Law of sines

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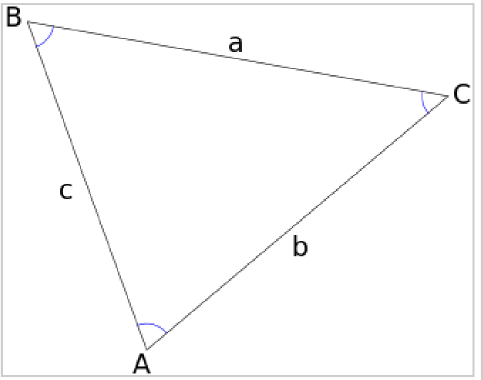
In trigonometry, the **law of sines**, **sine law**, **sine formula**, or **sine rule** is an equation relating the lengths of the sides of any shaped triangle to the sines of its angles. According to the law,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = d,$$

where a, b, and c are the lengths of the sides of a triangle, and A, B, and C are the opposite angles (see the figure to the right), while d is the diameter of the triangle's circumcircle. When the last of these equations is not used, the law is sometimes stated using the reciprocals;

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$



A triangle labelled with the components of the law of sines. Capital A, B and C are the angles, and lower-case a, b, c are the sides opposite them. (a opposite A, etc.)

The law of sines can be used to compute the remaining sides of a triangle when two angles and a side are known—a technique known as triangulation. Numerical calculation using this technique may result in a numerical error if an angle is close to 90 degrees. It can also be used when two sides and one of the non-enclosed angles are known. In some such cases, the triangle is not uniquely determined by this data (called the *ambiguous case*) and the technique gives two possible values for the enclosed angle.

The law of sines is one of two trigonometric equations commonly applied to find lengths and angles in scalene triangles, with the other being the law of cosines.

The law of sines can be generalized to higher dimensions on surfaces with constant curvature.[1]

Contents

Proof

Via a triangle area formula

The area T of any triangle can be written as one half of its base times its height. Depending on which side one chooses to be the base, the area can be written as any of

$$T = \frac{1}{2}b(c\sin A) = \frac{1}{2}c(a\sin B) = \frac{1}{2}a(b\sin C)\,.$$

Multiplying these by $2/abc$ gives

$$\frac{2T}{abc} = \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}\,.$$

Alternative proof

There are three cases to consider in proving the law of sines. The first is when all angles of the triangle are acute. The second is when one angle is a right angle. The third is when one angle is obtuse.

For acute triangles

We make a triangle with the sides a, b, and c, and angles A, B, and C. Then we draw the altitude from vertex B to side b; by definition it divides the original triangle into two right angle triangles: ABR and $R'BC$. Mark this line h_1.

Using the definition of $\sin\alpha$ = opposite / hypotenuse we see that for angle A on the right angle triangle ABR and C on $R'BC$ we have:

$$\sin A = \frac{h_1}{c};\ \sin C = \frac{h_1}{a}$$

Solving for h_1

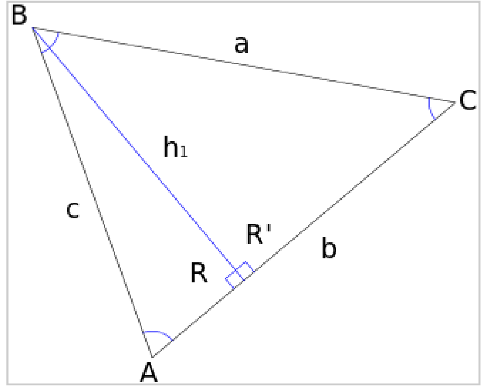
$$h_1 = c\sin A;\ h_1 = a\sin C$$

Equating h_1 in both expressions:

$$h_1 = c \sin A = a \sin C$$

Therefore:

$$\frac{a}{\sin A} = \frac{c}{\sin C}.$$



Triangle ABC with altitude from B drawn

Doing the same thing from angle A to side a we call the altitude h_2 and the two right angle triangles ABR and $AR'C$:

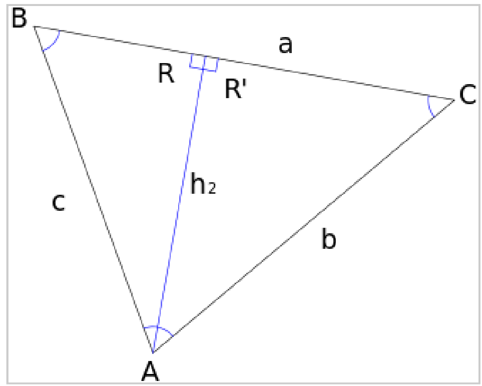
$$\sin B = \frac{h_2}{c};\ \sin C = \frac{h_2}{b}$$

Solving for h_2

$$h_2 = c \sin B;\ h_2 = b \sin C$$

Therefore:

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$



Triangle ABC with altitude from A drawn

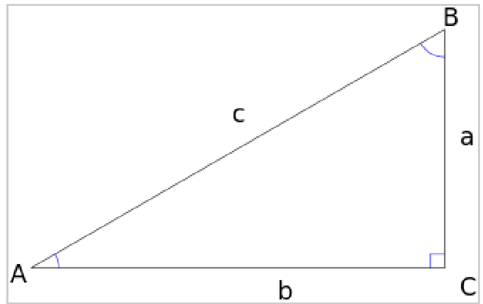
Equating the c / $\sin C$ terms in both expressions above we have:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

For right angle triangles

We make a triangle with the sides a, b, and c, and angles A, B, and C where C is a right angle.

Since we already have a right angle triangle we can use the definition of sine:



Triangle ABC with right angle C.

$$\sin A = \frac{a}{c};\ \sin B = \frac{b}{c}$$

Solving for c:

$$c = \frac{a}{\sin A};\ c = \frac{b}{\sin B}$$

Therefore:

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

For the remaining angle C we need to remember that it is a right angle and $\sin C = 1$ in this case. Therefore we can rewrite $c = c / 1$ as:

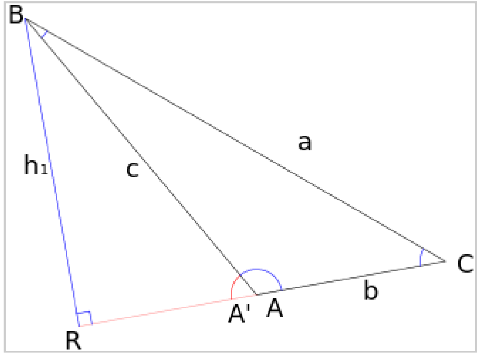
$$c = \frac{c}{\sin C}$$

Equating c in both the equations above we again have:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

For obtuse triangles

We make a triangle with the sides a, b, and c, and angles A, B, and C where A is an obtuse angle. In this case if we draw an altitude from any angle other than A the point where this line will touch the base of the triangle ABC will lie outside any of the lines a, b, or c. We draw the altitude from angle B, calling it h_1 and create the two extended right triangles RBA' and RBC.



Obtuse triangle ABC with altitude drawn from B.

From the definition of sine we again have:

$$\sin A' = \frac{h_1}{c};\ \sin C = \frac{h_1}{a}$$

We use identity $\sin \pi - \theta = \sin \theta$ to express $\sin A'$ in terms of $\sin A$. By definition we have:

$$A + A' = \pi$$

$$A = \pi - A'$$

$$\sin A = \sin(\pi - A') = \sin A'$$

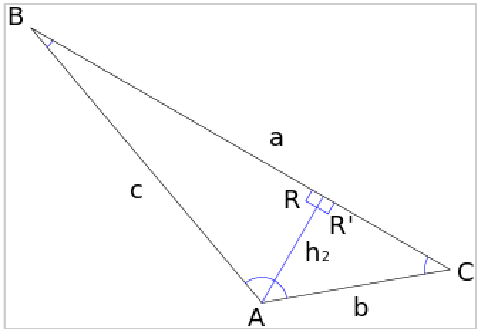
Therefore:

$$\sin A = \frac{h_1}{c};\ \sin C = \frac{h_1}{a}$$

and

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

We now draw an altitude from A calling it h_2 and forming two right triangles ABR and $AR'C$.



Obtuse triangle ABC with altitude from A

From this we straightforwardly get:

$$\sin B = \frac{h_2}{c};\ \sin C = \frac{h_2}{b}$$

and

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

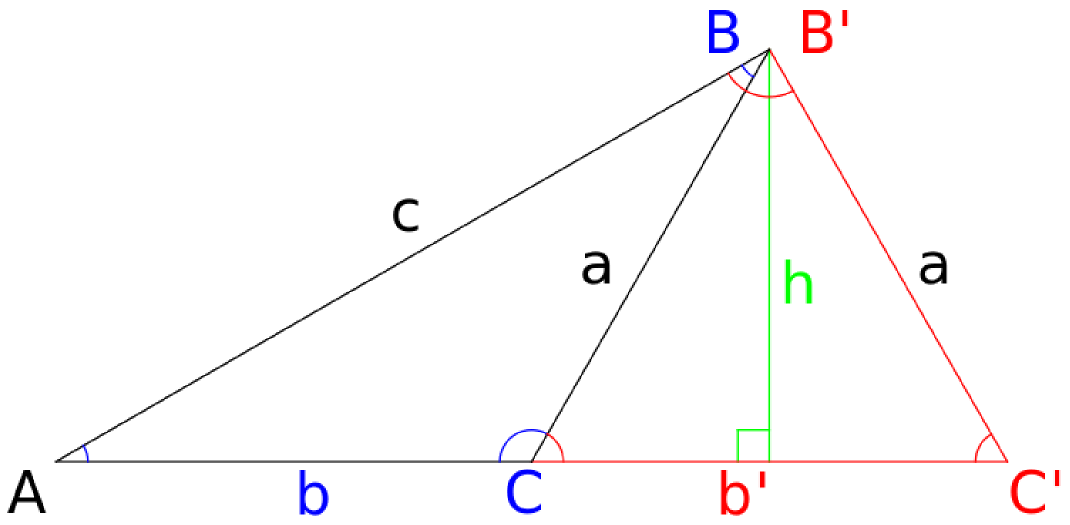
Equating the c / $\sin C$ in both equations above we again get:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Proving the theorem in all cases.

The ambiguous case of triangle solution

When using the law of sines to find a side of a triangle, an ambiguous case occurs when two separate triangles can be constructed from the data provided (i.e., there are two different possible solutions to the triangle). In the case shown below they are triangles ABC and $AB'C'$.



Given a general triangle the following conditions would need to be fulfilled for the case to be ambiguous:

- The only information known about the triangle is the angle A and the sides a and c.
- The angle A is acute (i.e., A < 90°).
- The side a is shorter than the side c (i.e., $a < c$).
- The side a is longer than the altitude h from angle B, where $h = c \sin A$ (i.e., $a > h$).

If all the above conditions are true, then both angles C or C' produce a valid triangle; meaning both of the following are true:

$$C' = \arcsin \frac{c \sin A}{a} \text{ or } C = \pi - \arcsin \frac{c \sin A}{a}$$

From there we can find the corresponding B and b or B' and b' if required, where b is the side bounded by angles A and C and b' bounded by A and C'.

Without further information it is impossible to decide which is the triangle being asked for.

Examples

The following are examples of how to solve a problem using the law of sines.

Example 1

Given: side $a = 20$, side $c = 24$, and angle $C = 40°$. Angle A is desired.

Using the law of sines, we conclude that

$$\frac{\sin A}{20} = \frac{\sin 40°}{24}.$$

$$A = \arcsin\left(\frac{20\sin 40^\circ}{24}\right) \approx 32.39^\circ.$$

Note that the potential solution $A = 147.61^\circ$ is excluded because that would necessarily give $A + B + C > 180^\circ$.

Example 2

If the lengths of two sides of the triangle a and b are equal to x, the third side has length c, and the angles opposite the sides of lengths a, b, and c are A, B, and C respectively then

$$A = B = \frac{180^\circ - C}{2} = 90^\circ - \frac{C}{2}$$
$$\sin A = \sin B = \sin(90^\circ - C/2) = \cos(C/2)$$
$$\frac{c}{\sin C} = \frac{a}{\sin A} = \frac{x}{\cos(C/2)}$$
$$\frac{c\ \cos(C/2)}{\sin C} = x$$

Relation to the circumcircle

In the identity

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

the common value of the three fractions is actually the diameter of the triangle's circumcircle.[2] It can be shown that this quantity is equal to

$$\begin{aligned}\frac{abc}{2T} &= \frac{abc}{2\sqrt{s(s-a)(s-b)(s-c)}} \\ &= \frac{2abc}{\sqrt{(a^2+b^2+c^2)^2 - 2(a^4+b^4+c^4)}},\end{aligned}$$

where T is the area of the triangle and s is the semiperimeter

$$s = \frac{a+b+c}{2}.$$

The second equality above readily simplifies to Heron's formula for the area.

Curvature

The law of sines takes on a similar form in the presence of curvature.

Spherical case

In the spherical case, the formula is:

$$\frac{\sin A}{\sin \alpha} = \frac{\sin B}{\sin \beta} = \frac{\sin C}{\sin \gamma}.$$

Here, α, β, and γ are the angles at the center of the sphere subtended by the three arcs of the spherical surface triangle a, b, and c, respectively. A, B, and C are the surface angles opposite their respective arcs.

It is easy to see how for small spherical triangles, when the radius of the sphere is much greater than the sides of the triangle, this formula becomes the planar formula at the limit, since

$$\lim_{\alpha \to 0} \frac{\sin \alpha}{\alpha} = 1$$

and the same for $\sin \beta$ and $\sin \gamma$.

See also Spherical law of cosines and Half-side formula.

Hyperbolic case

In hyperbolic geometry when the curvature is −1, the law of sines becomes

$$\frac{\sin A}{\sinh a} = \frac{\sin B}{\sinh b} = \frac{\sin C}{\sinh c}.$$

In the special case when B is a right angle, one gets

$$\sin C = \frac{\sinh c}{\sinh b}$$

which is the analog of the formula in Euclidean geometry expressing the sine of an angle as the opposite side divided by the hypotenuse.

See also hyperbolic triangle.

Unified formulation

Define a generalized sine function, depending also on a real parameter K:

$$\sin_K x = x - \frac{Kx^3}{3!} + \frac{K^2x^5}{5!} - \frac{K^3x^7}{7!} + \cdots.$$

The law of sines in constant curvature K reads as[1]

$$\frac{\sin A}{\sin_K a} = \frac{\sin B}{\sin_K b} = \frac{\sin C}{\sin_K c}.$$

By substituting $K = 0$, $K = 1$, and $K = -1$, one obtains respectively the Euclidean, spherical, and hyperbolic cases of the law of sines described above.

Let $p_K(r)$ indicate the circumference of a circle of radius r in a space of constant curvature K. Then $p_K(r) = 2\pi \sin_K r$. Therefore the law of sines can also be expressed as:

$$\frac{\sin A}{p_K(a)} = \frac{\sin B}{p_K(b)} = \frac{\sin C}{p_K(c)}.$$

This formulation was discovered by János Bolyai.[3]

Higher dimensions

For an n-dimensional simplex (i.e., triangle ($n = 2$), tetrahedron ($n = 3$), pentatope ($n = 4$), etc.) in n-dimensional Euclidean space, the absolute value of the polar sine of the normal vectors of the faces that meet at a vertex, divided by the hyperarea of the face opposite the vertex is independent of the choice of the vertex. For example, a tetrahedron has four triangular faces. The absolute value of the polar sine of the normal vectors to three of the faces (which share a vertex) divided by the area of the fourth face will not depend upon the choice of the vertex:

$$\frac{|\mathrm{psin}(\mathbf{n_2}, \mathbf{n_3}, \mathbf{n_4})|}{\mathrm{Area}_1} = \frac{|\mathrm{psin}(\mathbf{n_1}, \mathbf{n_3}, \mathbf{n_4})|}{\mathrm{Area}_2} = \frac{|\mathrm{psin}(\mathbf{n_1}, \mathbf{n_2}, \mathbf{n_4})|}{\mathrm{Area}_3} = \frac{|\mathrm{psin}(\mathbf{n_1}, \mathbf{n_2}, \mathbf{n_3})|}{\mathrm{Area}_4}\,.$$

History

According to Ubiratàn D'Ambrosio and Helaine Selin, the spherical law of sines was discovered in the 10th century. It is variously attributed to Abu-Mahmud Khojandi, Abu al-Wafa' Buzjani, Nasir al-Din al-Tusi and Abu Nasr Mansur.[4]

Ibn Muʿādh al-Jayyānī's *The book of unknown arcs of a sphere* in the 11th century introduced the general law of sines.[5] The plane law of sines was later described in the 13th century by Nasīr al-Dīn al-Tūsī. In his *On the Sector Figure*, he stated the law of sines for plane and spherical triangles, and provided proofs for this law.[6]

According to Glen Van Brummelen, "The Law of Sines is really Regiomontanus's foundation for his solutions of right-angled triangles in Book IV, and these solutions are in turn the bases for his solutions of general triangles."[7] Regiomontanus was a 15th-century German mathematician.

See also

- Gersonides
- Half-side formula – for solving spherical triangles
- Law of cosines
- Law of tangents
- Law of cotangents
- Mollweide's formula – for checking solutions of triangles
- Solution of triangles
- Surveying

References

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2. Coxeter, H. S. M. and Greitzer, S. L. *Geometry Revisited*. Washington, DC: Math. Assoc. Amer., pp. 1–3, 1967
3. Katok, Svetlana (1992). *Fuchsian groups*. Chicago: University of Chicago Press. p. 22. ISBN 0-226-42583-5.
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5. O'Connor, John J.; Robertson, Edmund F., "Abu Abd Allah Muhammad ibn Muadh Al-Jayyani", *MacTutor History of Mathematics archive*, University of St Andrews.
6. Berggren, J. Lennart (2007). "Mathematics in Medieval Islam". *The Mathematics of Egypt, Mesopotamia, China, India, and Islam: A Sourcebook*. Princeton University Press. p. 518. ISBN 978-0-691-11485-9.
7. Glen Van Brummelen (2009). "*The mathematics of the heavens and the earth: the early history of trigonometry (http://books.google.cz/books?id=bHD8IBaYN-oC&pg=&dq&hl=en#v=onepage&q=&f=false)*". Princeton University Press. p.259. ISBN 0-691-12973-8

External links

- Hazewinkel, Michiel, ed. (2001), "Sine theorem", *Encyclopedia of Mathematics*, Springer, ISBN 978-1-55608-010-4
- The Law of Sines (http://www.cut-the-knot.org/proofs/sine_cosine.shtml#law) at cut-the-knot
- Degree of Curvature (http://mysite.du.edu/~jcalvert/railway/degcurv.htm)
- Finding the Sine of 1 Degree (http://www.efnet-math.org/Meta/sine1.htm)

- Generalized law of sines to higher dimensions (http://mathworld.wolfram.com/GeneralizedLawofSines.html)
- Law of Sines - ProofWiki (http://proofwiki.org/wiki/Law_of_Sines)

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