$\boxed{4}$ (a) choose a discrete set of functions $\{\phi_i\}$ Such as that:

$$\hat{\psi}(r) = \sum_i \phi_i(r)\,\hat{a}_i$$

$$\int \phi_i^*(r)\phi_j(r)\,dr = \cancel{\delta(r-r')}\ \delta_{ij}$$

~~From~~

$$\delta(r-r') = [\psi(r), \psi^\dagger(r')] = \sum_{i,j} \phi_i(r)\phi_j(r')[a_i, a_j^\dagger]$$

Multiply by $\phi_k(r')$ and $\int \dots dr'$

$$\phi_k(r) = \sum_i \phi_i(r)\,[a_i, a_k^\dagger]$$

Multiply by $\phi_l^*(r)$ and integrate:

$$\delta_{lk} = [a_l, a_k^\dagger]$$

~~Find Let us find coherent states in each a_i~~

A coherent state of a_i is:

$$|\alpha_i) = e^{-\frac{|\alpha_i|^2}{2} + \alpha_i \hat{a}_i^\dagger}|0\rangle .$$

← Maybe should show how to find it, but we saw it in class.

$$|\{\alpha_i\}\rangle = \bigotimes_i |\alpha_i\rangle .$$

$$\psi(r)\,|\{\alpha_i\}\rangle = \sum_j \phi_j(r)\,a_j\,|\{\alpha_i\}\rangle = \sum_i \phi_j(r)\,\alpha_j\,|\{\alpha_i\}\rangle$$

So the ~~eigenfun~~ eigenstates of $\psi(r)$ are $|\{\alpha_i\}\rangle$ and the ~~eigenfunctions are~~ eigenvalues are $\psi_r = \sum_i \phi_i(r)\alpha_i$.

$$\langle\{\alpha_i\}|N|\{\alpha_i\}\rangle = \int \langle\{\alpha_i\}|\psi^\dagger(r)\psi(r)|\{\alpha_i\}\rangle\,dr =$$

$$= \sum_{i,j}\int \phi_i^*(r)\phi_j(r)\,\langle\{\alpha_i\}|a_i^\dagger a_j|\{\alpha_i\}\rangle\,dr = \sum_i \langle\{\alpha_i\}|a_i^\dagger a_j|\{\alpha_i\}\rangle =$$

$$= \sum_i |\alpha_i|^2 .$$

$$H = H_0 + V_{int}$$

$$H_0 = \int \psi^+(r) \overbrace{\left(-\frac{\hbar^2}{2m}\nabla^2 + U(r)\right)}^{h_0} \psi(r)\, dr$$

$$V_{int} = \frac{1}{2}\int \psi^+(r')\psi^+(r)\, V\, \psi(r)\psi(r')\, dr\, dr'$$

If we assume that $\{\phi_i\}$ are eigenfunctions of h_0 than

$H_0 = \sum_i \varepsilon_i a_i^+ a_i$. and general interaction is $V_{int} = \sum_{ijkl} V_{ijkl} a_i^+ a_j^+ a_l a_k$.

$$\langle \{\alpha_i\} | H_0 | \{\alpha_i\} \rangle = \sum_i \varepsilon_i |a_i|^2$$

$$\langle \{\alpha_i\} | V_{int} | \{\alpha_i\} \rangle = \sum V_{ijkl}\, \alpha_i^* \alpha_j^* \alpha_l \alpha_k .$$

$$V_{ijkl} = \langle ij | V | kl \rangle .$$

If V is diagonalized ~~with~~ in our basis then $\langle \{\alpha_i\} | V_{int} | \{\alpha_i\} \rangle = \sum V_{ij} |\alpha_i|^2 |\alpha_j|^2$.

Overall:

$$\langle \{\alpha_i\} | H | \{\alpha_i\} \rangle = \sum_i \varepsilon_i |\alpha_i|^2 + \sum_{ijkl} V_{ijkl}\, \alpha_i^* \alpha_j^* \alpha_l \alpha_k .$$

(b) a, a^+ are ladder operators, b, b^+ are creation & annihilation operators.

Both of them have the same commutation relations.

On the other side a, a^+ are operators in a Hilbert space of functions while b, b^+ are operators in Fock-Space.

Another differenca is that b, b^+ generate particles ~~exitatio~~ with certain enrgy while a, a^+ raise (or lower) the energy of one particle. Therefore, when working with s.p. it is easier to use a, a^+ while when working with ~~some~~ many body interactions b, b^+ are a lot more useful.