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Exotic Dynamics of Nonholonomic Roller Racer with Periodic Control

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Abstract—In this paper we consider the problem of the motion of the *Roller Racer*. We assume that the angle $\varphi(t)$ between the platforms is a prescribed function of time. We prove that in this case the acceleration of the *Roller Racer* is unbounded.
In this case, as the *Roller Racer* accelerates, the increase in the constraint reaction forces is also unbounded. Physically this means that, from a certain instant onward, the conditions of the rolling motion of the wheels without slipping are violated. Thus, we consider a model in which, in addition to the nonholonomic constraints, viscous friction force acts at the points of contact of the wheels. For this case we prove that there is no constant acceleration and all trajectories of the reduced system asymptotically tend to a periodic solution.

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1. INTRODUCTION

1. We describe some well-known models of a *wheeled vehicle* on a plane. We recall that if there is no slipping at the points of contact of the wheels with the support, then each wheel pair can be replaced with a knife edge which is situated at its center of mass and prevents sliding in the direction transverse to the plane of the wheels [8].

For this reason the vehicle with two fastened wheel pairs is equivalent to the *Chaplygin sleigh* problem [14] (a platform with a fastened knife edge and two smooth legs). The three-dimensional configuration space of the Chaplygin sleigh can be parameterized by the angle of rotation of the platform and the position of the point of contact of the sleigh on the plane. As a result, the problem of the inertial motion reduces to investigating a two-dimensional reduced system preserving the energy integral [5, 10, 15].

Another variation of the wheeled vehicle is the *Roller Racer*, a system consisting of two platforms (connected to each other by means of a cylindrical joint) with rigidly attached wheel pairs. We note that the name *Roller Racer* alludes to its similarity to a toy vehicle for children of the same name (see Fig. 1). We recall that the child sitting in it moves forward by oscillating the handlebars from side to side.

In contrast to the *Chaplygin sleigh*, this system possesses an additional configuration variable φ — the relative angle of rotation of the vehicle. The problem of the inertial motion of the *Roller Racer* reduces to investigating a three-dimensional reduced system.

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Fig. 1. *Roller Racer.*

A large number of parameters appear in the equations of motion of the *Roller Racer*, and so, as a rule, this system is considered under some restrictions on the position of the center of mass of both platforms [5, 19, 23]. In the most general case, the dynamics of the inertial motion of the *Roller Racer* is dealt with in [1].

Other models of a wheeled vehicle with two wheel pairs include the *snakeboard* [9, 11, 12, 18, 21], in which both wheel pairs can freely rotate. The dynamics of a wheeled vehicle consisting of three and more platforms is discussed in [13] and is still poorly understood. The dynamics of a vehicle with omniwheels is still poorly understood [7].

The *Chaplygin sleigh* and the *Roller Racer* are the most popular examples for testing various mathematical constructions in nonholonomic mechanics (see, e. g., [6]).

In both cases, when a known solution of the reduced system is used, the analysis of motion in absolute space reduces to investigating quadratures. Both systems possess limiting (as $t \to \pm\infty$) regimes of motion, specifically, the motion of the *Chaplygin sleigh* asymptotically tends to be rectilinear [10], and the motion of the *Roller Racer* tends, depending on the parameters, to rectilinear motion or motion in a circle [1].

2. This paper addresses the problem of the motion of the *Roller Racer* in which the angle $\varphi(t)$ between the platforms is a given periodic function of time. Equations of motion are obtained for a fairly general mass distribution of the platforms. A particular case of this system (for which the mass of one of the platforms is assumed to be negligibly small) is treated in [16]. The trajectories constructed numerically in this paper demonstrate a growth of the velocity of the *Roller Racer* at the initial stage.

We prove that an unbounded speed-up of the *Roller Racer* is observed in the general case for all trajectories both as $t \to +\infty$ and as $t \to -\infty$. We note that such behavior does not depend on the type of control $\varphi(t)$.

Another feature of the problem we consider here is an unbounded increase in the reaction forces of constraints as $t \to \pm\infty$. Physically this means that, from a certain instant onward, the condition of rolling of the wheels without slipping is violated. Therefore, attention is also given to a model in which, in addition to the nonholonomic constraints, the force of viscous friction acts at the point of contact of the wheels [16, 22]. It is proved that in this case there is no constant acceleration and all trajectories of the reduced system asymptotically tend to a periodic solution.

Numerical experiments show that the motion of the *Roller Racer* relative to the fixed coordinate system is complex (in contrast to the inertial motion) and can be quite different depending on the system parameters.

A special feature of the problem treated in this paper is that the reduced system reduces to one nonautonomous linear equation. For this reason, it is fairly simple to prove the presence of speeding-up trajectories. For example, for the Chaplygin sleigh with a given periodic action the proof of the presence of speeding-up trajectories is a separate problem [2–4].

We emphasize that in this paper we use a kinematic control of the system, which implies that one of the configuration variables of the system changes in a prescribed manner [17, 20]. In this case, we do not take into account the changes in the dynamical characteristics (mass distribution and gyrostatic torque) of the system which arise in real situations where the specified periodic change in the configuration variable is achieved.

The dynamics of the *Roller Racer* in which “dynamical” control is implemented by using external torque (gyrostatic torque) or by using the prescribed time dependence of the mass distribution has not been investigated.

2. EQUATIONS OF MOTION

Consider the problem of the motion of the simplest wheeled vehicle, the *Roller Racer*, on a plane. The *Roller Racer* consists of two coupled platforms which can freely rotate in a horizontal plane independently of each other (see Fig. 2). Each platform has a rigidly attached wheel pair consisting of two wheels lying on the same axis.

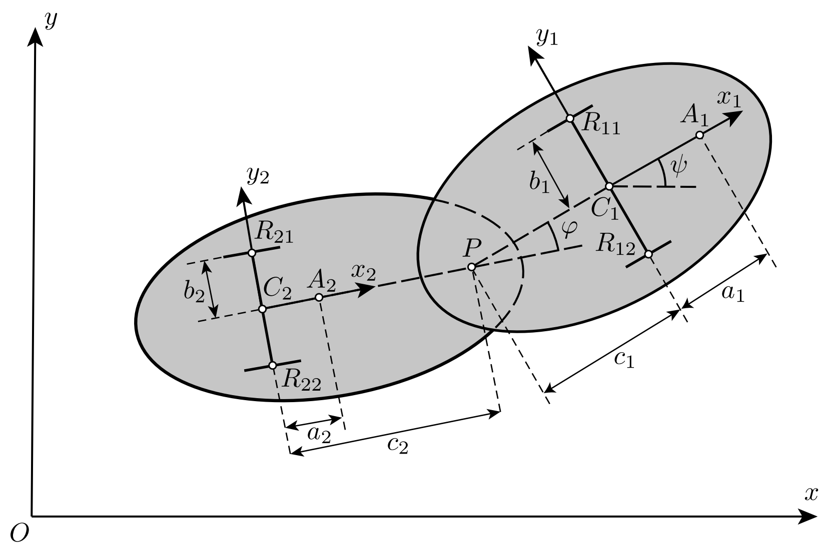


Fig. 2. *Roller Racer* on a plane.

We define three coordinate systems:

— *an inertial* coordinate system Oxy;

— *a noninertial* coordinate system $C_1x_1y_1$, attached to the *first platform*, with origin C_1 at the center of mass of the wheel pair. We assume that the axes C_1x_1 and C_1y_1 are directed, respectively, along a tangent and a normal to the plane of the wheels;

— *a noninertial* coordinate system $C_2x_2y_2$, attached to the *second platform*, with origin C_2 at the center of mass of the wheel pair. Again, we assume that the axes are directed along a tangent and a normal to the plane of the wheels.

The orientation of the platforms relative to each other is defined by the angle $\varphi \in [0, 2\pi)$ between the axes C_1x_1 and C_2x_2. We will assume that the angle $\varphi(t)$ is a prescribed periodic function of time (with period T).

Configuration variables. We now introduce generalized coordinates that define the position of each platform. Let $\boldsymbol{r} = (x, y)$ be the radius vector of the attachment point of the platforms P in the fixed coordinate system Oxy. We specify the orientation of the first platform by the angle ψ between the axes Ox and C_1x_1.

Thus, the configuration space $\mathcal{N}_{\boldsymbol{q}}$ is

$$\mathcal{N}_{\boldsymbol{q}} = \{\boldsymbol{q} = (x, y, \psi), \psi \bmod 2\pi\} \approx \mathbb{R}^2 \times \mathbb{S}^1.$$

Nonholonomic constraints. There is no slipping at the points R_{ij} of contact of the wheels with the plane, therefore, the following relations hold:

— for *the first* wheel pair:

$$\dot{\boldsymbol{r}} + \dot{\psi}(c_1\boldsymbol{n}_1 - b_1\boldsymbol{\tau}_1) + h_1\dot{\phi}_{11}\boldsymbol{\tau}_1 = 0,$$
$$\dot{\boldsymbol{r}} + \dot{\psi}(c_1\boldsymbol{n}_1 + b_1\boldsymbol{\tau}_1) + h_1\dot{\phi}_{12}\boldsymbol{\tau}_1 = 0,$$
$$\boldsymbol{\tau}_1 = \big(\cos\psi, \sin\psi\big), \quad \boldsymbol{n}_1 = \big(-\sin\psi, \cos\psi\big),$$

— for *the second* wheel pair:

$$\dot{\boldsymbol{r}} + (\dot{\varphi}(t) + \dot{\psi})(c_2\boldsymbol{n}_2 - b_2\boldsymbol{\tau}_2) + h_2\dot{\phi}_{21}\boldsymbol{\tau}_2 = 0,$$
$$\dot{\boldsymbol{r}} + (\dot{\varphi}(t) + \dot{\psi})(c_2\boldsymbol{n}_2 + b_2\boldsymbol{\tau}_2) + h_2\dot{\phi}_{22}\boldsymbol{\tau}_2 = 0,$$
$$\boldsymbol{\tau}_2 = \big(\cos(\varphi(t) + \psi), \sin(\varphi(t) + \psi)\big),$$
$$\boldsymbol{n}_2 = \big(-\sin(\varphi(t) + \psi), \cos(\varphi(t) + \psi)\big),$$

where ϕ_{ij} are the angles of rotation of the wheels, h_1 and h_2 are the radii of the wheels on the first and the second platform, b_1 and b_2 are the distances from the point of contact of the wheel to the center of mass of the corresponding wheel pair, and c_1 and c_2 are, respectively, the distances from the center of each wheel pair to the attachment point of the platforms (see Fig. 2).

Multiplying the equations for the first and the second wheel pairs by $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$, respectively, we obtain relations without the angles of rotation of the wheels:

$$\begin{gathered} -\dot{x}\sin\psi + \dot{y}\cos\psi + c_1\dot{\psi} = 0, \\ -\dot{x}\sin(\varphi(t) + \psi) + \dot{y}\cos(\varphi(t) + \psi) - c_2(\dot{\varphi}(t) + \dot{\psi}) = 0. \end{gathered} \tag{2.1}$$

These relations mean that the projection of the velocity of the center of mass of each wheel pair C_1 and C_2 onto the direction perpendicular to the plane of the corresponding wheel is equal to zero.

The angles of rotation of the wheels are defined in terms of quadratures by means of $x(t)$, $y(t)$, $\psi(t)$ [8]:

$$\begin{gathered} h_1\dot{\phi}_{11} = b_1\dot{\psi} - \dot{x}\cos\psi - \dot{y}\sin\psi, \quad h_1\dot{\phi}_{12} = -b_1\dot{\psi} - \dot{x}\cos\psi - \dot{y}\sin\psi, \\ h_2\dot{\phi}_{21} = b_2(\dot{\varphi}(t) + \dot{\psi}) - \dot{x}\cos(\varphi(t) + \psi) - \dot{y}\sin(\varphi(t) + \psi), \\ h_2\dot{\phi}_{22} = -b_2(\dot{\varphi}(t) + \dot{\psi}) - \dot{x}\cos(\varphi(t) + \psi) - \dot{y}\sin(\varphi(t) + \psi). \end{gathered} \tag{2.2}$$

These equations are obtained after multiplication of the constraints for the first and the second wheel pair by $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$, respectively.

As was shown in [8], we can ignore degrees of freedom describing the angles of rotation of the wheels when we consider the dynamics of wheeled platforms. Therefore, we will replace each wheel pair with a massless knife edge placed at the center of mass of the pair and governed by the constraint (2.1).

Kinetic energy. We will assume that the centers of mass of the first and second platforms (points A_1 and A_2) lie on the axes C_1x_1 and C_2x_2, respectively (i. e., the center of mass of each platform lies in the plane of the knife edge). The kinetic energy of the system can be represented as

$$\begin{gathered} T(\dot{\boldsymbol{q}}, \boldsymbol{q}) = \frac{1}{2}(m_1 + m_2)(\dot{x}^2 + \dot{y}^2) - m_1(a_1 + c_1)(\dot{x}\sin\psi - \dot{y}\cos\psi)\dot{\psi} \\ -m_2\,(a_2 - c_2)\,(\dot{x}\sin(\varphi(t) + \psi) - \dot{y}\cos(\varphi(t) + \psi))\,(\varphi(t) + \dot{\psi}) \\ +\frac{1}{2}I_1\dot{\psi}^2 + \frac{1}{2}I_2(\dot{\varphi}(t) + \dot{\psi})^2, \end{gathered}$$

where m_1, I_1 and m_2, I_2 are, respectively, the mass and the moment of inertia relative to the point P of the first and the second platform, a_1 is the distance from point C_1 to the center of mass of the first platform, and a_2 is the distance from point C_2 to the center of mass of the second platform.

The Rayleigh dissipation function. To analyze the influence of frictional forces on the dynamics of the *Roller Racer*, we consider the simplest model of viscous friction [16]. Let viscous friction forces act at the points of contact of the wheels with the plane with the Rayleigh function:

$$R = \frac{1}{2}\big(\kappa_1\dot\phi_{11}^2 + \kappa_1\dot\phi_{12}^2 + \kappa_2\dot\phi_{21}^2 + \kappa_2\dot\phi_{22}^2\big),$$

where κ_1 and κ_2 are the friction coefficients. Using relations (2.2) we represent the Rayleigh function in the form

$$R = k_1\big(b_1^2\dot\psi^2 + (\dot x\cos\psi + \dot y\sin\psi)^2\big) + k_2 b_2^2(\dot\varphi(t)+\dot\psi)^2$$
$$+k_2\big(\dot x\cos(\varphi(t)+\psi) + \dot y\sin(\varphi(t)+\psi)\big)^2, \quad k_1 = \frac{\kappa_1}{h_1^2}, \quad k_2 = \frac{\kappa_2}{h_2^2}.$$

Quasi-velocities and equations of motion. In this case, instead of the generalized velocities $\dot{\boldsymbol q} = (\dot x, \dot y, \dot\psi)$ it is more convenient to parameterize the tangent space $T\mathcal{N}_{\boldsymbol q}$ using the quasi-coordinates $\boldsymbol v = (v_1, v_2)$ and ω, where $\boldsymbol v$ is the velocity of point P referred to the axes of the moving coordinate system $C_1x_1y_1$, and ω is the absolute angular velocity of each of the platforms:

$$v_1 = \dot x\cos\psi + \dot y\sin\psi, \quad v_2 = -\dot x\sin\psi + \dot y\cos\psi, \quad \omega = \dot\psi.$$

The equations of motion in quasi-coordinates with undetermined multipliers can be represented in the following general form [10]:

$$\begin{gathered}
\frac{d}{dt}\left(\frac{\partial\widetilde T}{\partial v_1}\right) - \omega\frac{\partial\widetilde T}{\partial v_2} = \sum_{i=1}^{2}\lambda_i\frac{\partial f_i}{\partial v_1} - \frac{\partial\widetilde R}{\partial v_1},\\
\frac{d}{dt}\left(\frac{\partial\widetilde T}{\partial v_2}\right) + \omega\frac{\partial\widetilde T}{\partial v_1} = \sum_{i=1}^{2}\lambda_i\frac{\partial f_i}{\partial v_2} - \frac{\partial\widetilde R}{\partial v_2},\\
\frac{d}{dt}\left(\frac{\partial\widetilde T}{\partial \omega_1}\right) + v_1\frac{\partial\widetilde T}{\partial v_2} - v_2\frac{\partial\widetilde T}{\partial v_1} = \sum_{i=1}^{2}\lambda_i\frac{\partial f_i}{\partial \omega_1} - \frac{\partial\widetilde R}{\partial \omega},
\end{gathered} \tag{2.3}$$

where λ_1 and λ_2 are the undetermined multipliers, $\widetilde T = T\big(\boldsymbol q, \dot{\boldsymbol q}(\boldsymbol v,\omega)\big)$, $\widetilde R = R\big(\boldsymbol q, \dot{\boldsymbol q}(\boldsymbol v,\omega)\big)$, and f_1 and f_2 are the constraint equations in the new variables

$$f_1 = v_2 + c_1\omega_1 = 0, \quad f_2 = v_1\sin\varphi(t) - v_2\cos\varphi(t) + c_2\omega_2 = 0. \tag{2.4}$$

For the kinetic energy and the Rayleigh function we obtain

$$\widetilde T = \frac{1}{2}M(v_1^2+v_2^2) + m_1(a_1+c_1)v_2\omega + \frac{1}{2}I_1\omega^2$$
$$+m_2(a_2-c_2)(v_2\cos\varphi - v_1\sin\varphi)\big(\dot\varphi(t)+\omega\big) + \frac{1}{2}I_2\big(\dot\varphi(t)+\omega\big)^2,$$
$$\widetilde R = k_1(b_1^2\omega^2 + v_1^2) + k_2\big(b_2^2(\dot\varphi(t)+\omega)^2 + (v_1\cos\varphi + v_2\sin\varphi)^2\big)$$

where $M = m_1 + m_2$ is the mass of the whole system.

Let us solve the constraints (2.4) for the velocities ω and v_2. Then from the last two equations of the system (2.3) we find the undetermined multipliers as functions of the generalized coordinates $\boldsymbol q$ and velocity v_1:

$$\lambda_i = \Lambda_i^{(2)}(\varphi)v_1^2 + \Lambda_i^{(1)}(\varphi,\dot\varphi)v_1 + \Lambda_i^{(0)}(\varphi,\dot\varphi), \quad i = 1, 2, \tag{2.5}$$

where the functions $\Lambda_i^{(0)}(\varphi,\dot\varphi)$, $\Lambda_i^{(1)}(\varphi,\dot\varphi)$, and $\Lambda_i^{(2)}(\varphi)$ have no singularities in φ and $\dot\varphi$. Explicit expressions for these functions are rather cumbersome, but they can be obtained using any system of analytical computations, for example, Maple or Mathematica.

Reduced system. As a result, we obtain an equation describing the evolution of the linear velocity of the first platform v_1 in the form

$$\begin{gathered}\dot{v}_1=\big(A(t)-C(t)\big)v_1-B(t)+D(t),\\ A(t)=-\frac{\dot{\varphi}\sin\varphi(J_1S_2+\delta S_1)}{S_1(J_1\sin^2\varphi+MS_1^2)},\\ B(t)=\frac{\ddot{\varphi}\sin\varphi S_1(J_1c_2-J_2S_1)+\dot{\varphi}^2(J_1c_1c_2\sin^2\varphi-S_1(c_1\delta\cos\varphi+\varepsilon c_2S_1))}{S_1(J_1\sin^2\varphi+MS_1^2)}\\ C(t)=2\frac{(b_1^2k_1+k_2b_2^2)\sin^2\varphi+k_1S_1^2+k_2S_2^2}{J_1\sin^2\varphi+MS_1^2},\\ D(t)=2\frac{\dot{\varphi}\sin\varphi\big(c_1k_2(b_2^2-c_2^2)\cos\varphi-c_2(b_1^2k_1+k_2c_1^2)\big)}{J_1\sin^2\varphi+MS_1^2},\\ S_1=c_1\cos\varphi+c_2,\quad S_2=c_1+c_2\cos\varphi,\end{gathered}\tag{2.6}$$

where new variables are introduced:

$$\begin{gathered}J_1=I_1+I_2+m_1a_1^2+m_2(a_2^2+c_1^2-c_2^2),\quad J_2=I_2+m_2(a_2^2-c_2^2),\\ \delta=c_1a_2m_2-c_2a_1m_1,\quad \varepsilon=m_1a_1-c_1m_2.\end{gathered}$$

Reconstruction of trajectories. The evolution of the configuration variables is described by the system

$$\begin{gathered}\dot{\psi}=-\frac{v_1\sin\varphi(t)+c_2\dot{\varphi}(t)}{c_1\cos\varphi(t)+c_2},\\ \dot{x}=v_1\cos\psi-c_1\frac{v_1\sin\varphi(t)+c_2\dot{\varphi}(t)}{c_1\cos\varphi(t)+c_2}\sin\psi,\\ \dot{y}=v_1\sin\psi+c_1\frac{v_1\sin\varphi(t)+c_2\dot{\varphi}(t)}{c_1\cos\varphi(t)+c_2}\cos\psi.\end{gathered}\tag{2.7}$$

Thus, the systems (2.6) and (2.7) describe the motion of the *Roller Racer* in the case where the angle $\varphi(t)$ between the platforms is a prescribed function of time. We can see that in this case the nonautonomous equation describing the evolution of the velocity v_1 decouples from the system of equations of motion. In examples below we will use the control $\varphi(t)$ in the following form

$$\varphi(t)=\alpha\sin\Omega t+\beta.$$

3. NONHOLONOMIC MODEL. ACCELERATION

Consider the case when there is no friction force, i. e., when $C=0$ and $D=0$. The general solution of the linear equation (2.6) has the form

$$\begin{gathered}v_1(t)=\frac{c}{P(t)}-\frac{1}{P(t)}\int\limits_0^t B(\tau)P(\tau)d\tau,\qquad c=\mathrm{const},\\ P(t)=e^{-\int_0^t A\left(\varphi(\tau)\right)d\tau}.\end{gathered}\tag{3.1}$$

An important feature of this case is that for any (smooth) control $\varphi(t)$ the coefficient of the term with v_1 on the right-hand side of Eq. (2.6) has zero average value (zero average divergence):

$$\langle A\rangle=\frac{1}{T}\int\limits_0^T A(\varphi(t))dt=0.$$

Indeed, according to (2.6), we have $A = \dot{\varphi}\overline{A}(\varphi)$, where $\overline{A}$ depends only on φ. Let $P_A(\varphi)$ denote an antiderivative of the function $\overline{A}(\varphi)$ (that is, $P'_A(\varphi) = \overline{A}(\varphi)$). Then using $\varphi(t) = \varphi(t+T)$, we obtain

$$\langle A\rangle = \frac{1}{T}\big(P_A(\varphi(0)) - P_A(\varphi(T))\big) = 0.$$

Consequently, we can conclude that $P(t)$ is a periodic

$$P(t) = P(t+T)$$

and bounded function. In explicit form this function can be represented as

$$P(t) = \frac{\big(J_1\sin^2\varphi(t) + M(c_1\cos\varphi(t) + c_2)^2\big)^{\frac{1}{2}}}{c_1\cos\varphi(t) + c_2} e^{\delta\Delta\left(\varphi(t)\right)},$$

$$\Delta\big(\varphi(t)\big) = -\frac{1}{\sqrt{J_1M(c_1^2 - c_2^2) - J_1^2}}\arctan\left(\frac{2(Mc_1^2 - J_1)\cos\varphi(t) + 2Mc_1c_2}{2\sqrt{J_1M(c_1^2 - c_2^2) - J_1^2}}\right).$$

In (3.1) under the integral there is a function periodic in time. As is well known, in this case an integral can be represented as a sum of linear and periodic functions:

$$\int_0^t B(\tau)P(\tau)d\tau = kt + f(t),$$

$$k = \langle BP\rangle = \frac{1}{T}\int_0^T B(\tau)P(\tau)d\tau, \quad f(t) = f(t+T).$$

Finally, we find that the solution of Eq. (2.6) without friction can be represented as

$$v_1(t) = \frac{k}{P(t)}t + f(t), \tag{3.2}$$

where f, P are the T-periodic functions and $k = \text{const}$. So, we can conclude the following:

Proposition 1. *Without friction forces the absolute value of the linear velocity of the Roller Racer increases indefinitely* (*linearly*) *in time both as $t \to +\infty$ and as $t \to -\infty$.*

As follows from (2.5), the undetermined multipliers λ_1 and λ_2 depend polynomially on the velocity v_1. Consequently, when the system accelerates, constraint reaction forces are unbounded functions of time (see Fig. 3b). Physically, this means that at a certain instant of time slipping will start (i. e., the constraints (2.4) will be violated). However, in nonholonomic mechanics there exists an example — the Chaplygin sleigh with a rotor [4] — for which the reaction force remains a bounded function for constant speed-up.

Using (3.2), we rewrite the absolute angular velocity of the first platform in the form

$$\omega(t) = G_1(t)t + F_1(t),$$

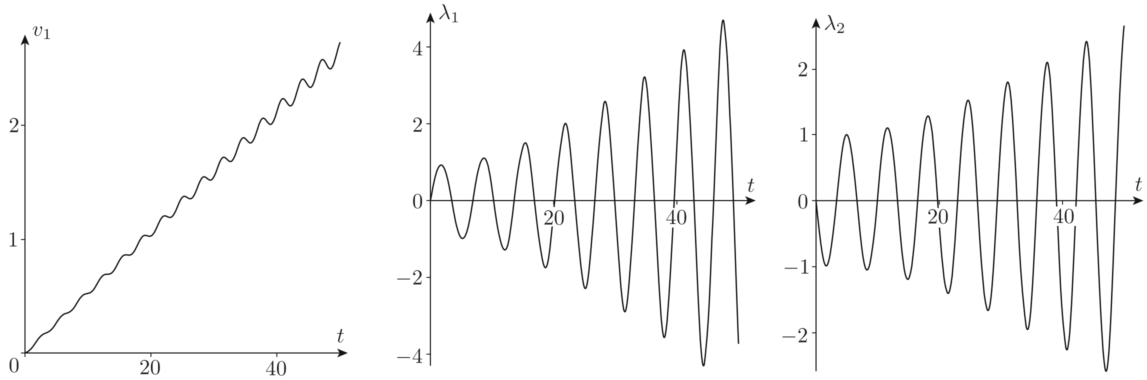
$$G_1(t) = \frac{k\sin\varphi(t)e^{\delta\Delta(\varphi(t))}}{\big(J_1\sin^2\varphi(t) + M(c_1\cos\varphi(t) + c_2)^2\big)^{\frac{1}{2}}},$$

$$F_1(t) = -\frac{f(t)\sin\varphi(t) + c_2\dot{\varphi}(t)}{c_1\cos\varphi(t) + c_2}.$$

Depending on the average value of the function $G_1(t)$ for period T

$$\langle G_1\rangle = \frac{1}{T}\int_0^T G_1(t)dt,$$

we can single out two cases:

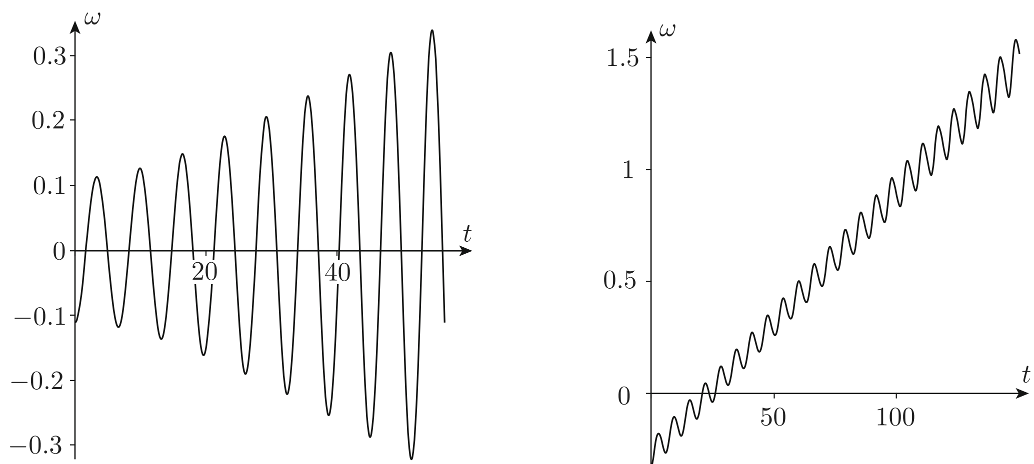


(a) (b)

Fig. 3. Typical dependences of the linear velocity v_1 and the undetermined multipliers λ_1 and λ_2 on time for $\varphi(t) = \alpha \sin \Omega t + \beta$ and for the fixed parameters $J_1 = 30, J_2 = 3, M = 3, c_1 = 2, c_2 = 1, \varepsilon = 1, \delta = 2, \alpha = \frac{1}{3}$, $\Omega = 1$, $\beta = 0$ and initial conditions $t = 0$, $v_2 = 0$.

— $\langle G_1 \rangle = 0$, then, as $t \to +\infty$, the angular velocity $\omega(t)$ has infinitely many zeros, the distance between which is equal to T (see Fig. 4a);

— $\langle G_1 \rangle \neq 0$, then the average value of the angular velocity $\omega(t)$ increases linearly for period T (see Fig. 4b).



(a) $\beta = 0$ (b) $\beta = \dfrac{3\pi}{2}$

Fig. 4. Typical dependences of the angular velocity ω on time for $\varphi(t) = \alpha \sin \Omega t + \beta$ and for the fixed parameters $J_1 = 30, J_2 = 3, M = 3, c_1 = 2, c_2 = 1, \varepsilon = 1, \delta = 2$, $\alpha = \frac{1}{3}$, $\Omega = 1$, $\beta = 0$ and initial conditions $t = 0$, $v_2 = 0$.

The expression which determines the angle of rotation of the first platform ψ is represented as

$$\psi(t) = \frac{1}{2}\langle G_1 \rangle t^2 + G_2(t)t + F_2(t),$$
$$G_2(t+T) = G_2(t), \quad F_2(t+T) = F_2(t).$$

For a complete description of the motion of the *Roller Racer* in the inertial coordinate system, it is necessary to analyze quadratures for the coordinates x and y of the attachment point of the platforms. This problem remains open.

Numerical experiments show that in the case $\langle G_1 \rangle = 0$ and at a small amplitude of oscillations of $\varphi(t)$, the *Roller Racer* starts from rest and performs for some time directed motion with an increasing oscillation amplitude. Further, when the velocity v_1 reaches some "critical" value, the *Roller Racer* turns around and moves in the opposite direction, changing the typical form of the trajectory (see Fig. 5). Then, at some time instant, the *Roller Racer* turns around again and moves back (see Fig. 6).

At a sufficiently large amplitude of oscillations of $\varphi(t)$, the *Roller Racer* performs no directional motion (see Fig. 7). Numerical experiments show that, if $\langle G_1\rangle \neq 0$, then the motion of the *Roller Racer* is bounded (see Fig. 8).

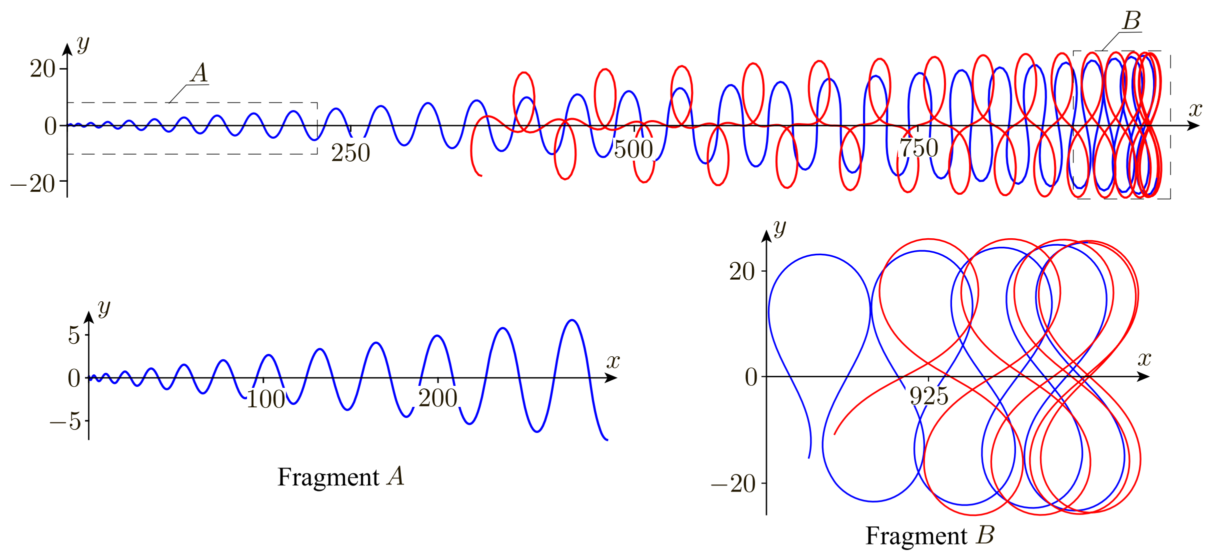


Fig. 5. Trajectory of point P on the plane in the case $\langle G_1\rangle = 0$ for $\varphi(t) = \alpha\sin\Omega t + \beta$ and for the fixed parameters $J_1 = 20, J_2 = 3, M = 3, c_1 = 2, c_2 = 2, \varepsilon = 1, \delta = 10,\ \alpha = \frac{1}{3},\ \Omega = 1,\ \beta = 0$ and initial conditions $t = 0,\ v_2 = 0$. Blue denotes the trajectory before turning, and red denotes the trajectory after turning.

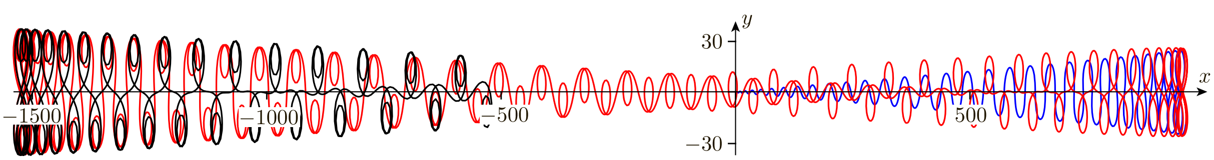


Fig. 6. Trajectory of point P on the plane for the parameters as in Fig. 5. Blue denotes the trajectory before turning, red denotes the trajectory after the first turning, and black denotes the trajectory after the second turning.

4. VISCOUS FRICTION

We now consider Eq. (2.6) in the case where $C, D \neq 0$. For brevity, we will write Eq. (2.6) in the form

$$\dot{v}_1 = -\Phi(t)v_1 + \Psi(t),$$

where $\Phi(t)$ and $\Psi(t)$ are periodic functions with the same period T.

Now, in contrast to the previous case, since $C(t) > 0$ at any t and positive k_1 and k_2, we have

$$\int_0^T \Phi(t)dt = \int_0^T C(t)dt > 0.$$

Consequently,

$$P(t) = e^{\int\limits_0^t \Phi(\varphi(\tau))d\tau}$$

is not a periodic function and $P(T) > 1$.

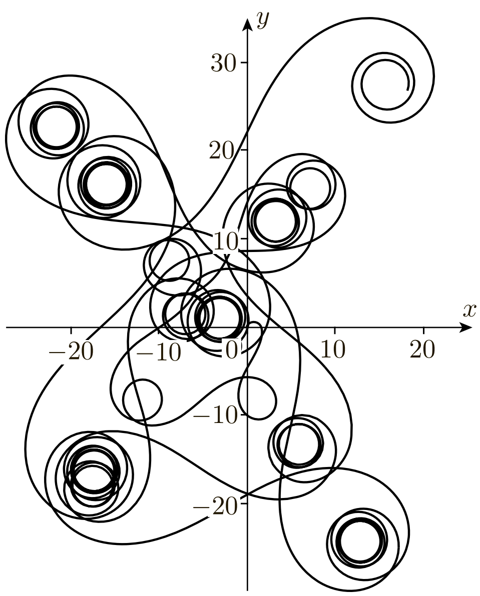


Fig. 7. Trajectory of point P on the plane in the case $\langle G_1\rangle = 0$ for $\varphi(t) = \alpha\sin\Omega t + \beta$ and for the fixed parameters $J_1 = 20$, $J_2 = 3$, $M = 3$, $c_1 = 2$, $c_2 = 2$, $\varepsilon = 1, \delta = 10$, $\alpha = \frac{3}{2}$, $\Omega = 1$, $\beta = 0$ and initial conditions $t = 0$, $v_2 = 0$.

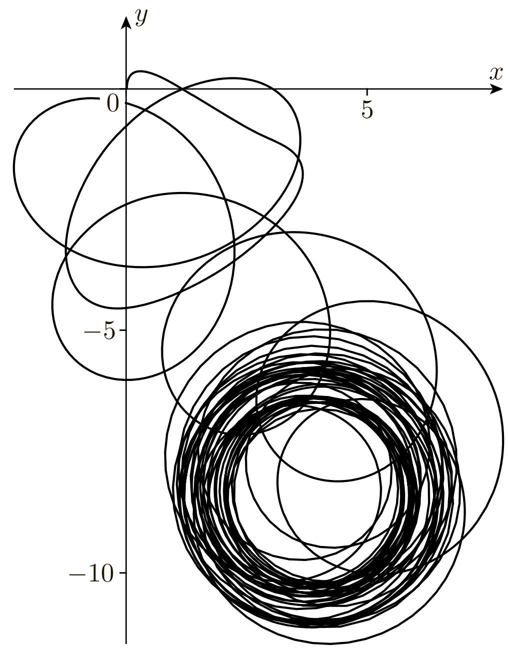


Fig. 8. Trajectory of point P on the plane in the case $\langle G_1\rangle \neq 0$ for $\varphi(t) = \alpha\sin\Omega t + \beta$ for the fixed parameters $J_1 = 2, J_2 = 3, M = 2, c_1 = 1, c_2 = 3, \varepsilon = 1$, $\delta = 2$, $\alpha = \frac{1}{3}$, $\Omega = 1, \beta = \frac{\pi}{2}$ and initial conditions $t = 0$, $v_2 = 0$, $\phi = 0, x = 0, y = 0$.

As is well known, in this case a general solution of this equation has the form

$$x(t) = \frac{c}{P(t)} + Q(t),$$

$$P(t) = e^{\int\limits_0^t \Phi(\tau)d\tau}, \quad c = \text{const},$$

where $Q(t)$ is a partial periodic solution:

$$Q(t) = \frac{1}{P(t)}\left(\frac{1}{(P(T)-1)}\int\limits_0^T \Psi(\tau)P(\tau)d\tau - \int\limits_0^t \Psi(\tau)P(\tau)d\tau\right), \tag{4.1}$$

$$Q(t+T) = Q(t).$$

Since $P(t) \to \infty$ as $t \to \infty$, the velocity v_1 tends to the periodic function (4.1). Thus, in the presence of viscous friction force the following proposition holds:

Proposition 2. *For positive* (*friction*) *coefficients* $k_1 > 0$ *and* $k_2 > 0$ *Eq.* (2.6) *has a partial solution which is periodic in time and to which all trajectories of Eq.* (2.6) *tend as* $t \to +\infty$.

Consequently, the linear velocity v_1 of the *Roller Racer* is a bounded function of time, and there is no acceleration in this case. Moreover, in this case the undetermined multipliers λ_1 and λ_2 are bounded functions of time (see Fig. 9). Different types of the trajectory of the attachment point of the platforms on the plane are shown in Fig. 10.

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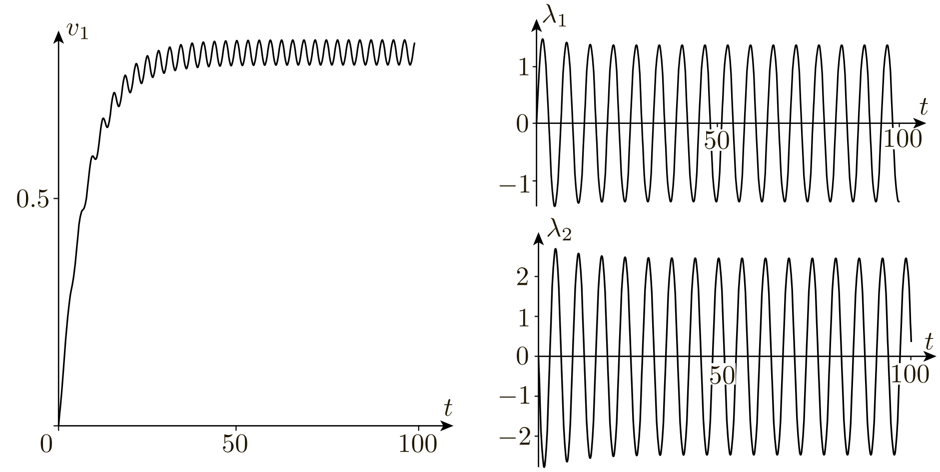

Fig. 9. Typical dependences of the linear velocity v_1 and the undetermined multipliers λ_1 and λ_2 on time for $\varphi(t) = \alpha \sin \Omega t + \beta$ for the fixed parameters $J_1 = 20, J_2 = 3, M = 3, c_1 = 1, c_2 = 2, \varepsilon = 1, \delta = 10, k_1 = k_2 = 0.1, b_1 = b_2 = 1$, $\alpha = \frac{1}{3}$, $\Omega = 1, \beta = \frac{\pi}{2}$ and initial conditions $t = 0, v_2 = 0, \phi = 0, x = 0, y = 0$.

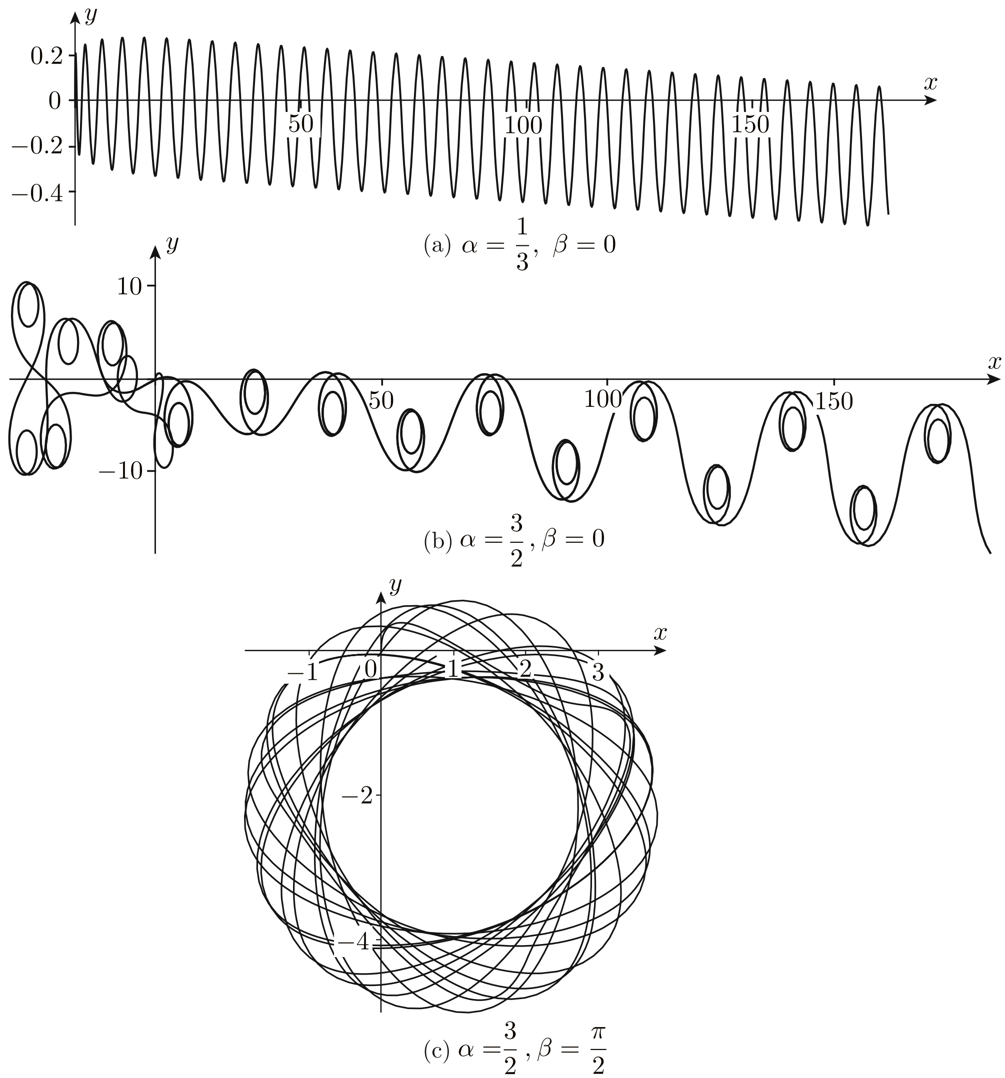

Fig. 10. Trajectory of point P for $\varphi(t) = \alpha \sin \Omega t + \beta$ for the fixed parameters $J_1 = 2, J_2 = 3, M = 3$, $c_1 = 1$, $c_2 = 3, \varepsilon = 1, \delta = 2, k_1 = k_2 = 0.1, b_1 = b_2 = 1$, $\alpha = \frac{1}{3}$, $\Omega = 1, \beta = \frac{\pi}{2}$ and initial conditions $t = 0, v_2 = 0$, $\phi = 0$, $x = 0$, $y = 0$.

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