt} = 0 \tag{137}
$$

cf. p. 61, in Ref.[7] or a less formal text - Ref.[8]. In Appendix 6.3 below we show the equivalence of the expressions (135) and (136).

5.2.2 Momentum of the EM radiation

In the absence of charged particles we can use $\nabla\cdot\hat{\mathbf{E}} = 0$ and replace $\hat{E}_j$ by the transversal $\hat{E}_{Tj}$ in the field part of the momentum in (135). The same can be done with $\hat{A}_j$ in it. Indeed the longitudinal part of $\hat{\mathbf{A}}$ can be written as a gradient $\mathbf{A}_L = \nabla\hat{\xi}$. Thus it contributes (after the replacement $\hat{E}_j \to \hat{E}_{Tj}$) the term

$$
\frac{\epsilon_0}{2}\sum_{j=1}^{3}\int d^3r\left\{\hat{E}_{Tj}(\mathbf{r})\nabla\hat{A}_{Lj}(\mathbf{r}) + h.c.\right\} = \frac{\epsilon_0}{2}\sum_{j=1}^{3}\int d^3r\left\{\hat{E}_{Tj}(\mathbf{r})\nabla\partial_j\hat{\xi}(\mathbf{r}) + h.c.\right\}
$$

in $\hat{\mathbf{P}}_{\text{field}}$. This term is however zero as can be seen by integrating by parts in the right hand side and using $\partial_j\hat{E}_{Tj} = 0$.

Thus we can write the momentum of the "pure" radiation as

$$
\hat{\mathbf{P}}_r = \frac{\epsilon_0}{2}\int d^3r\sum_{j=1}^{3}\left[\hat{E}_{T,j}(\mathbf{r})\nabla\hat{A}_{T,j}(\mathbf{r}) + h.c.\right] \tag{138}
$$

Inserting expressions for the operators $\mathbf{A}_T(\mathbf{r})$ and $\mathbf{E}_T(\mathbf{r})$ one obtains

$$\hat{\mathbf{P}}_r = \sum_{\mathbf{k}\alpha} \hbar\mathbf{k}\, \hat{a}^\dagger_{\mathbf{k}\alpha}\hat{a}_{\mathbf{k}\alpha} \tag{139}$$

where we used $\sum_{\mathbf{k}\alpha}\hbar\mathbf{k} = 0$[7]. As it should $\hat{\mathbf{P}}_r$ commutes with the Hamiltonian $\hat{H}_r$. Its eigenvalues are

$$\mathbf{P}_{\{N_{\mathbf{k}\alpha}\}} = \sum_{\mathbf{k}\alpha}\hbar\mathbf{k}\, N_{\mathbf{k}\alpha} \quad , \quad N_{\mathbf{k}\alpha} = 0, 1, 2, 3, ... \tag{140}$$

We can see that every state with $N_{\mathbf{k}\alpha}$ quanta has momentum $\hbar\mathbf{k}N_{\mathbf{k}\alpha}$ so that every energy quantum with $\epsilon_k = \hbar\omega_k$ carries momentum $\mathbf{p}_k = \hbar\mathbf{k}$. Using the dispersion relation $\omega_k = c|\mathbf{k}|$ of the (classal) light waves (EM normal modes) we find the energy-momentum relation of light quanta

$$\epsilon_k = c|\mathbf{p}_k| \tag{141}$$

i.e. of the massless particle moving with the light velocity.

5.3 Common States of Light

5.3.1 Number states

These are just the eigenstates $|\{N_{\mathbf{k}\alpha}\}\rangle$ of the $\hat{H}_r$, cf., Eq. (129). Although most natural from the formal point of view they are highly nonclassical and in fact are extremely hard to produce "on demand"[8]. Number states are states of well defined energy but not of well defined EM field. As an example consider a single mode of the electric fields, i.e. just one term with a given $\mathbf{k}, \alpha$ in the expression for $\mathbf{E}_T$ in (131) and calculate

$$\langle N_{\mathbf{k}\alpha}|\hat{\mathbf{E}}(\mathbf{r})|N_{\mathbf{k}\alpha}\rangle = i\boldsymbol{\lambda}_{\mathbf{k}\alpha}\left(\frac{\hbar\omega_k}{2\epsilon_0\Omega}\right)^{1/2}\langle N_{\mathbf{k}\alpha}|\left[\hat{a}_{\mathbf{k}\alpha}e^{i\mathbf{k}\cdot\mathbf{r}} - \hat{a}^+_{\mathbf{k}\alpha}e^{-i\mathbf{k}\cdot\mathbf{r}}\right]|N_{\mathbf{k}\alpha}\rangle = 0 \tag{142}$$

and

$$\begin{aligned}\Delta E &= \sqrt{\langle N_{\mathbf{k}\alpha}|\hat{\mathbf{E}}(\mathbf{r})\cdot\hat{\mathbf{E}}(\mathbf{r})|N_{\mathbf{k}\alpha}\rangle} = \\ &= \left(\frac{\hbar\omega_k}{2\epsilon_0\Omega}\right)^{1/2}\langle N_{\mathbf{k}\alpha}|\hat{a}_{\mathbf{k}\alpha}\hat{a}^+_{\mathbf{k}\alpha} + \hat{a}^+_{\mathbf{k}\alpha}\hat{a}_{\mathbf{k}\alpha}|N_{\mathbf{k}\alpha}\rangle = \left(\frac{\hbar\omega_k}{2\epsilon_0\Omega}\right)^{1/2}(2N_{\mathbf{k}\alpha}+1)^{1/2}\end{aligned} \tag{143}$$

[7]There is a subtle point here - this sum diverges and must be regularized by, say, assuming a cutoff at some large k_c.

[8]E.g. M. Oxborrow and A.G. Sinclair, Contemp. Phys. 46, 173 (2005).

so that the everage value of $\hat{\mathbf{E}}$ is zero while the fluctuations grow with the number of photons.

5.3.2 Quantum mechanics behind the classical EM field. Coherent states of light

The correct description of the world is quantum mechanical while the classical physics is just an approximation. So it is natural to ask what is the quantum mechanical state behind the classical EM field? Since the quantum mechanical operators of electric and magnetic components $\hat{\mathbf{E}}(\mathbf{r})$ and $\hat{\mathbf{A}}(\mathbf{r})$ of the field are non commuting there is no state in which they both have definite values.

Under these restrictions the appropriate quantum state $|\Psi(t)\rangle$ behind the classical EM field must be such that the averages, i.e. the expectation values of the field operators

$$\mathbf{E}(\mathbf{r},t) \equiv \langle\Psi(t)|\hat{\mathbf{E}}(\mathbf{r})|\Psi(t)\rangle \quad , \quad \mathbf{A}(\mathbf{r},t) \equiv \langle\Psi(t)|\hat{\mathbf{A}}(\mathbf{r})|\Psi(t)\rangle$$

will be developing in time as solutions of the classical Maxwell equation and be "classically large" i.e. much larger than the quantum uncertainties i.e. the standard deviations of these fields from the averages.

It is not hard to find the state with the above properties for a free EM field. Since such a field can be represented as a collection of independents modes it is useful to start by considering a simple case of just a single mode with a given wave number $\mathbf{k}$ and polarisation $\boldsymbol{\lambda}$. Concentrating on the electric field we have the operator

$$\hat{\mathbf{E}}(\mathbf{r}) = \frac{\boldsymbol{\lambda}_{\mathbf{k}}}{\sqrt{\Omega\epsilon_0}}\left[ck\hat{Q}_{\mathbf{k}}\cos(\mathbf{k}\cdot\mathbf{r}) - \hat{P}_{\mathbf{k}}\sin(\mathbf{k}\cdot\mathbf{r})\right] \tag{144}$$

The quantum mechanics of $\hat{\mathbf{E}}(\mathbf{r})$ and its non non commutativity with $\hat{\mathbf{A}}(\mathbf{r})$ is "encoded" in the canonical non commuting pair of the operators $\hat{P}_{\mathbf{k}}, \hat{Q}_{\mathbf{k}}$. Their dynamics (for a free field) is simple - just that of harmonic oscillator, cf., Eq. (123).

So the task is to find a quantum state of harmonic oscillator, i.e. solutions $|\psi(t)\rangle$ of the Schrödinger equation

$$i\hbar\frac{\partial|\psi(t)\rangle}{\partial t} = \hat{h}|\psi(t)\rangle \;\text{ with }\quad \hat{h} = \frac{1}{2}\left(\hat{p}^2 + \omega^2\hat{q}^2\right) \tag{145}$$

for which the averages

$$q(t) = \langle\psi(t)|\hat{q}|\psi(t)\rangle \quad , \quad p(t) = \langle\psi(t)|\hat{p}|\psi(t)\rangle$$

obey the classical equations of the harmonic oscillator

$$\dot{q} = p \ \ , \ \ \dot{p} = -\omega^2 q \tag{146}$$

and have smallest possible quantum uncertainties.

Such a state was first discussed by Schrödinger already in 1926 and has a name - coherent state. Its common formal definition is that it is an eigenstate of the annihilation operator

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \ \ \Rightarrow \ \ \left(\hbar\frac{\partial}{\partial q} + \omega\, q\right)\psi_\alpha(q) = \sqrt{2\hbar\omega}\alpha\psi_\alpha(q) \tag{147}$$

with eigenstates $|\alpha\rangle$ labeled by the eigenvalues α and where we used the coordinate representation of $\hat{a}$

$$\hat{a} = (i\hat{p} + \omega\,\hat{q})\,/\sqrt{2\hbar\omega}$$

Note that since $\hat{a}$ is non hermitian α's are in general complex valued[9]. Also note that for $\alpha = 0$ the coherent state is just a ground state of the harmonic oscillator

$$\psi_0(q) = Ae^{-\omega q^2/2\hbar} \ \ , \ \ A = (\omega/\pi\hbar)^{1/4} \tag{148}$$

The properties of the coherent state are discussed in the Appendix of this Chapter. It is shown there that coherent state is a wave packet the dynamics of which is such that the averages

$$q_0 \equiv \langle\alpha|\hat{q}|\alpha\rangle \ \ , \ \ p_0 \equiv \langle\alpha|\hat{p}|\alpha\rangle \tag{149}$$

move along the corresponding classical trajectories with uncertainties obeying the minimum uncertainties relation

$$\Delta q \Delta p = \hbar/2 \tag{150}$$

It is useful to schematically present this picture in the classical phase space as is explained in Fig. 5.

[9]There are many unfamiliar features of $|\alpha\rangle$ states as a result of this. Like non orthogonality at different α's or over completeness. This will be partly covered in the Appendix

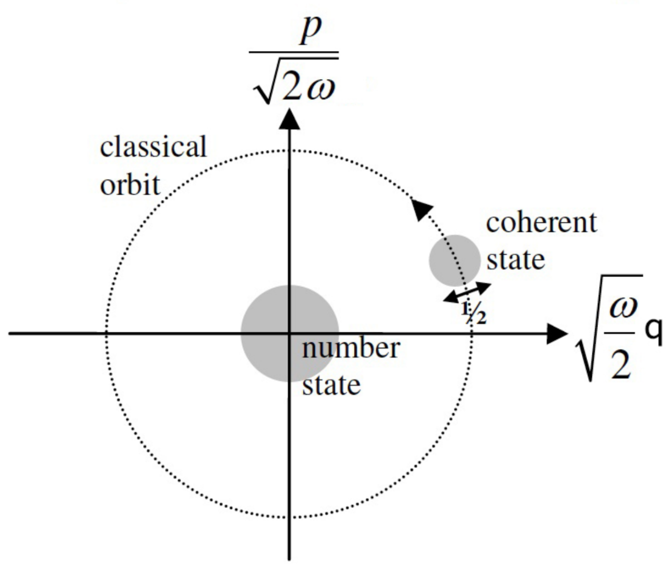

Figure 5: Schematic representation of a coherent state and its motion as a smeared distribution (like e.g. Wigner distribution) in a classical phase space, cf. Eq.(163). We use here $\hbar = 1$ units. For comparison also a number state $\psi_n(q) = \langle q|n\rangle$ centered at the phase space origin ($\langle n|\hat{q}|n\rangle = \langle n|\hat{p}|n\rangle = 0$) is shown schematically.

Let us briefly consider how the EM field "looks like" in a coherent state. Consider a single mode (144) written in terms of the photon and assume it is in a coherent state $|\alpha_{\mathbf{k}}(t)\rangle$. Then

$$\begin{aligned} \langle\alpha_{\mathbf{k}}(t)|\hat{\mathbf{E}}(\mathbf{r})|\alpha_{\mathbf{k}}(t)\rangle &= i\boldsymbol{\lambda}_{\mathbf{k}}\left(\frac{\hbar\omega_k}{2\epsilon_0\Omega}\right)^{1/2}\left[\alpha_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{r}} - \alpha^*_{\mathbf{k}}(t)e^{-i\mathbf{k}\cdot\mathbf{r}}\right] = \\ &= \boldsymbol{\lambda}_{\mathbf{k}}|\alpha_{\mathbf{k}}(0)|\left(\frac{2\hbar\omega_k}{\epsilon_0\Omega}\right)^{1/2}\sin(\omega_k t - \mathbf{k}\cdot\mathbf{r} - \phi_{\mathbf{k}}) \end{aligned} \tag{151}$$

where we used the results (168, 169) from the Appendix. This expression for the average $\hat{\mathbf{E}}(\mathbf{r})$ has the form of a classical field. Its amplitude is controlled by $|\alpha_{\mathbf{k}}(0)|$, cf., the radius of the classical trajectory in Fig. 5.

Calculating

$$\langle\alpha_{\mathbf{k}}(t)|\hat{\mathbf{E}}(\mathbf{r})\cdot\hat{\mathbf{E}}(\mathbf{r})|\alpha_{\mathbf{k}}(t)\rangle = \frac{\hbar\omega_k}{2\epsilon_0\Omega}\left[1 + 4|\alpha_{\mathbf{k}}(0)|^2\sin^2(\omega_k t - \mathbf{k}\cdot\mathbf{r} - \phi_{\mathbf{k}})\right] \tag{152}$$

we obtain for the quantum fluctuations

$$\Delta E \equiv \sqrt{\langle \alpha_{\mathbf{k}}(t)|\hat{\mathbf{E}}(\mathbf{r}) \cdot \hat{\mathbf{E}}(\mathbf{r})|\alpha_{\mathbf{k}}(t)\rangle - \langle \alpha_{\mathbf{k}}(t)|\hat{\mathbf{E}}(\mathbf{r})|\alpha_{\mathbf{k}}(t)\rangle^2} = \frac{\hbar\omega_k}{2\epsilon_0\Omega} \tag{153}$$

which is independent of the magnitude of the average, cf., again Fig. 5. So for the electric field $\gg$ than the quantum scale of the fluctuations ΔE the field can be viewed as classical.

5.3.3 Thermal light

Thermal radiation is radiation in thermal equilibrium, which means (as is usual in quantum statistical physics) that this radiation is described not by a wave function (or rather wave functional) but by the density matrix. This density matrix is diagonal in the eigenenergy basis

$$\rho = \sum_{\{N_{\mathbf{k}\alpha}\}} w(\{N_{\mathbf{k}\alpha}\})|\{N_{\mathbf{k}\alpha}\}\rangle\langle\{N_{\mathbf{k}\alpha}\}| \tag{154}$$

with the probabilities given by the Boltzmann factor

$$w(\{N_{\mathbf{k}\alpha}\}) = \frac{1}{Z(T)} \exp\left[-\frac{\mathcal{E}(\{N_{\mathbf{k}\alpha}\})}{T}\right] \quad , \quad Z(T) = \sum_{\{N_{\mathbf{k}\alpha}\}} \exp\left[-\frac{\mathcal{E}(\{N_{\mathbf{k}\alpha}\})}{T}\right]$$

which of course is equivalent to saying that the radiation power follows the Plank law. Just to remind - by using

$$\mathcal{E}(\{N_{\mathbf{k}\alpha}\}) = \sum_{\mathbf{k}\alpha} \hbar\omega_k N_{\mathbf{k}\alpha}$$

separating exponentials in $w(\{N_{\mathbf{k}\alpha}\})$ and $Z(T)$ into products with different $\mathbf{k}\alpha$ and summing over $N_{\mathbf{k}\alpha}$ for each $\mathbf{k}\alpha$ in $Z(T)$ one obtains

$$w(\{N_{\mathbf{k}\alpha}\}) = \prod_{\mathbf{k}\alpha} w(N_{\mathbf{k}\alpha}) \quad \text{with} \quad w(N_{\mathbf{k}\alpha}) = \left(1 - e^{-\beta\hbar\omega_k}\right) \exp\left(-\beta\hbar\omega_k N_{\mathbf{k}\alpha}\right) \tag{155}$$

and $\beta = (k_B T)^{-1}$. The average energy per mode is

$$\langle \mathcal{E} \rangle_{\mathbf{k}\alpha} = \sum_{N_{\mathbf{k}\alpha}} \left(\hbar\omega_k N_{\mathbf{k}\alpha}\right) w(N_{\mathbf{k}\alpha}) = \hbar\omega_k \langle N \rangle_{\mathbf{k}\alpha} = \frac{\hbar\omega_k}{e^{\beta\hbar\omega_k} - 1}$$

To obtain the Plank spectral energy density we sum over polarizations and the angles of $\mathbf{k}$ of photon states in a d^3k interval and unit space volume

$$dn = \sum_{\alpha} \int_{\text{angles}} \langle \mathcal{E} \rangle_{\mathbf{k}\alpha} \frac{d^3k}{(2\pi)^3} = \frac{\hbar\omega_k}{e^{\beta\hbar\omega_k} - 1} \int_{\gamma\in 4\pi} \frac{2k^2 dk d\gamma}{(2\pi)^3} = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{\beta h\nu} - 1} d\nu$$

with $\hbar\omega = h\nu = hck/2\pi$.

Such a spectrum is an idealization of a radiation spectrum emitted by matter sources which by themselves are in a thermal equilibrium and moreover the radiation which they emit "has enough time" inside the matter to reach equilibrium with it. The major factors "distorting" such spectra are layers of matter (like sun and earth atmospheres) between the equilibrated matter-radiation system and the observer. If such layers have different temperature and are too thin the light will "have no time" to re-equilibrate as it passes through them. The layers will just absorb some of the passing radiation at particular wave lengths depending on their chemical composition. This will produce the corresponding "absorption lines" in the radiation spectrum. Hot excited atoms, molecules, etc, inside the layers will also emit and add non equilibrated light at particular wavelengths producing the "emission lines". Example of the observed solar radiation spectrum, cf., Fig. 6 provides a good illustration of these features.

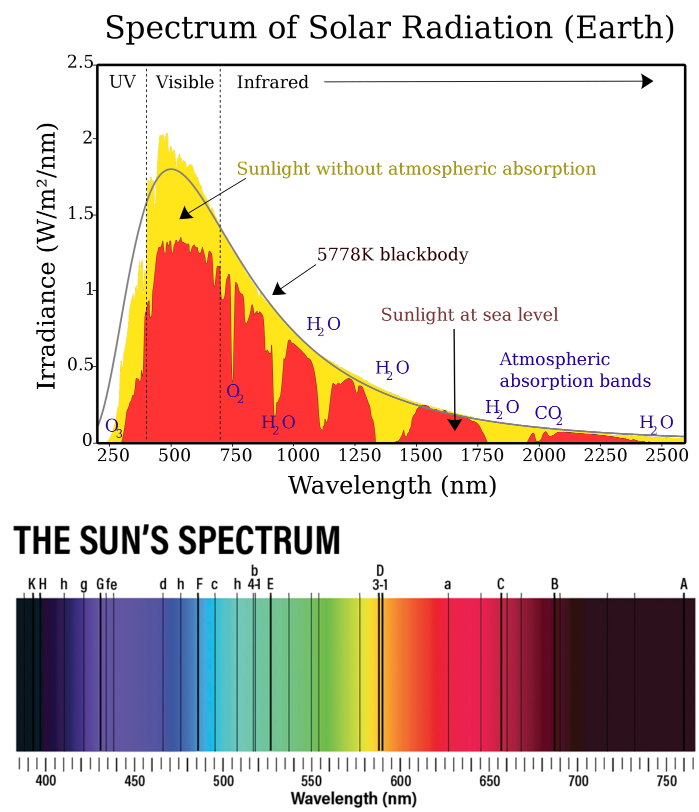

Figure 6: Above - the Plank black body spectrum and its modifications in real world. Below - discrete absorption lines on the background of the continuum solar light spectrum

5.4 Photon angular momentum and spin

Using the Maxwell and Newton equations (65), (67) together with the charge current and density (66) one can show (cf., Ch 1A in Ref.[7]) that the conserved angular momentum of the matter-field system is

$$\mathbf{J} = \sum_{a=1}^{N} \mathbf{r}_a \times m_a \mathbf{v}_a + \epsilon_0 \int d^3r\, \mathbf{r} \times [\mathbf{E}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})] \tag{156}$$

Comparing with the expression (136) for the matter-field momentum both terms have intuitively clear meaning.

It is important to note that as is usual with the definition of angular momentum the expression (156) refers to a specific point - the origin of the chosen coordinate system - with respect to which $\mathbf{J}$ is calculated. This of course can be easily changed by replacing $\mathbf{r}_a \to \mathbf{r}_a - \mathbf{r}_0$ and $\mathbf{r} \to \mathbf{r} - \mathbf{r}_0$ with an arbitrary vector $\mathbf{r}_0$ in both terms of $\mathbf{J}$ respectively. This change leads to a straightforward generalisation of the classical mechanics relation for such transformations of angular momenta

$$\mathbf{J} \quad \Rightarrow \quad \mathbf{J}' = \mathbf{J} - \mathbf{r}_0 \times \mathbf{P}$$

5.4.1 Generators of rotations in the matter-field system

Consider infinitesimal rotations of the coordinate system

$$\mathbf{r} \to \mathbf{r}' = \mathbf{r} + \delta\mathbf{r} = \mathbf{r} + \delta\boldsymbol{\alpha} \times \mathbf{r}$$

where as usual the magnitude of the vector $\delta\boldsymbol{\alpha}$ is equal the rotation angle and it is directed along the axis of rotation (right hand rule). We want to determine how the wave functional $\Psi[\mathbf{A}(\mathbf{r}), \mathbf{r}_1, ..., \mathbf{r}_N]$ changes under this transformation.

Let us start by recalling that a scalar field change obeys the intuitive rule

$$\phi(\mathbf{r}) \to \phi'(\mathbf{r}') = \phi(\mathbf{r})$$

saying that the values of the rotated field ϕ' at rotated points $\mathbf{r}'$ are the same as non rotated field ϕ in original points $\mathbf{r}$. Using $\mathbf{r} = \mathbf{r}' - \delta\mathbf{r}$ and dropping the prime in $\mathbf{r}'$ on both sides can write

$$\begin{aligned}
\phi'(\mathbf{r}) &= \phi(\mathbf{r} - \delta\mathbf{r}) \approx \phi(\mathbf{r}) - \delta\mathbf{r} \cdot \nabla\phi(\mathbf{r}) = \\
&= \phi(\mathbf{r}) - (\delta\boldsymbol{\alpha} \times \mathbf{r}) \cdot \nabla\phi(\mathbf{r}) = [1 - \delta\boldsymbol{\alpha} \cdot (\mathbf{r} \times \nabla)]\phi(\mathbf{r}) = \\
&= [1 - \frac{i}{\hbar}\delta\boldsymbol{\alpha} \cdot \hat{\mathbf{l}}]\phi(\mathbf{r}) \quad \text{with} \quad \hat{\mathbf{l}} = -i\hbar[\mathbf{r} \times \nabla]
\end{aligned} \tag{157}$$

For a vector field one also has to rotate the field itself

$$\mathbf{A}(\mathbf{r}) \to \mathbf{A}'(\mathbf{r}') = [1 + \delta\boldsymbol{\alpha}\times]\mathbf{A}(\mathbf{r}' - \delta\boldsymbol{\alpha} \times \mathbf{r})$$

which gives (after dropping the prime on $\mathbf{r}'$)

$$\begin{aligned}
\mathbf{A}'(\mathbf{r}) &\approx \mathbf{A}(\mathbf{r}) + \delta\mathbf{A}(\mathbf{r}) = \mathbf{A}(\mathbf{r}) + \delta\boldsymbol{\alpha} \times \mathbf{A}(\mathbf{r}) - [(\delta\boldsymbol{\alpha} \times \mathbf{r}) \cdot \nabla]\mathbf{A}(\mathbf{r}) = \\
&= \mathbf{A}(\mathbf{r}) + \delta\boldsymbol{\alpha} \times \mathbf{A}(\mathbf{r}) - [\delta\boldsymbol{\alpha} \cdot (\mathbf{r} \times \nabla)]\mathbf{A}(\mathbf{r})
\end{aligned} \tag{158}$$

As in the scalar field case the last term corresponds to the "orbital" rotation with $\hat{\mathbf{l}} = -i\hbar\mathbf{r} \times \nabla$ while in the Appendix below we show that the additional second term is (not surprisingly) a rotation of the components of the vector $\mathbf{A}$ with spin one matrices.

Let us now examine what happens to a wave functional $\Psi[\mathbf{A}(\mathbf{r})]$ when its argument is transformed as in (158). To simplify things we leave out the particle coordinates $\{\mathbf{r}_a\}$ aince the part of the rotation generator for them is obvious. Have

$$
\begin{aligned}
\Psi[\mathbf{A}(\mathbf{r})] \;&\rightarrow\; \Psi[\mathbf{A}(\mathbf{r}) + \delta\mathbf{A}(\mathbf{r})] \approx \Psi[\mathbf{A}(\mathbf{r})] + \int d^3r\,\delta\mathbf{A}(\mathbf{r}) \cdot \frac{\delta\Psi[\mathbf{A}(\mathbf{r})]}{\delta\mathbf{A}(\mathbf{r})} = \\
&= \left[1 + \int d^3r\,\delta\mathbf{A}(\mathbf{r}) \cdot \frac{\delta}{\delta\mathbf{A}(\mathbf{r})}\right] \Psi[\mathbf{A}(\mathbf{r})] = \\
&= \left[1 - \frac{i}{\hbar}\epsilon_0 \int d^3r\,\delta\mathbf{A}(\mathbf{r}) \cdot \mathbf{E}_{op}(\mathbf{r})\right] \Psi[\mathbf{A}(\mathbf{r})]
\end{aligned}
$$

where we used the expression for $\mathbf{E}_{op}(\mathbf{r})$ as defined in (81). Using the explicit form of $\delta\mathbf{A}$ from (158) we can write for the integral in the second term

$$
\begin{aligned}
\epsilon_0 \int d^3r\,\delta\mathbf{A}(\mathbf{r}) \cdot \mathbf{E}_{op}(\mathbf{r}) \;&=\; \epsilon_0 \int d^3r\,\left\{\delta\boldsymbol{\alpha} \times \mathbf{A}(\mathbf{r}) - [\delta\boldsymbol{\alpha} \cdot (\mathbf{r} \times \nabla)]\mathbf{A}(\mathbf{r})\right\} \cdot \mathbf{E}_{op}(\mathbf{r}) \\
&=\; \epsilon_0 \delta\boldsymbol{\alpha} \cdot \int d^3r\,\left\{\mathbf{A} \times \mathbf{E}_{op} - \sum_i [(\mathbf{r} \times \nabla)A_i(\mathbf{r})]\hat{E}_i\right\}
\end{aligned}
$$

From this we can read off the generator of rotations for the field part. It can be written as a sum of two parts - spin and orbital

$$
\hat{\mathbf{J}}_{field} = \hat{\mathbf{L}}_{field} + \hat{\mathbf{S}}_{field} \tag{159}
$$

with

$$
\begin{aligned}
\hat{\mathbf{L}}_{field} \;&=\; \epsilon_0 \int d^3r\,\sum_i \hat{E}_i(\mathbf{r} \times \nabla)\hat{A}_i \\
\hat{\mathbf{S}}_{field} \;&=\; \epsilon_0 \int d^3r\,[\mathbf{E}_{op} \times \mathbf{A}_{op}]
\end{aligned} \tag{160}
$$

where we indicated that $\mathbf{A}$ in this expression should be regarded as operator (although it is diagonal, $\mathbf{A}_{op} = \mathbf{A}$, in the representation of $\Psi[\mathbf{A}(\mathbf{r})]$.

Note that in the expressions for $\hat{\mathbf{L}}_{field}$ and $\hat{\mathbf{S}}_{field}$ we were free to commute $\hat{E}_i$ components to the left. Indeed in $\hat{\mathbf{L}}_{field}$ the commutator of $(\mathbf{r} \times \nabla)\hat{A}_i$ and $\hat{E}_i$ is proportional to the derivative of the delta function $\delta(\mathbf{r} - \mathbf{r}')$

$$\Big[(\mathbf{r} \times \nabla)\hat{A}_i(\mathbf{r}), \hat{E}_i(\mathbf{r}')\Big]_{\mathbf{r}=\mathbf{r}'} = -\frac{i\hbar}{\epsilon_0}(\mathbf{r} \times \nabla)\delta(\mathbf{r} - \mathbf{r}')\big|_{\mathbf{r}=\mathbf{r}'} = 0$$

which vanishes at $\mathbf{r} = \mathbf{r}'$. In $\hat{\mathbf{S}}_{field}$ only different i.e. commuting components $\hat{A}_i$ and $\hat{E}_j$ with $i \neq j$ enter in their vector product.

5.5 Photon parity and photon statistics

The vector potential $\mathbf{A}(\mathbf{r})$ is a polar vector - it changes it sign under parity transformation

$$\mathbf{A}(\mathbf{r}) \to -\mathbf{A}(-\mathbf{r}) \tag{161}$$

This property of $\mathbf{A}(\mathbf{r})$ is the basis of the statement that the photon, i.e. the quanta of the vibrations of $\mathbf{A}(\mathbf{r})$ have negative parity. We note that at this stage this is a fairly cryptic statement which becomes clear when photon emission and absorption by matter systems are studied (cf., later in the course).

Photons are bosons! This too is a somewhat cryptic statement at this stage. It will become more clear when we gain experience with quantum field description of bosonic matter systems in the framework of the second quantization formalism of the Schrödinger field. Here we only remark that one can have any number of photons in the same state, i.e. in the same mode characterized by $\mathbf{k}, \alpha$ quantum numbers.

One can calculate the commutator of the operators of the electric and magnetic fields $\hat{\mathbf{E}}_T(\mathbf{r})$ and $\hat{\mathbf{B}}(\mathbf{r}) = \nabla \times \hat{\mathbf{A}}_T(\mathbf{r})$ and find that they do not commute. This have all the usual quantum mechanical consequences. In fact in tutorials and home works we/you will deal with issues related to questions like "what is the electric/magnetic field of a photon?"

6 Appendix

6.1 Details of the standing to traveling waves transformation

6.1.1 What does the transformation Eq. (43) achieve

Let us start by noting that the Hamiltonian for a given k_ν has the same form in the new variables

$$H_\nu = \frac{1}{2}\sum_{i=1,2}(P_{i,\nu}^2 + \omega_\nu^2 Q_{i,\nu}^2) = \frac{1}{2}[(P_k^2 + \omega_k^2 Q_k^2) + (P_{-k}^2 + \omega_k^2 Q_{-k}^2)]$$

and due to their canonicity (cf., below) the dynamical equations for $Q_{\pm k}, P_{\pm k}$ are the same as for $Q_{i,\nu}, P_{i,\nu}, i = 1, 2$ so their solutions have the same form as in Eq. (42). Now both terms in this solution give traveling waves when inserted in the expansion (44). Indeed have for the first terms in these solutions when inserted in the expansion for $\phi(x)$

$$\sqrt{\frac{1}{L}}\sum_k Q_k(0)\,[\sin kx \cos\omega t - \cos kx \sin\omega t] = \sqrt{\frac{1}{L}}\sum_k Q_k 0)\sin(kx - \omega t)$$

and for the second terms

$$\sqrt{\frac{1}{L}}\sum_k \frac{P_k(0)}{\omega_\nu}\,[\sin kx \sin\omega t + \cos kx \cos\omega t] = \sqrt{\frac{1}{L}}\sum_k \frac{P_k(0)}{\omega}\cos(kx - \omega t)$$

These traveling waves are "running" in the positive or negative x-direction depending on the sign of k.

6.1.2 Verifying canonicity

Our transformation from the standing waves expansion (40) to the traveling waves (44) amounted to transforming from $Q_{i,\nu}, P_{i,\nu}$ phase space variables to $Q_{\pm k}, P_{\pm k}$, Eq. (43). Let us now check the canonicity of this transformation.

Let us recall that in a mechanical system described by a set of generalised coordinates and momenta $\{q, p\}$ the transformation to a canonically conjugate set $\{Q, P\}$ must satisfy

$$\sum_i p_i dq_i = \sum_k P_k dQ_k + dF$$

where dF denote a complete differential. In our case the set $\{q,p\}$ is $Q_{i,\nu}, P_{i,\nu}$ and we are transforming to $Q_{\pm k}, P_{\pm k}$. We obtain

$$\begin{aligned}
\sum_i P_{i,\nu} dQ_{i,\nu} &= \frac{1}{2}\left[(P_k - P_{-k})d(Q_k - Q_{-k}) - (Q_k + Q_{-k})d(P_k + P_{-k})\right] = \\
&= \frac{1}{2}[P_k dQ_k + P_{-k}dQ_{-k} - P_k dQ_{-k} - P_{-k}dQ_k - \\
&\quad - Q_k dP_k - Q_{-k}dP_{-k} - Q_k dP_{-k} - Q_{-k}dP_k] = \\
&= P_k dQ_k + P_{-k}dQ_{-k} - \frac{1}{2}(P_k dQ_k + P_{-k}dQ_{-k} + Q_k dP_k + Q_{-k}dP_{-k}) - \\
&\quad - \frac{1}{2}(P_k dQ_{-k} + P_{-k}dQ_k + Q_k dP_{-k} + Q_{-k}dP_k) = \\
&= P_k dQ_k + P_{-k}dQ_{-k} - \frac{1}{2}d(P_k Q_k + P_{-k}Q_{-k} - P_k Q_{-k} - P_{-k}Q_k)
\end{aligned}$$

It is instructive also to verify the canonicity of the general transformation (44). In this case the set $\{q,p\}$ is $\{\phi(x), \pi(x)\}$, the sum over i is integral over x and we are transforming to Q_k, P_k. So we have

$$\begin{aligned}
&\int_0^L dx \pi(x,t) \frac{\partial \phi(x,t)}{\partial t} = \\
= \ &\frac{1}{L}\sum_{kk'} \int_0^L dx \left[\sin k'x \ P_{k'}(t) - v|k'| \cos k'x \ Q_{k'}(t)\right] \times \\
\times \ &\left[\sin kx \ \dot{Q}_k(t) + \frac{1}{v|k|}\cos kx \ \dot{P}_k(t)\right] = \\
= \ &\sum_k \frac{1}{2}\left[P_k(t)\dot{Q}_k(t) - Q_k(t)\dot{P}_k(t)\right] = \\
= \ &\sum_k P_k(t)\dot{Q}_k(t) - \frac{1}{2}\sum_k \frac{d}{dt}[Q_k(t)P_k(t)]
\end{aligned}$$

which shows the canonicity of P_k and Q_k.

6.2 Details of the coherent states

6.2.1 Useful averages. Minimum uncertainty

It easy to find an explicit solution of the equation (147). But before doing that it is useful first to calculate the following averages

$$\begin{aligned} q_0 &\equiv \langle\alpha|\hat{q}|\alpha\rangle = \sqrt{\hbar/(2\omega)}\langle\alpha|(\hat{a}+\hat{a}^+)|\alpha\rangle = \sqrt{\hbar/(2\omega)}(\alpha+\alpha^*) = \sqrt{2\hbar/\omega}\ \mathrm{Re}\,\alpha \\ p_0 &\equiv \langle\alpha|\hat{p}|\alpha\rangle = i\sqrt{\hbar\omega/2}\langle\alpha|(\hat{a}^+-\hat{a})|\alpha\rangle = i\sqrt{\hbar\omega/2}(\alpha^*-\alpha) = \sqrt{2\hbar\omega}\ \mathrm{Im}\,\alpha \end{aligned} \tag{162}$$

which give

$$\alpha = \sqrt{\omega/2\hbar}\, q_0 + i\sqrt{1/(2\hbar\omega)}\, p_0 \tag{163}$$

Also have

$$\begin{aligned} \langle\alpha|\hat{q}^2|\alpha\rangle &= (\hbar/2\omega)\langle\alpha|\hat{a}^2+\hat{a}\hat{a}^+ + \hat{a}^+\hat{a} + (\hat{a}^+)^2|\alpha\rangle = \\ &= (\hbar/2\omega)\langle\alpha|\hat{a}^2 + 2\hat{a}^+\hat{a} + 1 + (\hat{a}^+)^2|\alpha\rangle = \\ &= (\hbar/2\,\omega)[(\alpha+\alpha^*)^2+1) = \langle\alpha|\hat{q}|\alpha\rangle^2 + \hbar/(2\omega) \end{aligned} \tag{164}$$

and

$$\begin{aligned} \langle\alpha|p^2|\alpha\rangle &= -(\hbar\omega/2)\langle\alpha|\hat{a}^{+2} - \hat{a}^+\hat{a} - \hat{a}\hat{a}^+ + \hat{a}^2|\alpha\rangle = \\ &= -(\hbar\omega/2)[(\alpha-\alpha^*)^2 - 1] = \langle\alpha|\hat{p}|\alpha\rangle^2 + \hbar\omega/2 \end{aligned} \tag{165}$$

which shows that the coordinate and momentum uncertainties in this state are independent of α

$$\begin{aligned} \Delta q &\equiv \sqrt{\langle\alpha|\hat{q}^2|\alpha\rangle - \langle\alpha|\hat{q}|\alpha\rangle^2} = \sqrt{\hbar/(2\omega)} \\ \Delta p &\equiv \sqrt{\langle\alpha|\hat{p}^2|\alpha\rangle - \langle\alpha|\hat{p}|\alpha\rangle^2} = \sqrt{\hbar\omega/2} \end{aligned} \tag{166}$$

which in turn means that for large (classical) values of q_0 and p_0, i.e. for large $|\alpha|$, cf., Eq. (163), the quantum uncertainties are negligible. The actual values of Δq and Δp show that $|\alpha\rangle$ is a minimum uncertainty state, Eq. (150).

6.2.2 Dynamics of coherent states

Let is now consider the dynamics of a coherent state, i.e. find

$$|\alpha(t)\rangle \equiv e^{-i\hat{h}t/\hbar}|\alpha\rangle \;\text{ with }\; \hat{h} = \hbar\omega(\hat{a}^+\hat{a}+1/2)$$

For this we use the Heisenberg representation $\hat{a}(t) = e^{i\hat{h}t/\hbar}\ \hat{a}\ e^{-i\hat{h}t/\hbar}$ of $\hat{a}$ and the corresponding Heisenberg equation which is easily solved

$$\begin{aligned} i\hbar\frac{\partial\hat{a}(t)}{\partial t} &= -\hat{h}\hat{a}(t) + \hat{a}(t)\hat{h} = \hbar\omega[-\hat{a}^{+}(t)\hat{a}(t)\hat{a}(t) + \hat{a}(t)\hat{a}^{+}(t)\hat{a}(t)] = \\ &= \hbar\omega\hat{a}(t) \quad \Rightarrow \quad \hat{a}(t) = \hat{a}e^{-i\omega t} \end{aligned} \tag{167}$$

This gives

$$\hat{a}|\alpha(t)\rangle = \hat{a}e^{-i\hat{h}t/\hbar}|\alpha\rangle = e^{-i\hat{h}t/\hbar}\hat{a}(t)|\alpha\rangle = e^{-i\hat{h}t/\hbar}\hat{a}e^{-i\omega t}|\alpha\rangle = \alpha e^{-i\omega t}|\alpha(t)\rangle \tag{168}$$

which shows that $|\alpha(t)\rangle$ remains a coherent state with

$$\alpha(t) = \alpha e^{-i\omega t} \tag{169}$$

In terms of the corresponding $q_0(t)$ and $p_0(t)$

$$\begin{aligned} q_0(t) &= \sqrt{2\hbar/\omega}\ \mathrm{Re}\ \alpha(t) = \sqrt{2\hbar/\omega}[\ \mathrm{Re}\ \alpha\cos\omega t + \ \mathrm{Im}\ \alpha\sin\omega t] = q_0\cos\omega t + (p_0/\omega)\sin\omega t \\ p_0(t) &= \sqrt{2\hbar\omega}\ \mathrm{Im}\ \alpha(t) = \sqrt{2\hbar\omega}[\ \mathrm{Im}\ \alpha\cos\omega t - \ \mathrm{Re}\ \alpha\sin\omega t] = p_0\cos\omega t - \omega q_0\sin\omega t \end{aligned}$$

which coincide with the solution of the classical equations (146).

6.2.3 Explicit expressions. Ground state of a shifted harmonic oscillator

Using the explicit expression (148) for the coherent state at $\alpha = 0$ it is easy to find solutions of Eq. (147) for a general α by using the decomposition (163) in (147)

$$\left(\hbar\frac{\partial}{\partial q} + \omega\ q\right)\psi_\alpha(q) = [\omega q_0 + i\ p_0]\psi_\alpha(q) \ \Rightarrow \ \left[\left(\hbar\frac{\partial}{\partial q} - ip_0\right) + \omega(q - q_0)\right]\psi_\alpha(q) = 0$$

and noticing that this equation is similar to the one with $\alpha = 0$, Eq. (148), but with a shift $q \to q - q_0$ and a p_0 dependent phase

$$\psi_\alpha(q) = A\exp\{-[\omega(q-q_0)^2 + ip_0]/\hbar\} \quad , \quad A = (\omega/\pi\hbar)^{1/4} \tag{170}$$

It clearly can be regarded as a ground state of a shifted harmonic oscillator, i.e. of

$$\hat{h} = \frac{1}{2}\left[(\hat{p} - p_0)^2 + \omega^2(\hat{q} - q_o)^2\right] \tag{171}$$

This observation is important for a qualitative discussion of the laser light.

Let us note that the coherent state can also be written as an expansion in a complete set of number states, i.e. the harmonic oscillator eigenstates $|n\rangle$

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$$

Acting with $\hat{a}$ we obtain

$$\begin{aligned}
\hat{a}|\alpha\rangle &= \sum_{n=0} c_n \hat{a}|n\rangle = \sum_{n=0}^{\infty} c_n \sqrt{n}|n-1\rangle = \alpha \sum_{n=0}^{\infty} c_n |n\rangle = \\
&= \alpha \sum_{k=1}^{\infty} c_{k-1}|k-1\rangle \quad \Rightarrow \quad \sqrt{n} c_n = \alpha c_{n-1} \\
&\Rightarrow c_n = \frac{\alpha^n}{\sqrt{n!}} c_0 \quad \Rightarrow \quad |\alpha\rangle = c_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle
\end{aligned}$$

Find c_0 from normalization

$$1 = \langle\alpha|\alpha\rangle = |c_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^2}{n!} = |c_0|^2 e^{|\alpha|^2} \qquad \Rightarrow \qquad c_0 = e^{-|\alpha|^2/2}$$

so

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{172}$$

It is also easy to calculate the overlap

$$\langle\alpha|\beta\rangle = e^{-|\alpha|^2/2} e^{-|\beta||^2/2} e^{\alpha^* \beta} \quad \Rightarrow \quad |\langle\alpha|\beta\rangle|^2 = e^{-|\alpha-\beta|^2}$$

showing non orthogonality of different $|\alpha\rangle$ states. The set $|\alpha\rangle$ is over-complete but satisfies a useful resolution of unity relation

$$\int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| = \sum_{n=0}^{\infty} |n\rangle\langle n| = \hat{1} \quad \text{with} \quad d^2\alpha = d\,\mathrm{Re}\,\alpha\; d\,\mathrm{Im}\,\alpha$$

which is easy to prove by using the expansion (172) and changing to polar coordinates $\alpha = re^{i\phi}, d^2\alpha = r dr d\phi$ in the integral.

6.3 More on the EM field momentum

6.3.1 Relation to the classical expressions for the matter-field momentum

Classical expression (136) can be written

$$\mathbf{P}_f = \epsilon_0 \int d^3r\, \mathbf{E}(\mathbf{r}) \times \mathbf{B}(\mathbf{r}) = \epsilon_0 \int d^3r\, \mathbf{E}(\mathbf{r}) \times \nabla \times \mathbf{A}(\mathbf{r}) \tag{173}$$

It is related to the integral of the Poynting vector, cf., the Feynman lectures, Ref. [8] for the physics discussion of this result.

Let us write this expression in components (using the Levi-Civita tensor and the summation convention)

$$\begin{aligned} (\mathbf{E} \times \nabla \times \mathbf{A})_i &= \epsilon_{ijk} E_j \epsilon_{klm} \partial_l A_m = \epsilon_{kij}\epsilon_{klm} E_j \partial_l A_m = \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{lj}) E_j \partial_l A_m = E_j \partial_i A_j - E_j \partial_j A_i \end{aligned} \tag{174}$$

so that

$$\epsilon_0 \int d^3r\, (\mathbf{E} \times \nabla \times \mathbf{A})_i = \epsilon_0 \int d^3r\, (E_j \partial_i A_j - E_j \partial_j A_i) = \epsilon_0 \int d^3r\, (E_j \partial_i A_j + \partial_j E_j A_i) \tag{175}$$

where we integrated by parts in the last equality. Using the Gauss law $\partial_j E_j = \rho/\epsilon_0$ this gives

$$(\mathbf{P}_f)_i = \epsilon_0 \int d^3r\, E_j(\mathbf{r}) \partial_i A_j(\mathbf{r}) + \int d^3r \rho(\mathbf{r}) A_i(\mathbf{r})) = \epsilon_0 \int d^3r\, E_j(\mathbf{r}) \partial_i A_j(\mathbf{r}) + \sum_{a=1}^{N} q_a A_i(\mathbf{r}_a) \tag{176}$$

where we used $\rho(\mathbf{r}) = \sum_{a=1}^{N} q_a \delta(\mathbf{r} - \mathbf{r}_a)$ to integrate.

Using this in the expression for the total momentum (136) (and restoring for better clarity the summation symbol for the repeated index j) we obtain

$$\mathbf{P} = \sum_{a=1}^{N} \mathbf{p}_a + \epsilon_0 \int d^3r\, \sum_{j=1}^{3} E_j(\mathbf{r}) \nabla A_j(\mathbf{r}) \tag{177}$$

with

$$\mathbf{p}_a = m_a \mathbf{v}_a + q_a \mathbf{A}(\mathbf{r}_a) \tag{178}$$

This coincides with the expression (135).

6.3.2 Field momentum in terms of the transverse components

Both terms in the expression (136) are separately gauge invariant. However the two terms in the transformed expression (177) are not. Only their sum is. We can repair this if we repeat the calculation (175) but first replacing $\mathbf{A}$ by $\mathbf{A}_T$ in the starting left hand side. This will lead to the same expression as (177) but with $\mathbf{A}_T$ replacing $\mathbf{A}$ in it

$$\mathbf{P} = \sum_{a=1}^{N}(m_a\mathbf{v}_a + q_a\mathbf{A}_T(\mathbf{r}_a)) + \epsilon_0 \int d^3r \sum_{j=1}^{3} E_j(\mathbf{r})\nabla A_{T,j}(\mathbf{r})$$

Now both terms are gauge invariant. We can moreover in the second term replace E by E_T. Indeed writing

$$E_j = E_{T,j} + E_{L,j} = E_{T,j} - \partial_j\phi$$

and using

$$\int d^3r \sum_{j=1}^{3} \partial_j\phi(\mathbf{r})\nabla A_{T,j}(\mathbf{r}) = -\int d^3r\phi(\mathbf{r})\nabla\left[\sum_{j=1}^{3}\partial_j A_{T,j}(\mathbf{r})\right] = 0$$

we express

$$\mathbf{P} = \sum_{a=1}^{N}(m_a\mathbf{v}_a + q_a\mathbf{A}_T(\mathbf{r}_a)) + \epsilon_0 \int d^3r \sum_{j=1}^{3} E_{T,j}(\mathbf{r})\nabla A_{T,j}(\mathbf{r}) \qquad (179)$$

In the absence of the charged matter (i.e. when all q_a's are zero) the field part of this momentum becomes the momentum of the free radiation as we have already derived in (138).

6.4 More on the EM field angular momentum

6.4.1 Relation to the classical expression

See Ref.[7], Complement B_I.

6.4.2 Spin 1 part of rotations of a vector field

We can write the 2nd term in Eq. (158) as

$$[\delta\boldsymbol{\alpha} \times \mathbf{A}]_j = \epsilon_{jkl}\delta\alpha_k A_l = -\frac{i}{\hbar}\delta\alpha_k s^k_{jl} A_l = -\frac{i}{\hbar}[\delta\boldsymbol{\alpha}\cdot\mathbf{s}]_{jl}A_l$$

where the matrices

$$s^k_{jl} = i\hbar\epsilon_{jkl}$$

are spin 1 matrices written in cartesian components basis $x_1 = x, x_2 = y, x_3 = z$ rather than in the more familiar spherical components basis ($x_m,\ m = \pm 1, 0$)

$$x_{+1} = -\frac{1}{\sqrt{2}}(x + iy) \quad , \quad x_{-1} = \frac{1}{\sqrt{2}}(x - iy) \quad , \quad x_0 = z$$

i.e. $x_m \sim rY_{1m}(\theta, \phi)$.

One can easily verify that the commutators indeed have the correct form

$$[s^i, s^j] = i\hbar\epsilon_{ijn}s^n \tag{180}$$

For this must prove that

$$[s^i, s^j]_{kl} = -\hbar^2[\epsilon_{kim}\epsilon_{mjl} - \epsilon_{kjm}\epsilon_{mil}]$$

is equal to

$$i\hbar\epsilon_{ijn}s^n_{kl} = i\hbar\epsilon_{ijn}i\hbar\epsilon_{knl} = -\hbar^2\epsilon_{ijn}\epsilon_{knl}$$

Have

$$\epsilon_{kim}\epsilon_{mjl} - \epsilon_{kjm}\epsilon_{mil} = (\delta_{kj}\delta_{il} - \delta_{kl}\delta_{ij}) - (\delta_{ki}\delta_{jl} - \delta_{kl}\delta_{ji}) = \delta_{kj}\delta_{il} - \delta_{ki}\delta_{jl}$$

which indeed is equal to

$$\epsilon_{ijn}\epsilon_{knl} = \epsilon_{ijn}\epsilon_{lkn} = \delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}$$

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