3. In this problem you are asked to discuss quantum states of a two dimensional electron in two dimensional "trap" subject to a perpendicular magnetic field.

   Start with pp. 2) and 3) of the homework no. 2. Add to the well $U(x)$ also a well $V(y)$ in the y direction, i.e have the electron totally confined in a finite area.

The second paragraph should read "Start with pp c) and d) of the problem 1 of the homework no. 2

From <https://mail.google.com/mail/u/1/#inbox/14b69ea493614590>

a) Assume that $U(x)$ and $V(y)$ are infinite potential wells. To have some physical intuition start by discussing (qualitatively) what are possible classical solutions in the resulting rectangular well $U(x) + V(y)$ and what you should expect "quantum mechanics will do to them". Start by easy cases of the relation between the parameters of the trap and magnetic field. Then try to go to harder cases.

Classically - if the well is big enough and the particle doesn't hit the walls:

$$m\frac{v^2}{R} = F_B = \frac{qvB}{c}$$
$$E = \frac{mv^2}{2}$$

b) Now switch to quantum mech. Use your understanding (from your homework and tutorial) of the solutions in the absence of $V(y)$ to find what will be the effect of adding $V(y)$. Conduct this discussion by considering mixing the "unperturbed" states by the added $V(y)$. Your task will be easier if you assume that the magnetic field is very strong. How do your results fit into your classical intuition.

$$H = \frac{1}{2m}\left[\vec{p} - \frac{q}{c}\vec{A}\right]^2 + U(x) + V(y) = \frac{1}{2m}\left[p_x - \frac{q}{c}A_x\right]^2 + \frac{1}{2m}\left[p_y - \frac{q}{c}A_y\right]^2 + U(x) + V(y)$$

Using the solution from Q1 HM2:

$$H = \frac{1}{2m}\left[p_x^2 + \left(p_y - \frac{q}{c}Bx\right)^2\right] + U(x) + V(y)$$

Inside the well $\boldsymbol{U(x) + V(y) = 0}$ (and outside $U(x) + V(y) = \infty$ so their effect manifests in applying bound boundary conditions, just as we solve a paticle in an infinite well):

$$H = \frac{1}{2m}\left[p_x^2 + \left(p_y - \frac{q}{c}Bx\right)^2\right]$$

I'll use the eigenfunction $\psi(x, y) = \phi(x)e^{ik_y y}$ as we used in class:

$$H\psi = E\psi$$

$$\left\{\frac{1}{2m}\left[p_x^2+\left(p_y-\frac{q}{c}Bx\right)^2\right]+U(x)+V(y)\right\}\phi(x)e^{ik_y y}=E\phi(x)e^{ik_y y}$$

- $\hat{p}_y\psi=-i\hbar\frac{\partial}{\partial y}\phi(x)e^{ik_y y}=i\hbar k_y\phi(x)e^{ik_y y}=ip_y\psi$
- $x_0=\frac{p_y}{m\omega_c},\ \omega_c\equiv\frac{qB}{mc}$

$$\left\{\frac{1}{2m}[p_x^2+(m\omega_c x_0-m\omega_c x)^2]+U(x)+V(y)\right\}\phi(x)e^{ik_y y}=E\phi(x)e^{ik_y y}$$

$$\left\{\frac{1}{2m}[p_x^2+m^2\omega_c^2(x_0-x)^2]+U(x)+V(y)\right\}\phi(x)=E\phi(x)$$

$$\left\{\frac{p_x^2}{2m}+\frac{m\omega_c^2}{2}(x_0-x)^2+U(x)+V(y)\right\}\phi(x)=E\phi(x)$$

Which is a 1D harmonic oscillator in a potential.
When $V(x)=0$ I'll define $x'\equiv x_0-x\Rightarrow\frac{\partial}{\partial x'}=\frac{\partial}{\partial x}$ and get $\phi(x)=\chi_n(x-x_0)$ when $\chi_n(x)$ is harmonic oscillator eigenfunctions.

Now applying the boundary conditions caused by the potentials:

$$U(x)=\begin{cases}0 & -a<x<a\\ \infty & else\end{cases}$$

$$V(y)=\begin{cases}0 & -b<y<b\\ \infty & else\end{cases}$$

$$\phi(-a)e^{ik_y y}=0$$

$$\phi(a)e^{ik_y y}=0$$

$$\phi(x)e^{-ik_y b}=0$$

$$\phi(x)e^{ik_y b}=0$$

$V(y)$ will just quantize the values of p_y.
The term for the standard deviation of harmonic oscillator we got in class:

$$\frac{m\omega_c^2}{2}\sqrt{\langle(x-x_0)^2\rangle}=l_B\sqrt{\frac{1}{2}+n}$$

$$l_B\equiv\sqrt{\frac{\hbar c}{qB}}$$

When $a\gg l_B\sqrt{\frac{1}{2}+n}$ the walls are far enough so the solution will be $\psi(x,y)=\chi_n(x-x_0)e^{ik_y y}$
Otherwise only $\chi_n(x-x_0)$ which obeys the boundary conditions will be manifested physically.

c) Now discuss an exactly solvable example of this problem by taking

$$U(x)+V(y)=\frac{m}{2}(\omega_x^2x^2+\omega_y^2y^2)$$

What are your expectations of the solution on the basis of classical mechanics.

$$U(x) + V(y) = \frac{m}{2}\left(\omega_x^2 x^2 + \omega_y^2 y^2\right)$$
$$\omega^2 = \frac{k}{m}$$

The classic solution - orbiting with precession:

$$\vec{F} = m\vec{a} = \vec{F}_B - k_x x - k_y y = \frac{q}{c}\vec{v} \times B - k_x x - k_y y$$
$$\ddot{x} = -\frac{\omega_x^2}{m}x + \frac{qB}{m}\dot{y}$$
$$\ddot{y} = -\frac{\omega_y^2}{m}y - \frac{qB}{m}\dot{x}$$

If $\boldsymbol{\omega_x = \omega_y}$ it can be solved with the trick of $z \equiv x + iy$:

$$\ddot{z} \equiv \ddot{x} + i\ddot{y} = \cdots = \frac{\omega^2}{m}\dot{z} - i\frac{qB}{m}z$$

…

If $\boldsymbol{\omega_x \neq \omega_y}$ then to diagonalize the Lagrangian (which we didn't write here):

$$a = \frac{x+y}{\sqrt{2}}$$
$$b = \frac{x-y}{\sqrt{2}}$$
$$\alpha \equiv \frac{qB}{m}$$

Dani's solution from Mathematica:

$$a = \exp\left(-\frac{t\omega^2}{\alpha+1}\right)$$
$$b = \exp\left(\frac{t\omega^2}{-1+\alpha}\right)$$

The plot appears in the dropbox, in Dani's wonderful Mathematica file "question3_c_classical_sol"

d) Now to quantum mech. Start with $\omega_x = \omega_y$. Such a potential is symmetric. What is the corresponding conserved quantity (in the presence of the magnetic field)? Choose appropriate gauge so that this symmetry can be easily exploited and variables separated. What are the qualitative solutions of the problem after the variable separation. Find a simple way of conducting this discussion. How do these considerations fit your classical intuition?

Using the symmetric gauge: $\vec{A} = \frac{B}{2}(x\hat{y} - y\hat{x})$

$$
\begin{aligned}
H &= \frac{1}{2m}\left[\vec{p}-\frac{q}{c}\vec{A}\right]^2 + U(x) + V(y) \\
&= \frac{1}{2m}\left(P_x - \frac{1}{2}\frac{e}{c}By\right)^2 + \frac{1}{2m}\left(P_y + \frac{1}{2}\frac{e}{c}Bx\right)^2 + \frac{1}{2}\omega^2(x^2+y^2) \\
&= P_x^2 + \left(\frac{1}{2}\frac{e}{c}B\right)^2 y^2 + P_y^2 + L_z\frac{e}{c}B + \left(\frac{1}{2}\frac{e}{c}B\right)^2 x^2 + \frac{1}{2}\omega^2(x^2+y^2)
\end{aligned}
$$

The angular momentum around the z axis is conserved: $[\boldsymbol{L_z}, \boldsymbol{H}] = \boldsymbol{0}$

Using polar coordinates (ρ, φ):

$$
\begin{aligned}
U(x) + V(y) &= \frac{m}{2}(\omega^2x^2 + \omega^2y^2) = \frac{m\omega^2}{2}\rho^2 \\
\vec{A} &= \frac{B}{2}(x\hat{y} - y\hat{x}) = \frac{B}{2}\rho\hat{\varphi} \\
\vec{p} &= -i\vec{\nabla} = \frac{\partial}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial}{\partial\varphi}\hat{\varphi} \\
L_z &= -i\hbar\frac{\partial}{\partial\theta} = xp_y - yp_x = rp_\theta
\end{aligned}
$$

$$
\begin{aligned}
H &= \frac{1}{2m}\left[p_\rho - \frac{q}{c}A_\rho\right]^2 + \frac{1}{2m}\left[p_\varphi - \frac{q}{c}A_\varphi\right]^2 + U(x) + V(y) \\
&= \frac{1}{2m}\left(p_r^2 - \hbar^2\frac{\partial_\varphi^2}{r^2} + \left(\left(\frac{eB}{2c}\right)^2 + m\omega^2\right)r^2 - \frac{eB}{c} - i\hbar\frac{\partial}{\partial\theta}\right)
\end{aligned}
$$

Using the conservation of L_z the solution should be $\psi = R(\rho)e^{il\phi}$:

$$
\begin{aligned}
&\frac{1}{2m}\left(-\hbar^2\left(e^{il\phi}\frac{1}{\rho}\partial_\rho\left(\rho\,\partial_\rho R(\rho)\right) - \frac{l^2}{\rho^2}R(\rho)e^{il\phi}\right) + \left(\frac{e}{c}B\rho\right)^2 R(\rho)e^{il\phi} + 2\hbar\frac{e}{c}BlR(\rho)e^{il\phi}\right)\psi \\
&\quad = ER(r)e^{il\phi}
\end{aligned}
$$

$$
\frac{1}{2m}\left(\left(\frac{-\hbar^2}{\rho}\left(\rho R''(\rho) + R'(\rho)\right) + \hbar^2\frac{l^2}{\rho^2}R(\rho)\right) + \left(\frac{e}{c}B\rho\right)^2 R(\rho) + 2\hbar\frac{e}{c}BlR(\rho)\right) = ER(r)
$$

The radial part is Laguerre polynomials, and the solution is:

$$
\boldsymbol{\psi = e^{il\phi}exp\left(-\frac{m\omega_c}{8\hbar}\rho^2\right)\rho^{|l|}L^{|l|}_{n+\frac{1}{2}(|l|+l)}\left(\frac{m\omega_c}{4\hbar}\rho^2\right)}
$$

e) Assume AB flux along the z-axis added at the origin. How will it change you results?

The addition of AB flux can be thought as a gauge (see EM Lecture notes eq. 39):

$$
\begin{aligned}
\vec{A}'(\vec{r},t) &= \vec{A}(\vec{r},t) + \vec{\nabla}\xi(\vec{r},t) \\
\xi(\vec{r}) &= \frac{\Phi}{2\pi}\varphi \\
\Phi_0 &\equiv \frac{\mathrm{hc}}{e}
\end{aligned}
$$

The "new" eigenfunctions will be gauged:

$$
\boldsymbol{\psi(r,\varphi,z)} = e^{i\frac{q}{\hbar c}\xi(\varphi)}\psi'(r,\varphi,z) = \boldsymbol{e^{i\frac{\Phi}{\Phi_0}\varphi}\psi'(r,\varphi,z)}
$$

When the eigenfunctions from the previous part, without the AB field:

$$\psi'(r,\varphi,z) = e^{il\phi} exp\left(-\frac{m\omega_c}{8\hbar}\rho^2\right)\rho^{|l|}L^{|l|}_{n+\frac{1}{2}(|l|+l)}\left(\frac{m\omega_c}{4\hbar}\rho^2\right)$$

f) (For X credit). Find exact solutions of the above problem. First do it for the symmetric case $\omega_x = \omega_y$. Then for XX credit - for the general $\omega_x \neq \omega_y$ case.

We did it in part (d) for $\omega_x = \omega_y$ case.