1

$$\beta H = \sum_{\langle i,j\rangle} -K S_i S_j + K_0$$

$$Z = \sum_{\{S_i\}} e^{NK_0 + K S_1 S_2 + K S_2 S_3 \cdots} = \sum e^{NK_0 + K S_2 (S_1 + S_3) + \cdots}$$

summing over every other spin:

$$= \sum_{\{S_i\}} \prod_{i=1}^{N/2} e^{NK_0} \left[2\cosh(K(S_i + S_{i+2})) + 1 \right]$$

$$\Rightarrow \quad e^{2K_0} \left(2\cosh(K(S_i + S_{i+2})) + 1 \right) = e^{K_0' + K' S_i S_{i+2}}$$

$-1,-1\,;\,+1\,+1 \rightarrow \quad e^{2K_0}(2\cosh 2K + 1) = e^{K_0' + K'}$

$0,0 \rightarrow \quad 3e^{2K_0} = e^{K_0'}$

$+1,-1\,;\,-1,+1 \rightarrow \quad 3e^{2K_0} = e^{K_0' - K'}$

$\Big\}\; K' = 0$ $\Downarrow$ No solution

$0\,\pm1\,;\,\pm1,0 \rightarrow \quad e^{2K_0}(2\cosh K + 1) = e^{K_0'}$

There is no solution with these 4 eq. and two variables. $\Rightarrow$ 2 more terms!

$$-\beta H = \sum k_0 + k_1 S_i S_{i+1} + k_2 (S_i^2 + S_{i+1}^2) + k_3 S_i^2 S_{i+1}^2$$

This is the simplest H that coserves inversion symmetry and has two additional terms. Lets hope it will be closed under decimation:

$$Z = \sum e^{2k_0 + k_1(S_1 S_2 + S_2 S_3 + \dots) + k_2(S_1^2 + S_2^2 + S_2^2 + S_3^2 \dots) + k_3(S_1^2 S_2^2 + S_2^2 S_3^2 + \dots)}$$

$$= \sum_{odd} e^{2k_0} \sum_{S_2} e^{k_1 S_2 (S_1 + S_3) + k_2 (S_1^2 + 2S_2^2 + S_3^2) + k_3 S_2^2 (S_1^2 + S_3^2)} \sum_{S_n} \dots\dots$$

$$= \sum_{odd} \prod_i^{N/2} e^{2k_0} \Big[e^{k_2(S_i^2 + S_{i+2}^2)} \qquad 0$$

$$+ e^{k_1(S_i + S_{i+2}) + k_2(S_i^2 + S_{i+2}^2 + 2) + k_3(S_i^2 + S_{i+2}^2)} \qquad +1$$

$$+ e^{-k_1(S_i + S_{i+2}) + k_2(S_i^2 + S_{i+2}^2 + 2) + k_3(S_i^2 + S_{i+2}^2)} \Big] \qquad -1$$

$$= \sum_{odd} \prod_i^{N/2} e^{2k_0} \Big[e^{k_2(S_i^2 + S_{i+2}^2)} + e^{k_2(S_i^2 + S_{i+2}^2 + 2) + k_3(S_i^2 + S_{i+2}^2)} \times 2\cosh(k_1(S_i + S_{i+2})) \Big]$$

$$= \sum_{odd} \prod_i^{N/2} e^{2k_0} e^{k_2(S_i^2 + S_{i+2}^2)} \Big[1 + 2e^{k_3(S_i^2 + S_{i+2}^2) + 2k_2} \cosh(k_1(S_i + S_{i+2})) \Big]$$

$$\Rightarrow e^{2k_0} e^{k_2(S_i^2+S_{i+2}^2)} \left[1+2e^{k_3(S_i^2+S_{i+2}^2)+2k_2}\cosh\left(k_1(S_i+S_{i+2})\right)\right]$$

$$=$$

$$e^{k_0'+k_1'S_iS_{i+2}+k_2'(S_i^2+S_{i+2}^2)+k_3'S_i^2S_{i+2}^2}$$

$-1,-1;+1\ +1 \xrightarrow{1} e^{2k_0}e^{2k_2}\left[1+2e^{2k_2+2k_3}\cosh 2k_1\right]=e^{k_0'+k_1'+2k_2'+k_3'}$

$0,0 \xrightarrow{2} e^{2k_0}\left[1+2e^{2k_2}\right]=e^{k_0'} \Rightarrow \underline{\underline{k_0'=2k_0+\ln(1+2e^{2k_2})}}$

$+1,-1;-1,+1 \xrightarrow{3} e^{2k_0}e^{2k_2}\left[1+2e^{2k_2+2k_3}\right]=e^{k_0'+2k_2'+k_3'-k_1'}$

$0\pm1;\pm1,0 \xrightarrow{4} e^{2k_0}e^{k_2}\left[1+2e^{2k_2+k_3}\cosh k_1\right]=e^{k_0'+k_2'}$

4 eq. 4 variables → might be solvable.

$\underline{1:3}$: $$\frac{\cancel{e^{2k_0}e^{2k_2}}\left[1+2e^{2k_2+2k_3}\cosh 2k_1\right]}{\cancel{e^{2k_0}e^{2k_2}}\left[1+2e^{2k_2+2k_3}\right]}=e^{2k_1'}$$

$$\underline{\underline{k_1'=\frac{1}{2}\ln\left[\frac{1+2e^{2k_2+2k_3}\cosh 2k_1}{1+2e^{2k_2+2k_3}}\right]}}$$

$\underline{4}$: $\cancel{2k_0}+k_2+\ln\left[1+2e^{2k_2+k_3}\cosh k_1\right]=\cancel{2k_0}+\ln(1+2e^{2k_2})+k_2'$

$$\underline{\underline{k_2'=k_2+\ln\left[\frac{1+2e^{2k_2+k_3}\cosh k_1}{1+2e^{2k_2}}\right]}}$$

⑤: $\cancel{2k_0} + \cancel{2k_2} + \ln\left[1+2e^{2k_2+2k_3}\right] = \cancel{2k_0} + \ln\left(1+2e^{2k_2}\right) + \cancel{2k_2}$

$$+ 2\ln\left[\frac{1+2e^{2k_2+k_3}\cosh k_1}{1+2e^{2k_2}}\right] + k_3'$$

$$- \frac{1}{2}\ln\left[\frac{1+2e^{2k_2+2k_3}\cosh 2k_1}{1+2e^{2k_2+2k_3}}\right]$$

$$\ln\left[1+2e^{2k_2+2k_3}\right] = \ln\left(1+2e^{2k_2}\right) + 2\ln\left(1+2e^{2k_2+k_3}\cosh k_1\right) - 2\ln\left(1+2e^{2k_2}\right)$$

$$-\frac{1}{2}\ln\left(1+2e^{2k_2+2k_3}\cosh 2k_1\right) + \frac{1}{2}\ln\left(1+2e^{2k_2+2k_3}\right) + k_3'$$

$$k_3' = \ln\left[\frac{\left(1+2e^{2k_2+2k_3}\right)^{1/2}\left(1+2e^{2k_2}\right)^{2}\left(1+2e^{2k_2+2k_3}\cosh 2k_1\right)^{1/2}}{\cancel{\left(1+2e^{2k_2}\right)}\left(1+2e^{2k_2+k_3}\cosh k_1\right)^2\,\cancel{\left(1+2e^{2k_2+2k_3}\right)^{1/2}}}\right] / \cdot 2$$

$$k_3' = \frac{1}{2}\ln\left[\frac{\left(1+2e^{2k_2+2k_3}\right)\left(1+2e^{2k_2}\right)^2\left(1+2e^{2k_2+2k_3}\cosh 2k_1\right)}{\left(1+2e^{2k_2+k_3}\cosh k_1\right)^4}\right]$$

$$e^{2k_3'} = \frac{\left(1+2e^{2k_2+2k_3}\right)\left(1+2e^{2k_2}\right)^2\left(1+2e^{2k_2+2k_3}\cosh 2k_1\right)}{\left(1+2e^{2k_2+k_3}\cosh k_1\right)^4}$$

$$e^{k_0'} = e^{2k_0}\left(1+2e^{2k_2}\right) \qquad e^{-k_0} = \left(1+2e^{2k_2}\right)$$

$$e^{2k_1'} = \frac{1+2e^{2k_2+2k_3}\cosh 2k_1}{1+2e^{2k_2+2k_3}}$$

$$e^{k_2'} = e^{k_2}\,\frac{1+2e^{2k_2+k_3}\cosh k_1}{1+2e^{2k_2}}$$

$$1+2e^{2k_2} = 1+2e^{2k_2+k_3}\cosh k_1$$

$$1 = e^{k_3}\cosh k_1$$

$\cosh k_1 \geq 1$ for all $k_1 \Rightarrow \boxed{k_1, k_3 = 0}$

$$e^{2k_1} + e^{2k_2+2k_3+2k_1} = 1+2e^{2k_2+2k_3}\cosh 2k_1$$

$$1 + e^{2k_2} = 1+2e^{2k_2} \checkmark$$

$$e^{2k_3'} = \frac{(1+2e^{2k_2+2k_3})(1+2e^{2k_2})^2(1+2e^{2k_2+2k_3}\cosh 2k_1)}{(1+2e^{2k_2+k_3}\cosh k_1)^4}$$

$$1 = \frac{(1+2e^{2k_2})(1+2e^{2k_2})^2(1+2e^{2k})}{\ldots\ldots} = 1 \checkmark$$

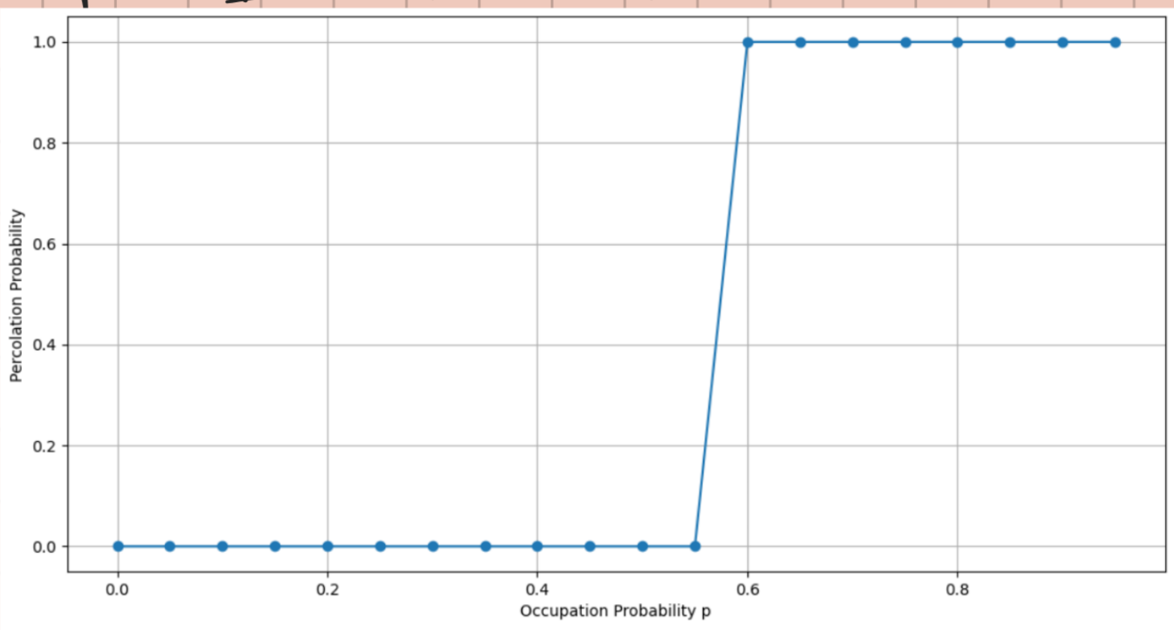
$$\Rightarrow \begin{cases} k_0 = -\ln(1+2e^{2k_2}) \\ k_1 = 0 \\ k_2 = k_2 \\ k_3 = 0 \end{cases}$$

<u>infinite solutions for arbitrary k_2</u>

(2) To get the percolation prob. I used DFS cuz it's simpler and run basically in the same duration.

psuedo code:

for each probability:
    for each realization:
        Find if there's a path
        if tere is increase counter by 1
    P = counter / number of realizations
plot P vs. Probabilities

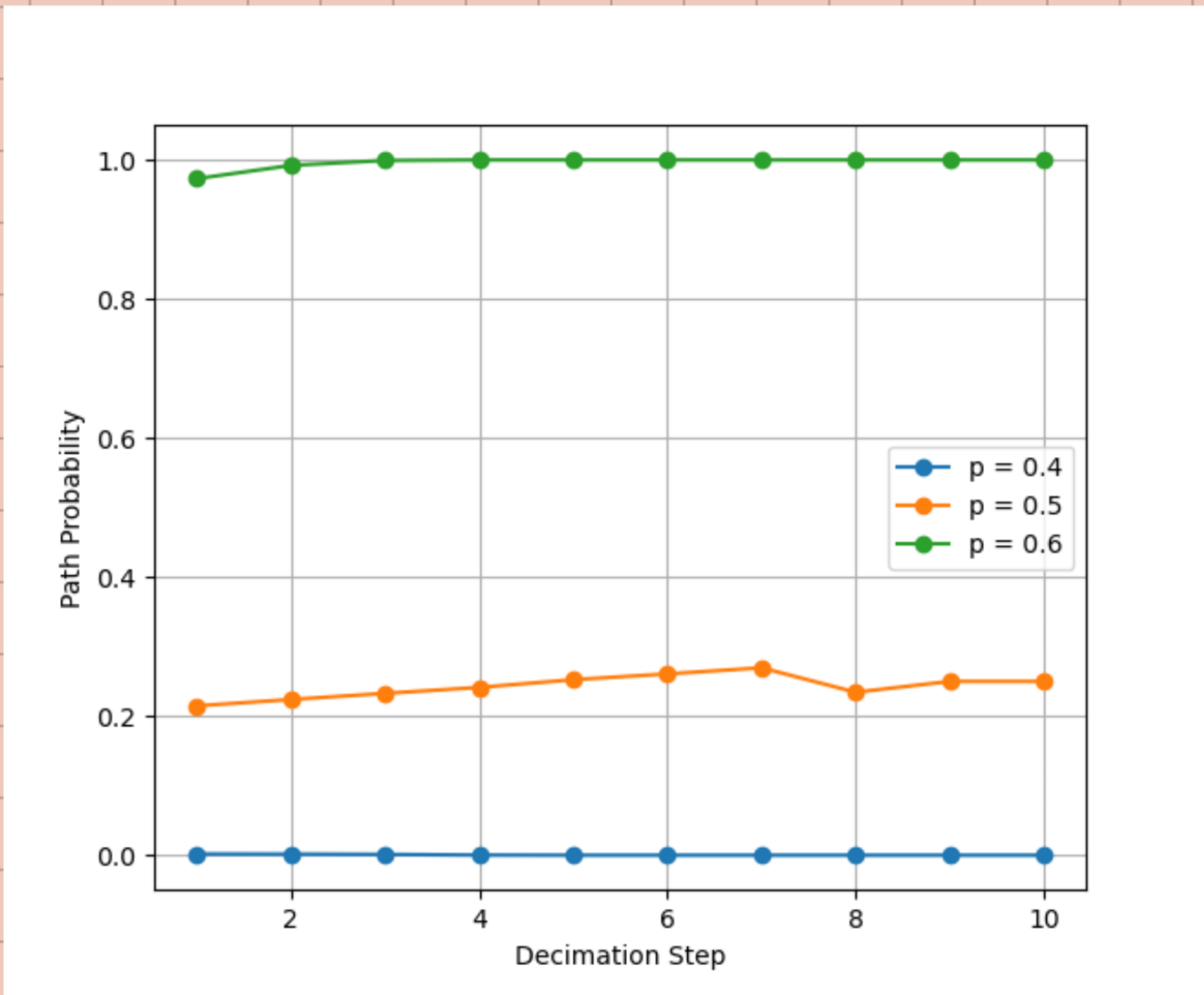


I found by running the code again and again that for $p = 0.5925$ P is persicly 0.5.

For the decimations, because we use one realization we can't initialize the lattice with binary values and see if theres a path. There will be no meaning to path Probability, the realization either has a path or it doesn't.

Therefore to initialize my single realization I used BFS. Seeding the top row, then adding cells to the rows bellow (iteratively) with probability P. Then path prob. is calculated by # of percolated blocks / lattice size.

For P_c we expect a self similar path probability (diferrent from 0 or 1). Or atleast close to self similar, because of the randomizing of the seed first row.



$p = 0.4$: No paths, almost no activated cells

$p = 0.6$: many paths, most cells are activated

$p = 0.5$: 1 path on average, number of activated cells per decimation is very similar => critical prob.

③ 1) $X \sim p(x)$ $\quad\quad$ $Y \sim p(y)$

$$Z = X + Y$$

X and Y are independent and drawn from the same distribution $p(x) = p(y)$.

To get $p(z)$ we need all the possibilities of the sum.

$$p(z) = \int p(\overbrace{z-y}^{x})\, p(y)\, dy$$

In other words for a fixed z we sum over all the possibilities of the pair (x,y) weighted by $p(x)\,p(y)$.

independent

$$V(z) = V(x) + V(y) + \cancel{2cov(x,y)} = 2\sigma^2$$

$V(x) = V(y) = \sigma$

$$\Rightarrow \sigma_z = \sqrt{V(z)} = \sqrt{2}\,\sigma$$

2) $\int S_{\sqrt{2}}[\rho(x)]dx = \int \sqrt{2}\,\rho(\sqrt{2}x)\,dx = \int \rho(u)\,du = 1$

3) $F(\sqrt{2}\rho(\sqrt{2}x)) = \sqrt{2}\int \rho(\sqrt{2}x)\, e^{-ikx}\,dx = \int \rho(u)\, e^{-i\frac{k}{\sqrt{2}}u}\,du$

$= \tilde{\rho}\left(\frac{k}{\sqrt{2}}\right)$

4) First the convolution sums over pairs of independent varisbles. This is the "zoom out" phase of the RG step. It already inheretely includes the part where you take into account both variables by weighting with $\rho(x)\rho(y)$.

This part integrates out 2 variables and replaces them with their combined effect.

Then there is the part of rescaling and renormalizing to be able to compare to the previous RG step.

$\rho^* = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{x^2}{2\sigma^2}}$

$$\sqrt{2}\int \rho(\sqrt{2}x-y)\rho(y)\,dy = \sqrt{2}\int \frac{1}{2\pi\sigma^2} e^{-\frac{(\sqrt{2}x-y)^2}{2\sigma^2}} e^{-\frac{y^2}{2\sigma^2}}\,dy$$

$$2x^2+y^2-2\sqrt{2}xy$$

$$= \frac{1}{\sqrt{2}\pi\sigma^2}\int e^{-\frac{x^2+y^2+\sqrt{2}xy}{\sigma^2}}\,dy = \frac{1}{\sqrt{2}\pi\sigma^2}\sqrt{\pi\sigma^2}\, e^{\frac{\frac{2x^2}{\sigma^4}-4\frac{1}{\sigma^2}\frac{x^2}{\sigma^2}}{\frac{4}{\sigma^2}}}$$

$$a=\frac{1}{\sigma^2},\ b=\frac{\sqrt{2}x}{\sigma^2},\ c=\frac{x^2}{\sigma^2}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} = \rho^*$$

5) T is convolution + rescaling

$$\Rightarrow \mathcal{F}[T(\rho(x))] = \tilde{\rho}\left(\frac{k}{\sqrt{2}}\right)^2$$

ρ^* is the fixed point of T, so appling twice:

$$\tilde{\rho}^*\left(\frac{k}{\sqrt{2}}\right)^2 = \tilde{\rho}^*(k) \quad (*)$$

$$\rho(x) = \rho^*(x) + \varepsilon f(x)$$

$$\downarrow \mathcal{F}$$

$$\tilde{\rho}(k) = \tilde{\rho}^*(k) + \varepsilon \tilde{f}(k)$$

$$\Rightarrow \tilde{T}[\rho] = \left[\tilde{\rho}^*\left(\frac{k}{\sqrt{2}}\right) + \varepsilon \tilde{f}\left(\frac{k}{\sqrt{2}}\right)\right]^2 \overset{(*)}{=} \tilde{\rho}^*(k) + 2\varepsilon\, \tilde{\rho}^*\left(\frac{k}{\sqrt{2}}\right)\tilde{f}\left(\frac{k}{\sqrt{2}}\right) + O(\varepsilon^2)$$

If instead we linearize T: $\tilde{T}[\rho] = \tilde{\rho}^*(k) + \varepsilon \tilde{L}[f]$

where L is the linearized operator.

$\Rightarrow$ The linearized RG transformation T in Fourier space acts on the perturbation: $\tilde{L}[F] = 2\tilde{\rho}^{*}(\frac{k}{\sqrt{2}})\tilde{f}(\frac{k}{\sqrt{2}})$

$$\Rightarrow \qquad \tilde{L}[F] = \lambda \tilde{f}(k) = 2\tilde{\rho}^{*}(\tfrac{k}{\sqrt{2}})\tilde{f}(\tfrac{k}{\sqrt{2}})$$

$$\tilde{f}(k) = \frac{2}{\lambda}\tilde{\rho}^{*}(\tfrac{k}{\sqrt{2}})\tilde{f}(\tfrac{k}{\sqrt{2}})$$

we already know (*) $\Longrightarrow$ $\tilde{\rho}^{*}(\frac{k}{\sqrt{2}}) = e^{-\frac{x^2}{4}}$

(we're taking $\sigma = 1$ for simplicity)

$$\tilde{f}_n(k) = \frac{2}{\lambda_n} e^{-\frac{x^2}{4}} \tilde{f}(\tfrac{k}{\sqrt{2}})$$

Now if we use (*) again we can easily guess:

$$\tilde{f}_n(k) = e^{-k^2/2} k^n \qquad \Leftarrow \text{eigenfunction}$$

$$\Rightarrow \qquad \cancel{e^{-k^2/2}}\,\cancel{k^n} = \frac{2}{\lambda_n}\cancel{e^{-\frac{x^2}{4}} e^{-\frac{k^2}{4}}}\left(\frac{k}{\sqrt{2}}\right)^n$$

$$1 = \frac{2}{\lambda_n}\frac{1}{2^{n/2}} \qquad \underline{\underline{\lambda_n = 2^{1-\frac{n}{2}}}} \qquad \Leftarrow \text{eigenvectors}$$

6) $\lambda_0 = 2$, $\lambda_1 = \sqrt{2}$, $\lambda_2 = 1$, $\lambda_3 = \frac{1}{\sqrt{2}}$, $\lambda_4 = \frac{1}{2}$, ...

First lets understand what $\tilde{f}_n(k)$ mean.

- $\tilde{f}(k)$ is the fourier transform of a perturbation (change) of a distribution from the Gaussian.

If we expand any distribution in fourier space:

$$\tilde{p}(k=0) = \tilde{p}(0) + \tilde{p}'(0)k + \frac{1}{2}\tilde{p}''(0)k^2 + \dots$$

and remember $\tilde{p}(k) = \int e^{-ikx} p(x)\,dx$:

$$\tilde{p}(k) = \underset{p(x) \text{ is normalized}}{1} + ik\langle x\rangle + \frac{k^2}{2}\langle x^2\rangle + \dots$$

$\Rightarrow$ $\tilde{f}_n(k)$ is the change of the n^{th} moment of the distribution from its gaussian value!

0^{th} moment - normalization, 1^{st} moment - mean

But our physical distribution has constraints on these moments, it must be normalized and have a certain mean (0 in our case).