Quantum Computation 101 for Physicists

Home exercise 3

1. Recall that we model a working quantum bomb using the following circuit:

A dud model is

You are given either a dud or a working bomb. Your goal is to successfully identify a working bomb (without exploding it). In class, we saw an algorithm which always identifies a dud as a dud, and it identifies a working bomb as such with probability $\frac{1}{4}$. The goal of this exercise is to increase the success probability for the working firework. Fill in the details in the following algorithm:

(a) Initialize the input qubit to the $|0\rangle$ state.

(b) Set $\theta =$ ___.

(c) Repeat n times:

   i. Apply $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ to the input qubit.

   ii. See if the potential bomb explodes when using the qubit as an input. If it does, declare failure.

(d) Measure the qubit in the standard basis. If the outcome is ___ , output "definitely a dud", if the outcome is ___, output "definitely working".

Prove the following:

(a) A dud is always identified as a dud.

(b) A working bomb is never identified as a dud, and that it explodes with probability at most $O(\frac{1}{n})$.

2. In this question you will get an example for the use of entanglement as a resource. This is the base for a quantum computation method called 'measurement-based quantum computing'.

We first introduce the **Bell basis** for two qubits:

$$|B_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|B_{01}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$|B_{10}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$$|B_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

Note that all the states in the Bell basis are maximally entangled states for two qubits, and that they obey (up to a global phase) $|B_{ab}\rangle = X_2^a Z_2^b |B_{00}\rangle$, where X_2, Z_2 are the X and Z operators acting on the leftmost qubit.

Assume we have a pair of entangled qubits in the state $|B_0\rangle = \frac{1}{\sqrt{2}}|00\rangle + |11\rangle$ (denoted as qubits 1 and 2), and a third qubit in the state $\alpha|0\rangle + \beta|1\rangle$ (denoted as qubit 3). Our goal is to apply the unitary U to the state $\alpha|0\rangle + \beta|1\rangle$, and we will do that by 'transferring' the state to qubit 1. We cannot apply U exactly all the time, so we will have to settle for the application of PU, where P is some unitary that we know and can therefore fix.

(a) Show that by measuring qubits 2 and 3 in the Bell basis, We get qubit 1 in the state $P(\alpha|0\rangle + \beta|1\rangle)$, where P is some operator we can deduce from the measurement outcome. It is useful to use

$$|B_{ab}\rangle = X^a Z^b \otimes \mathbb{I} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),$$

where X^a is X if $a = 1$ and $\mathbb{I}$ if $a = 0$.

(b) Use the result above to apply a unitary U on the state $\alpha|0\rangle + \beta|1\rangle$: Show that by measuring qubits 2 and 3 in the "rotated Bell basis": $|B(U)_{ab}\rangle = U^\dagger \otimes \mathbb{I}|B_{ab}\rangle$, we get qubit 1 in the state $PU(\alpha|0\rangle + \beta|1\rangle)$, where P is the same operator we got in (a).