Quantum Mechanics 1 - HW 4

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1 Question 1

The Hamiltonian for the electromagnetic (free) field is:

$$H = \frac{1}{2}\int d^3r \left[\vec{E}_T^2 + (\vec{\nabla} \times \vec{A}_T)^2\right] \tag{1}$$

Where we are considering only the transverse fields. We are also not going to discuss the charges as we are dealing specifically with the meaning of fields in terms of the photons.

We first go into momentum space so we can get rid of the curl of $\vec{A}$.

$$H = \frac{1}{2}\int d^3k \left[E_k^2 + k^2 A_k\right] \tag{2}$$

In the second term we have replaced the cross product $\vec{k} \times \vec{A}$ with a simple product since A is transverse[1].

In this space, for each momentum we have a Hamiltonian density:

$$\mathcal{H} = \frac{1}{2}(E_k^2 + k^2 A_k) \quad \rightarrow \quad \hat{\mathcal{H}} = \frac{1}{2}(\hat{P}^2 + \omega^2 \hat{Q}^2) \tag{3}$$

We have rewritten the Hamiltonian density in terms of the canonical momentum and position (these are not the physical momentum or position of the photon but only mathematical analogues of these quantities in the harmonic oscillator case). The frequency is simply $\omega = |k|$, which is the dispersion relation of a massless particle (when proper constants are returned).

It is easy to show that the commutation of $\hat{Q}$ and $\hat{P}$ are canonical quantum mechanical commutation relations, derived from the commutation relation of the field operators.

The single photon has a wavefunction of the h.o. type (Gaussian times Hermite polynomial) *in the space of Q and P*. This means that each photon has undetermined $\hat{Q}$ and $\hat{P}$, just like a quantum h.o. has undetermined position and momentum. Thus we can only discuss the mean field, the probability density of finding the photon as having a specific field value and the variance of the field.

The probability density of finding an electric field value $\hat{P}$ when considering one photon with momentum k is given by:

$$\rho(P) = \langle P \mid 1\rangle \propto e^{-P^2/2\omega} \mathrm{H}_1(P/\sqrt{2\omega}) \tag{4}$$

Where we have used the standard wavefunction of the h.o. in the first excited state (in P space representation).

Thus the fields $\vec{E}$ and $\vec{A}$ have probability amplitude (wavefunction) distributions:

$$\rho(E) = \sqrt[4]{\frac{\omega}{\pi}} e^{-\frac{E^2}{2\omega}} \cdot \frac{E}{\omega} \quad \text{and} \quad \rho(A) = \sqrt[4]{\frac{\omega}{\pi}} e^{-\frac{\omega A^2}{2}} \cdot \omega A \tag{5}$$

Notice that for a single photon, in Fourier space, the vector potential is $A = kB$ where $k = \omega$ so that the wavefunction distributions for E and B are

[1] I am using k to denote momentum, or mode specifier of the field, even though it is a continuous valued variable. I would prefer to use p but I am making an effort not to confuse it with P which is coming up as canonical momentum.

the same (up to factors of c which we do not include here) which is what we expect from Maxwell's equation for radiation.

A few words on polarization. We have used the fields above without referring to their vector nature. This is because we assume the photon is moving in on direction (assume z direction), and that the polarization is in another direction (either x or y for the electric field, and the other axis for the magnetic). Thus we can calculate the distribution of E on one axis and treat it as a scalar. We can generalize this to a photon in any direction or any mixed polarization, when a proper rotation of our axis will return to this same result.

If the photon is circularly polarized, we can treat each of the fields (lets take the electric field) as having the same distribution (5) in either axis, where the time dependence of the wavefunction would have a phase seperation of $\frac{\pi}{2}$. The oscillations of the electric field wavefunction are out of phase, which might affect phase sensitive interactions (like interactions with matter) but when we measure the electric field the probability in either axis is unaffected by phase, so the results would be the same. So any kind of polarization just requires a superposition of two orthogonal distributions, one for E_x and one for E_y, with any complex valued coefficients to each component.

The mean electric field of a single photon, is calculated by:

$$\langle 1 \ |\hat{P}| \ 1\rangle = \langle 1 \ |i\sqrt{\frac{\omega}{2}}(\hat{a} - \hat{a}^\dagger)| \ 1\rangle = 0 \tag{6}$$

As expected, as the field is spread out evenly its average value would vanish. The mean size of the field is gained by calculating the square of $\hat{P}$:

$$\langle 1 \ |\hat{P}^2| \ 1\rangle = \frac{3}{2}w \tag{7}$$

The field size is dependent on the frequency, which is proportional to the momentum of the photon. This is what we expect since we know photons with high energy have high frequency (the two are the same up to $\hbar$).

We get similar results for the vector potential, in the form of the canonical $\hat{Q}$:

$$\langle 1 \ |\hat{Q}^2| \ 1\rangle = \frac{3}{2}\frac{1}{\omega} \tag{8}$$

Since we still need to use the curl of $\vec{A}$ to get $\vec{B}$ and in the mode space that means multiplying by k, we would get:

$$\langle B^2\rangle = \langle k^2 A^2\rangle = k^2\langle Q^2\rangle = \omega^2\frac{3}{2}\frac{1}{\omega} = \frac{3}{2}\omega \tag{9}$$

Which is what we get from classical electrodynamics, that the field strengths are proportional at all times (equal up to factors of c).

2 Question 2

2.1 Setting up the system

The system is made out of two charges q_1 and q_2, that are free to rotate in the XY plane in a radius $R = L/2$ around the origin. The potential energy for moving the charges in any way other than rotational is so high these degrees of freedom never become excited. Thus we have an effective dependence of the wavefunction only on ϕ. The Shrödinger equation becomes:

$$E\psi = -\frac{\hbar^2}{2\mu}\nabla^2\psi = \frac{L_z^2}{2I}\psi(\phi) \tag{10}$$

Using the moment of inertia of the system:

$$I = 2m_q R^2 \tag{11}$$

and the angular momentum operator:

$$L_z = -i\hbar\frac{\partial}{\partial\phi} \tag{12}$$

The solution to the differential equation is simply:

$$\psi(\phi) = \tfrac{1}{\sqrt{2\pi}}e^{im\phi} \quad \text{with} \quad m = 0, \pm1, \pm2, \dots \tag{13}$$

The $m = 0$ is the groundstate where the wavefunction is just constant in space.

The wavefunction describes the position of one of the particles, where the position of the other is dependent on the first at all time (since there is only one degree of freedom we need only one wavefunction). The operators $\hat{\phi}$ and $\hat{L}_z$ are the canonical variables, where $\hat{\phi}$ is the operator of the position angle (using the regular $\hat{\phi}|\ \phi\rangle = \phi|\ \phi\rangle$) and the angular momentum replaces the regular momentum as

$$\hat{P}_\phi = -i\hbar\frac{\partial}{\partial\phi} = -i\hbar\frac{1}{R}\frac{\partial}{\partial\phi} = \frac{\hat{L}_z}{R} \tag{14}$$

Also works on the wavefunction of the single rod (rather than each particle separately).

2.2 The interaction

The interaction Hamiltonian is given by:

$$H_{\text{int}} = -\frac{e}{c}\int d^3r\ \vec{j}(\vec{r})\cdot\vec{A}(\vec{r}) \tag{15}$$

Or, in plane wave decomposition:

$$\hat{H}_{\text{int}} = -\frac{e}{c}\int d^3r \sum_{\vec{k},\vec{\lambda}}\left[\hat{A}_{\vec{k},\vec{\lambda}}\ \vec{j}(\vec{r})\cdot\vec{\lambda}e^{i\vec{k}\cdot\vec{r}} + \hat{A}^{\dagger}_{\vec{k},\vec{\lambda}}\ \vec{j}(\vec{r})\cdot\vec{\lambda}^{*}e^{-i\vec{k}\cdot\vec{r}}\right] \tag{16}$$

The $A_{\vec{k},\vec{\lambda}}$ elements allow upward transitions (and annihilation of a photon) while their conjugate is the downward transition (with a creation or emission of a photon).

We will assume the rotor is absorbing a photon so we can leave out the creation operator term, and get the matrix element:

$$\begin{aligned}\langle n;0,\dots,N_{\vec{k},\vec{\lambda}},\dots\ |\hat{H}_{\text{int}}|\ 0;0,\dots,N_{\vec{k},\vec{\lambda}},\dots\rangle = \\ = \sqrt{\frac{2\pi\hbar e^2}{\omega\Omega}}\langle N_{\vec{k},\vec{\lambda}}-1\ |\hat{a}_{\vec{k},\vec{\lambda}}|\ N_{\vec{k},\vec{\lambda}}\rangle\langle n\ |\int d^3r\left(\vec{j}(\vec{r})\cdot\vec{\lambda}\ e^{i\vec{k}\cdot\vec{r}}\right)|\ m\rangle\end{aligned} \tag{17}$$

Taking the decomposition of the current as:

$$\begin{aligned}\vec{j}(\vec{r}) = j_\phi(\phi) = \frac{eR}{I}\frac{1}{2}\Big[\hat{L}_z\delta(\phi-\hat{\phi}) + \delta(\phi-\hat{\phi})\hat{L}_z \\ \pm\ \Big(\hat{L}_z\delta(\phi-(\hat{\phi}+\pi)) + \delta(\phi-(\hat{\phi}+\pi))\hat{L}_z\Big)\Big]\end{aligned} \tag{18}$$

The delta function measures the position of the edge of the rod (as defined in the previous page) and the $\pm$ sign separates the two cases where the charges are of the same sign $(+)$ and opposite sign $(-)$. We add π to the operator $\hat{\phi}$ since it measures the position of the end of the rod and then adds 180° to give the position of the second particle. Notice we have used

$$\frac{p_\phi}{\mu} = \frac{L_z}{\mu R} = \frac{RL_z}{I} \tag{19}$$

Where μ is the system mass, and I the moment of inertia.

Looking only at the $\langle n\ |\dots|\ m\rangle$ expression in (17) we can act the angular momentum operators to either side, acting on the states n and m:

$$\begin{aligned}\langle n\ |\int d^3r\ \vec{j}(\vec{r})\cdot\vec{\lambda}\ e^{i\vec{k}\cdot\vec{r}}|\ m\rangle = \int d^3r\lambda_\phi\Big(\langle n\ |(-n\hbar)\big(\delta(\phi-\hat{\phi})\pm\delta(\phi-(\hat{\phi}+\pi))\big)+ \\ +\big(\delta(\phi-\hat{\phi})\pm\delta(\phi-(\hat{\phi}+\pi))\big)m\hbar|\ m\rangle\Big)e^{i\vec{k}\cdot\vec{r}} = \\ = \hbar(m-n)\int d^3r\lambda_\phi\Big(\delta(\phi-\hat{\phi})\pm\delta(\phi-(\hat{\phi}+\pi))\Big)e^{i\vec{k}\cdot\vec{r}}\end{aligned} \tag{20}$$

Now we can insert a position angle basis 'bra' and 'ket':

$$\int d^3r\lambda_\phi e^{i\vec{k}\cdot\vec{r}} \int d\phi' \langle n \mid \phi' \rangle \langle \phi' \mid \big(\delta(\phi - \hat{\phi}) \pm \delta(\phi - (\hat{\phi} + \pi))\big) \mid m \rangle \tag{21}$$

Now we can act the position angle operator on the left and get:

$$\int d^3r\lambda_\phi e^{i\vec{k}\cdot\vec{r}} \int d\phi' \langle n \mid \phi' \rangle \big(\delta(\phi - \phi') \pm \delta(\phi - (\phi' + \pi))\big) \langle \phi' \mid m \rangle \tag{22}$$

Using the wavefunction we defined earlier we get:

$$\int d^3r\lambda_\phi e^{i\vec{k}\cdot\vec{r}} \int d\phi' e^{-in\phi'} \big(\delta(\phi - \phi') \pm \delta(\phi - (\phi' + \pi))\big) e^{im\phi'} \tag{23}$$

And, finally, contracting the integration with the delta functions:

$$\int d^3r\lambda_\phi e^{i\vec{k}\cdot\vec{r}} \Big(e^{i(m-n)\phi} \pm e^{i(m-n)(\phi-\pi)}\Big) \tag{24}$$

Which brings us to some important results. If the particles are of opposite sign there are some values of $q \equiv m - n$ which give a null matrix element. For odd q the factor π in the exponent cancels the minus of opposite signs and we get a transition. For same sign charges we need to have even valued q or there is no transition. Thus the selection rules are:

$$\begin{cases} \text{same sign charges} & q \quad \text{odd} \\ \text{opposite sign charges} & q \quad \text{even} \end{cases} \tag{25}$$

Charges of opposite sign will therefore have an absorption amplitude for dipole transitions (where the angular momentum changes by one or any higher odd number) while the same charge system would only have transitions of quadropole or higher, since it can change only even number of angular momentum. Thus we need to solve the following integral:

$$V(\vec{k}, \vec{\lambda}) \equiv \int d^3r\lambda_\phi e^{i(\vec{k}\cdot\vec{r} + q\phi)} \tag{26}$$

Upon insertion of the parameters of the incoming photon we can integrate over ϕ and get a matrix element V to use in the following section.

2.3 Fermi Golden Rule

The total absorption is calculated by Fermi Golden Rule:

$$\Gamma = \frac{2\pi}{\hbar}|\langle n\ |H_{\text{int}}|\ m\rangle|^2\delta(\varepsilon_n - \hbar\omega_k - \varepsilon_m) \tag{27}$$

Using the matrix element in (17) and the delta function that restricts our k to allowed energy values for the transition.

For photons with the correct energy, the absorption rate is:

$$\Gamma_k = \frac{\pi^2\hbar^2R^2e^4N_{\vec{k},\vec{\lambda}}}{2\omega\Omega I^2}|V|^2 \tag{28}$$

Where the V term holds the dependence on the direction and size of $\vec{k}$ and the orientation of $\vec{\lambda}$ relative to the rotor.

2.4 Parametrization of $\vec{k}$ and $\vec{\lambda}$

Since the size of $\vec{k}$ is determined by the energy, it can be expressed by two angles, the azimuthal θ and equatorial φ:

$$\vec{k}(\varphi,\theta) = \begin{cases} k_x = k_0\sin\theta\cos\varphi \\ k_y = k_0\sin\theta\sin\varphi \\ k_z = k_0\cos\theta \end{cases} \tag{29}$$

And, similarly, we can express the vector $\vec{\lambda}$ as one parameter, since its size is $|\vec{\lambda}| = 1$ and we require orthogonality with $\vec{k}$. Defining the angle α between the $\vec{\lambda}$ and the plane of the z axis and $\vec{k}$:

$$\vec{\lambda}(\alpha) = \begin{cases} \lambda_x = -\cos\theta\cos\varphi\cos\alpha + \sin\varphi\sin\alpha \\ \lambda_y = -\cos\theta\sin\varphi\cos\alpha - \cos\varphi\sin\alpha \\ \lambda_z = \sin\theta\cos\alpha \end{cases} \tag{30}$$

It is easy to verify that indeed $\vec{k}\cdot\vec{\lambda} = 0$ and $|\vec{\lambda}| = 1$ using this parametrization.

Now we need to take the two vectors and find their components in the ϕ and R directions. By projecting λ_x and λ_y on the ϕ direction we get:

$$\lambda_\phi = \cos\theta\cos\alpha\sin(\phi-\varphi) - \sin\alpha\cos(\phi-\varphi) \tag{31}$$

Using

$$\vec{r} = x\hat{x} + y\hat{y} \tag{32}$$

We can calculate the scalar product $\vec{k}\cdot\vec{r}$ as a function of the angles ϕ,θ,φ:

$$\vec{k}\cdot\vec{r} = kR\sin\theta\cos(\phi-\varphi) \tag{33}$$

Recalling that the angles φ and θ are the parameters of the incoming photon momentum, the angle α is for the polarization relative to it, and the angle ϕ is the coordinate describing the orientation of the rod.

2.5 The integral for the matrix element

$$V(\theta,\varphi,\alpha)=\int d^3r\lambda_\phi e^{i(\vec{k}\cdot\vec{r}+q\phi)} \tag{34}$$

Since $\vec{k}\cdot\vec{r}$ is given by (33) and λ_ϕ is given by (31) we can write[2]:

$$\begin{aligned}V=R\int_{-\pi}^{\pi}d\phi\Big(\cos\theta\cos\alpha\sin(\phi-\varphi)-\sin\alpha\cos(\phi-\varphi)\Big)\\ \times\exp\Big[i\big(kR\sin\theta\cos(\phi-\varphi)+(m-n)\phi\big)\Big]\end{aligned} \tag{35}$$

First of all notice the angle φ appears in almost all places in the combination $(\phi-\varphi)$, where ϕ is being integrate over the entire circle. The last term in the exponent can be completed into the same form, by adding a total phase (which is irrelevant). So it is clear that there is no dependence of the result on the φ orientation of the photon.

So we can write $(\phi-\varphi)\to\phi$ and also the shorthands $\xi=kR\sin\theta$ and $q=m-n$ we can reduce the integral to:

$$V=R\int_{-\pi}^{\pi}d\phi\Big(\cos\theta\cos\alpha\sin\phi-\sin\alpha\cos\phi\Big)e^{i(\xi\cos\phi+q\phi)} \tag{36}$$

Which still does not look soluble but does remind of the Bessel integral

$$J_q(\xi)=\int_{-\pi}^{\pi}e^{-i(q\phi-\xi\sin\phi)}d\phi \tag{37}$$

So we translate the angle ϕ once more, this time by adding 90° so that $\sin\phi\to\cos\phi$ and $\cos\phi\to-\sin\phi$ with a factor of phase that changes nothing in the result.

$$V=R\int_{-\pi}^{\pi}d\phi\Big(\cos\theta\cos\alpha\cos\phi+\sin\alpha\sin\phi\Big)e^{-i(q\phi-\xi\sin\phi)} \tag{38}$$

Thus we can evaluate each term in the brackets by changing:

$$\cos\phi=\frac{e^{i\phi}+e^{-i}}{2}\quad\text{and}\quad\sin\phi=\frac{e^{i\phi}-e^{-i\phi}}{2i} \tag{39}$$

and joining the exponential terms (getting once $q+1$ and once $q-1$):

$$\begin{aligned}V=R\int d\phi\Big[\cos\theta\alpha\Big(e^{-i\left[(q+1)\phi-\xi\sin\phi\right]}+e^{-i\left[(q-1)\phi-\xi\sin\phi\right]}\Big)+\\ +\sin\alpha\Big(e^{-i\left[(q+1)\phi-\xi\sin\phi\right]}-e^{-i\left[(q-1)\phi-\xi\sin\phi\right]}\Big)\Big]\end{aligned} \tag{40}$$

[2]Referring to the definition of the current operator (18) we should have included terms of the form $\delta(r-R)$ and $\delta(z)$ to express the fact that the current is located on the edge of the rotor's trajectory. Thus the d^3r integration becomes $Rd\phi$ and the volume element Ω needs to be replaced with πL the length of the ring in which the current is localized.

Upon integration we get:

$$V = R\cos\theta\cos\alpha\Big(J_{q+1}(\xi) + J_{q-1}(\xi)\Big) - iR\sin\alpha\Big(J_{q+1} - J_{q-1}(\xi)\Big) \tag{41}$$

We can rearrange this and write our shorthands explicitly:

$$\begin{aligned} V = {} & RJ_{m-n+1}(kR\sin\theta)\big(\cos\theta\cos\alpha - i\sin\alpha\big) + \\ & + RJ_{m-n-1}(kR\sin\theta)\big(\cos\theta\cos\alpha + i\sin\alpha\big) \end{aligned} \tag{42}$$

2.6 Comments

The transitions when the charges have the same sign are allowed only when the angular momentum changes by two (or higher even numbers) so the transition to the nearest level is prohibited for this system. To asses which transition is stronger we need to look at some limits.

For a photon coming in the z direction the exponential term $e^{i\vec{k}\cdot\vec{r}}$ is unity and we do not need to consider the long wavelength limit. Since $\theta = 0$ in this limit we get, taking only the nearest level (in the cases of opposite charges):

$$V = -iR\sin\alpha(J_0(0) - J_2(0)) \tag{43}$$

But the only Bessel function to survive around $\xi = 0$ is the zero order one, which we can only get when discussing the transition between adjacent levels. Moreover, the expression for Γ is dependent on ω^{-1} so the closer the levels are, the stronger the transition (this is usually the case!). As we increase θ we see the Bessel functions all get comparable values, and a more rigorous analysis is required to check that this is indeed the strongest transition at all points.

When we check the long wavelength limit we can say that $kR \ll 1$ and the argument of the Bessel functions ξ goes to zero, and we get similar results (although there is still dependence on θ).

The case where the two charges have the same sign allows only for using $J_1(\xi)$ or higher order Bessel functions, all of which which have small contributions when $\xi \ll 1$. So the transitions for the equal sign charges are weak for the cases where either θ is small or when the wavelength is large compared with R. We might find some cases where the wavelength is short and the angle is right for a stronger transition for this system but most likely that the opposite charges, that have a strong dipole transition will be much more prominent.

If the photon is unpolarized we can say the transition will be equal to the average of transitions for two orthogonal polarization states (does not matter which two so long as they are mutually orthogonal). See discussion at the end of question 3.

3 Question 3

The electron is set in a tightly bound state of the atom, and put under a static uniform magnetic field B_0. This creates an energy difference between the two possible spin states of the electron:

$$\Delta E = g \cdot \mu_B B_0 \quad \text{with} \quad \mu_B = \frac{e\hbar}{2m_e c} \quad \text{and} \quad g \approx 2 \tag{44}$$

Assuming the incoming photon affects only the spin of the electron and not the energy level it is in (which happens when the field B_0 is weak and thus the level spacing of the atomic orbitals are much larger than the spin up - spin down level spacing) we can calculate the action of the magnetic field operator on the spin states only.

The transition rate is:

$$\begin{aligned}\Gamma = &\frac{2\pi}{\hbar}\delta(\Delta E - \hbar\omega_k)\times \\ &\times \langle \dots N_{\vec{k},\vec{\lambda}} - 1 \dots; n', \ell', m', m_s; \mid H_{\text{int}} \mid \dots N_{\vec{k},\vec{\lambda}} \dots; n, \ell, m, m_s\rangle\end{aligned} \tag{45}$$

Where the interaction Hamiltonian works only on the photon state space and the spin state space, not on the electron orbit states. Since the magnetic field operator is given by[3]:

$$\hat{\vec{B}} = i\sqrt{\frac{2\pi\omega}{\Omega}} \sum_{\vec{k},\vec{\lambda}} (\vec{k} \times \vec{\lambda}) e^{i\vec{k}\cdot\vec{r}} (\hat{a}_{\vec{k},\vec{\lambda}} - \hat{a}^\dagger_{\vec{k},\vec{\lambda}}) \tag{46}$$

and the interaction Hamiltonian is

$$H_{\text{int}} = g\frac{\mu_B}{\hbar}\vec{S} \cdot \vec{B} \tag{47}$$

We can write the matrix element as:

$$\begin{aligned}&\langle \dots N_{\vec{k},\vec{\lambda}} - 1 \dots; n', \ell', m', m'_s \mid H_{\text{int}} \mid \dots N_{\vec{k},\vec{\lambda}} \dots; n, \ell, m, m_s\rangle \\ &= \langle N_{\vec{k},\vec{\lambda}} - 1 \mid \hat{a}_{kl} \mid N_{kl}\rangle \frac{ig\mu_B}{\hbar}\sqrt{\frac{2\pi\omega}{\Omega}} \langle m'_s \mid (\vec{k} \times \vec{\lambda}) \cdot \vec{S} \mid m_s\rangle \langle n', \ell', m' \mid n, \ell, m\rangle e^{i\vec{k}\cdot\vec{r}}\end{aligned} \tag{48}$$

We have taken all terms outside of the n, ℓ, m bracket since they do not depend on the orbital part at all. The combination

$$\langle n', \ell', m' \mid n, \ell, m\rangle = 1 \tag{49}$$

through normalization and also

$$\langle \dots, N_{\vec{k},\vec{\lambda}} - 1, \dots \mid \hat{a}_{kl} \mid \dots, N_{kl}, \dots\rangle = \sqrt{N_{\vec{k},\vec{\lambda}}} \tag{50}$$

[3]We are going to use the annihilation operator term only, since we are discussing the absorption of a photon.

So we can focus on the matrix element:

$$\langle m_s' \,|(\vec{k}\times\vec{\lambda})\cdot\vec{S}\mid m_s\rangle = \langle m_s' \mid (\vec{k}\times\vec{\lambda})_x S_x + (\vec{k}\times\vec{\lambda})_y S_y \mid m_s\rangle \tag{51}$$

Notice the S_z component does not contribute to this transition since it implies a transition only when $m_s' = m_s$ which has zero energy, thus we can ignore these terms and assume $m_s' m_s$. To continue we use the $S_\pm$ shorthand:

$$\begin{cases} S_+ = S_x + iS_y & \to \quad S_+|\ -/2\rangle = |\ 1/2\rangle \\ S_- = S_x - iS_y & \to \quad S_-|\ 1/2\rangle = |\ -1/2\rangle \end{cases} \tag{52}$$

Then we can replace

$$S_x = \frac{1}{2}(S_+ + S_-) \quad \text{and} \quad S_y = \frac{1}{2i}(S_+ - S_-) \tag{53}$$

and get the matrix element:

$$\begin{aligned} \langle m_s' \mid \ \ldots\ \mid m_s\rangle &= \frac{1}{2i}\langle m_s' \mid i(\vec{k}\times\vec{\lambda})_x(S_+ + S_-) + (\vec{k}\times\vec{\lambda})_y(S_+ - S_-) \mid m_s\rangle \\ &= \frac{1}{2i}\left(\langle m_s' \mid (i(\vec{k}\times\vec{\lambda})_x + (\vec{k}\times\vec{\lambda})_y)S_+ + (i(\vec{k}\times\vec{\lambda})_x - (\vec{k}\times\vec{\lambda})_y)S_- \mid m_s\rangle\right) \end{aligned} \tag{54}$$

Since we are discussing absorption we know the electron is in the 'down' state, and we can discard the S_- operator as it cancels when acting on a groundstate 'ket'. If we wish to consider stimulated emission we can discuss the alternate case where the electron is in the 'up' state but the calculations would be very similar (only a minus changes between the x and y components of $(\vec{k}\times\vec{\lambda})$).

The absorption rate is then calculated by:

$$\Gamma_k = \frac{\pi^2\omega g^2\mu_B^2}{\hbar^3\Omega} N_{\vec{k},\vec{\lambda}} \left|\left((\vec{k}\times\vec{\lambda})_x + i(\vec{k}\times\vec{\lambda})_y\right) e^{i\vec{k}\cdot\vec{r}}\right|^2 \tag{55}$$

Notice the size of $\vec{k}$ is determined by the energy conservation requirement:

$$k = |\vec{k}| = \frac{g\mu_B B_0}{\hbar c} \tag{56}$$

Also notice the term $e^{i\vec{k}\cdot\vec{r}}$ does not contribute to the transition rate, as it is a global phase factor. This shows that the spatial dependence of the spin flip transition cancels out.

For any arbitrary photon trajectory $\vec{k}$ and polarization $\vec{\lambda}$ we can determine the transition rate using the above formula.

Since the size of $\vec{k}$ is determined by the energy, it can be expressed by two angles, the azimuthal θ and equatorial φ:

$$\vec{k}(\varphi,\theta)=\begin{cases}k_x=k_0\sin\theta\cos\varphi\\k_y=k_0\sin\theta\sin\varphi\\k_z=k_0\cos\theta\end{cases}\tag{57}$$

And, similarly, we can express the vector $\vec{\lambda}$ as one parameter, since its size is $|\vec{\lambda}|=1$ and we require orthogonality with $\vec{k}$. Defining the angle α between the $\vec{\lambda}$ and the plane of the z axis and $\vec{k}$:

$$\vec{\lambda}(\alpha)=\begin{cases}\lambda_x=-\cos\theta\cos\varphi\cos\alpha+\sin\varphi\sin\alpha\\\lambda_y=-\cos\theta\sin\varphi\cos\alpha-\cos\varphi\sin\alpha\\\lambda_z=\sin\theta\cos\alpha\end{cases}\tag{58}$$

It is easy to verify that indeed $\vec{k}\cdot\vec{\lambda}=0$ and $|\vec{\lambda}|=1$ using this parametrization. The vector products in (55) can be attained:

$$\begin{aligned}(\vec{k}\times\vec{\lambda})_x&=k(\cos\theta\cos\varphi\sin\alpha+\sin\varphi\cos\alpha)\\(\vec{k}\times\vec{\lambda})_y&=k(\cos\theta\sin\varphi\sin\alpha-\cos\varphi\cos\alpha)\end{aligned}\tag{59}$$

Thus the combination given above can be expressed as

$$(\vec{k}\times\vec{\lambda})_x+i(\vec{k}\times\vec{\lambda})_y=k(\cos\theta\sin\alpha-i\cos\alpha)e^{i\varphi}\tag{60}$$

Which shows that the choice of the equatorial angle φ is irrelevant to calculations of absorption rate (it is a phase angle that cancels when we take the absolute value square).

In special cases where the photon travels along the z axis or when it travels in the xy plane we can show what polarizations contribute to the absorption:

For photon in the xy plane, we can choose it to be moving with any angle φ (as we have shown, it does not change the result), and select $\theta=\frac{\pi}{2}$. The result is:

$$\Gamma_k=\frac{\pi^2\omega g^2\mu_B^2}{\hbar^3\Omega}N_{\vec{k},\vec{\lambda}}k^2\cos^2\alpha\tag{61}$$

Which means the photon must be polarized in the xy plane (corresponds to $\alpha=0$) for maximal absorption, and if the photon is polarized perpendicular to the plane there is no absorption. This is indeed what we expect since the direction of $\vec{\lambda}$ is the electric field vector orientation, and we want the magnetic field in the z axis to affect the z oriented electron spin.

For a photon moving in the z direction we set $\theta=0$ and get:

$$\Gamma_k=\frac{\pi^2\omega g^2\mu_B^2}{\hbar^3\Omega}N_{\vec{k},\vec{\lambda}}k^2|\sin\alpha-i\cos\alpha|^2=\frac{\pi^2\omega g^2\mu_B^2}{\hbar^3\Omega}N_{\vec{k},\vec{\lambda}}k^2\tag{62}$$

Which has no dependence on the polarization of the photon (as we expect as we already mentioned the system is symmetric in the φ direction and when $\theta=0$ the angle α corresponds to φ).

But this is not the largest value of absorption we can get!

If we allow $\vec{\lambda}$ to have imaginary values, and choose a circular polarization by letting[4] $\alpha = \frac{\pi}{4}$ and change $\sin\alpha \to \pm i \sin\alpha$[5]:

$$|\pm i\sin\alpha - i\cos\alpha|^2 = \sin^2\alpha + \cos^2\alpha \pm \sin 2\alpha\Big|_{\alpha=\frac{\pi}{4}} = 1 \pm 1 \tag{63}$$

So there is no absorption of a photon with left handed polarization for a photon moving with the magnetic field and there is twice as much absorption when it moves right handedly. If we switch the direcion of travel (by setting $\theta = \pi$ instead of $\theta = 0$ and flip sign of the $\cos\alpha$ term) we get opposite results. This agrees perfectly with the conservation of angular momenum, as the photon must give the electron a unit of $\hbar$ angular momentum to move it from spin $-1/2$ to spin $1/2$.

Which brings us to the last subject, which is unpolarized photons. If the photon is unpolarized we can say it has equal probability to be in any polarization, which is equivalent to having it in a linear superposition of *any* two orthogonal polarization states, regardless of basis. For the last example we saw that if we mix the left and right circular polarization we get an average of the two orthogonal states, and the absorption would be the average (between zero and two) which is one, which is the result we got from the linear polarization (which is a superposition of two circular states.

In the case of the photon moving in the xy plane, an unpolarized photon would have an absorption rate average of that of a photon polarized in the plane and perpendicular to the plane, namely half of the maximum absorption. This, once again, corresponds to the fact that the photon is statistically in either a polarization of maximum or minimal absorption, and the result is the average of both.

[4]We set the angle to $\alpha = 45°$ so it will be circular and not elliptical polarization, which is what we get if we let unequal amounts of the real axis and imaginary axis (which we identify with $\cos\alpha$ and $\sin\alpha$, respectively.

[5]The $\pm$ sign refers to our choice of right/lefthanded circular polarization.