Introduction to Particles and Nuclear Physics

Class Exercise 3

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13/06/2024

Today's Topics:

1. Scalar Field Theories

2. Matrix Element Calculation in ABC Model

Scalar Field Theories

From the last class, we saw that we may construct Lagrangian for a scalar field ϕ, such that the Euler-Lagrange equations come from the form:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0$$

We applied it to a massive Scalar field with the following Lagrangian:

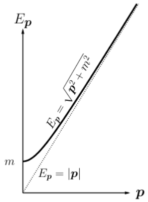
$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$$

Resulting in the Equation of Motion (EOM) known as the Klein-Gordon (KG) equation:

$$(\partial_\mu \partial^\mu + m^2)\phi = (\partial^2 + m^2)\phi = (\Box + m^2)\phi = 0$$

Which has the solution and dispersion relation:

$$\phi(t, \mathbf{x}) \propto e^{-i(E_p t - \mathbf{px})}, E_p^2 = m^2 + p^2$$



External Sources and ϕ^4 Theory

We now want to introduce interactions. The simplest way to do this is to have the scalar field interact with an external potential. This potential is described by a function known as a **source current** $J(x)$ which interacts with the field, giving a potential energy term $-J(x)\phi(x)$. The resulting Lagrangian is written:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - J\phi$$

Question 1: What is the general EOM for some source current $J(x)$?

Solution: We will begin by calculating: $\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi - J$. The second term in the E-L Equation is given by: $\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)} = \frac{1}{2}\frac{\partial(\partial_\mu\phi\partial^\mu\phi)}{\partial(\partial_\nu\phi)} = \frac{1}{2}\partial^\mu\phi\delta_{\mu\nu} + \frac{1}{2}\partial_\mu\phi\frac{\partial(\partial^\mu\phi)}{\partial(\partial_\nu\phi)} = \frac{1}{2}\partial^\nu\phi + \frac{1}{2}\partial_\mu\phi g^{\mu\alpha}\frac{\partial(\partial_\alpha\phi)}{\partial(\partial_\nu\phi)} =$
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Plugging all these back into our original E-L equation thus gives us: $\Box\phi(x) = J(x)$

A very famous toy model of a scalar field which can interact with itself is called the ϕ^4 theory, since we add an extra term to the Scalar field Lagrangian as the following:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

Which yields the (unsolvable) EOM: $(\Box + m^2)\phi = -\frac{\lambda}{3!}\phi^3$, meaning that we would need perturbation theory to get predictions from this model.

Two Scalar Fields

Here's another way to make particles interact. Why not have two different types of particle in the Universe, described by two fields $\phi_1(x)$ and $\phi 2(x)$? In our simple theory they have the same mass and will interact with themselves and each other via a potential energy $U(\phi_1, \phi_2) = g(\phi_1^2 + \phi_2^2)^2$ where g is the interaction strength here. Notice that multiplying out the bracket gives self interacting ϕ^4 terms and, crucially for us, a cross-term $2\phi_1^2\phi_2^2$ which forces the two types of field to interact.The resulting Lagrangian density is:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{1}{2}m^2\phi_1^2 - \frac{1}{2}m^2\phi_2^2 - g(\phi_1^2 + \phi_2^2)^2$$

By allowing ourselves to define the fields as the following: $\Phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$ we may rewrite our Lagrangian (prove to yourself!) in a more convenient fashion:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\Phi^*\partial^\mu\Phi - \frac{1}{2}m^2\Phi^*\Phi - g(\Phi^*\Phi)^2$$

Note that this is only true for the case of particles with the same mass. We may generalize to have different mass terms so that our Lagrangian now becomes:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\Phi^*\partial^\mu\Phi - \frac{1}{2}\Phi^*M^2\Phi - g(\Phi^*\Phi)^2, M = \begin{bmatrix} m_1 & & \\ & \ddots & \\ & & m_n \end{bmatrix}$$

We may even generalize this further to have cross mass terms: $M_{ij}\phi_i\phi_j$, however this is far more complicated. This means we cannot represent our interaction basis and our mass basis at the same time. This will have important implications later on.

Complex Scalar Fields

We can also make a transformation that simplifies the two-scalar-fields Lagrangian considered in the previous section. We define two new fields ψ and $\psi^\dagger$ given by:

$$\psi = \frac{1}{\sqrt{2}}[\phi_1 + i\phi_2]$$
$$\psi^\dagger = \frac{1}{\sqrt{2}}[\phi_1 - i\phi_2]$$

Which turns our Lagrangian into::

$$\mathcal{L} = \partial_\mu \psi^\dagger \partial^\mu \psi - m^2 \psi^\dagger \psi - g(\psi^\dagger \psi)^2$$

Note now our normalization for the kinetic term is 1. In general, the normalization for real scalar fields is $\frac{1}{2}$ and for Complex fields it's 1. That Lagrangian is invariant under phase transformations $\psi \to e^{-i\theta}\psi, \psi^\dagger \to \psi^\dagger e^{i\theta}$:

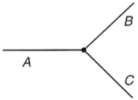
$$\mathcal{L}' = \partial_\mu \psi^\dagger e^{i\theta} \partial^\mu e^{-i\theta} \psi - m^2 \psi^\dagger e^{i\theta} e^{-i\theta} \psi - g(\psi^\dagger e^{i\theta} e^{-i\theta} \psi)^2 = \mathcal{L}$$

It's important to understand which transformations we can apply on our model which leaves it invariant. That means that these transformations can be called **Symmetries**, and according to Noether's Theorem, any mathematical symmetry is connected to some conserved physical size (Energy, momentum, particle number, particle charge, etc). By choosing which symmetries our model respects, we decide on our conservation laws, and thus our **forces**.

Matrix Element Calculation in ABC Model

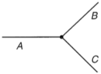
We introduce a simple toy-model (*"ABC theory"*) where:

- There are only three particles A, B, C, with m_a, m_b, m_c
- $Q_A = Q_B = 1$, $Q_C = 0$
- the coupling constant is g
- From the above, we conclude there is only one possible vertex:



Question 2:
As a first example, we will assume $m_a > m_b + m_c$ and calculate the Matrix element of the decay process. The leading order process for A decaying into lighter particles is given by $A \to BC$, which is exactly the vertex diagram above.

The Feynman rules for this theory, which are the steps we need to perform in order to derive $\mathcal{M}$, are as follows:



1. Denote the incoming and outgoing particle momenta by $p_1, p_2, \ldots, p_n$, and the propagator (internal lines/virtual particle) momenta by $q_1, q_2 \ldots, q_m$. It's advisable to put arrows next to each external line to keep track of the direction of momentum-flow.
2. For each vertex we multiply by a factor of the coupling constant $(-ig)$
3. For each internal line (propagator) we multiply by a factor of $\frac{i}{q^2 - m^2}$
4. For each vertex we enforce momentum conservation by adding a factor of $(2\pi)^4 \delta^4(\sum p_i)$, where p_i are the momenta of the three lines connected by the vertex. Outgoing particles momenta will get a $(-)$ sign.
5. For each internal line q_j, we add a factor of $\frac{d^4 q_j}{(2\pi)^4}$, and perform the integration over dq_j. The integral will cancel one of the δ functions.
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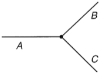

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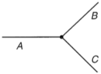

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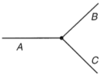

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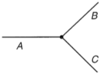

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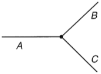

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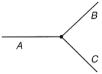

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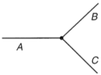

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$$\boxed{\mathcal{M} = g}$$

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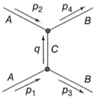
Calculate $\mathcal{M}$ for the process of $AA \to BB$ scattering via the following Feynman diagram:



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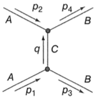

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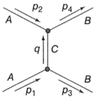

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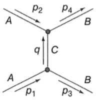

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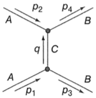

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$$\mathcal{M} = \left[(-ig)(2\pi)^4\delta^4(p_2 + q - p_4)\right] \times \left[(-ig)(2\pi)^4\delta^4(p_1 - q - p_3)\right] \frac{i}{q^2 - m_C^2}$$

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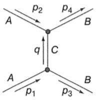

1. Denote the incoming and outgoing particle momenta by $p_1, p_2, \ldots, p_n$, and the propagator (internal lines/virtual particle) momenta by $q_1, q_2 \ldots, q_m$. It's advisable to put arrows next to each external line to keep track of the direction of momentum-flow.
2. For each vertex we multiply by a factor of the coupling constant $(-ig)$
3. For each internal line (propagator) we multiply by a factor of $\frac{i}{q^2 - m^2}$
4. For each vertex we enforce momentum conservation by adding a factor of $(2\pi)^4 \delta^4(\sum p_i)$, where p_i are the momenta of the three lines connected by the vertex. Outgoing particles momenta will get a $(-)$ sign.
5. For each internal line q_j, we add a factor of $\frac{d^4 q_j}{(2\pi)^4}$, and perform the integration over dq_j.
   The integral will cancel one of the δ functions.
6. The result will include a delta function $[(2\pi)^4 \delta^4(\sum p_i)]$ reflecting overall momentum conservation. Replace this remaining δ function (along with its $(2\pi)^4$ factor) with (i).

$$\mathcal{M} = \int \left[(-ig)(2\pi)^4 \delta^4(p_2 + q - p_4)\right] \times \left[(-ig)(2\pi)^4 \delta^4(p_1 - q - p_3)\right] \frac{i}{q^2 - m_c^2} \frac{d^4 q_j}{(2\pi)^4}$$

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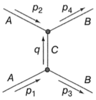

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$$\mathcal{M} = -g^2 \left[(2\pi)^4 \delta^4(p_1 - p_4 + p_2 - p_3)\right] \frac{i}{(p_4 - p_2)^2 - m_C^2}$$

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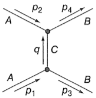

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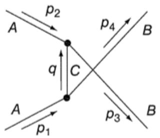
$$\boxed{\mathcal{M} = \frac{g^2}{(p_4 - p_2)^2 - m_c^2}}$$

Question 4:

Using the result of Q3, calculate M for this process in the COM frame under the assumption of $m_A = m_B = m$ and $m_C = 0$ to leading order in g.

Solution:

First, we note that there is another order g^2 diagram for this process:



We can repeat the same steps for calculating this diagram, but that won't be necessary, since the matrix element for this u-channel diagram is obtained by simply replacing $p_3 \leftrightarrow p_4$.

Thus we have for the amplitude:

$$\mathcal{M} = \frac{g^2}{(p_4 - p_2)^2 - m_c^2} + \frac{g^2}{(p_3 - p_2)^2 - m_c^2}$$

Note:

- $|M|$ is Lorentz-invariant

In the COM frame under the mass assumptions, we can write:

$$p_i = (2E_A, 0)\ ;\ p_f = (2E_B, 0)\ \Rightarrow\ E_A = E_B \Rightarrow |\mathbf{p}_A| = |\mathbf{p}_B| = |\mathbf{p}|$$

Using this we can find that:

$$(p_4 - p_2)^2 = 2m^2 - 2E_2E_4 + 2(\mathbf{p}_2 \cdot \mathbf{p}_4) = 2m^2 - 2(m^2 + \mathbf{p}^2) + 2\mathbf{p}^2 \cos\theta$$

$$\Rightarrow (p_4 - p_2)^2 = -2\mathbf{p}^2(1 - \cos\theta)$$

Since $\mathbf{p_3} = -\mathbf{p_4}$, this immediately also gives us:

$$\Rightarrow (p_3 - p_2)^2 = -2\mathbf{p}^2(1 + \cos\theta)$$

And so overall the matrix element is:

$$\mathcal{M} = -\left[\frac{g^2}{2\mathbf{p}^2(1 - \cos\theta)} + \frac{g^2}{2\mathbf{p}^2(1 + \cos\theta)}\right] = -\frac{g^2}{\mathbf{p}^2 \sin^2\theta}$$

And it is worth noting that this case is similar to Rutherford scattering in QED, with the Matrix Element divergent at the limit $\theta \to 0$.