Q-10

a.

$$\hat{h} = -\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\Omega^2 x^2$$

$$H = \int \psi^\dagger(x)\hat{h}\psi(x)dx + \int \psi^\dagger(x)\psi(x)\frac{1}{2}m\omega^2\left(x-y\right)^2\left(\xi_2^\dagger(y)\xi_1(y) + \xi_1^\dagger(y)\xi_2(y)\right)dxdy$$

write the fermion part as eigenstates of σ_x- $\eta_\pm^\dagger(y) = \frac{1}{\sqrt{2}}\left(\xi_1^\dagger(y) \pm \xi_2^\dagger(y)\right)$, than we guess a state of the form $|\phi_B\rangle \otimes |\phi_F\rangle$ and look at the fermion state of the form $|\phi_F\rangle = \eta_\pm^\dagger(y')\,|0\rangle$, only the interaction part acts on fermions and we can show that

$$\int \psi^\dagger(x)\psi(x)\frac{1}{2}m\omega^2\left(x-y\right)^2\left(\xi_2^\dagger(y)\xi_1(y) + \xi_1^\dagger(y)\xi_2(y)\right)dxdy = \int \psi^\dagger(x)\psi(x)\frac{1}{2}m\omega^2\left(x-y\right)^2\left(\eta_+^\dagger\eta_+ - \eta_-^\dagger\eta_-\right)dxdy$$

need to show anti commutation relation for $\eta_\pm, \eta_\pm^\dagger$.

and than a the states we had for fermions are eigenstates H_{int} such that $H_{int}\eta_\pm^\dagger(y')\,|0\rangle = \pm\eta_\pm^\dagger(y')\,|0\rangle$ so when we operate with H we get:

$$H\,|\phi_B\rangle \otimes \eta_\pm^\dagger(y')\,|0\rangle = \left(\int \psi^\dagger(x)\hat{h}\psi(x)dx \pm \int \psi^\dagger(x)\psi(x)\frac{1}{2}m\omega^2\left(x-y'\right)^2 dx\right)|\phi_B\rangle \otimes \eta_\pm^\dagger(y')\,|0\rangle$$

now we look at the boson part, we can write our hamiltonian after operating on the fermionc part as:

$$\int \psi^\dagger(x)\left(-\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\Omega^2 \pm \frac{1}{2}m\omega^2\left(x-y'\right)^2\right)\psi(x)dx$$

than we can write it as:

$$\frac{1}{2}m\Omega^2 \pm \frac{1}{2}m\omega^2\left(x-y'\right)^2 = \frac{1}{2}m\left(\Omega^2 \pm \omega^2\right)\left(x \pm x_0\right)^2 + C$$

and find the appropriate constants $x_0 = \frac{\omega^2 y'}{\Omega^2 \pm \omega^2}$. Than we can write $\psi(x) = \sum_i \phi_i\left(x \pm x_0\right)\hat{a}_i$ when $\phi_i\left(x \pm x_0\right)$ is the H.Osc. wavefunction around x_0 with energy $\epsilon_i = \hbar\sqrt{\Omega^2 \pm \omega^2}\left(i + \frac{1}{2}\right)$ (pay attention $\Omega^2 \pm \omega^2 > 0$ always!) . And than we can see that the eigen states of the bosonic part are

$$|\phi_B\rangle = \prod_i \frac{\left(a_i^\dagger\right)^{n_i}}{\sqrt{n_i!}}\,|0\rangle$$

when $M = \sum_i n_i$ is the total number of bosons.

b.

now for the fun part, we have:

$$|\phi_B\rangle \otimes |\phi_F\rangle = \prod_i \frac{\left(a_i^\dagger\right)^{n_i}}{\sqrt{n_i!}}\,|0\rangle \otimes A\left\{\eta_\pm^\dagger(y_1)\eta_\pm^\dagger(y_2)...\eta_\pm^\dagger(y_N)\right\}|0\rangle$$

when A is an antisymmetrizer from outer-space. in this case we need to 'drag' the $\eta_\pm(y)$ getting a delta function every time $\delta(y - y_i)$, so after using The Force we get:

$$H\,|\phi_B\rangle \otimes |\phi_F\rangle = \left(\int \psi^\dagger(x)\hat{h}\psi(x)dx + \int \psi^\dagger(x)\psi(x)\sum_{i=1}^{N}\frac{1}{2}m\sigma_i\omega^2\left(x-y_i\right)^2 dx\right)|\phi_B\rangle \otimes |\phi_F\rangle$$

when $\sigma_i = \pm 1$ as the state of the i'th η fermion.(e.g. for $\eta_+(y_3)$ we get $\sigma_3 = 1$). now we need to complete to square again, asumming $\Omega^2 > \sum_i \sigma_i\omega^2$ otherwise we get negative freq. So we get the Hamiltonian after operating on the fermion part:

$$\int \psi^\dagger(x)\left(-\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\Omega^2 + \sum_i \frac{1}{2}m\sigma_i\omega^2\left(x-y_i\right)^2\right)\psi(x)dx = \int \psi^\dagger(x)\left(-\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\left(\Omega^2 + \sum_i \sigma_i\omega^2\right)\left(x-x_0\right)^2 + C\right)\psi(x)dx$$

this time the constants are (i think...):

$$x_0 = \frac{\sum_i \omega^2\sigma_i y_i}{\Omega^2 + \sum_i \sigma_i\omega^2}$$

and than we get the same result as before with this new H. Osc. etc.. ...