$$\langle\psi|\rho|\psi\rangle = \left(\langle\psi_1|\alpha^* + \sqrt{1-|\alpha|^2}\,\langle\psi_2|\right)\left(w_1|\psi_1\rangle\langle\psi_1| + w_2|\psi_2\rangle\langle\psi_2|\right)\left(|\psi_1\rangle\alpha + \sqrt{1-|\alpha|^2}\,|\psi_2\rangle\right)$$

$$= |\alpha|^2 w_1 + (1+|\alpha|^2)w_2 = |\alpha|^2(w_1 - w_2) + w_2$$

ρ is mixed. For $|\alpha| = 1, 0$ we are in a pure state $|\psi_1\rangle, |\psi_2\rangle$ respectively, and for $|\alpha| = \frac{1}{2}$ we are in the maximaly mixed state.

b) Consider a linear combination $|\psi >= a|\psi_1 > +b|\psi_2 >$ and compare it with the density matrix $\rho = w_1|\psi_1 >< \psi_1| + w_2|\psi_2 >< \psi_2|$ above with $w_1 = |a|^2$ and $w_2 = |b|^2$. Do they describe the same state? What are the similarities? The differences? Explain in as many details as you can. Explore physically interesting limits of your answers.

$$|\psi\rangle\langle\psi| = \begin{pmatrix} |a|^2 & ab^* \\ a^*b & |b|^2 \end{pmatrix} \neq \rho = \begin{pmatrix} |a|^2 & 0 \\ 0 & |b|^2 \end{pmatrix}$$

$\hookrightarrow$ pure $\qquad\qquad$ $\swarrow$ mixed

$\swarrow$

by defenition ρ is unseperable and therefor mixed.

For $a = 1$, $b = 0$ or $b = 1$, $a = 0$ they are the same

For $a = b = \frac{1}{\sqrt{2}}$ it is the maximaly mixed state.

Here the off-diagonal elements of $|\psi\rangle\langle\psi|$

represent the fact that theres a phese relationship between $|\psi_1\rangle$ and $|\psi_2\rangle$. In contrast with ρ where there is none, there the system behaves as classical probabilities. Therefor in $|\psi\rangle\langle\psi|$ we can observe interferance between $|\psi_1\rangle$ and $|\psi_2\rangle$ and in ρ we can not.

② $$\rho = \sum_{lk} \rho_{nl;nk}\, \phi_{nl}\, \phi^{*}_{nk}$$

without degeneracies

$$\begin{pmatrix} e^{-\beta E_1} & & \\ & e^{-\beta E_2} & \\ & & \ddots \end{pmatrix}$$

with degeneracies

$$\begin{pmatrix} \begin{pmatrix} e^{-\beta E_1} & \# \\ \# & e^{-\beta E_1} \\ & & \ddots \end{pmatrix} & \#\# \\ \#\# & \begin{pmatrix} e^{-\beta E_2} & \# \\ \# & e^{-\beta E_2} \\ & & \ddots \end{pmatrix} \\ & & \ddots \end{pmatrix}$$

degenerate sub-space

diagonal block matrix (usually finite degeneracies)

T_1 wont change because the diagonal elements are basically the same (there are just more of them). T_2 is mor interesting we need to consider off diagonal elements within the same degenerate sub-space #, and off diagonal elements outside this subspace ##. Each sub-space is a fully mixed density matrix, therefore # = 0. As to ##, they can still hold any value and their phase is still ditermened by $\Delta_{nm} = E_n - E_m$ and by the same T_2.

③ $\prod_{\substack{N\\ \text{terms}}} 1 - i\varepsilon\hat{H}/\hbar - \frac{1}{2}(\varepsilon\hat{H}/\hbar)^2 \ldots$

With multiplying N terms of order $\mathcal{O}(\varepsilon)$ you get $\mathcal{O}(\varepsilon^N)$ and when you take $N\to\infty$ you get all orders.

Lets take just the first two terms for one sec:

$$N=2$$

$$\left[1 - i\varepsilon\hat{H}/\hbar - \frac{1}{2}(\varepsilon\hat{H}/\hbar)^2\right]\left[1 - i\varepsilon\hat{H}/\hbar - \frac{1}{2}(\varepsilon\hat{H}/\hbar)^2\right]$$

$$1 - i\varepsilon\hat{H}/\hbar - \frac{1}{2}(\varepsilon\hat{H}/\hbar)^2$$
$$-i\varepsilon\hat{H}/\hbar - (\varepsilon\hat{H}/\hbar)^2 + \frac{i}{2}(\varepsilon\hat{H}/\hbar)^3$$
$$-\frac{1}{2}(\varepsilon\hat{H}/\hbar)^2 + \frac{i}{2}(\varepsilon\hat{H}/\hbar)^3 + \frac{1}{4}(\varepsilon\hat{H}/\hbar)^4$$

We see that we get terms of the type $N^\alpha\varepsilon^k$. In the terms where $k\geq 2$ we get $\alpha\leq 2$ and therefor the term goes to zero when $\varepsilon\to 0, N\to\infty$ (they go at the same rate). And we are left wiht just

$[1 - i\varepsilon\hat{H}/\hbar]$.

b) $= U = e^{-iN\varepsilon\hat{H}/\hbar} = e^{-i\sum_{N\,\text{times}} \varepsilon\hat{H}/\hbar} = e^{-i\frac{1}{N}\sum_{N\,\text{times}} (t_f - t_i)\hat{H}/\hbar}$

$= e^{-i\int_{t_i}^{t_f} \hat{H}/\hbar \, dt}$

c) The form $[1 - i\hat{H}(t)\varepsilon/\hbar]$

because in the integral form

it is not clear at which time interval

$\hat{H}(t)$ acts in. Only in this form

we can make sure we dont get into

truble with the uncommutativety of $H(t_1)$ and

$H(t_2)$. If we want to use the integral

form we must use the $\hat{T}$, which will make sure we out with $H(t_i)$ in the correct order

and not get into truble.