Introduction to Particles and Nuclear Physics - Home Exercise 3

Question 1

Find the Equation of motion for a Scalar Field with the Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \sum_{n=1}^{\infty}\lambda_n\phi^{2n+2}$$

Question 2

1. For the Two Field Lagrangian shown in the TA session, write out the coupled equations of motion for each field.
2. Define: $\Phi = \begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix}, \Phi^* = \begin{pmatrix}\phi_1 & \phi_2\end{pmatrix}$, and rewrite the 2-fields Lagrangian. Then write the new EOM and confirm they give the same results.
3. Consider the Field transformation: $\phi_1 \to \phi_1', \phi_2 \to \phi_2'$ such that $\Phi' = \begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}\Phi$. Show that under this transformation, the Lagrangian you wrote in (2) remains Invariant.

Question 3

1. For a Scalar field of mass m as shown in class and TA session, show that you may write out the kinetic term explicitly to get the following expression:
$$\mathcal{L} = \frac{1}{2}(\dot{\phi})^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2$$
2. Define the conjugate momentum of the field as: $\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}}$ and find it.
3. Show the Hamiltonian may be written as: $\mathcal{H} = \frac{1}{2}(\dot{\phi})^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2$, hint: Recall the connection between the Hamiltonian and the Lagrangian.
4. Now we may define the **relativistic conjugate momentum**: $\Pi^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}$ and write out the 4-vector explicitly

Question 4

In the TA session we've seen that we may represent two scalar fields at the same time as a vector (doublet), or as a complex field. We've seen that for the complex field, our Lagrangian is invariant to phase transformations.
In Q2 you proved that the vector representation Lagrangian is invariant to rotation transformations.
In your own words: What conclusions about the transformations can you draw from comparing the two cases? (Hint: Draw a complex unit circle and mark out the phase on it, then draw a stick figure and rotate them by the same angle of the phase)

Question 5

In the ABC model: Calculate the Matrix Element of $AA \to BB$ in the center-of-mass frame, assuming $m_B = m_C = 0$. Give the answer in terms of the energy of each particle (E), the mass of particle A (m_a) and the scattering angle θ. Use the results obtained in class for ease of calculations.

Question 6

In the ABC model:

1. Draw the leading order diagrams and write an expression for the scattering amplitude $\mathcal{M}$ for the $AB \to AB$ process (there are 2 leading order diagrams).
2. Draw all the lowest order diagrams for the process $AA \to AA$ (there are six diagrams).
3. (**OPTIONAL**) Find the matrix element for this process assuming $m_B = m_C = 0$. Note that when you will arrive at the last remaining integral over the internal 4-momentum, there will no longer be a delta function including this internal momentum (there will be a $(2\pi)^4\delta^4(p_1 + p_2 + p_3 + p_4)$ term which should be swapped by i). Therefore, leave your answer as an integral over this remaining 4-momentum (q).

Question 7

In the ABC model, for the process $AA \to BB$ as seen in class, we know that the Scattering Amplitude $\mathcal{M}$ is related to the potential of the interactions of the incoming/outgoing particles that are mediated by our scalar by: $\mathcal{M} = \tilde{V}(q)$. Use the result given in class, and consider the non-relativistic limit and let $m_B = m_A$. (In this case: $q_\mu = (m, \vec{q})$.
Now, preform an inverse fourier transform to move from momentum space to real space, and show that this potential is given by: $V(r) = -\frac{g^2}{4\pi r}e^{-m_c r}$. This is known as the **Yukawa Potential**. Discuss what you understand about this potential: (Is it attractive or repulsive? What potential does it resemble if $m_c \to 0$? What are the differences between the Yukawa potential and this potential?)