(14) $H = \frac{P_x^2}{2m} + \frac{P_y^2}{2M} + \frac{1}{2}k(x^2+y^2) + \alpha xy \quad , \quad |\alpha| < k$

$M \gg m \rightarrow \quad v_y \equiv \frac{P_y}{M} \ll v_x \equiv \frac{P_x}{m} \rightarrow$ y is the "slow" coordinate.

$\hookrightarrow \quad H = \underset{\text{fast part}}{H_f} + \underset{\text{slow part}}{H_s} \qquad H_f \equiv \frac{P_x^2}{2m} + \frac{1}{2}kx^2 + \alpha xy$

$\qquad\qquad\qquad\qquad\qquad\qquad H_s = \frac{P_y^2}{2M} + \frac{1}{2}ky^2$

We can write: $H_f = \frac{P_x^2}{2m} + \frac{1}{2}k\left(x + \frac{\alpha y}{k}\right)^2 - \frac{\alpha^2 y^2}{2k} \leftarrow$ H.O. with coordinate $x + \frac{\alpha y}{k}$

The Eigenvalues of H_f: $\quad E_{f_n} = \hbar\omega_f(n+\frac{1}{2}) - \frac{\alpha^2 y^2}{2k} \quad , \quad \omega_f \equiv \frac{k}{m}$

The eigenstates are $\quad \Psi_n(z) \leftarrow$ eigenstates of H.O. with $z = x + \frac{\alpha y}{k}$

$$H_{B.O.} = \frac{P_y^2}{2M} + \frac{1}{2}ky^2 + \hbar\omega_f(n+\tfrac{1}{2}) - \frac{\alpha^2 y^2}{2k} =$$

$$= \frac{P_y^2}{2M} + \frac{1}{2}\left(k - \frac{\alpha^2}{k}\right)y^2 + \hbar\omega_f(n+\tfrac{1}{2})$$

So here the eigenstates are: $\xi_n(y)$

and eigenvalues: $\quad E_l = \hbar\omega_s(l+\frac{1}{2}) + \hbar\omega_f(n+\frac{1}{2}) \quad , \quad \omega_s \equiv \frac{k - \frac{\alpha^2}{k}}{M}$

The "potential surfaces":

Berry phase:

The eigenfunctions are real and therefore the "matrix element" $\langle \varphi_n | \frac{\partial}{\partial y} | \varphi_n \rangle$ is real so there is no Berry phase.

Also $\frac{\partial}{\partial y} \propto \frac{\partial}{\partial z} \propto P_z$ and for H.O. we have: $\langle \varphi_n | P_z | \varphi_m \rangle = i\sqrt{\frac{m\hbar\omega}{2}}\left(\sqrt{n+1}\,\delta_{n,m+1} - \sqrt{n}\,\delta_{n,m-1}\right)$

so $\langle \varphi_n | \frac{\partial}{\partial y} | \varphi_n \rangle = 0$.

Generally, $H_{B.O.}$ is written as:

$$H^{B.O.} = -\frac{\hbar^2}{2M}\left(\nabla_Q^2 + 2\langle\varphi_m|\nabla_Q|\varphi_m\rangle\nabla_Q + \left(\langle\varphi_m|\nabla_Q|\varphi_m\rangle\right)^2\right)$$

The criteria for obtaining this expression is that the off diagonal elements of $\langle\varphi_m|\nabla_Q^2|\varphi_n\rangle$ are neglegible (compare to the digonal elements).

$$\langle\varphi_n|\frac{\partial^2}{\partial Q^2}|\varphi_n\rangle\xi_n(Q) = \langle\varphi_n|\frac{\partial^2}{\partial Q^2}|\varphi_n\rangle\xi_n(Q) + 2\langle\varphi_m|\frac{\partial}{\partial Q}|\varphi_m\rangle\frac{\partial\xi_n}{\partial Q} + \langle\varphi_n|\varphi_m\rangle\frac{\partial^2\xi}{\partial Q^2}$$

In our case $Q \equiv y$ and φ_n's are H.O. eigenstates.

as we've shown $\langle\varphi_n|\frac{\partial}{\partial y}|\varphi_n\rangle = 0$

$$\langle\varphi_n|\frac{\partial}{\partial y}|\varphi_m\rangle = i\sqrt{\frac{\hbar\sqrt{km}}{2}}\left(\sqrt{n+1}\,\delta_{n,m+1} + \sqrt{n}\,\delta_{n,m-1}\right)$$

Also $\langle\varphi_n|\frac{\partial^2}{\partial Q^2}|\varphi_n\rangle = \sum_k\langle\varphi_m|\frac{\partial}{\partial Q}|\varphi_k\rangle\langle\varphi_k|\frac{\partial}{\partial Q}|\varphi_n\rangle \propto$

$$\propto i\,\frac{\hbar\sqrt{km}}{2}\left(\sqrt{k+1}\,\delta_{m,k+1} + \sqrt{k}\,\delta_{m,k-1}\right)\left(\sqrt{n+1}\,\delta_{k,n+1} + \sqrt{n}\,\delta_{k,n-1}\right)$$

So the diagonal term is $\delta_{mn}\frac{\partial^2}{\partial y^2}\cdot\left(-\frac{\hbar^2}{2M}\right)$

and the off digonal terms are proportional to $\frac{\sqrt{km}}{M}$ or $\frac{(km)^{\frac{1}{4}}}{M}$ which are neglegible in comparison to $\frac{1}{M}$ so the off-digonal elements are indeed neglegible.

1b Exact solution:

$$V = \tfrac{1}{2}kx^2 + \tfrac{1}{2}ky^2 + \alpha xy$$

$\downarrow$

$$\ddot{x} = -\frac{k}{m}x - \frac{\alpha}{m}y$$
$$\ddot{y} = -\frac{k}{M}y - \frac{\alpha}{M}x$$

$\rightarrow$ $$\ddot{\vec{x}} = -A\vec{x} \qquad \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} \frac{k}{m} & \frac{\alpha}{m} \\ \frac{\alpha}{M} & \frac{k}{M} \end{pmatrix}$$

$$U^{-1}\ddot{\vec{x}} = -U^{-1}AUU^{-1}\vec{x}$$

$$\ddot{\vec{z}} = -D\vec{z}, \quad \vec{z} = U^{-1}\vec{x} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad D = U^{-1}AU \leftarrow \text{diagonal}$$

Diagonalize A:

$$\omega_1^2 = \tfrac{1}{2}k\left(\frac{1}{m} + \frac{1}{M}\right) + \left(\frac{k^2}{4}\left(\frac{1}{m} - \frac{1}{M}\right)^2 + \frac{\alpha^2}{mM}\right)^{\frac{1}{2}}$$

$$\omega_2^2 = \tfrac{1}{2}k\left(\frac{1}{m} + \frac{1}{M}\right) - \left(\frac{k^2}{4}\left(\frac{1}{m} - \frac{1}{M}\right)^2 + \frac{\alpha^2}{mM}\right)^{\frac{1}{2}}$$

To compare to the B.O. approx use $m \ll M$:

$$\omega_{1,2}^2 \simeq \frac{k}{2}\left(\frac{1}{m} + \frac{1}{M}\right) \pm \frac{k}{2m}\left(1 - \frac{m}{M} + \frac{2\alpha^2}{k^2}\frac{m}{M}\right)$$

$\hookrightarrow$ $$\omega_1^2 \simeq \frac{k}{m}$$

$$\omega_2^2 \simeq \frac{k}{2M} - \frac{k}{\cancel{2m}}\frac{\cancel{2}\alpha^2}{k^{\cancel{2}}}\frac{\cancel{m}}{M} + \frac{k}{\cancel{2m}}\cdot\frac{\cancel{m}}{M} = \frac{k - \frac{\alpha^2}{k}}{M}$$

This is the same result as for the B.O. approx.

2) Take a 2D electron in (x,y) with magnetic field $B\hat{z}$ and external potential $U(x)$.

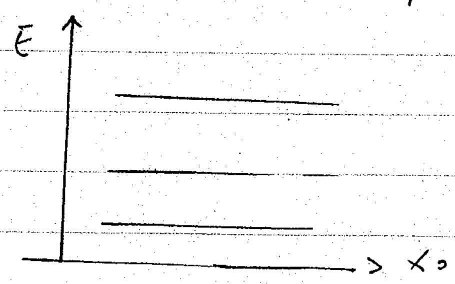
What are the symmetries? What do they imply about degeneracies?

<u>Answer:</u>

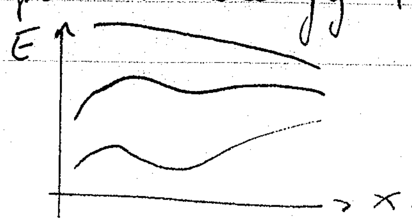
The symmetry is a translation invariance along the y-axis.

The generator of this translation is $\hat{p}_y$. This means we can diagonalize $\hat{X}_0$ by choosing the gauge such that $\hat{X}_0 \propto \hat{p}_y$. We will therefore have a quantum number x_0 which shifts the harmonic oscillator along the x-axis. Since nothing is known about $U(x)$, there is not necessarily any degeneracy in the problem.

However, X_0 is still a good quantum number.
In the case of $U=0$, the energy would not depend on ~~k~~ x_0, so one could plot



In our case, however, the landau levels, which are not necessarily h.o. eigenfunctions have x_0-dependent energy in general.



S.R.

- Consider $U(x) = \begin{cases} 0 & , x < 0 \\ U_0 & , x \geq 0 \end{cases}$. Write a 1-D Schrödinger equation.

Plot the resulting potential. Draw qualitatively the Landau levels for $U_0 \gg \hbar\omega_c$.

Answer: Let's choose a gauge which will diagonalize p_y.

$$A_x = 0 \quad , \quad A_y = Bx \qquad A_z = 0$$

check: $B = \nabla \times \bar{A} = \hat{z}\left(\partial/\partial x \, A_y - \partial/\partial y \, A_x\right) = B\hat{z}$

We then have

$$\hat{X}_0 = \hat{x} + \frac{\hat{v}_y}{\omega_c} \qquad , \quad \omega_c = \frac{eB}{mc}$$

$$= \hat{x} + \frac{1}{m\omega_c}\left(\hat{p}_y - \frac{e}{c}A_y\right) = \frac{1}{m\omega_c}\hat{p}_y$$

as expected

The Hamiltonian is

$$\hat{H} = \frac{m\hat{v}_x^2}{2} + \frac{m\hat{v}_y^2}{2} + U(x)$$

$$= \frac{\hat{p}_x^2}{2m} + \frac{1}{2m}\left(\hat{p}_y - \frac{eBx}{c}\right)^2 + U(x)$$

We indeed observe that $H \neq H(y)$, so $[H, \hat{X}_0] = 0$.

Let's then choose eigenstates of $\hat{p}_y$ as our basis:

$$\psi(x,y) = e^{ik_y y}\chi(x)$$

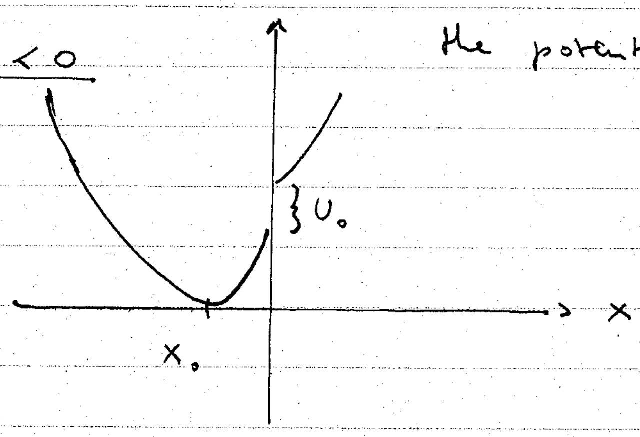
$$\Rightarrow H\psi = \left[\frac{p_x^2}{2m} + \frac{m\omega_c^2}{2}(x - X_0)^2 + U(x)\right]\psi = E\psi$$

where $x_0 = \frac{\hbar k_y}{m\omega_c}$. The Schrödinger eq. thus becomes

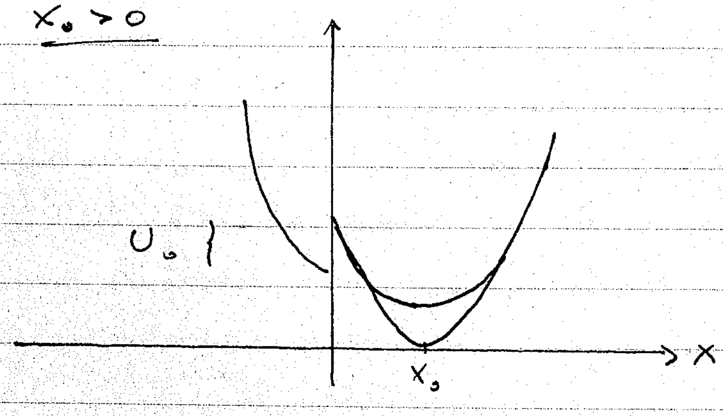
$$-\frac{\hbar^2}{2m}\frac{\partial^2 \chi(x)}{\partial x^2} + \frac{m\omega_c^2}{2}(x-x_0)^2\chi + [U(x) - E]\chi = 0$$

$$\Rightarrow \begin{cases} -\frac{\hbar^2}{2m}\partial_x^2 \chi + \frac{m\omega_c^2}{2}(x-x_0)^2\chi - E\chi = 0 , & x < 0 \\ -\frac{\hbar^2}{2m}\partial_x^2 \chi + m\omega_c^2 (x-x_0)^2\chi + (U_0 - E)\chi = 0 , & x > 0 \end{cases}$$

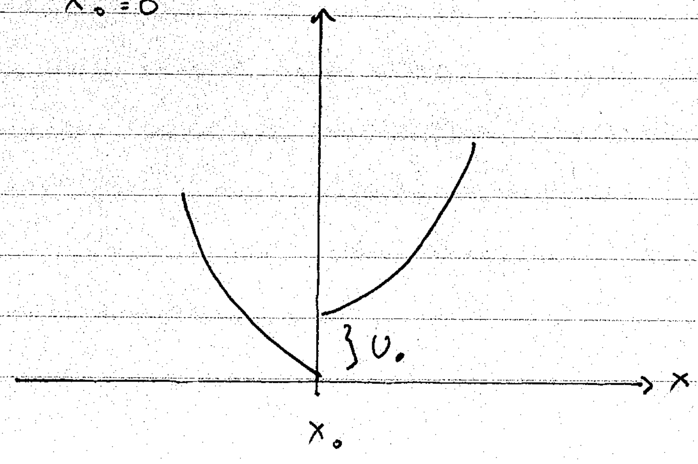
So for $\underline{x_0 < 0}$ the potential is



$\underline{x_0 > 0}$

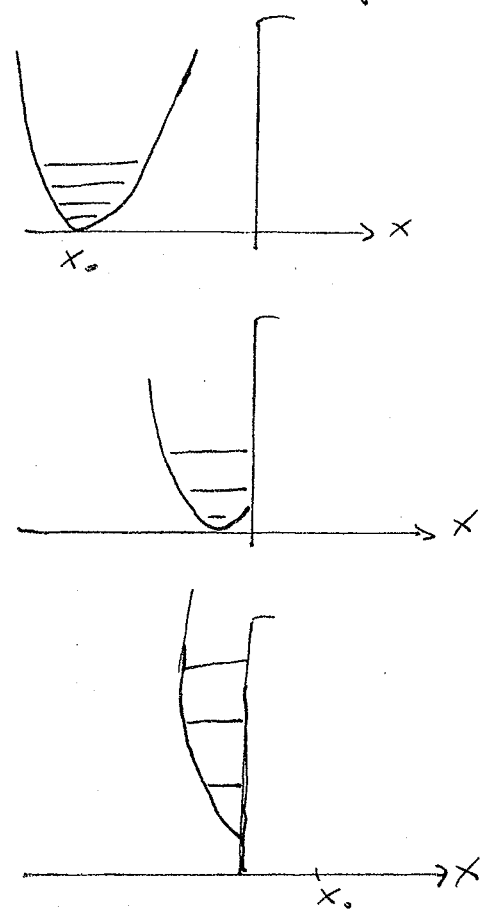


$x_0 = 0$



When $V_0 \gg \hbar\omega_c$, the spacing between the landau levels is much smaller than the potential wall.

Let's draw the landau levels for various x_0.



→ minimum at $x=0$!

He

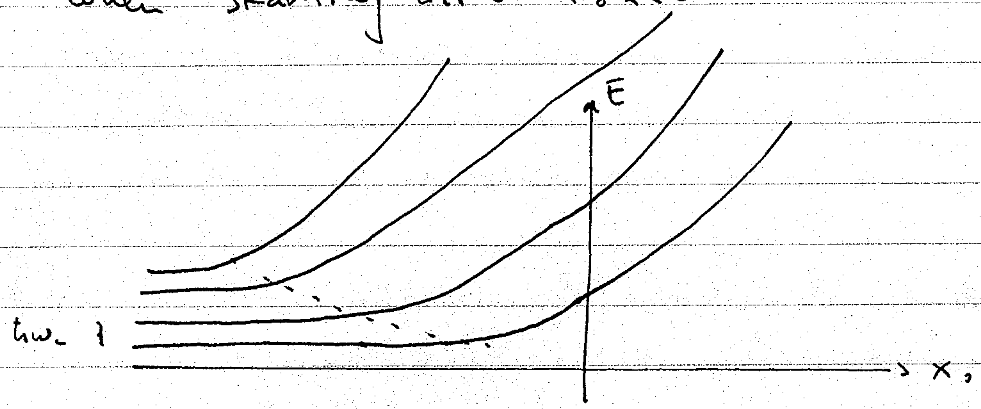
Explanation: for $x_0 \ll 0$, the lowest energy levels do not feel the wall. Therefore we'll have the regular h.o. wavefunctions with normal h.o. energies.

However, as x_0 approaches 0, more and more levels are starting to feel the wall, which confines the wavefunctions spatially. According to location-momentum uncertainty, this will increase the energies of the affected wavefunctions.

This can also be seen with the Bohr-Sommerfeld rule:

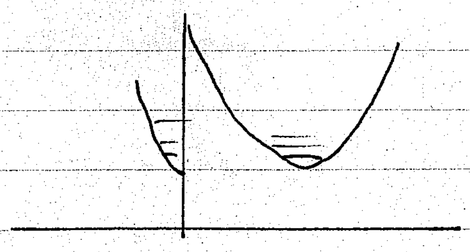
$$\int_{x_i}^{x_f} \sqrt{2m(E-U)} = hn$$

So in order to stay in the same level (same n) when the path $x_f - x_i$ gets shorter, we need increased E. Let's plot the behavior of the energy levels when starting with $x_0 \ll 0$



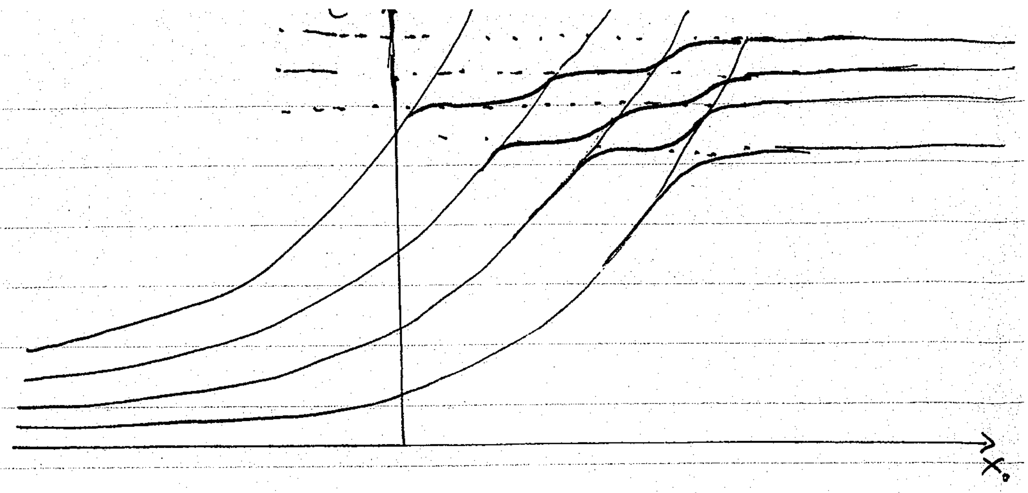
If we start from $x_0 \gg 0$, the minimum will again be at x_0, and we will have regular Landau levels with spacing $\hbar\omega_c$.

<u>However</u>, when $\frac{m\omega^2 x_0^2}{2} \sim U_0$, we will have two minima at about the same energies.



In that case, there won't be any level crossings, since the probability of tunneling lifts the degeneracy. This leads to level repulsion.

Summarizing:



- Make the electron climb the step adiabatically. Use an electric field in a time-dependent gauge. When does the adiabatic approximation break down?

Answer: We add an electric field in the $\hat{y}$ direction, so that if we start with $x_0 \ll 0$, the field will shift x_0 to the right, until we end up with $x_0 \gg 0$.

Since $\bar{E} = -\frac{1}{c}\partial A_t = E\hat{y}$ (A_0 is still taken to be zero) we can choose

$$A_x = 0 \quad , \quad A_y = Bx - cEt \quad , \quad A_z = 0$$

Note that this change does not affect $\bar{B} = \bar{\nabla} \times \bar{A}$.

$$\Rightarrow \hat{x}_0 = \hat{x} + \frac{1}{m\omega_c}\left(\hat{p}_y - \frac{e}{c}A_y\right) = \frac{1}{m\omega_c}\hat{p}_y + \frac{c}{B}Et$$

Note that since we still have $[H, \hat{x}_0] = 0$ we can still choose $\psi(x,y) = e^{ik_y y}\chi(x)$

$$\Rightarrow H\psi = \left[\frac{p_x^2}{2m} + \frac{1}{2m}\left(p_y - \frac{eBx}{c} + eEt\right)^2\right]\psi = E\psi$$

$$\Rightarrow -\frac{\hbar^2}{2m}\partial_x^2\chi + \frac{m\omega_c^2}{2}(x-x_0)^2\chi + (U-E)\chi = 0$$

where $x_0 = \frac{\hbar k_y}{m\omega_c} + \frac{c}{B}Et$

We see that the potential moves to the right with speed $\frac{cE}{B}$.

Note that in the adiabatic approximation we solve the time-independent equation $H\psi = E\psi$ with t as a <u>parameter</u>. This neglects the fact that we might have transitions from one level to another as time passes by.

1'

דרך שנייה למצוא
את $\rho(t)$ מופיעה
בשאלה 3.

$\boxed{3}$ $H=\begin{pmatrix}\varepsilon_1 & v\\ v & \varepsilon_2\end{pmatrix}$

$$\rho=\frac{1}{2}\left(\mathbb{1}+a_x\sigma_x+a_y\sigma_y+a_z\sigma_z\right)$$

$$-i\hbar\frac{\partial\rho}{\partial t}=[\rho,H]$$

$$H=\frac{\varepsilon_1+\varepsilon_2}{2}\mathbb{1}+\frac{\varepsilon_1-\varepsilon_2}{2}\sigma_z+v\sigma_x$$

$$[H,\rho]=\frac{1}{2}\Big[\frac{\varepsilon_1-\varepsilon_2}{2}a_x[\sigma_z,\sigma_x]+\frac{\varepsilon_1-\varepsilon_2}{2}a_y[\sigma_z,\sigma_y]+va_y[\sigma_x,\sigma_y]+$$

$$+va_z[\sigma_x,\sigma_z]\Big]=$$

$$=i\left[\frac{\varepsilon_1-\varepsilon_2}{2}a_x\sigma_y-\frac{\varepsilon_1-\varepsilon_2}{2}a_y\sigma_x+va_y\sigma_z-va_z\sigma_y\right].$$

(1) $\frac{1}{2}\hbar\dot{a}_x=-\frac{\varepsilon_1-\varepsilon_2}{2}a_y$

(2) $\frac{1}{2}\hbar\dot{a}_y=\frac{\varepsilon_1-\varepsilon_2}{2}a_x-va_z$

(3) $\frac{1}{2}\hbar\dot{a}_z=va_y$

$\Downarrow$

(1)+(3) $\Rightarrow$ $\frac{1}{2}\hbar\dot{a}_x=-\frac{\varepsilon_1-\varepsilon_2}{2}\left(\frac{\hbar}{2v}\dot{a}_z\right)$

(1) $\dot{a}_x=-\frac{\varepsilon_1-\varepsilon_2}{2v}\dot{a}_z$

(2) $\frac{1}{2}\hbar\left(\frac{1}{2}\frac{\hbar}{v}\ddot{a}_z\right)=\frac{\varepsilon_1-\varepsilon_2}{2}a_x-va_z$

נגזור לפי t ונציב את (1)

$$\frac{\hbar^2}{4v}\dddot{a}_z=-\frac{(\varepsilon_1-\varepsilon_2)^2}{4v}\dot{a}_z-v\dot{a}_z$$

$$\frac{\hbar^2}{4v}\ddot{a}_z=\left[-\frac{(\varepsilon_1-\varepsilon_2)^2}{4v}-v\right]a_z+C$$

ונקבל $a_z(t)=Ae^{ikt}+Be^{-ikt}+\frac{C}{k^2}$ כאשר

$$k^2=\frac{1}{\hbar^2}\left[(\varepsilon_1-\varepsilon_2)^2+4v^2\right].$$

וכך ניתן לפתור את המשוואה.

$$\cancel{\dot a_x = -\frac{\varepsilon_1-\varepsilon_2}{2V}\dot a_z = -ik\frac{\varepsilon_1-\varepsilon_2}{2V}\left[Ae^{+ikt}-Be^{-ikt}\right]}$$

$$a_y = \frac{\hbar}{2V}\dot a_z = ik\frac{\hbar}{2V}\left[Ae^{ikt}-Be^{-ikt}\right].$$

$$a_x = \frac{2}{\varepsilon_1-\varepsilon_2}\left[Va_z + \frac{1}{2}\hbar\dot a_y\right] = \frac{2}{\varepsilon_1-\varepsilon_2}\left[V\left(Ae^{ikt}+Be^{-ikt}+\frac{C}{k^2}\right)+\right.$$

$$\left. -\frac{k^2\hbar^2}{4V}\left[Ae^{ikt}+Be^{ikt}\right]\right] =$$

$$= \frac{2}{\varepsilon_1-\varepsilon_2}\left[\left(V-\frac{k^2\hbar^2}{4V}\right)\left(Ae^{ikt}+Be^{-ikt}\right)+\frac{VC}{k^2}\right].$$

תנאי ההתחלה הינו

$$\rho(0) = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \Rightarrow a_x = a_y = 0\ ,\ a_z = 1.$$

$\underline{a_x=0:}$ $\quad \frac{VC}{k^2} = \left(\frac{k^2\hbar^2}{4V}-V\right)(A+B)$

$\underline{a_y=0:}$ $\quad A-B=0 \Rightarrow \boxed{A=B}$

$$C = \left(\frac{k^4\hbar^2}{4V^2}-k^2\right)2A$$

$\underline{a_z=1:}$ $\quad 1 = 2A + \frac{C}{k^2} \Rightarrow C = k^2(1-2A)$

$$k^2(1-2A) = \left(\frac{k^4\hbar^2}{4V^2}-k^2\right)2A$$

$$\boxed{A = \frac{2V^2}{\hbar^2k^2} = B}$$

$$C = k^2(1-2A) = k^2\left(1-\frac{4V^2}{k^2\hbar^2}\right)$$

$$a_x(t) = \frac{2}{\varepsilon_1-\varepsilon_2}\left[2\left[\frac{2V^3}{\hbar^2k^2}-\frac{V}{2}\right]\cos(kt) + V\left(1-\frac{4V^2}{k^2\hbar^2}\right)\right] =$$

$$= \frac{2}{\varepsilon_1-\varepsilon_2}\left(\frac{4V^3}{\hbar^2k^2}-V\right)\left[\cos(kt)-1\right].$$

$$a_z(t) = \frac{4V^2}{\hbar^2k^2}\cos(kt) + 1 - \frac{4V^2}{k^2\hbar^2} = \frac{4V^2}{\hbar^2k^2}\left(\cos(kt)-1\right)+1.$$

$$a_y(t) = -\frac{2V}{\hbar k}\sin(kt).$$

הסיכוי להמצאו במצב היסוד: דרך II

נלכסן את H:

$$\bar{H} = H - \frac{\varepsilon_1+\varepsilon_2}{2}\mathbb{1} = \begin{pmatrix} \frac{\varepsilon_1-\varepsilon_2}{2} & v \\ v & -\frac{\varepsilon_1-\varepsilon_2}{2} \end{pmatrix} = \begin{pmatrix} \tilde{\varepsilon} & v \\ v & -\tilde{\varepsilon} \end{pmatrix}$$

בגלל שמורים: $\mathrm{Tr}\,\bar{H}$, $\det(\bar{H})$

לכן $\bar{H}$ מלוכסנת היא:

$$\bar{H} = \begin{pmatrix} \sqrt{\tilde{\varepsilon}^2+v^2} & 0 \\ 0 & -\sqrt{\tilde{\varepsilon}^2+v^2} \end{pmatrix} \qquad \tilde{\varepsilon} = \frac{\varepsilon_1-\varepsilon_2}{2}.$$

נחשב את $|+\rangle$:

$$\left(\bar{H} - \sqrt{\tilde{\varepsilon}^2+v^2}\,\mathbb{1}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\tilde{\varepsilon} - \sqrt{\tilde{\varepsilon}^2+v^2}\right)x + vy = 0$$

$$-\left(\tilde{\varepsilon} + \sqrt{\tilde{\varepsilon}^2+v^2}\right)y + vx = 0$$

נציב בשנייה $y=v$ ונקבל:

$$y = v,\ x = \tilde{\varepsilon} + \sqrt{\tilde{\varepsilon}^2+v^2}$$

וצריך לנרמל את הווקטור $|+'\rangle = \begin{pmatrix} \tilde{\varepsilon}+\sqrt{\tilde{\varepsilon}^2+v^2} \\ v \end{pmatrix}$ כלומר:

$$|+\rangle = N_+\begin{pmatrix} \tilde{\varepsilon}+\sqrt{\tilde{\varepsilon}^2+v^2} \\ v \end{pmatrix}, \qquad \cancel{N_+^2 = \frac{1}{\langle +'|+'\rangle} = \frac{1}{2(\tilde{\varepsilon}^2+v^2+\sqrt{\tilde{\varepsilon}^2+v^2})}}$$

$$N_+^2 = \frac{1}{\langle +'|+'\rangle} = \frac{1}{2(\tilde{\varepsilon}^2+v^2+\tilde{\varepsilon}\sqrt{\tilde{\varepsilon}^2+v^2})}$$

$$|-\rangle = N_-\begin{pmatrix} \tilde{\varepsilon}-\sqrt{\tilde{\varepsilon}^2+v^2} \\ v \end{pmatrix}$$ באותה מידה

$$N_- = \frac{1}{2(\tilde{\varepsilon}^2+v^2-\tilde{\varepsilon}\sqrt{\tilde{\varepsilon}^2+v^2})}$$

$$\rho(0)=\frac{1}{2}\begin{pmatrix} 1+\frac{\tilde{\varepsilon}}{\sqrt{\ }} & -\frac{v}{\sqrt{\ }} \\ -\frac{v}{\sqrt{\ }} & 1-\frac{\tilde{\varepsilon}}{\sqrt{\ }} \end{pmatrix}$$

בגלל שמטריצת המעבר עבור וקטורים עצמיים מנורמלים היא:

$$P=\begin{pmatrix} V_+^1 & V_-^1 \\ V_+^2 & V_-^2 \end{pmatrix} \qquad P^{-1}=\begin{pmatrix} V_-^2 & -V_-^1 \\ -V_+^2 & V_+^1 \end{pmatrix}$$

$$|+\rangle=\begin{pmatrix} V_+^1 \\ V_+^2 \end{pmatrix} \qquad |-\rangle=\begin{pmatrix} V_-^1 \\ V_-^2 \end{pmatrix}.$$

ניתן לרשום $\rho(0)=|\psi\rangle\langle\psi|$ כאשר

$$|\psi\rangle=\begin{pmatrix} 1 \\ 0 \end{pmatrix}=\frac{\frac{1}{N_+}|+\rangle-\frac{1}{N_-}|-\rangle}{2\sqrt{\tilde{\varepsilon}^2+v^2}}$$

לכן בהסתברות 1 יהיה לנו מצב $|\psi\rangle$, ולכן אנחנו ב- pure state.

~~אפשר לראות שההתפתחות בזמן היא~~

~~מכיוון ש:~~ $\cancel{\rho(0)=|\psi\rangle\langle\psi|}$,

(מכיוון ש: $\rho(0)=|\psi\rangle\langle\psi|$, ניתן לכתוב $\cancel{\rho(t)=\langle\psi(t)|}$

$$\rho(t)=|\psi(t)\rangle\langle\psi(t)|$$

ולכן הצורה הכללית היא

$$\rho(t)=\begin{pmatrix} 1+\frac{\tilde{\varepsilon}}{\sqrt{\ }} & -\frac{v}{\sqrt{\ }}e^{-2i\sqrt{\ }t} \\ -\frac{v}{\sqrt{\ }}e^{2i\sqrt{\ }t} & 1-\frac{\hat{\varepsilon}}{\sqrt{\ }} \end{pmatrix}$$

אינו יוצאים

$$\rho=|\psi\rangle\langle\psi|$$

$$|\psi(t)\rangle=\frac{\frac{1}{N_+}|+\rangle e^{iE_+t/\hbar}-\frac{1}{N_-}|-\rangle e^{iE_-t/\hbar}}{2\sqrt{\tilde{\varepsilon}^2+v^2}}$$

ולכן $|\langle-|\psi(t)\rangle|^2$ קבוע בזמן וזוהי ההסתברות להיות ב- g.s.

$$\sigma = \frac{\varepsilon_1 - \varepsilon_2}{2V} F$$

$$A + B = 0$$

$$F = \frac{2A(\varepsilon_1 - \varepsilon_2)}{i\sqrt{}}$$

$$2A \cdot \frac{2V}{i\sqrt{}} + \frac{2A(\varepsilon_1 - \varepsilon_2)^2}{iV2\sqrt{}} = \frac{1}{2}$$

עם תנאי ההתחלה שנתנו מה הסיכוי למצוא את המערכת ב g.s בזמן t?

התשובה ½. זה לא הבסיס של הפתרון שלנו שלכסן. הסיכוי לא תלוי ב t. כי יש לנו pure state שמתפתח בזמן. להזהר לא להתבלבל עם ρ.

במערכת המבודדת שאנחנו יודעים שזה pure state עקבנו את הדרך ללכסן את המטריצה הזו.

<u>Relaxation time approx</u> ומפגל

לפני שפונים לקירוב דרך להסתכל על מה שקורה כאן באמת. שמצמידים את המערכת לאמבט. ונחכה שמספיק זמן האמבט המערכת תגיע לש"מ. ובש"מ אנחנו יודעים להגיד הכל על המערכת הסתברויות וכאלה.

$$\rho_\infty = \frac{e^{\beta H}}{tr(e^{\beta H})} \rightarrow \text{density matrix in eq.}$$

אבל הצגנו את המערכת בת.ה שלא קשור בכלל. ניתן לפתור במדויק ולתאר את כל הדינמיקה של המערכת זה משהו קשה לתאר באופן שמכליל את הצימוד. לכן נבוא לעשות קירוב: ניקח פרמטר פנומנולוגי שמתאר כמה זמן לוקח לנו להגיע לש"מ.

אם המערכת לא מצומדת: $i\,\frac{d\rho}{dt} = [H,\rho]$

הקירוב: נוסיף את המשוואה

$$\Rightarrow \frac{d\rho}{dt} + i[H,\rho] = -\frac{1}{\tau}(\rho-\rho_\infty)$$

שיתאר כמה מהר מתקרבים ל-ρ_∞ ← [illegible].

למשוואה הזו יש פתרון

$$\boxed{\rho(t) = \rho_\infty + e^{-iHt}(\rho(0)-\rho_\infty)\,e^{+iHt}\,e^{-t/\tau}}$$

בספרות נהוג לסמן ρ_0. (זה בעצם ρ בש"מ)

במקרה שלנו $[\rho_0, H]=0 \Rightarrow [\rho_0, e^{\pm iHt}]=0$

<u>הערה</u>: כל מה שרשום פה מתאר את המערכת הקטנה הדבר היחיד שקושר אותנו למערכת הגדולה הוא הטמפ' ו-τ הגודל הפנומנולוגי.

בדיקה:

$$\rho(t=0) = \rho(0) \quad \checkmark$$

$$\frac{\partial\rho}{\partial t} = -iH(\rho(t)-\rho_\infty) + i(\rho(t)-\rho_\infty)H - \frac{1}{\tau}(\rho(t)-\rho_\infty)$$

$$\frac{d\rho}{dt} + i[H,\rho] = -\frac{1}{\tau}(\rho(t)-\rho_\infty)$$

המערכת שלנו אכן מקיימת את המשוואה.

$$\rho_\infty = \frac{1}{Z}\begin{pmatrix} e^{-\beta\epsilon_+} & \\ & e^{-\beta\epsilon_-}\end{pmatrix}$$

בבסיס H מלוכסן הדבר נראה כך:

$$\rho_0 = \frac{1}{2}\begin{pmatrix} 1+\frac{\tilde{\epsilon}}{\sqrt{\ }} & -\frac{V}{\sqrt{\ }} \\ -\frac{V}{\sqrt{\ }} & 1-\frac{\tilde{\epsilon}}{\sqrt{\ }}\end{pmatrix}$$

$$\rho(t) = e^{-iHt}\rho(0)\,e^{iHt}e^{-t/\tau} + \rho_0(1-e^{-t/\tau}).$$

נעבוד בבסיס שבו H מלוכסן.

$$e^{iHt} = \begin{pmatrix} e^{iE_+t} & 0 \\ 0 & e^{+iE_-t}\end{pmatrix}$$

$$e^{-iHt}\rho(0)e^{iHt} = \frac{1}{2}\begin{pmatrix} 1+\frac{\tilde{\varepsilon}}{\sqrt{}} & -\frac{V}{\sqrt{}}e^{i(E_+-E_-)t} \\ -\frac{V}{\sqrt{}}e^{-i(E_+-E_-)t} & 1-\frac{\tilde{\varepsilon}}{\sqrt{}} \end{pmatrix}$$

$$E_\pm = \frac{\varepsilon_1+\varepsilon_2}{2} \pm \sqrt{\left(\frac{\varepsilon_1-\varepsilon_2}{2}\right)^2+V^2}$$

$$\rho(t) = \begin{pmatrix} \frac{e^{-\beta E_+}}{Z}\left(1-e^{-t/\tau}\right)+\frac{1}{2}\left(1+\frac{\tilde{\varepsilon}}{\sqrt{\tilde{\varepsilon}^2+V^2}}\right)e^{-t/\tau} & -\frac{1}{2}\frac{V}{\sqrt{\tilde{\varepsilon}^2+V^2}}e^{-i(E_+-E_-)t}e^{-t/\tau} \\ -\frac{1}{2}\frac{V}{\sqrt{}}e^{-i(E_+-E_-)t}e^{-t/\tau} & \frac{e^{-\beta E_-}}{Z}\left(1-e^{-t/\tau}\right)+\frac{1}{2}\left(1-\frac{\tilde{\varepsilon}}{\sqrt{\tilde{\varepsilon}^2+V^2}}\right)e^{-t/\tau} \end{pmatrix}$$

אפשר לבדוק שה-trace $=1$.
בזמן $t=0$ מצבים טהורים. שבבסיס הזה חייב
להיות תערובת של $|+\rangle$ ו-$|-\rangle$.
בזמן $t\to\infty$ סכום לא קוהרנטי של מצבים
עם משקלים מתאימים. יש סיכוי בולצמני להיות ב-$|+\rangle$
וסיכוי בולצמני להיות במצב אחר. עבור זמן t
המערכת היא תערובת של שני מצבים.
לאחר זמן מסוים ~~פשוט~~ המצבים הסטטיסטית דועכים לכוון שמצוינים (+) או (-)?
מצב שהוא תערובת $|+\rangle$ ו-$|-\rangle$ ואין סיכוי להיות
במצב הטהור. ~~(כלומר ... של הקוהרנטיות?)~~.
(בזמן אינסוף המערכת שנמצאת בהתפלגות בולצמן).

$\boxed{4}$ (a) choose a discrete set of functions $\{\phi_i\}$ Such as that:

$$\hat{\psi}(r) = \sum_i \phi_i(r)\,\hat{a}_i$$

$$\int \phi_i^*(r)\phi_j(r)\,dr = \delta_{ij} \quad \& \quad \cancel{\sum \phi_i(r)\phi_j^*(r)}$$

~~From~~

$$\delta(r-r') = [\psi(r), \psi^+(r')] = \sum_{i,j} \phi_i(r)\phi_j(r')[a_i, a_j^+]$$

Multiply by $\phi_k(r')$ and $\int \dots dr'$

$$\phi_k(r) = \sum_i \phi_i(r)[a_i, a_k^+]$$

Multiply by $\phi_\ell^*(r)$ and integrate:

$$\delta_{\ell k} = [a_\ell, a_k^+]$$

~~Find~~ ~~Let us find coherent states in each a_i~~

~~A~~ coherent state ~~in~~ of a_i is:

$$|\alpha_i) = e^{-\frac{|\alpha_i|^2}{2} + \alpha_i \hat{a}_i^+}|0\rangle.$$

← Maybe should show how to find it, but we saw it in class.

$$|\{\alpha_i\}\rangle = \bigotimes_i |\alpha_i\rangle.$$

$$\psi(r)\,|\{\alpha_i\}\rangle = \sum_j \phi_j(r)\,a_j\,|\{\alpha_i\}\rangle = \sum_i \phi_j(r)\alpha_j\,|\{\alpha_i\}\rangle$$

So the ~~eigenfun~~ eigenstates of $\psi(r)$ are $|\{\alpha_i\}\rangle$ and the ~~eigenfunctions are~~ eigenvalues are $\psi_r = \sum_i \phi_i(r)\alpha_i$.

$$\langle\{\alpha_i\}|N|\{\alpha_i\}\rangle = \int \langle\{\alpha_i\}|\psi^+(r)\psi(r)|\{\alpha_i\}\rangle\,dr =$$

$$= \sum_{i,j}\int \phi_i^*(r)\phi_j(r)\,\langle\{\alpha_i\}|a_i^+a_j|\{\alpha_i\}\rangle\,dr = \sum_i \langle\{\alpha_i\}|a_i^+a_j|\{\alpha_i\}\rangle =$$

$$= \sum_i |\alpha_i|^2.$$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

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[illegible]

[illegible]

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[illegible]

[illegible]

[illegible]

$$H = H_0 + V_{int}$$

$$H_0 = \int \psi^+(r) \underbrace{\left(-\frac{\hbar^2}{2m}\nabla^2 + U(r)\right)}_{h_0} \psi(r)\, dr$$

$$V_{int} = \frac{1}{2}\int \psi^+(r')\psi^+(r)\, V\, \psi(r)\psi(r')\, dr\, dr'$$

If we assume that $\{\phi_i\}$ are eigenfunctions of h_0 than

$H_0 = \sum_i \varepsilon_i a_i^+ a_i$. and general interaction is $V_{int} = \sum_{ijkl} V_{ijkl} a_i^+ a_j^+ a_l a_k$.

$$\langle \{\alpha_i\} | H_0 | \{\alpha_i\} \rangle = \sum_i \varepsilon_i |a_i|^2$$

$$\langle \{\alpha_i\} | V_{int} | \{\alpha_i\} \rangle = \sum V_{ijkl}\, \alpha_i^* \alpha_j^* \alpha_l \alpha_k .$$

$$V_{ijkl} = \langle ij | V | kl \rangle .$$

If V is diagonalized ~~with~~ in our basis then $\langle \{\alpha_i\} | V_{int} | \{\alpha_i\} \rangle = \sum V_{ij} |\alpha_i|^2 |\alpha_j|^2$.

Overall:

$$\langle \{\alpha_i\} | H | \{\alpha_i\} \rangle = \sum_i \varepsilon_i |\alpha_i|^2 + \sum_{ijkl} V_{ijkl}\, \alpha_i^* \alpha_j^* \alpha_l \alpha_k .$$

(b) a, a^+ are ladder operators, b, b^+ are creation & annihilation operators.

Both of them have the same commutation relations.

On the other side a, a^+ are operators in a Hilbert space of functions while b, b^+ are operators in Fock-Space.

Another difference is that b, b^+ generate particles ~~exitatio~~ with certain energy while a, a^+ raise (or lower)

the energy of one particle. Therefore, when working with s.p. it is easier to use a, a^{+} while when working with ~~some~~ many body interactions b, b^{+} are a lot more useful.

Question 5

(a) $V = \begin{cases} V_0 & r \le a \\ 0 & a < r \end{cases}$ (get $a^3 V_0$ is constant !)

To solve this problem we'll assume that since a is small, ψ is approximately constant in the sphere i.e $\psi(r \le a) \approx \psi(0)$

We use Lippman-Schwinger before large distance approx. (Sakurai 7.1.22)

$$\psi(0) = \frac{1}{(2\pi)^{3/2}} - \frac{2m}{\hbar^2} \int d^3x' \frac{e^{-ik|\bar{x}'|}}{4\pi|\bar{x}'|} V(\bar{x}')\psi(\bar{x}') = \frac{1}{(2\pi)^{3/2}} - \frac{2mV_0}{\hbar^2} \int_0^a dr' r' e^{-ikr'} \psi(r')$$

$$\int_0^a dr' r' e^{-ikr'} = \left[\frac{ire^{-ikr}}{k} \Big|_0^a - \frac{i}{k} \int_0^a e^{-ikx} dr \right] = \frac{iae^{-ika}}{k} + \frac{1}{k^2}(e^{-ika} - 1)$$

$$\Rightarrow \psi(0) \approx \frac{1}{(2\pi)^{3/2}} - \frac{2mV_0}{\hbar^2 k^2}\left((1 + ika)e^{-ika} - 1\right) \cdot \psi(0)$$

$$\overset{ka \ll 1}{\approx} \frac{1}{(2\pi)^{3/2}} - \frac{2mV_0}{\hbar^2 k^2}((1+ika)(1-ika) - 1)\,\psi(0) = \frac{1}{(2\pi)^{3/2}} - \frac{2mV_0 \cancel{k^2} a^2}{\hbar^2 \cancel{k^2}}\psi(0)$$

$$\Rightarrow \psi(0)\left(1 + \frac{2ma^2V_0}{\hbar^2}\right) = \frac{1}{(2\pi)^{3/2}} \Rightarrow \psi(0) = \frac{\hbar^2}{(2\pi)^{3/2}(\hbar^2 + 2ma^2V_0)}$$

Finally we can calculate $f(k,k')$ (Sakurai 7.1.34)

$$f(k',k) = \frac{1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3 \int d^3x' \frac{e^{-i\bar{k}\bar{x}'}}{(2\pi)^{3/2}} V(\bar{x}')\psi(\bar{x}')$$

$$\approx \frac{1}{4\pi}\frac{2m}{\hbar^2}\frac{(2\pi)^3}{(2\pi)^{3/2}}\psi(0) V_0 2\pi \int_{-1}^{1} d\cos\theta \int dr\, r^2 e^{-ikr\cos\theta}$$

$$\overset{ka \ll 1}{\approx} \frac{mV_0}{\hbar^2 + 2ma^2V_0} \int \frac{i\,dr\,r^{\cancel{2}}}{\cancel{k}\cancel{r}}(e^{-ikr} - e^{ikr}) = \frac{mV_0}{k(\hbar^2 + 2ma^2V_0)}\, 2\int_0^a r\sin(kr)\, dr$$

$$= \frac{2mV_0}{\hbar^2 + 2ma^2V_0}\left[\frac{\sin(kr) - kr\cos(kr)}{k^3}\right]_0^a = \frac{2mV_0}{\hbar^2 + 2ma^2V_0}\left[\frac{\sin(ka) - ka\cos(ka)}{k^3}\right]$$

$$\approx \frac{2ma^3V_0}{3(\hbar^2 + 2ma^2V_0)} = \frac{a\,2m\,a^3V_0}{3(\hbar^2 a + 2ma^3V_0)} \xrightarrow[\substack{a^3V_0 \\ \text{is const.}}]{a \to 0} \frac{a\,\cancel{2m a^3 V_0}}{3(\cancel{2ma^3V_0})} \approx$$

$f(k) \approx \frac{a}{3}$ this is almost like hard sphere $\int f^2(k)\,d\Omega = \frac{4\pi a^2}{9}$

[If somone finds that $f \simeq \frac{a}{2}$ and not $\frac{a}{3}$ please notify!!!]

Obviously $\frac{d\sigma}{d\Omega} = \frac{a^2}{9}$ is not angle dependent,

We tend to think that this is because the potential has spherical symmetry, and is very small (think of hard spheres and δ function potentials)

Note that the result we obtained is $\sigma_{tot} = \frac{4a^2\pi}{9}$

We know that hard sphere result is $\sigma_{tot} = \pi a^2$ (the particle "sees" impenetrable barier of 2-dim sphere. πa^2)

In our case the total cross-section is somewhat smaller since the limit $a \to 0$ while keeping $a^3 V_0$ const although qualitatively similar, does not exceeds quantitavely hard sphere result - the particle has finite probability not to be scattered at all!

Q.5

(a) The integral Lippmann Schwinger Equation

$$\psi^{\pm}(x) = \phi(x) - \frac{2m}{\hbar^2}\int d^3x' \frac{e^{\pm ik|x-x'|}}{4\pi|x-x'|} U_0 \delta(x')\psi^{\pm}(x')$$

$$\psi^{\pm}(x) = \phi(x) - \frac{2m}{\hbar^2}\frac{e^{\pm ik|x|}}{|x|}\frac{U_0}{4\pi}\psi^{\pm}(0)$$

In the presence of the delta function $\psi(0) = 0$.

So the solution is

$$\psi(\vec{x}) = \begin{cases} 0 & \vec{x} = 0 \\ \phi(\vec{x}) & \vec{x} \neq 0 \end{cases}$$

$$\phi(\vec{x}) = \frac{e^{i\vec{k}\vec{x}}}{(2\pi\hbar)^{3/2}}$$

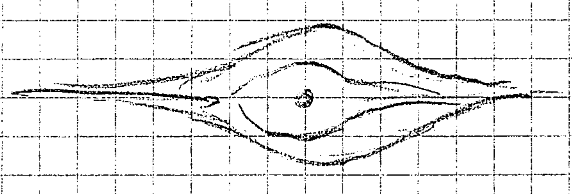
For $x > 0$

$$\psi(\vec{x}) = \frac{e^{i\vec{k}\vec{x}}}{(2\pi\hbar)^{3/2}}$$

So $f(k', k) = 0$!

Independent of the scattering angle.

Since the scatterer in placed at one point the wave can "pass around" it



So it is not scattered any where

The total cross section is zero as well since no scattering occurs.

(b) In the first Born approximation

$$f^{(1)}(k',k) = -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int d^3x'\, e^{i(k-k')x'}\, V(x')$$

Which is just the fourier trans. of V in the variable $\bar{k}'-\bar{k}$.

We examine:

$$U(r) = A\,\frac{e^{-r/R_s}}{r} \qquad A \equiv Z_1 Z_2 e^2$$

It is the Green's function for $\left(\nabla^2 - \frac{1}{R_s^2}\right)$ so:

$$\left(\nabla^2 - \frac{1}{R_s^2}\right)U(r) = A\,\delta(r)\cdot 4\pi$$

The eq in Fourier space is

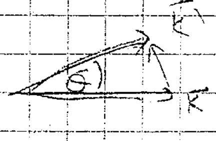
$$\left(-k^2 - \frac{1}{R_s^2}\right)U(\bar{k}) = A\cdot 4\pi$$

$$\Downarrow$$

$$U(\bar{k}) = \frac{-A}{k^2 + \frac{1}{R_s^2}}\cdot 4\pi$$

So

$$f^{(1)}(k',k) = +\frac{2m}{\hbar^2}\,\frac{A}{(\bar{k}-k')^2 + \frac{1}{R_s^2}}$$



$$|\bar{k}-\bar{k}'|^2 = 2k^2(1-\cos\theta)$$

So

$$f^{(1)}(\vec{k}', \vec{k}) = + \frac{2m}{\hbar^2} \frac{A}{2k^2(1-\cos\theta) + \frac{1}{R_S^2}}$$

Ruthorford scattering is the scattering of the Culomb potential in which

$$\frac{d\sigma}{d\Omega} = \left(\frac{Z_1 Z_2 e^2}{4E}\right)^2 \frac{1}{(1-\cos\theta)^2} \qquad E \equiv \frac{\hbar^2 k^2}{2m}$$

This is exactly the case for $R_S \to \infty$ where we have the Culomb potential.
In the case of the screened potential we don't have the divergence at $\theta = 0$

$$\left.\frac{d\sigma}{d\Omega}\right|_{\theta=0} \propto \left.\frac{1}{\left[2k^2(1-\cos\theta) + \frac{1}{R_S^2}\right]^2}\right|_{\theta=0} = R_S^4$$

At what range around the zero does it really make a difference?

$$1 - \cos\theta \sim \frac{1}{2R_S^2 k^2} = \frac{\hbar^2}{mE} \frac{1}{R_S^2}$$

$$\cos\theta \sim 1 - \frac{\hbar^2}{mE R_S^2}$$

We get a cone around the forward scattering in which the scattering differs from the Ruthoford scattering.

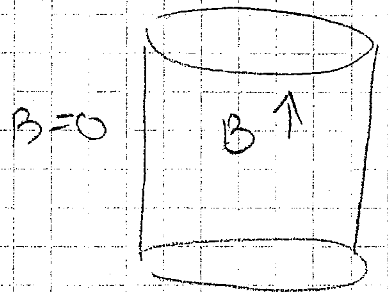
$$\sigma = 2\pi \int_{-1}^{1} \frac{d(\cos\theta)}{(1-\cos\theta)^2} \left(\frac{Z_1 Z_2 e^2}{4E}\right)$$

$$= 2\pi \left(\frac{Z_1 Z_2 e^2}{4E}\right) \frac{1}{1-\cos\theta}\Bigg|_{\cos\theta=-1}^{\cos\theta=1} = \infty$$

The physical reason for divergence comes from the fact that the culomb potential is not localized in space. $(1/r^2)$

We can see that this divergence does not exist in this potential because the differential cross-section itself does not diverge. This happens since not the potential is very localized around zero and decays exponentialy.

$$H = \frac{1}{2m}\left(p - \frac{e}{c}A\right)$$ .6



$$B = \begin{cases} B\hat{z} & r<R \\ 0 & r>R \end{cases} \qquad A = \begin{cases} \frac{Br}{2}\hat{\varphi} & r<R \\ \frac{BR^2}{2r} & r>R \end{cases}$$

<u>Inside</u> $A_r = 0 \quad A_z = 0 \quad A_\varphi = \frac{Br}{2}\hat{\varphi}$

$$\frac{1}{2m}\left(-\hbar^2\nabla^2 + \frac{e}{c}2i\hbar\,\bar{A}\cdot\bar{\nabla} + \left(\frac{e}{c}\right)^2 A^2\right)\psi = E\psi \qquad (\nabla\cdot A = 0)$$

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + i\cancel{2}\frac{e}{c}\frac{\hbar}{\cancel{2}m}\frac{Br}{2r}\frac{\partial}{\partial\varphi} + \left(\frac{e}{c}\right)^2\frac{B^2r^2}{2\cdot 4m}\right)\psi = E\psi$$

$$\left(-\frac{\hbar^2\nabla^2}{2m} + i\hbar\frac{\omega}{2}\frac{\partial}{\partial\varphi} + \frac{m\omega^2 r^2}{8}\right)\psi = E\psi$$

<u>Outside</u> $A_r = 0 \quad A_z = 0 \quad A_\varphi = \frac{BR^2}{2r}$

$$\frac{1}{2m}\left(-\hbar^2\nabla^2 + \frac{\cancel{2}e}{c}i\hbar\frac{BR^2}{\cancel{2}r}\frac{1}{r}\frac{\partial}{\partial\varphi} + \left(\frac{e}{c}\right)^2\left(\frac{BR^2}{2r}\right)^2\right)\psi = E\psi$$

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + \frac{i\omega\hbar}{2}\frac{R^2}{r^2}\frac{\partial}{\partial\varphi} + \frac{m\omega^2}{8}\frac{R^4}{r^2}\right)\psi = E\psi$$

נחפש פתרון בהפרדת משתנים.

$$\psi = \rho(r)\cdot e^{i\alpha\varphi}\cdot e^{ikz}$$

$$\left(\begin{aligned} \bar{\nabla}\cdot &= \frac{1}{r}\frac{\partial(r\cdot A_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial F_z}{\partial z} \\ \bar{\nabla}^2 &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2} \end{aligned}\right)$$

1

והמשוואה הרדיאלית:

$$\left(-\frac{\hbar^2}{2m}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)-\frac{\alpha^2}{r^2}\right)+i\hbar\frac{\omega}{2}\,i\alpha+\frac{m\omega^2r^2}{8}\right)\psi=E'\psi$$

$$\left(-\frac{\hbar^2}{2m}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)-\frac{\alpha^2}{r^2}\right)-i\omega\hbar\frac{R^2\alpha}{2r^2}+\frac{m\omega^2}{8}\frac{R^4}{r^2}\right)\rho(r)=E'\rho$$

$$\left\{\left(-\frac{\hbar^2}{2m}\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\right]\right)+\left(\frac{\hbar^2\alpha^2}{2mr^2}-\frac{\hbar\alpha\omega}{2}+\frac{m\omega^2r^2}{8}\right)\right\}\rho=E'\rho$$

$$\left\{\left(-\frac{\hbar^2}{2m}\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\right]\right)+\left(\frac{\hbar^2\alpha^2}{2mr^2}-\frac{\hbar\omega\alpha}{2}\frac{R^2}{r^2}+\frac{m\omega^2}{8}\frac{R^4}{r^2}\right)\right\}\rho=E'\rho$$

$$u(r)=\sqrt{r}\,\rho(r)$$

$$\frac{u(r)}{\sqrt{r}}=\rho(r)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\frac{u(r)}{\sqrt{r}}\right]=\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{u'(r)}{\sqrt{r}}-\frac{u(r)}{2r^{3/2}}\right)\right]=$$

$$=\frac{1}{r}\frac{\partial}{\partial r}\left[\sqrt{r}\,u'(r)-\frac{u(r)}{2\sqrt{r}}\right]=\frac{1}{r}\left[\cancel{\frac{1}{2\sqrt{r}}u'(r)}+\sqrt{r}\,u''(r)\right.$$

$$\left.-\cancel{\frac{u'(r)}{2\sqrt{r}}}+\frac{u(r)}{2\cdot 2\,r^{3/2}}\right]$$

$$=\frac{u''(r)}{\sqrt{r}}+\frac{u(r)}{4\,r^{5/2}}$$

בחוץ:

Outside:

$$-\frac{\hbar^2}{2m}\frac{u''(r)}{\sqrt{r}}+\frac{\hbar^2}{2m}\frac{u(r)}{4r^{5/2}}+\frac{\hbar^2\alpha^2}{2m}\frac{u(r)}{r^{5/2}}-\frac{\hbar\omega_\alpha R^2}{2r^{5/2}}+\frac{m\omega^2R^4}{8r^{5/2}}$$

$$=E'\frac{u(r)}{\sqrt{r}}$$

$$-\frac{\hbar^2}{2m}u''(r)+\left(-\frac{\hbar^2}{2m}\frac{1}{4r^2}+\frac{\hbar^2\alpha^2}{2m}\frac{1}{r^2}-\frac{\hbar\omega\alpha R^2}{2r^2}+\frac{m\omega^2R^4}{8r^2}\right)u$$

$$=E'u$$

קבלנו פוטנציאל אפק מסוף:

$$V_{out}(r) = \left(\frac{\hbar^2}{2m4} + \frac{\hbar^2\alpha^2}{2m} - \frac{\hbar^2\omega\alpha R^2}{2} + \frac{m\omega^2R^4}{8}\right)\frac{1}{r^2}$$

$$\boxed{V_{out} = \left[\frac{\hbar^2}{2m}\left(-\frac{1}{4}+\alpha^2\right) + \left(\frac{m\omega^2R^4}{8} - \frac{\hbar^2\omega\alpha R^2}{2}\right)\right]\frac{1}{r^2}}$$

עכשיו נציב לפוטנציאל הפנימי:

$$-\frac{\hbar^2}{2m}\frac{u''(r)}{\cancel{\sqrt{r}}} = \frac{\hbar^2}{2m}\frac{u(r)}{4r^2}\frac{1}{\cancel{\sqrt{r}}} + \left(\frac{\hbar^2\alpha^2}{2mr^2} - \frac{\hbar\alpha\omega}{2} + \frac{m\omega^2r^2}{8}\right)\frac{u(r)}{\cancel{\sqrt{r}}}$$

$$= E'\frac{u(r)}{\cancel{\sqrt{r}}}$$

$$-\frac{\hbar^2}{2m}u''(r) + \left(\frac{\hbar^2}{2m}\frac{1}{4r^2} + \frac{\hbar^2\alpha^2}{2mr^2} + \frac{m\omega^2r^2}{8} - \frac{\hbar\alpha\omega}{2}\right)u(r)$$

$$= E'u(r)$$

$$\boxed{V_{in}(r) = \frac{\hbar^2}{2m}\left(-\frac{1}{4}+\alpha^2\right)\frac{1}{r^2} + \left(\frac{m\omega^2r^2}{8} - \frac{\alpha\hbar\omega}{2}\right)}$$

$$V = \begin{cases} \dfrac{a_\alpha}{r^2} + \dfrac{bR^4}{r^2} - \dfrac{c_\alpha R^2}{r^2} & r>R \\[2ex] \dfrac{a_\alpha}{r^2} + br^2 - c_\alpha & r<R \end{cases}$$

עכשיו בעצם רוצים להוכיח שב $r<R$ הערך
המינימלי הוא חיובי כך שיש לנו בעצם
False vacuum

נגזור את הפוטנציאל עבור $r<R$

$$V' = -\frac{2a_\alpha}{r^3} + 2br = 0$$

$$\boxed{r^4 = \frac{a_\alpha}{b}}$$

$$r_{min} = \left(\frac{a_\alpha}{b}\right)^{1/4}$$

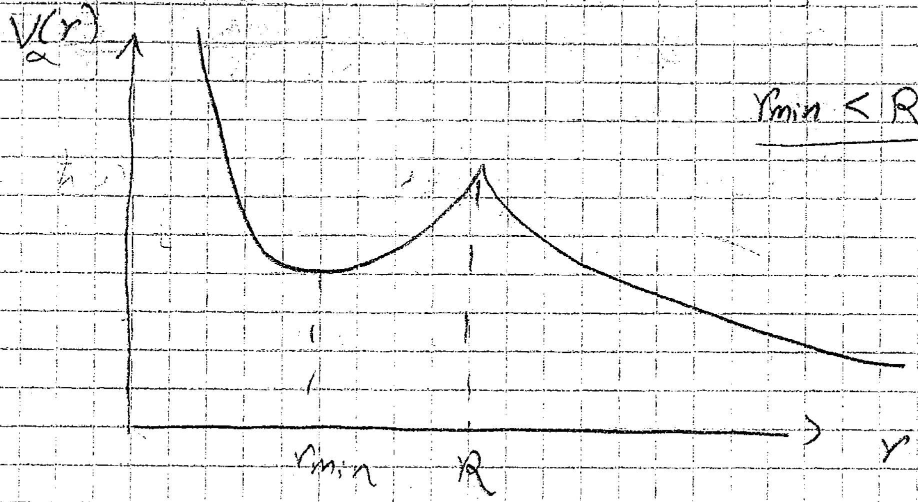
$$V(r_{min}) = \frac{a_\alpha}{\sqrt{\frac{a_\alpha}{b}}} + b\sqrt{\frac{a_\alpha}{b}} - C_\alpha =$$

$$= 2\sqrt{a_\alpha b} - C_\alpha = 2\cdot\sqrt{\left(\alpha^2 - \frac{1}{4}\right)\cdot\frac{\cancel{m}\omega^2}{8}\frac{\hbar^2}{2\cancel{m}}} - \frac{\alpha\hbar\omega}{2}$$

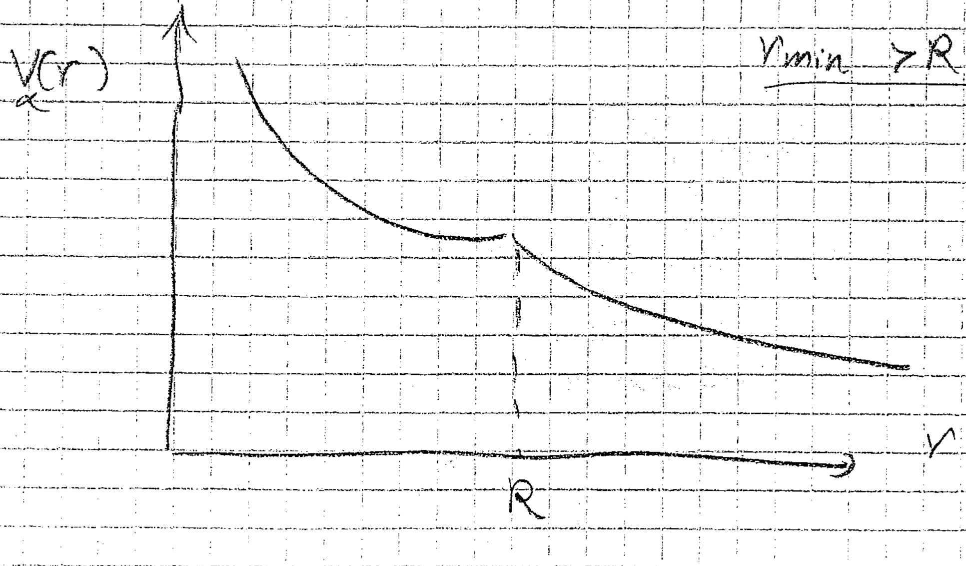
$$= \frac{2}{4}\hbar\omega\sqrt{\left(\alpha^2-\frac{1}{4}\right)} - \frac{\alpha\hbar\omega}{2} = \frac{\hbar\omega}{2}\left(\sqrt{\alpha^2-\frac{1}{4}} - \alpha\right)$$

הערך של הפוטנציאל במינימום יהיה חיובי עבור אלפות שליליות

עבור $r_{min} < R$



עבור $r_{min} > R$

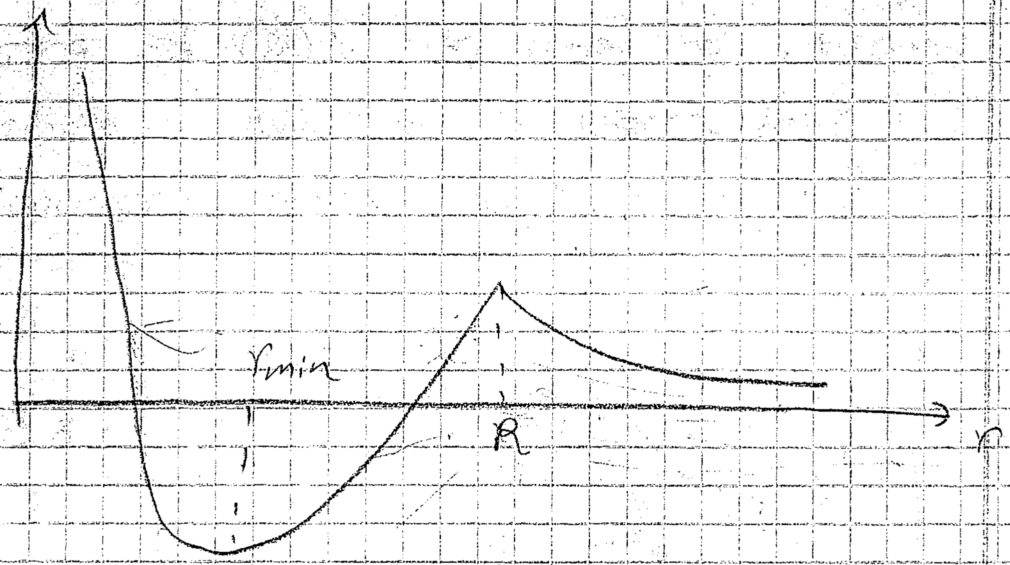


כך שיש לנו רק False vacuum ולכן עבור

α-ות שליליות לא יתכן מצב קשור שכן הוא

ודעך לבסוף.

עבור α-ות חיוביות הפוטנציאל נראה כך:



אם עומק הבור אינו עמוק ביחס לגובה המחסום גם
לא יתקיימו מצבים קשורים.

נקרב סביבה של הפוטנציאל ב-r_{min}

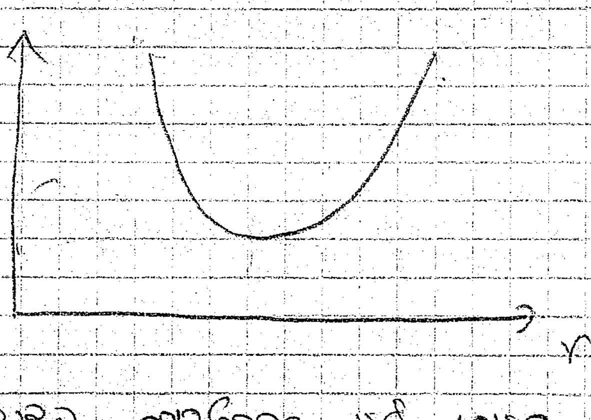
$$m\omega_0^2 = V'' = \left(+\frac{3\cdot 2 a_\alpha}{r^4} + 2b\right)\Bigg|_{r=r_{min}} = \frac{6a_\alpha}{r_{min}^4} + 2b$$

$$= \frac{6a_\alpha}{a_\alpha/b} + 2b = 8b = m\omega^2$$

$$\left|\frac{\hbar\omega}{2}\left(\sqrt{\alpha^2-\frac{1}{4}} - \alpha\right)\right| < \frac{\hbar\omega_0}{2}$$

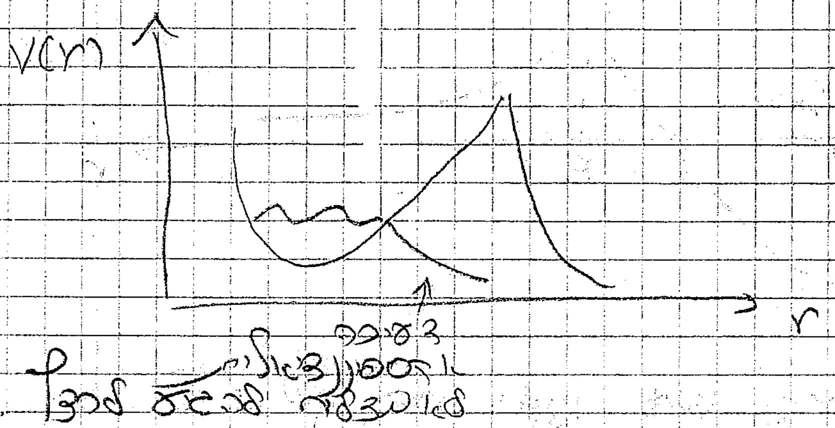
הבור שטוח שכזה ולכן רמת היסוד מעל 0 - כך שאין
מצבים קשורים גם במקרה זה.
כל שאר המקרים הנוספים נופלים תחת אותו שיקול
עבור $R \to \infty$ אנחנו מקבלים פוטנציאל:

$\alpha \neq 0$



מה שנותן לנו ספקטרום בדיד בהכרח.

כשעובר מרצף למס' דיסקרטי של רמות אנרגיה
הערך של V(R) ולכן יגדל רק שהמצבים הבודדים
הולכים ומתרחקים שהאפשרות לרצף
והופכת לרמה דיסקרטית אקספוננציאלית.



<u>הניוון</u> - נובע שרק כשהפכו מצבי רצף להיות מצבי
שכנים דיסקרטיים ולכן חייב להיות ניוון ששונה מאשר
בערך האנרגיה. כנראה יש ניוון בדרך זו.

(8) $H = \sigma_x \sum_\alpha \lambda_\alpha (a_\alpha^\dagger + a_\alpha) + \sum_\alpha \varepsilon_\alpha a_\alpha^\dagger a_\alpha \cdot \mathbb{1}_{2\times2}$

For a certain α (ommiting the α label):

$$\hookrightarrow \begin{pmatrix} \varepsilon a^\dagger a & \lambda(a^\dagger + a) \\ \lambda(a^\dagger + a) & \varepsilon a^\dagger a \end{pmatrix} \xrightarrow{\text{diagonalize}} \begin{pmatrix} \varepsilon a^\dagger a + \lambda(a^\dagger + a) & 0 \\ 0 & \varepsilon a^\dagger a - \lambda(a^\dagger + a) \end{pmatrix} =$$

$$= \varepsilon \begin{pmatrix} \left(a^\dagger + \frac{\lambda}{\varepsilon}\right)\left(a + \frac{\lambda}{\varepsilon}\right) - \frac{\lambda^2}{\varepsilon^2} & 0 \\ 0 & \left(a^\dagger - \frac{\lambda}{\varepsilon}\right)\left(a - \frac{\lambda}{\varepsilon}\right) - \frac{\lambda^2}{\varepsilon^2} \end{pmatrix} \equiv$$

$$\equiv \varepsilon \begin{pmatrix} b^\dagger b - \frac{\lambda^2}{\varepsilon^2} & 0 \\ 0 & c^\dagger c - \frac{\lambda^2}{\varepsilon^2} \end{pmatrix}$$

$b = a + \frac{\lambda}{\varepsilon}$, $c = a - \frac{\lambda}{\varepsilon}$

$b^\dagger b$ and $c^\dagger c$ are nuber operator:

$b^\dagger b \, |n_b\rangle = n_b |n_b\rangle$, $c^\dagger c \, |n_c\rangle = n_c |n_c\rangle$

the eigenvector of the matrix are:

$|+\rangle \equiv \frac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\right)$, $|-\rangle = \frac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}\right)$

Restoring the α indice we have:

summation convention

$$H = \varepsilon \begin{pmatrix} b_\alpha^\dagger b_\alpha - \frac{\lambda_\alpha \lambda_\alpha}{\varepsilon^2} & 0 \\ 0 & c_\alpha^\dagger c_\alpha - \frac{\lambda_\alpha \lambda_\alpha}{\varepsilon^2} \end{pmatrix}$$

define $b_\alpha^\dagger b_\alpha \left(\prod_{\otimes \alpha} |n_b^\alpha\rangle\right) \equiv n_b |n_b\rangle$

same for c.

$$\boxed{\begin{aligned} &\text{eigenstates: } |\Psi_b\rangle \equiv |n_b\rangle \otimes |+\rangle , \quad H|\Psi_b\rangle = \left(\varepsilon n_b - \frac{\lambda_\alpha \lambda_\alpha}{\varepsilon}\right)|\Psi_b\rangle \\ &|\Psi_c\rangle \equiv |n_c\rangle \otimes |-\rangle \quad H|\Psi_c\rangle = \left(\varepsilon n_c - \frac{\lambda_\alpha \lambda_\alpha}{\varepsilon}\right)|\Psi_c\rangle \end{aligned}}$$

summation convention

8. interaction between 2-level system and a bosonic system

$$H = \underset{\substack{\uparrow\\ \text{pauli-matrix (acting on the 2-level system states)}}}{\sigma_x} \sum_\alpha \lambda_\alpha (a_\alpha + a_\alpha^+) + \sum_\alpha \varepsilon_\alpha a_\alpha^+ a_\alpha$$

boson operators

$\lambda_\alpha, \varepsilon_\alpha$ - positive constants.

$$H = \sum_\alpha \varepsilon_\alpha \Big(a_\alpha^+ a_\alpha + \sigma_x \frac{\lambda_\alpha}{\varepsilon_\alpha} a_\alpha + \sigma_x \frac{\lambda_\alpha}{\varepsilon_\alpha} a_\alpha^+ + \underbrace{\sigma_x^2}_{\mathbb{1}_{2\times 2}} \Big(\frac{\lambda_\alpha}{\varepsilon_\alpha}\Big)^2 \Big) + - \sum_\alpha \frac{\lambda_\alpha^2}{\varepsilon_\alpha} =$$

$$= \sum_\alpha \varepsilon_\alpha \Big(a_\alpha^+ + \sigma_x \frac{\lambda_\alpha}{\varepsilon_\alpha}\Big)\Big(a_\alpha + \sigma_x \frac{\lambda_\alpha}{\varepsilon_\alpha}\Big) - \sum_\alpha \frac{\lambda_\alpha^2}{\varepsilon_\alpha}$$

denote: $A_\alpha = a_\alpha + \sigma_x \frac{\lambda_\alpha}{\varepsilon_\alpha}$

$A_\alpha^+ = a_\alpha^+ + \sigma_x \frac{\lambda_\alpha}{\varepsilon_\alpha}$

we have: $[A_\alpha, A_\beta] = [a_\alpha + \sigma_x \frac{\lambda_\alpha}{\varepsilon_\alpha}, a_\beta + \sigma_x \frac{\lambda_\beta}{\varepsilon_\beta}] = 0$

$[A_\alpha^+, A_\beta^+] = 0$

and $[A_\alpha, A_\beta^+] = [a_\alpha + \sigma_x \frac{\lambda_\alpha}{\varepsilon_\alpha}, a_\beta^+ + \sigma_x \frac{\lambda_\beta}{\varepsilon_\beta}] =$

$= [a_\alpha, a_\beta^+] = \delta_{\alpha\beta}$

$\Rightarrow$ A_α, A_α^+ obey the bosonic commutation relation of annihilation and creation operators,

$$\boxed{H = \sum_\alpha \varepsilon_\alpha A_\alpha^+ A_\alpha - \sum_\alpha \frac{\lambda_\alpha^2}{\varepsilon_\alpha}}$$

define the fock space:

let $|+\rangle_B$ be the normalised state of the bosonic system such that $a_\alpha |+\rangle_B = \frac{\lambda_\alpha}{\varepsilon_\alpha} |+\rangle_B$ for every α.
(כמו מצב קוהרנטי)

let $|\downarrow\rangle_2$ be the normalised 2-level system state such that $\sigma_x |\downarrow\rangle_2 = -|\downarrow\rangle_2$. define $|\phi_+\rangle = |+\rangle_B \otimes |\downarrow\rangle_2$

then for every α: $A_\alpha |\phi_+\rangle = a_\alpha |\phi_+\rangle + \frac{\lambda_\alpha}{\varepsilon_\alpha} \sigma_x |\phi_+\rangle =$

$$= \left(\frac{\lambda_\alpha}{\varepsilon_\alpha} - \frac{\lambda_\alpha}{\varepsilon_\alpha} \right) |\phi_+\rangle = 0$$

we can define another state $|\phi_-\rangle$ such that $A_\alpha |\phi_-\rangle = 0 \quad \forall \alpha$:

let $|-\rangle_B$ be the state of the bosonic system such that $a_\alpha |-\rangle_B = -\frac{\lambda_\alpha}{\varepsilon_\alpha} |-\rangle_B$ for every α.

let $|\uparrow\rangle_B$ be the 2-level system state such that $\sigma_x |\uparrow\rangle_2 = |\uparrow\rangle_2$. define $|\phi_-\rangle = |-\rangle_B \otimes |\uparrow\rangle_2$

$\Rightarrow \forall \alpha \quad A_\alpha |\phi_-\rangle = 0.$

now for every series $\{n_\alpha\} \subseteq \mathbb{N}$

$$|\{n_\alpha\}_+\rangle = \prod_\alpha \frac{1}{\sqrt{n_\alpha!}} (A_\alpha^+)^{n_\alpha} |\phi_+\rangle$$ is an eigenstate of H with eigenvalue $\sum_\alpha \varepsilon_\alpha n_\alpha - \sum_\alpha \frac{\lambda_\alpha^2}{\varepsilon_\alpha}$

and $$|\{n_\alpha\}_-\rangle = \prod_\alpha \frac{1}{\sqrt{n_\alpha!}} (A_\alpha^+)^{n_\alpha} |\phi_-\rangle$$ is also an eigenstate of H with the same eigenvalue.

(every superposition of $|\{n_\alpha\}_+\rangle$ and $|\{n_\alpha\rangle_-\rangle$ (for the same series $\{n_\alpha\}$) will be an eigenstate of H)

$\Rightarrow$ the eigenstates of H are:

$a|\{n_\alpha\}_+\rangle + b|\{n_\alpha\}_-\rangle$ where $a, b \in \mathbb{C}$ $\{n_\alpha\} \subseteq \mathbb{N}$

with eigenenergies: $\sum_\alpha \varepsilon_\alpha n_\alpha - \sum_\alpha \frac{\lambda_\alpha^2}{\varepsilon_\alpha}$

(in case of more degeneracies, i.e. if there exist $\{n_\alpha\} \neq \{m_\alpha\}$ such that [illegible]

Problem 9

Consider fermionic Hamiltonian

$$\mathcal{H} = \sum_{i,j} h_{ij} a_i^+ a_j + \frac{1}{2} \sum_{ijkl} V_{ijkl} a_i^+ a_j^+ a_l a_k$$

h_{ij} is single particle hamiltonian $h(\bar{r}_i) = \frac{\bar{P}_i^2}{2m} + V(\bar{r}_i)$

and $V(|\bar{r}-\bar{r}'|) = V_0$

(a) Find ground state of M electrons:

denote $|\Psi_0\rangle$ - single particle ground state and write
$\hookrightarrow h|\Psi_i\rangle = \varepsilon_i|\Psi_i\rangle$

$\mathcal{H}$ in coordinate representation:

$$\mathcal{H} = \sum_{i=1}^{M} h(\bar{r}_i) + \sum_{i<j} V_0 = \sum_{i=1}^{M} h(\bar{r}_i) + \frac{1}{2} M(M-1) V_0$$

recall Ψ are fermions and define anti-symm product

$$\boxed{\Psi(\bar{r}_1, \ldots, \bar{r}_M) = A\left[\Psi_0(\bar{r}_1)\Psi_1(\bar{r}_2)\cdot\ldots\cdot\Psi_{M-1}(\bar{r}_M)\right]}$$

[can also define $A[\Psi_{0,\uparrow}(\bar{r}_1)\Psi_{0,\downarrow}(\bar{r}_2)\cdot\ldots]$]

act with $\mathcal{H}$ to obtain:

$\boxed{E_0 = \sum_{i=1}^{M}\varepsilon_i + \frac{1}{2}M(M-1)V_0}$, when $h|\Psi_i\rangle = \varepsilon_i|\Psi_i\rangle$;

(b) Heisenberg eq:

$$\frac{da_m^+}{dt} = i[\mathcal{H}, a_m^+] = i\left(\sum_{ij} h_{ij}[a_i^+ a_j, a_m^+] + \frac{1}{2}\sum_{ijkl} V_{ijkl}[a_i^+ a_j^+ a_l a_k, a_m^+]\right)$$

We use the following facts: (easily verified)

1. $\{a_i, a_j^+\} = \delta_{ij}$

2. $[AB, C] = A\{B, C\} - \{A, C\}B$

3. $\{AB, C\} = A\{B, C\} - \{A, C\}B + 2CAB$

$$\frac{da_m^+}{dt} \overset{2}{=} i\Big(\sum_{ij} h_{ij}\, a_i^+ \underbrace{\{a_j, a_m^+\}}_{\delta_{jm}} + \frac{1}{2}\sum_{ijkl} V_{ijkl}\big(\underbrace{a_i^+ a_j^+ \{a_l a_k, a_m^+\}}_{*} - \underbrace{\{a_i^+ a_j^+, a_m^+\} a_l a_k}_{**}\big)\Big)$$

$$* \overset{3}{=} a_i^+ a_j^+ \big(a_l \underbrace{\{a_k, a_m^+\}}_{\delta_{km}} - \underbrace{\{a_l, a_m^+\}}_{\delta_{lm}} a_k + 2a_m^+ a_l a_k\big)$$

$$** = -\big(a_i^+ \underbrace{\{a_j^+, a_m^+\}}_{0} - \underbrace{\{a_i^+, a_m^+\}}_{0} a_j^+ + 2a_m^+ a_i^+ a_j^+\big) a_l a_k$$

$$\Rightarrow * + ** = a_i^+ a_j^+ a_l \delta_{km} - a_i^+ a_j^+ \delta_{lm} a_k$$

(note we $a_i^+ a_j^+ a_m^+ = a_m^+ a_i^+ a_j^+$ since we exchange twice)

summing up over the δ's we obtain:

$$\frac{da_m^+}{dt} = i\Big(\sum_i h_{im} a_i^+ + \frac{1}{2}\Big(\sum_{ijl} V_{ijml}\, a_i^+ a_j^+ a_l - \sum_{ijk} V_{ijkm}\, a_i^+ a_j^+ a_k\Big)\Big)$$

$$V_{ijkl} = \langle ij|V|kl\rangle = \int dr\,dr' \langle ij|V|rr'\rangle\langle rr'|kl\rangle$$

$$= \frac{1}{(2\pi)^6}\int dr\,dr'\, V_0\, e^{-ik_i r} e^{-ik_j r'} e^{ik_k r} e^{ik_l r'} = \frac{V_0}{(2\pi)^6}\int dr\,dr'\, e^{i(k_l - k_j)r'} e^{i(k_k - k_i)r} = V_0 \delta_{lj}\delta_{ki}$$

* note that all along the way i,k,j,l denote k-values.

$$\Rightarrow \frac{da_m^+}{dt} = i\Big(\sum_i h_{im} a_i^+ + \frac{V_0}{2}\Big(\sum_{ijl} \delta_{lj}\delta_{mi}\, a_i^+ a_j^+ a_l - \sum_{ijk} \delta_{mj}\delta_{ki}\, a_i^+ a_j^+ a_k\Big)\Big)$$

$$= i\Big(\sum_i h_{im} a_i^+ + \frac{V_0}{2}\Big(\sum_j a_m^+ a_j^+ a_j - \sum_i a_i^+ a_m^+ a_i\Big)\Big) = i\Big(\sum_i h_{im} a_i^+ + V_0 \sum_i a_m^+ a_i^+ a_i\Big)$$

Finally we can write:

(summation convention)

$$\boxed{\begin{cases} \dfrac{da_m^*}{dt} = i\left[a_i^* h_{im} + V_0 a_m^* a_i^* a_i\right] \\[2ex] \dfrac{da_m}{dt} = -i\left[h_{mi} a_i + V_0 a_i^* a_i a_m\right] = -i\left[h_{mi}^* a_i + V_0 a_m a_i^* a_i\right] \end{cases}}$$

To solve take unitary U_{ij} s.t $U_{ik}^* h_{im} U_{mj} = \varepsilon_k \delta_{kj}$, $U_{ji}^* U_{jl} = \delta_{il}$

consider b_i s.t $a_i = U_{ij} b_j$ and solve:

$$\frac{da_m}{dt} = \frac{d}{dt}(U_{mj} b_j) = -i\left[h_{mi} U_{ij} b_j + V_0 U_{mj} b_j U_{ik}^* b_k^* U_{il} b_l\right] \quad \backslash \cdot U_{mn}^*$$

$$\frac{d}{dt}\delta_{nj} b_j = -i\left[\varepsilon_n \delta_{nj} b_j + V_0 \delta_{nj} \delta_{kl} b_j b_k^* b_l\right]$$

$$\frac{db_n}{dt} = -i\left[\varepsilon_n b_n + V_0 b_n b_k^* b_k\right] = -i\left[\varepsilon_n + V_0 b_k^+ b_k\right] b_n$$

$$\Rightarrow \boxed{\begin{aligned} b_n &= b_n(0)\, e^{-i(\varepsilon_n + V_0 b_k^+ b_k)t} \\ b_n^+ &= e^{i(\varepsilon_n + V_0 b_k^+ b_k)t}\, b_n^+(0) \end{aligned}}$$

(can get the solution to a_n by $U_{nj} b_j$)

since $b_n(t) = e^{+iHt} b_n(0) e^{-iHt}$, act with $|M\rangle$:

$$* \quad b_n(0)\, e^{-i(\varepsilon_n + V_0 b_k^+ b_k)t} |M\rangle = e^{-i(\varepsilon_n + V_0 M)t}\, b_n(0)|M\rangle = e^{-i(\varepsilon_n + V_0 M)} |M-1\rangle$$

(Here we assume $|M\rangle$ has fermion in n-state, otherwise it's zero)

$$* \quad e^{+iHt} b_n(0) e^{-iHt} |M\rangle = e^{+iHt} b_n(0) e^{-iE_M t} |M\rangle = e^{-iE_M t} e^{+iHt} |M-1\rangle = e^{i(E_n - E_{M-1})t} |M-1\rangle$$

$$\Rightarrow \quad E_M - E_{M-1} = \varepsilon_n + V_0 M \quad , \quad E_{M+1} - E_M = \varepsilon_n + V_0(M+1)$$

This makes sense, the energy lost due to annihilation by b_n is the energy of the particle, ε_n, plus the energy of interaction. This defines the spectrum of the excitations.

The wave functions are obviously $b_n \underbrace{\prod_{i \in M} b_i |0\rangle}_{|M\rangle}$ for $|M-1\rangle$ and $b_n^+ \prod_{i \in M} b_i |0\rangle$ for $|M+1\rangle$.

(c) for the particle-hole operator $a_k^+ a_i$ (we use b_i's again)

$$b_k^+ b_i(t) = e^{i(\varepsilon_k + V_0 b_l^+ b_l)t} b_k^+(0)\, b_i(0)\, e^{-i(\varepsilon_i + V_0 b_l^+ b_l)t} = e^{iHt} b_k^+(0) b_i\, e^{-iHt}$$

$$* \quad b_k^+ b_i(t)|M\rangle = e^{i(\varepsilon_k - \varepsilon_i + V_0(M-M))t} |M\rangle$$

(again we assume that there was fermion in i, and wasn't fermion in k)

$$* \quad e^{iHt} b_k^+(0) b_i(0) e^{-iHt} |M\rangle = e^{-i(E(i) - E(k))t}$$

$\Rightarrow \quad E(k) - E(i) = \varepsilon_k - \varepsilon_i$, since interaction term is constant, we get this simple expression, which gives us the spectrum

the wave function is $b_k^+ b_i \prod_{l \in M} b_l^+ |0\rangle$.

(d) Hartree-Fock :

$$H = \sum_{ij} \langle i|h|j\rangle a_i^\dagger a_j + \frac{1}{2}\sum_{ijkl} \langle ij|V|kl\rangle a_i^\dagger a_j^\dagger a_k a_l$$

The wave function $|\xi\rangle = \prod_i a_i^\dagger |0\rangle$. (Here i is totaly general)

recall what was done in class:

$$\Rightarrow \langle \xi|H|\xi\rangle = \sum_i \langle i|h|i\rangle n_i + \frac{1}{2}\sum_{ij}\Big[\overbrace{\langle ij|V|ij\rangle}^{\text{direct integral}} - \overbrace{\langle ij|V|ji\rangle}^{\text{exchange integral}}\Big] n_i n_j$$

For fermions $n_i = 0, 1$. In position space: $\xi = \prod_i \varphi_i(r_i)$

$$\lambda_i \varphi_i(\bar r) = h\varphi_i(\bar r) + \int V(|\bar r - \bar r'|) \sum_j |\varphi_j(\bar r')|^2 d^3\bar r' \varphi_i(\bar r) - \int d^3\bar r' V(|\bar r - \bar r'|) \sum_j \varphi_j^*(\bar r')\varphi_j(\bar r)\varphi_i(\bar r')$$

$$= V_0 \overbrace{\int \sum_j |\varphi_j(\bar r')|^2 d^3\bar r'}^{M} \varphi_i(\bar r) - V_0 \sum_j \overbrace{\int \varphi_j^*(\bar r')\varphi_i(\bar r') d^3\bar r'}^{\delta_{ij}} \varphi_j(\bar r) + h\varphi_i(\bar r)$$

$$\Rightarrow \big(h + V_0(M-1)\big)\varphi_i(\bar r) = \lambda_i \varphi_i(\bar r)$$

we get that the φ_i's are eigenstates of h. Thus, they are the same as the ψ_i's from (a).

$$h\varphi_i = (\lambda_i - V_0(M-1))\varphi_i = \epsilon_i \varphi_i$$

Then

$$\langle \xi|H|\xi\rangle = \sum_i \int \varphi_i^* h \varphi_i + \frac{1}{2}\sum_{ij}\Big[\iint d\bar r\, d\bar r'\, \varphi_i^*(\bar r)\varphi_j^*(\bar r') \underbrace{V(\bar r, \bar r')}_{= V_0} \varphi_i(\bar r)\varphi_j(\bar r')$$

$$- \iint d\bar r\, d\bar r'\, \varphi_i^*(\bar r)\varphi_j^*(\bar r') V(\bar r, \bar r') \varphi_i(\bar r')\varphi_j(\bar r)\Big]$$

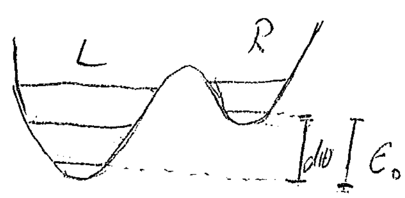
$$= \sum_i \epsilon_i + \frac{1}{2} V_0 \sum_{ij}(1 - \delta_{ij})$$

$$= \underline{\underline{\sum_i \epsilon_i + \frac{M^2 - M}{2} V_0}}$$

we get the same energies.

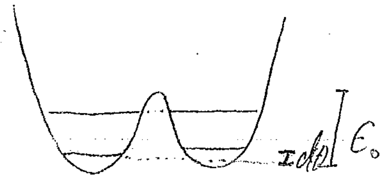
Problem 10

Initialy the ground state is $|L\rangle$ which is approximately the single well ground state of the left well.

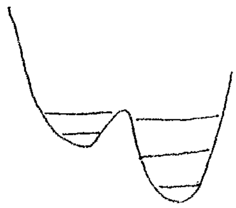


1

Let ϵ_0 be the initial distance between the bottom of the two wells and $d(t)$ the distance between the bottom of the Right well to the bottom of the left well (so in fig. 3 it is negative).



2



3

we define $R(t) = \frac{\epsilon_0 - d(t)}{\epsilon_0}$ so that $R(t)$ goes from 0 to 2 going from 1 to 3

Now, first we are going to assume that $R(t)$ is changing mush slower than the distance between the single well levels

$$\hbar \dot{R}(t) << E_{L,1} - E_{L,0} \sim \hbar\omega_L \qquad \left(\omega \simeq \sqrt{\tfrac{1}{2}V''(x)}\right)$$

$$\hbar \dot{R}(t) << E_{R,1} - E_{R,0} \sim \hbar\omega_R$$

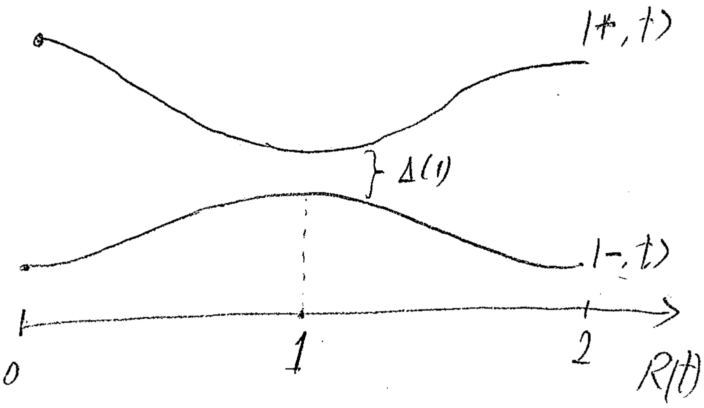
By the addiebatic approximation this means that the state will not undergo transition to other states localized in the same well! This means we can express the state of the particle, at any time as a linear combination of the ground states of the single wells $|R\rangle$ and $|L\rangle$

$$|\Psi(t)\rangle = C_L(t)\, e^{-i/\hbar \int_0^t E_L(t')dt'}|L\rangle + C_R(t)\, e^{-i/\hbar \int_0^t E_R(t')dt'}|R\rangle$$

So we effectively reduced the problem to a two-state system.

Initialy $\quad C_L(t_0) = 1 \quad C_R(t_0) = 0 \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

when we start lowering the right well $E_R - E_L$ is still much larger than $\hbar \dot{R}$ so there won't be any transitions. The interesting region is when the bottom of the right well gets very close to the bottom of the left one. Because of avoided level crossing the instantaneous eigenstates of the system denoted by $|-,t\rangle$ and $|+,t\rangle$ will look like this as a function of $R(t)$



No crossing

$|-,t\rangle$ and $|+,t\rangle$ are the Adiabatic basis.

There is no degeneracy when $R(t) = 1$ and the well is symmetric because tunneling effects split the levels

$$|-, R(t)=1\rangle = \frac{1}{\sqrt{2}}\left(|L\rangle + |R\rangle\right) \qquad \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$|+, R(t)=1\rangle = \frac{1}{\sqrt{2}}\left(|L\rangle - |R\rangle\right) \qquad \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

$\Delta(1)$ is given by the tunneling amplitude: $\Delta(1) = \frac{\hbar}{\pi}\sqrt{\omega_L \omega_R}\, e^{-\frac{1}{\hbar}\int \underbrace{\sqrt{2m(V-E)}}_{K}}$ $\rightarrow$ over the barrier

$K \rightarrow$ barrier integral

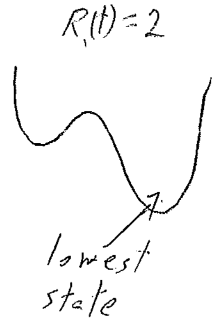
We see that if we change R slow enough so

$$\hbar \dot{R} \ll \Delta(1)$$ → This is what slow enough means

then by the adiabatic theorem the partice which started at

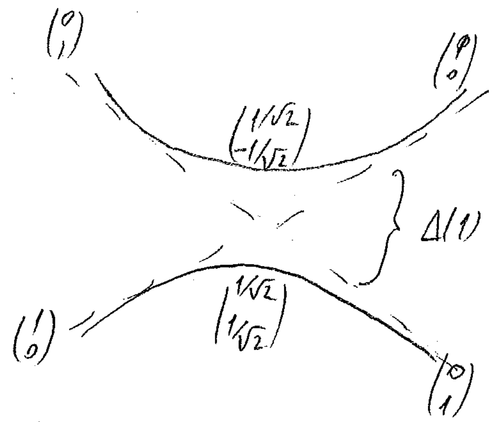
$$|-, R(t_0)=0\rangle = |L\rangle$$

must stay in the adiabatic state $|-, R(t)\rangle$.
But we know that $|-, R(t)=2\rangle = |R\rangle$
so the particle moves from the left well to the right well.



In the $|L\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ $|R\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ basis

(1) $$H = \begin{pmatrix} E_L(R) & \Delta(R)/2 \\ \Delta(R)/2 & E_R(R) \end{pmatrix}$$



with eigenvalues

$$E_\pm(R) = \frac{E_L + E_R}{2} \pm \frac{1}{2}\sqrt{(E_L - E_R)^2 + \Delta^2}$$

we see again that $\Delta(1)$ is the gap when $E_R = E_L$

If we solve Schrodinger eq. with H from (1) we expect the solution $|-, t\rangle$ to go from $\begin{pmatrix}1\\0\end{pmatrix}$ to $\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$ to $\begin{pmatrix}0\\1\end{pmatrix}$ as in the figure, so the particle goes from the left well to the right.

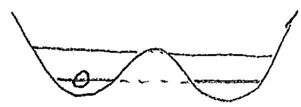
(b) In the simplest semiclassical approximation of levels in each well the energies are given by the Bohr somerfeld quantization of periodic orbits

$$\int_{q_{L-}}^{q_{L+}} dq \sqrt{2m(E-V)} = (n_l + \tfrac{1}{2})\hbar$$

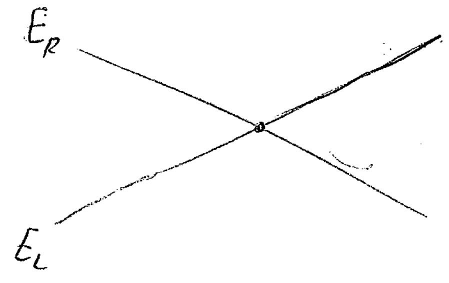
$$\int_{q_{R-}}^{q_{R+}} dq \sqrt{2m(E-V)} = (n_r + \tfrac{1}{2})\hbar$$

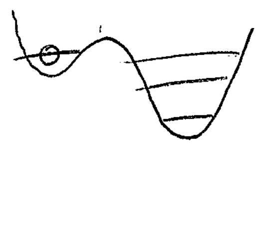
where the limits are the classical turning points

It is clear that in this case, when the double well is symmetric, we will have exactly the same energy levels in each well.



Around $R \simeq 1$ then the levels will look like





Since this simple semiclassical approx. does not consider tunneling at all, the particle would remain in the left well. In order to recover the above result we would have to include tunneling effects (using WKB or instantons), which would cause the particle to "leak" into the right well.

שאלה 11

(a) ניקח שדה אמ"ג קלאסי המתאר קרינת אנטנה, אפשר לפרק למונוכרומטיות

של גלים מישוריים כמו פתרון שדה כללי!

נגדיר מצב קוהרנטי:

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

מצב עצמי של אופרטור הכמות

$$\hat{a}|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \sqrt{n}|n-1\rangle$$

$$= \alpha e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^{n-1}}{\sqrt{(n-1)!}} |n-1\rangle = \alpha e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle = \alpha|\alpha\rangle$$

$$\langle\alpha|a^\dagger = \alpha^*$$

$$|\alpha_{k,\alpha}\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n_{k\alpha}\rangle \otimes |n_q = 0 \quad \forall \vec{q} \neq \vec{k}\rangle$$

$$\vec{E}(\vec{r},t) = i \sum_{\vec{k},\alpha} \sqrt{\frac{\hbar\omega_k}{\Omega}}\, \vec{\lambda}_\alpha \left(\hat{a}_k e^{i(\vec{k}\cdot\vec{r}+\omega_k t)} - \hat{a}_k^\dagger e^{-i(\vec{k}\cdot\vec{r}+\omega_k t)}\right)$$ (ש"ס 10)

נבדוק את הקירוב!

על התוחלת של מצב קוהרנטי $|\alpha_{q,\sigma}\rangle$:

$$\langle\alpha_{q,\sigma}|\vec{E}(\vec{r},t)|\alpha_{q,\sigma}\rangle = \langle\alpha_{q,\sigma}| i \sum_{\vec{k},\alpha} \sqrt{\frac{\hbar\omega_k}{\Omega}}\, \vec{\lambda}_\alpha \left(\hat{a}_k e^{i(\vec{k}\cdot\vec{r}+\omega_k t)} - \hat{a}_k^\dagger e^{-i(\vec{k}\cdot\vec{r}+\omega_k t)}\right)|\alpha_{q,\sigma}\rangle$$

$$= i \sum_{\vec{k},\alpha} \sqrt{\frac{\hbar\omega_k}{\Omega}}\, \vec{\lambda}_\alpha\, \delta_{\vec{q},\vec{k}}\delta_{\alpha,\sigma} \left(\alpha e^{i(\vec{k}\cdot\vec{r}+\omega_k t)} - \alpha^* e^{-i(\vec{k}\cdot\vec{r}+\omega_k t)}\right)$$

$$= i \sqrt{\frac{\hbar\omega_q}{\Omega}}\, \vec{\lambda}_\sigma\, 2i\, \mathrm{Im}\left(\alpha e^{i(\vec{q}\cdot\vec{r}+\omega_q t)}\right)$$

$$\underset{\alpha=|\alpha|e^{i\varphi}}{=} -2\sqrt{\frac{\hbar\omega_q}{\Omega}}\, \vec{\lambda}_\sigma\, |\alpha| \sin(\vec{q}\cdot\vec{r}+\omega_q t+\varphi) \,//$$

נבדוק מקרה אחר של משהו הקלאסי כי נבדוק שונות:

$$E^2(\vec{r},t) = -\sum_{\vec{k},\alpha}\sum_{\vec{q},\sigma} \frac{\hbar}{\Omega}\sqrt{\omega_k\omega_q}\, \vec{\lambda}_\alpha\vec{\lambda}_\sigma \Big(a_k a_q e^{i(\vec{k}\cdot\vec{r}+\vec{q}\cdot\vec{r}+\omega_k t+\omega_q t)} +$$

$$+ a_k^\dagger a_q^\dagger e^{-i(\vec{k}\cdot\vec{r}+\vec{q}\cdot\vec{r}+\omega_k t+\omega_q t)} - a_k a_q^\dagger e^{i(\vec{k}\cdot\vec{r}-\vec{q}\cdot\vec{r}+\omega_k t-\omega_q t)} - a_k^\dagger a_q e^{-i(\vec{k}\cdot\vec{r}-\vec{q}\cdot\vec{r}+\omega_k t-\omega_q t)}$$

$$[a_{k,\alpha}, a_{q,\sigma}^\dagger] = \delta_{k,q}\delta_{\alpha,\sigma}$$

$$\langle\alpha_{k',\alpha'}|E^2(\vec{r},t)|\alpha_{k',\alpha'}\rangle = -\sum_{\substack{\vec{k},\alpha\\ \vec{q},\sigma}} \frac{\hbar}{\Omega}\sqrt{\omega_k\omega_q}\, \vec{\lambda}_\alpha\vec{\lambda}_\sigma \Big[\delta_{kk'}\delta_{qk'}\delta_{\alpha\alpha'}\delta_{\sigma\alpha'}\Big(\alpha^2 e^{i((\vec{k}+\vec{q})\cdot\vec{r}+(\omega_k+\omega_q)t)} +$$

$$+ (\alpha^*)^2 e^{-i((\vec{k}+\vec{q})\cdot\vec{r}+(\omega_k+\omega_q)t)} - \delta_{kq}\delta_{\alpha\sigma}\, e^{i((\vec{k}-\vec{q})\cdot\vec{r}+(\omega_k-\omega_q)t)} - |\alpha|^2\delta_{kk'}\delta_{qk'}\delta_{\alpha\alpha'}\delta_{\sigma\alpha'}\Big($$

$$\Big(e^{i((\vec{k}-\vec{q})\cdot\vec{r}+(\omega_k-\omega_q)t)} + e^{-i((\vec{k}-\vec{q})\cdot\vec{r}+(\omega_k-\omega_q)t)}\Big)\Big]$$

$$= -\frac{\hbar}{\Omega}\omega_{k'}\Big[\alpha^2 e^{2i(\vec{k}'\cdot\vec{r}+\omega_{k'}t)} - 2|\alpha|^2 + (\alpha^*)^2 e^{-2i(\vec{k}'\cdot\vec{r}+\omega_{k'}t)}\Big] + \sum_{\vec{k},\alpha}\frac{\hbar}{\Omega}\omega_k$$

$$= -\frac{\hbar}{\Omega}\omega_{k'}\Big[\alpha e^{i(\vec{k}'\cdot\vec{r}+\omega_{k'}t)} - \alpha^* e^{-i(\vec{k}'\cdot\vec{r}+\omega_{k'}t)}\Big]^2 + \sum_{\vec{k},\alpha}\frac{\hbar}{\Omega}\omega_k$$

$$= \left(\langle\alpha_{k'\alpha'}|\vec{E}(\vec{r},t)|\alpha_{k'\alpha'}\rangle\right)^2 + \sum_{\vec{k},\alpha}\frac{\hbar}{\Omega}\omega_k$$

$$\langle \Delta E^2 \rangle = \langle E^2 \rangle - \langle E \rangle^2$$

$$= \sum_{\vec{k},\alpha} \frac{\hbar}{\Omega} \omega_k \longrightarrow$$ [illegible] [illegible]..

$$\frac{\langle \Delta E \rangle^2}{\langle E \rangle^2} = \frac{\hbar/_{\Omega}\, \omega_k}{2 \frac{\hbar \omega_k}{\Omega} |\alpha|^2} \xrightarrow[\alpha \to \infty]{} 0$$

(b) constant in time magnetic field, with existance of an external current. The new hamiltonian:

$$H = \sum_{\vec{k},\alpha} \hbar\omega_k \hat{a}^{+}_{\vec{k},\alpha} \hat{a}_{\vec{k},\alpha} - \sum_{\vec{k},\alpha} \left(\frac{\hbar}{\omega_k \Omega}\right)^{1/2} (\vec{J}_{-\vec{k}} \cdot \vec{\lambda}_\alpha) [\hat{a}_{\vec{k}\alpha} + a^{+}_{-\vec{k}\alpha}]$$

where $\vec{J}$ is the current which is time constant.

$$\vec{J}_{\vec{k}} = \int d^3r\, \vec{j}(\vec{r})\, e^{-i\vec{k}\cdot\vec{r}} \quad , \quad \vec{J}_{-\vec{k}} = \int d^3r\, \vec{j}(\vec{r})\, e^{i\vec{k}\cdot\vec{r}}$$

$$H = \sum_{\vec{k},\alpha} \left[\hbar\omega_k \hat{a}^{+}_{\vec{k},\alpha} \hat{a}_{\vec{k},\alpha} - \left(\frac{\hbar}{\Omega\omega_k}\right)^{1/2} (\vec{J}_{-\vec{k}} \cdot \vec{\lambda}_\alpha) a_{\vec{k},\alpha} - \left(\frac{\hbar}{\omega_k \Omega}\right)^{1/2} (\vec{J}_{\vec{k}} \cdot \vec{\lambda}_\alpha) a^{+}_{\vec{k},\alpha} \right]$$

$$= \sum_{\vec{k},\alpha} \hbar\omega_k \left(\hat{a}^{+}_{k,\alpha} - \left(\frac{\hbar}{\omega_k\Omega}\right)^{1/2} \frac{1}{\hbar\omega_k} (\vec{J}_{-\vec{k}} \cdot \vec{\lambda}_\alpha) \right) \left(\hat{a}_{\vec{k}\alpha} - \left(\frac{\hbar}{\omega_k\Omega}\right)^{1/2} \frac{1}{\hbar\omega_k} (\vec{J}_{\vec{k}} \cdot \vec{\lambda}_\alpha) \right)$$

$$- \sum_{\vec{k}\alpha} \cancel{\hbar\omega_k} \left(\frac{\cancel{\hbar}}{\omega_k\Omega}\right) \frac{1}{(\cancel{\hbar\omega_k})^{\cancel{2}}} (\vec{J}_{-\vec{k}} \cdot \vec{\lambda}_\alpha)(\vec{J}_{\vec{k}} \cdot \vec{\lambda}_\alpha)$$

define:
$$\hat{A}^{+}_{k,\alpha} = \hat{a}^{+}_{k,\alpha} - \left(\frac{\hbar}{\omega_k\Omega}\right)^{1/2} \frac{1}{\hbar\omega_k} (\vec{J}_{-\vec{k}} \cdot \vec{\lambda}_\alpha)$$

$$\hat{A}_{k,\alpha} = \hat{a}_{k,\alpha} - \left(\frac{\hbar}{\omega_k\Omega}\right)^{1/2} \frac{1}{\hbar\omega_k} (\vec{J}_{\vec{k}} \cdot \vec{\lambda}_\alpha)$$

$$\vec{J}_{\vec{k}}^{\,*} = \vec{J}_{\vec{k}}$$

and they satisfy commutation relations of creation & annihilation op.

$$H = \sum_{\vec{k}\alpha} \hbar\omega_k \hat{A}^{+}_{\vec{k},\alpha} \hat{A}_{k,\alpha} - \sum_{\vec{k},\alpha} \frac{1}{\omega_k^2 \Omega} \vec{J}_{-\vec{k}} \cdot \vec{J}_{\vec{k}}$$

now, we can define a new Fock space of which $A_{\vec{k},\alpha}|0\rangle = 0 \quad \forall \alpha$

and the states are defined by $A^{+}_{\vec{k},\alpha}|0\rangle$:→ defines $|\{n_{\vec{k},\alpha}\}\rangle$

$$A_{\vec{k}\alpha}|0\rangle = 0 \Rightarrow \left[a_{\vec{k},\alpha} - \left(\frac{\hbar}{\omega_k\Omega}\right)^{1/2} \frac{1}{\hbar\omega_k} (\vec{J}_{\vec{k}} \cdot \vec{\lambda}_\alpha) \right] |0\rangle = 0 \Rightarrow$$

$$a_{\vec{k},\alpha}|0\rangle = \left(\frac{\hbar}{\omega_k\Omega}\right)^{1/2} \frac{1}{\hbar\omega_k} (\vec{J}_{\vec{k}} \cdot \vec{\lambda}_\alpha) |0\rangle$$ (like a coherent state!)

$$\langle 0 | a^{+}_{\vec{k},\alpha} = \langle 0 | \left(\frac{\hbar}{\omega_k\Omega}\right)^{1/2} \frac{1}{\hbar\omega_k} \vec{J}_{-k} \cdot \vec{\lambda}_\alpha$$

The magnetic field:

$$\vec B = \vec k \times \vec E$$

$$= i \sum_{\vec k \alpha} \left(\frac{\hbar\omega_k}{\Omega}\right)^{1/2} \underbrace{\vec k \times \vec\lambda_\alpha}_{\cdot\forall\alpha,\ \vec\lambda'_\alpha \perp \vec\lambda_\alpha,\ \vec\lambda''_\alpha} \left(\hat a_k(t) e^{i\vec k\cdot\vec r} - \hat a_k^\dagger(t) e^{-i\vec k\cdot\vec r}\right)$$

This is very similar to $\vec E$ only with λ_α in a different direction!

$$\langle\{n_{k\alpha}\}|A^\dagger_{\vec k,\alpha}|\{n_{\vec k,\alpha}\}\rangle = \langle\{n_{\vec k,\alpha}\}|A_{\vec k,\alpha}|\{n_{\vec k,\alpha}\}\rangle = 0 \quad \text{מאחר ;}$$

מה שרציתי של התחלתיים

$$\langle\{n_{\vec k\alpha}\}|\vec B|\{n_{\vec k\alpha}\}\rangle = \langle\{n_{\vec k\alpha}\}| i\sum_{\vec k'\alpha'}\left(\frac{\hbar\omega_{k'}}{\Omega}\right)^{1/2} \vec k'\times\vec\lambda_{\alpha'}\left(\left(\hat A_{k'\alpha'}(t) + \left(\frac{\hbar}{\omega_{k'}\Omega}\right)^{1/2}\frac{1}{\hbar\omega_{k'}}\vec j_{k'}\cdot\vec\lambda_{\alpha'}\right)e^{i\vec k'\cdot\vec r} -\right.$$

$$\left. - \left(\hat A^\dagger_{k'\alpha'} + \left(\frac{\hbar}{\omega_{k'}\Omega}\right)^{1/2}\frac{1}{\hbar\omega_{k'}}\vec j_{-k'}\cdot\vec\lambda_{\alpha'}\right)e^{-i\vec k'\cdot\vec r}\right)|\{n_{\vec k\alpha}\}\rangle$$

$$= i\left(\frac{\cancel{\hbar\omega_k}}{\Omega}\right)^{1/2}\vec k\times\vec\lambda_\alpha\left(\frac{\cancel{\hbar}}{\cancel{\omega_k}\Omega}\right)^{1/2}\frac{1}{\cancel{\hbar\omega_k}}\left(\vec j_k\cdot\vec\lambda_\alpha e^{i\vec k\cdot\vec r} - \underbrace{\vec j_{-k}}_{=\vec j_k^*}\cdot\vec\lambda_\alpha e^{-i\vec k\cdot\vec r}\right)$$

$$= i\frac{1}{\Omega\omega_k}\vec k\times\vec\lambda_\alpha\, 2i\,\mathrm{Im}\left(\vec j_k\cdot\vec\lambda_\alpha e^{i\vec k\cdot\vec r}\right) \rightarrow \text{ממשי!}$$

מקבל רק תלות שלא תלויה בזמן כמו שרצינו, קיבלנו משהו ויש במשך זמן קווי, רק

$\alpha \leftarrow \vec j_k$, השדה נקבע רק ע"י הזרמים

כשמפעילים את $\vec B$ אז תוחלת של השדה הזה של $A_{k\alpha}$ (שהוא קוהרנטי של $a_{k\alpha}$) נקבל מה

שקיבלנו ב-(a) כלומר, נקבל משהו קלאסי שזור $\vec B$ (שמקיים את משוואות מקסוול),

(ונקבל גם שינוי שהוא לא שוב עבור ערכים עצמיים של ההרח-)

כל מה שצריך זה H שנתון לנו $\vec B$ שלא תלוי בזמן, כי:

$$\frac{d}{dt}\langle\hat B\rangle = \frac{1}{i\hbar}\langle[B,H]\rangle$$

$$= \frac{1}{i\hbar}\langle\{n_{k\alpha}\}|BH - HB|\{n_{k\alpha}\}\rangle$$

$$= \frac{1}{i\hbar}\langle\{n_{k,\alpha}\}|BE_{k\alpha} - E_{k,\alpha}B|\{n_{k\alpha}\}\rangle = 0.$$

<u>שדה חשמלי קבוע בזמן:</u> (שמקיים $\nabla\cdot E=\rho$), אומר ש- $E = E_T + E_L$, $\nabla\cdot E_T = 0$ ו-

$\nabla\cdot E_L = \rho$, ועם המונח הזה קיבלנו את האינטראקציה שקשור לחומר: $H \leftarrow \frac12\int E_L^2 d^3r = \frac12\sum\frac{e_i e_j}{|r_i - r_j|} = H_{matter}$

כלומר השדה המשתנה (הקוונטי) רק ע"י החומר, ולכן נעסוק שיתקיים שם

התוחלת של E_T מאפס, כלומר אפשר לקחת מצב של H כך ש: $\langle\{n_{k\alpha}\}|E_T|\{n_{k\alpha}\}\rangle = 0$

ואז: $\langle\vec E\rangle = \langle E_T\rangle + \langle E_L\rangle = \langle E_L\rangle \rightarrow$ קבוע בזמן

לא ציינת ש גירויים חסר.

12. electric dipole transitions, $\Delta m = -1$. (hydrogen)

* <u>one electron atom</u>; final state: $|n_f, l_f, m_f\rangle = |0,0,0\rangle$

$\Delta m = -1 \Rightarrow m_i = 1$, from the dipole selection rules we have: $|l_i - 1| \leq l_f = 0 \Rightarrow l_i = 1$.

$\Rightarrow$ we have transition only from initial states of the form $|n, 1, 1\rangle$.

denote the direction of the emitted photon by (θ, φ). $(\hat{R} = \sin\theta\cos\varphi\,\hat{x} + \sin\theta\sin\varphi\,\hat{y} + \cos\theta\,\hat{z})$.

from the derivation in class we know that the transition rate $\Gamma_{k,\alpha}$ will be proportional to $|\langle 0,0,0|\vec{\lambda}_\alpha \cdot \vec{R}|n,1,1\rangle|^2 =$

$$= \Bigg| \underbrace{\left[\int_0^\infty dr'\,r'^3 R_{n,1}(r') R^*_{0,0}(r')\right]}_{\text{indendent on } \vec{\lambda} \;\Downarrow\; \text{Independent on } \theta,\varphi} \underbrace{\left[\int_0^{2\pi} d\varphi' \int_0^\pi d\theta' \sin\theta' (\vec{\lambda}_\alpha \cdot \hat{r}') Y_{1,1}(\theta',\varphi') Y^*_{00}(\theta',\varphi')\right]}_{I} \Bigg|^2$$

calculation of I:

$$\vec{\lambda} \cdot \hat{r}' = \lambda_x \sin\theta'\cos\varphi' + \lambda_y \sin\theta'\sin\varphi' + \lambda_z \cos\varphi'$$

use:
$$\begin{cases} Y^*_{1,0}(\theta',\varphi') = \sqrt{\dfrac{3}{4\pi}} \cos\theta' \\ Y^*_{1,-1}(\theta',\varphi') = \sqrt{\dfrac{3}{8\pi}} (\sin\theta'\cos\varphi' + i\sin\theta'\sin\varphi') \\ Y^*_{1,1}(\theta',\varphi') = -\sqrt{\dfrac{3}{8\pi}} (\sin\theta'\cos\varphi' - i\sin\theta'\sin\varphi') \end{cases}$$

$\Updownarrow$

$$\begin{cases} \cos\theta' = \sqrt{\dfrac{4\pi}{3}} Y_{10}(\theta',\varphi') \\ \sin\theta'\cos\varphi' = \sqrt{\dfrac{2\pi}{3}} \left(Y^*_{1,-1}(\theta',\varphi') - Y^*_{1,1}(\theta',\varphi')\right) \\ \sin\theta'\sin\varphi' = -i\sqrt{\dfrac{2\pi}{3}} \left(Y^*_{1,-1}(\theta',\varphi') + Y^*_{1,1}(\theta',\varphi')\right) \end{cases}$$

$$\Rightarrow \vec{\lambda} \cdot \hat{r}' = \sqrt{\frac{2\pi}{3}} \left(\lambda_x (Y^*_{1,-1} - Y^*_{1,1}) - i\lambda_y (Y^*_{1,-1} + Y^*_{1,1}) + \sqrt{2}\,\lambda_z Y^*_{10}\right)$$

$$= \sqrt{\frac{2\pi}{3}}\left(Y^*_{1,-1}(\lambda_x - i\lambda_y) - Y^*_{1,1}(\lambda_x + i\lambda_y) + Y^*_{10}\cdot(\sqrt{2}\,\lambda_z)\right)$$

$$\Downarrow$$

$$I = \frac{1}{\sqrt{4\pi}}\sqrt{\frac{2\pi}{3}}\int_0^{2\pi} d\varphi' \int_0^{\pi} d\theta' \sin\theta' \Big[(\lambda_x - i\lambda_y)\,Y^*_{1,-1}(\theta',\varphi') - (\lambda_x + i\lambda_y)\,Y^*_{1,1}(\theta',\varphi') +$$

$$+\sqrt{2}\,\lambda_z\, Y^*_{10}(\theta',\varphi')\Big]\, Y_{11}(\theta',\varphi') =$$

$$Y^*_{0,0} = \frac{1}{\sqrt{4\pi}}$$

$$= -\frac{1}{\sqrt{4\pi}}\sqrt{\frac{2\pi}{3}}\,(\lambda_x + i\lambda_y) = -\frac{1}{\sqrt{6}}(\lambda_x + i\lambda_y)$$

$$\int_0^{2\pi} d\varphi' \int_0^{\pi} d\theta \sin\theta'\, Y^*_{1m}(\theta',\varphi')\, Y_{11}(\theta',\varphi') = \delta_{1m}$$

now for a given direction (θ,φ) of $\vec{k}$ we choose

$\vec{\lambda}_1 = \cos\theta\cos\varphi\,\hat{x} + \cos\theta\sin\varphi\,\hat{y} - \sin\theta\,\hat{z}$

$\vec{\lambda}_2 = -\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y}$

(so that $\vec{\lambda}_1\cdot\vec{\lambda}_1 = 1$, $\vec{\lambda}_2\cdot\vec{\lambda}_2 = 1$, $\vec{\lambda}_1\cdot\vec{\lambda}_2 = 0$, $\vec{\lambda}_1\cdot\vec{k} = \vec{\lambda}_2\cdot\vec{k} = 0$)

$$\Downarrow$$

$$\Gamma_{\hat{k}} = \sum_\alpha \Gamma_{\hat{k},\alpha} = \Gamma_{\hat{k},1} + \Gamma_{\hat{k},2} \propto \left(\left|-\frac{1}{\sqrt{6}}(\lambda_{1x} + i\lambda_{1y})\right|^2 + \left|-\frac{1}{\sqrt{6}}(\lambda_{2x} + i\lambda_{2y})\right|^2\right)$$

$$= \frac{1}{6}\left[(\cos\theta\cos\varphi)^2 + (\cos\theta\sin\varphi)^2 + \sin^2\varphi + \cos^2\varphi\right] =$$

$$= \frac{1}{6}\left[\cos^2\theta\cos^2\varphi + \cos^2\theta\sin^2\varphi + 1\right] =$$

$$= \frac{1}{6}\left[1 + \cos^2\theta\right]$$

ט ה-2 עמודים האלו היו קצת מיותרים כי לא עקרה
פרטי של מה שיהיה בהמשך, אבל נראה לנו שהוא
אולי הכוון שנעשה גם את זה.

for a more complicated atom, we assume that the matter states are eigenstates of the (total) angular momentum operator $\Rightarrow$ we consider states of the form: $|\beta, j, m\rangle$

from the derivation in class: we know that

$\Gamma_{k,\alpha}$ is proportional to $|\langle\beta, j, m| \vec{d}\cdot\vec{\lambda}_\alpha |\beta', j', m'\rangle|^2$

when $\vec{d}$ is the dipole moment: $\vec{d} = \sum_{i=1}^{N} e\vec{r}_i$
sum over the electrons.

$\vec{d}$ is a vector $\Rightarrow$ $d_{q=\pm1} = \mp \frac{d_x \pm i d_y}{\sqrt{2}}$

$$d_{q=0} = d_z$$

are spherical tensors of rank 1.

$\Rightarrow$ by wigner-eckart theorem

$\langle\beta, j, m| d_q |\beta', j', m'\rangle = \langle 1 q\, j' m' | j m\rangle \times$ number independent of m, m' & q.

but the clebsch-Gordan coefficient vanishes unless

$m' + q = m \Rightarrow q = m - m' = -\Delta m = -1$

$\Rightarrow \langle\beta, j, m| d_1 |\beta', j', m'\rangle = \langle\beta, j, m| d_0 |\beta', j', m'\rangle = 0$

$\langle\beta, j, m| \vec{d}\cdot\vec{\lambda}_\alpha |\beta', j', m'\rangle = \langle\beta, j, m| \sum_q d_q \lambda^*_{\alpha q} |\beta', j', m'\rangle =$

$= \sum_q \lambda^*_{\alpha q} \langle\beta, j, m| d_q |\beta', j', m'\rangle =$

$\lambda^*_{\alpha -1} \langle\beta, j, m| d_{-1} |\beta', j', m'\rangle = \frac{\lambda_{\alpha,x} + i\lambda_{\alpha,y}}{\sqrt{2}} \langle\beta, j, m| d_{-1} |\beta', j', m'\rangle$

$\Rightarrow \Gamma_{k,\alpha} \propto |\lambda_{\alpha,x} + i\lambda_{\alpha,y}|^2 = \lambda_{\alpha,x}^2 + \lambda_{\alpha,y}^2$

$\Rightarrow$ we get the same angular dependence as in the hydrogen atom case:

$\Gamma_k = \sum \Gamma_{k,\alpha} = \Gamma_{k,1} + \Gamma_{k,2} \propto \lambda_{1,x}^2 + \lambda_{1,y}^2 + \lambda_{2,x}^2 + \lambda_{2,y}^2 =$

$$= (\cos\theta\cos\varphi)^2 + (\cos\theta\sin\varphi)^2 + \sin^2\varphi + \cos^2\varphi = 1 + \cos^2\theta$$

פתרון:

$$\mathcal{H} = \frac{P_x^2}{2m} + \frac{P_y^2}{2M} + \frac{k(x^2+y^2)}{2} + \alpha xy \,, \quad |\alpha|<k$$

$m \ll M \Rightarrow$ y - החלק האיטי, x - החלק המהיר.

ראשית נפתור עבור החלק המהיר: $h(p_x, x; y)\phi_n(x;y) = \varepsilon_n(y)\phi_n(x;y)$

(y הוא פרמטר במשו' זו).

$$\Rightarrow \left[\frac{P_x^2}{2m} + \frac{kx^2}{2} + \alpha xy\right]\phi_n(x;y) = \varepsilon_n(y)\phi_n(x;y)$$

לאחר השלמה לריבוע: $h(P_x, x; y) = \frac{P_x^2}{2m} + \frac{k}{2}\left(x + \frac{\alpha}{k}y\right)^2 - \frac{\alpha^2}{2k}y^2$

זהו H.O מוזז ב-$\frac{\alpha}{k}y$ עם תדירות $\omega_f^2 = \frac{k}{m}$ והזזה באנרגיה של $\frac{\alpha^2}{2k}y^2$.

פונ' גל של H.O בתדר ω_f

$$\Rightarrow \boxed{\begin{aligned} \phi_n(x;y) &= n^{\omega_f}\left(x + \frac{\alpha}{k}y\right) \\ \varepsilon_n(y) &= \hbar\omega\left(n+\frac{1}{2}\right) - \frac{\alpha^2}{2k}y^2 \end{aligned}}$$

$n = 0, 1, 2 \ldots$

אפשר להשתמש בתכונות של נגזרות פונקציות הרמיט ותכונות אורתוגונליות

נבדוק אם יש פאזת ברי בבעיה!

מתאפס אם מצב שניתן לבחור ϕ_m ממשי.

$$A_m(y) = i\left\langle \phi_m(x;y) \right| \frac{\partial}{\partial y} \left| \phi_m(x;y) \right\rangle$$ פוטנציאל

ידוע שלפונ' גל של H.O יש זוגיות מוגדרת. כמו כן $\frac{\partial}{\partial y} = -\Pi\frac{\partial}{\partial y}\Pi$ זוגי.

$$\Rightarrow \frac{A_m(y)}{i} = \langle \phi_m(x,y)| \overbrace{\Pi\Pi}^{1} \frac{\partial}{\partial y} \overbrace{\Pi\Pi}^{1} |\phi_m(x;y)\rangle = -\langle \phi_m(x;y)|\frac{\partial}{\partial y}|\phi_m(x;y)\rangle$$

אם הזוגיות $1 \leftarrow 1 = 1^2$, ואם $-1 \leftarrow (-1)^2 = 1$

$\Rightarrow A_m(y) = 0 \Rightarrow$ אין Berry Phase בבעיה זו.

אם כן:

$$H_m^{BO} = \frac{P_y^2}{2M} + \frac{ky^2}{2} - \frac{\alpha^2}{2k}y^2 + \hbar\omega\left(m+\frac{1}{2}\right) =$$

$$= \frac{P_y^2}{2M} + \frac{1}{2}k\left(1 - \left(\frac{\alpha}{k}\right)^2\right)y^2 + \hbar\omega^f\left(m+\frac{1}{2}\right)$$

ושוב התקבל H.O עם $\omega_s^2 = \frac{k\left(1-\left(\frac{\alpha}{k}\right)^2\right)}{M}$

$$\Rightarrow \boxed{\begin{aligned} \xi_{mn}(y) &= n^{\omega_s}(y) \\ E_{mn} &= \hbar\omega^f\left(m+\frac{1}{2}\right) + \hbar\omega^s\left(n+\frac{1}{2}\right) \end{aligned}}$$

$n = 0, 1, 2 \ldots$

ג) סעיף א' (המשך) האם קירוב B.O מוצדק:

$$\omega_s^2=\frac{k\left(1-\left(\frac{a}{k}\right)^2\right)}{M} \qquad \omega_f^2=\frac{k}{m}$$

$M>>m$ וכן $1-\left(\frac{a}{k}\right)^2<1$ ← $\omega_f^2>>\omega_s^2$ ולכן התוצאה קונסיסטנטית עם ההנחה ש-x הוא החלק המהיר ו-y הוא החלק האיטי ← מוצדק להזניח את הנגזרות $\langle\phi_n|\frac{\partial}{\partial y}|\phi_m\rangle$ כי ההשתנות בפוטנציאל איטית ב-y.

איך אפשר להצדיק אחרי? מחוץ לאלכסון! $\frac{\partial}{\partial x}\sim\frac{\partial}{\partial y}$ כשפועלים על $\phi_m(x)$. $\langle\phi_n|\frac{\partial}{\partial x}|\phi_m\rangle\sim\sqrt{\frac{m\omega}{\hbar}}\left[\sqrt{n+1}\,\delta_{n,m+1}-\sqrt{n}\,\delta_{n,m-1}\right]$ ← מתאים אבל צריך לחלק ב-M ← $\frac{\sqrt{m}}{M}$ קטן ← הזנחה ← קירוב טוב.

פתרון מדויק: אפשר לעבור לקואורדינטות נורמליות:

$$|m_{ik}\overset{\omega^2}{\lambda}-k_{ik}|=\begin{vmatrix}m\lambda-k & -\frac{a}{2}\\ -\frac{a}{2} & M\lambda-k\end{vmatrix}=(m\lambda-k)(M\lambda-k)-\frac{a^2}{4}=$$

$$=mM\lambda^2-\lambda k(m+M)+\left(k^2-\frac{a^2}{4}\right)$$

$$\omega_{1,2}^2=\lambda_{1,2}=\frac{k(m+M)\pm\sqrt{k^2(m+M)^2-4mM\left(k^2-\frac{a^2}{4}\right)}}{2mM}\ \underset{M>>m}{\simeq}$$

$$\simeq\frac{k}{2m}\pm\sqrt{\frac{k^2}{4m^2}-\frac{\left(k^2-\frac{a^2}{4}\right)}{mM}}=$$

$$=\frac{k}{2m}\pm\frac{k}{2m}\sqrt{1-\frac{m}{M}\frac{4\left(k^2-\frac{a^2}{4}\right)}{k^2}}\simeq$$

$$\simeq\frac{k}{2m}\pm\left[\frac{k}{2m}-\frac{k}{M}\left(1-\left(\frac{a}{2k}\right)^2\right)\right]$$

$$\Rightarrow\omega_+\simeq\frac{k}{m}\to\omega_f \text{ כמו}$$

$$\omega_-=\frac{k}{M}\left(1-\left(\frac{a}{2k}\right)^2\right)\to\omega_s \text{ כמעט כמו}$$

פקטור 2 מסמנן ← זו בגלל ש-B.O נותן פתרון קירובי שמתאים אך לא זהה לגמרי.

ניתן להראות שהקואורד' הנורמליות מאוד קרובות ל-x ו-y.