Fundamentals of Quantum Technology
Homework Sheet 7

1. In this exercise you will complete the details of the derivation of the exact Jaynes-Cummings energy spectrum. Recall that the quantization of the single-mode electric field that interacts with the two-level system (TLS) produces the Hamiltonian

$$\hat{\mathcal{H}} = \frac{\hbar\omega_0}{2}\hat{\sigma}_z + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar\lambda\left(\hat{\sigma}_+ + \hat{\sigma}_-\right)\left(\hat{a} + \hat{a}^\dagger\right).$$

   (a) The transition to the interaction picture is done by applying the unitary $\hat{U} = \exp\left(it\hat{K}/\hbar\right)$ to the state $|\psi(t)\rangle$, which then evolves according to the effective Hamiltonian $\hat{\mathcal{H}}_{\text{eff}} = \hat{U}\left(\hat{\mathcal{H}} - \hat{K}\right)\hat{U}^\dagger$. Show that the choice $\hat{K} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\hat{\sigma}_z\right)$, along with the rotating wave approximation, yield

$$\hat{\mathcal{H}}_{\text{eff}} = \frac{\hbar\Delta}{2}\hat{\sigma}_z + \hbar\lambda\left(\hat{\sigma}_+\hat{a} + \hat{a}^\dagger\hat{\sigma}_-\right),$$

   where $\Delta = \omega_0 - \omega$.

   (b) The effective Hamiltonian we have derived has a 2×2 block structure, where the subspace of each block is spanned by $\{|e,n\rangle, |g,n+1\rangle\}$ (apart from a single 1×1 block which corresponds to the eigenstate $|g,0\rangle$). Write the 2×2 Hamiltonian of each block, and show that its two eigenvalues are given by $E_{n,\pm}^{(\text{eff})} = \pm\frac{\hbar}{2}\Omega_n(\Delta)$, where $\Omega_n(\Delta) = \sqrt{\Delta^2 + 4\lambda^2(n+1)}$.

   (c) Show that the eigenstates corresponding to these eigenvalues are

$$|n,+\rangle = \cos\left(\frac{\Phi_n}{2}\right)|e,n\rangle + \sin\left(\frac{\Phi_n}{2}\right)|g,n+1\rangle,$$
$$|n,-\rangle = -\sin\left(\frac{\Phi_n}{2}\right)|e,n\rangle + \cos\left(\frac{\Phi_n}{2}\right)|g,n+1\rangle,$$

   with $\tan(\Phi_n) = 2\lambda\sqrt{n+1}/\Delta$.

   (d) Return to the Schrödinger picture to show that the true energies of these eigenstates are given by $E_{n,\pm} = \hbar\omega\left(n + \frac{1}{2}\right) \pm \frac{\hbar}{2}\Omega_n(\Delta)$.
   Hint: The effective energies $E_{n,\pm}^{(\text{eff})}$ that you have derived imply that, within the interaction picture, the eigenstates remain stationary and their time evolution simply amounts to an overall time-dependent phase $\exp\left(-iE_{n,\pm}^{(\text{eff})}t/\hbar\right)$. The transition back to the Schrödinger picture is done by applying $\hat{U}^\dagger$ to the interaction picture states. This does not change the fact that they are stationary, but it does change the overall time-dependent phase, and from this phase you can read out the true energies of the eigenstates.

2. Suppose that the assumption of the dispersive regime – i.e., $|\Delta| \gg \lambda\sqrt{n}$ – applies. In class we saw that applying the Schrieffer-Wolf (SW) transformation $e^{\hat{S}}\hat{\mathcal{H}}_{\text{eff}}e^{-\hat{S}}$ (with $\hat{S} = \frac{\lambda}{\Delta}\left(\hat{\sigma}_+\hat{a} - \hat{a}^\dagger\hat{\sigma}_-\right)$) to the effective Hamiltonian leads to

$$e^{\hat{S}}\hat{\mathcal{H}}_{\text{eff}}e^{-\hat{S}} \approx \frac{\hbar}{2}\left(\Delta + \frac{\lambda^2}{\Delta}\right)\hat{\sigma}_z + \frac{\hbar\lambda^2}{\Delta}\hat{a}^\dagger\hat{a}\hat{\sigma}_z + \frac{\hbar\lambda^2}{2\Delta}.$$

We observed that the bare states are the eigenstates of this transformed Hamiltonian, and also calculated their energies.

(a) Show that the same energies are obtained starting from the exact spectrum $E_{n,\pm}$ whenever $\lambda\sqrt{n} \ll |\Delta|$. Assume for simplicity that $\Delta > 0$.

(b) To go back to the interaction picture from the transformed picture, we need to apply $e^{-\hat{S}}$ to the state of the system. Show that

$$e^{-\hat{S}} |e, n\rangle = \cos(\theta_n) |e, n\rangle + \sin(\theta_n) |g, n+1\rangle ,$$
$$e^{-\hat{S}} |g, n+1\rangle = -\sin(\theta_n) |e, n\rangle + \cos(\theta_n) |g, n+1\rangle ,$$

where $\theta_n = \lambda\sqrt{n+1}/\Delta$. These are the true eigenstates of the JC Hamiltonian in the dispersive regime.

(c) In class we claimed that in the dispersive regime we can indirectly measure the TLS state by measuring the oscillator frequency, but this relied on the structure of the Hamiltonian following the SW transformation; as you saw in the previous item, this measurement is not exact because the SW transformation slightly mixes the levels of the TLS. Let us denote by $P(\sigma|\sigma')$ the probability that a TLS in the state σ is measured to be in the state σ'. We define the **fidelity** F of the measurement to be $F = 1 - P(e|g) - P(g|e)$; this is a measure of its accuracy, and for a perfect measurement we will have $F = 1$.
Calculate the fidelity of the measurement of the TLS state that we described in class. Assume for simplicity that the average photon number $\bar{n}$ is large, $\bar{n} \gg 1$, and that the distribution of n is very narrowly peaked about $\bar{n}$, such that we can use $n \approx \bar{n}$.