EL2520 - Control Theory and Practice
Project: The four-tank process
Group 30

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Abstract

This report describes the experiment performed in the final project for the course EL2520 Control Theory and Practice, Advanced Course. The purpose of this experiment is to investigate how a four-tank system process is controlled. This is done by first performing the modeling and the manual control which determines the numerical values of the important parameters that will be needed for model based controllers. Both minimum and non-minimum phases control design will then be investigated and evaluated. The performed model number is number 18.

Modeling

For the model number, we took number 18 and not 30!

Exercise 2.1.1

Each tank can be analysed by:

$$A\frac{dh}{dt} = q_{in} - q_{out} \tag{1}$$

For q_{out}, Bernoulli's law holds:

$$q_{out} = a\sqrt{2gh} \tag{2}$$

As for q_{in},this flow is then divided according to:

$$\begin{aligned} &q_L = \gamma k u, \\ &q_U = (1-\gamma)ku, \qquad \gamma \in [0,1] \end{aligned}$$

According to the provided figure of the four-tank process, we can get equation set for each of the tank as:

$$\begin{aligned} A_1\frac{dh_1}{dt} &= -q_{Tank1,out} + q_{Tank3,out} + q_{Tank1,Lower}, \\ A_2\frac{dh_2}{dt} &= -q_{Tank2,out} + q_{Tank4,out} + q_{2,Lower}, \\ A_3\frac{hd_3}{dt} &= -q_{Tank3,out} + q_{Tank2,Upper}, \\ A_4\frac{hd_4}{dt} &= -q_{Tank4,out} + q_{Tank1,Upper}. \end{aligned} \tag{3}$$

By replacing q, the new equation set becomes:

$$\begin{aligned} \frac{dh_1}{dt} &= -\frac{a_1}{A_1}\sqrt{2gh_1} + \frac{a_3}{A_1}\sqrt{2gh_3} + \frac{\gamma_1 k_1}{A_1}u_1, \\ \frac{dh_2}{dt} &= -\frac{a_2}{A_2}\sqrt{2gh_2} + \frac{a_4}{A_2}\sqrt{2gh_4} + \frac{\gamma_2 k_2}{A_2}u_2, \\ \frac{dh_3}{dt} &= -\frac{a_3}{A_3}\sqrt{2gh_3} + \frac{(1-\gamma_2)k_2}{A_3}u_2, \\ \frac{dh_4}{dt} &= -\frac{a_4}{A_4}\sqrt{2gh_4} + \frac{(1-\gamma_1)k_1}{A_4}u_1. \end{aligned} \tag{4}$$

Exercise 2.1.2

The output voltages y_i are proportional to the water levels h_i:

$$y_i = k_c h_i \tag{5}$$

An equilibrium point can be calculated by setting the derivative value of the equation set (4) equal to 0, which gives us:

$$
\begin{aligned}
-\frac{a_1}{A_1}\sqrt{2gh_1^0}+\frac{a_3}{A_1}\sqrt{2gh_3^0}+\frac{\gamma_1 k_1}{A_1}u_1^0&=0,\\
-\frac{a_2}{A_2}\sqrt{2gh_2^0}+\frac{a_4}{A_2}\sqrt{2gh_4^0}+\frac{\gamma_2 k_2}{A_2}u_2^0&=0,\\
-\frac{a_3}{A_3}\sqrt{2gh_3^0}+\frac{(1-\gamma_2)k_2}{A_3}u_2^0&=0,\\
-\frac{a_4}{A_4}\sqrt{2gh_4^0}+\frac{(1-\gamma_1)k_1}{A_4}u_1^0&=0,\\
y_1^0-k_c h_1^0&=0,\\
y_2^0-k_c h_2^0&=0.
\end{aligned}
\tag{6}
$$

Exercise 2.1.3

Deviations from an equilibrium can be denoted as:

$$
\begin{aligned}
\Delta u_i &= u_i - u_i^0,\\
\Delta h_i &= h_i - h_i^0,\\
\Delta y_i &= y_i - y_i^0.
\end{aligned}
$$

At the same time, the introduced vector will be:

$$
u=\begin{bmatrix}\Delta u_1\\ \Delta u_2\end{bmatrix},\qquad x=\begin{bmatrix}\Delta h_1\\ \Delta h_2\\ \Delta h_3\\ \Delta h_4\end{bmatrix},\qquad y=\begin{bmatrix}\Delta y_1\\ \Delta y_2\end{bmatrix}
\tag{7}
$$

The linearized model is then given by calculating the Jacobian matrix, and f is assigned to be the equation set (4):

$$
\begin{aligned}
f&=\begin{bmatrix}\frac{dh_1}{dt}\\ \frac{dh_2}{dt}\\ \frac{dh_3}{dt}\\ \frac{dh_4}{dt}\end{bmatrix},\\
\left.\frac{dx}{dt}\right|_{x^0,u^0}&=\left.\begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_4}\\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial x_4}\\
\frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} & \frac{\partial f_3}{\partial x_4}\\
\frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial x_2} & \frac{\partial f_4}{\partial x_3} & \frac{\partial f_4}{\partial x_4}
\end{bmatrix}\right|_{x^0,u^0} x+\left.\begin{bmatrix}
\frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2}\\
\frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2}\\
\frac{\partial f_3}{\partial u_1} & \frac{\partial f_3}{\partial u_2}\\
\frac{\partial f_4}{\partial u_1} & \frac{\partial f_4}{\partial u_2}
\end{bmatrix}\right|_{x^0,u^0} u\\
y&=\begin{bmatrix}k_c & 0 & 0 & 0\\ 0 & k_c & 0 & 0\end{bmatrix}x
\end{aligned}
\tag{8}
$$

After simplification, we can get

$$
\begin{aligned}
\frac{dx}{dt}&=\begin{bmatrix}
-\frac{1}{T_1} & 0 & \frac{A_3}{A_1T_3} & 0\\
0 & -\frac{1}{T_2} & 0 & \frac{A_4}{A_2T_4}\\
0 & 0 & -\frac{1}{T_3} & 0\\
0 & 0 & 0 & -\frac{1}{T_4}
\end{bmatrix}x+\begin{bmatrix}
\frac{\gamma_1 k_1}{A_1} & 0\\
0 & \frac{\gamma_2 k_2}{A_2}\\
0 & \frac{(1-\gamma_2)k_2}{A_3}\\
\frac{(1-\gamma_1)k_1}{A_4} & 0
\end{bmatrix}u\\
y&=\begin{bmatrix}k_c & 0 & 0 & 0\\ 0 & k_c & 0 & 0\end{bmatrix}x
\end{aligned}
\tag{9}
$$

where

$$T_i = \frac{A_i}{a_i}\sqrt{\frac{2h_i^0}{g}}. \tag{10}$$

Exercise 2.1.4

For a linear time independent model

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned}$$

we can compute the transfer matrix using $G(s) = C(sI - A)^{-1}B$ which gives us

$$\begin{aligned} G(s) &= \begin{bmatrix} k_c & 0 & 0 & 0 \\ 0 & k_c & 0 & 0 \end{bmatrix} \left(\begin{bmatrix} s & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & s \end{bmatrix} - \begin{bmatrix} -\frac{1}{T_1} & 0 & \frac{A_3}{A_1T_3} & 0 \\ 0 & -\frac{1}{T_2} & 0 & \frac{A_4}{A_2T_4} \\ 0 & 0 & -\frac{1}{T_3} & 0 \\ 0 & 0 & 0 & -\frac{1}{T_4} \end{bmatrix} \right)^{-1} \begin{bmatrix} \frac{\gamma_1 k_1}{A_1} & 0 \\ 0 & \frac{\gamma_2 k_2}{A_2} \\ 0 & \frac{(1-\gamma_2)k_2}{A_3} \\ \frac{(1-\gamma_1)k_1}{A_4} & 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{\gamma_1 k_1 c_1}{1+sT_1} & \frac{(1-\gamma_2)k_2c_1}{(1+sT_3)(1+sT_1)} \\ \frac{(1-\gamma_1)k_1c_2}{(1+sT_4)(1+sT_2)} & \frac{\gamma_2 k_2 c_2}{1+sT_2} \end{bmatrix}, \end{aligned} \tag{11}$$

where $c_i = T_i k_c / A_i$.

Exercise 2.1.5

According to the definition on zeros polynomials for MIMO system, we can compute the zeros of $G(s)$ from the maximal minor of $G(s)$ normed by the pole polynomial:

$$\begin{aligned} \det G(s) &= \frac{\gamma_1\gamma_2 k_1 k_2 c_1 c_2}{\Pi_{i=1}^4 (1+sT_i)} \left[(1+sT_3)(1+sT_4) - \frac{(1-\gamma_1)(1-\gamma_2)}{\gamma_1\gamma_2} \right] \\ &= \frac{\gamma_1\gamma_2 k_1 k_2 c_1 c_2 T_3 T_4}{\Pi_{i=1}^4 (1+sT_i)} \left[s^2 + \frac{T_3+T_4}{T_3T_4}s + \frac{\gamma_1+\gamma_2-1}{\gamma_1\gamma_2 T_3 T_4} \right]. \end{aligned} \tag{12}$$

Next, we can determine whether the system is minimum phase or non-minimum phase by determining the location of the zeros and poles. From equation (12), it is clear that all poles have negative real part. Hence, we only need to investigate the sign of the zeros. We can apply Veda's theorem to equation $s^2 + \frac{T_3+T_4}{T_3T_4}s + \frac{\gamma_1+\gamma_2-1}{\gamma_1\gamma_2T_3T_4}$ to analyze the positive and negative case of its roots, and we find that:

$$1 < \gamma_1 + \gamma_2 \leq 2 \tag{13}$$

holds for zeros with negative real part and

$$0 < \gamma_1 + \gamma_2 \leq 1 \tag{14}$$

holds for zeros with non-negative real part. Therefore, if $1 < \gamma_1 + \gamma_2 \leq 2$, that means the system is minimum phase. However, if $0 < \gamma_1 + \gamma_2 \leq 1$, that means the system is non-minimum phase.

Exercise 2.1.6

The relative gain array is given by

$$\begin{aligned}\mathrm{RGA}(G) &= G.*(G^{-1})^T \\ &= \frac{1}{g_{11}g_{22}-g_{12}g_{21}}\begin{bmatrix} g_{11}g_{22} & -g_{12}g_{21} \\ -g_{12}g_{21} & g_{11}g_{22}\end{bmatrix},\end{aligned} \tag{15}$$

where $.*$ indicate element-wise multiplication. Here we have defined a function $\lambda(G)$:

$$\lambda(G) = \frac{1}{1-\frac{g_{12}g_{21}}{g_{11}g_{22}}}, \tag{16}$$

we have

$$\mathrm{RGA}(G) = \begin{bmatrix} \lambda(G) & 1-\lambda(G) \\ 1-\lambda(G) & \lambda(G)\end{bmatrix}. \tag{17}$$

By evaluating $\lambda(G)$ at $s=0$ we obtain

$$\lambda(G(0)) = \frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2-1}. \tag{18}$$

For the minimum phase we have $\gamma_1=\gamma_2=0.625$, thus $\lambda=1.5625$. The corresponding RGA matrix will be given as

$$\mathrm{RGA}(G_{mp}(0)) = \begin{bmatrix} 1.5625 & -0.5625 \\ -0.5625 & 1.5625\end{bmatrix}. \tag{19}$$

Meanwhile for the non-minimum phase case we have $\gamma_1=\gamma_2=0.375$, thus $\lambda=-0.5625$. The corresponding RGA matrix will be given as

$$\mathrm{RGA}(G_{nmp}(0)) = \begin{bmatrix} -0.5625 & 1.5625 \\ 1.5625 & -0.5625\end{bmatrix}. \tag{20}$$

Exercise 2.1.7

Parameters k_1 and k_2 are constants for tank1 and tank2 respectively. In order to identify k_1, we can first close the output of tank1 and tank3, and adjust the pump1 to the specific valve so that all the water from pump1 will goes into tank1, which means $\gamma_1=1$, rather than other value, which will pump some water to Tank 3. As for Tank 2, it behaves the same. Then the equation (4) can be simplified:

$$\begin{aligned}\frac{dh_1}{dt} &= -\frac{a_1}{A_1}\sqrt{2gh_1}+\frac{a_3}{A_1}\sqrt{2gh_3}+\frac{k_1}{A_1}u_1 = \frac{k_1}{A_1}u_1 \\ \frac{dh_2}{dt} &= -\frac{a_2}{A_2}\sqrt{2gh_2}+\frac{a_4}{A_2}\sqrt{2gh_4}+\frac{k_2}{A_2}u_2 = \frac{k_2}{A_2}u_2\end{aligned} \tag{21}$$

As a result, parameter k can be calculated by known A, u, and Δh by equation (21)
$k_1 = 4.32cm^3/(sV)$, $k_2 = 3.74cm^3/(sV)$ by MATLAB.

Exercise 2.1.8

In order to identify the outlet hole areas a_i for the minimum phase system, we can manually adjust the input u to make the system stable at the equilibrium state. From the simulation, we find that when $u_2^0 = 7.5V$, $tank_3$ will reach the equilibrium state with $h_3^0 = 13.9cm$. Thus, according to the equation 6 we can calculate a_3 by:

$$a_3 = \frac{(1-\gamma_2)\,k_2 u_2^0}{\sqrt{2gh_3^0}} = 0.0637 \text{ cm}^2 \tag{22}$$

For the $tank4$, when the $u_1^0 = 7.5V$, $tank3$ will reach the equilibrium state with $h_4^0 = 23.5cm$. Then, a_4 for the minimum phase case will be:

$$a_4 = \frac{(1-\gamma_1)\,k_1 u_1^0}{\sqrt{2gh_4^0}} = 0.0566 \text{ cm}^2 \tag{23}$$

By substituding a_3 and a_4, we can get a_1 and a_2

$$\begin{aligned} a_1 &= \frac{(1-\gamma_2)k_2 u_2^0}{\sqrt{2gh_1^0}} + \frac{\gamma_1 k_1}{\sqrt{2gh_1^0}} u_1^0 = 0.1563 \text{ cm}^2 \\ a_2 &= \frac{(1-\gamma_1)k_1 u_1^0}{\sqrt{2gh_2^0}} + \frac{\gamma_2 k_2}{\sqrt{2gh_2^0}} u_2^0 = 0.1283 \text{ cm}^2 \end{aligned} \tag{24}$$

u_1^0	u_2^0	h_1^0	h_2^0	h_3^0	h_4^0
$7.5V$	$7.5V$	$19.775cm$	$27.325cm$	$13.9cm$	$23.5cm$

Table 1: value for minimum phase

For the non-minimum phase case, performing the calculation in the same way as minimum phase, we can obtain:

$$\begin{aligned} a_1 &= 0.1538cm^2 \\ a_2 &= 0.1331cm^2 \\ a_3 &= 0.1617cm^2 \\ a_4 &= 0.1504cm^2 \end{aligned} \tag{25}$$

u_1^0	u_2^0	h_1^0	h_2^0	h_3^0	h_4^0
$7.5V$	$7.5V$	$19cm$	$27.25cm$	$6cm$	$9.25cm$

Table 2: value for non-minimum phase

Manual control

Exercise 2.2.1

For the minimum phase:

$value$	h_1^0	h_2^0	h_3^0	h_4^0
$simulation$	$20.3cm$	$27.2cm$	$13.3cm$	$22.15cm$
$calculated$	$19.775cm$	$27.325cm$	$13.9cm$	$23.5cm$

Table 3: Minimum phase case when both u1 and u2 are 7.5V.

For the non-minimum phase:

$value$	h_1^0	h_2^0	h_3^0	h_4^0
$simulation$	$18.89cm$	$27.2cm$	$5.8cm$	$10.15cm$
$calculated$	$19cm$	$27.25cm$	$6cm$	$9.25cm$

Table 4: Non-minimum phase case when both u1 and u2 are 7.5V.

Exercise 2.2.2

The four figures below gives the four tank output, in terms of two phases: minimum and non-minimum, input 1 and input 2. We can take the minimum phase case as an example. The figure 1 is the responses when we set input1 to 7.5 V, and input 2 to 0 V. From the figure 1 we can find that when input1 is active, the tank4 and tank1 will be significantly affected. That means tank1 and tank4 are significantly coupled. We can also get this conclusion from the RGA matrix in equation 19.

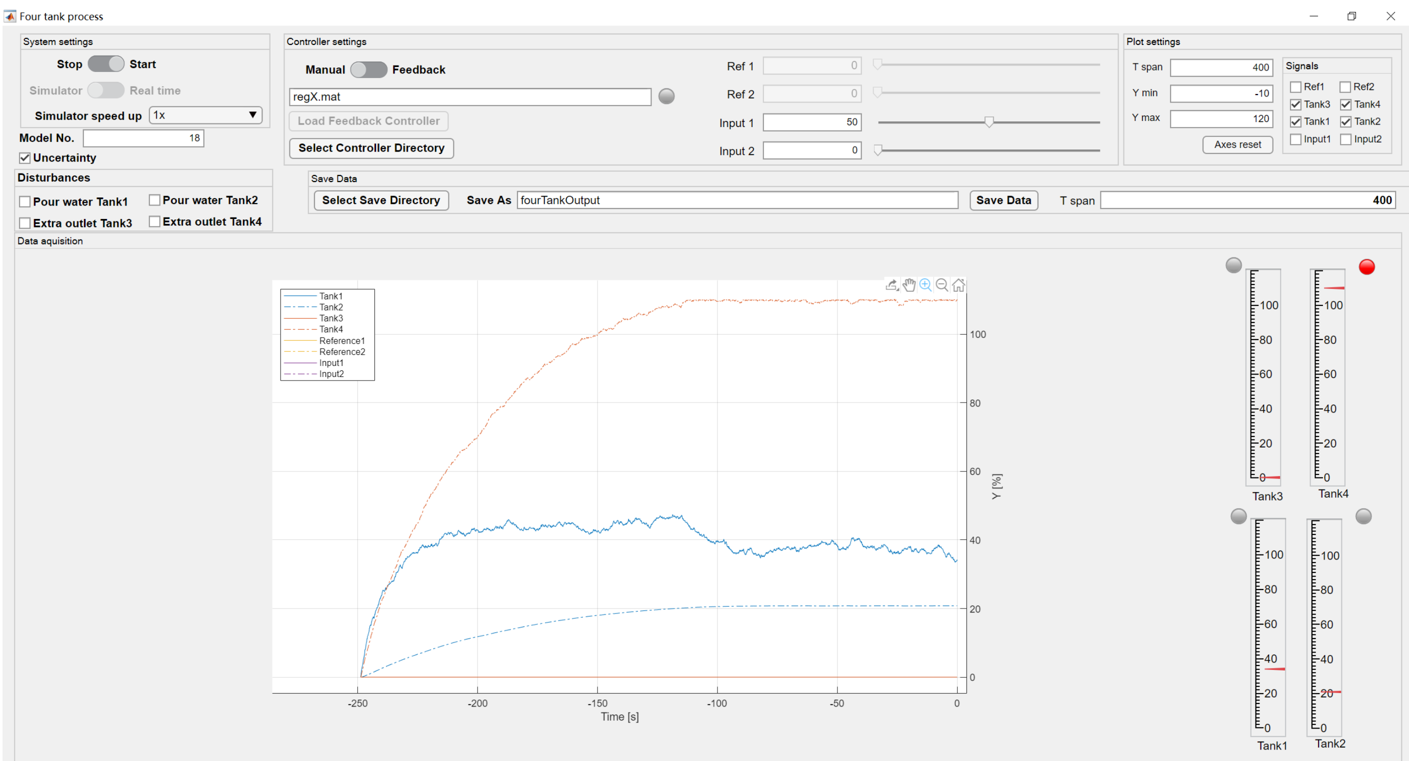

Figure 1: input1=50,input2=0,minimum

Similarly, the figure 2 means that for the minimum phase the tank2 and tank3 are significantly coupled.

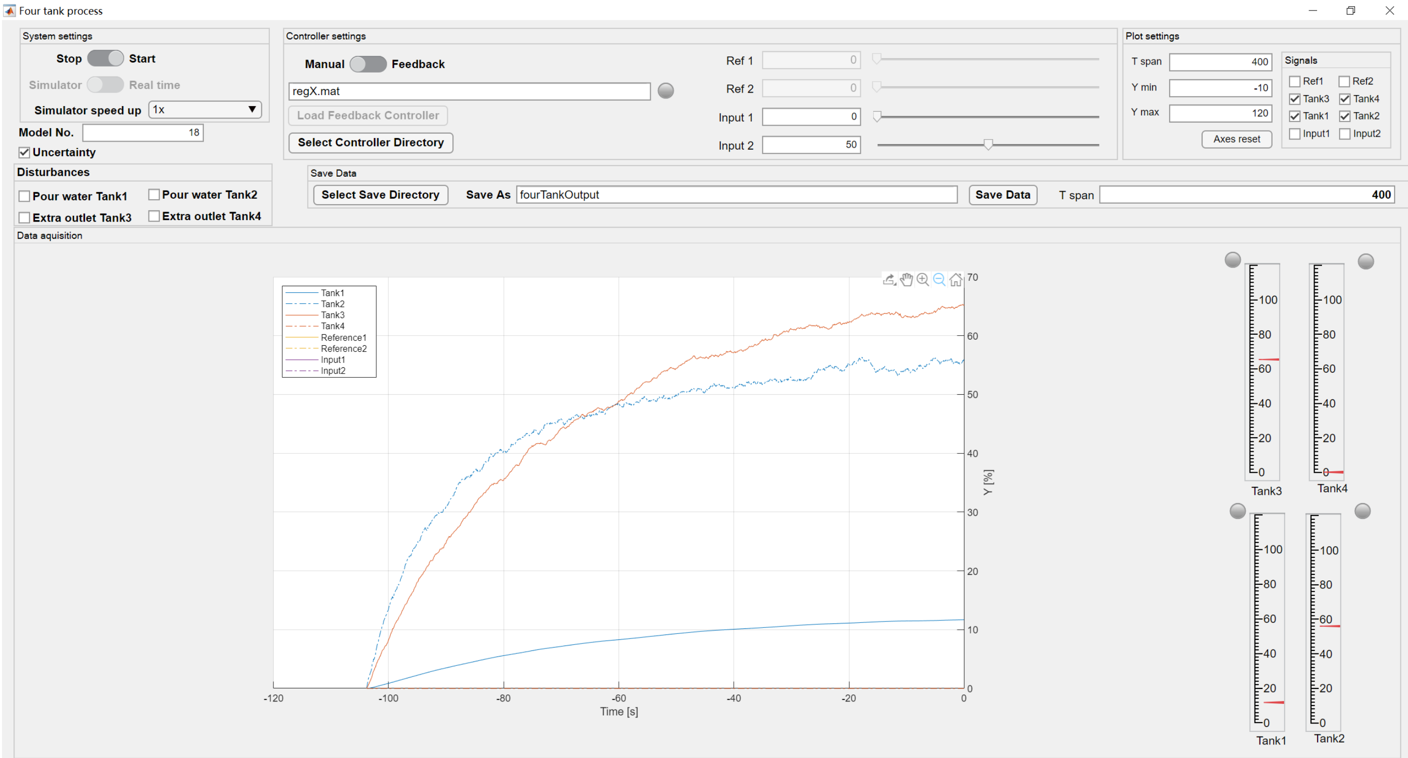

Figure 2: input1=0,input2=50,minimum

For the non-minimum phase, the figure 3 shows tank2 and tank4 are significantly coupled.

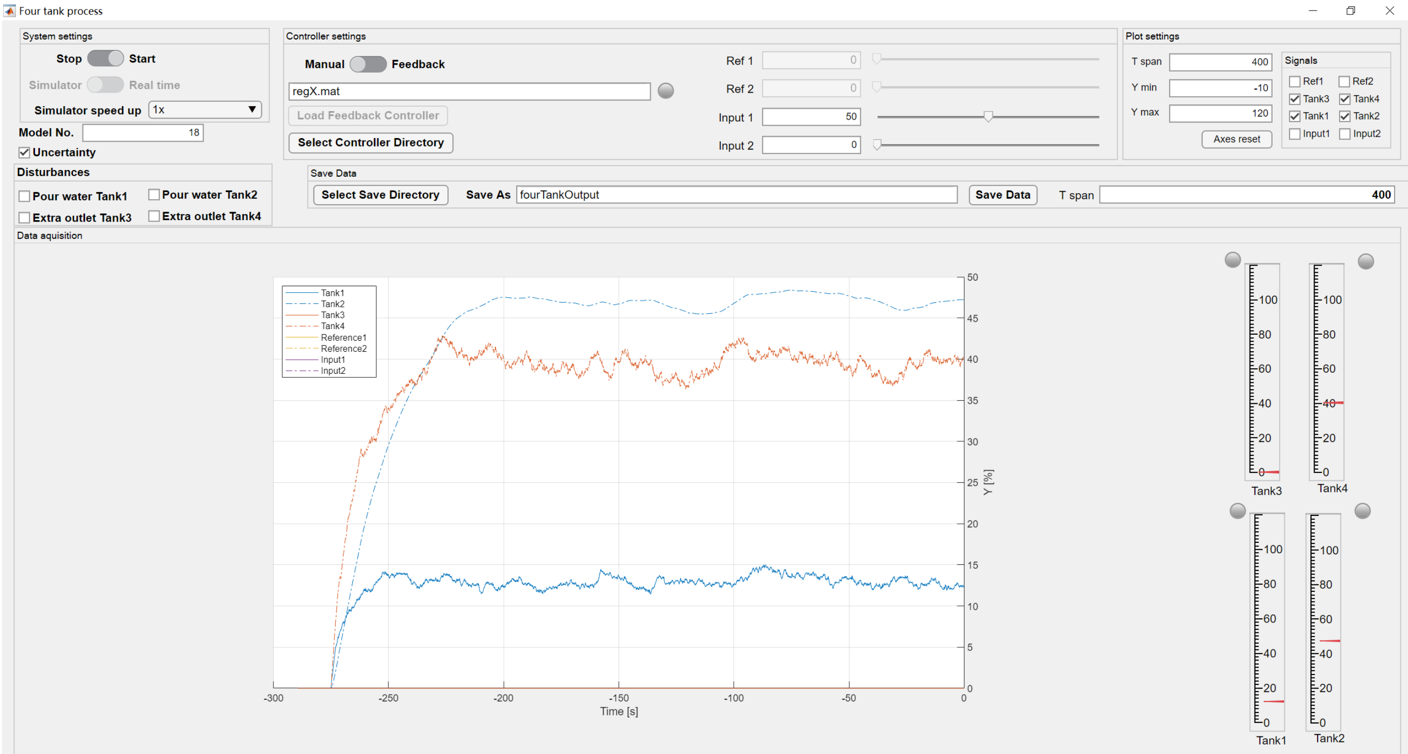

Figure 3: input1=50,input2=0,non-minimum

For the non-minimum phase, the figure 4 shows tank1 and tank3 are significantly coupled.

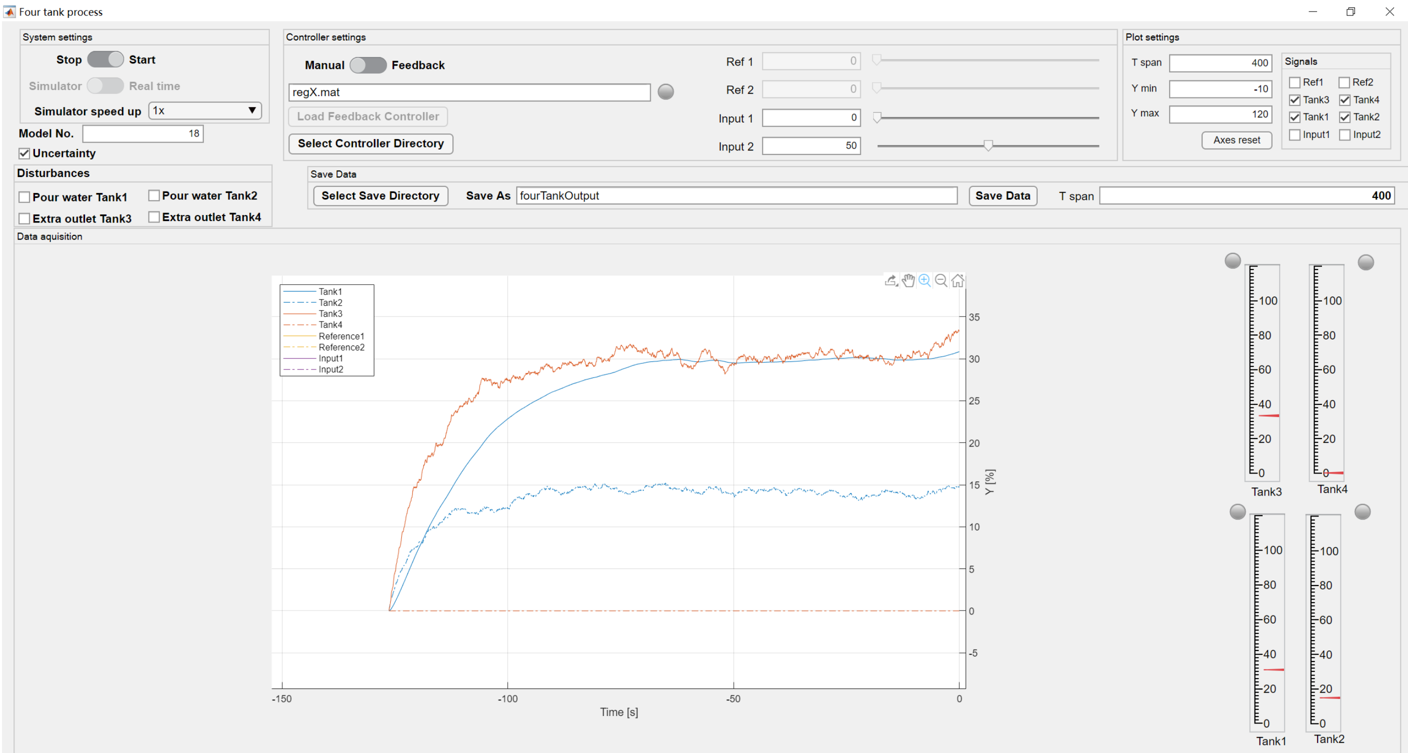

Figure 4: input1=0,input2=50,non-minimum

Exercise 2.2.3

For the minimum phase, by setting input1 to 6.6v and input2 to 3.6V manually, we can make the two lower tanks stable at about 15 cm. The result is shown in the figure 5. From the figure we can get that the transient time is about 220 seconds.

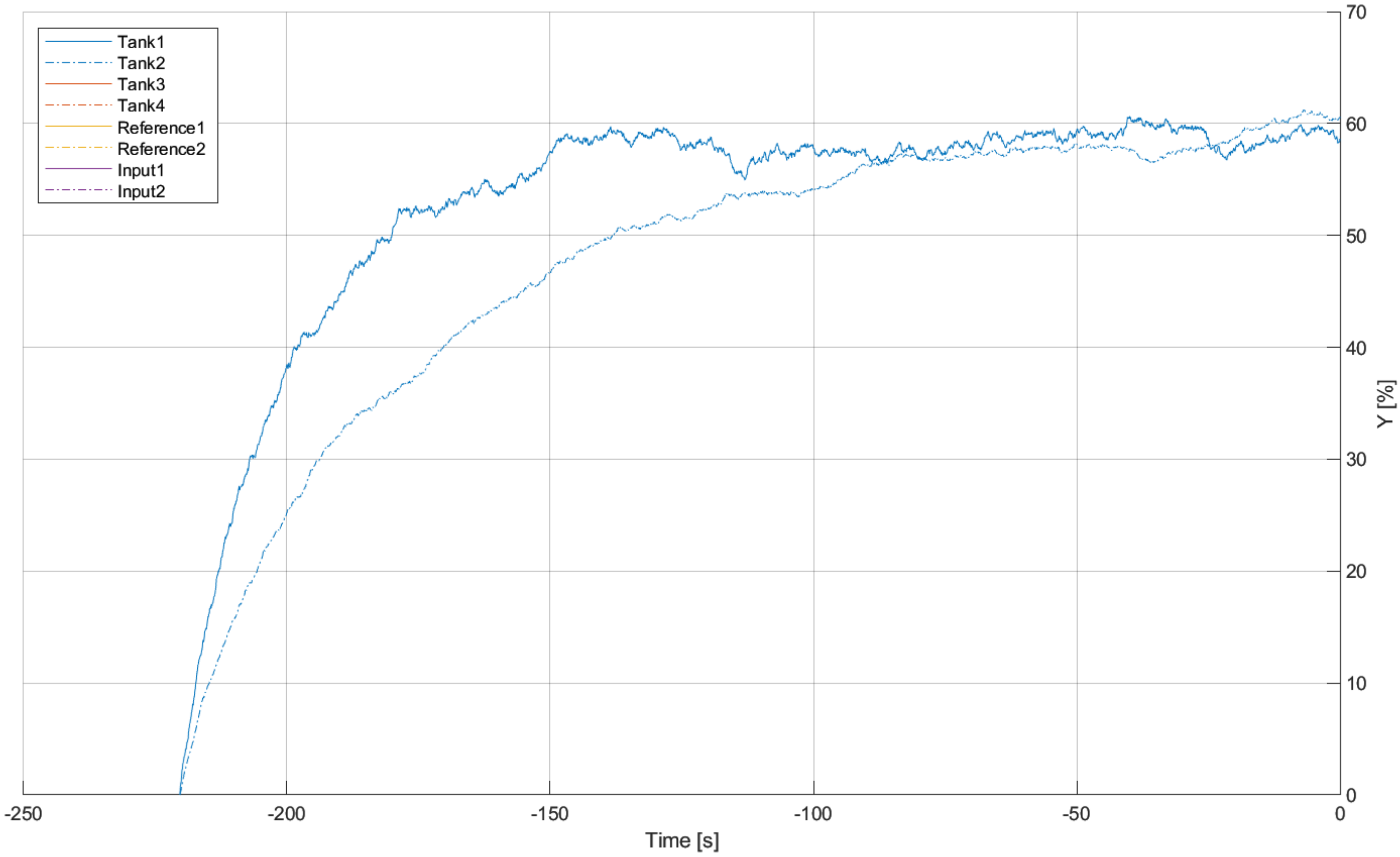

Figure 5: input1=44%,input2=24%,minimum

For the non-minimum phase case, we can keep the tank1 and tank2 at 15cm water level, by changed the input1 to and input2 to .The result is shown in the figure 6. From the figure we can get that the transient time is about 78 seconds.

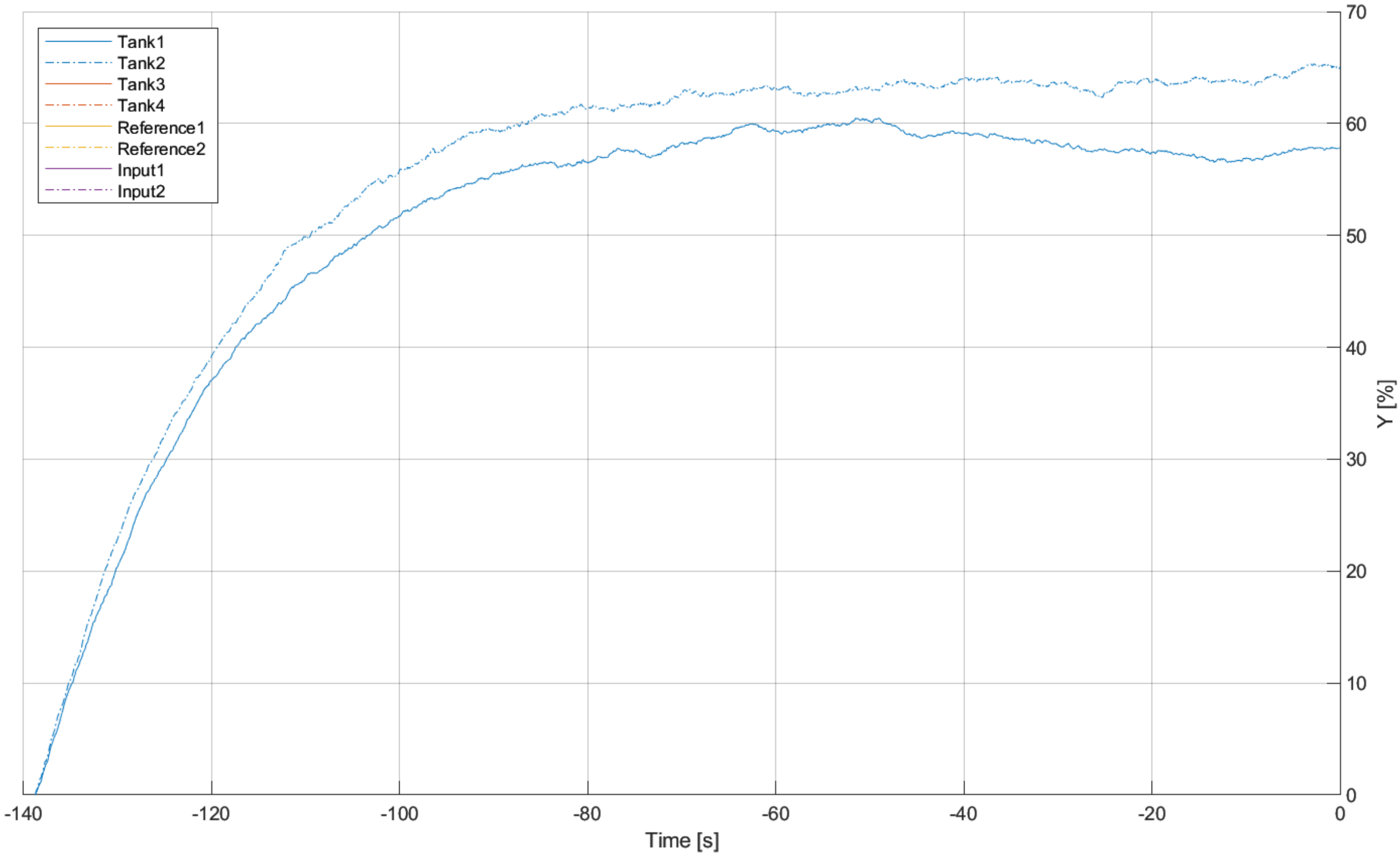

Figure 6: input1=32%,input2=49%,non-minimum

Exercise 2.2.4

For the above exercises, the most important differences between the minimum phase and the non-minimum phase we fond are: Their physical models are different. In other words, a_3 and a_4 are not the same, which differs their transfer functions. According to the previous calculations, the non-minimum phase will have a RHP zero, and this could cause limitations and bad effects to the system. It can be found in section 2.2.3, where Non-minimum are more difficult to be manually controlled, compared with the minimum one. Moreover, their pump input behaves differently.

Decentralized control

Exercise 4.1.1

Previous designed decentralized controllers are used, and transferred into state-space form matrix A,B,C,D, for both the minimum phase case and the non-minimum one.

For example, in order to verify the minimum phase case decentralized controller's performance, reference 1 and 2 are set at 60% for the system for response observation. After the system is stable, disturbances, pour water tank1 and tank2, are introduced to observe the system robustness. Similarly, non minimum phase case has the same process. The results are given in the figure 7 for minimum, figure 8 for the non minimum, and for both cases in table 5.

$phase$	$rise\ time1/2$	$overshoot1/2$	$recovery\ time1/2$
$minimum$	$6.3s$	$15.17\%/17.11\%$	$46s$
$non-minimum$	$29.4s/32.6s$	$6.67\%/27.98\%$	$85.75/104.8s$

Table 5: decentralized controller for minimum phase and non-minimum phase, ref1=60,ref2=60

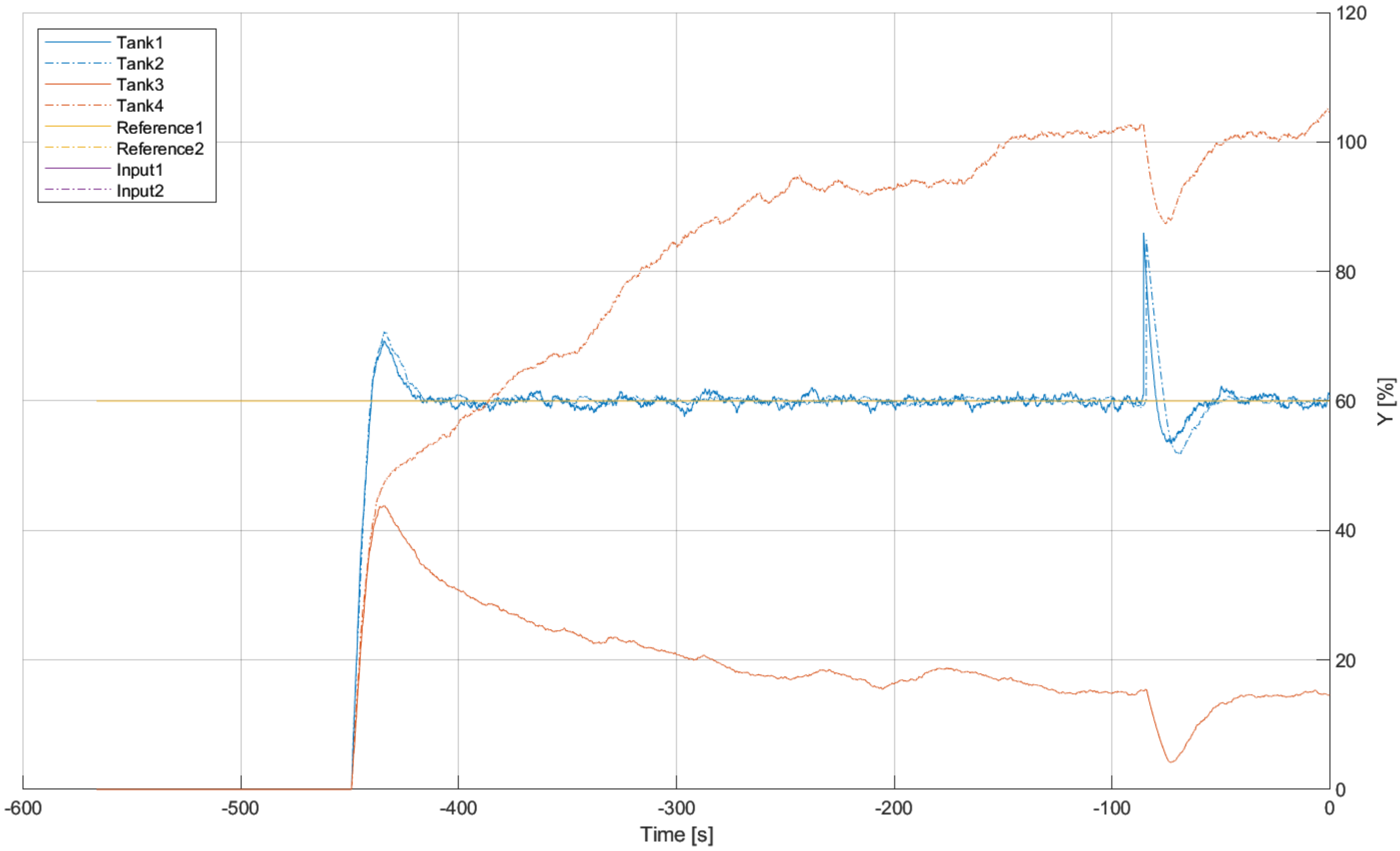

Figure 7: ref1=60%,ref2=60% for minimum phase case.

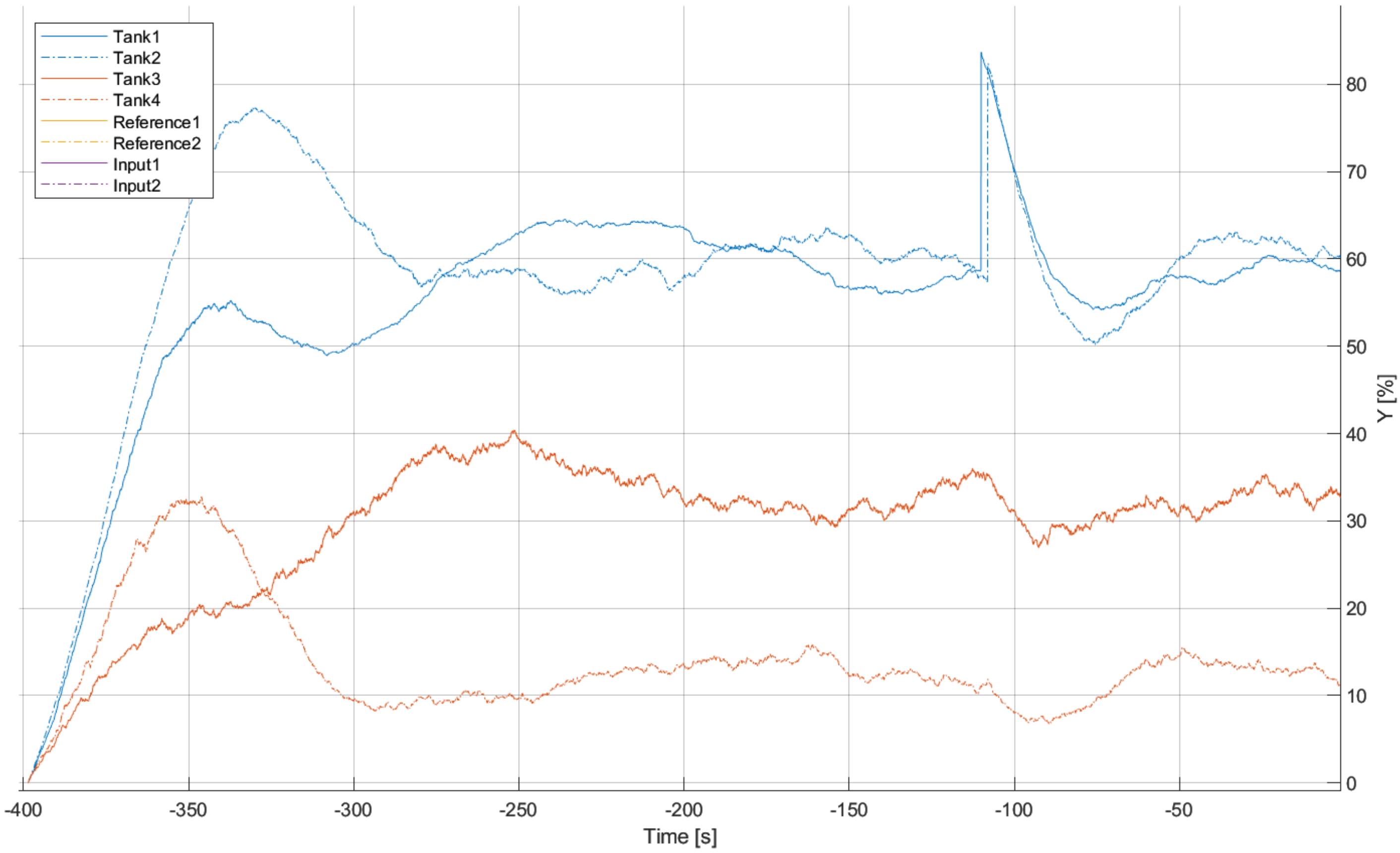

Figure 8: ref1=60%,ref2=60% for non-minimum phase case.

Exercise 4.1.2

For the minimum phase case decentralized PI controller, the results are given in the figure 7 for minimun case, figure 8 for the non minimum case, and their parameter table 5. The figures and table show that the non-minimum phase system has worse performance in robustness, compared with the minimum one, since the non-minimum one has lower response reaction and higher recovery time after introducing disturbances pour water 1, 2. Moreover, the non-minimum phase case's oscillation amplitude and rise time are always larger than the minimum one, whether disturbances are introduced to this system.

Robust control

Exercise 4.2.1

Previous designed Glover-McFarlane controllers in lab4 are used, and transferred into state-space form matrix A,B,C,D, for both the minimum phase case and the non-minimum one.

For example, in order to verify the minimum phase case decentralized controller's performance, reference 1 and 2 are set at 60% for the system for response observation. After the system is stable, disturbances are introduced by pouring water into tank1 and tank2, which let us observe the system's robustness. Similarly, non minimum phase case has the same process. The results are given in the figure 9 for minimun, figure 10 for the non minimum, and for both cases in table 6.

$phase$	$rise\ time1/2$	$overshoot1/2$	$recovery\ time1/2$
$minimum$	$6.6s$	$11.63\%/15.58\%$	$27.15s$
$non-minimum$	$103.1s/81.5s$	$0.83\%/1.78\%$	$83.89s$

Table 6: Glover-McFarlane controller for minimum phase and non-minimum phase, ref1=60, ref2=60.

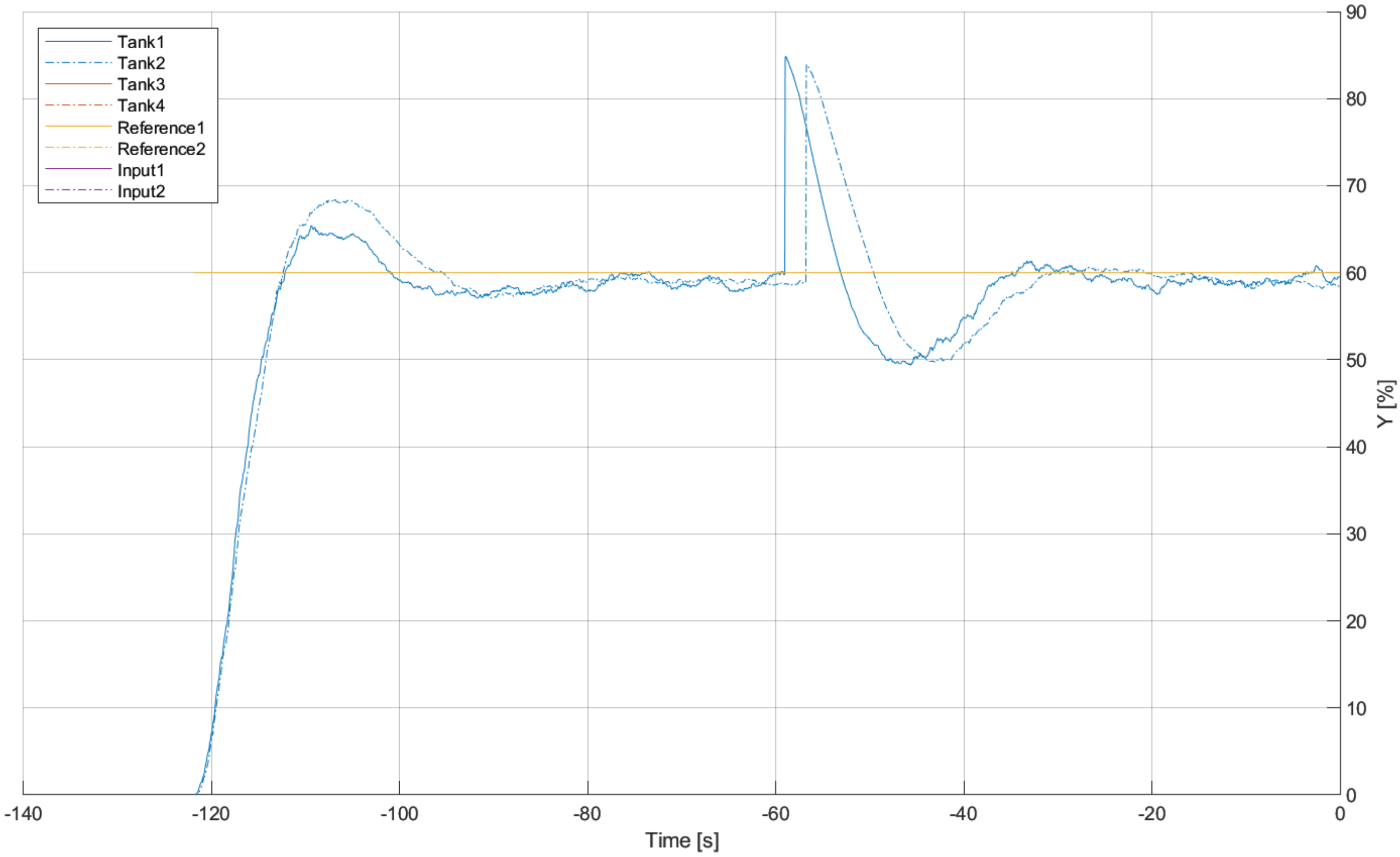

Figure 9: ref1=60%, ref2=60% for minimum phase case.

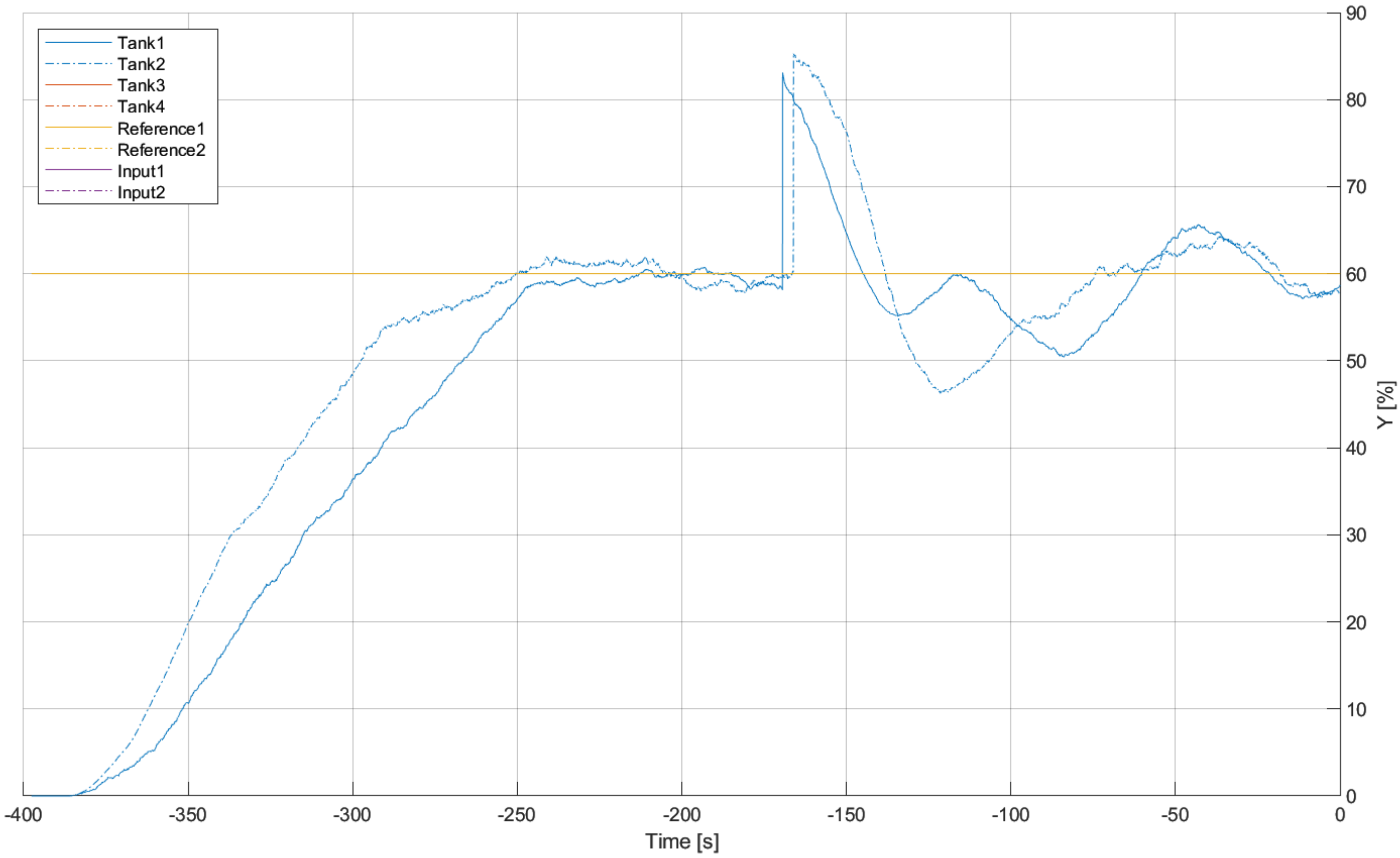

Figure 10: ref1=60%,ref2=60% for non-minimum phase case.

Exercise 4.2.2

When comparing table 5 by decentralized controller with table 6 by Glover-McFarlane controller, we can see that Glover-McFarlane have smaller overshoot but have a longer rise time as a result. This is shown much more significantly for non-minimum phase case. However, Gloever-MacFarlane guarantee robust stabilization which makes it attenuate the disturbance faster as you can see in the recovery time column.

Exercise 4.2.3

For the minimum phase case Glover-McFarlane controller, the results are given in the figure 9. The results of non-minimum case is shown in figure 10. The parameter of both cases are shown in the table 6. The figures and table shows that the non-minimum phase system has worse performance in robustness, compared with the minimum one, since the non-minimum one has lower response reaction and higher recovery time after introducing disturbances by pouring water into tank1 and tank2. Moreover, the non-minimum phase case's rise time are always larger than the minimum one, whether disturbances are introduced to this system or not. However, the minimum phase system has much larger overshoot value. As a result, the non-minimum phase system has worse overall performance.

Conclusions

This report concludes the theoretical and the practical analysis in order to control the experimental four-tank system. Both the minimum phase and non-minimum phase control design was evaluated by by performing: modelling, manual control, designing controllers by decentralization and robust control.