Sinusoidal current and voltage in circuit components.

The three basic circuit components are resistors (R), inductors (L) and capacitors (C). When the **current** in each of these elements is a sinusoid, we have:

Element	Voltage for general i	Voltage for $i = I_M \sin(\omega t)$
Resistor	$v_R = Ri$	$v_R = RI_M \sin(\omega t)$
Inductor	$v_L = L\dfrac{di}{dt}$	$v_L = \omega L I_M \cos(\omega t)$
Capacitor	$v_C = \dfrac{1}{C}\int i dt$	$v_C = \dfrac{I_M}{\omega C}(-\cos(\omega t))$

When the **voltage** across each of these elements is a sinusoid, we have:

Element	Current for general v	Current for $v = V_M \sin(\omega t)$
Resistor	$i_R = \dfrac{v}{R}$	$i_R = \dfrac{V_M}{R}\sin(\omega t)$
Inductor	$i_L = \dfrac{1}{L}\int v dt$	$i_L = \dfrac{V_M}{\omega L}(-\cos(\omega t))$
Capacitor	$i_C = C\dfrac{dv}{dt}$	$i_C = \omega C V_M \cos(\omega t)$

Impedance.

The impedance (Z) of a circuit element (or combination of circuit elements) is the ratio of the voltage to the current:

(General case of Ohm's Law)

$$\boxed{\text{Impedance} = \frac{\text{Voltage function}}{\text{Current function}};\ Z = \frac{V}{I}}$$

Where the voltage and current functions vary with time, for example as a sinusoid, this ratio will have a magnitude and angle. The angle between the voltage and current is called the ***phase angle.***

Example. An electric hair dryer is designed to be used with the domestic mains voltage (V_{RMS}=240V A.C.) and has a heater element with a resistance of 100Ω. What is the current in the heater element? Sketch the current & power waveform.

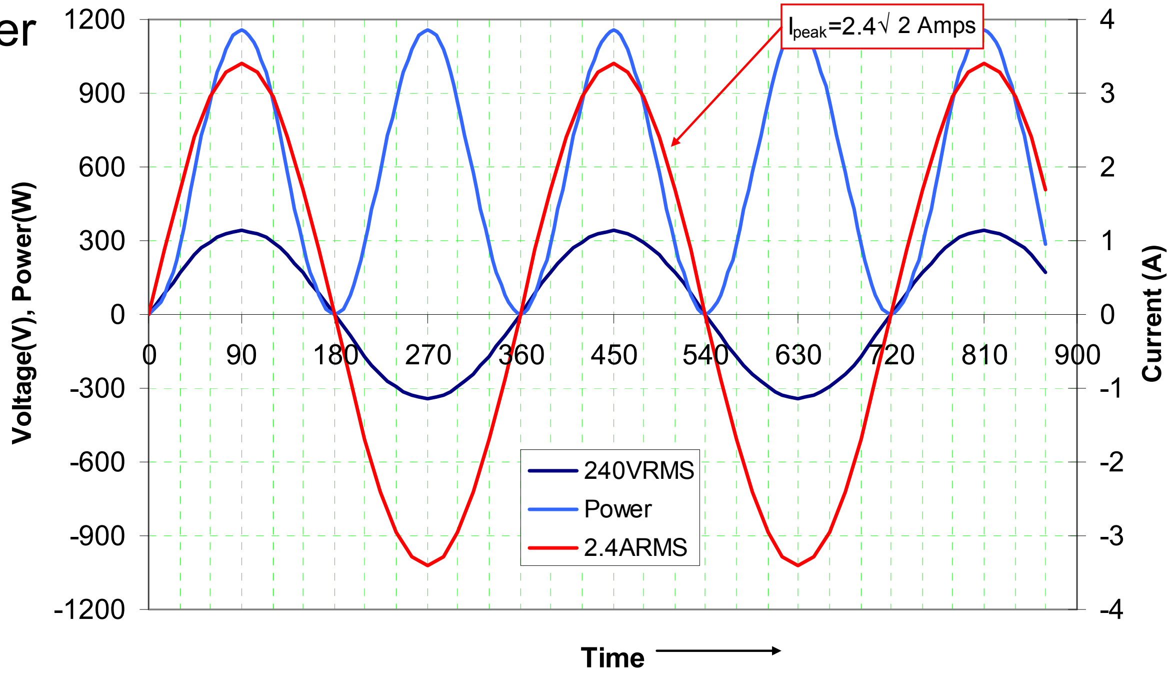
$$Z = \frac{V}{I} \quad => I = \frac{V}{Z} = \frac{240}{100} = 2.4A$$

Power

$$= V \times I = 240 \times 2.4 = 576W$$

$$= \frac{V^2}{R} = \frac{240^2}{100} = 576W$$

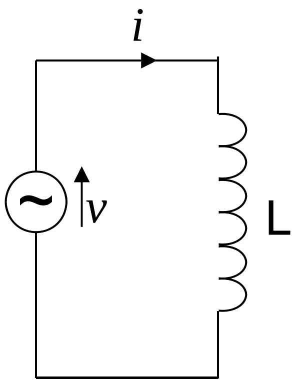
$$= I^2R = 2.4^2 \times 100 = 576W$$

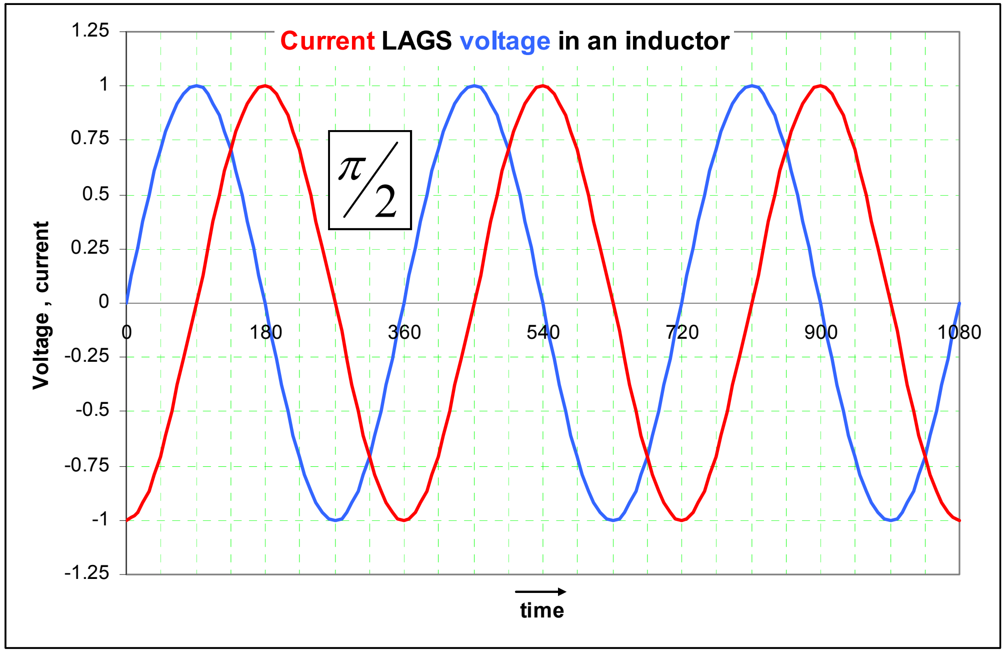


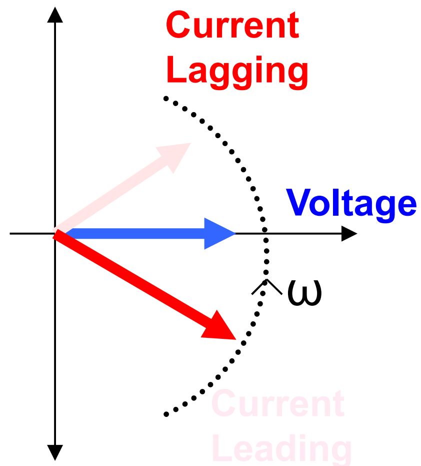
The resistor is a special case where the voltage and current are exactly in phase, i.e., the phase angle is zero.

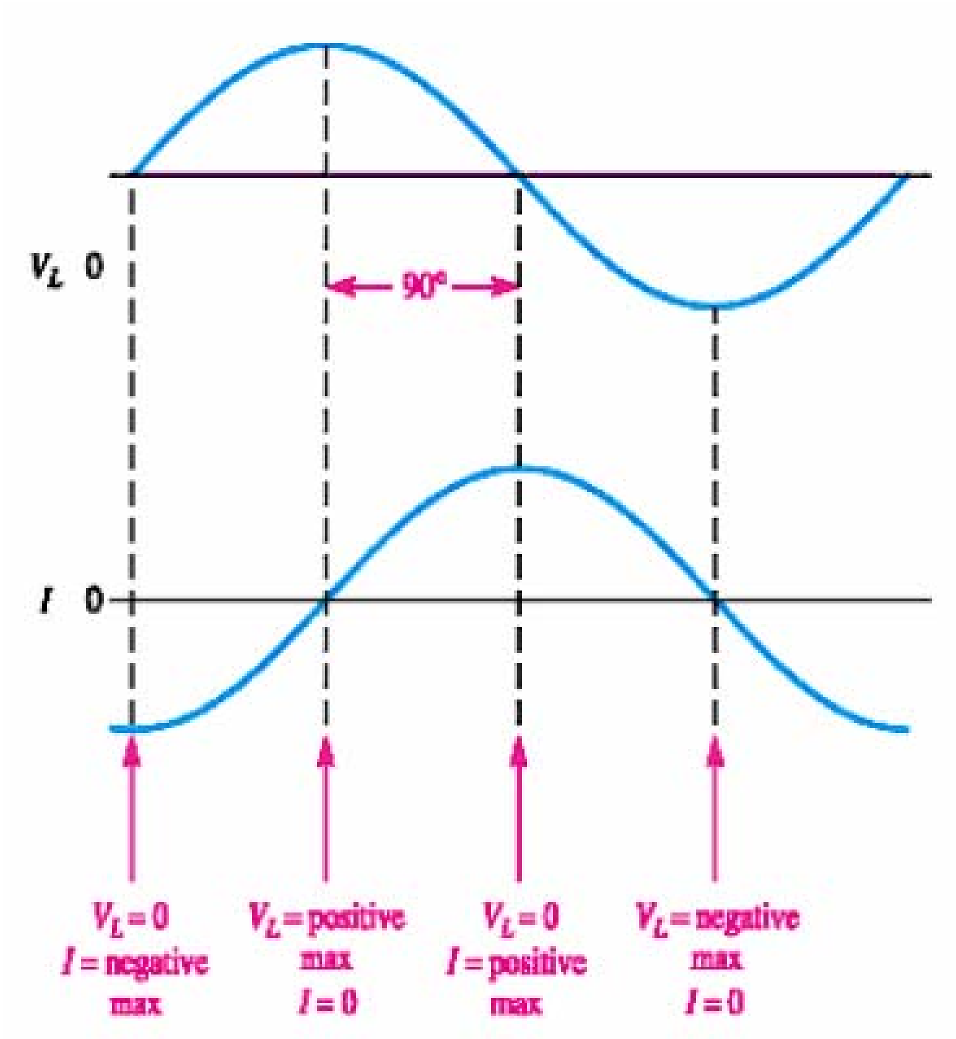
When talking about the phase angle, the accepted convention is that we use the voltage as the reference and examine where the current is in relation to it. There are three possibilities: behind, superimposed or ahead. In electrical terminology this is called “lagging” or “leading”.

For a pure inductor, the current in the circuit lags the applied voltage by 90^0 ($\pi/2$).



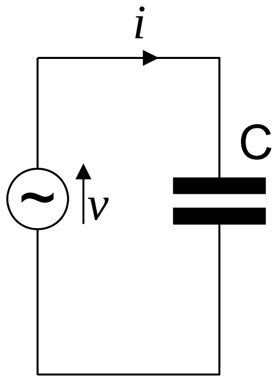


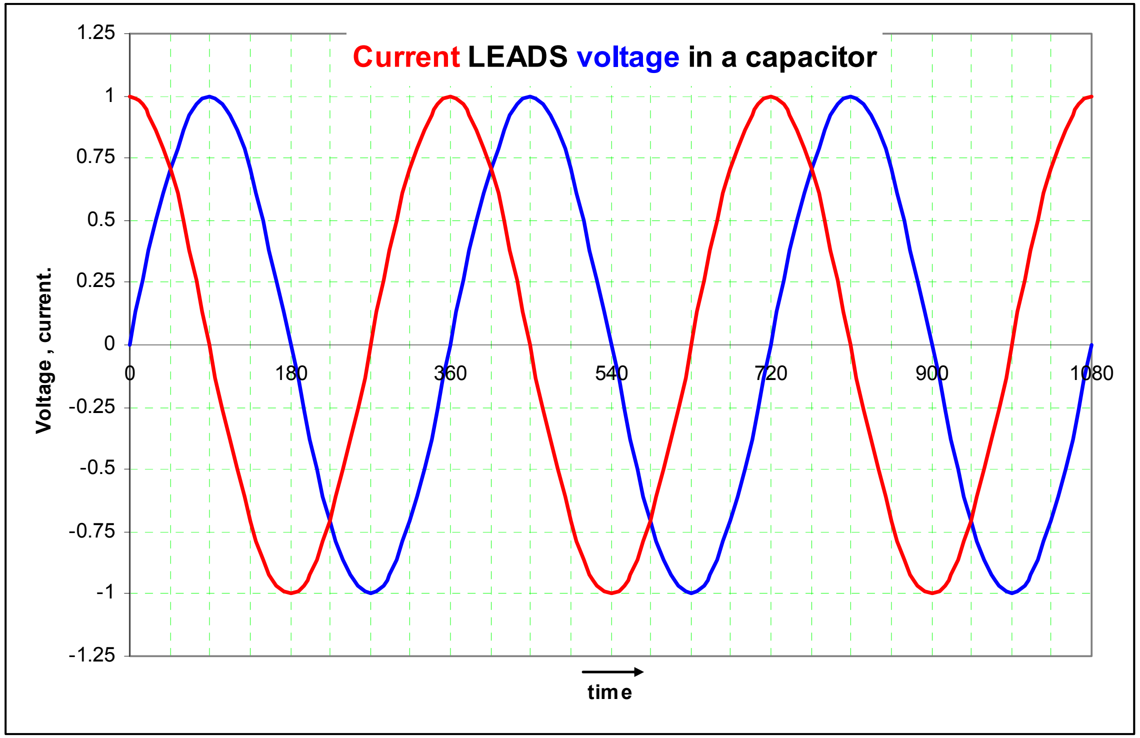




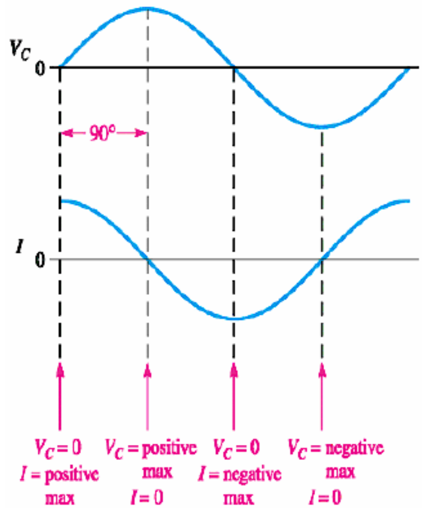
$$v_L = L\frac{di}{dt}$$

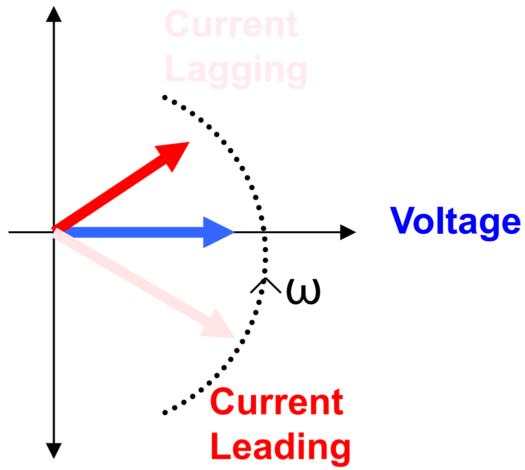
For a pure capacitor, the current in the circuit <u>leads</u> the applied voltage by 90^0 ($\pi/2$)





$$i_C = C\frac{dv}{dt}$$





For the case of the resistor, this circuit component **resists** the flow of current through it. The behaviour is the same regardless of whether the voltage applied is static (DC, as in a car battery) or time-varying (AC, as in mains electricity). Inductors and capacitors **react** against the flow of current through them, and hence they have **reactance**. The behaviour of inductors and capacitors varies with the frequency of the applied voltage.

Capacitive Reactance, X_c

Remember that for a capacitor, $I = C\dfrac{dv}{dt}$

If $v = \hat{V}\sin(\omega t),\ i = C\omega\hat{V}\cos(\omega t)$

If we disregard the phase angle and consider only the magnitude,

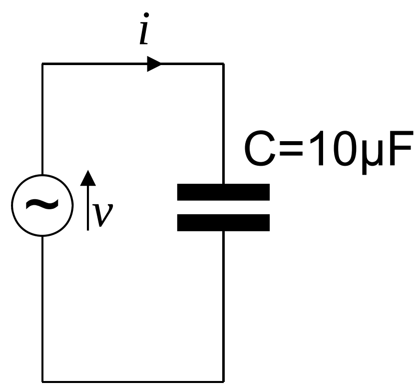
$$i = C\omega\hat{V}, \text{but } \mathrm{Z} = \frac{\mathrm{V}}{\mathrm{I}}, \Rightarrow \mathrm{Z} = \frac{1}{\omega \mathrm{C}}$$

Reactance is given the symbol X, and capacitive reactance is $\boxed{X_c = \dfrac{1}{\omega C}, \text{units} = \text{ohms}\ (\Omega)}$

Example.

A capacitor is placed across the output of a signal generator. The signal generator is adjusted to give an output frequency of 100Hz. What is the capacitive reactance?

$$X_c = \frac{1}{\omega C},\ \omega = 2\pi f \quad \Rightarrow X_c = \frac{1}{2\pi \times 100 \times 10^{-5}} = 159\Omega$$



If the output frequency is increased to 1KHz, what is the new reactance?

$$X_c = \frac{1}{\omega C},\ \omega = 2\pi f \quad \Rightarrow X_c = \frac{1}{2\pi \times 1000 \times 10^{-5}} = 15.9\Omega$$

If the generator's output voltage is set to 5V (rms), what is the current i at 1KHz?

Method 1.

$$Z = \frac{V}{I} \quad \Rightarrow I = \frac{V}{Z} \quad = \frac{5}{15.9} = 314mA$$

Method 2.

$$I = C\frac{dv}{dt} \quad C = 10^{-5}F;\ v = 5\sqrt{2}\sin(2000\pi t)$$

$$\frac{dv}{dt} = 2000\pi \times 5\sqrt{2}$$

$$\hat{i} = 10^{-5} \times 2000\pi \times 5\sqrt{2} = 444mA\,(\text{peak})\text{ or }314mA\,(\text{rms})$$

Inductive Reactance, X_L.

For an inductor, $v_L = L\frac{di}{dt}$

If $i = \hat{I}\sin(\omega t), v_L = \omega L\hat{I}\cos(\omega t)$

Again, if we disregard the phase angle and consider only the magnitude,

$$v_L = \omega L\hat{I}, \quad \text{and } Z = \frac{V}{I} \quad \Rightarrow Z = \frac{v_L}{\hat{I}} = \omega L$$

So the inductive reactance is $\boxed{X_L = \omega L, \quad \text{units} = \text{ohms}\ (\Omega)}$

If we use the integral forms $v_C = \frac{1}{C}\int idt$ and $i_L = \frac{1}{L}\int vdt$ we get the same result:

$$\text{Let } i = \hat{I}\sin(\omega t) \quad v_C = \frac{1}{C}\int \hat{I}\sin(\omega t)dt$$

$$= \frac{\hat{I}}{\omega C}(-\cos(\omega t))$$

$$\Rightarrow \frac{v_C}{\hat{I}} = \frac{1}{\omega C} = X_C$$

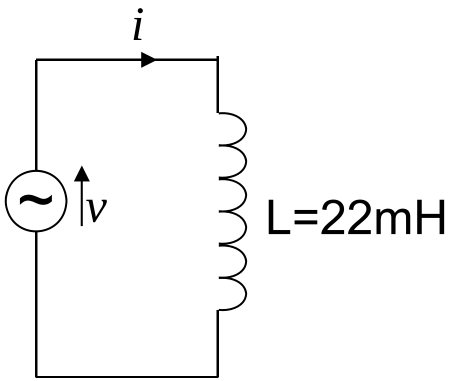
$$\text{Let v} = \hat{V}\sin(\omega t) \quad i_L = \frac{1}{L}\int \hat{V}\sin(\omega t)dt$$

$$= \frac{\hat{V}}{\omega L}(-\cos(\omega t))$$

$$\Rightarrow \frac{\hat{V}}{i_L} = \omega L = X_L$$

Example (contd).

The capacitor is replaced by an inductor of 22mH. The signal generator's voltage is set to 10V (rms) and the frequency set to 5000Hz. What is the current, i ?



Method 1.

$$X_L = \omega L, \quad \omega = 2\pi f \quad \Rightarrow X_L = 2\pi \times 5000 \times 22 \times 10^{-3} = 691\Omega$$

$$Z = \frac{V}{I} \quad \Rightarrow I = \frac{V}{Z} \quad = \frac{10}{691} = 14.47mA$$

Method 2.

$$i_L = \frac{1}{L}\int v dt, \quad \mathrm{L} = 22\mathrm{mH}, \quad \mathrm{v} = 10\sqrt{2}\sin(10000\pi t)$$

$$\hat{i}_L = \frac{10\sqrt{2}}{10000\pi \times 22 \times 10^{-3}} = 0.02046A(\mathrm{peak}) = 14.47mA\,(\mathrm{rms})$$

Both inductors and capacitors behave like frequency-dependent resistors.

$X_L = \omega L.$ As $\omega \to 0$, $X_L \to 0$. In other words, at DC the inductor does not oppose the steady flow of current and behaves as a short circuit. (A real inductor has a small resistor though.)

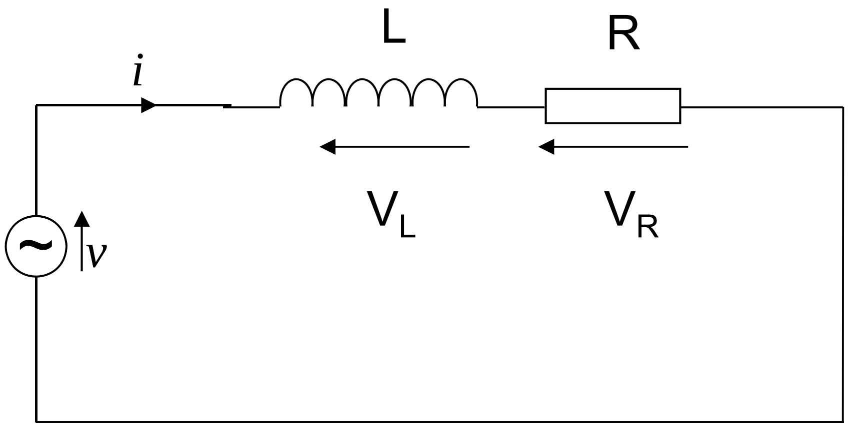
As $\omega \to \infty$, $X_L \to \infty$ In other words, at high frequency the inductor does not allow any current to flow and behaves as an open circuit.

$X_C = \dfrac{1}{\omega C}$. As $\omega \to 0$, $X_C \to \infty$ In other words, at DC the capacitor does not allow any current to flow and behaves as an open circuit.

As $\omega \to \infty$, $X_C \to 0$ In other words, at high frequency the capacitor does not oppose the flow of current and behaves as a short circuit.

RL Series circuit.

A real inductor comprises a copper wire wound around a permeable core. The wire has a resistance R. Hence the “real” inductor is a combination of an ideal resistor and an ideal inductor.



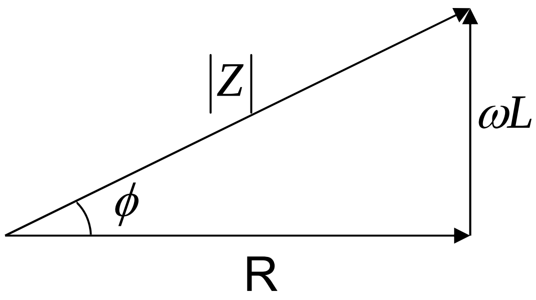
Let $i = \hat{I}\sin(\omega t)$

$(Use\ V = L\frac{di}{dt})$

$$V = V_R + V_L = R\hat{I}\sin(\omega t) + \omega L\hat{I}\cos(\omega t)$$

Remember that for an inductor, the current <u>lags</u>* the voltage by 90^0.

Thus we can draw this as:

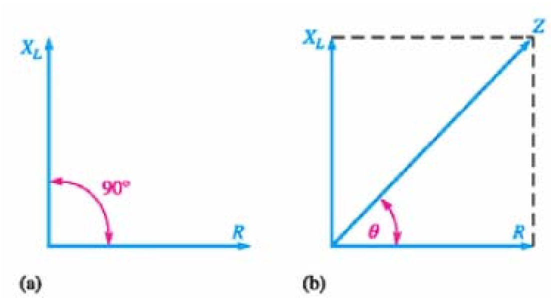


$$|Z| = \sqrt{R^2 + (\omega L)^2}$$

$$\phi = \tan^{-1}\left(\frac{\omega L}{R}\right)$$

* The convention is that a lag corresponds to a positive rotation.

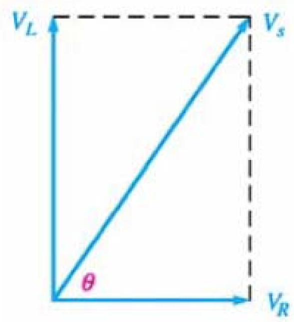
Impedance



$$Z = R + jX_L$$

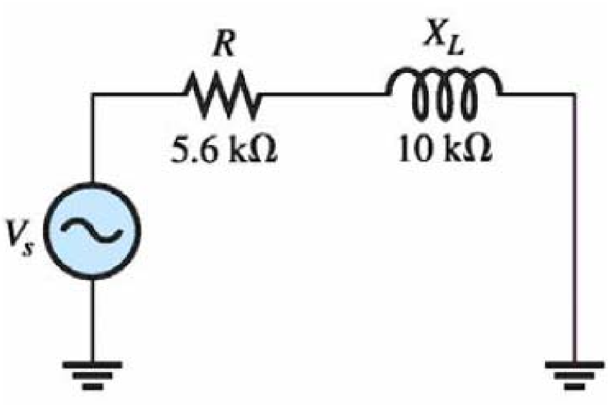
$$Z = \sqrt{R^2 + X_L^2}$$

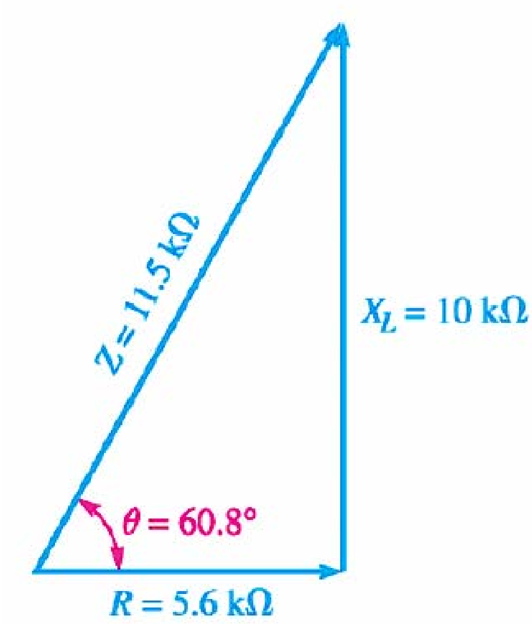
Voltage



$$V_S = V_R + jV_L$$

$$V_S = \sqrt{V_R^2 + V_L^2}$$



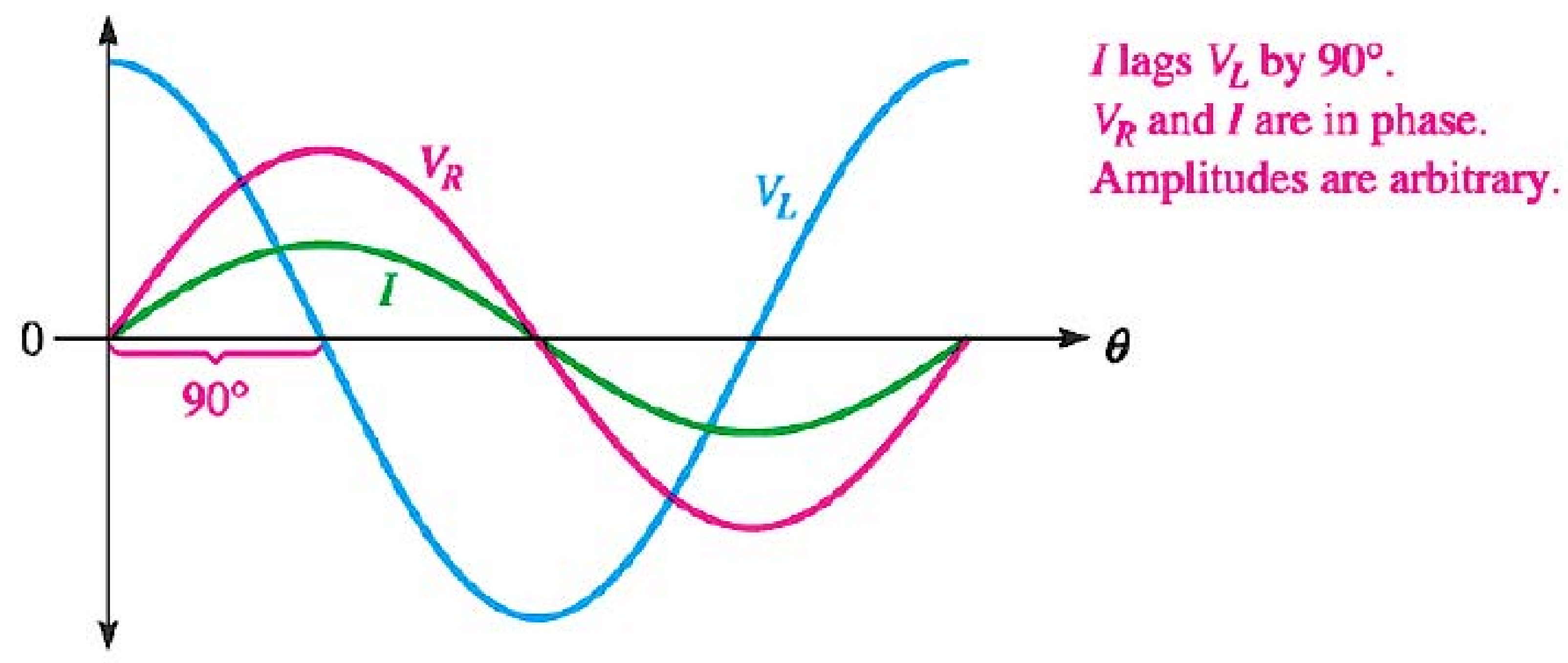


$$|Z| = \sqrt{R^2 + (\omega L)^2}$$

$$\phi = \tan^{-1}\left(\frac{\omega L}{R}\right)$$

$$Z_T = 5600 + j\,10000$$

$$Z_T = 11500 \angle 60.8°$$



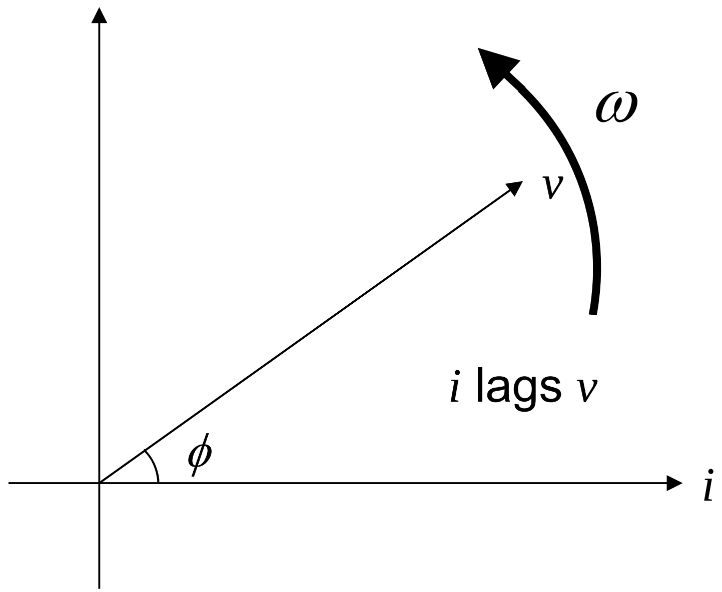
Remember - CIVIL ← L – V leads I

C – I leads V

$$|Z| = \sqrt{R^2 + (\omega L)^2} \qquad\qquad \phi = \tan^{-1}\left(\frac{\omega L}{R}\right)$$

$If\ \ R >> \omega L \text{ then } \dfrac{\omega L}{R} \to 0 \text{ and } \phi \to 0.$ In other words, the current and voltage are in phase; a pure resistance.

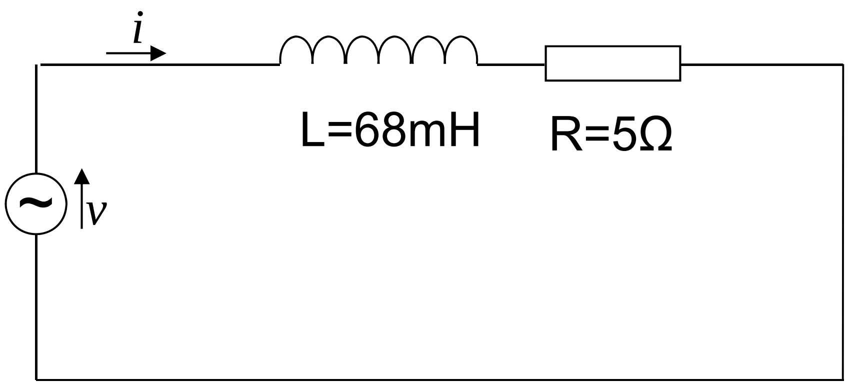
$If\ \ R << \omega L \text{ then } \dfrac{\omega L}{R} \to \infty \text{ and } \phi \to \dfrac{\pi}{2}.$ In other words, the current and voltage are out of phase by $\pi/2$; a pure inductance.



In a series combination of R and L, the current will lag the voltage by some angle ϕ between 0^0 and 90^0, depending on the relative magnitudes of R and ωL.

Note that the resistance is never negative and hence lies in the first quadrant.

Example.



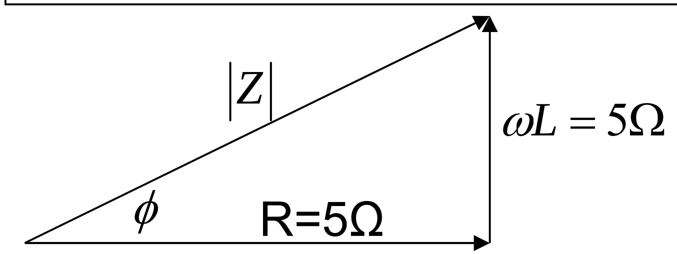
At what frequency does the magnitude of the inductive reactance equal the value of the resistor?

$$X_L = \omega L = 5\Omega$$

$$=> \omega = \frac{5}{L} = \frac{5}{68\times10^{-3}} = 73.5 \text{ radians per second.}$$

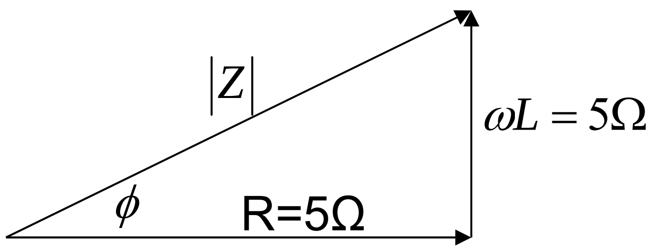
$$\omega = 2\pi f \quad => f = \frac{\omega}{2\pi} = \frac{73.5}{2\pi} = 11.7\text{Hz.}$$

What is the phase angle between the voltage and current at this frequency?



$$\phi = \tan^{-1}\left(\frac{\omega L}{R}\right) = \tan^{-1}(1) = 45^o \; or \; \pi/4.$$

Example (contd).
If the voltage at this frequency has a value of 10V (rms), how much power is dissipated in the resistor and what is the current through it?



$$Z = \frac{V}{I} \quad => I = \frac{V}{Z}$$

$$|Z| = \sqrt{R^2 + (\omega L)^2} = \sqrt{5^2 + 5^2} = 5\sqrt{2}\Omega$$

$$=> I = \frac{10}{5\sqrt{2}} \quad = \sqrt{2} \text{ Amps}$$

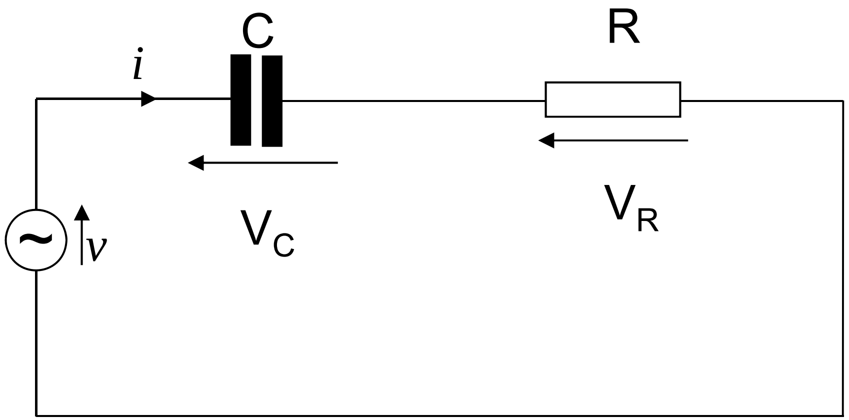
$\text{Power} = I^2 R = \left(\sqrt{2}\right)^2 \times 5 = 10\text{W}$ (Note that no power is dissipated in the inductor.)

If the frequency is reduced to zero and the voltage is kept at 10V (note not RMS!!) what is the power in the resistor?

$\text{Power} = \frac{V^2}{R} = \frac{10^2}{5} = 20\text{W}$ (Inductor goes to zero ohms – a short circuit.)

RC Series circuit.

A real capacitor has wire leads connecting the capacitive element to the outside world. These wires have a resistance R. Hence the "real" capacitor is a combination of an ideal resistor and an ideal capacitor.



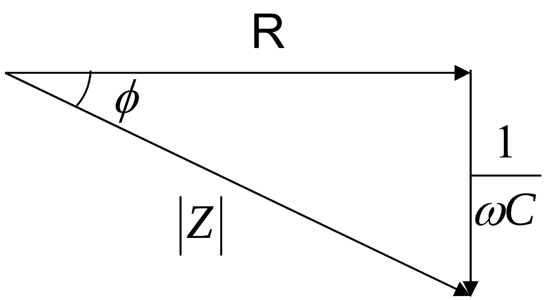
$$\text{Let } i = \hat{I}\cos(\omega t)$$

$$\left(Use\ \ V_C = \frac{1}{C}\int i dt\right)$$

$$V = V_R + V_C = R\hat{I}\cos(\omega t) + \frac{1}{\omega C}\hat{I}\sin(\omega t)$$

Remember that for an capacitor, the current leads* the voltage by 90^0.

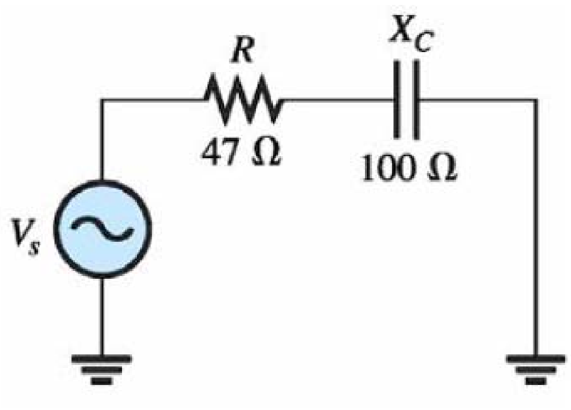
Thus we can draw this as:

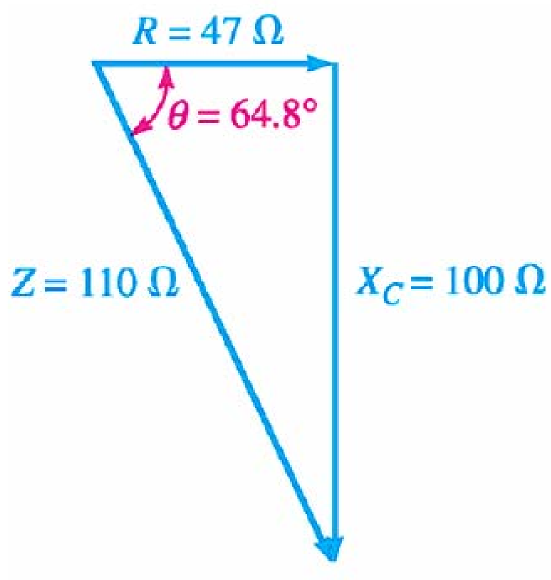


$$|Z| = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}$$

$$\phi = \tan^{-1}\left(\frac{1}{\omega CR}\right)$$

* The convention is that a lead corresponds to a negative rotation.



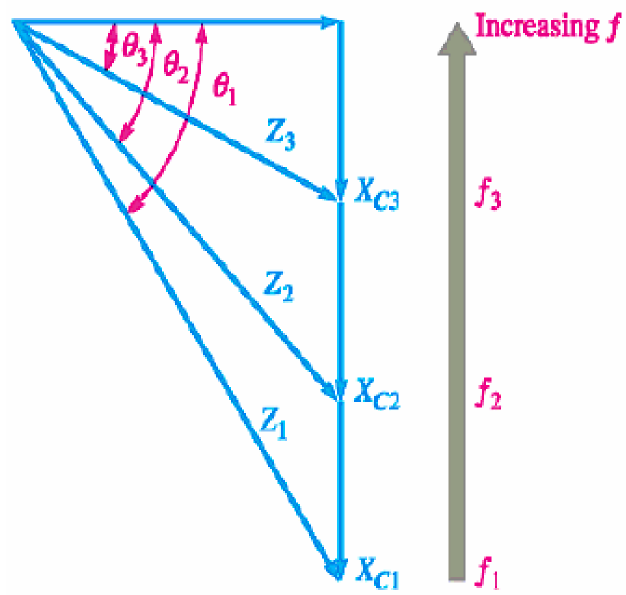


Rectangular
Z = 47 – j100

Polar
Z = 110 ∠ -64.8°

In a series RC circuit; as frequency increases:

- Xc decreases
- Z decreases
- θ decreases
- R remains constant



$$|Z| = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2} \qquad\qquad \phi = \tan^{-1}\left(\frac{1}{\omega CR}\right)$$

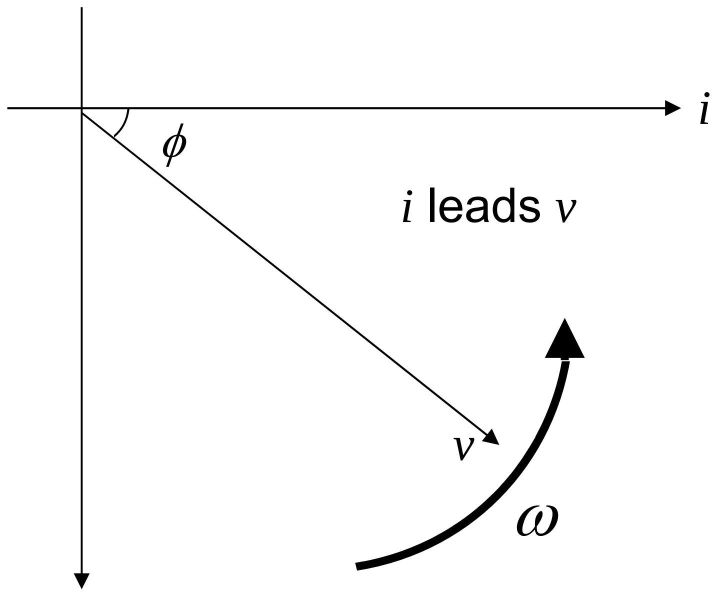
$If\ \ R >> \dfrac{1}{\omega C}$ then $\dfrac{1}{\omega CR} \to 0$ and $\phi \to 0$. In other words, the current and voltage are in phase; a pure resistance.

$If\ \ R << \dfrac{1}{\omega C}$ then $\dfrac{1}{\omega CR} \to \infty$ and $\phi \to \dfrac{\pi}{2}$. In other words, the current and voltage are out of phase by $\pi/2$; a pure capacitance.

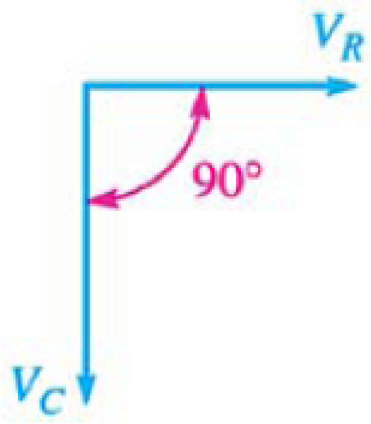
ϕ

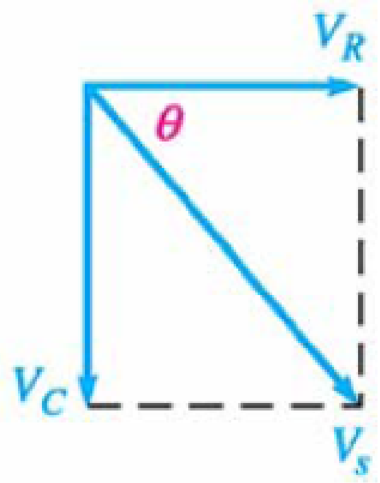
In a series combination of R and C, the current will lead the voltage by some angle
between 0^0 and 90^0, depending on the relative magnitudes of R and $1/\omega C$.

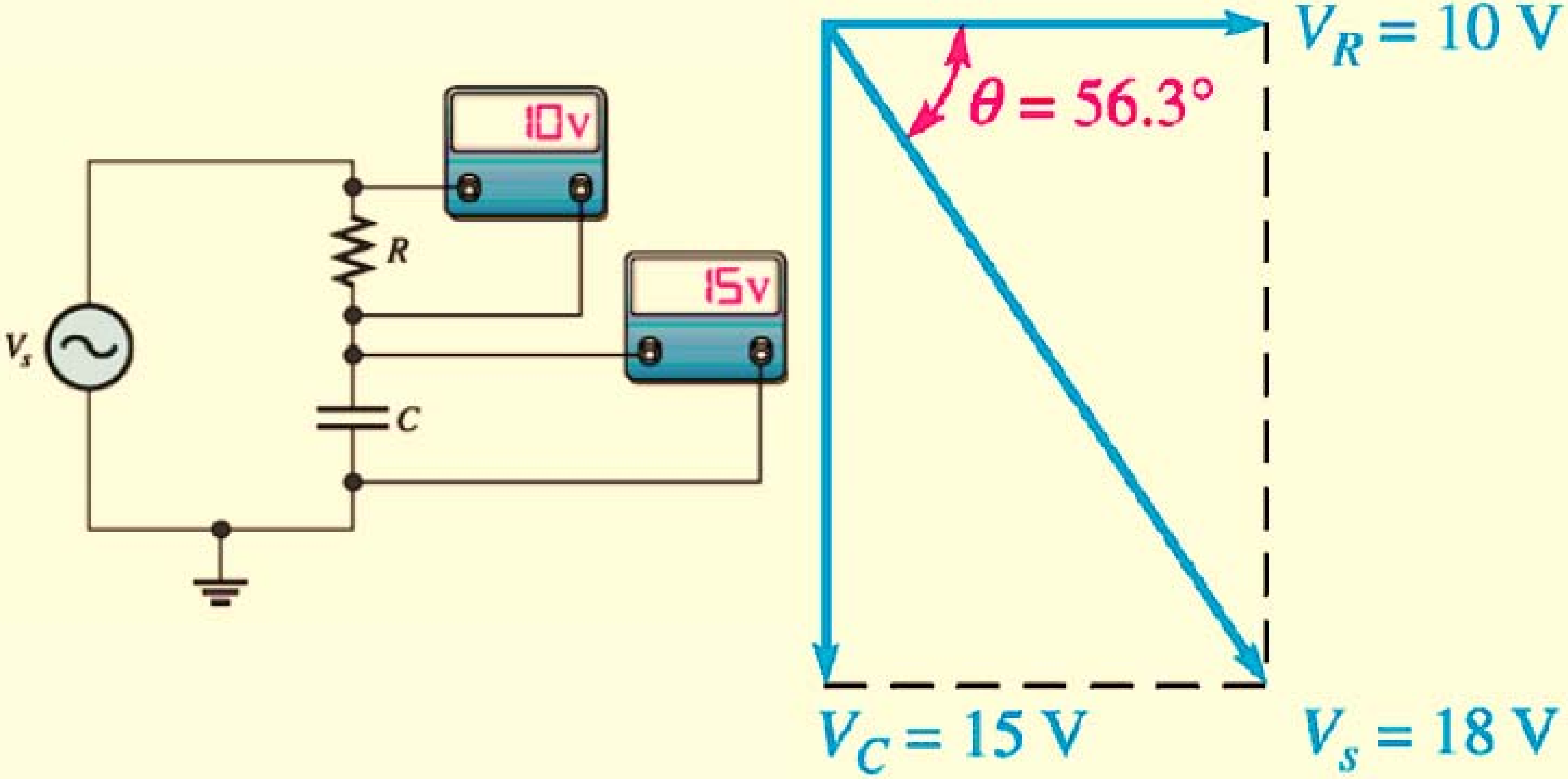
Note that the resistance is never negative and hence   lies in the fourth quadrant.



- From KVL, the sum of the voltage drops must equal the applied voltage (V_S)
- Since V_R and V_C are 90° out of phase with each other, they must be added as phasor quantities
- Magnitude of source voltage: $$V_S = \sqrt{V_R^2 + V_C^2}$$
- Phase angle between V_R and V_S: $$\theta = \tan^{-1}\left(\frac{-V_C}{V_R}\right)$$



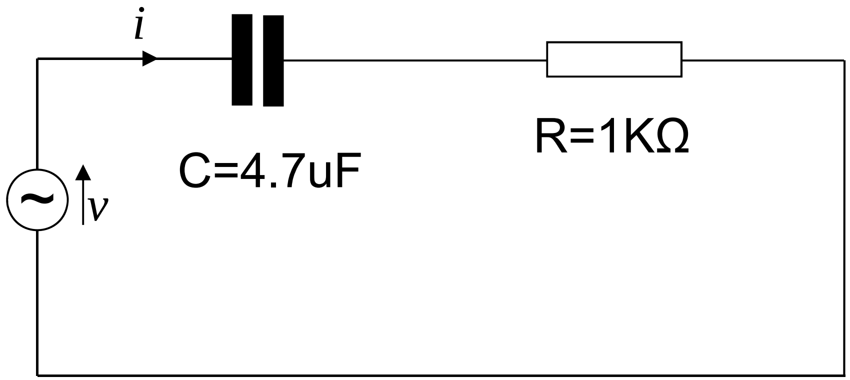




$V_S = 10 - j15$

$V_S = 18 \angle -56.3°$

Example.

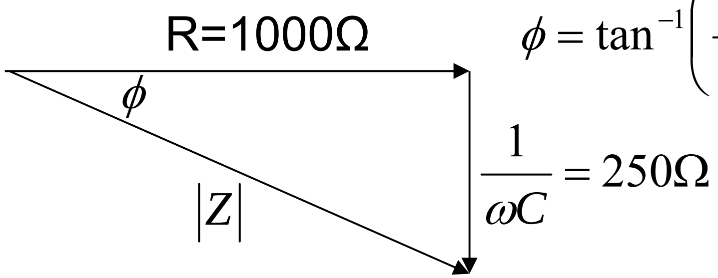


At what frequency does the capacitive reactance equal 250Ω?

$$X_C = \frac{1}{\omega C} = 250\Omega \quad => \omega = \frac{1}{250\times \mathrm{C}} = \frac{1}{250\times 4.7\times 10^{-6}} = 851 \text{ radians per second}$$

$$\omega = 2\pi f \quad => f = \frac{\omega}{2\pi} = \frac{851}{2\pi} = 135.5\mathrm{Hz}$$

What is the phase angle between the voltage and current at this frequency?

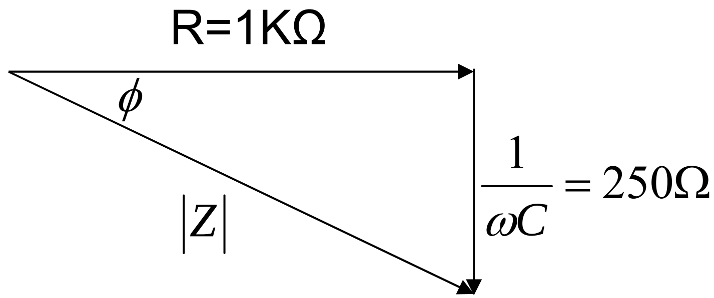


$$\phi = \tan^{-1}\left(\frac{1}{\omega CR}\right) = \tan^{-1}\left(\frac{1}{851\times 4.7\times 10^{-6}\times 1000}\right) = \tan^{-1}(1/4) = 14^{o}.$$

Example (contd).
If the voltage at this frequency has a value of 5V (rms), what is the current in the resistor?

$$|Z| = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2} = \sqrt{1000^2 + 250^2} = 1031\Omega$$

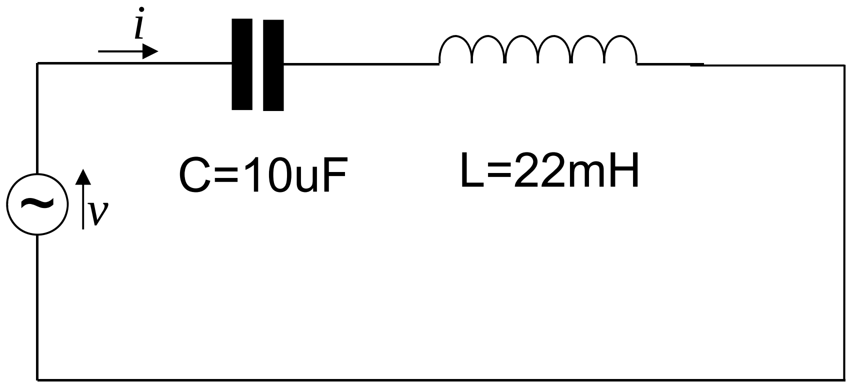
$$Z = \frac{V}{I} \quad => I = \frac{V}{Z} = \frac{5}{1031} = 4.85mA$$



If the frequency is reduced to zero what is the power in the resistor?

As $f \to 0, X_C \to \infty \;=> i \to 0 \;=> P = 0$ watts.

Example.



At what frequency are the capacitive and inductive reactances equal?

What is their value?

$$X_L = \omega L, \quad X_C = \frac{1}{\omega C}$$

When $X_L = X_C$, $\omega L = \dfrac{1}{\omega C}$

$$=> \omega^2 L = \frac{1}{C}$$

$$=> \omega^2 = \frac{1}{LC}$$

$$=> \omega = \frac{1}{\sqrt{LC}}$$

This is called the *natural oscillation frequency*.

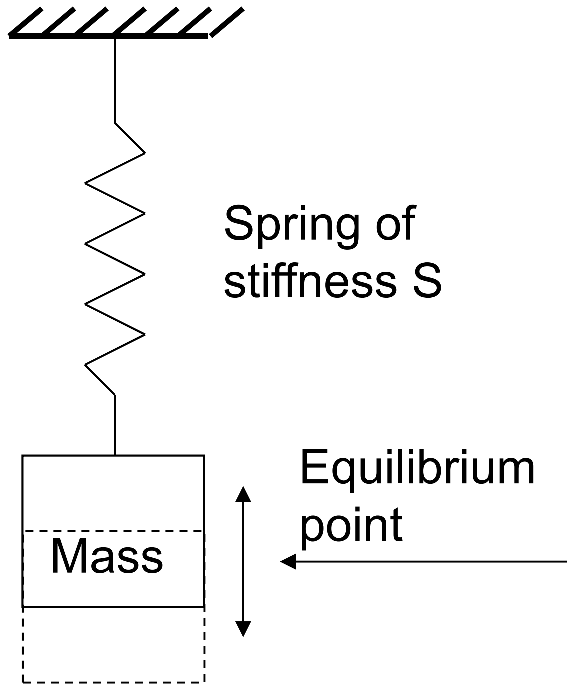
$$= \frac{1}{\sqrt{22\times10^{-3}\times10^{-5}}} = 2132 \text{ radians per second } (339\text{Hz})$$

$$X_L = 2132\times22\times10^{-3} = 46.9\Omega \qquad X_C = \frac{1}{2132\times10^{-5}} = 46.9\Omega$$

Natural frequency of oscillation.

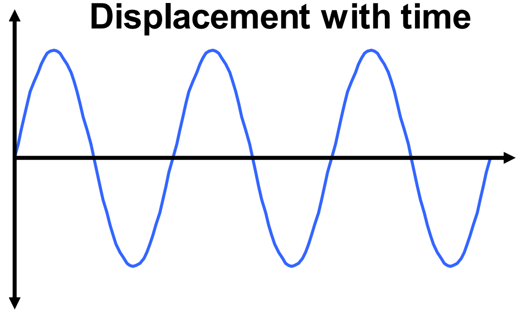
Consider a mass hanging from a spring of stiffness S. If the mass is displaced and then released, it will oscillate about the equilibrium point. When the mass is at its highest position, the energy in the system is stored as potential energy in the mass. When the mass is at its lowest position, the energy is stored in the spring. The frequency at which the mass oscillates is known as its ***natural frequency***, f_0 and is given by:

$$\boxed{f_0 = \frac{1}{2\pi}\sqrt{\frac{S}{M}}}$$



If we put compliance, (C), = 1 / stiffness and $\omega=2\pi f$, we have:

$$\boxed{\omega_0 = \frac{1}{\sqrt{MC}}}$$



The mass's displacement about the equilibrium point varies sinusoidally with time.

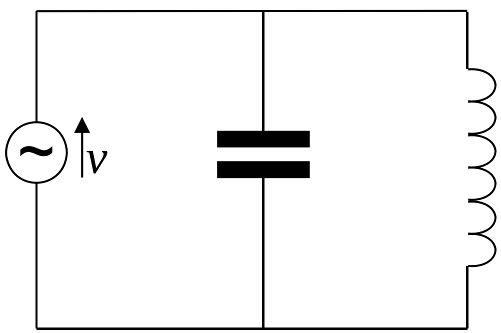
Tuning forks are a variation on the mass/spring system and they are designed to "ring" at a particular frequency.

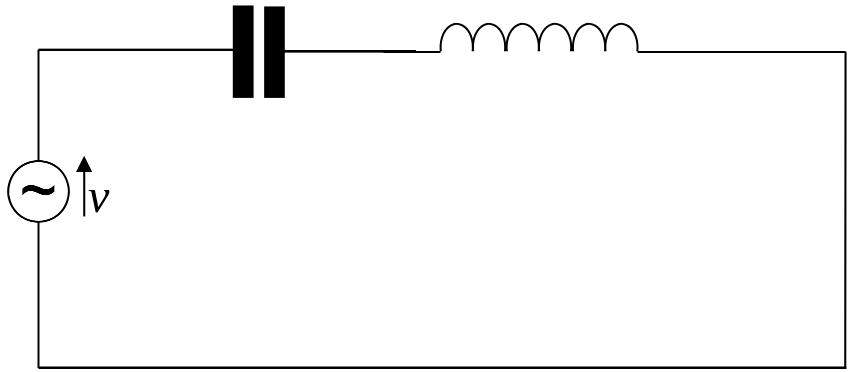
Natural frequency of oscillation of electrical circuits.

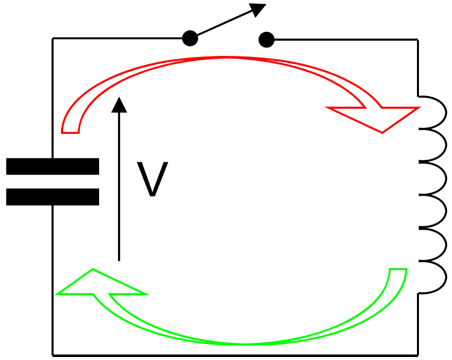
In a circuit with inductors and capacitors, energy will oscillate between the inductor and the capacitor in an analogous way to the mass and spring system. The inductor behaves like the mass, and the capacitor like the spring. For the L-C system, the natural frequency of oscillation is given by:

$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}}, \quad f_0 = \frac{1}{2\pi\sqrt{LC}}}$$

This is true whether the inductor and capacitor are connected in parallel or series.







Consider the parallel circuit shown. Let the capacitor be charged up to some voltage V. The energy in the capacitor is $E = \frac{1}{2}CV^2$ (Stretched spring)

When the switch is closed, current starts to flow (mass starts to rise) in the inductor and the energy transferred from the capacitor is stored in the inductor's magnetic field. (Mass's momentum.)

Eventually all the energy from the capacitor is transferred to the inductor's magnetic field (mass at maximum velocity).
The inductor's stored energy is $E = \frac{1}{2}LI^2$

The inductor now tries to maintain the flow of current through it, and the voltage on the capacitor starts to become increasingly negative (mass starts to slow down). The energy building in the capacitor is drawn from the magnetic field until all the energy has been transferred back to the capacitor (mass has highest P.E., zero momentum).

The capacitor's voltage now starts to drive up the current in the inductor again as energy flows from the capacitor to the inductor's magnetic field (mass starts to drop), until the original starting conditions are returned (stretched spring, mass at lowest point).

The angular frequency at which this oscillation takes place is $\boxed{\omega_0 = \frac{1}{\sqrt{LC}}, \quad f_0 = \frac{1}{2\pi\sqrt{LC}}}$

The voltage across the capacitor (or inductor) varies sinusoidally with time.