Simplex

Linear programs

- A solution is feasible iff it satisfies all the constraints
  - Feasible region = set of feasible solutions
- An optimal solution will be corner of the feasible region that intersects first/last with lines of the form z=c (lines with gradient of ratio of coefficients in z)
- An LP is in standard form iff all constraints are equality constraints, all variables are non-negative, and the objective function is maximization
  - Objective function is multiplied by -1 to turn minimization into maximization — <u>don't forget to reverse at end!</u>
  - Slack variables are added to turn ≤ into =
  - Excess variables are subtracted to turn ≥ into =
  - In this module, excess and slack variables draw from the same pool of subscripts
  - If some x_n is unrestricted in sign, substitute $x_n=x_n'-x_n''$ where x_n' and x_n'' are both non-negative

Simplex principles

- **A basic solution to $A_{mxn}x = b$ is obtained by picking n-m non-basic variables to set equal to zero and solving for the remaining m basic variables**
  - Variables can be isolated by reducing A to row echelon form
- High-level overview of simplex:
  0. Pick the basic (non-zero) variables
  1. Solve for the basic variables
  2. (Check if solution is feasible, if not pick different basic variables)
  3. Substitute variables into objective function
  4. Check if solution is optimal, if not move a non-basic variable into basis
     - Optimal iff all non-basic variables have non-positive coefficients in objective function

Simplex tableau

0. Find an initial basic feasible solution e.g. all decision variables non-basic if all constraints are ≥
1. **Check for feasibility by checking that RHS is non-negative in every row**
2. **Check for optimality by checking all non-basic variables have non-negative** (as they have been moved to LHS) **coefficients in objective function row**
3. **If feasible but not optimal:**
    a. **Move non-basic variable with most negative coefficient in row 0 (objective function) into basis**
        i. **Find the row with smallest positive RHS/(coefficient in that row of variable moving into basis)**
        ii. **Get new basic variable a coefficient of 1 in that row and a coefficient of 0 in all other rows**
4. **If feasible and optimal:**
    a. **A feasible optimal solution is: Non-basic variables equal to 0, basic variables equal to RHS of row they have a 1 in <u>— gives an objective function value of its RHS</u>**

- Provided initial tableau is feasible, it will not become infeasible
- *If a non-basic variable has a coefficient of zero in the objective function row, then there exists another optimal solution where it is basic*

Duality

Constructing dual using matrices

- Any LP can be written in the form max $c^T x$ s.t. $Ax \leq b \wedge x \geq 0$
  - Such an LP has dual min $b^T x$ s.t. $A^T x \geq c \wedge y \geq 0$

Intuition

- *Consider the problem of trying to find an upper bound on the objective function value of a feasible solution to max $z=a_1x_1+\ldots+a_nx_n$ s.t.*
  $B_{11}x_1+b_{12}x_2+\ldots+b_{1n}x_n \leq c_1$
  …
  $B_{n1}x_1+b_{n2}x_2+\ldots+b_{nn}x_n \leq c_n$
  $x_1, \ldots, x_n \geq 0$

  Idea: Write z as sum of scalar multiples of the inequalities
  Picking $y_1,\ldots,y_n \geq 0$ s.t. $y_1b_{11}+\ldots+y_nb_{n1} \geq a_1 \wedge \ldots \wedge y_nb_{1n}+\ldots+y_nb_{nn} \geq a_n$ and thus deduce that $z \leq y_1c_1+\ldots+y_nc_n$
  In matrix language we have max $z = a^Tx$ s.t. $Bx \leq c \wedge x \geq 0 \Rightarrow z \leq c^Ty$ for all y s.t. $B^Ty \geq c \wedge y \geq 0$
  <u>(Note that the sign on the constraints flipped except for the sign constraint)</u>

  Now consider trying to find the least upper bound (i.e. something providing a good bound for an optimal solution (it turns out this will give the exact value for an optimal solution)), this leads us to $z \leq z^*$ where z^* is determined by solving the following LP
  min $z^* = c^Ty$ s.t.
  $B^Ty \geq c \wedge y \geq 0$

Constructing dual

- Each decision variable of primal corresponds to a constraint of dual and each constraint of primal corresponds to a decision variable of dual
- **Type of objective of dual is opposite to that of primal**
- **Coefficient of the ith dual variable in the dual objective function is the RHS of the ith constraint (ignoring sign restrictions) in primal**
  - RHS of the ith constraint (ignoring sign restrictions) in dual is the coefficient of the ith variable in primal objective function
- **The coefficient of the ith dual variable in the jth dual constraint is the coefficient of the jth primal variable in the ith primal constraint**
- **<u>When going from max to min:</u> Inequality of ith constraint in dual is the same as sign restriction of ith variable in primal except no sign restriction goes to equality**
  - **<u>When going from max to min:</u> Sign restriction of ith variable in dual is opposite of inequality of ith constraint in primal with equality going to no sign restriction**
- **<u>When going from min to max:</u> Inequality of ith constraint in dual is the opposite as sign restriction of ith variable in primal with equality going to no sign restriction**
  - **<u>When going from max to min:</u> Sign restriction of ith variable in dual is same as inequality of ith constraint in primal except un sign restriction goes to equality**

Weak and strong duality

- **Weak duality theorem: If a pair of feasible solutions to primal and dual give objective function values Z_P and Z_D respectively then $Z_P \leq Z_D$**
- Strong duality theorem: Exactly one of the following is true:
  - Primal and dual are both feasible (set of feasible solutions is non-empty) problems and when both at an optimal solution $Z_P = Z_D$
  - Primal and dual are both infeasible (set of feasible solutions is empty) problems
  - One of primal and dual is unbounded (there exists a sequence of feasible solutions for which objective function tends to $+\infty$ for a maximization problem and $-\infty$ for a minimization problem) and the other is infeasible
- **Consequences of strong duality:**
  - **One of primal and dual has an optimal solution (problem is feasible and bounded) ⇒ both have optimal solutions (and values of objective function are the same at optimal solutions)**
  - **One of primal and dual is infeasible ⇒ other is infeasible or unbounded**
  - **One of primal and dual is unbounded ⇒ other is infeasible**

Complementary slackness

- Complementary slackness theorem: Let s_i be the slack/excess variable of the ith constraint in the primal and e_i be the excess/slack variable of the ith constraint in the dual.
  Let x_i be the ith decision variable in the primal and u_i be the the ith decision variable in the dual.
  Then, x and u are optimal solutions iff the following complementary slackness conditions hold:
  $u \cdot s$ = zero vector
  $e \cdot x$ = zero vector
  - **Corollary: In an optimal solution, if the slack/excess variable for the ith row of the dual is non-zero, then the ith primal decision variable is zero** — this is useful if dual has fewer decision variables than primal as can solve dual more easily and can from that obtain a solution to primal

Transport problems

Transportation problems (TPs)

- A TP consists of variables s_i, d_j, c_{ij} the amounts that can be shipped from each supply node and to each demand node and the costs of doing so and decision variables x_{ij} the actual amounts shipped
- We are only taught how to solve balanced TPs (total supply = total demand), to solve an unbalanced problem we must first balance by adding a dummy node:
  - **If total supply < total demand, problem is infeasible unless we are allowed to not meet demand subject to paying some finite fine per unit demanded not delivered — add a dummy supply node with supply = total supply - total demand and “transportation” cost of the fine**
  - **If total supply > total demand, add a dummy demand node with demand = total demand — total supply and “transportation” cost of 0**

Finding a basic feasible solution

- As all decision variables zero is not a feasible solution we can't fill out a simplex tableau with our LP straight away
- **Transportation tableau = a table with jth column headed by d_j, ith row headed by s_i, and cell ij x_{ij}**
- **North-west-corner method: Start at top left corner of the tableau. Set current cell to minimum of column and row and update row and column headers accordingly.** If column was picked, move right. If row was picked, move down
    - **If column=row, move <u>either</u> right or down**
- **Minimum-cost method: Create a transportation tableau with c_{ij} in the corner of each cell. Pick cell with lowest cost. Set current cell to minimum of column and row and update row and column headers accordingly.** Ignore column/row once it is headed by zero

Solving a TP

1. **Balance**
2. **Find initial basic feasible solution**
3. **Calculate dual variables $u_1, \dots, u_m, v_1, \dots, v_n$ using:**
   $u_1=0$
   $v_j=c_{ij}-u_i$ for all i, j <u>s.t. x_{ij} is basic</u>
4. **Calculate slacks $s_{ij}=u_i+v_j-c_{ij}$** (which for some bizarre reason are called d_{ij} in lectures making them look at a glance like demands) **for all i, j <u>s.t. x_{ij} is non-basic</u>**
5. **If all calculated slacks are all non-positive, solution is optimal**
6. **If solution is not optimal, move $x'=x_{ij}$ with largest s_{ij} into basis**
   a. **Draw a graph of supply nodes and demand nodes with edges where $x_{ij}\neq 0$**
   b. **Add a dashed edge x'**
   c. **Spot the cycle (i.e. no repetition of edges allowed) that starts by following x' out of d_j and ends by returning to d_j**
      i. **Label edges as odd and even respectively starting with odd**
      ii. **Subtract the even variable with the smallest value from every other even variable and add it to every odd variable** — this causes this variable to leave the basis and x' to enter it
   d. **Return to step 3**

Sensitivity analysis of TPs

- **If x_{ab} is non-basic** (no goods are shipped from s_a to d_b) **in the optimal solution, then c_{ab} can change without affecting the optimality of the solution iff the non-positivity of $s_{ab}' = u_a+v_b-c_{ab}'$ is not affected**
    - As x_{ab} is non-basic, we know that we won't affect the non-positivity of other slacks inside this range
    - Hence, c_{ab} can increase by any amount and decrease by up to $|s_{ab}|$
- **If x_{ab} is basic in the optimal solution, then c_{ab} can change without affecting the optimality of the solution iff the non-positivity of all $s_{ib}' = u_i+v_b-c_{ib}'$ is not affected — <u>as when solving, only consider s_{ib}' for i s.t. x_{ib} is non-basic</u>**
    - As when solving, **use $v_b=c_{ab}'-u_a$ for each of these**
    - We won't affect the non-positivity of other slacks inside this range
    - c_{ab} can decrease by any amount and increase by up to a certain amount

(Zero-sum matrix) Game theory

Zero-sum matrix games

- Row player and column player pick without knowledge of what the other has picked — **selected cell = amount row player gains** = amount column player loses
- Strategy = a complete description of how to play a game — no ingenuity required by executor of strategy
    - Pure strategy = A strategy that results in the same choices every time
    - Mixed strategy = A strategy that assigns probabilities to choices which are used to randomly pick
- Solution of a game = an optimal strategy

Reduction

- A row r is strictly dominated iff there exists another row r’ such that the element in every column of r’ is greater than that of r
- A row r is weakly dominated iff there exists another row r’ such that the element in every column of r’ is greater than or equal to that of r and in at least one column r’ is greater than that of r
- A row is dominated iff it is weakly dominated or strongly dominated
- There is an dual notion of column domination with less than
- Game reduction = the process of removing domainted rows and columns to produce a reduced matrix
    - Any solution to the reduced matrix is a solution to the original matrix
- If the reduced matrix is 1x1 we have obtained a solution (in pure strategies), otherwise we analyse the reduced matrix

Looking for a solution in pure strategies: Minimax

- Idea: Calculate the worst case scenario that could occur from each choice you could make, pick the choice with the best worst case scenario
- For row player: Find the minimum of each row and pick the row with the greatest minimum
- For column player: Find the maximum of each column and pick the column with the least maximum
- If max row min = min column max, this pair of strategies is a saddle point (if a player knew what the other had played they would not wish to play differently to what the strategy suggests) and thus a solution
  - **Value of game = max row min = min column max**
- If max row min ≠ min column max, this pair of strategies is not a solution — need to use a mixed strategy instead

Finding for a solution in mixed strategies: Graphing

- *Only works on pen and paper for a matrix with a dimension of 2 — in this module a matrix that requires a mixed strategy will always reduce to one with a dimension of 2*

1. **Act as the player who only has two choices, assign one a probability of p and hence the other a probability of 1-p**
2. **For each choice opponent has, multiply payoff under each choice you can make by its probability and sum them and plot the line — this results in a line for each choice opponent has**
3. **Draw line that is the minimum/maximum of all the lines and find the value of p that this line attains maximum/minimum at**
4. **Other player will only want to play their strategies that correspond to lines that intersected — same process for them with only these**

- **Value of game = value of y at chosen value of p — this will be the same for each player — a useful check**
- *If there a solution in pure strategies, this will give it, but this module expects checking for such a solution before resorting to graphing*

Solving a family of games

- **Find for what values of parameter domination occurs for and solve for each possible domination case and for the no domination case**