Introduction to Particles and Nuclear Physics

Class Exercise 5

Matan Parnas

Tel Aviv University

27/06/2024

Today's Topics:

1. Decays

2. Cross-Section

Decays

The wave-function of a free particle is given by $\psi(x^\mu) \propto e^{-ip^\mu x_\mu}$, which in the center-of-mass frame becomes: $\psi(t) \propto e^{-im_0 t}$

For a non-stable particle we want $|\psi(t)|^2$, which is proportional to the probability of finding the particle in time t, to vanish when $t \to \infty$.

This can be done by adding a term to the wave function:

$$\psi(t) \propto e^{-im_0 t} e^{-\frac{\Gamma}{2} t}$$

Decays

To understand the implications on the mass of a decaying particle, we apply Fourier Transformation and look at the wave-function in the energy domain:

$$\psi(m) = \int_0^\infty \left[e^{-im_0 t} e^{-\frac{\Gamma}{2}t}\right] e^{imt} dt = \frac{i}{(m - m_0) + i\frac{\Gamma}{2}}$$

We now have a mass distribution for the particle, which reflects the probability of measuring it with some mass m.

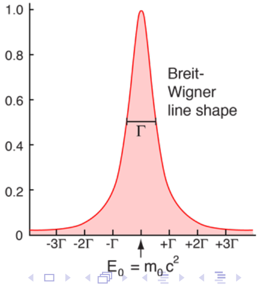
The probability density (up to a normalization constant) is obtained by taking the norm:

$$|\psi(m)|^2 = \frac{1}{(m - m_0)^2 + \frac{\Gamma^2}{4}}$$

This is known as **Breit-Wigner** distribution.

- It peaks at $m = m_0$
- It's width is determined by Γ
- The relation to the particle's *lifetime* is:

$$\tau = \frac{1}{\Gamma}$$



Decays

A few comments on the width of the distribution Γ:

- Γ is a characteristic of the **particle**, not the **measurement**
- Stable particles like an electron or proton do not have a width
- In experiments, new particles can be discovered by measuring the energy of the decay products and observing a "**resonance**"
- A particle can decay in a number of different ways ("channels") each with a different probability. The total width takes all of them into account $\Gamma = \sum_i \Gamma_i$

Question 1:

Consider the decay of a muon to an electron: $\mu^- \rightarrow e^- \bar{\nu}_e \nu_\mu$

a) What is the energy of the electron as a function of the neutrino energies ($E_\nu, E_{\bar{\nu}}$) and the angle between the neutrinos ($\theta_{\nu\bar{\nu}}$)? Work in the muon's rest frame, and assume neutrinos are massless.

We begin by writing a general expression for conservation of the norm squared of 4-momenta in the initial and final states, which is valid in any reference frame.

$$p_i^\mu = (E_\mu, \vec{p}_\mu)$$
$$p_f^\mu = (E_e + E_\nu + E_{\bar{\nu}}, \vec{p}_e + \vec{p}_\nu + \vec{p}_{\bar{\nu}})$$

Equating the squares of p_i^μ and p_f^μ gives us our first equation:

$$\mathrm{M}_\mu^2 = \textstyle\sum_{i,j}(E_i E_j - \vec{p}_i \cdot \vec{p}_j)$$
$$\mathrm{M}_\mu^2 = E_e^2 + E_\nu^2 + E_{\bar{\nu}}^2 + 2E_e(E_\nu + E_{\bar{\nu}}) + 2E_\nu E_{\bar{\nu}} - p_e^2 - p_\nu^2 - p_{\bar{\nu}}^2 - 2\vec{p}_e \cdot (\vec{p}_\nu + \vec{p}_{\bar{\nu}}) - 2\vec{p}_\nu \vec{p}_{\bar{\nu}}$$

Massless neutrino assumption gives $|p_\nu| = E_\nu$, and the electron mass obviously satisfies $M_e^2 = E_e^2 - p_e^2$. The dot product of neutrino 3-momenta gives the angle between the neutrinos ($\vec{p}_\nu \cdot \vec{p}_{\bar{\nu}} = |p_\nu||p_{\bar{\nu}}| \cos\theta_{\nu\bar{\nu}}$). Plugging these three identities into the above equation allows us to simplify it to read:

$$\mathrm{M}_\mu^2 = M_e^2 + 2E_e(E_\nu + E_{\bar{\nu}}) - 2\vec{p}_e \cdot (\vec{p}_\nu + \vec{p}_{\bar{\nu}}) + 2E_\nu E_{\bar{\nu}}(1 - \cos\theta_{\nu\bar{\nu}})$$

Question 1:

Consider the decay of a muon to an electron: $\mu^- \to e^- \bar{\nu}_e \nu_\mu$

a) What is the energy of the electron as a function of the neutrino energies ($E_\nu, E_{\bar{\nu}}$) and the angle between the neutrinos ($\theta_{\nu\bar{\nu}}$)? Work in the muon's rest frame, and assume neutrinos are massless.

This expression holds in any reference frame. Now we specifically work as instructed in the muon rest frame, where the following holds: $\vec{p}_e = -(\vec{p}_\nu + \vec{p}_{\bar{\nu}})$ and $E_\nu + E_{\bar{\nu}} = M_\mu - E_e$. We plug those into the equation and isolate the electron's energy:

$$\mathrm{M}_\mu^2 = M_e^2 + 2E_e M_\mu + 2E_\nu E_{\bar{\nu}}(1 - \cos\theta_{\nu\bar{\nu}}) - 2(E_e^2 - p_e^2)$$

The last term is just M_e^2 again, and after isolating and reordering the final expression for E_e reads:

$$\mathrm{E}_e = \frac{1}{2M_\mu}\left[M_\mu^2 + M_e^2 - 2E_\nu E_{\bar{\nu}}(1 - \cos\theta_{\nu\bar{\nu}})\right]$$

Question 1:

Consider the decay of a muon to an electron: $\mu^- \to e^- \bar{\nu}_e \nu_\mu$

b) Assuming the muon has a 3-momentum of $|\vec{p}| = 30\text{GeV}$, what distance will the muon travel before it decays (give the answer in "regular" SI units)?

In the **muon** rest frame it lives a typical time of $\tau = 2.2\mu s$
In the lab frame this time equals to $t = \gamma\tau$

So the typical distance a muon will travel is:

$$d = \beta ct = \beta c\gamma\tau = \frac{\gamma m_\mu \beta cc}{m_\mu c^2} c\tau = \frac{p_1 c}{m_\mu c^2} c\tau$$
$$= \frac{29.8\ MeV}{105.6\ MeV} \times 3 \cdot 10^8 \frac{m}{sec} .2 \cdot 10^{-6} sec \approx 190m$$

Cross-Section

The cross-section, σ, for a process can be defined as:

$$\sigma = \frac{\text{No. of particles scattered from a target per unit time}}{\text{Flux of particles hitting the target}}$$

- Flux of particles hitting the target = no. of incident particles per unit area per unit time
- We can see that σ has units of area $\left[cm^2\right]$
  - σ can be thought of as the effective cross-sectional area of the target as "seen" by an incident particle (though it's not necessarily related to the actual size of the target)
- σ is a characteristic of the **process**, not the incident beam or the target
- The total cross section σ is Lorentz-invariant.
- For different processes, The total cross section is additive, e.g. $\sigma_{pp\to X} = \sum_i \sigma_{pp\to X_i}$

Often we discuss the differential cross section, $\frac{d\sigma}{d\Omega}$

$$\frac{d\sigma}{d\Omega} = \frac{\text{No. of particles scatterd from a target per unit time into } d\Omega}{\text{Flux of particles hitting the target}}$$

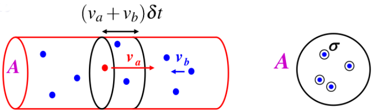
- Or more generally, we can have $\frac{d\sigma}{d\dots}$
- $\sigma = \int \frac{d\sigma}{d\Omega} d\Omega$

Cross-Section

Let's try to understand this better with an example:[1]

Consider a particle of type a with velocity v_a traversing a region of area A containing n_b particles of type b per unit volume, moving with (average) velocity v_b in the opposite direction.

In time δt a particle of type a traverses a region with $\delta N = n_b(v_a + v_b)A\delta t$ particles of type b



Interaction probability can be obtained from the effective cross sectional area σ of the δN particles, divided by the area A.

$$\delta P = \frac{\delta N \sigma}{A} = \frac{n_b(v_a + v_b)A\sigma\delta t}{A} = n_b \sigma v \delta t \qquad (v \equiv v_a + v_b)$$

For a beam of particles a, with density number n_a, confined to a volume V, the *rate* of interactions is:

$$R = \frac{\delta P}{\delta t}(n_a V) = (n_b v \sigma)(n_a V) = (n_a v)(n_b V)\sigma = \Phi_a N_b \sigma$$

As expected:

$$\sigma = \frac{\text{Rate of interactions}}{\text{incident flux} \times \text{number of targets}}$$

[1] For more details see Thomson Ch. 3

Cross-Section

Luminosity:
We saw that the rate of an interaction X is proportional to the cross-section for that interaction.

$$R = \frac{dN_X}{dt} = \Phi_a N_b \sigma_X$$

We can define the luminosity, $\mathcal{L} = \Phi_a N_b$, so that we have:

$$\frac{dN_X}{dt} = \mathcal{L}\sigma_X$$

$\mathcal{L}$ is something we can control (or determine) in an experiment. σ_X is controlled by the underlying physics - "by nature".

To see the total number of interactions in a given experiment we integrate over time to get the *integrated luminosity*:

$$L = \int \mathcal{L} dt$$

- The relevant units for cross section are *barn*: $b = 10^{-24} cm^2$. Though most processes we are interested in are much smaller and are typically of orders $nb \sim fb$
- The units for Luminosity are $cm^{-2} sec^{-}1$ and for integrated luminosity cm^{-2} or more often fb^{-1}

Question 2:

a) In 2018, the LHC delivered an integrated luminosity of $80fb^{-1}$. Given the total cross section for proton-proton collisions of $\sigma_{pp\to all} = 100mb$, find the average collision rate during this year.

b) Given the cross section for $t\bar{t}H$ production $\sigma_{t\bar{t}H} = 0.5pb$ in the LHC, find the minimal required (instant) luminosity such that in a year enough events occur to reach a statistical uncertainty of 5%.

Solution:

a) The average luminosity is $\mathcal{L} = \frac{L_{int}}{1year}$, we write this in units of $\frac{1}{fb \cdot sec}$ to get that: $\mathcal{L} = \frac{80fb^{-1}}{365 \cdot 24 \cdot 60 \cdot 60} \approx 2.5 \cdot 10^{-6}\ fb^{-1}s^{-1}$. The event (collision) rate is then simply given by:

$$\frac{dN}{dT} = \mathcal{L}\sigma = \mathcal{L} \cdot 10^2\,[mb] \cdot 10^{12}\left[\frac{fb}{mb}\right] \approx 2.5 \cdot 10^9\ Hz$$

b) The number of events N is poisson-distributed, so its uncertainty is $\sqrt{N}$. A 5% uncertainty is thus equivalent to the requirement $\frac{\Delta N}{N} = \frac{1}{\sqrt{N}} \leq \frac{1}{20}$, so we must require $N \geq 400$. This means we must have $L_{int} \cdot \sigma \geq 400$, so $L_{int} \geq \frac{400}{500fb} = 0.8\ fb^{-1}$. As before, $\mathcal{L} = \frac{L_{int}}{1year}$, so we plug in the numbers to get

$$\mathcal{L} \geq 2.5 \cdot 10^{-8}\ fb^{-1}s^{-1}$$

For the sake of comparison, the LHC peak luminosity is approximately $2.5 \cdot 10^{-5}\ fb^{-1}s^{-1}$

Fermi's Golden Rule

Fermi's golden rule states that the rate of a process (i.e. transition rate between two quantum states) is given by:

$$Rate \; = |M|^2 \times (phase\text{-}space)$$

- $|M|$ - Matrix element: holds information about the strength of the interaction, the particles involved, the initial and final momenta. It has units of momentum raised to the power of 4-n, with n the total number of on-shell particles in the process **dim(M)=**$[mv]^{4-n}$
- *phase-space* - holds information about the number of available quantum states for the final state particles (kinematics)

In class you used Fermi's golden rule to derive the differential width for a general $1 \to 2$ **decay** in the C.O.M frame from a general expression for the width to be:

$$d\Gamma = \frac{1}{32\pi^2} \frac{S \; |M|^2 \; |p|}{m_1^2} \; d\Omega$$

In the case of a decaying scalar particle, you found the total width to be:

$$\Gamma = \frac{1}{8\pi} \frac{S \; |M|^2 \; |p|}{m_1^2} \; d\Omega$$

Fermi's Golden Rule

For a scattering process, we are interested in calculating the cross-section. The general formula for the cross section of a $2 \to n$ scattering process is given by:

$$\sigma = \frac{S}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2)^2}} \int |M|^2 (2\pi)^4 \delta^4(p_1 + p_2 - \sum_{j=3}^{N} p_j) \prod_{k=3}^{N} 2\pi\delta(p_k^2 - m_k^2)\Theta(p_k^0) \frac{d^4 p_k}{(2\pi)^4}$$

Which in the $2 \to 2$ case in the COM system, using a similar derivation as you did in class, reduces to the expression:

$$d\sigma = \frac{1}{(8\pi)^2} \frac{S\ |M|^2}{s} \frac{p_f}{p_i}\ d\Omega$$

- S - combinatorical factor, related to the number of identical particles in the process. For each set of k identical particles in the final state, S receives a factor of $\frac{1}{k!}$
- $s = (E_1 + E_2)^2$ - mandelstam variable
- p_f,p_i - initial and final state momenta

Try deriving this result at home!

Question 3:

Assume that the π^0 is an elementary scalar, and approximate its lifetime for decay into two photons using Fermi's golden rule, using the fact that each photon contributes to the matrix element a factor of $\sqrt{\alpha}$.

Solution:

Since dim(M)=$[mv]^{4-n}$ and n=3 in this case, then dim(M)=$[mv]$.
Photons are massless and the π^0 is at rest, so there's only one mass and one velocity which can be used in M to get the proper dimensions. Therefore we can conclude that $M = \alpha m_\pi c$, plug this into the formula for the width with $S = \frac{1}{2}$ to get:

$$\Gamma = \frac{|p|}{16\pi m_\pi c\hbar}(\alpha m_\pi c)^2$$

But since $|p_\gamma| = \frac{E_\gamma}{c}$ and the momenta must be equal and opposite in sign, then $2E_\gamma = m_\pi c^2$ and so $|p_\gamma| = \frac{1}{2}m_\pi c$, meaning we can write:

$$\Gamma = \frac{\alpha^2 c^2 m_\pi}{32\pi\hbar} = \frac{1}{\tau}$$

And plugging in the numbers gives:

$$\tau_\pi \approx 9.2 \cdot 10^{-18} s$$

Which is an order of magnitude smaller than the experimental value.

Question 4:

Consider the process $1+2 \to 3+4$ of particle 1 scattering off a stationary target (particle 2) - in the rest frame of particle 2 (i.e. the lab frame), and producing two massless particles $m_3 = m_4 = 0$. Derive the expression for $\frac{d\sigma}{d\Omega}$ in this case.

Solution:

I will denote the spatial momentum vector in bold in this derivation. We first simplify the general formula by performing the p_k^0 integrals. Using the delta function property:

$$\delta(x^2 - a^2) = \frac{1}{2|a|}\left[\delta(x-a) + \delta(x+a)\right]$$

We can rewrite:

$$\delta(p^2 - m^2) = \delta\left[(p^0)^2 - \mathbf{p}^2 - m^2\right] = \frac{1}{2\sqrt{\mathbf{p}^2 + m^2}}\left[\delta(p^0 - \sqrt{\mathbf{p}^2 + m^2}) + \delta(p^0 + \sqrt{\mathbf{p}^2 + m^2})\right]$$

Using the fact that the Theta function will drop the second delta function gives us:

$$\Theta(p^0)\delta(p^2 - m^2) = \frac{1}{2\sqrt{\mathbf{p}^2 + m^2}}\delta(p^0 - \sqrt{\mathbf{p}^2 + m^2})$$

So we can plug this in and perform the integral over all the temporal momentum components. This fixes the values of the final state energies to $p^0 = \sqrt{\mathbf{p}^2 + m^2}$, and simplifies the general formula to:

$$\sigma = \frac{S}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2)^2}} \int |M|^2 (2\pi)^4 \delta^4(p_1 + p_2 - \sum_{j=3}^{N} p_j) \prod_{k=3}^{N} \frac{d^3 p_k}{2\sqrt{\mathbf{p}_k^2 + m_k^2}} \frac{1}{(2\pi)^3}$$

We can now start using the info in our particular problem statement. We start by calculating the denominator in the fraction outside the integral in the p_2 rest frame. In this frame, $p_1 = (E_1, \mathbf{p}_1)$ and $p_2 = (m_2, 0)$. So:

$$p_1 \cdot p_2 = E_1 m_2 \;\Rightarrow\; \sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2)^2} = \sqrt{E_1^2 m_2^2 - m_1^2 m_2^2} = \sqrt{m_2^2(E_1^2 - m_1^2)} = \sqrt{m_2^2 p_1^2} = m_2 |\mathbf{p}_1|$$

We now use $m_3 = m_4 = 0$ to fix $p_3^0 = |\mathbf{p}_3|$ and $p_4^0 = |\mathbf{p}_4|$ and take the various 2π factors out of the integral:

$$\sigma = \frac{S}{4|p_1|m_2}\left(\frac{1}{4\pi}\right)^2 \int |M|^2 \delta(E_1 + E_2 - |\mathbf{p}_3| - |\mathbf{p}_4|)\delta^3(\mathbf{p_1} + \mathbf{p_2} - \mathbf{p_3} - \mathbf{p_4}) \frac{d^3\mathbf{p}_3}{|\mathbf{p}_3|}\frac{d^3\mathbf{p}_4}{|\mathbf{p}_4|}$$

Now with $\mathbf{p}_2 = 0$ and $E_2 = m_2$, we use the $\delta^3(\mathbf{p_1} + \mathbf{p_2} - \mathbf{p_3} - \mathbf{p_4})$ to perform the integral over $d^3\mathbf{p}_4$ to get:

$$\sigma = \frac{S}{64\pi^2|p_1|m_2} \int |M|^2 \delta(E_1 + m_2 - |\mathbf{p}_3| - |\mathbf{p_1} - \mathbf{p_3}|) \frac{d^3\mathbf{p}_3}{|\mathbf{p}_3||\mathbf{p_1} - \mathbf{p_3}|}$$

Writing $d^3\mathbf{p}_3 = |\mathbf{p}_3|^2 d|p_3| d\Omega$ and $(\mathbf{p}_1 - \mathbf{p}_3)^2 = |\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 - 2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta$ (with θ the scattering angle of particle 3):

$$\frac{d\sigma}{d\Omega} = \frac{S}{64\pi^2|p_1|m_2} \int |M|^2 \frac{\delta(E_1 + m_2 - |\mathbf{p}_3| - \sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 - 2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta})|\mathbf{p}_3|^2}{|\mathbf{p}_3|\sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 - 2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta}} d|\mathbf{p}_3|$$

To solve this integral, we substitute variables to $z = |\mathbf{p}_3| + \sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 - 2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta}$ and so:

$$\frac{dz}{d|\mathbf{p}_3|} = 1 + \frac{1}{2}\left(\frac{2|\mathbf{p}_3| - 2|\mathbf{p}_1|\cos\theta}{\sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 - 2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta}}\right) = \frac{z - |\mathbf{p}_1|\cos\theta}{\sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 - 2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta}}$$

Therefore we can conclude that $\frac{d|\mathbf{p}_3|}{\sqrt{|\mathbf{p}_1|^2+|\mathbf{p}_3|^2-2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta}} = \frac{dz}{z-|\mathbf{p}_1|\cos\theta}$ and rewrite our integral as:

$$\frac{d\sigma}{d\Omega} = \frac{S}{64\pi^2|p_1|m_2}\int |M|^2\delta(E_1 + m_2 - z)\frac{|\mathbf{p}_3|dz}{z - |\mathbf{p}_1|\cos\theta}$$

And use the delta function to easily integrate over z, which simply sets $z = E_1 + m_2$ and gives us the final expression for the differential cross section:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left[\frac{S|M|^2|\mathbf{p}_3|}{m_2|\mathbf{p}_1|(E_1 + m_2 - |\mathbf{p}_1|\cos\theta)}\right]$$

At home you will derive the formula for the general case of $m_3 \neq m_4 \neq 0$.