Fundamentals of Quantum Technology
Week 1: Quantum Beam Splitters

1 Technical details

- Teaching assistant: Shachar Fraenkel (shacharf@mail.tau.ac.il)
- Recommended literature: The course covers various topics with no single textbook to unite them all. I will recommend relevant sources for each new subject we learn (these recommendations will appear in my notes on Moodle). During the first half of the course, the main source will be *Introductory Quantum Optics* by C. Gerry and P. Knight.
- Office hour: Wednesday at 11:00 (write me an email first).
- Homework: A new exercise sheet will be uploaded each week on Sunday, following the recitation. Submission is due by Thursday of the following week, at 11:55pm. Your solutions should be uploaded in a pdf format. Each homework sheet you submit adds 1 point to your final grade (90 points of your final grade are calculated from your score in the final exam).

2 And now for the real stuff

Recommended literature: Gerry, Knight (ch. 3.1-3.2, 6)

In class you saw two different descriptions of a lossless beam splitter. In the case of a **classical beam splitter**, two complex parameters r and t relate the amplitude of the incident field to the amplitudes of the reflected and transmitted beams,

$$\mathcal{E}_2 = r\mathcal{E}_1, \quad \mathcal{E}_3 = t\mathcal{E}_1.$$

The total intensity must be conserved, $|\mathcal{E}_1|^2 = |\mathcal{E}_2|^2 + |\mathcal{E}_3|^2$, and therefore $|r|^2 + |t|^2 = 1$.

In the case of a **quantum beam splitter**, we would like to associate with each beam annihilation (and creation) operators[1] $\hat{a}_i$. It turns out that, in order to properly describe the beam splitter, we must also take into account $\hat{a}_0$, the mode which is classically vacant. The transformation of field operators can then be generally written as

$$\begin{array}{l} \hat{a}_2 = t'\hat{a}_0 + r\hat{a}_1 \\ \hat{a}_3 = t\hat{a}_1 + r'\hat{a}_0 \end{array} \longrightarrow \begin{pmatrix} \hat{a}_2 \\ \hat{a}_3 \end{pmatrix} = \begin{pmatrix} t' & r \\ r' & t \end{pmatrix} \begin{pmatrix} \hat{a}_0 \\ \hat{a}_1 \end{pmatrix}.$$

[1] Soon we will see that a quantized field mode with frequency ω is equivalent to a quantum harmonic oscillator with the same frequency.

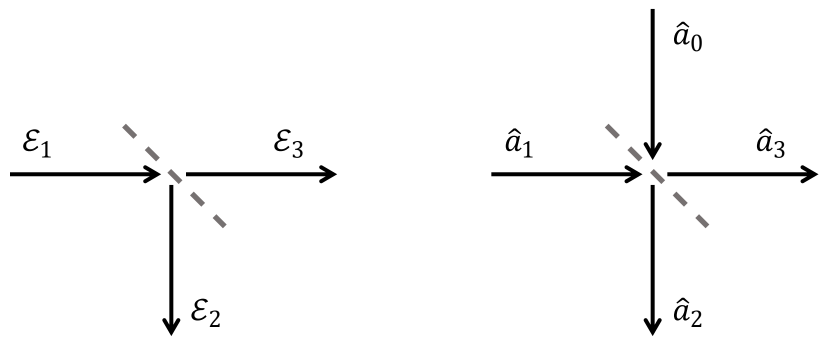

Figure 1: Left panel: A classical beam splitter. Right panel: A quantum beam splitter.

Exercise

For the output modes to be proper independent field modes, their field operators must satisfy the following commutation relations (for $i, j = 2, 3$):

$$[\hat{a}_i, \hat{a}_j] = \left[\hat{a}_i^\dagger, \hat{a}_j^\dagger\right] = 0, \quad \left[\hat{a}_i, \hat{a}_j^\dagger\right] = \delta_{ij}.$$

What are the conditions that r, r', t, t' must satisfy to ensure that these relations apply?

Solution

Since the incident modes 0 and 1 are obviously independent, they satisfy the relations above for $i, j = 0, 1$. The conclusion that $[\hat{a}_i, \hat{a}_j] = \left[\hat{a}_i^\dagger, \hat{a}_j^\dagger\right] = 0$ for $i, j = 2, 3$ is therefore immediate (and independent of the choice of r, r', t, t'). We must then require

$$\begin{aligned}
0 &\stackrel{!}{=} \left[\hat{a}_2, \hat{a}_3^\dagger\right] = \left[t'\hat{a}_0 + r\hat{a}_1, t^*\hat{a}_1^\dagger + r'^*\hat{a}_0^\dagger\right] = t'r'^* \left[\hat{a}_0, \hat{a}_0^\dagger\right] + rt^* \left[\hat{a}_1, \hat{a}_1^\dagger\right] = t'r'^* + rt^*, \\
1 &\stackrel{!}{=} \left[\hat{a}_2, \hat{a}_2^\dagger\right] = \left[t'\hat{a}_0 + r\hat{a}_1, t'^*\hat{a}_0^\dagger + r^*\hat{a}_1^\dagger\right] = |t'|^2 \left[\hat{a}_0, \hat{a}_0^\dagger\right] + |r|^2 \left[\hat{a}_1, \hat{a}_1^\dagger\right] = |t'|^2 + |r|^2, \\
1 &\stackrel{!}{=} \left[\hat{a}_3, \hat{a}_3^\dagger\right] = \ldots = |t|^2 + |r'|^2,
\end{aligned}$$

implying in total that the matrix

$$S \equiv \begin{pmatrix} t' & r \\ r' & t \end{pmatrix}$$

is *unitary*, $S^\dagger = S^{-1}$.

3 Classical light in a quantum beam splitter

A crucial element of any quantum theory is its ability to recreate results of classical physics when the appropriate classical limit is taken. We shall see throughout this semester that for quantized field modes this can be achieved by looking at a special set of states, called **coherent states**. These states are defined with respect to a certain field mode $\hat{a}$ (or, equivalently, with respect to a quantum harmonic oscillator) in the following way.

We denote by $|\alpha\rangle$ the right eigenstate of $\hat{a}$ such that

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle \quad \text{(equivalently: } \langle\alpha|\,\hat{a}^\dagger = \alpha^*\,\langle\alpha|\text{)}.$$

Since $\hat{a}$ is not Hermitian, α is generally a (*dimensionless*) complex number. At home you will show that there is a coherent state $|\alpha\rangle$ corresponding to *any* complex number α, given explicitly by

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle ,$$

where

$$|n\rangle \equiv \frac{1}{\sqrt{n!}} \left(\hat{a}^\dagger\right)^n |0\rangle$$

is a number state of the mode.

Coherent states are considered to be the "most classical" states of a quantum harmonic mode. We will repeat this claim several times in the coming weeks, giving more and more sense to it as we go along.

Basic properties (with proofs left as homework)

1. $\langle\alpha| \hat{n} |\alpha\rangle = |\alpha|^2$, with $\hat{n} = \hat{a}^\dagger \hat{a}$ being the photon number operator.
2. Let us define the displacement operator $\hat{D}(\alpha) = \exp\left(\alpha \hat{a}^\dagger - \alpha^* \hat{a}\right)$. Then $\hat{D}(\alpha)|0\rangle = |\alpha\rangle$.
3. $\hat{D}^\dagger(\alpha) = \hat{D}(-\alpha) = \hat{D}^{-1}(\alpha)$.

Reminder: A single photon in a 50:50 beam splitter

A 50:50 beam splitter is described by the output modes

$$\hat{a}_2 = \frac{1}{\sqrt{2}}\left(\hat{a}_0 + i\hat{a}_1\right), \quad \hat{a}_3 = \frac{1}{\sqrt{2}}\left(i\hat{a}_0 + \hat{a}_1\right).$$

In class you saw that a single photon will choose one of the paths with equal probability,

$$|0\rangle_0 |1\rangle_1 \xrightarrow{\text{BS}} \frac{1}{\sqrt{2}}\left(i|1\rangle_2 |0\rangle_3 + |0\rangle_2 |1\rangle_3\right).$$

Exercise

Suppose that the light incident on the 50:50 beam splitter is in a coherent state $|\alpha\rangle$. What is the output state?

Solution

Let us write the input state in the following way:

$$|0\rangle_0 |\alpha\rangle_1 = \hat{D}_1(\alpha)|0\rangle_0 |0\rangle_1 = \exp\left(\alpha\hat{a}_1^\dagger - \alpha^*\hat{a}_1\right)|0\rangle_0 |0\rangle_1 .$$

Inverting the relation between input and output states, we have $\hat{a}_1 = \frac{1}{\sqrt{2}}\left(-i\hat{a}_2 + \hat{a}_3\right)$, and therefore

$$\begin{aligned}
|0\rangle_0 |\alpha\rangle_1 \xrightarrow{\text{BS}} & \exp\left[\frac{1}{\sqrt{2}}\left(i\alpha\hat{a}_2^\dagger + i\alpha^*\hat{a}_2^\dagger + \alpha\hat{a}_3^\dagger - \alpha^*\hat{a}_3\right)\right]|0\rangle_2 |0\rangle_3 \\
&= \exp\left[\frac{i\alpha}{\sqrt{2}}\hat{a}_2^\dagger - \frac{-i\alpha^*}{\sqrt{2}}\hat{a}_2\right]\exp\left[\frac{\alpha}{\sqrt{2}}\hat{a}_3^\dagger - \frac{\alpha^*}{\sqrt{2}}\hat{a}_3\right]|0\rangle_2 |0\rangle_3 \\
&= \hat{D}_2\left(\frac{i\alpha}{\sqrt{2}}\right)\hat{D}_3\left(\frac{\alpha}{\sqrt{2}}\right)|0\rangle_2 |0\rangle_3 \\
&= \left|\frac{i\alpha}{\sqrt{2}}\right\rangle_2 \left|\frac{\alpha}{\sqrt{2}}\right\rangle_3 .
\end{aligned}$$

We may observe that this is the behavior expected from classical light: the input mode has an average photon number of $|\alpha|^2$, while each output mode is characterized by an average photon number of $\frac{1}{2}|\alpha|^2$; the incident intensity is exactly split in half.

Comment Note that the single photon result is *not* a small-α limit of the coherent state result. In particular, the output state for a coherent state input does not feature entanglement between the two output modes.

Exercise

Suppose that the input state in a 50:50 beam splitter is $|0\rangle_0 |N\rangle_1$, i.e. a number state. What would be the output state?

Solution

We shall use the result obtained for the case of a coherent input state, by expanding both the input and the output states according to the number-state basis:

$$\begin{aligned}
|0\rangle_0 |\alpha\rangle_1 &= e^{-|\alpha|^2/2} \sum_{N=0}^{\infty} \frac{\alpha^N}{\sqrt{N!}} |0\rangle_0 |N\rangle_1 , \\
\left|\frac{i\alpha}{\sqrt{2}}\right\rangle_2 \left|\frac{\alpha}{\sqrt{2}}\right\rangle_3 &= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\left(i\alpha/\sqrt{2}\right)^n \left(\alpha/\sqrt{2}\right)^m}{\sqrt{n!m!}} |n\rangle_2 |m\rangle_3 \\
&= e^{-|\alpha|^2/2} \sum_{N=0}^{\infty} \frac{\alpha^N}{\sqrt{N!}} \left\{ \frac{1}{2^{N/2}} \sum_{n=0}^{N} i^n \left[\frac{N!}{n!\,(N-n)!}\right]^{1/2} |n\rangle_2 |N-n\rangle_3 \right\},
\end{aligned}$$

where in the final line we rewrote the original double sum as a sum over states with a fixed total number of photons in the two output modes. Since the beam splitter conserves the number of photons, we may identify

$$|0\rangle_0 |N\rangle_1 \xrightarrow{\text{BS}} \frac{1}{2^{N/2}} \sum_{n=0}^{N} i^n \left[\frac{N!}{n!\,(N-n)!}\right]^{1/2} |n\rangle_2 |N-n\rangle_3 .$$

This result corresponds to a *binomial distribution* of the N incoming photons over the two output modes,

$$P(n, N-n) = \frac{1}{2^N} \binom{N}{n}.$$

This is again a manifestation of the particle nature of photons: the binomial distribution is the one expected assuming that each one of the (indistinguishable) photons chooses a path independently, and with probability $\frac{1}{2}$ for each path.