mber

4 free

final exam 2004 QM I

(~~2009-5 free~~)

2010 - 9 - 8

① ⓐ To find the coherent states of $\hat{\psi}(\vec{r})$ we will use a discrete orthonormal ~~sets~~ complete set of functions $\xi_i(\vec{r})$ and use the creation and anihilation operators a_i, a_i^+ that creates and anihilates particles having these wave functions.

$$\hat{a}_i = \int \xi_i^*(\vec{r}) \hat{\psi}(\vec{r}) d^3r \qquad \hat{\psi}(\vec{r}) = \sum_i \xi_i(\vec{r}) \hat{a}_i$$

$$\hat{a}_i^+ = \int \xi_i(\vec{r}) \psi^+(\vec{r}) d^3r \qquad \hat{\psi}^+(\vec{r}) = \sum_i \xi_i^*(\vec{r}) \hat{a}_i^+$$

$$\int d^3r\, \xi_j^*(\vec{r}) \xi_i(\vec{r}) = \delta_{ij}$$

$$\sum_i \xi_i(\vec{r}) \xi_i^*(\vec{r}') = \delta(\vec{r}-\vec{r}')$$

We know we have a Boson field:

$$[\hat{\psi}(\vec{r}), \hat{\psi}^+(\vec{r}')] = \delta(\vec{r}-\vec{r}')$$

$$\Rightarrow \delta(\vec{r}-\vec{r}') = \sum_i \sum_j \xi_i(\vec{r}) \xi_j^*(\vec{r}') [\hat{a}_i, \hat{a}_j^+]$$

| applying on both sides $\int \xi_k(\vec{r}')(\text{term}) d^3r'$

$$\xi_k(\vec{r}) = \sum_i \xi_i(\vec{r}) [\hat{a}_i, \hat{a}_k^+]$$

| applyin on both s $\int \xi_\ell^*(\vec{r}) d^3r\,(\text{term}$

$$\delta_{\ell k} = [\hat{a}_\ell, \hat{a}_k^+]$$

We now build coherent states for $\hat{a}_i$ using the number product states $|n_1, n_2 .. n_k -->$

Later we will build coherent states of $\hat{\psi}(\vec{r})$ using the basis of the coherent states of $\hat{a}_i$.

we want coheren state:

Since $\hat{a}_i$ "work" (change) only the states ξ_i then it does not matter the accupation of the other states ξ_j. $j \neq i$.

So we can look only on the occupation of the ith state.

$$\hat{a}_i |\alpha_i\rangle \underset{\text{demand}}{=} \alpha_i |\alpha_i\rangle$$

. α_i is any complex number.

$$|\alpha_i\rangle = \sum_{n=0}^{\infty} C_n |n_i\rangle$$

$$\Rightarrow \sum_{n=1}^{\infty} \sqrt{n_i}\, C_n |n_i - 1\rangle = \sum_{n=0}^{\infty} \alpha_i C_n |n\rangle$$

$$\Rightarrow \sqrt{n_i + 1}\, C_{n+1} = \alpha_i C_n \Rightarrow C_{n+1} = \frac{\alpha_i}{\sqrt{n_i+1}} C_n$$

$$\Rightarrow C_n = \frac{\alpha_i^n}{\sqrt{n!}} C_0$$

We can see that this is true using induction:

$n=0$ $\quad C_0 = C_0$ ✓

assuming for n and proving for $n+1$:

$$C_{n+1} = \frac{\alpha}{\sqrt{n+1}} \frac{\alpha^n}{\sqrt{n!}} C_0 = \frac{\alpha_i}{\sqrt{n+1}} C_n \quad ✓$$

$$1 = \langle \alpha_i | \alpha_i \rangle = \sum_{n=0}^{\infty} \frac{\alpha_i^{*n}}{\sqrt{n!}} c_0^* \sum_{m=0}^{\infty} \frac{\alpha_i^m}{\sqrt{m!}} c_0 \langle n_i | m_i \rangle =$$

$$= |c_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} = |c_0|^2 e^{|\alpha|^2}$$

$$\Rightarrow \quad c_0 = e^{-\frac{1}{2}|\alpha|^2}$$

So we find the coherent states:

$$|\alpha_i\rangle \equiv |0, 0, .. \alpha_i, 0 0\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^{n_i}}{\sqrt{n_i!}} |n_i\rangle$$

From this we can build an arbitrary product coherent state:

$$|\alpha_1, \alpha_2 .. \alpha_i, ..\rangle \equiv \bigotimes_i |\alpha_i\rangle$$

We will show that we can build any number state from coheren states - and this will show that the coherent states are a complete states:

We will show: $\alpha_i = r e^{i\theta}$

$$\frac{1}{2\pi} \frac{\sqrt{n!}}{r^n} e^{\frac{1}{2}r^2} \int_0^{2\pi} d\theta \, e^{-in\theta} |r e^{i\theta}\rangle =$$

$$= \frac{1}{2\pi} \frac{\sqrt{n!}}{r^n} e^{\frac{1}{2}r^2} \int_0^{2\pi} d\theta \, e^{-in\theta} \, e^{-\frac{1}{2}r^2} \sum_{m=0}^{\infty} \frac{r^m e^{im\theta}}{\sqrt{m!}} |m\rangle =$$

$$= \frac{1}{2\pi} \frac{r^m}{r^n} \frac{\sqrt{n!}}{\sqrt{m!}} \sum_{m=0}^{\infty} \underbrace{\int_0^{2\pi} d\theta \, e^{i(m-n)\theta}}_{2\pi \delta_{nm}} |m\rangle = |n_i\rangle$$

Now going back to the original problem we will show that the product coherent states

$\{|\alpha_1, \alpha_2 .. \alpha_i ..\rangle\}$

They form a complete ~~~~ set we don't have to look for more coherent states (although there might be a superposition of them that form also a coherent state of $\hat{\psi}(\vec{r})$)

$$\hat{\psi}(\vec{r})\,|\{\alpha_i\}\rangle = \sum_j \xi_j(\vec{r})\,\hat{a}_j\,|\{\alpha_i\}\rangle = \sum_j \xi_j(\vec{r})\,\alpha_j\,|\{\alpha_i\}\rangle$$

so by denoting $\psi_{\vec{r}} = \sum_j \xi_j(\vec{r})\alpha_j$

we get: $|\psi_{\vec{r}}\rangle \equiv |\{\alpha_i\}\rangle$

$$\hat{\psi}(\vec{r})\,|\psi_{\vec{r}}\rangle = \psi_{\vec{r}}\,|\psi_{\vec{r}}\rangle$$

$$\langle\{\alpha_i\}|\,\hat{N}\,|\{\alpha_i\}\rangle = \sum_j \langle\{\alpha_i\}|\,\hat{N}_j\,|\{\alpha_i\}\rangle = \sum_j \langle\{\alpha_i\}|\,\hat{a}_j^{+}\hat{a}_j\,|\{\alpha_i\}\rangle =$$

$$= \sum_j |\alpha_j|^2 .$$

And in another way:

$$\langle\{\alpha_i\}|\,\hat{N}\,|\{\alpha_i\}\rangle = \int d^3r\,\langle\{\alpha_i\}|\,\hat{\psi}^{+}(\vec{r})\,\hat{\psi}(\vec{r})\,|\{\alpha_i\}\rangle =$$

$$= \int d^3r \sum_{i,j} \alpha_i^* \xi_i^*(\vec{r})\,\alpha_j \xi_j(\vec{r}) =$$

$$= \sum_{i,j} \alpha_i^*\alpha_j \int d^3r\, \xi_i^*(\vec{r})\xi_j(\vec{r}) = \sum_{i,j} \delta_{ij}\,\alpha_i^*\alpha_j = \sum_i |\alpha_i|^2$$

The Hamiltonian ~~~~ is

$$H = H_g + V_d$$

Suppose $\{\xi_i\}$ are the eigenfunction of the single particle Hamiltonian h. Then we can write:

$$H_0 = \sum_i \varepsilon_i \hat{a}^\dagger_i \hat{a}_i$$

The two body interaction will be:

$$V = \sum_{ijkl} \langle ij|V|kl\rangle \hat{a}^\dagger_i \hat{a}^\dagger_j \hat{a}_l \hat{a}_k \equiv \sum_{ijkl} V_{ijkl} \hat{a}^\dagger_i \hat{a}^\dagger_j \hat{a}_l \hat{a}_k$$

and than:

$$\langle\{\alpha_i\}| H_0 |\{\alpha_i\}\rangle = \sum_i \varepsilon_i |\alpha_i|^2$$

$$\langle\{\alpha_i\}| V |\{\alpha_i\}\rangle = \sum_{ijkl} V_{ijkl} \alpha^*_i \alpha^*_j \alpha_l \alpha_k$$

$$\Rightarrow \langle\{\alpha_i\}| H |\{\alpha_i\}\rangle = \sum_i \varepsilon_i |\alpha_i|^2 + \sum_{ijkl} V_{ijkl} \alpha^*_i \alpha^*_j \alpha_l \alpha_k$$

b) The single particle Hamiltonian of H.O. is

$$h = \hbar\omega(a^\dagger a + \tfrac{1}{2})$$

and its eigen functions are $u_n(x)$

In second quatization we consider the sch. equ:

$$i\hbar \frac{\partial}{\partial t}\psi = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2\right)\psi$$

... field equ of motion which can be ...

The normal modes of h we get

$$H = \sum_n \varepsilon_n \hat{b}_n^+ \hat{b}_n$$

$$\varepsilon_n = \hbar\omega(n+\tfrac{1}{2})$$

The operators $\hat{b}_n^+, \hat{b}_n$ creates and anihilates particles from eigenstate n of h.

While the operators a^+, a

transform an eigenstate n of h to be the eigenstate $n\pm$ of h. So a^+, a works on the indices of the number product state while b_n^+, b_n works on the numbers.

For example:

$$b_j^+ |0 \ldots 0, n_j, 0 \ldots\rangle = \sqrt{n_j+1}\,|0, \ldots n_j{+}1\, 0 \ldots\rangle$$

$$a^+|j\rangle = \sqrt{j+1}\,|j+1\rangle$$

The way to use them both is to let a, a^+ work on the indexes (states of h_i) and let b_n^+, b_n work on the number states

$$\hat{O} = \sum_{ij} \langle i | f(a, a^+) | j \rangle b_i^+ b_j$$

Question 1

(a) We choose a discrete orthogonal set of functions $\varphi_k(\vec r)$ and develop $\Psi(\vec r)$ with regard to them.

$$\begin{cases} \hat\Psi(r) = \sum_k \varphi_k(r)\,\hat a_k \\ \hat\Psi^{+}(r) = \sum_k \varphi_k^{*}(r)\,\hat a_k^{+} \end{cases}$$

$$\begin{cases} \hat a_k = \int dr\, \varphi_k(r)\,\hat\Psi(r) \\ \hat a_k^{+} = \int dr\, \varphi_k^{*}(r)\,\hat\Psi^{+}(r) \end{cases}$$

Now we treat the $\hat a_k, \hat a_k^{+}$ operators which have the meaning of annihilation and creation operators since

$$[a_k, a_{k'}] = \int dr \int dr'\, \varphi_k(r)\,\varphi_{k'}(r')\, \underbrace{[\Psi(r), \Psi(r')]}_{0} = 0$$

$$[a_k^{+}, a_{k'}^{+}] = \int dr \int dr'\, \varphi_k^{*}(r)\,\varphi_{k'}^{*}(r')\, \underbrace{[\Psi^{+}(r), \Psi^{+}(r')]}_{0} = 0$$

$$[a_k, a_{k'}^{+}] = \int dr \int dr'\, \varphi_k(r)\,\varphi_{k'}^{*}(r')\, \underbrace{[\Psi(r), \Psi^{+}(r')]}_{\delta}$$

$$= \int dr\, \varphi_k(r)\,\varphi_{k'}^{*}(r) = \delta_{k,k'}$$

Therefore we can work in the space of excitation numbers with regarding to these operators $|\ldots n_k \ldots\rangle$

Denote an eigenstate of a_k by $|\alpha\rangle_k$ (not w... in the above space - we'll do the convention

$$a_k |\alpha\rangle_k = \alpha\, |\alpha\rangle_k$$

The operator Ψ is a sum of commuting operators and the eigenstate is a product of eigenstates of each component

$$|\alpha\rangle_k = \sum_n C_n(\alpha)\, |n\rangle_k$$

$$a\,|\alpha\rangle_k = \sum_{n=1}^{\infty} C_n(\alpha)\sqrt{n}\,|n-1\rangle_k = \sum_{n=0}^{\infty} \alpha\, C_n(\alpha)\,|n\rangle_k$$

$$\Rightarrow \quad C_{n+1}(\alpha) = \frac{\alpha}{\sqrt{n+1}}\, C_n(\alpha)$$

$$C_n(\alpha) = \frac{\alpha^n}{\sqrt{n!}}\, C_0(\alpha)$$

From normalization

$$1 = \sum_n |C_n(\alpha)|^2 = |C_0(\alpha)|^2 \sum_n \frac{|\alpha|^{2n}}{n!} = |C_0(\alpha)|^2 e^{|\alpha|^2}$$

$$\Rightarrow \quad |\alpha\rangle_k = \exp\left(-\frac{|\alpha|^2}{2}\right) \sum_n \frac{\alpha^n}{\sqrt{n!}}\, |n\rangle_k$$

For each complex number α and mode number k, we've found a coherent state $|\alpha\rangle_k$, we can express it in the way described above in the space of excitation numbers with all now $n_{k'} = 0$ for $k' \neq k$. The following state is an $\mathcal{H}$! eigenstate of $\psi(r)$, since

$$\psi(r)\prod_{k'}|\alpha\rangle_k = \sum_{k'} \varphi_{k'}(r)\, a_{k'}\prod|\alpha\rangle_k = \sum_k \varphi_k(r)\cdot\alpha\cdot|\alpha\rangle_k$$

and the eigenvalue is $\alpha\cdot\varphi_k(r)$ need to consider product of coherent states

$$N = \int dr\, \psi^\dagger(r)\psi(r) = \int dr \sum_k\sum_{k'} \varphi_k^*(r)\varphi_{k'}(r)\, a_k^\dagger a_{k'} =$$

$$= \sum_k\sum_{k'} \delta_{k,k'}\, a_k^\dagger a_{k'} = \sum_k a_k^\dagger a_{k'} = \sum_k N_k$$

$${}_k\langle\alpha|N|\alpha\rangle_k = \exp(-|\alpha|^2) \sum_n\sum_{n'} \frac{\alpha^{*n}}{\sqrt{n!}}\frac{\alpha^{n'}}{\sqrt{n'!}}\, {}_k\langle n|N|n'\rangle_k =$$

$$= \exp(-|\alpha|^2) \sum_n\sum_{n'} \frac{\alpha^{*n}\alpha^{n'}}{\sqrt{n!\,n'!}}\, {}_k\langle n|N_k|n'\rangle_k =$$

$$= \exp(-|\alpha|^2) \sum_n \frac{|\alpha|^{2n}}{n!}\, n = \exp(-|\alpha|^2) \sum_n \frac{\alpha^{2n}}{(n-1)!} = |\alpha|^2$$

We can also now treat the more general

This state is a direct product of states mentioned above, but with different α and k values. now ok

$$a(r) |\ldots \alpha_k \ldots\rangle = \sum_{k'} \varphi_{k'}(r)\, a_{k'} |\ldots \alpha_k \ldots\rangle =$$
$$= \left(\sum_{k'} \alpha_{k'} \varphi_{k'}(r)\right) |\ldots \alpha_k \ldots\rangle$$

So that the eigenvalue is $\sum_{k'} \alpha_{k'} \varphi_{k'}(r)$

The expectation value of N of that state is:

$$\langle \ldots \alpha_k \ldots | N | \ldots \alpha_k \ldots \rangle = \sum_{k'} {}_k\langle \alpha | N_k | \alpha \rangle_{k'} =$$
$$= \exp(-|\alpha|^2) \sum_{k'} \sum_n \frac{\alpha^{2n}}{(n-1)!} = \sum_k |\alpha_k|^2$$

The hamiltonian of the two body interaction is

$$H = \sum_{k'} \varepsilon_{k'} N_{k'} + \sum_{ijlm} U_{ijlm}\, a_i^+ a_j^+ a_l a_m$$

The expectation value of the first term is:

$$\langle \ldots \alpha_k \ldots | \sum_{k'} \varepsilon_{k'} N_{k'} | \ldots \alpha_k \ldots \rangle = \sum_{k'} \varepsilon_{k'}\, {}_k\langle \alpha | N_{k'} | \alpha \rangle_{k'} =$$
$$= \sum_{k'} \varepsilon_{k'} \exp(-|\alpha|^2) \sum_n \sum_{n'} \frac{\alpha^n}{\sqrt{n!}} \frac{\alpha^{n'}}{\sqrt{n'!}}\, {}_k\langle n | \varepsilon_{k'} N_{k'} | n' \rangle_k$$
$$= \exp(-|\alpha|^2) \sum_{n'} \sum_{k'} \frac{\alpha_{k'}^{2n}}{(n-1)!} \varepsilon_{k'}$$

the 2nd term:

$$\langle \ldots \alpha_k \ldots | \sum_{ijlm} U_{ijlm}\, a_i^+ a_j^+ a_l a_m | \ldots \alpha_k \ldots \rangle =$$
$$= \sum_{l,m} \alpha_l \alpha_m \sum_{ij} U_{ijlm} \langle \ldots \alpha_k \ldots | a_i^+ a_j^+ | \ldots \alpha_k \ldots \rangle$$
$$= \sum_{l,m} \alpha_l \alpha_m \sum_{ij} U_{ijlm} \Big[\delta_{ij} \exp(-|\alpha|^2) \sum_n \sum_{n'} \frac{\alpha^n \alpha^{n'}}{\sqrt{n!}\sqrt{n'!}} \delta_{n,n'+2}$$
$$+ (1-\delta_{ij}) \Big[\exp(-|\alpha|^2) \sum_n \sum_{n'} \frac{\alpha^n \alpha^{n'}}{\sqrt{n!}\sqrt{n'!}} \delta_{n,n'+1}\Big]^2 \Big] =$$
$$= \sum_{l,m} \alpha_l \alpha_m \sum_{ij} U_{ijlm} \Big[\delta_{ij} \exp(-|\alpha|^2) \sum_n \frac{\alpha^{2n+2}}{\sqrt{n!\,(n+2)!}} + (1-\delta_{ij}) \exp(-|\alpha|^2) \sum_n \frac{\alpha^{2n+1}}{\sqrt{n!\,(n+1)!}}$$

(b) The process of second quantization of the harmonic oscillator is as following:
take the sch. eq.:

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{1}{2}m\omega^2x^2\right)\psi=E\psi$$

whose solutions are:

$$\varphi_n(x)=\frac{1}{\sqrt{2^n n!}}\cdot\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{1}{2}\frac{m\omega}{\hbar}x^2\right)\cdot H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)$$

with energies $E_n=\hbar\omega\left(n+\frac{1}{2}\right)$

Now treat ψ as a field, and both quantize it and expand it in modes of $\varphi_n(x)$, namely:

$$\hat{\psi}(x)=\sum_n\varphi_n(x)\cdot\hat{b}_n$$

$$\hat{\psi}^{+}(x)=\sum_n\varphi_n^*(x)\cdot\hat{b}_n^{+}$$

✓

The commutation relations between the new operators $\hat{b}_n,\hat{b}_n^{+}$ are the same as the commutation relations between the regular a, a^{+} operators of the harmonic oscillator, as was already proved (for the general case) in section (a).

The meaning of those operators is different: b_n^{+}, b_n create and annihilate another particle whose excitation is like the nth excitation of the harmonic oscillator whereas, a^{+}, a just change the state (raise/lower the energy) of a single particle which is in ...

Therefore, a^+, a are good to describe a single particle hamiltonian but to describe many particles hamiltonian (either of free particles or interacting particles) we need to use the b^+, b operators.

48/50