Fundamentals of Quantum Technology
Week 6: Quantization of the Electromagnetic Field

Recommended literature:

1. Gerry, Knight (ch. 2.1–2.4, 2.7, 3.1–3.3).
2. P. Carruthers and M. M. Nieto, *Phase and angle variables in quantum mechanics*, Review of Modern Physics **40**, pp. 422–432 (1968).

1 The phase variable

The classical description of an electromagnetic field requires the knowledge of both its intensity and its complex phase. In the quantum regime, the intensity is captured by the number of photons, represented by the Hermitian number operator $\hat{n} = \hat{a}^\dagger \hat{a}$. However, a corresponding Hermitian operator $\hat{\phi}$ representing the phase does not exist. A possible solution is to define the normalized ladder operators

$$\hat{E} = \sum_{n=0}^{\infty} |n\rangle \langle n+1| \,, \quad \hat{E}^\dagger = \sum_{n=0}^{\infty} |n+1\rangle \langle n| \,,$$

which are the analogues of $\exp(\pm i\phi)$, respectively. In analogy to the number states $|n\rangle$, we define the "eigenstates" of $\hat{E}$ which are given by

$$|\phi\rangle = \sum_{n=0}^{\infty} e^{in\phi} |n\rangle \longrightarrow \hat{E} |\phi\rangle = e^{i\phi} |\phi\rangle \quad (0 \le \phi < 2\pi) \,.$$

These states are not normalizable, but they enable us to define a **phase distribution** $\mathcal{P}(\phi)$ for any state $\hat{\rho}$ (be it pure or mixed) of the field:

$$\mathcal{P}(\phi) = \frac{1}{2\pi} \langle \phi| \hat{\rho} |\phi\rangle \,.$$

At home you will show that this distribution is indeed normalized, meaning that $\int_0^{2\pi} \mathcal{P}(\phi)\, \mathrm{d}\phi = 1$ for any $\hat{\rho}$.

Exercise

Calculate the phase distributions of a number state $|n\rangle$ and of a coherent state $|\alpha\rangle$ (assume $|\alpha| \gg 1$).

Solution

For the number state we have $\hat{\rho} = |n\rangle \langle n|$, and thus

$$\boxed{\mathcal{P}(\phi) = \frac{1}{2\pi} |\langle n \,|\, \phi\rangle|^2 = \frac{1}{2\pi} \left|e^{in\phi}\right|^2 = \frac{1}{2\pi}.}$$

This implies that for any number state (including the vacuum $n=0$) the phase distribution is uniform, so that the phase is completely undetermined.

On the other hand, for the coherent state $\hat{\rho} = |\alpha\rangle\langle\alpha|$ we use

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

to obtain

$$\begin{aligned}\mathcal{P}(\phi) &= \frac{1}{2\pi}|\langle\phi\,|\,\alpha\rangle|^2 = \frac{e^{-|\alpha|^2}}{2\pi}\left|\sum_{n=0}^{\infty}\frac{e^{-in\phi}\alpha^n}{\sqrt{n!}}\right|^2\\ &= \frac{e^{-|\alpha|^2}}{2\pi}\left|\sum_{n=0}^{\infty}\frac{\left(|\alpha|\,e^{i(\mathrm{Arg}(\alpha)-\phi)}\right)^n}{\sqrt{n!}}\right|^2 = \frac{1}{2\pi}\left|\sum_{n=0}^{\infty}\sqrt{P_n(\alpha)}e^{in(\mathrm{Arg}(\alpha)-\phi)}\right|^2,\end{aligned}$$

where

$$P_n(\alpha) \equiv |\langle n\,|\,\alpha\rangle|^2 = e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!}$$

is the probability to measure n photons given the coherent state $|\alpha\rangle$. If we recall that the average number of photons for this state is $\bar{n} = \langle\alpha|\,\hat{n}\,|\alpha\rangle = |\alpha|^2$, we may observe that this is simply the Poisson distribution,

$$P_n(\alpha) = e^{-\bar{n}}\frac{\bar{n}^n}{n!}.$$

Now, it is a known fact[1] that for $\bar{n} \gg 1$ (which agrees with the assumption $|\alpha| \gg 1$), the Poisson distribution is well-approximated by the normal distribution with $\bar{n}$ being both the mean and the variance[2], that is

$$e^{-\bar{n}}\frac{\bar{n}^n}{n!} \approx \frac{1}{\sqrt{2\pi\bar{n}}}\exp\left[-\frac{(n-\bar{n})^2}{2\bar{n}}\right],$$

and thus

$$\begin{aligned}\mathcal{P}(\phi) &\approx \frac{1}{2\pi\sqrt{2\pi\bar{n}}}\left|\sum_{n=0}^{\infty}\exp\left[-\frac{(n-\bar{n})^2}{4\bar{n}} + i\left(\mathrm{Arg}(\alpha)-\phi\right)n\right]\right|^2\\ &\approx \frac{1}{2\pi\sqrt{2\pi\bar{n}}}\left|\int_{-\infty}^{\infty}\mathrm{d}x\exp\left[-\frac{(x-\bar{n})^2}{4\bar{n}} + i\left(\mathrm{Arg}(\alpha)-\phi\right)x\right]\right|^2\\ &= \frac{1}{2\pi\sqrt{2\pi\bar{n}}}\left|\sqrt{4\pi\bar{n}}\cdot\exp\left[-\bar{n}\left(\phi-\mathrm{Arg}(\alpha)\right)^2\right]\right|^2,\end{aligned}$$

where we used the following identity for Gaussian integrals

$$\int_{-\infty}^{\infty}\mathrm{d}x\exp\left[-ax^2+bx\right] = \sqrt{\frac{\pi}{a}}\exp\left[\frac{b^2}{4a}\right].$$

[1] Here I add some details for those who are interested. The Poisson distribution can be obtained as the limit of the binomial distribution $B(n,p)$ for $n\to\infty$ and $p\to 0$ with $np=\bar{n}$ kept constant. On the other hand, for large n the binomial distribution approaches the normal distribution with mean np and variance $np(1-p)$; this can be seen as a result of the central limit theorem, as a binomial variable is by definition a sum of n independent coin tosses (Bernoulli variables) with probability p for success; for $p\to 0$ we again get that the mean and variance are equal.

[2] The fact that, within a coherent state, the variance for the number of photons is $\bar{n}$ entails that the standard deviation of the photon number is $\Delta n = \sqrt{\bar{n}}$, so that the fractional uncertainty $\Delta n/\bar{n}$ decreases as the average photon number increases.

So now we have that

$$\boxed{\mathcal{P}(\phi) \approx \sqrt{\frac{2\bar{n}}{\pi}} \cdot \exp\left[-2\bar{n}\,(\phi - \mathrm{Arg}(\alpha))^2\right].}$$

This result implies that when the average photon number $\bar{n}$ is very large, the phase distribution of a coherent state is well-localized about $\mathrm{Arg}(\alpha)$ (deviations from $\mathrm{Arg}(\alpha)$ are exponentially suppressed). As the phase of the field amplitude should be a well-defined feature in the classical limit, this is another property of coherent states which leads to us treating them as "the most classical pure quantum states" of the electromagnetic field. $\square$

In the lecture you also saw definitions for Hermitian operators that correspond to $\cos\phi$ and $\sin\phi$:

$$\hat{C} = \frac{1}{2}\left(\hat{E} + \hat{E}^\dagger\right), \quad \hat{S} = \frac{1}{2i}\left(\hat{E} - \hat{E}^\dagger\right).$$

They do not commute with $\hat{n}$, attesting to the fact that the number and phase variables cannot be precisely determined simultaneously for a single mode; the corresponding commutation relations yield

$$\left[\hat{C}, \hat{n}\right] = i\hat{S}, \quad \left[\hat{S}, \hat{n}\right] = -i\hat{C}.$$

However, the phase *difference* between two independent modes can be determined simultaneously with the *total* number of photons in the two modes.

Suppose that the Hamiltonian features two independent modes,

$$\begin{aligned}\hat{H} &= \hbar\omega_1\left(\hat{a}_1^\dagger\hat{a}_1 + \frac{1}{2}\right) + \hbar\omega_2\left(\hat{a}_2^\dagger\hat{a}_2 + \frac{1}{2}\right)\\ &= \hbar\omega_1\left(\hat{n}_1 + \frac{1}{2}\right) + \hbar\omega_2\left(\hat{n}_2 + \frac{1}{2}\right).\end{aligned}$$

Following the identities for the cosine and sine of a phase difference,

$$\begin{aligned}\cos(\phi_1 - \phi_2) &= \cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2,\\ \sin(\phi_1 - \phi_2) &= \sin\phi_1\cos\phi_2 - \sin\phi_2\cos\phi_1,\end{aligned}$$

we define the Hermitian operators

$$\begin{aligned}\hat{C}_{12} &= \hat{C}_1\hat{C}_2 + \hat{S}_1\hat{S}_2,\\ \hat{S}_{12} &= \hat{S}_1\hat{C}_2 - \hat{S}_2\hat{C}_1.\end{aligned}$$

Exercise

Show that $\hat{C}_{12}$ and $\hat{S}_{12}$ commute with $\hat{n} = \hat{n}_1 + \hat{n}_2$.

Solution

We will show this for $\hat{C}_{12}$, where the same result for $\hat{S}_{12}$ may be obtained in a similar manner. Using the identity

$$\left[\hat{A}\hat{B}, \hat{D}\right] = \hat{A}\left[\hat{B}, \hat{D}\right] + \left[\hat{A}, \hat{D}\right]\hat{B},$$

we observe that

$$\begin{aligned}\left[\hat{C}_1\hat{C}_2, \hat{n}\right] &= \hat{C}_1\left[\hat{C}_2, \hat{n}\right] + \left[\hat{C}_1, \hat{n}\right]\hat{C}_2\\ &= \hat{C}_1\left[\hat{C}_2, \hat{n}_2\right] + \left[\hat{C}_1, \hat{n}_1\right]\hat{C}_2\\ &= i\hat{C}_1\hat{S}_2 + i\hat{S}_1\hat{C}_2,\end{aligned}$$

and similarly

$$\left[\hat{S}_1\hat{S}_2,\hat{n}\right] = \hat{S}_1\left[\hat{S}_2,\hat{n}_2\right] + \left[\hat{S}_1,\hat{n}_1\right]\hat{S}_2$$
$$= -i\hat{S}_1\hat{C}_2 - i\hat{C}_1\hat{S}_2.$$

Therefore,

$$\left[\hat{C}_{12},\hat{n}\right] = \left[\hat{C}_1\hat{C}_2 + \hat{S}_1\hat{S}_2,\hat{n}\right] = 0,$$

as required. Consequently, we can find eigenstates of $\hat{C}_{12}$ (and of $\hat{S}_{12}$) that have a fixed *total* photon number n. $\square$

2 Thermal field

In the lecture you discussed the **thermal state** of a single-mode electromagnetic field with frequency ω. This is a mixed equilibrium state, and its density operator is given by

$$\hat{\rho}_{\mathrm{Th}} = \frac{1}{Z}\sum_{n=0}^{\infty}\exp\left(-\beta n\hbar\omega\right)|n\rangle\langle n|,$$

where β is the inverse temperature and $Z = \left(1-\exp\left(-\beta\hbar\omega\right)\right)^{-1}$.

Exercise

Show that $\langle\hat{n}\rangle = \left(\exp\left(\beta\hbar\omega\right)-1\right)^{-1}$.

Solution

Recall that, for a general state $\hat{\rho}$, the expectation value of an operator $\hat{O}$ is defined as $\mathrm{Tr}\left(\hat{\rho}\hat{O}\right)$. Therefore

$$\langle\hat{n}\rangle = \mathrm{Tr}\left(\hat{\rho}_{\mathrm{Th}}\hat{n}\right) = \sum_{n=0}^{\infty}\langle n|\,\hat{\rho}_{\mathrm{Th}}\hat{n}\,|n\rangle$$
$$= \sum_{n=0}^{\infty} n\,\langle n|\,\hat{\rho}_{\mathrm{Th}}\,|n\rangle = \frac{1}{Z}\sum_{n=0}^{\infty} n\exp\left(-\beta n\hbar\omega\right).$$

To evaluate the sum we first observe that for any $x<0$,

$$\sum_{n=0}^{\infty} ne^{nx} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\sum_{n=0}^{\infty}e^{nx}\right] = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{1-e^x}\right] = \frac{e^x}{\left(1-e^x\right)^2},$$

and thus

$$\langle\hat{n}\rangle = \frac{1}{Z}\cdot\frac{e^{-\beta\hbar\omega}}{\left(1-e^{-\beta\hbar\omega}\right)^2} = \frac{1}{e^{\beta\hbar\omega}-1},$$

which is the known Bose-Einstein distribution factor. $\square$