I wrote the corrections from my previous version in red. There might be more mistakes, feel free to find them (but don't tell me, I don't have time to correct anymore).

9. a)Find eigenfunctions and eigenvalues of the boson operator $\psi_{op}(\mathbf{r})$. Can you do the same for $\psi^{+}_{op}(\mathbf{r})$? For the fermionic field operator?

<u>Bosonic case</u>

States of $\boldsymbol{\psi(\vec{r})}$

The field operator can be written $\psi(\vec{r}) = \sum_i \phi_i(\vec{r})\hat{a}_i$ when $\hat{a}_i$ are annihilation operators. I will define their eigenstates $|\alpha_i\rangle$:

$$\hat{a}_i|\alpha_i\rangle = \alpha_i|\alpha_i\rangle$$

We know that $|\alpha_i\rangle$ are coherent states:

$$|\alpha_i\rangle = e^{-\frac{|\alpha_i|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha_i^n}{\sqrt{n!}}|n\rangle = e^{-\frac{|\alpha_i|^2}{2}}\sum_{n=0}^{\infty}\frac{\left(\alpha_i\hat{a}_i^\dagger\right)^n}{\sqrt{n!}}|0\rangle = e^{-\frac{|\alpha_i|^2}{2}}e^{\alpha_i\hat{a}_i^\dagger}|0\rangle$$

Because $\psi(\vec{r})|\alpha_i\rangle = \phi_i(\vec{r})\alpha_i|\alpha_i\rangle$ I will define:

$$\Phi(\vec{r}) \equiv \sum_i \alpha_i\phi_i(\vec{r})$$

$$|\Phi(\vec{r})\rangle \equiv \prod_i |\alpha_i\rangle$$

$$\boldsymbol{\psi(\vec{r})|\Phi(\vec{r})\rangle = \sum_i \phi_i(\vec{r})\alpha_i|\alpha_i\rangle = \Phi(\vec{r})|\Phi(\vec{r})\rangle}$$

To make it general with no dependence on $\phi_i(\vec{r})$:

$$|\Phi(\vec{r})\rangle \equiv \prod_i|\alpha_i\rangle = \prod_i e^{-\frac{|\alpha_i|^2}{2}}e^{\alpha_i\hat{a}_i^\dagger}|0\rangle$$

$$\psi(\vec{r}) = \sum_i \phi_i(\vec{r})\hat{a}_i$$

$$\psi^\dagger(\vec{r}) = \sum_i \phi_i^*(\vec{r})\hat{a}_i^\dagger$$

$$\hat{a}_i^\dagger = \int \psi^\dagger(\vec{r})\phi_i(\vec{r})d^3r$$

$$|\boldsymbol{\Phi(\vec{r})}\rangle = \prod_i e^{-\frac{|\alpha_i|^2}{2}}e^{\alpha_i\int\psi^\dagger(\vec{r})\phi_i(\vec{r})d^3r}|0\rangle = \prod_i e^{-\frac{|\alpha_i|^2}{2}}\prod_i e^{\alpha_i\int\psi^\dagger(\vec{r})\phi_i(\vec{r})d^3r}|0\rangle$$

$$= e^{-\sum_i\frac{|\alpha_i|^2}{2}}e^{\sum_i\alpha_i\int\psi^\dagger(\vec{r})\phi_i(\vec{r})d^3r}|0\rangle = e^{-\frac{1}{2}\int|\Phi(\vec{r})|^2d^3r}e^{\int\psi^\dagger(\vec{r})\sum_i\alpha_i\phi_i(\vec{r})d^3r}|0\rangle =$$

Using $\sum_i\frac{|\alpha_i|^2}{2} = \int|\Phi(\vec{r})|^2d^3r$:

$$\boldsymbol{= e^{-\frac{1}{2}\int|\Phi(\vec{r})|^2d^3r}e^{\int\psi^\dagger(\vec{r})\Phi(\vec{r})d^3r}|0\rangle}$$

I previously had a mistake:

$$\boldsymbol{= e^{-\frac{1}{2}\int|\Phi(\vec{r})|^2d^3r}e^{\int\psi^\dagger(\vec{r})\psi(\vec{r})d^3r}|0\rangle}$$

Why $\boldsymbol{\psi^\dagger(\vec{r})}$ has no eigenstates

$|\phi_N\rangle = \prod_{n,i} f_i |n_i\rangle$ is a Fock state when n_i is the number of particles in i state.
$|\phi_N\rangle$ is with $N = \sum_i n_i$ particles in total.
Assuming $|\phi\rangle = \sum_{N=0}^{\infty} c_N |\phi_N\rangle$ is an eigenstate of $\psi^\dagger(\vec{r})$:

$$\psi^\dagger(\vec{r})|\phi\rangle = \phi|\phi\rangle$$

$$\psi^\dagger(\vec{r})|\phi\rangle = \psi^\dagger(\vec{r})(c_0|\phi_0\rangle + c_1|\phi_1\rangle + \cdots) = c_0|\phi_1\rangle + c_1|\phi_2\rangle + \cdots = \phi|\phi\rangle$$
$$= \phi(c_0|\phi_0\rangle + c_1|\phi_1\rangle + \cdots)$$
$$|\phi_1\rangle \neq |\phi_0\rangle \Rightarrow c_0 = 0$$
$$\phi c_1 = c_0 = 0$$

…
Thus $\forall N: c_N = 0$
Creation operators have no eigenstates !
For annihilation operators it is different because there is no -1 particle:

$$\psi(\vec{r})|\phi\rangle = \psi(\vec{r})(c_0|\phi_0\rangle + c_1|\phi_1\rangle + \cdots) = c_1|\phi_0\rangle + c_2|\phi_1\rangle + \cdots = \phi|\phi\rangle$$
$$= \phi(c_0|\phi_0\rangle + c_1|\phi_1\rangle + \cdots)$$

There is a possible solution…

Fermionic case
The result about creation operators having no eigenstates holds.
Coherent states of fermions are much more complicated because the anti-commutation of the operators demand anti-commutation of the **eigenvalues** and ordinary numbers cannot be used - but Grassman variables…

b) Return back to the case of the eigenstate of the bosonic $\psi_{op}(\mathbf{r})$ and calculate the average of H_{op} (with the two body interaction $V(\mathbf{r}-\mathbf{r'})$) in this state. What is the average of N_{op}?

Help : to facilitate the understanding of how to deal with the first question write momentarily $\psi_{op}(\mathbf{r})$ in terms of an expansion in some complete set. But then reformulate your answer for the "unexpanded" $\psi_{op}(\mathbf{r})$, i.e. give it with no relation to the set into which you have expanded. (it was anyways arbitrary, wasn't it?).

$$\hat{H} = \int d^3r \psi^\dagger(\vec{r})\left(-\frac{\hbar^2}{2m}\vec{\nabla}_{\vec{r}}^2 + U(\vec{r})\right)\psi(\vec{r}) + \frac{1}{2}\int d^3r d^3r' \psi^\dagger(\vec{r})\psi^\dagger(\vec{r}')V(\vec{r}-\vec{r}')\psi(\vec{r}')\psi(\vec{r})$$

$$\langle\Phi(\vec{r})|H|\Phi(\vec{r})\rangle = \int d^3r \langle\Phi(\vec{r})|\psi^\dagger(\vec{r})\left(-\frac{\hbar^2}{2m}\vec{\nabla}_{\vec{r}}^2 + U(\vec{r})\right)\psi(\vec{r})|\Phi(\vec{r})\rangle$$
$$+\frac{1}{2}\int d^3r d^3r' \langle\Phi(\vec{r})|\psi^\dagger(\vec{r})\psi^\dagger(\vec{r}')V(\vec{r}-\vec{r}')\psi(\vec{r}')\psi(\vec{r})|\Phi(\vec{r})\rangle =$$

Using $a|\alpha\rangle = \alpha|\alpha\rangle \Rightarrow \langle\alpha|a^\dagger = \alpha^*\langle\alpha|$:

$$= \int d^3r \langle\Phi(\vec{r})|\Phi^*(\vec{r})\left(-\frac{\hbar^2}{2m}\vec{\nabla}_{\vec{r}}^2 + U(\vec{r})\right)\Phi(\vec{r})|\Phi(\vec{r})\rangle$$
$$+\frac{1}{2}\int d^3r d^3r' \Phi^*(\vec{r})\langle\boldsymbol{\Phi}(\vec{\mathbf{r}})|\boldsymbol{\psi}^\dagger(\vec{\mathbf{r}}')V(\vec{r}-\vec{r}')\boldsymbol{\psi}(\vec{\mathbf{r}}')|\boldsymbol{\Phi}(\vec{\mathbf{r}})\rangle\Phi(\vec{r}) =$$

Using a result **which is proved below** $\psi(\vec{r}')|\Phi(\vec{r})\rangle = \Phi(\vec{r}')|\Phi(\vec{r})\rangle$:

$$= \int d^3r \langle \Phi(\vec{r}) | \Phi^*(\vec{r}) \left(-\frac{\hbar^2}{2m} \vec{\nabla}^2_{\vec{r}} + U(\vec{r}) \right) \Phi(\vec{r}) | \Phi(\vec{r}) \rangle$$
$$+ \frac{1}{2} \int d^3r d^3r' \langle \Phi(\vec{r}) | \Phi^*(\vec{r}) \Phi^*(\vec{r}') V(\vec{r} - \vec{r}') \Phi(\vec{r}') \Phi(\vec{r}) | \Phi(\vec{r}) \rangle$$

Previously I wrote:

$$= \int d^3r \left(-\frac{\hbar^2}{2m} \vec{\nabla}^2_{\vec{r}} + U(\vec{r}) \right) |\Phi(\vec{r})|^2 + \frac{1}{2} \int d^3r d^3r' V(\vec{r} - \vec{r}') |\Phi(\vec{r})|^2 |\Phi(\vec{r}')|^2$$

It is a mistake, because it is not known to us that the Hamiltonian does not act on $\Phi(\vec{r})$ or $|\Phi(\vec{r})\rangle$. Or at least that's what I suspect…

$$\hat{N}(\vec{r}) = \int \boldsymbol{\rho}(\vec{\boldsymbol{r}}) d\vec{r} = \int \psi^\dagger(\vec{r}) \psi(\vec{r}) d\vec{r}$$
$$\langle \Phi(\vec{r}) | \hat{N}(\vec{r}) | \Phi(\vec{r}) \rangle = \int d^3r \psi^\dagger(\vec{r}) \psi(\vec{r}) |\Phi(\vec{r})|^2$$

I will prove that $\boldsymbol{\psi}(\vec{\boldsymbol{r}}') | \boldsymbol{\Phi}(\vec{\boldsymbol{r}}) \rangle = \boldsymbol{\Phi}(\vec{\boldsymbol{r}}') | \boldsymbol{\Phi}(\vec{\boldsymbol{r}}) \rangle$:

$$|\Phi(\vec{r})\rangle = e^{-\frac{1}{2}\int |\Phi(\vec{r})|^2 d^3r} e^{-\int \psi^\dagger(\vec{r})\psi(\vec{r}) d^3r} |0\rangle$$
$$\psi(\vec{r}') |\Phi(\vec{r})\rangle = e^{-\frac{1}{2}\int |\Phi(\vec{r})|^2 d^3r} e^{\int \psi^\dagger(\vec{r})\psi(\vec{r}) d^3r} \psi(\vec{r}') |0\rangle$$
$$+ \left[\psi(\vec{r}'), e^{-\frac{1}{2}\int |\Phi(\vec{r})|^2 d^3r} e^{\int \psi^\dagger(\vec{r})\psi(\vec{r}) d^3r} \right] |0\rangle$$
$$= e^{-\frac{1}{2}\int |\Phi(\vec{r})|^2 d^3r} \left[\psi(\vec{r}'), e^{\int \psi^\dagger(\vec{r})\psi(\vec{r}) d^3r} \right] |0\rangle$$

$$e^{-\int \psi^\dagger(\vec{r})\psi(\vec{r}) d^3r} = \sum_{n=0}^{\infty} \frac{\left(\int \psi^\dagger(\vec{r})\psi(\vec{r}) d^3r\right)^n}{n!}$$

$$\left[\psi(\vec{r}'), \int \psi^\dagger(\vec{r})\psi(\vec{r}) d^3r \right] = \int [\psi(\vec{r}'), \psi^\dagger(\vec{r})\psi(\vec{r})] d^3r$$
$$= \int \left([\psi(\vec{r}'), \psi^\dagger(\vec{r})] \psi(\vec{r}) + \psi^\dagger(\vec{r}) [\psi(\vec{r}'), \psi(\vec{r})] \right) d^3r = \int \psi^\dagger(\vec{r}) \delta(\vec{r} - \vec{r}') d^3r$$
$$= \psi^\dagger(\vec{r}')$$

Using $[A, B] = \lambda \Rightarrow [A, B^n] = n\lambda B^{n-1}$:

$$\left[\psi(\vec{r}'), e^{\int \psi^\dagger(\vec{r})\psi(\vec{r}) d^3r} \right] = \left[\psi(\vec{r}'), \sum_{n=0}^{\infty} \frac{\left(\int \psi^\dagger(\vec{r})\psi(\vec{r}) d^3r\right)^n}{n!} \right]$$
$$= \sum_{n=1}^{\infty} \frac{n\left(\int \psi^\dagger(\vec{r})\psi(\vec{r}) d^3r\right)^{n-1}}{n!} \psi(\vec{r}') = \sum_{n=1}^{\infty} \frac{\left(\int \psi^\dagger(\vec{r})\psi(\vec{r}) d^3r\right)^{n-1}}{(n-1)!} \psi(\vec{r}')$$
$$= \sum_{n=0}^{\infty} \frac{\left(\int \psi^\dagger(\vec{r})\psi(\vec{r}) d^3r\right)^n}{n!} \psi(\vec{r}') = e^{\int \psi^\dagger(\vec{r})\psi(\vec{r}) d^3r} \psi(\vec{r}')$$