Fundamentals of Quantum Technology
Homework Sheet 4

1. In class you saw that, for the semiclassical Rabi model, if $C_g\left(t=0\right)=1$ then the occupation probability of the excited state is given by

$$P_e\left(t\right)=\frac{\mathcal{V}^2}{\hbar^2\Omega_{\mathrm{R}}^2}\sin^2\left(\frac{\Omega_{\mathrm{R}}t}{2}\right).$$

The perturbative solution, on the other hand, yields

$$P_e\left(t\right)=\frac{\mathcal{V}^2}{\hbar^2\Delta^2}\sin^2\left(\frac{\Delta t}{2}\right).$$

Show that the perturbative result is recovered from the Rabi result, either when the driving field has been turned on only for a short duration of time, or when the interaction amplitude $\mathcal{V}$ is small. Define what constitutes a "short duration of time" and a "small amplitude" in the aforementioned cases.

2. Recall the Bloch vector representation of the density matrix for a two-level system,

$$\rho=\frac{1}{2}\left[\mathbb{I}+\mathbf{s}\cdot\overrightarrow{\sigma}\right]=\frac{1}{2}\begin{pmatrix}1+s_3 & s_1-is_2\\ s_1+is_2 & 1-s_3\end{pmatrix}.$$

Compute the two eigenvalues of ρ in terms of the components of $\mathbf{s}$. Based on this calculation, show the following:

   (a) In order for ρ to be a valid density matrix, we must require $|\mathbf{s}|\leq 1$.
   Hint: Recall Question 2a in Homework 3.

   (b) For ρ representing a pure state, $|\mathbf{s}|=1$.

   (c) For ρ representing a mixed state, $|\mathbf{s}|<1$.

3. A general pure state of a two-level system can be written as

$$|\psi\rangle=e^{i\varphi}\left(\cos\left(\frac{\theta}{2}\right)|e\rangle+e^{i\phi}\sin\left(\frac{\theta}{2}\right)|g\rangle\right),$$

with $0\leq\phi<2\pi$ and $0\leq\theta\leq\pi$, and with $e^{i\varphi}$ an overall phase factor. Show that in the Bloch vector representation this corresponds to

$$\mathbf{s}=\left(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta\right).$$

4. This exercise will show how the Hadamard gate, given by

$$U_H=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}=\frac{1}{\sqrt{2}}\left(\sigma_x+\sigma_z\right),$$

can be implemented using our control of the Rabi Hamiltonian,

$$\mathcal{H} = \frac{\hbar\omega_0}{2}\sigma_z + (\mathcal{V}\sigma_+ + \mathcal{V}^*\sigma_-)\cos(\omega t) + \frac{B}{2}\sigma_z.$$

As in class, we define the basis states of the qubit such that $|0\rangle \equiv |e\rangle$ and $|1\rangle \equiv |g\rangle$.

(a) Let $\hat{n}$ be a unit vector. Use the Pauli identity $\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}\mathbb{I}$ to show that $(\hat{n}\cdot\overrightarrow{\sigma})^2 = \mathbb{I}$. Use this result to show that, for any real value of φ,

$$e^{-i\varphi\hat{n}\cdot\overrightarrow{\sigma}} = \cos(\varphi)\,\mathbb{I} - i\sin(\varphi)\,(n_x\sigma_x + n_y\sigma_y + n_z\sigma_z)\,.$$

(b) Recall that on resonance ($\omega = \omega_0$) we obtained the following interaction picture Hamiltonian:

$$\mathcal{H}_I = \frac{1}{2}\begin{pmatrix} B & \mathcal{V} \\ \mathcal{V}^* & -B \end{pmatrix} = \frac{1}{2}\left[\mathrm{Re}(\mathcal{V})\,\sigma_x - \mathrm{Im}(\mathcal{V})\,\sigma_y + B\sigma_z\right].$$

Recall also that, within the interaction picture, the state evolves according to $|\psi_I(t)\rangle = e^{-i\mathcal{H}_I t/\hbar}|\psi_I(0)\rangle$.

i. Assume that $\mathcal{V}$ is imaginary, $\mathcal{V} = \mathrm{Im}(\mathcal{V})$. Show that the Hadamard gate may be implemented by first applying only the Rabi driving field (with $B = 0$, assuming $\mathrm{Im}(\mathcal{V}) > 0$) for a time duration of $t_1 = \pi\hbar/2\mathrm{Im}(\mathcal{V})$, and then applying only the B field (with $\mathcal{V} = 0$, assuming $B > 0$) for a time duration of $t_2 = \pi\hbar/B$. You do not need to mind an overall phase factor multiplying the unitary operator.

ii. Assume that $\mathcal{V}$ is real, $\mathcal{V} = \mathrm{Re}(\mathcal{V})$. Choose an appropriate value of B and a time duration t such that, when both fields $\mathcal{V}$ and B are applied simultaneously, we get $U_H = e^{-i\mathcal{H}_I t/\hbar}$. Assume that $\mathcal{V} > 0$. Again, ignore overall phase factors.

Hint: It may be helpful to first express the Hadamard gate as $U_H = e^{-i\varphi\hat{n}\cdot\overrightarrow{\sigma}}$ using an appropriate choice of φ and $\hat{n}$.