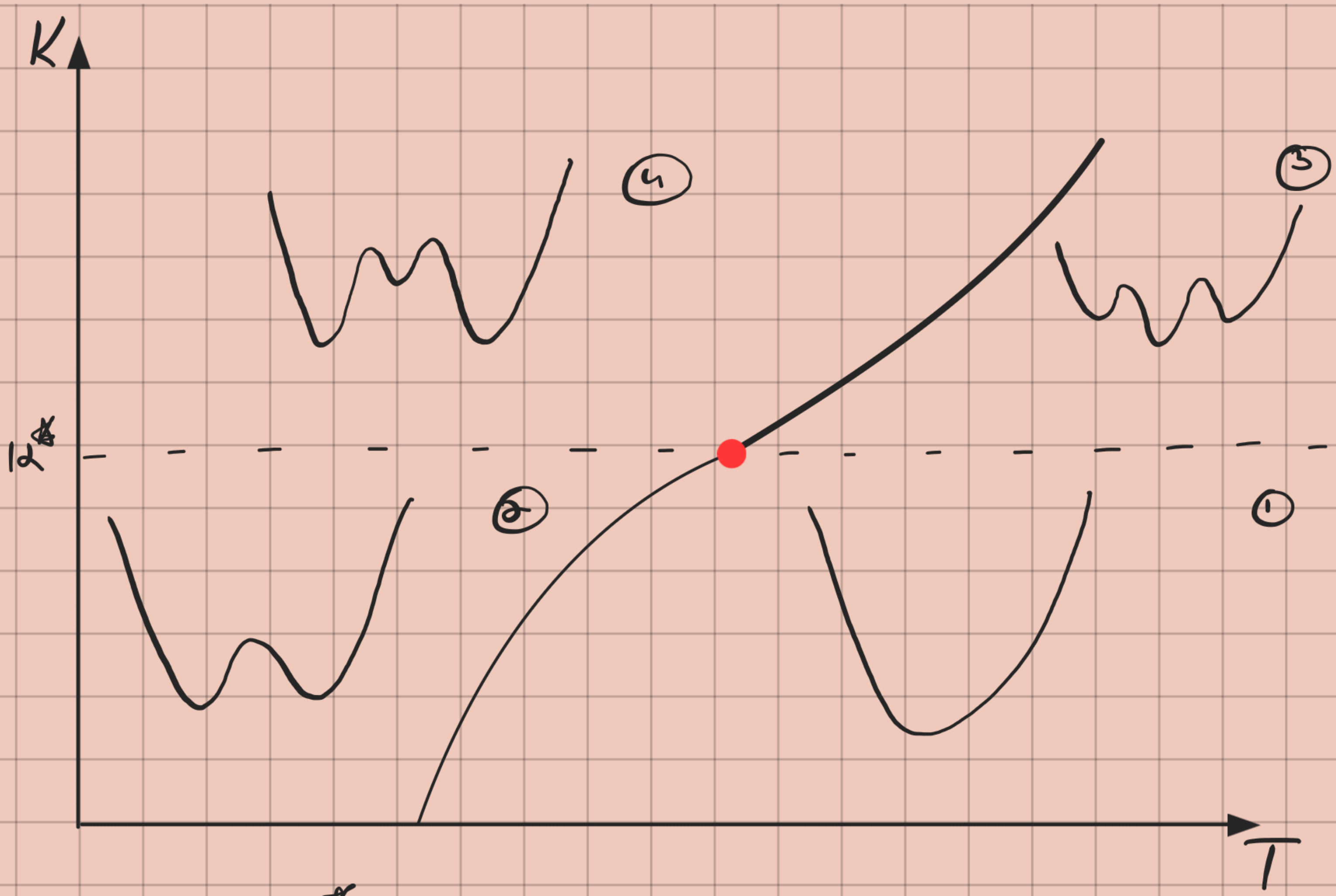
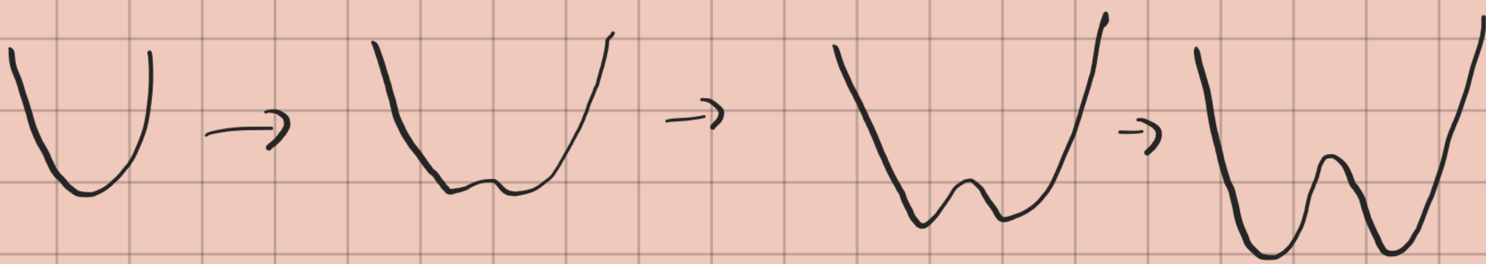


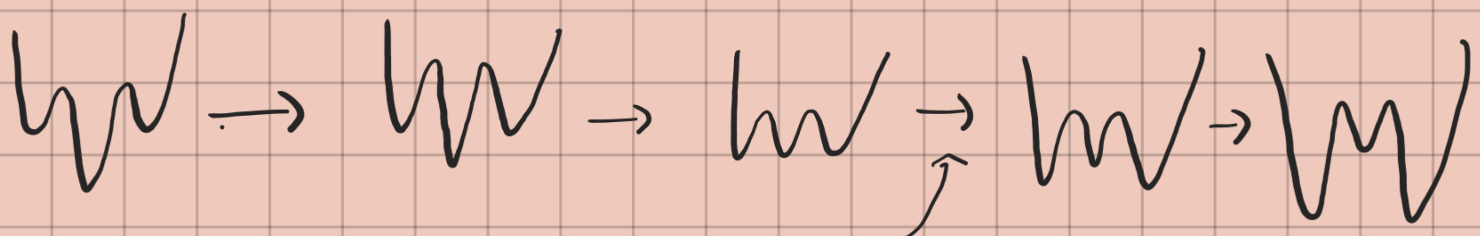
1 → 2 : $K < K^*$

As T lowers



The $m \neq 0$ solutions change continuously to larger and larger m values, second order transition.

③ → ④: $K > K^*$



The solutions ($m \neq 0$) jumps from $m=0$ to $m \neq 0$,
hence first order phas transition.

$$F = a m^2 + b m^4 + c m^6$$

C is always positive, so there wont be a case were the solution is $m \to \infty$, which is not physical.

For $K < K^*$ need to get basically the Ising model, so $b > 0$. And for $K > K^*$ $b < 0$ and we get 5 extrema.

$\Rightarrow$ Close to K^* we can approx.: $b = b_0 (K^* - K)$

Now if we set b thusly, for $b>0$ we need the Ising model behavior.

So we just take a to be like in the Ising model: $a = a_0(T-T_c)$ but now $T_c \to T_c(K)$ so $a = a_0(T - T_c(K))$

$$\Rightarrow \quad F = a_0(T - T_c(K))m^2 + b_0(K^* - K)m^4 + cm^6$$

$\underline{\beta:}$

$$\frac{dF}{dm} = 2a_0(T - T_c)m + \cancelto{K=K^*}{4b_0(K^*-K)m^3} + 6cm^5 \overset{!}{=} 0$$

$$m\left(a_0(T-T_c) + 3cm^4\right) = 0$$

$$m = 0, \pm\left(\frac{a_0}{3c}(T_c - T)\right)^{1/4}$$

$$\beta = 1/4$$

And β of the Ising model is $\frac{1}{2}$.

$\underline{\alpha}$

$$C_V = -T_c \left.\frac{\partial^2 F}{\partial T^2}\right|_{m=m'}$$

$$m' = \left(\frac{a_0}{3c} (T_c - T) \right)^{1/4} \qquad \text{Still } h = h^*$$

$$F\Big|_{m'} = a_0 (T - T_c) \left(\frac{a_0}{3c} (T_c - T) \right)^{1/2} + c \left(\frac{a_0}{3c} (T_c - T) \right)^{3/2}$$

$$= (T_c - T)^{3/2} \left(c \left(\frac{a_0}{3c} \right)^{3/2} - a_0 \left(\frac{a_0}{3c} \right)^{1/2} \right)$$

$$\Rightarrow \quad C_V \propto |t|^{-1/2} \quad \Rightarrow \quad \alpha = \frac{1}{2}$$

And for the Ising model $\alpha = 0$

$$F = a\,(\bar{S}\cdot\bar{S}) + b\,(\bar{S}\cdot\bar{S})^2 + c\,\bar{B}\cdot\bar{S}$$

Like befor $b>0$ always such that $\bar{S}$ is always finite.

$a = a_0\,(T - T_c)$ so it would flip sign when crossing T_c.

$c = -1$, Minus so F will be minimal when $\bar{B} /\!/ \bar{S}$, and 1 cuz we can always change the magnitude of $\bar{B}$. So we will shove all the constants to $|\bar{B}|$.

We can write explicitely $\bar{S}\cdot\bar{S} = S_x^2 + S_y^2$

β:

$$\Rightarrow \nabla F\Big|_{\bar{B}=0} = \begin{pmatrix} 2a_0(T-T_c)S_x + 4b(S_x^2+S_y^2)S_x \\ 2a_0(T-T_c)S_y + 4b(S_x^2+S_y^2)S_y \end{pmatrix} = 0$$

$$\Rightarrow 2a_0(T-T_c)\bar{S} + 4b|\bar{S}|^2\bar{S} = 0$$

$$\Rightarrow \quad 2a_0(T-T_c) + 4b|\bar{S}|^2 = 0$$

$$|\bar{S}|_{min} = \pm\sqrt{\frac{a_0}{2b}(T_c-T)}$$

$$\Rightarrow \quad \beta = \frac{1}{2} \quad \text{like in the regular IM}$$

$\underline{\alpha}$:

$$F\Big|_{S_{min}} = a_0(T-T_c)\frac{a_0}{2b}(T_c-T) + b\left(\frac{a_0}{2b}(T_c-T)\right)^2$$

$$\propto |t|^2$$

$$\Rightarrow \quad C_v \propto \text{const} \quad \Rightarrow \alpha = 0$$

$\underline{\delta}$:

At T_c $\quad F = b(\bar{S}\cdot\bar{S})^2 - \bar{B}\cdot\bar{S}$

$$\nabla F = \begin{pmatrix} 4b(S_x^2+S_y^2)S_x - B_x \\ 4b(S_x^2+S_y^2)S_y - B_y \end{pmatrix} \stackrel{!}{=} 0$$

But we can set $\bar{B} = B_x\hat{x}$

$$\Rightarrow \quad S_{y_{min}} = 0$$

$$\Rightarrow \quad 4bS_x^3 - B_x = 0$$

$$\Rightarrow \quad \delta = \frac{1}{3}$$

$\underline{\gamma}$:

Now for γ the last question suggests we should split in to two cases, $T > T_c$, $T < T_c$:

For $T > T_c$ we don't need the $(\bar{S} \cdot \bar{S})^2$ term to describe the free energy (close to T_c)

$$\Rightarrow \quad \nabla F = \begin{pmatrix} 2a_0(T-T_c)S_x - B_x \\ 2a_0(T-T_c)S_y - B_y \end{pmatrix} = 0$$

$$\Rightarrow \quad S_{x,y} = \frac{B_{x,y}}{2a_0(T-T_c)}$$

$$\Rightarrow \gamma_{T>T_c} = 1$$

For $T < T_c$ we can not neglect the $(\bar{S} \cdot \bar{S})^2$ term, even for $B=0$ $|\bar{S}| \neq 0$.

So we can expand around $|\bar{S}|_{min} \equiv S'$:

$$\bar{S}\cdot\bar{S} = |\bar{S}|^2$$

$$F = a_0(T-T_c)|s'+\delta s|^2 + b|s'+\delta s|^4 - \bar{B}\cdot(\bar{s}'+\delta\bar{s})$$

$$\frac{\delta F}{\delta s} = 2a_0(T-T_c)|s'+\delta s| + 4b|s'+\delta s|^3 - \bar{B}$$

up to δs : $|s'+\delta s|^3 = s'^3 + 3s'^2\delta s$

$$= 2a_0(T-T_c)(s'+\delta s) + 4b(s'^3+3s'^2\delta s) - \bar{B}$$

But we know s', so we only care

about the terms with δs

$$2a_0(T-T_c)\delta s + 12b\, s'^2\delta s - B = 0$$

$$s'^2 = \frac{a_0}{2b}(T-T_c)$$

$$8a_0(T-T_c)\delta s - B = 0$$

$$\Rightarrow \chi_{T<T_c} = \frac{1}{8a_0(T-T_c)} \Rightarrow \gamma_{T<T_c} = 1$$

And we saw before that $\chi_{T>T_c} = \frac{1}{2a_0(T-T_c)}$

$$\Rightarrow \frac{\chi_{T>T_c}}{\chi_{T<T_c}} = \frac{A_+}{A_-} = 4$$

(3) For the infinite tree its pretty obvious that we would get the mean field results. Because every spin is tied to all other spins through going up the tree.

The key observation is the fact that if we sum over the root node we get two identical trees.

$(K = \beta J)$

$$Z = \sum_{\{\sigma\}} e^{K(\sigma_1\sigma_2 + \sigma_1\sigma_3)} e^{K(\sigma_2\sigma_4 + \sigma_2\sigma_5)} e^{K(\sigma_3\sigma_6 + \sigma_3\sigma_7)} \cdots$$

Now "truncating" means suming over σ_1:

$$Z = \sum_{\{\sigma\}} \left(e^{K(\sigma_2+\sigma_3)} + e^{-K(\sigma_2+\sigma_3)}\right) e^{K(\sigma_2\sigma_4 + \sigma_2\sigma_5)} e^{K(\sigma_3\sigma_6 + \sigma_3\sigma_7)} \cdots$$

$$Z_+ = \sum_{\substack{\{\sigma\} \\ \neq \sigma_1}} \underbrace{e^{K\sigma_2} e^{K(\sigma_2\sigma_4 + \sigma_2\sigma_5)} \cdots}_{\sigma_2 \text{ tree}} \underbrace{e^{K\sigma_3} e^{K(\sigma_3\sigma_6 + \sigma_3\sigma_7)} \cdots}_{\sigma_3 \text{ tree}}$$

$Z_- \to$ same but with $e^{-K\sigma_2}$ and $e^{-K\sigma_3}$

Now the two $\sigma_{2,3}$ trees are identical:

$$Z_+ = \Big(\sum_{\substack{\{\sigma\} \\ \neq \sigma_1}} e^{K\sigma_2} e^{K\sigma_2(\sigma_4+\sigma_5)} \cdots \Big)^2$$

$$= \Big(e^{K} \underbrace{\sum_{\substack{\{\sigma\} \\ \neq \sigma_{1,2}}} e^{K(\sigma_4+\sigma_5)} \cdots}_{Z_+} + e^{-K} \underbrace{\sum_{\substack{\{\sigma\} \\ \neq \sigma_{1,2}}} e^{K(\sigma_4+\sigma_5)} \cdots}_{Z_-} \Big)^2$$

$$\Rightarrow \qquad Z_+ = \left(e^{K} Z_+ + e^{-K} Z_- \right)^2$$

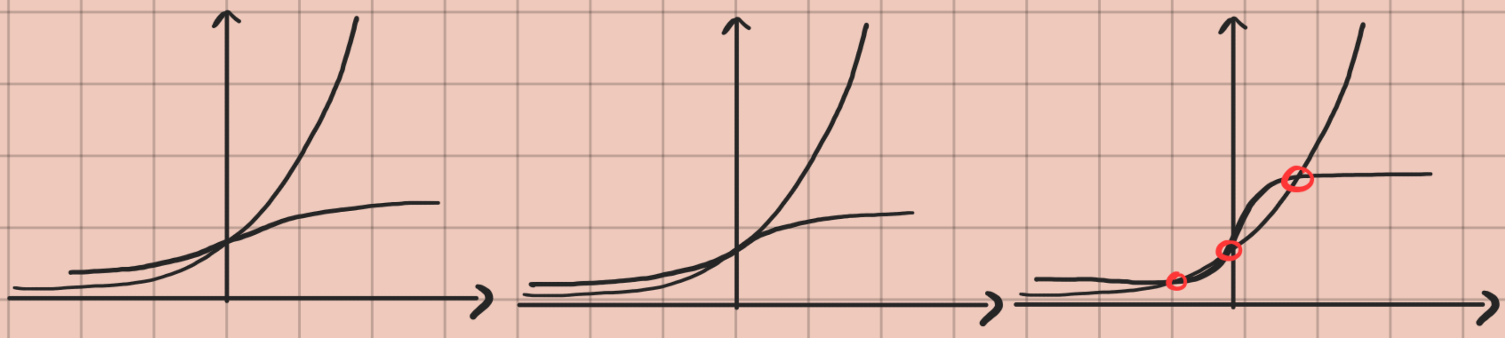
$$Z_- = \left(e^{-K} Z_+ + e^{K} Z_- \right)^2$$

Now we define:

$$\frac{Z_+}{Z_-} = \frac{\left(e^{K} Z_+ + e^{-K} Z_- \right)^2}{\left(e^{-K} Z_+ + e^{K} Z_- \right)^2} = \frac{\left(e^{K} + e^{-K} \frac{Z_-}{Z_+} \right)^2}{\left(e^{-K} + e^{K} \frac{Z_-}{Z_+} \right)^2} \equiv e^{2h}$$

$$\Rightarrow \quad \frac{e^{K} + e^{-K-2h}}{e^{-K} + e^{K-2h}} = \frac{e^{K+h} + e^{-K-h}}{e^{-K+h} + e^{K-h}} = \frac{\cosh(K+h)}{\cosh(K-h)}$$

$$\Rightarrow \quad e^{u} = \frac{\cosh(K+u)}{\cosh(K-u)} \qquad \frac{\sinh(K+u)\cosh(K-u) + \cosh(K+u)\sinh(K-u)}{\cosh^2(K-u)}$$



$K < K_c$ $\qquad$ $K = K_c$ $\qquad$ $K > K_c$

equal slope

$$\Rightarrow \quad 1 \overset{u=0}{=} \frac{2\sinh(K_c)}{\cosh(K_c)} = 2\tanh(K_c)$$

$$\Rightarrow \quad K_c = \tanh^{-1}\left(\frac{1}{2}\right) = 0.5493 = \beta_c J = \frac{J}{k_B T_c}$$

$$\underline{\underline{T_c = \frac{J}{k_B}\,\frac{1}{0.5493}}}$$

critical exponents

$$m = \langle \sigma_1 \rangle = \frac{1}{Z}\left(1 Z_+ + (-1) Z_-\right) = \frac{Z_+ - Z_-}{Z_+ + Z_-}$$

$$\Rightarrow m = \tanh u \sim u$$

around k_c

$$u = \ln\left(\frac{\cosh(k+h)}{\cosh(k-h)}\right) \approx \underbrace{\ln\left(\frac{\cosh(k_c+h)}{\cosh(k_c-h)}\right)}_{\text{const}} + (k-k_c)\left(\tanh(k_c+h) - \tanh(k_c-h)\right) + \dots$$

$$\Rightarrow u^2 \propto t \leftarrow k - k_c$$

$$u \propto t^{1/2}$$

$$\Rightarrow m \propto t^{1/2}$$

$$\Rightarrow \beta = \frac{1}{2}$$

α:

also function of u

$$Z = Z_+ + Z_- = Z_+\left(1 + \frac{Z_-}{Z_+}\right) = Z_+\left(1 + e^{-2u}\right)$$

Z is analytical at T_c $\Rightarrow$ $F = -\ln Z$ is also analytical

$\Rightarrow$ no jumps in 1st and 2nd derivative

$$\Rightarrow C_V = \text{const} \Rightarrow \alpha = 0$$

We got two critical exponents like in the Ising MF. From the following eq. the other two (γ, δ) will be the same:

$$2 = \alpha + 2\beta + \gamma$$

$\leftarrow$ Wikipedia

$$\gamma = \beta(\delta - 1)$$