(2) Question 2

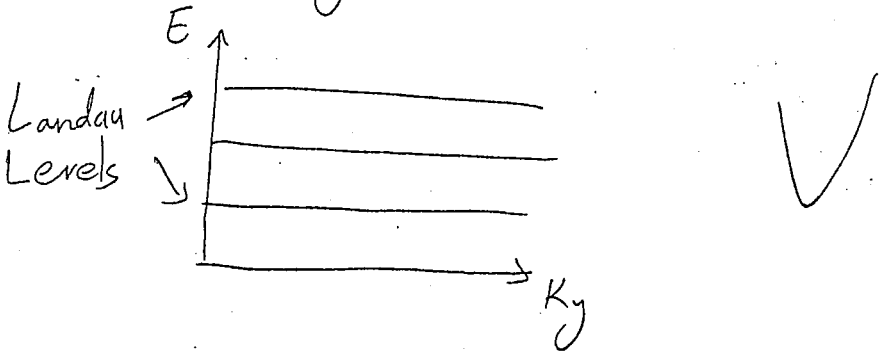
- Symmetries of the problem:
  Translations in the y axis
  → K_y conserved
  (Without U(x) also have symmetry for x translations, and conservation of K_x)

  Implications about degeneracies:
  In the absence of U(x) we saw in class that the x-y translational symmetry leads to degeneracy in the energy → ~~K_y and K_x are both conserved quantum numbers (though not mutually! can take the energies as dependent on~~

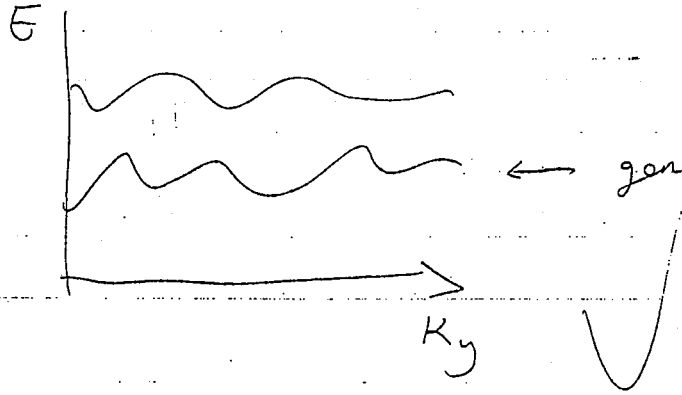
  → Due to the symmetry $\hat{X}_0$, and Y_0 (the guiding center coordinates) both commute with H (but not with one another)
  The eigenstates where dependent on 2 quantum numbers, (n, K_x or K_y), but the energies only on n.

  In our case, Theres translational symmetry in only one direction → Expect only one of X_0 and Y_0 to commute with H (probably X_0, saw that depends on K_y)
  Therefore probably won't be a degeneracy, though can't tell for sure for a general U(x).

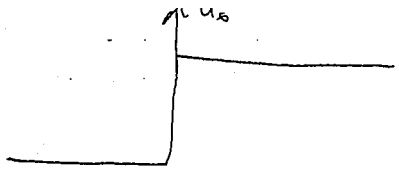
In the absence of $U(x)$ K_x and K_y are both continuum conserved quantum numbers (again - but not together!) So the energy could be plotted vs one of them. Since the energy is degenerate in K_y, (or K_x) would be only straight lines:



- With non zero $U(x)$, K_x is no longer conserved, but K_y is conserved, so the energy levels may still be plotted vs K_y (but not K_x). However, the degeneracy might be removed, and the energy dependent on K_y



- $U(x) = \begin{cases} 0 & ; \ x < 0 \\ U_0 & ; \ x > 0 \end{cases}$



Choosing the X dependent gauge, since the potential is also x-dependent (following the tutorial):

$\vec{A}(\vec{r},t) = (0, Bx, 0)$

$\hat{X}_0 = \hat{X} + \frac{\hat{v}_y}{\omega_c} = \frac{1}{m\omega_c}\hat{p}_y \xrightarrow{\hat{p}_y = -i\hbar\frac{\partial}{\partial y}}$ Conserved, since $\hat{p}_y$ is conserved due to the symmetry

- $H = \frac{1}{2m}(p_x^2 + (p_y - Bx)^2) + U(x)$ ✓

(Indeed, $\hat{y}$ doesn't appear in H, so $[H, X_0] = [H, p_y] = 0$)

Taking eigenstates of $\hat{X}_0$: $\psi(x,y) = e^{\frac{iX_0 y}{\ell^2}}\phi(x)$

$(\ell = \sqrt{\frac{\hbar c}{eB}})$

~~$(\hat{H} - E)\psi(x,y) \Rightarrow \phi''(x) + \frac{2m}{\hbar^2}(E - \frac{m\omega_c^2}{2}(x - X_0)^2)\phi = 0$~~

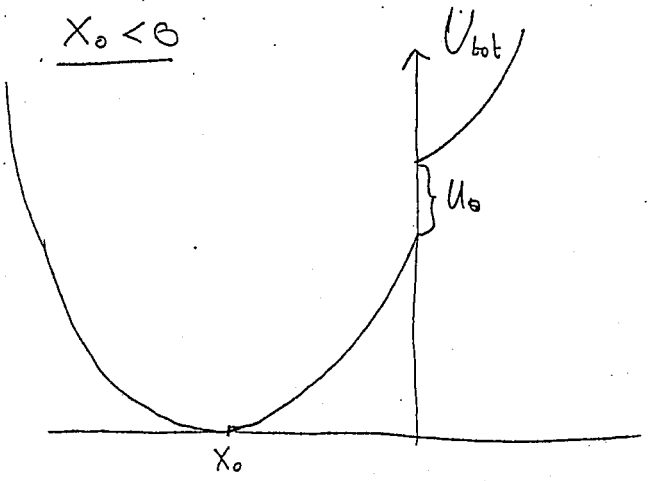
- $(\hat{H} - E)\psi(x,y) \Rightarrow$ ✓

$$\phi''(x) + \frac{2m}{\hbar^2}\left(E - U(x) - \frac{m\omega_c^2}{2}(x - X_0)^2\right)\phi = 0$$

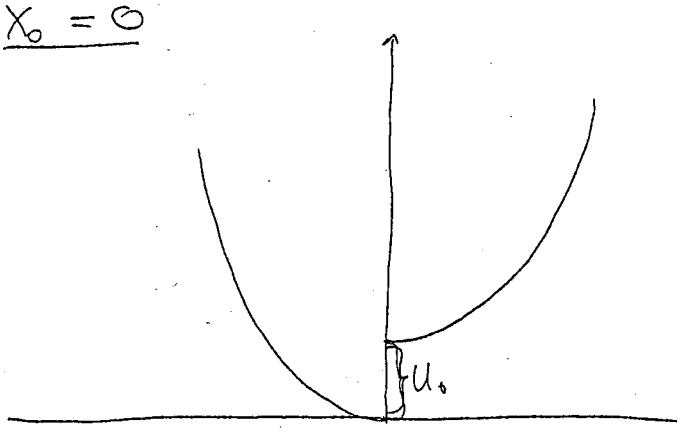
$\Rightarrow$ Sum of 1d harmonic oscillator centered at X_0, and a step function

The potential for various X_0:

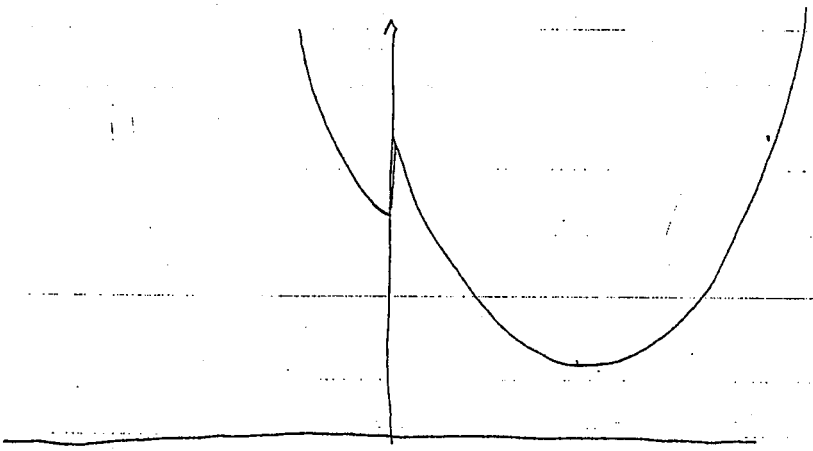
$X_0 < 0$



$X_0 = 0$



$X_0 > 0$

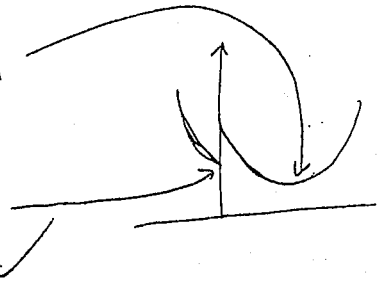


- Make the electrons climb the potential step:
for $x_0 < 0$ the wavefunctions will be centered around x_0: At very negative x_0 the eigenstates will be almost exact h.o eigenstate → centered around x_0.

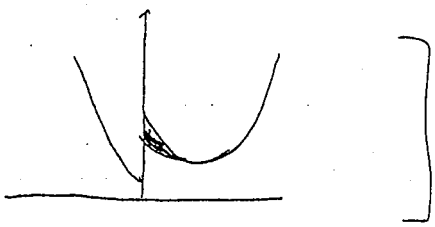
At very large x_0, again the wave functions will be almost h.o wave functions centered around x_0.

Very large here means $\frac{m\omega^2 x_0^2}{2} \gg U_0$

(i.e - the minimum of the parabola is much lower than the local minima at $x=0$)



[For $\frac{m\omega^2 x_0^2}{2} < U_0$, $x_0 > 0$ the absolute minimum will be at $x=0$



]

So – taking the electrons across the potential step can be done by starting with an eigenstate with x_0 a large negative number, and ending with x_0 a large positive number

→ Put electric field in y direction

$$\vec{E} = E\hat{e}_y$$

Working in the time-dependent gauge:

$\vec{A} = (0, +Bx - cEt, 0) \quad A_0 = 0$

$\vec{v}_{y_{op}} = \frac{1}{m}(p_y - \frac{e}{c}Bx + eEt) \qquad \vec{v}_{x_{op}} = \frac{1}{m}p_x$

$\hat{X}_0 = \hat{x} + \hat{v}_y \frac{mc}{eB} = \frac{c}{eB}(p_y + eEt)$

$H = \frac{m}{2}(v_x^2 + v_y^2)$

$= \frac{1}{2m}(p_x^2 + (p_y - \frac{e}{c}Bx + eEt)^2)$ ✓

$\Rightarrow$ Still have $[X_0, H] = 0$

* Working in the extreme adiabatic approx, so t is taken as a parameter ✓

Taking the eigenfunctions of X_0 as before, $\varphi(y) = e^{i\frac{X_0 y}{\ell^2}}$

$X_0 \varphi(y) = (-i\ell^2 \frac{\partial}{\partial y} + \frac{cE}{B}t) e^{i\frac{X_0 y}{\ell^2}} = (X_0 + c\frac{E}{B}t)\varphi(y)$

$\Rightarrow$ the center of the harmonic potential moves right at speed $c\frac{E}{B}$

(take small E if want the process to be adiabatic)

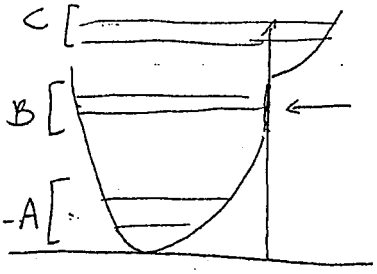
Note: There will ~~always~~ be a local minimum for $x_0 > 0$ at $x = 0$ ✓

$\longrightarrow$ a meta stable state.

But as x_0 will slowly go to the right, ~~this~~ the ~~energy of this state~~ electrons will tunnel out to the absolut minimum at $x = x_0$

Evaluating how the energy levels will look as a function of X_0, in the extreme adiabatic approx:

At $X_0 < 0$:



~h.o levels →

Looking at the development of energy level n:
at low enough X_0 will be in the h.o part of the potential → $E_n \sim \hbar\omega(n+\frac{1}{2})$ (denoted A in the picture)
(~~0~~ closer to 0 than before)

At some point $X_0 V$, the energy level will be in part B of the drawing (i.e - have a "straigh wall" on the right)

→ Using Bohr-Zommerfeld quanization rule

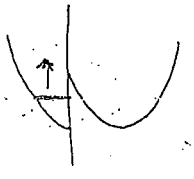
$$\int_{x_-}^{x_+} \sqrt{2m(E-U)}\,dx = 2\pi\hbar(n+\tfrac{1}{2})$$

we can see that the energy will be higher than before (Same potential U as the harmonic case, but integration over shorter path) ✓

[* Assume we're looking at an energy level $E_n < U_0$, so that it won't be in part C of the drawing]

the energy level will keep growing. ✓

If we neglect tunneling at first, than this energy level will continue growing also after $X_0 > 0$,
(until it will be higher than the barrier)

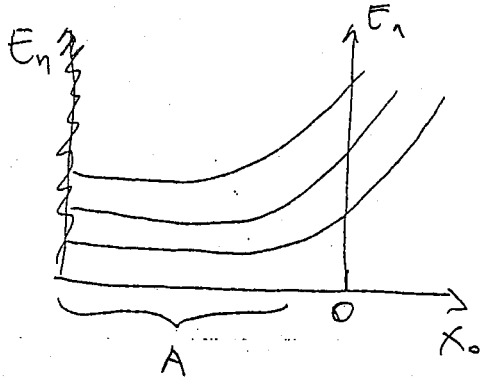


Schematic drawing

Note: If take U_0 very large, than only at large X_0 the absolute minimum will be at X_0 and not at 0.

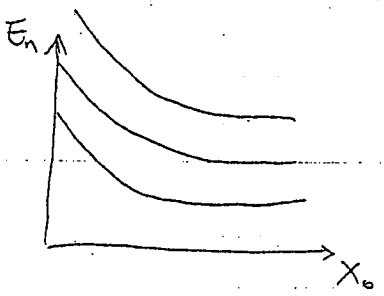
(as was said before: only at $\frac{m\omega^2 X_0^2}{2} > U_0$)

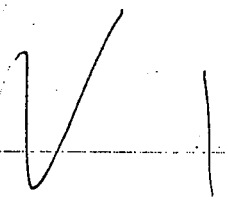
Schematic drawing of these levels



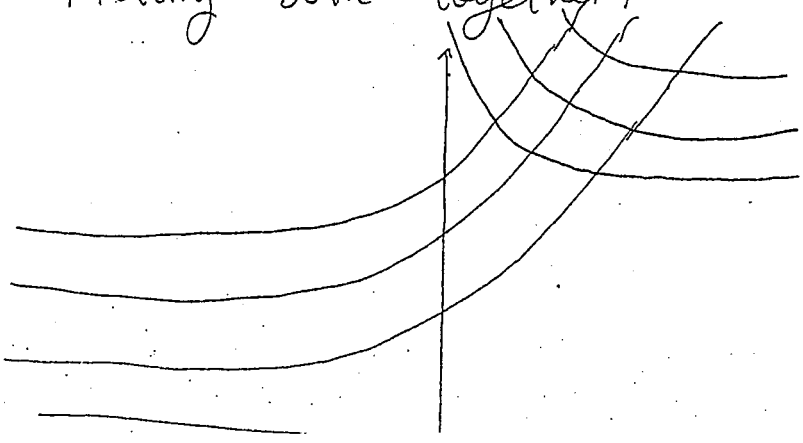
(n.o energies)

When $X_0 > 0$ there is a minima at X_0 (local at first and later global) the behavior of energy states of the states centered in X_0 will be:

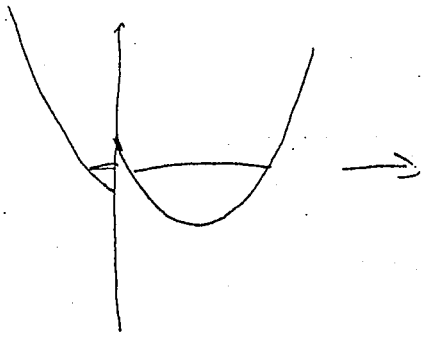




Plotting both together:

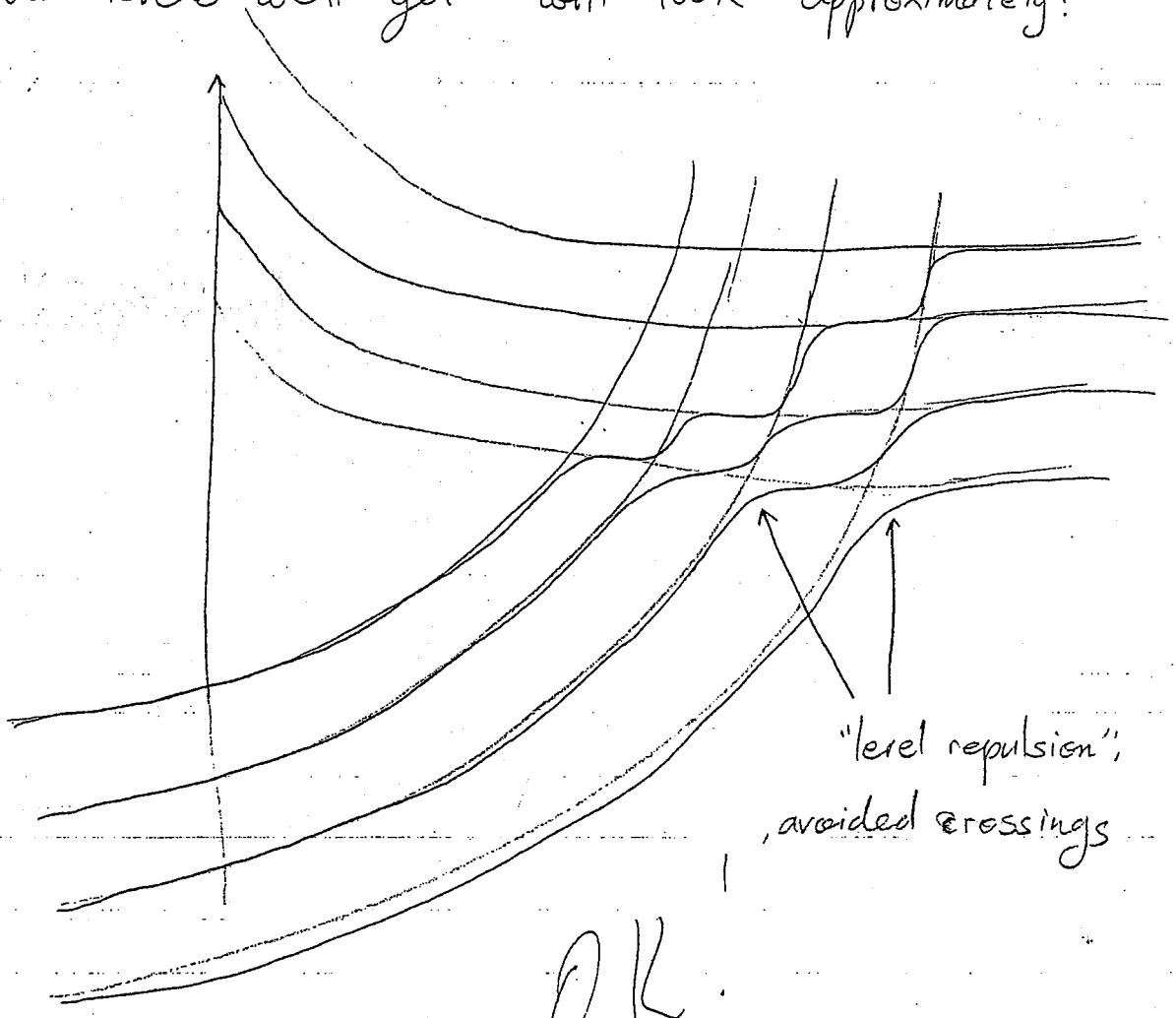


But we know in the adiabatic case there won't really be level crossings, due to tunneling



→ Won't really be 2 separate degenerate levels in both minima, but 2 states with different energies spread in both

So the levels we'll get will look approximately:



OK.

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QM-1 **Final Exam** 2007-2008

Solve two of the following three problems

1. Consider an electron moving in a two dimensional (x,y) plane and placed in a uniform magnetic field B which is directed perpendicular to the plane. In addition a potential $U(x)$ (which depends only on x) is acting on the electron.

   - What are the symmetries of this problem? What do they imply about possible degeneracies of the energy eigenstates? In the absence of $U(x)$ one could plot energy levels vs a continuum conserved quantum number. Will this be true with non zero $U(x)$? What will change in comparison with the original Landau levels?
   - Consider $U(x)$ in the form of a step

     $$U(x) = 0 \quad \text{for} \quad x < 0\,; \quad U(x) = U_0 \quad \text{for} \quad x > 0$$

     with a constant U_0. Use the symmetry considerations and reduce the Schroedinger equation to a one dimensional form. Plot the resulting potential function for different locations of the eigenstates relative to the position of the potential step.
   - You are asked to make the electrons climb across the potential step by adding electric field in an appropriate direction. Explain how this can be done. ((Hint : The simplest way of modelling this seems to take the electric field in a time dependent gauge and work in the extreme adiabatic limit).

2. Consider the following Hamiltonian

$$H = \int \psi^+(x) \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + U(x) \right) \psi(x)\, dx + \int \psi^+(x)\psi(x) V(x-y)\mu(y)\, dxdy\,,$$

where $\psi(x)$ and $\psi^+(x)$ are boson operators,

$$\mu(y) = \frac{1}{2}\sum_{\alpha,\beta=1}^{2} \xi^+_\alpha(y)\sigma^x_{\alpha\beta}\xi_\beta(y)\,,$$

(2)

Suppose $x_0 < 0$ at $t = t_0$ and we want to transport electrons to the right, over the step-func.

We take $\vec{A}(x,t) = (0, -Bx + BE(t), 0) = (-Bx + B \cdot E \cdot t)\hat{y}$

$\vec{B} = \nabla \times \vec{A} = -B\hat{z}$ as before.

$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} = -\frac{1}{c}(BE)\hat{y}$

so for $E, B > 0$, we apply in the $-\hat{y}$ direction an elec. field $|\vec{E}| = \frac{BE}{c}$

$$H = \frac{1}{2m}P_x^2 + \frac{1}{2m}\left(P_y - \frac{e}{c}(-Bx + BEt)\right)^2 + U(x)$$

$$= \frac{1}{2m}P_x^2 + \frac{1}{2m}\left(P_y + \frac{eB}{c}x - \frac{eB}{c}Et\right)^2 + U(x)$$

Define: $\hat{x}_0 = x - \frac{v_y}{\omega_c} = x - \frac{1}{\omega_c m}\left(P_y + \frac{eB}{c}x - \frac{eBE}{c}t\right) =$

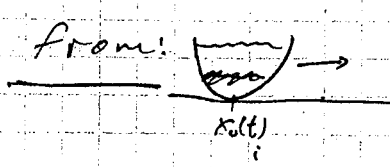
$$= x - \frac{P_y}{\omega_c m} - \frac{eB}{mc}\frac{1}{\omega_c}x + \frac{eBE}{mc\omega_c}t$$

$$= -\frac{P_y}{m\omega_c} + E \cdot t \quad \checkmark$$

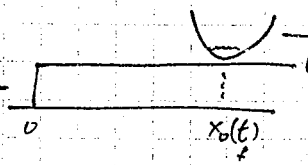
So, how do the levels look as a function of x_0?

In the extreme adiabatic limit ($E \ll 1$) we proceed as we did before and we find that the bottom of the well is $x_0(t) = x_0 + Et$

so eventually it will climb the step and the wave function will slowly transform back to a standard $\chi_n(x)$ form (solve h.o.) in our reduced (x-dependent) problem, the farther it moves from the step. Having a wave function that transforms smoothly from:



U_0 to:



means the electron is very likely to be found in the $x > 0$ region in the end.

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When we solve for $\psi(x)$ in the $x<0$ region, our equations look like those of an h.o. In the $x>0$ region, we have a h.o. with a step. According to the WKB procedure we know ~~approximately~~ exactly the nature of the wave functions. We know that the energy is quantized via the Born-Sommerfeld condition $\oint p\,dq = 2\pi\hbar(n+\frac{1}{2})$ so we know that each x_0 has a descrete spectrum of $\psi_n(x)$ with $\{E_n\}$ energy spectrum between turning points, so for any given x_0 we can describe the $\psi_n(x)$ dependence and write down the (equations for) E_{n,x_0} energy spectrum.

Why only?

$$\psi(x) \sim \frac{C_+}{\sqrt{p}} e^{\frac{i}{\hbar}\int_a^x p(x)dx} + \frac{C_-}{\sqrt{p}} e^{-\frac{i}{\hbar}\int_a^x p(x)dx}$$

$\hat{H}(x_0)\psi_n(x) = E_{n,x_0}\psi_n(x) \Rightarrow E_{n,x_0}$

$p(x) = \sqrt{2m(E-V(x))}$

Since E_{n,x_0} heavily depend on x_0 quantum number (since it finkers with turning points) we see the lifting of the degeneracy in the other quantum number.

$\leftarrow\rightarrow$ Meaning, $x_0 \neq x_0' \Rightarrow E_{n,x_0} \neq E_{n,x_0'}$ [perhaps not always, but in general it is true].

I didn't understand this

Since x_0 is continuous for an infinite volume, fine-tuning this parameter allows for easy production of equal-energy wave-functions (Large freedom).

Now $U(x) = \begin{cases} 0 & x<0 \\ U_0 & x>0 \end{cases}$

and we use prev. conclusions to separate the x and y dependence of $\psi(x,y)$:

$\psi_{x_0}(x,y) = e^{-i\frac{x_0 y}{\ell^2}}\phi(x)$, $\ell \equiv \sqrt{\frac{\hbar c}{eB}}$ as usual.

$H\psi_{x_0} = E\psi = \left[\frac{1}{2m}(-\hbar^2\partial_x^2) + \frac{1}{2m}\left(-i\hbar\partial_y + \frac{eB}{c}x\right)^2 + U(x)\right]\psi_{x_0} = E\psi_{x_0}$

$= \frac{-\hbar^2}{2m} e^{-i\frac{x_0 y}{\ell^2}}\phi'' + \frac{1}{2m}\left(-\frac{\hbar x_0}{\ell^2} + \frac{eB}{c}x\right)^2 e^{-i\frac{x_0 y}{\ell^2}}\phi + U(x)e^{-i\frac{x_0 y}{\ell^2}}\phi = E\psi_{x_0}$

$\Rightarrow -\frac{\hbar^2}{2m}\phi'' + \frac{1}{2m}\left(\frac{eB}{c}x - \frac{eB}{c}x_0\right)^2\phi + U(x)\phi = E\phi$

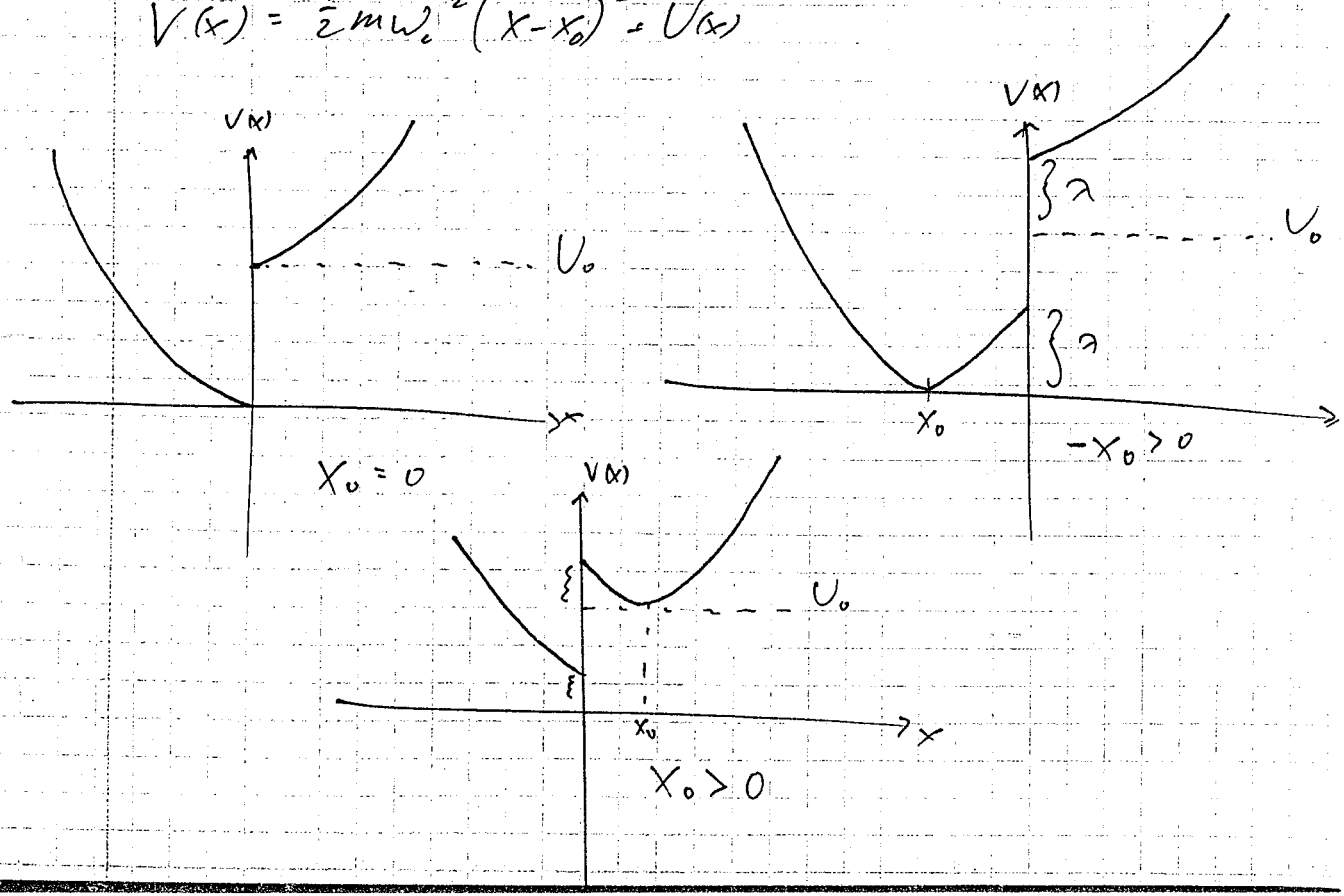
$-\frac{\hbar^2}{2m}\phi'' + \frac{1}{2}m\omega_c^2(x-x_0)^2\phi = (E-U(x))\phi$

✓ which is just the h.o. with an added step-func.

We recall x_0 was chosen when we denoted ψ_{x_0}.

✓ Every choice of x_0 moves the bottom of the well:

$V(x) = \frac{1}{2}m\omega_c^2(x-x_0)^2 + U(x)$



For any bounded-from-below $U(x)$ we could solve the reduced problem using WKB and we are guaranteed a discrete Energy spectrum. Now, changing x_0 will change the turning points continuously and our spectrum will continuously shift as well.

? Here, when $U(x) \neq 0$, any energy level is not compatible, necessarily, with a continuous x_0 value, unlike in the $U(x)=0$ case, where it is!

The original Landau levels were evenly spaces. Now, for $U(x) \neq 0$, the quantization condition is such that ~~may~~ shift these levels unevenly, though there will always be spacing between them for x_0 fixed.

More details follow next:

next page.

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1. $\vec{B} = -B\hat{z}$. $U = U(x)$. $\vec{r} = (x, y)$.

~~Take~~ ~~$\vec{A}(x,t) = (Bx + c)\hat{y}$~~

$$H = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 + U(x)$$

Since we have explicit x dependence, we take the gauge: $\vec{A} = -Bx\hat{y}$ ✓

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 0 & -Bx & 0 \end{vmatrix} = \hat{k}(-B) = -B\hat{k}$$ ✓

$$H = \frac{1}{2m}\left(p_x\right)^2 + \frac{1}{2m}\left(p_y + \frac{eB}{c}x\right)^2 + U(x)$$

~~Define: $\hat{x}_0 = \hat{x} - \frac{\hat{v}_y}{\omega_c}$~~

Define: $\omega_c = \frac{eB}{mc}$, $v_x = \frac{p_x}{m}$, $v_y = \frac{p_y + \frac{eB}{c}x}{m}$

Define: $\hat{x}_0 = \hat{x} - \frac{\hat{v}_y}{\omega_c} = x - \frac{mc}{eB}\left(\frac{1}{m}p_y + \frac{eB}{mc}x\right) = -\frac{mc}{eB}\hat{p}_y = +\frac{\hbar}{\omega_c m}\partial_y$

$$\hat{y}_0 = \hat{y} + \frac{\hat{v}_x}{\omega_c} = y + \frac{mc}{eB}\left(\frac{p_x}{m}\right) = \hat{y} + \frac{\hat{p}_x}{m\omega_c}$$

We notice that $[H, p_y] = 0 \Rightarrow [H, \hat{x}_0] = 0$ ✓

so $\hat{x}_0$ is a good guiding center and quantum number.

Conclusion: $\dot{p}_y = 0$ so $p_y \not| (\hat{x}_0)$ is conserved, so we can say we have translational invariance ✓ in the $\hat{y}$ direction. (Continuos in an infinite volume space

We will soon see how we can reduce the problem to a 1 dim problem once $\hat{x}_0$ is known. In the reduced problem, $U(x)$ will simply be an added potential to a h.o. centered around x_0.