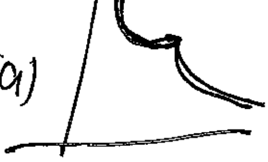
Question 8 outline

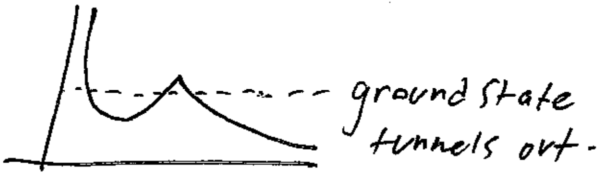
- write $\hat{H}$ in cylindrical coordinates. add $A_\theta(r)$ term.
- use $e^{ik_\varphi \cdot \varphi}$ and $e^{ik_z \cdot z}$ to get rid of those derivatives. we get $\psi = \chi(r) e^{ik_\varphi \varphi} e^{ik_z z}$ and
- $E\chi = (\underset{\downarrow}{\quad}) \chi(r)$
  differential operator (ugly)

- change variables to get from $\frac{1}{r}\partial_r r \partial_r \chi(r)$ to $\frac{\partial^2 u(r)}{\partial r^2}$. solution $\chi = \frac{u}{\sqrt{r}}$
- Shrödinger's equation is $\boxed{\frac{\partial^2 u}{\partial r^2} + V_{eff}(r) u = E u}$
- Show that $V_{eff}(r)$ has a minimum but either:

(a) the minimum is outside R, and there V_{eff} no longe has a minimum.

(b) it has a minimum but even then the lowest point $V_{min} > -\frac{\hbar \omega_{eff}}{2}$ is ~~[illegible]~~ not deep enough and the ground state always has enough energy to tunnel out.

(a) 

(b) 

Question 8

$$H = \frac{P_r^2}{2m} + \frac{\left(P_\theta - \frac{e}{c}A_\theta\right)^2}{2m} + \frac{P_z^2}{2m} \qquad (1)$$

choose $\vec{A}$ only in $\hat{\theta}$ direction, with r dependence:

$$A_\theta(r)\begin{cases} \frac{1}{2}B\cdot r & r<R \\ \frac{1}{2}\frac{BR^2}{r} & r>R \end{cases}$$

$2\pi r\cdot A_\theta = \pi r^2\cdot B$

$2\pi r\cdot A_\theta = \pi R^2 B$

$$\boxed{P_\theta = -i\hbar\frac{\partial}{r\partial\theta}}$$

Use Ampere Law:

$$\oint_C A\,dl = \int \nabla\times A\,dS = \int_S B\,dS$$

$$H = \frac{P_r^2}{2m} + \frac{P_\theta^2}{2m} - 2\frac{e}{c}\frac{P_\theta}{2m}A_\theta + \frac{e^2}{2mc^2}A^2 + \frac{P_z^2}{2m}$$

commutative since $A_\theta(r) \neq A(\theta)$

$$= \underbrace{\frac{P_r^2}{2m} + \frac{P_\theta^2}{2m} + \frac{P_z^2}{2m}}_{} - \frac{e}{mc}AP_\theta + \frac{e^2}{2mc^2}A^2$$

$$= -\frac{\hbar^2\nabla^2}{2m} + i\frac{\hbar e}{mc}\frac{A_\theta}{r}\cdot\partial_\theta + \frac{e^2}{2mc^2}A^2$$

$$= -\frac{\hbar^2}{2m}\left(\frac{1}{r}\partial_r r\,\partial_r + \frac{1}{r^2}\partial_\theta^2 + \partial_z^2\right) + i\frac{\hbar e}{mc}\frac{A}{r}\partial_\theta + \frac{e^2}{2mc^2}A^2.$$

$E\Psi = H\Psi$ set $\Psi = \chi(r) e^{ik_\theta \theta} e^{ik_z z}$

* I will denote $k_\theta \equiv \beta$ and $k_z \equiv K$ *

$$E\chi = \left[-\frac{\hbar^2}{2m}\left(\frac{1}{r}\partial_r r \partial_r - \frac{1}{r^2}\beta^2 - k^2\right) - \beta\frac{\hbar e}{mc}\frac{A}{r} + \frac{e^2}{2mc^2}A^2\right]\chi$$

notice $\frac{1}{r}\partial_r \cdot r \cdot \partial_r \chi$ is not a good kinetic term.

$$\chi = u \cdot r^n \rightarrow \frac{1}{r}\partial_r r \partial_r (u \cdot r^n) = \frac{1}{r}\partial_r r (u' r^n + n u r^{n-1})$$

$$= \frac{1}{r}\partial_r (u' r^{n+1} + n u r^n) = \frac{1}{r}(u'' r^{n+1} + (n+1) u' r^n + n u' r^n + n^2 u r^{n-1})$$

$$= u'' r^n + \underline{(n+1) u' r^{n-1}} + \underline{n u' r^{n-1}} + n^2 u r^{n-2}$$

for this to cancel we need

$$n+1+n = 0 \rightarrow n = -1/2$$

thus

$$\chi(r) = \frac{u(r)}{\sqrt{r}} \Rightarrow \frac{1}{r}\partial_r r \partial_r \chi = \frac{u''}{\sqrt{r}} + \frac{u}{4r^{2.5}} = \frac{1}{\sqrt{r}}\left(u'' + \frac{u}{4r^2}\right)$$

set into $E\chi = \cdots$ get

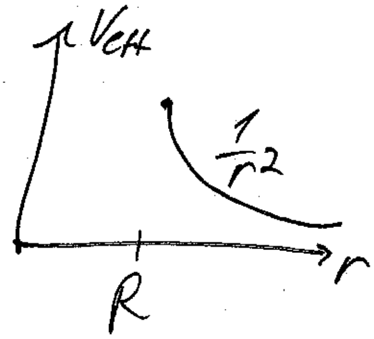
$$E u = \left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{4r^2} - \frac{\beta^2}{r^2} - k^2\right) - \beta\frac{\hbar e}{mc}\frac{A}{r} + \frac{e^2}{2mc^2}A^2\right]u$$

$$E u = -\frac{\hbar^2}{2m}\frac{\partial^2 u}{\partial r^2} + V_{eff}(r)\cdot u$$

$$V_{eff} = -\frac{\hbar^2}{8mr^2} + \frac{\hbar^2\beta^2}{2mr^2} + \frac{\hbar^2 k^2}{2m} - \beta\frac{\hbar e}{mc}\frac{A}{r} + \frac{e^2}{2mc^2}A^2$$

for $r > R$ $\quad \frac{A}{r} = \frac{BR^2}{r^2} \quad A^2 = \frac{B^2R^4}{r^2}$

$$V_{eff}(r>R) = \underbrace{\frac{1}{r^2}\left(-\frac{\hbar^2}{8m} + \frac{\hbar^2\beta^2}{2m} - \beta\frac{\hbar e\cdot BR^2}{mc} + \frac{e^2B^2R^4}{2mc^2}\right)}_{\sim\frac{1}{r^2}\text{ term}} + \underbrace{\frac{\hbar^2k^2}{2m}}_{\text{const}}$$



for $r < R$ $\quad \frac{A}{r} = B \quad A^2 = B^2 r^2$

$$\boxed{\omega_c = \frac{eB}{mc}}$$

$$V_{eff}(r<R) = \underbrace{-\frac{\hbar^2}{8mr^2} + \frac{\hbar^2\beta^2}{2mr^2}}_{\sim\frac{1}{r^2}\text{ term}} + \underbrace{\frac{\hbar^2k^2}{2m} - \beta\hbar\frac{eB}{mc}}_{\text{const term}} + \underbrace{\frac{1}{2m}\frac{e^2B^2}{c^2}r^2}_{\sim r^2\text{ term}}$$

$$= \frac{1}{r^2}\left(-\frac{\hbar^2}{8m} + \frac{\hbar^2\beta^2}{2m}\right) + \frac{\hbar^2k^2}{2m} - \hbar\beta\omega_c + \underbrace{\frac{1}{2}m\omega_c^2 r^2}_{\text{h.o. term!}}$$

find minimum of $V_{eff}(r<R)$: ⑤

$$\frac{\partial V}{\partial r} = -\frac{\hbar^2}{r^3}\left(\frac{\beta^2}{2m} - \frac{1}{8m}\right) + m\omega_c^2 r = 0$$

$$\frac{\hbar^2}{mr^3}\left(\beta^2 - \frac{1}{4}\right) = m\omega_c^2 r$$

$$r^4 = \frac{\hbar^2}{m^2\omega_c^2}\left(\beta^2 - \frac{1}{4}\right) \quad r_{min} = \sqrt[4]{\frac{\hbar^2}{m^2\omega_c^2}\left(\beta^2 - \frac{1}{4}\right)}$$

$$V_{eff}(r_{min}) = \frac{1}{\sqrt{\frac{\hbar^2}{m^2\omega_c^2}\left(\beta^2 - \frac{1}{4}\right)}}\left(\beta^2 - \frac{1}{4}\right)\frac{\hbar^2}{2m} + \frac{\hbar^2 k^2}{2m} - \hbar\beta\omega_c + \frac{1}{2}m\omega_c^2\sqrt{\frac{\hbar^2}{m^2\omega_c^2}\left(\beta^2 - \frac{1}{4}\right)}$$

$$= \sqrt{\beta^2 - \frac{1}{4}}\,\frac{\hbar\omega_c}{2} + \frac{\hbar^2 k^2}{2m} - \hbar\beta\omega_c + \frac{\hbar\omega_c}{2}\sqrt{\beta^2 - \frac{1}{4}} =$$

$$= \frac{\hbar^2 k^2}{2m} - \hbar\beta\omega_c + \hbar\omega_c\sqrt{\beta^2 - \frac{1}{4}}$$

if $k=0$ and $\beta > 0$ we might get a negative potential. But how deep?

$$V_{min} = -\hbar\omega_c \cdot \underset{\substack{\uparrow \\ \text{integer}}}{\beta} + \hbar\omega_c\sqrt{\beta^2 - \frac{1}{4}}$$

integer (from $e^{i\beta\theta}$ wavefunction periodic boundary conditions)

$$\frac{\partial^2 V_{eff}}{\partial r^2} = m\omega_c^2 + \frac{3\hbar^2}{m r^4}\left(\beta^2 - \frac{1}{4}\right)\Big|_{r=r_{min}} =$$

$$= m\omega_c^2 + \frac{3\hbar^2}{m}\left(\beta^2 - \frac{1}{4}\right) \Big/ \frac{\hbar^2}{m^2\omega_c^2}\left(\beta^2 - \frac{1}{4}\right) = 4m\omega_c^2$$

therefore $m\omega_{eff}^2 = 4m\omega_c^2$ in V_{eff}

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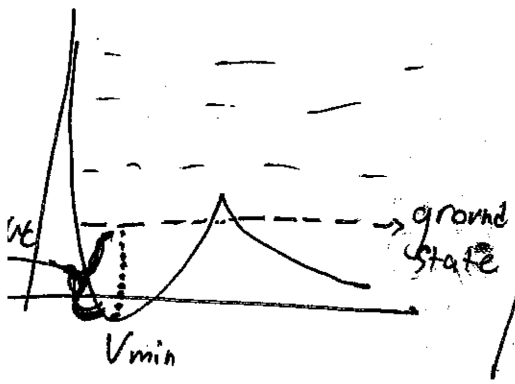
this means the ground state (in h.o. approximation) will have energy $E_0 \sim 2\hbar\omega_c$

we saw before that

$V_{min} = \hbar\omega_c\left(-\beta + \sqrt{\beta^2 - \frac{1}{4}}\right)$. for $\beta = 0$ there is no minimum



for $\beta < 0$ V_{min} is positive

for $\beta = 1$, $V_{min} = \hbar\omega_c\left(\sqrt{\frac{3}{4}} - 1\right)$

but still not deep enough because

$2\hbar\omega_c > \left|\hbar\omega_c\left(1 - \sqrt{\frac{3}{4}}\right)\right|$

for $\beta > 1$ the potential well is negative but getting smaller as $\beta \gg 1$, $\sqrt{\beta^2 - \frac{1}{4}} \simeq \beta$. Thus we always get ~~ground state energy~~ decay into a continuum.