Fundamentals of Quantum Technology
Week 2: Optical Tests of Quantum Mechanics; Optical Realizations of Quantum Gates

Recommended literature: Gerry, Knight (ch. 9, 11)

1 Bell's inequality

"What is proved by impossibility proofs is lack of imagination."

– J. S. Bell, *On the impossible pilot wave*, Foundations of Physics **12**, pp. 989-999 (1982).

Entanglement is a type of non-local quantum correlations that defies classical intuition. **Bell's inequality** is a general name given to several tests that can experimentally distinguish between a correlation that is genuinely non-local and a correlation which is controlled by a local variable of which we are not aware: a **local hidden variable**. Today we will construct such an inequality for entangled polarization states following the original suggestion by John Bell[1].

We assume that we can produce the Bell state given by

$$\left|\Psi^-\right\rangle = \frac{1}{\sqrt{2}}\left(|H\rangle_1 |V\rangle_2 - |V\rangle_1 |H\rangle_2\right),$$

and then measure each photon in a polarization basis $\left\{|\theta\rangle, \left|\theta^\perp\right\rangle\right\}$, related to the standard polarization directions according to

$$\begin{cases} |\theta\rangle = \cos\theta\,|H\rangle + \sin\theta\,|V\rangle \\ \left|\theta^\perp\right\rangle = -\sin\theta\,|H\rangle + \cos\theta\,|V\rangle \end{cases} \longrightarrow \begin{cases} |H\rangle = \cos\theta\,|\theta\rangle - \sin\theta\,\left|\theta^\perp\right\rangle \\ |V\rangle = \sin\theta\,|\theta\rangle + \cos\theta\,\left|\theta^\perp\right\rangle \end{cases}.$$

We choose independently the directions θ and ϕ for the polarization measurements of photons 1 and 2, so that before the measurement we can write $|\Psi^-\rangle$ in this new basis as

$$\begin{aligned} \left|\Psi^-\right\rangle = {} & \frac{\sin(\phi-\theta)}{\sqrt{2}}\left(|\theta\rangle_1 |\phi\rangle_2 - \left|\theta^\perp\right\rangle_1 \left|\phi^\perp\right\rangle_2\right) \\ & + \frac{\cos(\theta-\phi)}{\sqrt{2}}\left(|\theta\rangle_1 \left|\phi^\perp\right\rangle_2 - \left|\theta^\perp\right\rangle_1 |\phi\rangle_2\right). \end{aligned}$$

We then define a variable $A(\theta)$ such that $A(\theta) = +1$ (-1) if photon 1 is measured in $|\theta\rangle_1$ $\left(\left|\theta^\perp\right\rangle_1\right)$, and a similar variable $B(\phi)$ for photon 2. We further define the correlation function

$$C(\theta,\phi) = \text{Average}\left[A(\theta)\,B(\phi)\right],$$

where the average is taken over many experimental runs. One can check that the Bell state gives

$$C(\theta,\phi) = -\cos(2\theta - 2\phi).$$

[1]See: J. S. Bell, *On the Einstein Podolsky Rosen paradox*, Physics **1**, pp. 195-200 (1964).

Exercise

Show that, assuming the existence of a local hidden variable, the following inequality holds[2]:

$$|C(\theta,\phi) - C(\theta,\phi')| \le 1 + C(\phi,\phi').$$

Find a choice of θ, ϕ, ϕ' that violates this inequality.

Solution

The assumption that the results of our measurements are determined by a local hidden variable λ amounts to writing $C(\theta,\phi)$ in the following form:

$$C(\theta,\phi) = \int A(\theta,\lambda) B(\phi,\lambda) \rho(\lambda) \, \mathrm{d}\lambda,$$

where $\rho(\lambda)$ is a probability density, obeying in particular $\int \rho(\lambda)\,\mathrm{d}\lambda = 1$, and $A, B \in \{\pm 1\}$. The main point to understand here is that under the integral we do *not* have a function $D(\theta,\phi,\lambda)$ that cannot be decomposed into a product of one function that is independent of ϕ and another that is independent of θ: the choice of measurement basis for one of the photons cannot affect the result of measuring the other photon.

Now, that local hidden variable theory must recreate the results predicted by quantum mechanics, and specifically for the Bell state $|\Psi^-\rangle$ we have that $C(\theta,\theta) = -1$. This is possible only if $B(\theta,\lambda) = -A(\theta,\lambda)$ for all λ, and so we may write

$$C(\theta,\phi) = -\int A(\theta,\lambda) A(\phi,\lambda) \rho(\lambda) \, \mathrm{d}\lambda.$$

We now note that

$$\begin{aligned}
|C(\theta,\phi) - C(\theta,\phi')| &= \left| \int A(\theta,\lambda) \left[A(\phi,\lambda) - A(\phi',\lambda) \right] \rho(\lambda) \, \mathrm{d}\lambda \right| \\
&= \left| \int A(\theta,\lambda) A(\phi,\lambda) \left[1 - A(\phi,\lambda) A(\phi',\lambda) \right] \rho(\lambda) \, \mathrm{d}\lambda \right| \\
&\le \int |A(\theta,\lambda) A(\phi,\lambda)| \cdot |1 - A(\phi,\lambda) A(\phi',\lambda)| \, \rho(\lambda) \, \mathrm{d}\lambda \\
&= \int \left[1 - A(\phi,\lambda) A(\phi',\lambda) \right] \rho(\lambda) \, \mathrm{d}\lambda = 1 + C(\phi,\phi').
\end{aligned}$$

where we used a basic fact from calculus, $\left|\int f(x)\,\mathrm{d}x\right| \le \int |f(x)|\,\mathrm{d}x$.

Recall, on the other hand, that for the Bell state $|\Psi^-\rangle$ we have $C(\theta,\phi) = -\cos(2\theta - 2\phi)$, and thus we must have

$$|C(\theta,\phi) - C(\theta,\phi')| = 2\left|\sin(2\theta - \phi - \phi') \sin(\phi - \phi')\right|$$

and

$$1 + C(\phi,\phi') = 2\sin^2(\phi - \phi').$$

A local hidden variable theory therefore requires that if $\sin(\phi - \phi') \neq 0$ then

$$|\sin(2\theta - \phi - \phi')| \le |\sin(\phi - \phi')|,$$

but, for example, the choice $\theta = 0, \phi = \frac{\pi}{6}, \phi' = \frac{\pi}{3}$ violates this requirement.

The CHSH version of Bell's inequality improves the original version by not requiring the specific relation $B(\theta,\lambda) = -A(\theta,\lambda)$. The CHSH inequality applies to any entangled two-photon state, not just states where there is perfect anti-correlation (or perfect correlation) between the results of the two measurements, and therefore it is much more realistic to devise experiments which will verify its violation.

[2] Think, for example, of two angles ϕ, ϕ' which are very anti-correlated, giving $C(\phi,\phi') \approx -1$; then this inequality suggests that the correlation of an angle θ with ϕ must be *very* similar to its correlation with ϕ'.

2 Optical realization of the C-NOT gate

The **controlled-not gate**, or C-NOT gate, is an important two-qubit gate where one qubit serves as the *control* and the other serves as the *target*. The transformation is given by

$$\hat{U}_{\text{CNOT}} \left|x\right\rangle_c \left|y\right\rangle_t = \left|x\right\rangle_c \left|\text{mod}_2 \left(x+y\right)\right\rangle_t ,$$

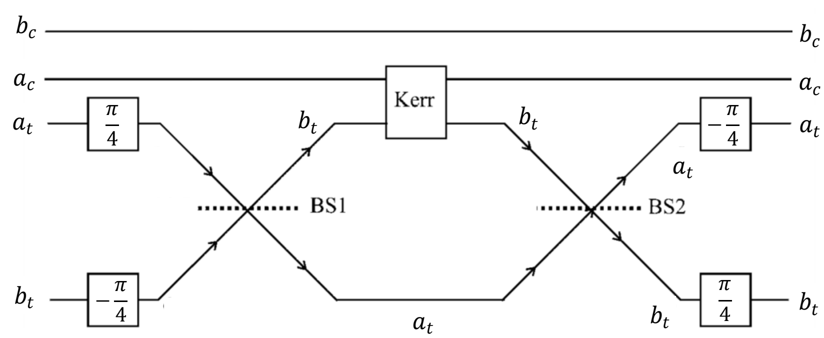
or more explicitly

$$\hat{U}_{\text{CNOT}} \left|0\right\rangle_c \left|0\right\rangle_t = \left|0\right\rangle_c \left|0\right\rangle_t ; \quad \hat{U}_{\text{CNOT}} \left|0\right\rangle_c \left|1\right\rangle_t = \left|0\right\rangle_c \left|1\right\rangle_t ;$$
$$\hat{U}_{\text{CNOT}} \left|1\right\rangle_c \left|0\right\rangle_t = \left|1\right\rangle_c \left|1\right\rangle_t ; \quad \hat{U}_{\text{CNOT}} \left|1\right\rangle_c \left|1\right\rangle_t = \left|1\right\rangle_c \left|0\right\rangle_t .$$

As usual, we use single-photon states of two photonic modes to define a single qubit. The control qubit, for example, is defined by two modes $\hat{a}_c$ and $\hat{b}_c$ such that

$$\left|0\right\rangle_c \equiv \left|0\right\rangle_{a_c} \left|1\right\rangle_{b_c} , \quad \left|1\right\rangle_c \equiv \left|1\right\rangle_{a_c} \left|0\right\rangle_{b_c} .$$

The target qubit is defined is a similar way with respect to $\hat{a}_t$ and $\hat{b}_t$. We will now examine the following optical setup – containing two 50:50 beam splitters, a cross-Kerr interaction component and phase shifters – and see how it realizes the C-NOT gate:



At home you showed that a beam splitter is generally equivalent to the transformation

$$\hat{U}_{\text{BS}} \left(\theta\right) = \exp\left[i\frac{\theta}{2}\left(\hat{a}_t^{\dagger}\hat{b}_t + \hat{b}_t^{\dagger}\hat{a}_t\right)\right].$$

Here we choose $\hat{U}_{\text{BS1}} = \hat{U}_{\text{BS}}\left(\frac{\pi}{2}\right)$ and $\hat{U}_{\text{BS2}} = \hat{U}_{\text{BS}}\left(-\frac{\pi}{2}\right) = \hat{U}_{\text{BS1}}^{\dagger}$, giving us 50:50 beam splitters. The transformation due to the cross-Kerr interaction is described by

$$\hat{U}_{\text{Kerr}} \left(\eta\right) = \exp\left[i\eta\hat{a}_c^{\dagger}\hat{a}_c\hat{b}_t^{\dagger}\hat{b}_t\right].$$

Exercise

Show that the given optical setup realizes the C-NOT gate for $\eta = \pi$.

Solution

At home you will show that the 3 components in the middle generally amount to

$$\begin{aligned}\hat{U}_F\left(\eta\right) &\equiv \hat{U}_{\text{BS1}}^{\dagger}\hat{U}_{\text{Kerr}}\left(\eta\right)\hat{U}_{\text{BS1}} \\ &= \exp\left[i\frac{\eta}{2}\hat{a}_c^{\dagger}\hat{a}_c\right]\exp\left[\frac{\eta}{2}\hat{a}_c^{\dagger}\hat{a}_c\left(\hat{a}_t^{\dagger}\hat{b}_t - \hat{b}_t^{\dagger}\hat{a}_t\right)\right].\end{aligned}$$

The first term simply yields a phase factor that will be corrected by the phase shifters. If the $\hat{a}_c$ mode is empty – meaning that the control qubit is in the $|0\rangle$ state – then the second term is trivial, doing nothing to the target qubit (as we should expect). Now suppose that the control qubit is in the $|1\rangle$ state, such that $\langle \hat{a}_c^\dagger \hat{a}_c \rangle = 1$. We then want to examine the effect of $\exp\left[\frac{\eta}{2}\left(\hat{a}_t^\dagger \hat{b}_t - \hat{b}_t^\dagger \hat{a}_t\right)\right]$ on the target qubit.

Let us check what this unitary does to $|0\rangle = |0\rangle_a |1\rangle_b$:

$$\exp\left[\frac{\eta}{2}\left(\hat{a}^\dagger \hat{b} - \hat{b}^\dagger \hat{a}\right)\right] |0\rangle_a |1\rangle_b = \sum_{n=0}^{\infty} \frac{\left(\frac{\eta}{2}\right)^n}{n!} \left(\hat{a}^\dagger \hat{b} - \hat{b}^\dagger \hat{a}\right)^n |0\rangle_a |1\rangle_b .$$

We notice the fact that

$$\begin{aligned} \left(\hat{a}^\dagger \hat{b} - \hat{b}^\dagger \hat{a}\right) |0\rangle_a |1\rangle_b &= |1\rangle_a |0\rangle_b , \\ \left(\hat{a}^\dagger \hat{b} - \hat{b}^\dagger \hat{a}\right) |1\rangle_a |0\rangle_b &= - |0\rangle_a |1\rangle_b , \end{aligned}$$

and therefore

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\left(\frac{\eta}{2}\right)^n}{n!} \left(\hat{a}^\dagger \hat{b} - \hat{b}^\dagger \hat{a}\right)^n |0\rangle_a |1\rangle_b &= \sum_{n=0}^{\infty} \frac{\left(\frac{\eta}{2}\right)^{2n}}{(2n)!} (-1)^n |0\rangle_a |1\rangle_b + \sum_{n=0}^{\infty} \frac{\left(\frac{\eta}{2}\right)^{2n+1}}{(2n+1)!} (-1)^n |1\rangle_a |0\rangle_b \\ &= \cos\left(\frac{\eta}{2}\right) |0\rangle_a |1\rangle_b + \sin\left(\frac{\eta}{2}\right) |1\rangle_a |0\rangle_b . \end{aligned}$$

Setting $\eta = \pi$ we obtain

$$\exp\left[\frac{\pi}{2}\left(\hat{a}^\dagger \hat{b} - \hat{b}^\dagger \hat{a}\right)\right] |0\rangle_a |1\rangle_b = |1\rangle_a |0\rangle_b ,$$

and similarly

$$\exp\left[\frac{\pi}{2}\left(\hat{a}^\dagger \hat{b} - \hat{b}^\dagger \hat{a}\right)\right] |1\rangle_a |0\rangle_b = - |0\rangle_a |1\rangle_b .$$

This is exactly what we wanted: a transformation that flips the state of the target qubit if and only if the control qubit is in the state $|1\rangle$. The unwanted sign is corrected by choosing different phase shifters for the two modes.

In total, when including the phase shifters, the operator representation of the C-NOT gate is given by

$$\boxed{\hat{U}_{\text{CNOT}} = \exp\left[-i\frac{\pi}{4}\hat{a}_t^\dagger \hat{a}_t\right] \exp\left[i\frac{\pi}{4}\hat{b}_t^\dagger \hat{b}_t\right] \hat{U}_F(\pi) \exp\left[i\frac{\pi}{4}\hat{a}_t^\dagger \hat{a}_t\right] \exp\left[-i\frac{\pi}{4}\hat{b}_t^\dagger \hat{b}_t\right] .}$$