13) GPE:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + U(\mathbf{r}-\mathbf{r}_0-\mathbf{v}t)\psi + g\left|\psi\right|^2\psi$$

Let's plug in the following:

$$\psi = \rho^{\frac{1}{2}}e^{i\chi}$$

$$\rightarrow i\hbar\left[\tfrac{1}{2\rho^{\frac{1}{2}}}\tfrac{\partial\rho}{\partial t} + i\rho^{\frac{1}{2}}\tfrac{\partial\chi}{\partial t}\right]e^{i\chi} = \tfrac{-\hbar^2}{2m}\left[-\tfrac{1}{4\rho^{\frac{3}{2}}}\left(\vec{\nabla}\rho\right)^2 + \tfrac{1}{2\rho^{\frac{1}{2}}}\nabla^2\rho + \tfrac{i}{\rho^{\frac{1}{2}}}\vec{\nabla}\rho\cdot\vec{\nabla}\chi + i\rho^{\frac{1}{2}}\nabla^2\chi - \rho^{\frac{1}{2}}\left(\vec{\nabla}\chi\right)^2\right]e^{i\chi} -$$

$$i\left(\tfrac{\hbar}{2\rho^{\frac{1}{2}}}\tfrac{\partial\rho}{\partial t}\right) - \left(\hbar\rho^{\frac{1}{2}}\tfrac{\partial\chi}{\partial t}\right) = -i\left(\tfrac{\hbar^2}{2m\rho^{\frac{1}{2}}}\vec{\nabla}\rho\cdot\vec{\nabla}\chi + \tfrac{\hbar^2\rho^{\frac{1}{2}}}{2m}\nabla^2\chi\right) + \left(\tfrac{\hbar^2}{8m\rho^{\frac{3}{2}}}\left(\vec{\nabla}\rho\right)^2 - \tfrac{\hbar^2}{4m\rho^{\frac{1}{2}}}\nabla^2\rho + U(\mathbf{r}-\mathbf{r}_0-\mathbf{v}\right.$$

And taking the real and imaginary parts separately:

$$\tfrac{\partial\rho}{\partial t} = -\tfrac{\hbar}{m}\vec{\nabla}\rho\cdot\vec{\nabla}\chi - \tfrac{\hbar\rho}{m}\nabla^2\chi$$

$$\tfrac{\partial\chi}{\partial t} = -\tfrac{\hbar}{8m\rho^2}\left(\vec{\nabla}\rho\right)^2 + \tfrac{\hbar}{4m\rho}\nabla^2\rho - \tfrac{1}{\hbar}g\rho - U(\mathbf{r}-\mathbf{r}_0-\mathbf{v}t) - \tfrac{\hbar}{2m}\left(\vec{\nabla}\chi\right)^2$$

The uniform solution is given in the abscence of u by $\nabla\rho = 0\,,\, \nabla\chi = 0$:

$$\tfrac{\partial\rho}{\partial t} = 0$$

$$\rightarrow \rho = \rho_0$$

$$\tfrac{\partial\chi}{\partial t} = -\tfrac{1}{\hbar}g\rho_0$$

$$\rightarrow \chi = \chi_0 - \tfrac{1}{\hbar}g\rho_0 t$$

$$\psi = \sqrt{\rho_0}exp\left[-\frac{i}{\hbar}V_0\rho_0 t + \chi_0\right]$$

Now, in order to obtain equations for small oscillations around this solution we take $\rho = \rho_0 + \delta\rho$ and $\chi = \chi_0 - \frac{1}{\hbar}V_0\rho_0 t + \delta\chi$ and get:

$$\frac{\partial\delta\rho}{\partial t} = -\frac{\hbar}{m}\nabla\delta\rho\cdot\nabla\delta\chi - \frac{\hbar\left(\rho_0+\delta\rho\right)}{m}\nabla^2\delta\chi \approx -\frac{\hbar\rho_0}{m}\nabla^2\delta\chi$$

$$\frac{\partial\delta\chi}{\partial t} = -\frac{\hbar}{8m\rho_0^2}\left(\vec{\nabla}\delta\rho\right)^2 + \frac{\hbar}{4m\rho_0}\nabla^2\delta\rho - \frac{\hbar}{2m}\left(\vec{\nabla}\delta\chi\right)^2 - \frac{1}{\hbar}g\delta\rho - U(\mathbf{r}-\mathbf{r}_0-\mathbf{v}t) \approx \frac{\hbar}{4m\rho_0}\nabla^2\delta\rho - \frac{1}{\hbar}g\delta\rho - U(\mathbf{r}-\mathbf{r}_0-\mathbf{v}$$

In other terms:

$$\frac{\partial\rho}{\partial t} = -\frac{\hbar\rho_0}{m}\nabla^2\chi$$

$$\frac{\partial\chi}{\partial t} = \frac{\hbar}{4m\rho_0}\nabla^2\rho - \frac{1}{\hbar}g\rho - U(\mathbf{r}-\mathbf{r}_0-\mathbf{v}t)$$

$$\ddot{\rho} = -\tfrac{\hbar\rho_0}{m}\nabla^2\left(\tfrac{\hbar}{4m\rho_0}\nabla^2\rho - \tfrac{1}{\hbar}g\rho - U(\mathbf{r}-\mathbf{r}_0-\mathbf{v}t)\right)$$

Fourier:

$$\rho = \int \rho_k e^{ikr} dk$$
$$U = \int U_k e^{ikr} dk$$
$$U_k = \int e^{-ikr} U(r) dr = U_0 \int e^{-ikr} \delta(r - r_0 - vt) dr = U_0 e^{-ik(r_0 + vt)}$$

WE get:

$$\ddot{\rho_k} = -\frac{\hbar\rho_0}{m} \nabla^2 \left(\frac{\hbar}{4m\rho_0} \nabla^2 \rho - \frac{1}{\hbar} g\rho - U(\mathbf{r} - \mathbf{r}_0 - \mathbf{v}t) \right)$$
$$\ddot{\rho_k} = -\frac{\hbar\rho_0}{m} \left(\frac{\hbar}{4m\rho_0} \nabla^2 \nabla^2 \rho - \frac{1}{\hbar} g \nabla^2 \rho - \nabla^2 U(\mathbf{r} - \mathbf{r}_0 - \mathbf{v}t) \right)$$
$$-\frac{\hbar\rho_0}{m} \left(\frac{\hbar}{4m\rho_0} k^4 \rho + \frac{1}{\hbar} g k^2 \rho + k^2 U_0 e^{-ik(r_0 + vt)} \right)$$
$$\ddot{\rho_k} = -\frac{\hbar^2}{4m^2} k^4 \rho_k - \frac{\rho_0 g}{m} k^2 \rho - \frac{\hbar\rho_0}{m} U_0 k^2 e^{-ik(r_0 + vt)}$$

Like shaked now:

$$\ddot{\rho_k} = -\frac{\hbar^2}{4m^2} k^4 \rho_k - \frac{\rho_0 g}{m} k^2 \rho_k - \frac{\hbar\rho_0}{m} k^2 U_0 e^{-ik(r_0 + vt)}$$

Now, in time:

$$\rho_{k,t} = \int e^{iwt} \rho_{k,w} dw$$
$$U_{k,t} = \int e^{iwt} U_{k,w} dw$$
$$U_{k,w} = \int e^{-iwt} U_{k,t} dt = e^{-ikr_0} \int dt e^{-it(kv + w)} = e^{-ikr_0} \delta(kv + w)$$

$$w^2 \rho_{k,w} = \frac{\hbar^2}{4m^2} k^4 \rho_{k,w} + \frac{\rho_0 g}{m} k^2 \rho_{k,w} + \frac{\hbar\rho_0}{m} k^2 U_0 e^{-ikr_0} \delta(kv + w)$$
$$\rho_{k,w} = \frac{\frac{\hbar\rho_0}{m} k^2 U_0 e^{-ikr_0} \delta(kv + w)}{w^2 - \frac{\hbar^2}{4m^2} k^4 - \frac{\rho_0 g}{m} k^2}$$

Fourier back:

$$\rho_{r,t} = \int dk dw e^{iwt + ikr} \frac{\frac{\hbar\rho_0}{m} k^2 U_0 e^{-ikr_0} \delta(kv + w)}{w^2 - \frac{\hbar^2}{4m^2} k^4 - \frac{\rho_0 g}{m} k^2}$$
$$\int dk e^{-ikvt + ikr} \frac{\frac{\hbar\rho_0}{m} k^2 U_0 e^{-ikr_0}}{k^2 v^2 - \frac{\hbar^2}{4m^2} k^4 - \frac{\rho_0 g}{m} k^2}$$
$$= \int dk e^{-ikvt + ikr - ikr_0} \frac{\frac{\hbar\rho_0}{m} U_0}{v^2 - \frac{\hbar^2}{4m^2} k^2 - \frac{\rho_0 g}{m}}$$

The singularities:

$$k^2 - \frac{4m^2}{\hbar^2} v^2 + \frac{4m\rho_0 g}{\hbar^2} = 0$$
$$k^2 = \frac{4m^2}{\hbar^2} v^2 - \frac{4m\rho_0 g}{\hbar^2}$$
$$k_\pm = \pm\sqrt{\frac{4m^2}{\hbar^2} v^2 - \frac{4m\rho_0 g}{\hbar^2}}$$

We will get real solutions for:

$$\frac{4m^2}{\hbar^2}v^2 > \frac{4m\rho_0 g}{\hbar^2}$$
$$v > \sqrt{\frac{\rho_0 g}{m}}$$

the result of the integral:

$$res = q/p' = e^{ik(r-vt-r_0)} \frac{\frac{\hbar\rho_0}{m}U_0}{-\frac{\hbar^2}{2m^2}k^2}_{|k=k_\pm}$$

$$I = 2\pi i \quad \frac{\frac{\hbar\rho_0}{m}U_0}{-\frac{\hbar^2}{2m^2}\left(\frac{4m^2}{\hbar^2}v^2 - \frac{4m\rho_0 g}{\hbar^2}\right)}\left(e^{ik_+(r-vt-r_0)} + e^{ik_-(r-vt-r_0)}\right)$$

for v greater than speed of sound ($= \frac{\rho_0 g}{m}$, from looking at the linearized dispersion relation in the absence of potential) we get real solution, thus waves that will carry energy.

for v smaller than speed of sound we get imaginary solutions which are decaying waves -> superfluidity.

* We need to check which contour to take, jordan's lemma and so on..