Question 14

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2M} + \frac{k(x^2+y^2)}{2} + \alpha xy, \qquad |\alpha| < k, \qquad M \gg m$$

BOA steps:

1. Identify fast coordinate as x, separate the Hamiltonian: $H = H_o(p_y, y) + h(p_x, x; y)$
2. Write $\psi(x,y) = \sum_n \xi_n(y)\phi_n(x;y)$, Solve $h\phi_n(x;y) = \varepsilon_n(y)\phi_n(x;y)$
3. Solve $-\frac{\hbar^2}{2m}\left(\partial_y - iA_n(y)\right)^2 \xi_n(y) + V_{eff}(y)\xi_n(y) = E\xi_n(y)$
   where $A_n y = \langle \phi_n | \partial_y \phi_n \rangle$ and $V_{eff}(y) = V_0(y) + \varepsilon_n(y)$

Step 1

$M \gg m$ means the slow coordinate is y, x is "fast"

$$H = \frac{p_y^2}{2M} + \frac{1}{2}ky^2 + \frac{p_x^2}{2m} + \frac{kx^2}{2} + \alpha xy = \underbrace{\frac{p_y^2}{2M} + \frac{1}{2}ky^2 - \frac{1}{2}\frac{\alpha^2}{k}y^2}_{H_o(p_y,y)} + \underbrace{\frac{p_x^2}{2m} + \frac{1}{2}k\left(x + \frac{\alpha}{k}y\right)^2}_{h(p_x,x;y)}$$

$$H_o(p_y, y) = \frac{p_y^2}{2M} + \frac{1}{2}k\left(1 - \frac{\alpha^2}{k^2}\right)y^2$$

$$h(p_x, x; y) = \frac{p_x^2}{2m} + \frac{1}{2}m\omega^2(x + x_0(y))^2, \qquad where:\ \omega = \sqrt{\frac{k}{m}},\ \ x_0(y) = \frac{\alpha}{k}y$$

Step 2

Look for a solution: $\psi(x,y) = \sum_n \xi_n(y)\phi_n(x;y)$

h describes a shifted H-O with energy levels and corresponding wavefunctions:

$$\varepsilon_n = \hbar\omega\left(n + \tfrac{1}{2}\right), \qquad \phi_n(x;y) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left(-\frac{m\omega(x + x_0(y))^2}{2\hbar}\right) H_n\left(\sqrt{\frac{m\omega}{\hbar}}(x + x_0(y))\right)$$

Step 3

$$-\frac{\hbar^2}{2m}\left(\partial_y - iA_n(y)\right)^2 \xi_n(y) + V_{eff}(y)\xi_n(y) = E\xi_n(y)$$

$A_n(y) = \langle \phi_n | \partial_y \phi_n \rangle$, and $A_n(y)$ is pure imaginary, here $\phi_n(x;y)$ are real $\Longrightarrow A_n(y) = 0$

$$V_{eff}(y) = V_0(y) + \varepsilon_n(y) = \frac{1}{2}k\left(1 - \frac{\alpha^2}{k^2}\right)y^2 + \hbar\omega\left(n + \frac{1}{2}\right) = \frac{1}{2}M\Omega^2 y^2 + \hbar\omega\left(n + \frac{1}{2}\right)$$

Define: $\Omega = \sqrt{\frac{k\left(1 - \frac{\alpha^2}{k^2}\right)}{M}}$

$$\rightarrow \left(\frac{p_y^2}{2m} + \frac{1}{2}M\Omega^2 y^2 + \hbar\omega\left(n + \frac{1}{2}\right)\right)\xi_n(y) = E\xi_n(y)$$

We get a H-O for y and shifted spectrum by constant $\hbar\omega\left(n + \frac{1}{2}\right)$.

The total energy spectrum and corresponding wavefunctions are therefore:

$$E_{n,l} = \hbar\Omega\left(l+\frac{1}{2}\right)+\hbar\omega\left(n+\frac{1}{2}\right), \qquad l,n\in\mathbb{N}$$

$$\psi_{n,l} = \frac{1}{\sqrt{2^l l!}}\left(\frac{M\Omega}{\pi\hbar}\right)^{\frac{1}{4}}\exp\left(-\frac{M\Omega y^2}{2\hbar}\right)H_l\left(\sqrt{\frac{M\Omega}{\hbar}}y\right)$$
$$\cdot\frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\exp\left(-\frac{m\omega(x+x_0(y))^2}{2\hbar}\right)H_n\left(\sqrt{\frac{m\omega}{\hbar}}(x+x_0(y))\right)$$

*Recalling that: $\omega=\sqrt{\frac{k}{m}},\ x_0(y)=\frac{\alpha}{k}y,\ \Omega=\sqrt{\frac{k\left(1-\frac{\alpha^2}{k^2}\right)}{M}}$

Exact:
Here basically I need to diagonalize 2 matrices simultaneously but its tricky.
Trace back to the Lagrangian because working with $p's$ might be wrong.

$$\dot{x}=\frac{\partial H}{\partial p_x}=\frac{p_x}{m}\to p_x=m\dot{x}, \qquad \dot{y}=\frac{\partial H}{\partial p_y}=\frac{p_y}{M}\to p_y=M\dot{y}$$

$$L=m\dot{x}^2+M\dot{y}^2-\frac{m^2\dot{x}^2}{2m}-\frac{M^2\dot{y}^2}{2M}-\frac{k(x^2+y^2)}{2}-\alpha xy=\frac{1}{2}m\dot{x}^2+\frac{1}{2}M\dot{y}^2-\frac{k(x^2+y^2)}{2}-\alpha xy$$

First choose to work with coordinates: $\tilde{x}=\sqrt{m}x,\ \ \tilde{y}=\sqrt{M}y$ (The reason for this is to first make the kinetic term invariant to any orthogonal transformation, in my experience its extremely hard to try to nail the kinetic and potential matrices at once without doing this)

$$L=\frac{1}{2}\dot{\tilde{x}}^2+\frac{1}{2}\dot{\tilde{y}}-\frac{1}{2}\frac{k}{m}\tilde{x}^2-\frac{1}{2}\frac{k}{M}\tilde{y}^2-\frac{\alpha}{\sqrt{mM}}\tilde{x}\tilde{y}$$

Define: $\eta=\binom{\tilde{x}}{\tilde{y}}$

$$L=\frac{1}{2}\dot{\eta}^T\begin{pmatrix}1&0\\0&1\end{pmatrix}\dot{\eta}-\frac{1}{2}\eta^T\underbrace{\begin{pmatrix}\frac{k}{m}&\frac{\alpha}{\sqrt{mM}}\\\frac{\alpha}{\sqrt{mM}}&\frac{k}{M}\end{pmatrix}}_{K}\eta$$

Find e.vals, e.vecs of K and define Orthogonal transformation with normalized e.vecs.

$$0=\begin{vmatrix}\frac{k}{m}-\lambda&\frac{\alpha}{\sqrt{mM}}\\\frac{\alpha}{\sqrt{mM}}&\frac{k}{M}-\lambda\end{vmatrix}=\left(\frac{k}{m}-\lambda\right)\left(\frac{k}{M}-\lambda\right)-\frac{\alpha^2}{mM}=\lambda^2-k\frac{M+m}{mM}\lambda+\frac{(k^2-\alpha^2)}{mM}$$

$$\lambda_\pm = \frac{1}{2}\left(k\frac{M+m}{mM} \pm \sqrt{k^2\frac{(M+m)^2}{(mM)^2} - 4\frac{(k^2-\alpha^2)}{mM}}\right)$$
$$= \frac{1}{2mM}\left(k(M+m) \pm \sqrt{k^2(M+m)^2 - 4(k^2-\alpha^2)mM}\right)$$
$$= \frac{1}{2mM}\left(k(M+m) \pm \sqrt{k^2(M-m)^2 + 4\alpha^2 mM}\right)$$

E.vec for λ_+:

$$\begin{pmatrix} \frac{k}{m} - \lambda_+ & \frac{\alpha}{\sqrt{mM}} \\ \frac{\alpha}{\sqrt{mM}} & \frac{k}{M} - \lambda_+ \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \ \rightarrow \quad a = \frac{\alpha}{\sqrt{mM}}, \qquad b = \lambda_+ - \frac{k}{m}$$

$$v_+ = \frac{1}{\sqrt{\left(\lambda_+ - \frac{k}{m}\right)^2 + \frac{\alpha^2}{mM}}}\begin{pmatrix} \frac{\alpha}{\sqrt{mM}} \\ \lambda_+ - \frac{k}{m} \end{pmatrix}$$

E.vec for λ_-:

$$\begin{pmatrix} \frac{k}{m} - \lambda_- & \frac{\alpha}{\sqrt{mM}} \\ \frac{\alpha}{\sqrt{mM}} & \frac{k}{M} - \lambda_- \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \ \rightarrow \quad a = \lambda_- - \frac{k}{M}, \qquad b = \frac{\alpha}{\sqrt{mM}}$$

$$v_- = \frac{1}{\sqrt{\left(\lambda_- - \frac{k}{M}\right)^2 + \frac{\alpha^2}{mM}}}\begin{pmatrix} \lambda_- - \frac{k}{M} \\ \frac{\alpha}{\sqrt{mM}} \end{pmatrix}$$

Defining the orthogonal matrix O:

$$O = (v_+, v_-), \qquad O^T = \begin{pmatrix} v_+^T \\ v_-^T \end{pmatrix} = O^{-1}$$

Note: v_+, v_- are orthogonal because they are e.vecs of different e.values. They are also normalized hence orthornormal, thus $O^T = O^{-1}$. Define yet another set of coordinates: q_1, q_2

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = O^{-1}\eta = \begin{pmatrix} \dfrac{\frac{\alpha}{\sqrt{mM}}\tilde{x} + \left(\lambda_+ - \frac{k}{m}\right)\tilde{y}}{\sqrt{\left(\lambda_+ - \frac{k}{m}\right)^2 + \frac{\alpha^2}{mM}}} \\ \dfrac{\left(\lambda_- - \frac{k}{M}\right)\tilde{x} + \frac{\alpha}{\sqrt{mM}}\tilde{y}}{\sqrt{\left(\lambda_- - \frac{k}{M}\right)^2 + \frac{\alpha^2}{mM}}} \end{pmatrix}$$

The Lagrangian in these coordinates:

$$L = \frac{1}{2}\dot{q}_1^2 + \frac{1}{2}\dot{q}_2^2 - \frac{1}{2}\lambda_+ q_1^2 - \frac{1}{2}\lambda_- q_2^2$$

Defining the canonical conjugates: $p_1 = \frac{\partial L}{\partial \dot{q}_1} = \dot{q}_1,\ p_2 = \frac{\partial L}{\partial \dot{q}_2} = \dot{q}_2$

$$H = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + \frac{1}{2}\lambda_+ q_1^2 + \frac{1}{2}\lambda_- q_2^2$$

At last, two decoupled H-O's. with spectrum and corresponding wavefunctions:

$$E_{n,l} = \hbar\sqrt{\lambda_+}\left(n + \frac{1}{2}\right) + \hbar\sqrt{\lambda_-}\left(l + \frac{1}{2}\right), \qquad l, n \in \mathbb{N}$$

$$\psi_{n,l} = \psi_{n,l} = \frac{1}{\sqrt{2^l l!}}\left(\frac{\sqrt{\lambda_-}}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left(-\frac{\sqrt{\lambda_-}q_2^2}{2\hbar}\right) H_l\left(\sqrt{\frac{\sqrt{\lambda_-}}{\hbar}}q_2\right)$$

$$\cdot \frac{1}{\sqrt{2^n n!}}\left(\frac{\sqrt{\lambda_+}}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left(-\frac{\sqrt{\lambda_+}q_1^2}{2\hbar}\right) H_n\left(\sqrt{\frac{\sqrt{\lambda_+}}{\hbar}}q_1\right)$$

<u>Comparison to BOA</u>:

Approximate $\lambda_\pm$ in the limit $M \gg m$

$$\lambda_\pm = \frac{1}{2mM}\left(k(M+m) \pm \sqrt{k^2(M-m)^2 + 4\alpha^2 mM}\right)$$

$$\approx \frac{1}{2mM}\left(kM + km \pm \sqrt{k^2M^2 - 2k^2mM + 4\alpha^2 mM}\right)$$

$$= \frac{k}{2m} + \frac{k}{2M} \pm \frac{k}{2m}\sqrt{1 - 2\frac{m}{M} + 4\frac{\alpha^2}{k^2}\frac{m}{M}} \approx \frac{k}{2m} + \frac{k}{2M} \pm \frac{k}{2m}\left(1 - \frac{m}{M} + 2\frac{\alpha^2}{k^2}\frac{m}{M}\right)$$

$$= \frac{k}{2m} \pm \frac{k}{2m} + \frac{k}{2M} \pm \frac{k}{2M}\left(2\frac{\alpha^2}{k^2} - 1\right)$$

$$\lambda_+ \approx \frac{k}{m} + \frac{\alpha^2}{Mk} \approx \frac{k}{m} = \omega^2$$

$$\lambda_- \approx \frac{k}{M} - \frac{\alpha^2}{Mk} = \frac{k}{M}\left(1 - \frac{\alpha^2}{k^2}\right) = \Omega^2$$

Plugging these approximate values back to the exact spectrum indeed it is the same as the BOA:

$$E_{n,l} \approx \hbar\omega\left(n + \frac{1}{2}\right) + \hbar\Omega\left(l + \frac{1}{2}\right)$$

*Its possible to look at the wavefunctions as well and approximate q_1 and q_2 but this would be very tidius.