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The Inertial Motion of a *Roller Racer*

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Abstract—This paper addresses the problem of the inertial motion of a *roller racer*, which reduces to investigating a dynamical system on a (two-dimensional) torus and to classifying singular points on it. It is shown that the motion of the *roller racer* in absolute space is asymptotic. A restriction on the system parameters in which this motion is bounded (compact) is presented.

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1. INTRODUCTION

This paper is concerned with the problem of the inertial motion of a wheeled vehicle consisting of two coupled platforms with one wheeled pair fastened on each of them. This model is used, for example, in a toy vehicle, the *roller racer* (see Fig. 1). Its special feature is that a child sitting on it can move forward by oscillating the handle bar from side to side. For this reason, we shall call this system a *roller racer* in what follows.

Fig. 1. Roller racer.

The *roller racer* features many parameters in equations of motion, therefore, this system is usually considered under some restrictions on the position of the center of mass of both wheeled platforms. For example, Y. Rocard [14] showed the asymptotic stability of straight-line motions for the case where the center of mass is at the attachment point of the platforms. A similar model relating to the motion of mobile robots is treated in [13]. The authors of [8] address control and reduction problems in the case where the centers of mass of the wheeled platforms coincide with the attachment points of the wheeled pairs.

If we place one wheeled pair of the *roller racer* at the attachment point of the platforms, we obtain the problem of a platform with one fixed axis (trailer), which is dealt with, for example, in [4, 5].

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This paper presents equations of motion obtained for the *roller racer* in a fairly general case (which includes those mentioned above). The problem reduces to investigating a reduced system on the torus $\mathbb{T}^2$ and to classifying singular points on it.

Attention is also given to the motion in absolute space, which exhibits asymptotic behavior. In particular, a restriction on the system parameters is shown which implies that the motion of the *roller racer* is restricted (compact). The conclusion discusses various generalizations of the system of interest which lead to the addition of potential forces.

We note that the *roller racer* is an immediate generalization of the problem of a Chaplygin sleigh [6], since, according to [1], each wheeled pair can be replaced without loss of generality by a weightless skate (knife edge). We note that the motion of the Chaplygin sleigh in absolute space (as opposed to that of the *roller racer*) is always unbounded [10] and asymptotically tends to straight-line motion.

We also note that the analysis of the system (which involves the search for tensor invariants) agrees with the general scheme of the hierarchy of dynamical behavior of nonholonomic systems, which is discussed and illustrated in [2, 3] by the rolling motion of a rigid body on a plane and a sphere.

2. EQUATIONS OF MOTION

Consider the problem of the motion of the simplest wheeled vehicle, the *roller racer*, on a plane. The *roller racer* consists of two coupled platforms which are rigid bodies and can freely rotate in a horizontal plane independently of each other (see Fig. 2). Each platform has a rigidly attached wheeled pair consisting of two wheels lying on the same axis.

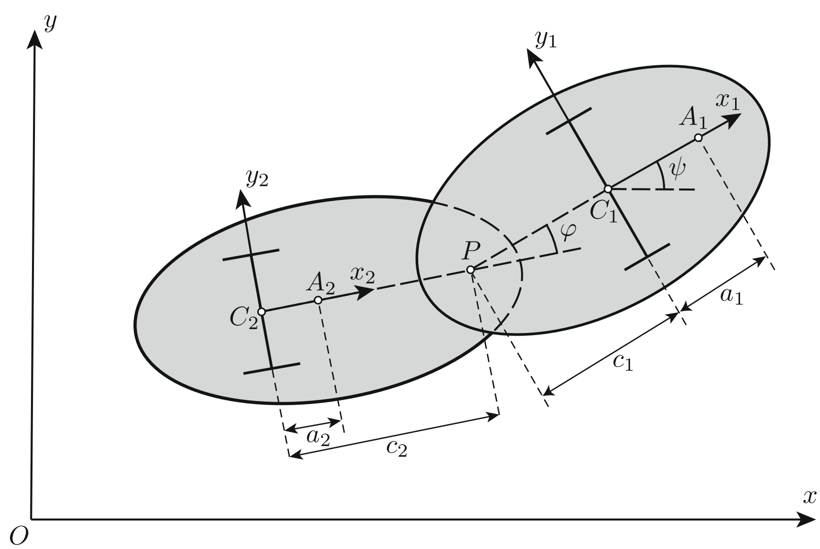


Fig. 2. *Roller racer* on a plane.

We define three coordinate systems:

— *an inertial* coordinate system Oxy;

— *a noninertial* coordinate system $C_1x_1y_1$ attached to the *first platform*, with origin C_1 at the center of mass of the wheeled pair. We assume that the axes C_1x_1 and C_1y_1 are directed, respectively, along a tangent and a normal to the plane of the wheels;

— *a noninertial* coordinate system $C_2x_2y_2$ attached to the *second platform*, with origin C_2 at the center of mass of the wheeled pair. Again, we assume that the axes are directed along a tangent and a normal to the plane of the wheels.

There is no slipping at the points of contact of the wheels with the plane. In [1] it is shown that in this case the constraints for the wheels to roll without slipping can be replaced with the *Chaplygin constraints*, namely, that the velocity of the center of mass of each wheeled pair, C_1 and C_2, in the direction perpendicular to the plane of the wheels be zero and that the degrees of freedom describing the rotation angles of the wheels be ignored. Therefore, in what follows we replace *each wheeled pair with a weightless knife edge* (*skate*) *located at its center of mass.*

Let us introduce the generalized coordinates parameterizing the configuration space of the system. Let $\boldsymbol{r} = (x, y)$ be the radius vector of the attachment point of the platforms P in the fixed coordinate system Oxy.

We specify the orientation of the first platform by the angle ψ between the axes Ox and Ox_1 and the orientation of the second platform by the angle φ between the axes Ox_1 and Ox_2.

Thus, the configuration space $\mathcal{N}$ is

$$\mathcal{N} = \{\boldsymbol{q} = (x, y, \psi, \varphi), \psi, \varphi \bmod 2\pi\} \approx \mathbb{R}^2 \times \mathbb{T}^2.$$

The constraint equations in these variables can be represented as

$$-\dot{x}\sin\psi + \dot{y}\cos\psi + c_1\dot{\psi} = 0,$$
$$-\dot{x}\sin(\psi + \varphi) + \dot{y}\cos(\psi + \varphi) - c_2(\dot{\psi} + \dot{\varphi}) = 0,$$

where c_1 and c_2 are, respectively, the distances from the center of each wheeled pair to the attachment point of the platforms.

The Lagrangian of the system has the standard form

$$L = T(\dot{\boldsymbol{q}}, \boldsymbol{q}) - U,$$

where T is the kinetic energy of the system and $U = U(\varphi, \psi)$ is the potential energy of external forces, which we assume to be dependent on φ and ψ.

Suppose that *the center of mass of each platform lies in the plane of the knife edge*. Then the kinetic energy can be represented as

$$T(\dot{\boldsymbol{q}}, \boldsymbol{q}) = \frac{1}{2}(m_1 + m_2)(\dot{x}^2 + \dot{y}^2) - m_1(a_1 + c_1)(\dot{x}\sin\psi - \dot{y}\cos\psi)\dot{\psi}$$
$$-m_2\,(a_2 - c_2)\,(\dot{x}\sin(\psi + \varphi) - \dot{y}\cos(\psi + \varphi))\,(\dot{\psi} + \dot{\varphi}) + \frac{1}{2}I_1\dot{\psi}^2 + \frac{1}{2}I_2(\dot{\psi} + \dot{\varphi})^2,$$

where m_1, I_1 and m_2, I_2 are, respectively, the mass and the moment of inertia relative to the point P of the first and the second platform and a_1 is the distance from the point C_1 to the center of mass of the first platform and a_2 is the distance from the point C_2 to the center of mass of the second platform.

In this case, instead of the generalized velocities $\dot{\boldsymbol{q}} = (\dot{x}, \dot{y}, \dot{\psi}, \dot{\varphi})$ it is more convenient to parameterize the tangent space $T\mathcal{N}_{\boldsymbol{q}}$ using the quasi-coordinates $\boldsymbol{v} = (v_1, v_2)$ and $\boldsymbol{\omega} = (\omega_1, \omega_2)$, where $\boldsymbol{v}$ is the velocity of the point P referred to the axes of the moving coordinate system $C_1x_1y_1$, and $\boldsymbol{\omega}$ is the absolute angular velocity of each of the platforms:

$$v_1 = -\dot{x}\sin\psi + \dot{y}\cos\psi, \quad v_2 = \dot{x}\cos\psi + \dot{y}\sin\psi, \quad \omega_1 = \dot{\psi}, \quad \omega_2 = \dot{\psi} + \dot{\varphi}.$$

In order to obtain the equations of motion, we rewrite the previous relations as

$$\dot{\boldsymbol{q}} = v_1\boldsymbol{E}_1^{(v)} + v_2\boldsymbol{E}_2^{(v)} + \omega_1\boldsymbol{E}_1^{(\omega)} + \omega_2\boldsymbol{E}_2^{(\omega)},$$

where $\boldsymbol{E}_i^{(v)}$ and $\boldsymbol{E}_i^{(\omega)}$ are the vector fields which in the coordinate basis have the form

$$\boldsymbol{E}_1^{(v)} = -\sin\psi\frac{\partial}{\partial x} + \cos\psi\frac{\partial}{\partial y}, \quad \boldsymbol{E}_2^{(v)} = \cos\psi\frac{\partial}{\partial x} + \sin\psi\frac{\partial}{\partial y},$$
$$\boldsymbol{E}_1^{(\omega)} = -\frac{\partial}{\partial \varphi} + \frac{\partial}{\partial \psi}, \quad \boldsymbol{E}_2^{(\omega)} = \frac{\partial}{\partial \varphi}.$$

The nonzero commutation relations of these vector fields are

$$[\boldsymbol{E}_1^{(v)}, \boldsymbol{E}_1^{(\omega)}] = -\boldsymbol{E}_2^{(v)}, \quad [\boldsymbol{E}_2^{(v)}, \boldsymbol{E}_1^{(\omega)}] = \boldsymbol{E}_1^{(v)}.$$

This implies that the equations of motion for this nonholonomic system can be represented in the following general form (see, e. g., [2]):

$$\begin{gathered}
\frac{d}{dt}\left(\frac{\partial \widetilde{L}}{\partial v_1}\right)+\sin\psi\frac{\partial \widetilde{L}}{\partial x}-\cos\psi\frac{\partial \widetilde{L}}{\partial y}+\omega_1\frac{\partial \widetilde{L}}{\partial v_2}=\sum_{i=1}^{2}\lambda_i\frac{\partial f_i}{\partial v_1},\\
\frac{d}{dt}\left(\frac{\partial \widetilde{L}}{\partial v_2}\right)-\cos\psi\frac{\partial \widetilde{L}}{\partial x}-\sin\psi\frac{\partial \widetilde{L}}{\partial y}-\omega_1\frac{\partial \widetilde{L}}{\partial v_1}=\sum_{i=1}^{2}\lambda_i\frac{\partial f_i}{\partial v_2},\\
\frac{d}{dt}\left(\frac{\partial \widetilde{L}}{\partial \omega_1}\right)+\frac{\partial \widetilde{L}}{\partial \varphi}-\frac{\partial \widetilde{L}}{\partial \psi}-v_1\frac{\partial \widetilde{L}}{\partial v_2}+v_2\frac{\partial \widetilde{L}}{\partial v_1}=\sum_{i=1}^{2}\lambda_i\frac{\partial f_i}{\partial \omega_1},\\
\frac{d}{dt}\left(\frac{\partial \widetilde{L}}{\partial \omega_2}\right)-\frac{\partial \widetilde{L}}{\partial \varphi}=\sum_{i=1}^{2}\lambda_i\frac{\partial f_i}{\partial \omega_2},
\end{gathered}\tag{2.1}$$

where λ_1 and λ_2 are the undetermined multipliers, $\widetilde{L}=L\big(\boldsymbol{q},\dot{\boldsymbol{q}}(\boldsymbol{\omega},\boldsymbol{v})\big)$, and f_1 and f_2 are the constraint equations in the new variables

$$f_1=v_1+c_1\omega_1=0,\quad f_2=v_1\cos\varphi-v_2\sin\varphi-c_2\omega_2=0.\tag{2.2}$$

For the Lagrangian function we obtain

$$\begin{gathered}
\widetilde{L}=\frac{1}{2}(m_1+m_2)(v_1^2+v_2^2)+m_1(a_1+c_1)v_1\omega_1\\
+m_2(a_2-c_2)(v_1\cos\varphi-v_2\sin\varphi)\omega_2+\frac{1}{2}I_1\omega_1^2+\frac{1}{2}I_2\omega_2^2-U(\varphi,\psi).
\end{gathered}$$

Let us solve the constraints (2.2) for the angular velocities ω_1 and ω_2. Then from the last two equations of the system (2.1) we find the undetermined multipliers as functions of the generalized coordinates $\boldsymbol{q}$ and velocities v_1, v_2.

Further, solving the first two equations of the system (2.1) for the first derivatives, we represent the equations of motion on $\mathcal{M}^6=\{v_1,v_2,\varphi,\psi,x,y\}$ as

$$\begin{gathered}
\Phi(\varphi)\dot{\boldsymbol{v}}=\frac{1}{c_2^2}\left(\frac{v_1}{c_1}+\frac{v_1\cos\varphi-v_2\sin\varphi}{c_2}\right)\mathbf{V}(\varphi)\boldsymbol{v}+\frac{\delta}{c_2}v_1\boldsymbol{\Omega}(\varphi)\boldsymbol{v}-\boldsymbol{M},\\
\Phi(\varphi)=(j_1+\mu c_1^2)(j_2+\mu c_2^2)-j_1j_2\cos^2\varphi,\\
\dot{\varphi}=\left(\frac{v_1}{c_1}+\frac{v_1\cos\varphi-v_2\sin\varphi}{c_2}\right),\quad \dot{\psi}=-\frac{v_1}{c_1},\\
\dot{x}=-v_1\sin\psi+v_2\cos\psi,\quad \dot{y}=v_1\cos\psi+v_2\sin\psi,
\end{gathered}\tag{2.3}$$

where $\mathbf{V}$ and $\boldsymbol{\Omega}$ are the matrices

$$\mathbf{V}=\begin{pmatrix}(\nu+\mu c_2^2)j_2c_1^2\sin\varphi\cos\varphi & (\nu+\mu c_2^2)j_2c_1^2\cos^2\varphi-\nu c_1^2(j_2+\mu c_2^2)\\ j_2(j_1c_2^2+(\nu+\mu c_2^2)c_1^2)\cos^2\varphi-c_2^2(j_1+\mu c_1^2)(j_2-\nu) & -j_2(j_1c_2^2+(\nu+\mu c_2^2)c_1^2)\sin\varphi\cos\varphi\end{pmatrix},$$

$$\boldsymbol{\Omega}=\begin{pmatrix}j_2\sin\varphi\cos\varphi & -j_2\sin^2\varphi-\mu c_2^2\\ j_2\cos^2\varphi+\mu c_2^2+j_1\frac{c_2^2}{c_2^2} & -j_2\sin\varphi\cos\varphi\end{pmatrix},$$

and $\boldsymbol{M}=(M_1,M_2)$ is the angular momentum due to external forces:

$$\begin{gathered}
M_1=c_1\big(j_2\sin^2\varphi+\mu c_2(c_1\cos\varphi+c_2)\big)\frac{\partial U}{\partial\varphi}-c_1(j_2\sin^2\varphi+\mu c_2^2)\frac{\partial U}{\partial\psi},\\
M_2=\sin\varphi\big(j_2c_1\cos\varphi-c_2(j_1+\mu c_1^2)\big)\frac{\partial U}{\partial\varphi}-j_2c_1\cos\varphi\sin\varphi\frac{\partial U}{\partial\psi}.
\end{gathered}$$

In the system (2.3) we have introduced new parameters which are related to the previous ones by

$$j_1 = I_1 - 2m_1c_1(a_1 + c_1), \quad j_2 = I_2 + 2m_2c_2(a_2 - c_2),$$
$$\mu = m_1 + m_2, \quad \delta = m_1a_1c_2 - m_2a_2c_1, \quad \nu = m_2c_2(a_2 - c_2).$$

The system (2.3) admits the energy integral

$$E = \frac{\mu}{2}(v_1^2 + v_2^2) + \frac{j_1}{2c_1^2}v_1^2 + \frac{j_2}{2c_2^2}(v_1\cos\varphi - v_2\sin\varphi)^2 + U.$$

If $U = U(\varphi)$, the equations of motion (2.3) are invariant under the group of motions of the plane $SE(2)$, and hence the closed (reduced) system of equations governing the evolution of v_1, v_2 and φ decouples. For it to be integrable by the Euler – Jacobi theorem, we also need an invariant measure.

For an arbitrary potential $U(\varphi, \psi)$, a system of equations for the variables v_1, v_2, φ and ψ decouples, for its integrability we need, in addition to an invariant measure, an additional integral.

In what follows, it will be shown that in the general case there is no smooth invariant measure in the system (2.3), since the reduced system can have asymptotically stable fixed points.

3. INERTIAL MOTION

Consider the motion in the absence of an external field ($U = 0$). In this case, it can be shown that the level set of the energy integral $E = \mathrm{const}$ in three-dimensional space given by the variables (v_1, v_2, φ) is the two-dimensional torus $\mathbb{T}^2$. We introduce on it local coordinates as follows:

$$v_1 = u\sin\vartheta, \quad v_2 = u\cos\vartheta.$$

Let us fix the level set of the energy integral $E = h$:

$$\left(\mu + \frac{j_1\sin^2\vartheta}{c_1^2} + \frac{j_2\sin^2(\vartheta - \varphi)}{c_2^2}\right)\frac{u^2}{2} = h,$$

and, taking into account that $u > 0$ (for $h \neq 0$), we find from this equation the unique solution for u.

Finally, rescaling time as $dt = u(\varphi, \vartheta)d\tau$, we obtain a system on the torus $\mathbb{T}^2$ whose trajectories in the angular variables $\boldsymbol{x} = (\varphi, \vartheta) \bmod 2\pi$ are given by the equations

$$\frac{d\varphi}{d\tau} = \frac{\sin\vartheta}{c_1} + \frac{\sin(\vartheta - \varphi)}{c_2}, \quad \Phi(\varphi)\frac{d\vartheta}{d\tau} = \left(\frac{\sin\vartheta}{c_1} + \frac{\sin(\vartheta - \varphi)}{c_2}\right)\Theta_1 - \delta\Theta_2\sin\vartheta,$$
$$\Theta_1 = j_1\sin\vartheta\big(j_2\sin\varphi\cos(\vartheta - \varphi) - \nu\sin\vartheta\big) - \frac{c_1^2}{c_2^2}j_2(\nu + \mu c_2^2)\sin^2(\vartheta - \varphi) + \mu c_1^2(j_2 - \nu), \tag{3.1}$$
$$\Theta_2 = \frac{j_1c_2}{c_1^2}\sin^2\vartheta + \frac{j_2}{c_2}\sin^2(\vartheta - \varphi) + \mu c_2.$$

The system (3.1) possesses an involution (i. e., it is reversible):

$$R : \vartheta \to \vartheta + \pi, \quad t \to -t.$$

Consequently, all trajectories on the thorus turn out to be symmetric relative to $\vartheta = \pi$ (and the motion reverses direction).

From the known solutions $\varphi(\tau)$ and $\vartheta(\tau)$ of (3.1) the motion in the fixed coordinate system Oxy is determined by quadratures:

$$\frac{d\psi}{d\tau} = -\frac{1}{c_1}\sin\vartheta(\tau), \quad \frac{dx}{d\tau} = \cos\big(\vartheta(\tau) + \psi\big), \quad \frac{dy}{d\tau} = \sin\big(\vartheta(\tau) + \psi\big). \tag{3.2}$$

The properties of the reduced system (3.1) largely determine the properties of the system dynamics in absolute space. We consider them in more detail.

3.1. Dynamics of a Balanced Roller Racer

If $\delta = 0$, the system (3.1) possesses two degenerate families of equilibrium points (see Fig. 3), which on the thorus $\mathbb{T}^2$ are given by the curves

$$\Sigma_1 = \left\{ \varphi = \vartheta_0 + \arcsin\left(\frac{c_2}{c_1}\sin\vartheta_0\right), \quad \vartheta = \vartheta_0 \right\}, \quad \vartheta_0 \in (0, 2\pi],$$

$$\Sigma_2 = \left\{ \varphi = \vartheta_0 - \pi - \arcsin\left(\frac{c_2}{c_1}\sin\vartheta_0\right), \quad \vartheta = \vartheta_0 \right\}, \quad \vartheta_0 \in (0, 2\pi].$$

It can be shown that for the the above-mentioned equilibrium points in the general case $\mathrm{Tr}\mathbf{A}^{(L)} \neq 0$, where $\mathbf{A}^{(L)}$ is the linearization matrix. Consequently, the necessary condition for the existence of a smooth invariant measure, as presented in [11], is not satisfied.

However, this is no obstruction to the existence of an invariant measure with singular density, which has a singularity in Σ_1 and Σ_2:

$$\rho = \left(\frac{\sin\vartheta}{c_1} + \frac{\sin(\vartheta - \varphi)}{c_2}\right)^{-1} \frac{\sqrt{\Phi}}{u(\varphi, \vartheta)}.$$

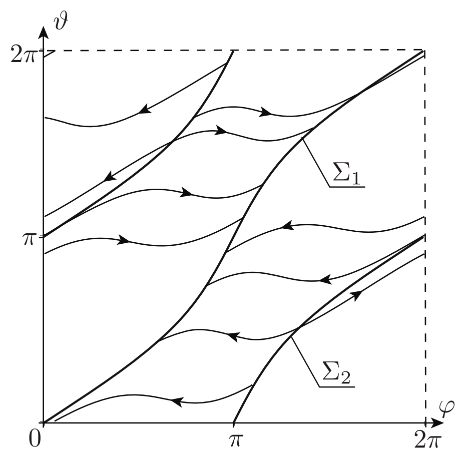


Fig. 3. Phase portrait (3.1) in the case $\delta = 0$, $\nu = 0.3$, $j_1 = 2$, $j_2 = 3$, $c_1 = 2$, $c_2 = 1$, $\mu = 1$.

For the above-mentioned equilibrium points we obtain from (3.2):

$$\psi(\tau) = -\frac{\sin\vartheta_0}{c_1}\tau + \psi_0, \quad \big(x(\tau) - x_0\big)^2 + \big(y(\tau) - y_0\big)^2 = \frac{c_1}{\sin^2\vartheta_0}, \tag{3.3}$$

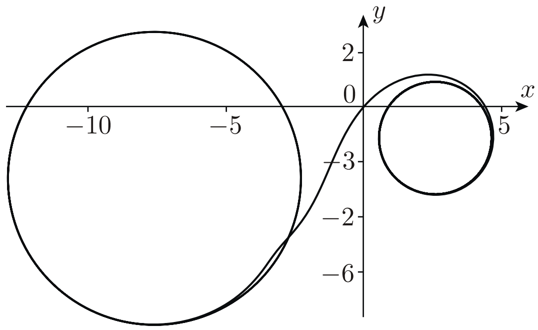
where ψ_0, x_0 and y_0 are the values of ψ, x and y at time $\tau = 0$. Thus, the following relations hold:

Proposition 1. *If $\delta = 0$, the following holds for the roller racer:*

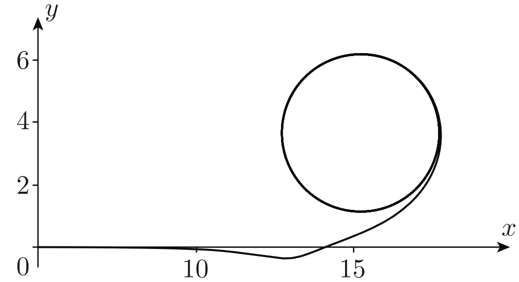
1) *The motion in the interval $\tau \in (-\infty, +\infty)$ as $\tau \to -\infty$ "starts" with the unstable steady-state solution* (3.1) *and then, as $\tau \to +\infty$, tends to the stable steady-state solution* (3.1).

2) *The above-mentioned steady-state solutions correspond to motion in a circle* (3.3) *except for the cases $\vartheta_0 = 0, \pi$ in which the roller racer moves in a straight line* (*see Fig. 4b*)*. Hence, the motion of the roller racer is, as a rule, bounded* (*see Fig. 4a*).

Remark 1. For the case at hand, in the initial system (2.3), after rescaling time by $dt \to \left(\frac{\sin\vartheta}{c_1} + \frac{\sin(\vartheta - \varphi)}{c_2}\right)^{-1} dt$, a system that is linear in the velocities v_1 and v_2 and has coefficients periodic in φ decouples. This linear system possesses, in addition to the energy integral E, another additional integral. In the initial system (2.3) it is not defined in the region for which the above-mentioned time rescaling has a singularity (i. e., the integral is defined "semilocally").



a) $\vartheta(0)=4,\ \varphi(0)=1,$
$\psi(0)=0,\ x(0)=0,\ y(0)=0.$



b) $\vartheta(0)=\pi,\ \varphi(0)=\pi,$
$\psi(0)=0,\ x(0)=0,\ y(0)=0.$

Fig. 4. Trajectory of the point P for the fixed parameters $\delta=0$, $\nu=0.3$, $j_1=2$, $j_2=3$, $c_1=2$, $c_2=1$, $\mu=1$.

3.2. Dynamics of an Unbalanced Roller Racer

If $\delta\neq 0$, the system (3.1) possesses isolated equilibrium points (corresponding to the case $\vartheta_0=0,\pi$ of the previous section). We divide them into two groups, depending on the symmetry relative to involution R (i. e., $R(\boldsymbol{x}^{(i)})=\boldsymbol{x}^{(i+1)}$):

— $\boldsymbol{x}^{(1)}=(0,0)$ and $\boldsymbol{x}^{(2)}=(0,\pi)$. The characteristic polynomial for $\boldsymbol{x}^{(1)}$ has the form

$$P^{(1)}(\lambda)=\lambda^2+p_1^{(1)}\lambda+p_0^{(1)},$$
$$p_1^{(1)}=\frac{(j_1+\delta)c_2^2+\nu c_1^2+(\mu c_1c_2+\nu-j_2)c_1c_2}{c_2(j_1c_2^2+j_2c_1^2+\mu c_1^2c_2^2)},\quad p_0^{(1)}=\frac{\delta}{j_1c_2^2+j_2c_1^2+\mu c_1^2c_2^2};$$

— $\boldsymbol{x}^{(3)}=(\pi,0)$ and $\boldsymbol{x}^{(4)}=(\pi,\pi)$. In this case, the characteristic polynomial for $\boldsymbol{x}^{(3)}$ can be represented as

$$P^{(3)}(\lambda)=\lambda^2+p_1^{(3)}\lambda+p_0^{(3)},$$
$$p_1^{(3)}=-\frac{(j_1-\delta)c_2^2+\nu c_1^2+(\mu c_1c_2-\nu-j_2)c_1c_2}{c_2(j_1c_2^2+j_2c_1^2+\mu c_1^2c_2^2)},\quad p_0^{(3)}=-\frac{\delta}{j_1c_2^2+j_2c_1^2+\mu c_1^2c_2^2}.$$

Numerical experiments (see Fig. 5) show that the phase portrait on the thorus $\mathbb{T}^2$ in this case consists only of equilibrium points $\boldsymbol{x}^{(i)}$ $i=1,\ldots,4$ and trajectories asymptotically tending to them. However, the absence of limit cycles on the thorus $\mathbb{T}^2$ remains unproved. This issue is complicated by the fact that the system (3.1) has a large number of parameters.

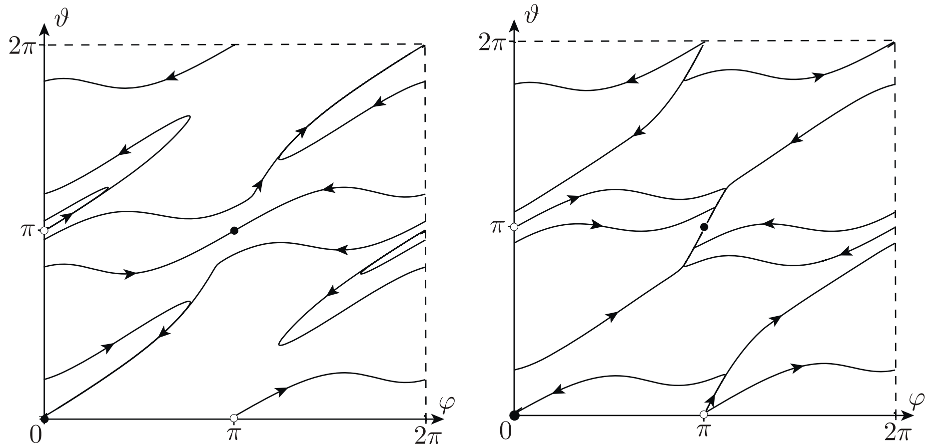
At the above-mentioned equilibrium points, $\psi=\psi_0=\mathrm{const}$, therefore, they correspond to the straight-line motion of the *roller racer* (see Fig. 6):

$$x(\tau)=\pm\cos\psi_0\tau+x_0,\quad y(\tau)=\pm\sin\psi_0\tau+y_0,$$

where the upper sign corresponds to $\boldsymbol{x}^{(1)}$, $\boldsymbol{x}^{(2)}$ and the lower sign corresponds to $\boldsymbol{x}^{(3)}$, $\boldsymbol{x}^{(4)}$. Thus, the following proposition holds:

Proposition. *All trajectories of the roller racer are not compact when* $\delta\neq 0$ *and, as* $\tau\to+\infty$ (*or* $\tau\to-\infty$), *asymptotically tend to straight-line motion.*

Remark. If this hypothesis is correct, then from the known solution $\vartheta(\tau)$ of the reduced system (3.1) with $\delta\neq 0$, the system (3.2) defines the scattering map (see, e. g., [7, 9]), which is apparently regular (since the reduced system describes the vector field on the thorus $\mathbb{T}^2$). But the question of explicit expression for the scattering angle $\Delta\Psi$ remains open. For the Chaplygin sleigh this angle has been found in [10].



a) $\delta = 0.2$ *b*) $\delta = -0.2$

Fig. 5. A typical phase portrait of (3.1) for $\delta \neq 0$ and the fixed parameters $\nu = 0.3$, $j_1 = 2$, $j_2 = 3$, $c_1 = 2$, $c_2 = 1$, $\mu = 1$. In case a the singular points $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}$ are of focus type and $\boldsymbol{x}^{(3)}, \boldsymbol{x}^{(4)}$ are of saddle type, while in case b the points $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}$ are of saddle type and $\boldsymbol{x}^{(3)}, \boldsymbol{x}^{(4)}$ are of node type.

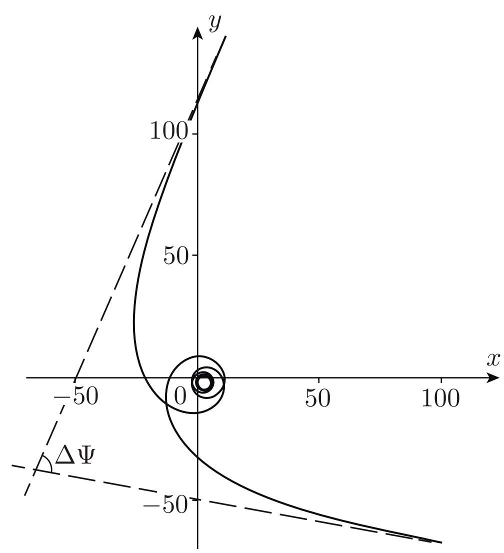


Fig. 6. Trajectory of the point P for the fixed parameters $\delta = 1.2$, $\nu = 0.8$, $j_1 = 1$, $j_2 = 2$, $c_1 = 3$, $c_2 = 4$, $\mu = 1$ and the initial conditions $\varphi(0) = 0$, $\vartheta = 4.11$, $\psi(0) = 0$, $x(0) = 0$, $y(0) = 0$.

4. CONCLUSION

To conclude, we discuss possible generalizations of the problem considered in this paper. First of all, we present the generalizations leading to the addition of a potential:

1) At the attachment point P of the platforms, a spring with stiffness k is placed (see, e. g., [12]):

$$U(\varphi) = \frac{k\varphi^2}{2}.$$

In this case, the problem reduces again to investigating the vector field on $\mathbb{T}^2$ and to classifying singular points on it.

2) Motion of the *roller racer* on an inclined plane. We direct the axis Ox along the line of steepest descent and let

$$U(\varphi, \psi, x) = g \sin \chi\big((m_1 + m_2)x - m_1(a_1 + c_1)\cos\psi - m_2(a_2 - c_2)\cos(\psi + \varphi)\big),$$

where χ is the angle of inclination of the plane to the horizontal. It can be shown that then the closed system describing the evolution of $(v_1, v_2, \varphi, \psi)$ decouples. For it to be integrable by quadratures, one needs two additional integrals and an invariant measure. A point of interest in this case is the asymptotic motion. Asymptotic chaos, featuring sensitivity to initial conditions, for a Chaplygin sleigh on an inclined plane is discussed in [10].

Another generalization can be made by increasing the number of platforms (parts) of the vehicle. The case of three platforms moving by inertia in which the phase flow of the reduced system is three-dimensional is of special interest from the viewpoint of qualitative behavior.

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