Fundamentals of Quantum Technology
Week 7: The Jaynes-Cummings Model

Recommended literature:

1. Gerry, Knight (ch. 4.5, 4.6, 4.8).
2. J. Larson and T. Mavrogordatos, *The Jaynes-Cummings model and its descendants*, arXiv:2202.00330, pp. 11–14 (2022).

1 General model

The Jaynes-Cummings (JC) model is the fully-quantized version of the semi-classical Rabi model. The previously classical electric field is replaced with a single-mode photonic ladder,

$$\hat{\mathcal{H}} = \frac{\hbar\omega_0}{2}\hat{\sigma}_z + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar\lambda\left(\hat{\sigma}_+ + \hat{\sigma}_-\right)\left(\hat{a} + \hat{a}^\dagger\right).$$

Moving to the interaction picture[1] and applying the rotating wave approximation, the resultant effective Hamiltonian is

$$\hat{\mathcal{H}}_{\text{eff}} = \frac{\hbar\Delta}{2}\hat{\sigma}_z + \hbar\lambda\left(\hat{\sigma}_+\hat{a} + \hat{a}^\dagger\hat{\sigma}_-\right),$$

with $\Delta = \omega_0 - \omega$. In class, you saw that this Hamiltonian can be decomposed into a 2×2 block structure, with each block corresponding to a subspace spanned by $\{\left|e,n\right\rangle, \left|g,n+1\right\rangle\}$ (the *bare states*). The energy eigenstates are then given by *dressed states*,

$$\begin{aligned}\left|n,+\right\rangle &= \cos\left(\frac{\Phi_n}{2}\right)\left|e,n\right\rangle + \sin\left(\frac{\Phi_n}{2}\right)\left|g,n+1\right\rangle,\\ \left|n,-\right\rangle &= -\sin\left(\frac{\Phi_n}{2}\right)\left|e,n\right\rangle + \cos\left(\frac{\Phi_n}{2}\right)\left|g,n+1\right\rangle,\end{aligned}$$

corresponding to the energies

$$E_{n,\pm} = \hbar\omega\left(n + \frac{1}{2}\right) \pm \frac{\hbar}{2}\Omega_n\left(\Delta\right); \quad \Omega_n\left(\Delta\right) \equiv \sqrt{\Delta^2 + 4\lambda^2\left(n+1\right)},$$

with Φ_n defined such that $\tan\left(\Phi_n\right) = 2\lambda\sqrt{n+1}/\Delta$. We immediately observe that the energy eigenstates are, in general, states where the two-level system and the harmonic oscillator are **entangled**.

2 The dispersive regime

The above spectrum is the general exact solution for the JC Hamiltonian. An important limit of this model is the limit of **large detuning** Δ, also called the *dispersive regime* because direct atomic

[1] The unitary that we apply to the state in order to obtain this specific form of the effective Hamiltonian is $\hat{U} = \exp\left[-i\omega t\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\hat{\sigma}_z\right)\right]$.

transitions do not occur (since the coherent exchange of energy quanta between the TLS and the oscillator is not resonant), but dispersive interactions (through **virtual** photon processes) are still present.

The regime $|\Delta| \gg \lambda\sqrt{n}$ can indeed be examined by taking the appropriate limit of the exact result that we already have (as you will do at home). It is, however, instructive to examine this case also through the *Schrieffer-Wolf transformation*, which is a tool that turns out to be useful much more generally.

Exercise

Apply the unitary $\hat{U} = e^{\hat{S}}$ with $\hat{S} = \frac{\lambda}{\Delta}\left(\hat{\sigma}_+\hat{a} - \hat{a}^\dagger\hat{\sigma}_-\right)$ to $\hat{\mathcal{H}}_{\text{eff}}$ and obtain the spectrum and the form of the eigenstates in the limit of large detuning.

Solution

We will use the identity

$$\hat{U}\hat{\mathcal{H}}_{\text{eff}}\hat{U}^\dagger = \hat{\mathcal{H}}_{\text{eff}} + \left[\hat{S}, \hat{\mathcal{H}}_{\text{eff}}\right] + \frac{1}{2!}\left[\hat{S}, \left[\hat{S}, \hat{\mathcal{H}}_{\text{eff}}\right]\right] + \ldots,$$

omitting terms which are of a sufficiently high order. One may check that

$$\left[\hat{S}, \frac{\hbar\Delta}{2}\hat{\sigma}_z\right] = -\hbar\lambda\left(\hat{\sigma}_+\hat{a} + \hat{a}^\dagger\hat{\sigma}_-\right),$$

meaning that

$$\hat{\mathcal{H}}_{\text{eff}} + \left[\hat{S}, \frac{\hbar\Delta}{2}\hat{\sigma}_z\right] = \frac{\hbar\Delta}{2}\hat{\sigma}_z.$$

In other words, our choice of $\hat{S}$ eliminated the term that is first-order in λ/Δ, our perturbation parameter. Next, we observe that

$$\left[\hat{S}, \hbar\lambda\left(\hat{\sigma}_+\hat{a} + \hat{a}^\dagger\hat{\sigma}_-\right)\right] = \frac{2\hbar\lambda^2}{\Delta}\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\hat{\sigma}_z + \frac{\hbar\lambda^2}{\Delta},$$

finally yielding

$$\begin{aligned}\hat{U}\hat{\mathcal{H}}_{\text{eff}}\hat{U}^\dagger &= \frac{\hbar}{2}\left(\Delta + \frac{\lambda^2}{\Delta}\right)\hat{\sigma}_z + \frac{\hbar\lambda^2}{\Delta}\hat{a}^\dagger\hat{a}\hat{\sigma}_z + \frac{\hbar\lambda^2}{2\Delta} + \mathcal{O}\left(\frac{\lambda^3}{\Delta^2}\right)\\ &\approx \frac{\hbar}{2}\left(\Delta + \frac{\lambda^2}{\Delta}\right)\hat{\sigma}_z + \frac{\hbar\lambda^2}{\Delta}\hat{a}^\dagger\hat{a}\hat{\sigma}_z + \frac{\hbar\lambda^2}{2\Delta}.\end{aligned}$$

The second-order shift of the TLS frequency is the well-known Lamb shift, and it is a result of interactions with the vacuum of the electromagnetic mode. The second term, which couples the mode and the TLS, is called the *Stark shift.* This coupling is very different from the coupling that appears in the original JC Hamiltonian, since now the eigenstates are simply given by the bare states $|e,n\rangle, |g,n\rangle$, with energies

$$\begin{aligned}|e,n\rangle: \quad E &= \hbar\omega n + \frac{\hbar\lambda^2}{2\Delta} + \frac{\hbar}{2}\left[\omega_0 + \frac{2\lambda^2}{\Delta}\left(n + \frac{1}{2}\right)\right]\\ &= \hbar\left(\omega + \frac{\lambda^2}{\Delta}\right)n + \frac{\hbar\omega_0}{2} + \frac{\hbar\lambda^2}{\Delta},\\ |g,n\rangle: \quad E &= \hbar\omega n + \frac{\hbar\lambda^2}{2\Delta} - \frac{\hbar}{2}\left[\omega_0 + \frac{2\lambda^2}{\Delta}\left(n + \frac{1}{2}\right)\right]\\ &= \hbar\left(\omega - \frac{\lambda^2}{\Delta}\right)n - \frac{\hbar\omega_0}{2}\end{aligned}$$

Exercise

Compare the structure of the energy eigenstates in the dispersive regime to that in the case of perfect resonance, $\Delta = 0$. Draw both spectra schematically.

Solution

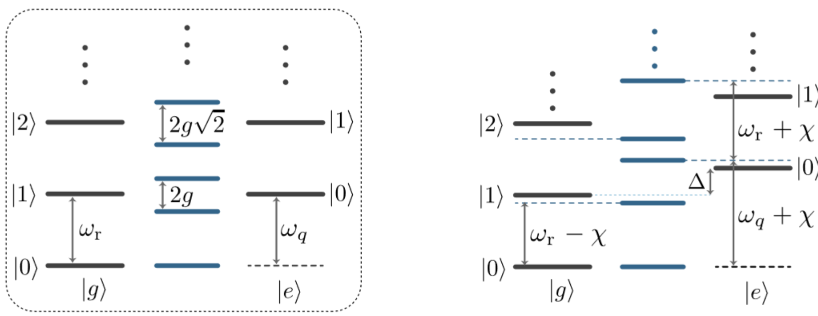
On resonance we have $\Phi_n = \pi/2$ for all n, and so the eigenstates are given by

$$|n, \pm\rangle = \frac{1}{\sqrt{2}} \left(|e, n\rangle \pm |g, n+1\rangle\right).$$

These states manifest maximal entanglement between the harmonic mode and the TLS. The control over the TLS (with which we are familiar from our acquaintance with the Rabi model) therefore comes at the expense of the TLS being in a well-defined pure state.

On the other hand, in the dispersive regime the harmonic mode and the TLS are much more weakly entangled. Indeed, the eigenstates of the effective Hamiltonian are $|e, n\rangle$ and $|g, n\rangle$. Note that the true eigenstates do feature entanglement, because the unitary transformation $e^{\hat{S}}$ which we have applied mixes TLS and harmonic excitations; however, since $\hat{S} \sim \frac{\lambda}{\Delta}$, this only amounts to a slight dressing of the disentangled states.

The energy spectra in both cases are shown schematically below:



The JC energy spectrum on resonance (left) and in the dispersive regime (right). In both figures the black lines represent the uncoupled spectrum, while blue lines represent the dressed spectrum. Here ω_r is the frequency of the Harmonic mode, ω_q is the transition frequency of the TLS, g is the coupling parameter (λ in our notation), and $\chi = g^2/\Delta$.
[Taken from Blais et al., *Circuit quantum electrodynamics*, Rev. Mod. Phys. **93**, 025005 (2021).]

On resonance, the uncoupled spectrum features a twofold degeneracy of all energy levels (apart from $|g, 0\rangle$), which the coupling lifts by $2\lambda\sqrt{n+1}$. In the dispersive regime, the transition frequency of the TLS (originally ω_0) is shifted by λ^2/Δ, and moreover the frequency of the harmonic oscillator becomes dependent on the state of the TLS, being shifted to $\omega \pm \lambda^2/\Delta$ when $\langle\hat{\sigma}_z\rangle = \pm 1$.

Exercise

Explain how the state of the TLS can be measured indirectly within the dispersive regime.

Solution

Because the shift of the frequency of the harmonic mode is dependent on the TLS state – becoming $\omega \pm \lambda^2/\Delta$ when $\langle\hat{\sigma}_z\rangle = \pm 1$ (respectively) – then the measurement of the frequency of the mode indicates to us the state of the TLS. Moreover, we recall that the effective Hamiltonian in this regime is given (up to an insignificant constant) by

$$\hat{\mathcal{H}}'_{\text{eff}} \approx \frac{\hbar\Delta'}{2}\hat{\sigma}_z + \frac{\hbar\lambda^2}{\Delta}\hat{a}^\dagger\hat{a}\hat{\sigma}_z.$$

The measured observable $\hat{\sigma}_z$ commutes with $\mathcal{H}_{\text{eff}}$, and therefore this measurement is a **quantum non-demolition measurement**, meaning that it does nothing to the measured system other than collapsing it to one of its eigenstates; that is, if the TLS is in a superposition state $C_g |g\rangle + C_e |e\rangle$ before the measurement, then the measurement of the oscillator frequency will cause it to collapse to $|g\rangle$ or to $|e\rangle$ with probability $|C_g|^2$ or $|C_e|^2$, respectively. The mark of a non-demolition measurement is that any subsequent measurement will not yield a random result, but rather will reproduce the result of the first measurement. Note that this measurement is performed without directly interacting with the TLS.