QM-1 **Problem Set 6** **2012-13**

1. Calculate the norm of the following states for both bosons and fermions

$$a) \ \sum_{ij} C_{ij} a_i^+ a_j^+ |0> \quad , \quad b) \ a_i^+ a_j^+ a_k^+ |0> \ .$$

2. Suppose that in the general many body Hamiltonian

$$H = \sum_\alpha \varepsilon_\alpha a_\alpha^+ a_\alpha + \frac{1}{2} \sum_{\alpha\beta\gamma\delta} V_{\alpha\beta\gamma\delta} a_\alpha^+ a_\beta^+ a_\delta a_\gamma$$

   the interaction is such that only $V_{\alpha\beta\alpha\beta}$ and $V_{\alpha\beta\beta\alpha}$ are not zero. In the questions below consider both the fermion and the boson cases.

   (a) What are the special symmetries of the above Hamiltonian and what are their generators? Use this knowledge to completely solve the problem, i.e. to find the exact eigenvalues and eigenfunctions of this Hamiltonian.

   b) Now derive and solve the Hartree Fock equations for this Hamiltonian. Do the solutions reproduce the exact ones?

   c) For this Hamiltonian and the fermions case only derive and solve the Heisenberg equations of motion for the operators a_i and a_i^+. Use the solutions to find the spectrum and the wave functions of excitations with M+1 and M-1 particles. Compare with the s. p. energies of the H.-F. problem.

3. Consider a time dependent single particle equation for $\psi(\mathbf{r}, t)$ in an external potential $U(\mathbf{r})$. Represent ψ as $\sqrt{\rho}\exp(i\chi)$ with real ρ and χ and write coupled equations for ρ and χ. Introduce the "velocity field" $\mathbf{v} = \nabla\chi$ and show that the equation for ρ is just the continuity equation. Now turn to the equation for χ and (by taking gradient of it) write it as an equation for $\mathbf{v}$. Show that it is equivalent to Euler equation in fluid mechanics - consult for example the link `http://en.wikipedia.org/wiki/Euler_equations_(fluid_dynamics)`

   Identify the pressure term. What is its physical interpretation? What part of it would you call quantum pressure? What is the equation of state (i.e. the dependence of the pressure on the density) of this fluid?

(for X credit) What are sound waves in this fluid?

4. a) Consider the time dependent version of the Gross-Pitaevsky (GP) equation without the external potential and with the short range two body interaction
$$V(\mathbf{r}-\mathbf{r}') = V_0\delta(\mathbf{r}-\mathbf{r}')$$
Transform this equation to the variables
$$\psi(\mathbf{r},t) = \sqrt{\rho(\mathbf{r},t)}\exp[i\chi(\mathbf{r},\mathbf{t})].$$
Show that the variables $\rho(\mathbf{r})$ and $\chi(\mathbf{r})$ are canonical. Find a uniform (space independent) stationary solution. Derive equations for small oscillations of ρ and χ around this solution and solve them (hint - try plane waves). Show that the behavior of the resulting dispersion relation of these oscillations (the so called Bogoliubov spectrum) at $k \to 0$ corresponds to the long wave length oscillations of the phase χ. Explain how is this related to the breaking of global gauge symmetry by the uniform solution. Compare to the non interacting case $V_0 = 0$. (see previous problem).

   b) Discuss the non relativistic version of the so called Higgs mechanism - assume that the bose liquid is charged i.e. in addition to the short range interaction $V_0\delta(\mathbf{r}-\mathbf{r}')$ there is a Coulomb interaction between the particles of the liquid. Discuss what happens in this case with the Bogoliubov spectrum paying special attention to the $k \to 0$ (long wave length) end of the spectrum. (Hint - it is easy to do the above using coupled GP-Poisson equations. Do not forget to place the liquid in an inert uniform neutralizing background.)