Question 7

Consider a pendulum of charge q and mass m attached on a rigid rod length L, in static electric field E_0.We will be interested in an emission from an exicted state. We will consider the probelm in $2D$ (Where Shimon noted that for full credit one should slove for $3D$).

1. Lets wirte the Hamiltonian of the problem:

$$H_0 = \frac{\vec{p}^2}{2m} + mg\left(L - \sqrt{L^2 - r^2}\right) + q\vec{E}_0 \cdot \vec{r}$$

   Where we work in spherical coordinates with center on $\hat{z}$ axis at level of rotation (see diagram) , and $\vec{E_0}$ at $\hat{x}$ (if E_0 has a component in $\hat{z}$ we will get an effective $g_{eff} = \frac{mg+qE_0}{m}$). In order to find the eigenstates of this Hamiltonian we consider $p_z = 0$:

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + qE_0x + mg\left(L - \sqrt{L^2 - x^2 - y^2}\right)$$

2. The general expression for the probability of emission is given by:

$$\Gamma_{i\to f} = \frac{2\pi}{\hbar}\left|\langle f\left|V_{int}\right|i\rangle\right|^2 \delta\left(\hbar\omega_k - \varepsilon_f - \varepsilon_i\right)$$

   Where $|i\rangle, |f\rangle$ are the eigenstates of H_0 , and $\hbar\omega_k$ is the energy of the emitted photon.

3. Under small amplitudes and long wave approximation we can write H_0 as:

$$H_0 = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}m\omega^2\left(x^2 + y^2\right) + qE_0x$$

   Where $\omega^2 = \frac{g_{eff}}{L}$, completing the square:

$$H_0 \approx \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}m\omega^2y^2 + \frac{1}{2}m\omega^2\left(x + \frac{eE}{m\omega^2}\right)^2 - \frac{(eE)^2}{2m\omega^2}$$

   Which is simply two H.O (one of them is shifted in energy and in coordinate).

$$H_0|n,\mu\rangle = \left\{\hbar\omega\left(n + \mu + 1\right) - \frac{(eE)^2}{2m\omega^2}\right\}|n,\mu\rangle$$

   Note that

$$\begin{aligned} V_{int} &= -\frac{i\omega}{\hbar}\vec{\lambda}_{k,\alpha}\cdot\vec{d} \\ &= -\frac{i\omega e}{\hbar}\vec{\lambda}_{k,\alpha}\cdot\vec{r} \\ &= -\frac{ie}{\hbar}\sqrt{\frac{\omega\hbar}{2m}}\vec{\lambda}_{k,\alpha}\cdot\begin{pmatrix} a_x + a_x^\dagger \\ a_y + a_y^\dagger \end{pmatrix} \end{aligned}$$

Here we only discuss emission , so we drop $a^\dagger$ terms. So in order to calculate the transfer rates , we first need to calculate:

$$\langle f \left| V_{int} \right| i \rangle = -\frac{ie}{\hbar}\sqrt{\frac{\omega\hbar}{2m}}\vec{\lambda}_{k,\alpha} \cdot \left(\begin{array}{c} \sqrt{n}\delta_{n',n-1} \\ \sqrt{\mu}\delta_{\mu',\mu-1} \\ 0 \end{array} \right)$$

So we have 3 possible transitions , corresponding to $\{\Delta n = 1, \Delta\mu = 0\}$ or $\{\Delta n = 0, \Delta\mu = 1\}$ or $\{\Delta n = 1, \Delta\mu = 1\}$.

4. In order to calculate the angular distribution we will use the polariztion that we have used in class. We will get different distribution depending on polarization and the transition we are at (intial and final state of the pendulum). For example lets look at the transition $\{\Delta n = 1, \Delta\mu = 0\}$:

$$\begin{aligned} \Gamma_{2\to 1} &= \sum_{\alpha} \frac{\pi}{\hbar^2}\frac{\omega e^2}{m}\left| \vec{\lambda}_{k,\alpha} \cdot \left(\begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right) \right|^2 \delta\left(\hbar\omega_k - \varepsilon_2 - \varepsilon_1\right) \\ &= \frac{\pi}{\hbar^2}\frac{\omega e^2}{m}\left\{ \cos^2\theta_k \cos^2\varphi_k + \sin^2\varphi_k \right\} \end{aligned}$$