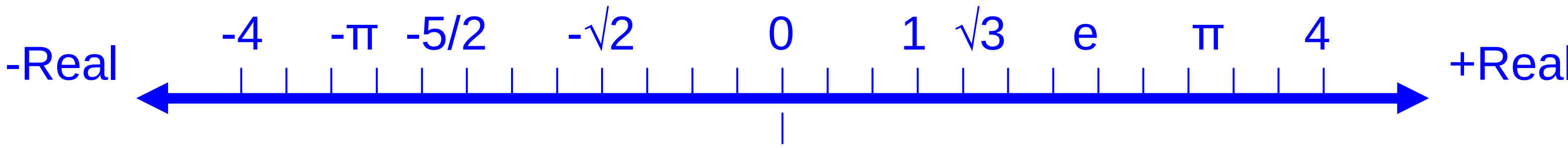
Real Numbers

The set of real numbers includes all rational and irrational numbers. The set has a one to one correspondence with the set of all points on a straight line.



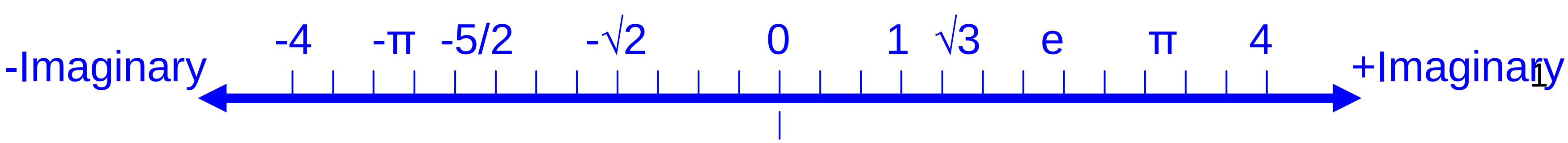
The operations of addition, subtraction, multiplication and division can be performed at any point on the line. The square root of positive real numbers exists on the line but the square root of of a negative number does not exist in the set of real numbers.

Imaginary numbers.

The square root of a negative real number is called a pure imaginary number, e.g., $\sqrt{-1}$, $\sqrt{-3}$, $\sqrt{-8}$, etc. By definition, j (or i) = $\sqrt{-1}$. j is preferred because i is used frequently to denote current in an electrical circuit.

From this definition, $j^2 = -1$, $j^3 = j.j^2 = (-1)j = -j$, $j^4 = j^2 . j^2 = (-1)(-1) = 1$, *and so on.*

The set of all imaginary numbers can also be represented by points on a straight line. “Imaginary” is an unfortunate name for this set of numbers since plainly they exist. However, they cannot be plotted on the real number line.

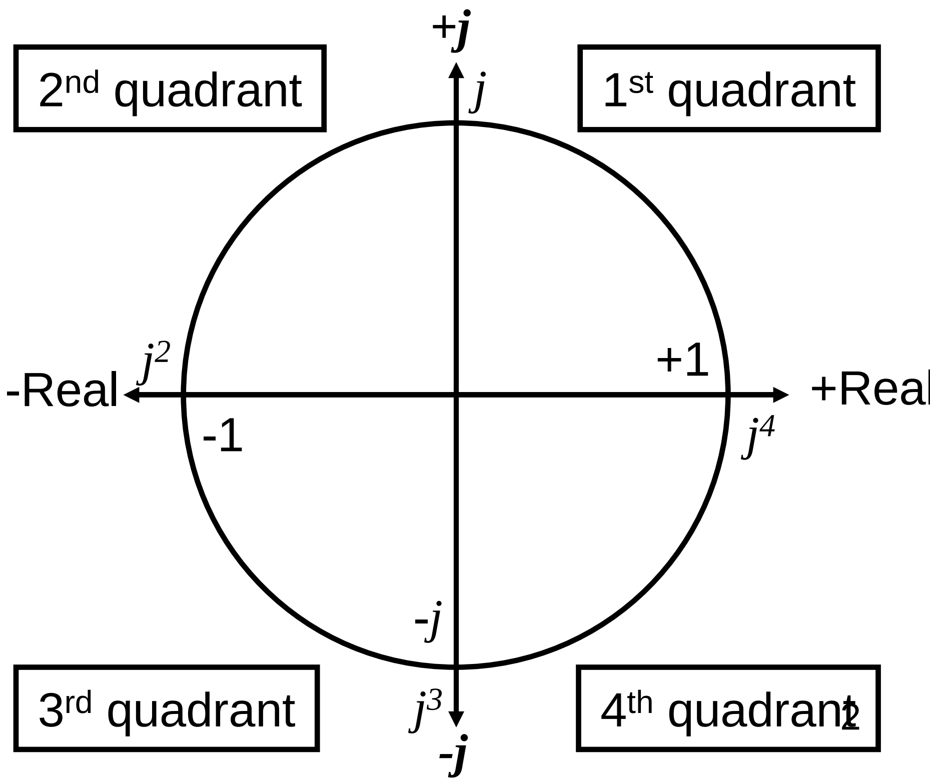


Complex Numbers.

A complex number (Z) is a number of the form $x+jy$, where x & y are real numbers and $j=\sqrt{-1}$. The first term, x, is called the real part. The second term, jy, is called the imaginary part. When $x = 0$, the complex number is a pure imaginary and corresponds to a point on the j axis. When $jy = 0$, the complex number is a real number and corresponds to a point on the real axis. Thus complex numbers include the sets of real and imaginary numbers.

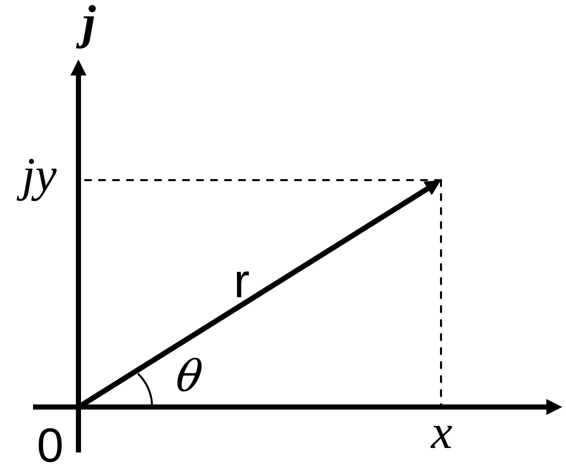
The complex numbers $p+jq$ and $r+js$ are equal if and only if $p = r$ and $q = s$.

Complex numbers are plotted on an Argand Diagram, where real numbers lie on the ‘x’ axis and imaginary numbers lie on the ‘y’ axis.



There are four ways complex numbers can be represented:

1. Rectangular form: $z = x + jy$
2. Polar form: $z = r\angle\theta$
3. Exponential form: $z = re^{j\theta}$
4. Trigonometric form: $z = r(\cos\theta + j\sin\theta)$



These four forms are all interchangeable and which version is easiest to work with often depends on the problem to be solved. For instance, the polar form is usually employed in circuit analysis.

With reference to the diagram, $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Hence the complex number $z = x + jy = r(\cos(\theta) + j\sin(\theta))$, where $r = \sqrt{x^2 + y^2}$ (the modulus or absolute value of z) and angle $\theta = \tan^{-1}({}^{y}/_{x})$, (the amplitude or argument of z).

Euler's formula, $e^{j\theta} = (\cos(\theta)+j\sin(\theta))$ permits the exponential form:

$z = r\cos(\theta) + j\sin(\theta) = re^{j\theta}$.

The polar form is written $r\angle\theta$, where θ is usually in degrees.

Conjugate of a complex number.

The conjugate z^* of a complex number $z = x + jy$ is the complex number $z^* = x - jy$.

For example, conjugate complex pairs are:

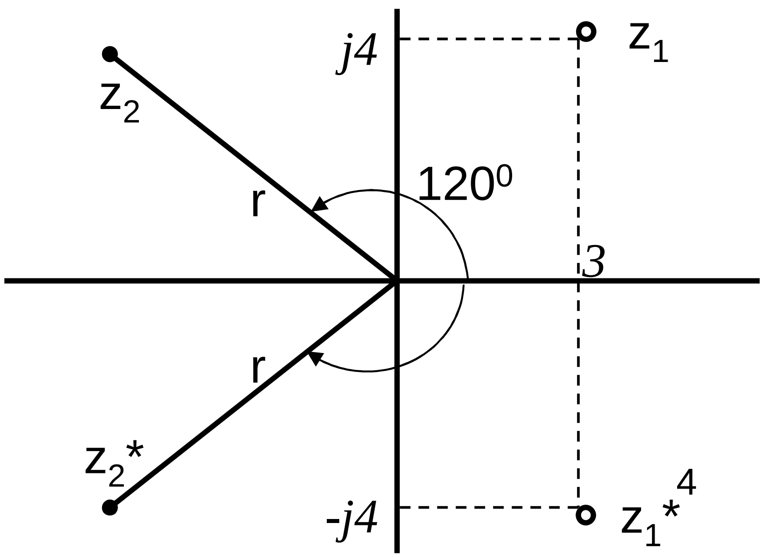
3 + j4	3 – j4
6 – j1	6 + j1
-4 + j2	-4 – j2
-8 – j6	-8 + j6

In the polar form, the conjugate of $z = r\angle\theta$ is $z^* = r\angle -\theta$. Since $\cos(-\theta) = \cos(\theta)$, and $\sin(-\theta) = -\sin(\theta)$, the conjugate of $z = r(\cos(\theta) + j\sin(\theta))$ is $z^* = r(\cos(\theta) - j\sin(\theta))$.

For example, the conjugate of $z = 0.4\angle 25^0$ is $z^* = 0.4\angle -25^0$.

The conjugate z^* of a complex number z is always the image of z with respect to the axis of reals.

$z_1 = 3 + j4$, $z_1^* = 3 - j4$.
$z_2 = r\angle 120^0$, $z_2^* = r\angle -120^0$.



There are four ways of writing a complex number and its conjugate:

$$z = x + jy \qquad z^* = x - jy$$

$$z = r\angle\theta \qquad z^* = r\angle\text{-}\theta$$

$$z = re^{j\theta} \qquad z^* = re^{-j\theta}$$

$$z = r(\cos\theta + j\sin\theta) \qquad z^* = r(\cos\theta - j\sin\theta)$$

Addition and subtraction of complex number.

To add or subtract two complex numbers, treat the real and imaginary parts separately. As noted previously, the most convenient form of a complex number often depends on the situation in which it is used. For addition and subtraction the rectangular form is simplest:

E.g. z_1 = 6 + j4, z_2 = 5 – j8, z_3 = -1 – j3.

$z_1 + z_2$ = (6 + 5) + j(4 – 8) = 11 – j4

$z_1 - z_2$ = (6 – 5) + j(4 + 8) = 1 + j12

$z_2 - z_1$ = (5 – 6) + j(-8 - -4) = -1 – j12

$z_1 + z_2 + z_3$ = (6 + 5 – 1) + j(4 – 8 – 3) = 10 – j7.

Multiplication of Complex Numbers.

The product of two complex numbers when both are in exponential form follows directly from the laws of exponents:

$$z_1 = r_1 e^{j\theta_1}, \; z_2 = r_2 e^{j\theta_2}$$

$$z_1 z_2 = (r_1 e^{j\theta_1})(r_2 e^{j\theta_2}) = r_1 r_2 e^{j(\theta_1+\theta_2)}$$

When in polar form, the product is evident by reference to the exponential form:

$$z_1 = r_1 \angle \theta_1, \; z_2 = r_2 \angle \theta_2$$

$$z_1 z_2 = (r_1 \angle \theta_1)(r_2 \angle \theta_2) = r_1 r_2 \angle (\theta_1 + \theta_2)$$

In rectangular form the product can be found by treating the complex numbers as binomials:

$$z_1 = x_1 + jy_1, \quad z_2 = x_2 + jy_2$$

$$\begin{aligned} z_1 z_2 &= (x_1 + jy_1)(x_2 + jy_2) \\ &= (x_1 x_2 + jy_1 x_2 + x_1 jy_2 + j^2 y_1 y_2) \\ &= (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1) \end{aligned}$$

Examples:

$$z_1 = 4e^{j\frac{\pi}{3}}, \;\; z_2 = 3e^{-j\frac{\pi}{2}}$$

$$z_1 z_2 = \left(4e^{j\frac{\pi}{3}}\right)\left(3e^{-j\frac{\pi}{2}}\right) = 4\times 3e^{j(\frac{\pi}{3}-\frac{\pi}{2})} = \underline{12e^{-j\frac{\pi}{6}}}$$

$$z_1 = 2.5\angle 45°, \quad z_2 = 2\angle -30°$$

$$z_1 z_2 = (2.5\angle 45°)(2\angle -30°) = 2.5\times 2\angle(45° - 30°) = \underline{5\angle 15°}$$

$$z_1 = 6 + j3, \quad z_2 = 4 - j5$$

$$z_1 z_2 = (6 + j3)(4 - j5) = (6\times 4) + j(3\times 4) + (6\times - j5) + (j3\times - j5)$$

$$= 24 + j12 - j30 - j^2 15$$

$$= \underline{39 - j18}$$

Division of Complex Numbers

The quotient of two complex numbers when both are in exponential form follows directly from the laws of exponents:

$$z_1 = r_1 e^{j\theta_1}, \;\; z_2 = r_2 e^{j\theta_2}$$

$$\frac{z_1}{z_2} = \frac{(r_1 e^{j\theta_1})}{(r_2 e^{j\theta_2})} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}$$

When in polar form, the quotient is evident by reference to the exponential form:

$$z_1 = r_1 \angle \theta_1, \;\; z_2 = r_2 \angle \theta_2$$

$$\frac{z_1}{z_2} = \frac{r_1 \angle \theta_1}{r_2 \angle \theta_2} = \frac{r_1}{r_2} \angle (\theta_1 - \theta_2)$$

Division of Complex Numbers (contd)

In rectangular form the quotient can be found by multiplying the numerator and denominator by the conjugate of the denominator:

$$z_1 = x_1 + jy_1, \quad z_2 = x_2 + jy_2$$

$$\frac{z_1}{z_2} = \frac{x_1 + jy_1}{x_2 + jy_2} \times \frac{x_2 - jy_2}{x_2 - jy_2}$$

$$= \frac{(x_1 x_2 + y_1 y_2) + j(x_1 y_2 + x_2 y_1)}{{x_2}^2 + {y_2}^2}$$

[This method is seldom used.]

Examples:

$$z_1 = 4e^{j\frac{\pi}{3}}, \; z_2 = 3e^{-j\frac{\pi}{4}}$$

$$\frac{z_1}{z_2} = \frac{4e^{j\frac{\pi}{3}}}{3e^{-j\frac{\pi}{4}}} = \frac{4}{3}e^{j(\frac{\pi}{3}+\frac{\pi}{4})} = \frac{4}{3}e^{j\frac{7\pi}{12}}$$

$$z_1 = 3\angle 45°, \quad z_2 = -2\angle -60°$$

$$\frac{z_1}{z_2} = \frac{3\angle 45°}{-2\angle -60°} = -1.5\angle 105°$$

$$z_1 = 2 + j3, \quad z_2 = 4 - j5$$

$$\frac{z_1}{z_2} = \frac{2+j3}{4-j5} \times \frac{4+j5}{4+j5} \quad = \frac{8+j12+j10+j^2 15}{16-j20+j20-j^2 25} \quad = \frac{-7+j22}{41}$$

Roots of a Complex Number.

Any complex number $z = re^{j\theta}$ may be written as $z = re^{j(\theta+2\pi N)}$, N = 0, 1, 2, 3, …

Similarly, $z = r\angle\theta$ may be written as $z = r\angle(\theta + N\times 360°)$

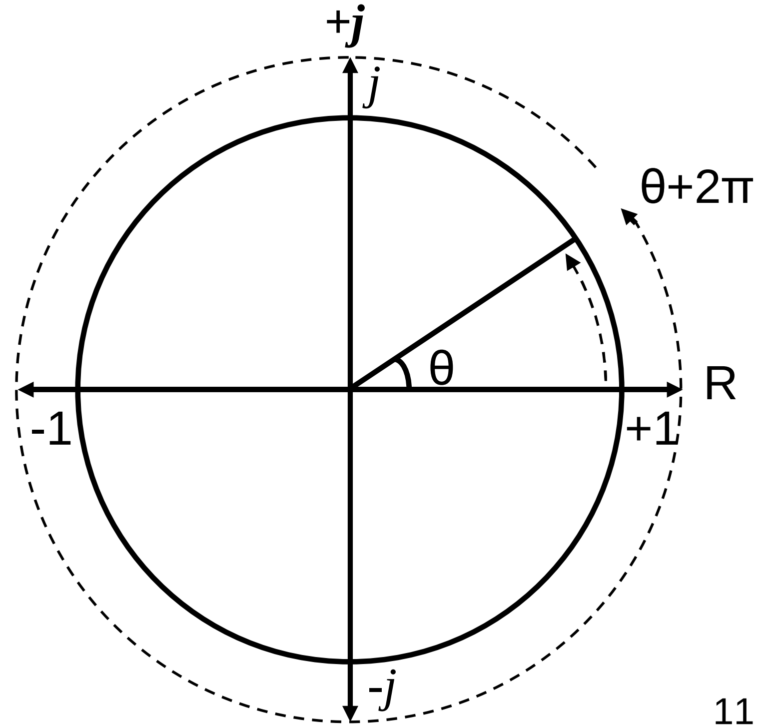
$$z = r\angle\theta = r\angle(\theta + N\times 360°)$$

$$\Rightarrow \sqrt[k]{z} = \sqrt[k]{r\angle(\theta + N\times 360°)} = \sqrt[k]{r}\angle\frac{(\theta + N\times 360°)}{k}$$

$$z = re^{j\theta} = re^{j(\theta+2\pi N)}$$

$$\Rightarrow \sqrt[k]{z} = \sqrt[k]{re^{j(\theta+2\pi N)}} = \sqrt[k]{r}e^{j\frac{(\theta+2\pi N)}{k}}$$

Now the k distinct k^{th} roots of the complex number can be obtained by assigning N to the values 0, 1, 2, 3, …



Example.

Find the three cube roots of z = 8∠60⁰ and plot them on the Argand diagram.

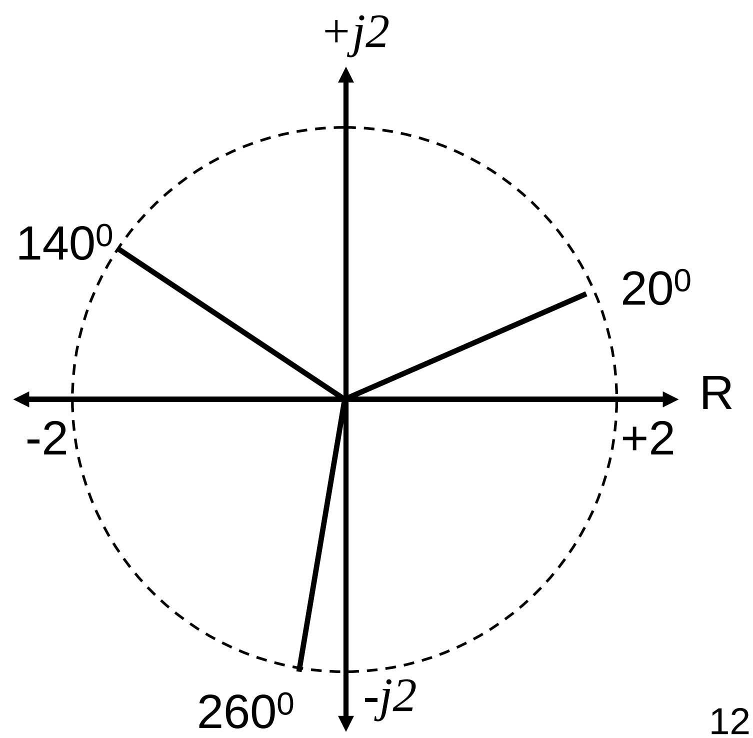
$$z = 8\angle 60° \;\Rightarrow \sqrt[3]{z} = \sqrt[3]{8\angle(60° + n\times 360°)}. \quad (\text{Let } z = 8\angle(60° + n\times 360°))$$

$$\text{Then } \sqrt[3]{z} = \sqrt[3]{8}\;\angle(60° + n\times 360°)/3$$

$$\text{For } n = 0:\; \sqrt[3]{8}\angle(60°)/3 = 2\angle 20°$$

$$\text{For } n = 1:\; \sqrt[3]{8}\;\angle(60° + 360°)/3 = 2\angle 140°$$

$$\text{For } n = 2:\; \sqrt[3]{8}\angle(60° + 2\times 360°)/3 = 2\angle 260°$$



Example.

Find the five fifth roots of 1. (Remember $e^0 = 1$.)

$$\text{Let } z = 1 = 1e^{j2\pi n}$$

$$\Rightarrow \sqrt[5]{1} = \sqrt[5]{1e^{j2\pi n}} = \sqrt[5]{1}e^{j(2\pi n/5)} = 1e^{j(2\pi n/5)}$$

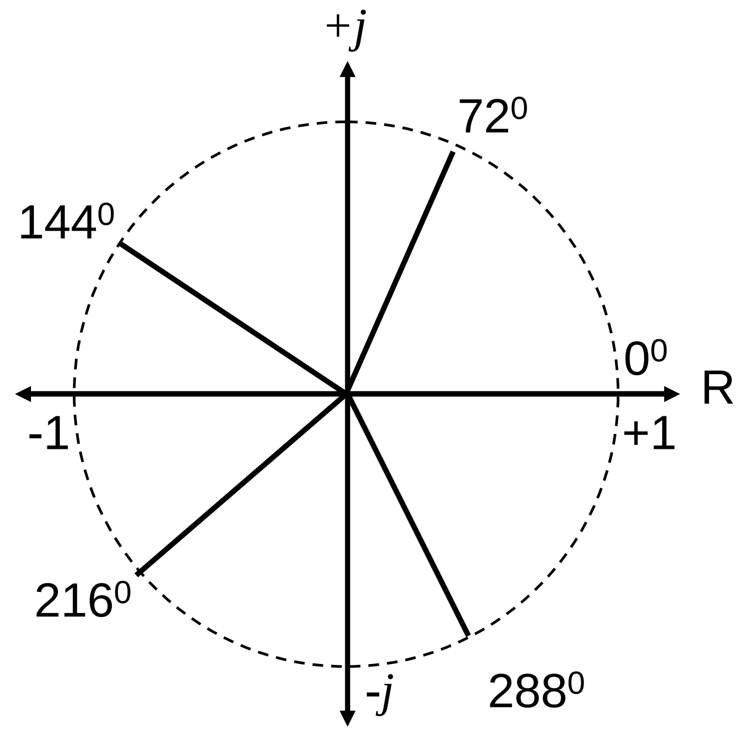
$$n = 0: \quad z = e^{j(0)} \quad = 1\angle 0°$$

$$n = 1: \quad z = e^{j(2\pi/5)} \quad = 1\angle 72°$$

$$n = 2: \quad z = e^{j(4\pi/5)} \quad = 1\angle 144°$$

$$n = 3: \quad z = e^{j(6\pi/5)} \quad = 1\angle 216°$$

$$n = 4: \quad z = e^{j(8\pi/5)} \quad = 1\angle 288°$$



Logarithm of a Complex Number.

If we use the exponential form of a complex number, finding its natural logarithm is very straightforward:

$$\text{Let } z = re^{j\theta} = re^{j(\theta+2\pi n)}$$

$$\text{Then } \ln(z) = \ln(re^{j(\theta+2\pi n)})$$

$$= \ln(r) + \ln(e^{j(\theta+2\pi n)})$$

$$= \ln(r) + j(\theta + 2\pi n)$$

Note that the result is not unique. However, the principal value, when n = 0, is almost always used.

Example.

Find the natural logarithm of the complex number $z = 3e^{j\left(\frac{\pi}{6}\right)}$

$$z = 3e^{j\left(\frac{\pi}{6}\right)} \Rightarrow \ln(z) = \ln\left(3e^{j\left(\frac{\pi}{6}\right)}\right)$$

$$= \ln(3) + \ln\left(e^{j\left(\frac{\pi}{6}\right)}\right)$$

$$= \ln(3) + j\frac{\pi}{6} = 1.099 + j0.524$$