Fundamentals of Quantum Technology
Homework Sheet 5

1. Consider the semiclassical Rabi model in the case where $\mathcal{V}$ is real and $\omega = \omega_0$. Recall that, in this case, we saw that the terms of the density matrix follow the equations of motion

$$\dot{\rho}_{11} = \frac{\mathcal{V}}{\hbar}\mathrm{Im}\left[\rho_{21}\right],$$
$$\dot{\rho}_{21} = -i\frac{\mathcal{V}}{\hbar}\left(\rho_{11} - \frac{1}{2}\right).$$

   (a) Use the representation of the density matrix in terms of the expectation values $\langle\hat{\sigma}_z\rangle$ and $\langle\hat{\sigma}_\pm\rangle$ to show that the following equations of motion apply for these expectation values:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{\sigma}_z\rangle = -\frac{i\mathcal{V}}{\hbar}\left[\langle\hat{\sigma}_+\rangle - \langle\hat{\sigma}_-\rangle\right],$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{\sigma}_\pm\rangle = \mp\frac{i\mathcal{V}}{2\hbar}\langle\hat{\sigma}_z\rangle.$$

   (b) When including effects of dissipation, the equations of motion read[1]

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{\sigma}_z\rangle = -\frac{i\mathcal{V}}{\hbar}\left[\langle\hat{\sigma}_+\rangle - \langle\hat{\sigma}_-\rangle\right] - \frac{1}{T_1}\left[\tanh\left(\frac{\beta\hbar\omega_0}{2}\right) + \langle\hat{\sigma}_z\rangle\right],$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{\sigma}_\pm\rangle = \mp\frac{i\mathcal{V}}{2\hbar}\langle\hat{\sigma}_z\rangle - \frac{1}{T_2}\langle\hat{\sigma}_\pm\rangle.$$

   Show that the steady-state solution (when the derivatives of the left hand sides are 0) is

$$\langle\hat{\sigma}_z\rangle = -\frac{\hbar^2\tanh\left(\frac{\beta\hbar\omega_0}{2}\right)}{\hbar^2 + \mathcal{V}^2 T_1 T_2},$$
$$\langle\hat{\sigma}_\pm\rangle = \pm i\frac{\hbar\mathcal{V}T_2\tanh\left(\frac{\beta\hbar\omega_0}{2}\right)}{2\hbar^2 + 2\mathcal{V}^2 T_1 T_2}.$$

   In particular, the off-diagonal terms do not completely vanish in the steady state, and this state still has some coherence.

2. In this exercise we will investigate the dynamics of a two-level system (TLS) that is coupled to a harmonic bath through a Hamiltonian of the form

$$\hat{\mathcal{H}} = \frac{\hbar\omega_0}{2}\hat{\sigma}_z + \hbar\sum_\lambda \Omega_\lambda \hat{B}_\lambda^\dagger \hat{B}_\lambda + \hbar\hat{\sigma}_z\sum_\lambda\left[\mu_\lambda\hat{B}_\lambda^\dagger + \mu_\lambda^*\hat{B}_\lambda\right],$$

   where μ_λ are coupling constants which are assumed to be small. As in class, we will work in the Heisenberg picture.

[1] If you wonder where is the $\pm i\omega_0\langle\hat{\sigma}_\pm\rangle$ term in the second equation, it no longer appears because in the Rabi problem we start from the interaction picture, and treat all the operators that were transformed to the interaction picture as if they were the Schrödinger picture operators. The difference is only a time-dependent phase, so it is just a matter of convenience and does not change the physics.

(a) Show that the equations of motion produced by the Heisenberg equation are

$$\begin{aligned}
\dot{\hat{\sigma}}_z &= 0, \\
\dot{\hat{B}}_\lambda &= -i\Omega_\lambda \hat{B}_\lambda - i\mu_\lambda \hat{\sigma}_z, \\
\dot{\hat{\sigma}}_- &= -i\left\{\omega_0 + \sum_\lambda \left[\mu_\lambda \hat{B}_\lambda^\dagger + \mu_\lambda^* \hat{B}_\lambda\right]\right\}\hat{\sigma}_-.
\end{aligned}$$

The first equation entails, of course, that $\hat{\sigma}_z(t) = \hat{\sigma}_z(0) = |e\rangle\langle e| - |g\rangle\langle g|$. Check that the solution the second equation is then

$$\begin{aligned}
\hat{B}_\lambda(t) &= \hat{B}_\lambda(0)\,e^{-i\Omega_\lambda t} - i\mu_\lambda\hat{\sigma}_z\int_0^t e^{-i\Omega_\lambda(t-\tau)}\mathrm{d}\tau \\
&= \hat{B}_\lambda(0)\,e^{-i\Omega_\lambda t} - \frac{\mu_\lambda}{\Omega_\lambda}\left(1 - e^{-i\Omega_\lambda t}\right)\hat{\sigma}_z.
\end{aligned}$$

(b) Define $\hat{\eta}_-(t) = e^{i\omega_0 t}\hat{\sigma}_-(t)$ and show that

$$\dot{\hat{\eta}}_- = -i\sum_\lambda\left[\mu_\lambda\hat{B}_\lambda^\dagger + \mu_\lambda^*\hat{B}_\lambda\right]\hat{\eta}_-.$$

This transformation simplifies the derivation that follows, and from the solution for $\hat{\eta}_-$ we can easily return to $\hat{\sigma}_-$.

(c) The solution for $\hat{\eta}_-(t)$ can be written as a self-consistent integral expression (which yields the correct equation of motion, as you can easily check),

$$\hat{\eta}_-(t) = \hat{\eta}_-(0) - i\int_0^t \sum_\lambda\left[\mu_\lambda\hat{B}_\lambda^\dagger(\tau) + \mu_\lambda^*\hat{B}_\lambda(\tau)\right]\hat{\eta}_-(\tau)\,\mathrm{d}\tau.$$

Note that $\hat{\eta}_-$ appears also in the integral, so this is not a closed expression. Substituting this solution into $\hat{\eta}_-(\tau)$ in the integral, we obtain

$$\begin{aligned}
\hat{\eta}_-(t) = \hat{\eta}_-(0) &- i\int_0^t \sum_\lambda\left[\mu_\lambda\hat{B}_\lambda^\dagger(\tau) + \mu_\lambda^*\hat{B}_\lambda(\tau)\right]\hat{\eta}_-(0)\,\mathrm{d}\tau \\
&- \int_0^t \sum_\lambda\left[\mu_\lambda\hat{B}_\lambda^\dagger(\tau) + \mu_\lambda^*\hat{B}_\lambda(\tau)\right]\mathrm{d}\tau\int_0^\tau\sum_{\lambda'}\left[\mu_{\lambda'}\hat{B}_{\lambda'}^\dagger(\tau') + \mu_{\lambda'}^*\hat{B}_{\lambda'}(\tau')\right]\hat{\eta}_-(\tau')\,\mathrm{d}\tau'.
\end{aligned}$$

Show that, when substituting the solutions for $\hat{B}_\lambda$ and $\hat{B}_\lambda^\dagger$, keeping terms only up to second order in the coupling constants, and taking again the time derivative of both sides, we arrive at the equation

$$\begin{aligned}
\dot{\hat{\eta}}_- \approx &-i\sum_\lambda\left[\mu_\lambda\hat{B}_\lambda^\dagger(0)\,e^{i\Omega_\lambda t} + \mu_\lambda^*\hat{B}_\lambda(0)\,e^{-i\Omega_\lambda t}\right]\hat{\eta}_-(0) + 2i\sum_\lambda\frac{|\mu_\lambda|^2}{\Omega_\lambda}\left(1 - \cos(\Omega_\lambda t)\right)\hat{\eta}_-(0) \\
&- \sum_{\lambda,\lambda'}\int_0^t \mathrm{d}\tau\left[\mu_\lambda\mu_{\lambda'}\hat{B}_\lambda^\dagger(0)\,\hat{B}_{\lambda'}^\dagger(0)\,e^{i\Omega_\lambda t + i\Omega_{\lambda'}\tau} + \mu_\lambda^*\mu_{\lambda'}^*\hat{B}_\lambda(0)\,\hat{B}_{\lambda'}(0)\,e^{-i\Omega_\lambda t - i\Omega_{\lambda'}\tau}\right]\hat{\eta}_-(0) \\
&- \sum_{\lambda,\lambda'}\int_0^t \mathrm{d}\tau\left[\mu_\lambda\mu_{\lambda'}^*\hat{B}_\lambda^\dagger(0)\,\hat{B}_{\lambda'}(0)\,e^{i\Omega_\lambda t - i\Omega_{\lambda'}\tau} + \mu_\lambda^*\mu_{\lambda'}\hat{B}_\lambda(0)\,\hat{B}_{\lambda'}^\dagger(0)\,e^{-i\Omega_\lambda t + i\Omega_{\lambda'}\tau}\right]\hat{\eta}_-(0).
\end{aligned}$$

(d) Take the lower limit of each integral to $-\infty$ and average over the bath. Show that this leads to

$$\dot{\hat{\eta}}_- \approx 2i\sum_\lambda \frac{|\mu_\lambda|^2}{\Omega_\lambda}\left(1-\cos\left(\Omega_\lambda t\right)\right)\hat{\eta}_-(0)$$
$$-\sum_\lambda |\mu_\lambda|^2 \int_{-\infty}^{t} \mathrm{d}\tau \left[f_{\mathrm{BE}}(\Omega_\lambda)\, e^{i\Omega_\lambda(t-\tau)} + \left(1+f_{\mathrm{BE}}(\Omega_\lambda)\right) e^{-i\Omega_\lambda(t-\tau)}\right]\hat{\eta}_-(0)\,.$$

(e) Moving to a continuous representation of the bath, we can write

$$\dot{\hat{\eta}}_- \approx 2i\int \mathrm{d}\Omega\varrho(\Omega)\frac{|\mu(\Omega)|^2}{\Omega}\left(1-\cos(\Omega t)\right)\hat{\eta}_-(0)$$
$$-\int \mathrm{d}\Omega\varrho(\Omega)\,|\mu(\Omega)|^2 \int_{-\infty}^{t} \mathrm{d}\tau \left[f_{\mathrm{BE}}(\Omega)\, e^{i\Omega(t-\tau)} + \left(1+f_{\mathrm{BE}}(\Omega)\right) e^{-i\Omega(t-\tau)}\right]\hat{\eta}_-(0)\,.$$

Since all terms are already of second order in the coupling constants, we may replace $\hat{\eta}_-(0)$ with $\hat{\eta}_-(t)$ in the right hand side. Use the identity

$$\int_0^\infty \mathrm{d}\tau e^{\pm i\omega\tau} = \pi\delta(\omega) \pm i\mathcal{P}\frac{1}{\omega}$$

to show that

$$\dot{\hat{\eta}}_- = -\left(i\nu(t)+\Gamma\right)\hat{\eta}_-,$$

where $\nu(t)$ is some real function of t that you do **not** need to write down explicitly, and Γ is a decay rate given by

$$\Gamma = \pi\varrho(0)\,|\mu(0)|^2\left(1+2f_{\mathrm{BE}}(0)\right).$$

Of course, $f_{\mathrm{BE}}(\omega)$ diverges at $\omega=0$, so we need to assume that $\rho(0)\,|\mu(0)|^2=0$ and think of the expression for Γ as $\Gamma = \pi\lim_{\Omega\to 0}\varrho(\Omega)\,|\mu(\Omega)|^2\left(1+2f_{\mathrm{BE}}(\Omega)\right)$.

(f) Finally, by ignoring the $\nu(t)$ term, return to $\hat{\sigma}_-$ (and its conjugate $\hat{\sigma}_+$) to obtain the equations of motion

$$\dot{\hat{\sigma}}_\pm = \left(\pm i\omega_0 - \Gamma\right)\hat{\sigma}_\pm.$$

The effect of $\nu(t)$ is simply to slightly shift the frequency ω_0 (a Lamb shift), and we can absorb it into the original definition of ω_0. This result implies that, in the Schrödinger picture, off-diagonal terms of the density matrix decay exponentially with a typical rate of Γ.