Electrolitic
Exercise 2

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Question 1.2

Show that the requirement that the Lorentz transformation Λ preserves the norm of the 4-vector is given by the expression:

$$g = \Lambda^T g \Lambda.$$

Where g is the metric tensor.

Solution:
Firstly we'll note that $g_{\mu\nu} = g_{\nu\mu}$ for g is symetric.

$$x'^{\mu} x'_{\mu} = g_{\mu\alpha} x'^{\mu} x'^{\alpha} = g_{\mu\alpha} \underbrace{\Lambda^{\mu}{}_{\beta}}_{\left[\Lambda^{\beta}{}_{\mu}\right]^T} x^{\beta} \Lambda^{\alpha}{}_{\nu} x^{\nu} = \left[\Lambda^{\beta}{}_{\mu}\right]^T g_{\mu\alpha} \Lambda^{\alpha}{}_{\nu} x^{\beta} x^{\nu} \stackrel{!}{=} g_{\mu\alpha} x^{\mu} x^{\alpha}.$$

$$\Rightarrow \boxed{\left[\Lambda^{\beta}{}_{\mu}\right]^T g_{\mu\alpha} \Lambda^{\alpha}{}_{\nu} = g_{\mu\alpha}}.$$

Question 1.2

Show explicitly that the Boost transformation Λ_B preserves the norm.

Solution:
we'll look at a boost in the x axis Λ_x:

$$\begin{aligned}
\Lambda_x^T g \Lambda_x &= \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ -\beta\gamma & -\gamma & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} \gamma^2 - \beta^2\gamma^2 & \beta\gamma^2 - \beta\gamma^2 & 0 & 0 \\ \beta\gamma^2 - \beta\gamma^2 & \beta^2\gamma^2 - \gamma^2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.
\end{aligned}$$

But $\gamma^2\left(1-\beta^2\right)=1$, so:

$$\Lambda_x^T g \Lambda_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g.$$

And with what we showed in Part 1 it proves that Λ preserves the norm.

Question 2

$$\mathcal{L} = \frac{k}{2}\partial_t\theta\partial_x\theta - \frac{m}{2}\left(\partial_x\theta\right)^2.$$

Where $m, k > 0$ real constant.

Part 1

Find the equations of motion using E-L.

Solution:

$$\frac{\partial\mathcal{L}}{\partial\theta} = \partial_\nu \frac{\partial\mathcal{L}}{\partial\left(\partial_\nu\theta\right)}.$$

$$\Downarrow$$

$$\begin{aligned} \frac{k}{2}\frac{\partial}{\partial t}\left(\partial_x\theta\right) + \frac{\partial}{\partial x}\left(\frac{k}{2}\partial_t\theta - m\partial_x\theta\right) &= 0 \\ k\partial_t\partial_x\theta &= m\partial_x^2\theta. \end{aligned}$$

Part 2

Find the equation of motion using the Hemiltion principle.

Solution:

$$\begin{aligned}\delta S &= S\left(\theta+\delta\theta, \partial_x\theta+\delta\left(\partial_x\theta\right), \partial_t\theta+\delta\left(\partial_t\theta\right)\right) - S\left(\theta, \partial_x\theta, \partial_t\theta\right)\\ &= \int \frac{\partial\mathcal{L}}{\partial\theta}\delta\theta + \frac{\partial\mathcal{L}}{\partial\left(\partial_x\theta\right)}\delta\left(\partial_x\theta\right) + \frac{\partial\mathcal{L}}{\partial\left(\partial_t\theta\right)}\delta\left(\partial_t\theta\right)\\ &= \int \frac{\partial\mathcal{L}}{\partial\theta}\delta\theta + \frac{\partial\mathcal{L}}{\partial\left(\partial_x\theta\right)}\partial_x\left(\delta\theta\right) + \frac{\partial\mathcal{L}}{\partial\left(\partial_t\theta\right)}\partial_t\left(\delta\theta\right).\end{aligned}$$

Integrating by parts:

$$= \int \left[\frac{\partial\mathcal{L}}{\partial\theta} - \frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_x\theta\right)}\right) - \frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_t\theta\right)}\right)\right]\delta\theta.$$

And the Hemilton principle states that the physical trajectory is governed by $\delta S = 0$.
Hence:

$$\frac{\partial\mathcal{L}}{\partial\theta} = \partial_\nu \frac{\partial\mathcal{L}}{\partial\left(\partial_\nu\theta\right)}.$$

And the rest is the same as in the previous part.

Part 3

Find the canonical momentum.

Solution:

$$\Pi \equiv \frac{\partial\mathcal{L}}{\partial\left(\partial_t\theta\right)} = \frac{k}{2}\partial_x\theta.$$

Part 4

Find $\mathcal{H}$.

Solution:

$$\mathcal{H} \equiv \Pi\frac{\partial\theta}{\partial t} - \mathcal{L} = \frac{m}{2}\left(\partial_x\theta\right)^2.$$

Question 3.1

Show that:

$$\frac{\mathrm{d}E_k}{\mathrm{d}t} = \vec{v} \cdot \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}.$$

Solution:

$$\begin{aligned} E_k \equiv E - E_{rest} &= \gamma m_0 c^2 - m_0 c^2 = (\gamma - 1)\, m_0 c^2 \\ \Rightarrow \frac{\mathrm{d}E_k}{\mathrm{d}t} &= m_0 c^2 \frac{\mathrm{d}}{\mathrm{d}t}(\gamma - 1)\,. \end{aligned}$$

We'll note that:

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{\gamma^3}{c^2}\vec{a}\cdot\vec{v}.$$

$$\Rightarrow \frac{\mathrm{d}E_k}{\mathrm{d}t} = m_0 \cancel{c^2}\frac{\gamma^3}{\cancel{c^2}}\vec{a}\cdot\vec{v}.$$

On the other hand:

$$\begin{aligned} \vec{v}\cdot\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} &= \vec{v}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(m_0\gamma\vec{v}\right) \\ &= m_0\vec{v}\cdot\left(\gamma\vec{a} + \vec{v}\frac{\gamma^3}{c^2}\left(\vec{a}\cdot\vec{v}\right)\right) \\ &= m_0\gamma\left(\vec{a}\cdot\vec{v}\right) + m_0\frac{\gamma^3}{c^2}\left(\vec{a}\cdot\vec{v}\right)v^2 \\ &= \gamma m_0\left(\vec{a}\cdot\vec{v}\right)\underbrace{\left(1+\gamma^2\beta^2\right)}_{\gamma^2} = \gamma^3 m_0\left(\vec{a}\cdot\vec{v}\right). \end{aligned}$$

Hence:

$$\frac{\mathrm{d}E_k}{\mathrm{d}t} = \vec{v}\cdot\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \gamma^3 m_0\left(\vec{a}\cdot\vec{v}\right) \tag{1}$$

Question 3.2

Show that the 4-acceleration $a^\nu \equiv \frac{dU^\nu}{d\tau}$ is orthogonal to the 4-velocity.

Solution:

$$\begin{aligned} \frac{dU^\nu}{d\tau} &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\gamma c, \gamma\vec{v}\right) \\ &= \left(c\frac{\mathrm{d}\gamma}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau}, \gamma\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} + \frac{\mathrm{d}\gamma}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau}\vec{v}\right). \end{aligned}$$

We'll note that:

$$
\begin{aligned}
\frac{\mathrm{d}t}{\mathrm{d}\tau} &= \gamma \\
\frac{\mathrm{d}\gamma}{\mathrm{d}t} &= \frac{\gamma^3}{c^2}\vec{a}\cdot\vec{v} \\
\Rightarrow a^\nu = \frac{dU^\nu}{d\tau} &= \left(\gamma^4\frac{\vec{a}\cdot\vec{v}}{c}, \gamma^2\vec{a} + \gamma^4\frac{\vec{a}\cdot\vec{v}}{c^2}\vec{v}\right) \\
\Rightarrow a^\nu U_\nu &= \gamma^4\frac{\vec{a}\cdot\vec{v}}{c}\gamma c - \left(\gamma^2\vec{a} + \gamma^4\frac{\vec{a}\cdot\vec{v}}{c^2}\vec{v}\right)\gamma\vec{v} \\
&= \gamma^5\left(\vec{a}\cdot\vec{v}\right) - \gamma^3\left(\vec{a}\cdot\vec{v}\right) - \gamma^5\left(\vec{a}\cdot\vec{v}\right)\beta^2 \\
&= \gamma^{\cancelto{3}{5}}\left(\vec{a}\cdot\vec{v}\right)\cancelto{=\gamma^2}{\left(1-\beta^2\right)} - \gamma^3\left(\vec{a}\cdot\vec{v}\right) \\
&= \boxed{0}.
\end{aligned}
$$

Question 3.3

Find the 4 components $f^\nu = \frac{\mathrm{d}p^\nu}{\mathrm{d}\tau}$. What's the relation between f^0 and $\vec{f}$?

Solution:

$$
\begin{aligned}
\frac{\mathrm{d}p^\nu}{\mathrm{d}\tau} &= \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}p^\nu}{\mathrm{d}t} = \gamma\frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma m_0 c, \gamma m_0\vec{v}\right) \\
&= \boxed{\left(\gamma\frac{1}{c}\gamma^3 m_0\left(\vec{a}\cdot\vec{v}\right), \gamma\frac{1}{c^2}m_0\gamma^3\left(\vec{a}\cdot\vec{v}\right)\vec{v} + \gamma^2 m_0\vec{a}\right)} \\
(1) \Rightarrow f_0 &= \frac{\gamma}{c}\left(\vec{v}\cdot\frac{\mathrm{d}\vec{p}}{\mathrm{d}t}\right) = \frac{\gamma}{c}\left(\vec{v}\cdot\vec{f}\right) \quad \left(= \frac{\gamma}{c}\frac{\mathrm{d}E_k}{\mathrm{d}t}\right).
\end{aligned}
$$