Fundamentals of Quantum Technology
Homework Sheet 1

1. We have defined a coherent state $|\alpha\rangle$ of a quantized harmonic oscillation mode as a right eigenstate of the annihilation operator, $\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$.

   (a) Show that the expansion of $|\alpha\rangle$ in the basis of number states $|n\rangle$ is given by

   $$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\,|n\rangle\,.$$

   Note that this implies in particular that a coherent state $|\alpha\rangle$ exists for *any* complex α.

   (b) Show that $\langle\alpha|\,\hat{n}\,|\alpha\rangle = |\alpha|^2$, where $\hat{n} = \hat{a}^\dagger\hat{a}$.

   (c) Show that, with respect to a coherent state $|\alpha\rangle$, the number fluctuation defined as $\Delta n \equiv \sqrt{\langle\hat{n}^2\rangle - \langle\hat{n}\rangle^2}$ is given by $\Delta n = |\alpha|$.

2. The displacement operator $\hat{D}(\alpha)$ was defined in class as $\hat{D}(\alpha) = \exp\left(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right)$, for any complex number α.

   (a) Show that $\hat{D}(\alpha)\,|0\rangle = |\alpha\rangle$, where $|0\rangle$ is the vacuum state.
   Hint: Recall that for two operators $\hat{A}$ and $\hat{B}$ such that $\left[\hat{A},\left[\hat{A},\hat{B}\right]\right] = \left[\hat{B},\left[\hat{B},\hat{A}\right]\right] = 0$, the following identity applies:
   $$e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-\frac{1}{2}\left[\hat{A},\hat{B}\right]}.$$

   (b) Show that $\hat{D}(\alpha)$ is a unitary operator,

   $$\hat{D}(\alpha)\,\hat{D}^\dagger(\alpha) = \hat{D}^\dagger(\alpha)\,\hat{D}(\alpha) = 1.$$

   (c) Show that $\hat{D}(\alpha)\,\hat{D}(\beta) = e^{i\mathrm{Im}(\alpha\beta^*)}\hat{D}(\alpha+\beta)$.

3. The operation of the quantum beam splitter can be thought of as the evolution in the Heisenberg picture of the mode operators $\hat{a}_i$. That is, we can define a unitary $\hat{U}$ such that

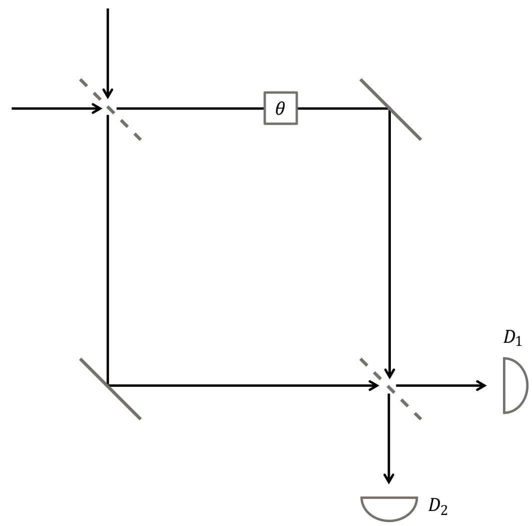
   $$\hat{a}_2 = \hat{U}^\dagger\hat{a}_0\hat{U}, \quad \hat{a}_3 = \hat{U}^\dagger\hat{a}_1\hat{U}.$$

   (a) Consider the operator
   $$\hat{U}(\theta) = \exp\left[i\frac{\theta}{2}\left(\hat{a}_0^\dagger\hat{a}_1 + \hat{a}_1^\dagger\hat{a}_0\right)\right].$$
   Obtain the transformation of the operators, and relate θ to the parameters r, r', t, t'. What choice of θ will give us the 50:50 beam splitter?
   Hint: Recall the Baker-Hausdorff lemma, $e^{i\hat{G}}\hat{A}e^{-i\hat{G}} = \hat{A} + i\left[\hat{G},\hat{A}\right] + \frac{i^2}{2!}\left[\hat{G},\left[\hat{G},\hat{A}\right]\right] + \ldots$.

   (b) Explain why this formulation of the operator transformation is equivalent to a Heisenberg picture evolution.

4. Recall the setup presented in class of a Mach-Zehnder interferometer (MZI) with a relative phase shift of θ in the clockwise arm. You saw that in the event of a single incident photon, the operation of the MZI is given by

$$|0\rangle_0 |1\rangle_1 \xrightarrow{\text{MZI}} \frac{1}{2}\left[\left(e^{i\theta}-1\right)|0\rangle_{D_1}|1\rangle_{D_2} + i\left(e^{i\theta}+1\right)|1\rangle_{D_1}|0\rangle_{D_2}\right].$$



(a) Suppose that a *classical* light wave with an amplitude $\mathcal{E}$ is incident upon the MZI. What would be the output amplitude at each detector?

(b) Obtain the output state given a coherent input state $|0\rangle_0 |\alpha\rangle_1$ (note that, in the quantum mechanical description, the phase shift is given by the action of the operator $\exp\left[i\theta\hat{a}^\dagger\hat{a}\right]$). Compare this with the results for a single photon and for classical light.