

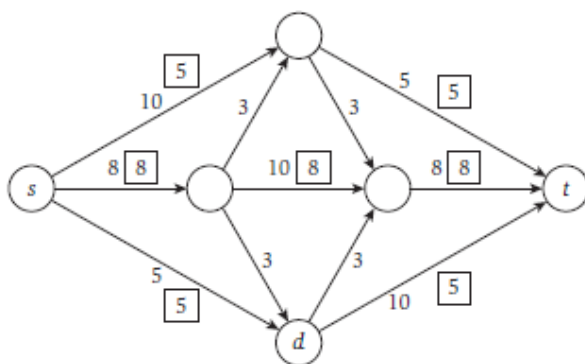
Answer the questions in the boxes provided on the question sheets. If you run out of room for an answer, add a page to the end of the document.

Name: \_\_\_\_\_

Wisc id: \_\_\_\_\_

## Network Flow

1. *Kleinberg, Jon. Algorithm Design (p. 415, q. 3a)* The figure below shows a flow network on which an  $s - t$  flow has been computed. The capacity of each edge appears as a label next to the edge, and the flow is shown in boxes next to each edge. An edge with no box has no flow being sent down it.



- (a) What is the value of this flow?

- (b) Please draw the **residual graph** associated with this flow.

- (c) Is this a maximum  $s - t$  flow in this graph? If not, describe an augmenting path that would increase the total flow.

2. *Kleinberg, Jon. Algorithm Design (p. 419, q. 10)* Suppose you are given a directed graph  $G = (V, E)$ . This graph has a positive integer capacity  $c_e$  on each edge, a source  $s \in V$ , a sink  $t \in V$ . You are also given a maximum  $s - t$  flow through  $G$ :  $f$ . You know that this flow is *acyclic* (no cycles with positive flow all the way around the cycle), and every flow  $f_e \in f$  has an integer value.

Now suppose we pick an edge  $e^*$  and reduce its capacity by 1 unit. Show how to find a maximum flow in the resulting graph  $G^*$  in time  $O(m + n)$ , where  $n = |V|$  and  $m = |E|$ .

3. *Kleinberg, Jon. Algorithm Design (p. 420, q. 11)* A friend of yours has written a very fast piece of code to calculate the maximum flow based on repeatedly finding augmenting paths. However, you realize that it's not always finding the maximum flow. Your friend never wrote the part of the algorithm that uses backward edges! So their program finds only augmenting paths that include all forward edges, and halts when no more such augmenting paths remain. (Note: We haven't specified *how* the algorithm selects forward-only augmenting paths.)

When confronted, your friend claims that their algorithm may not produce the maximum flow every time, but it is guaranteed to produce flow which is within a factor of  $b$  of maximum. That is, there is some constant  $b$  such that no matter what input you come up with, their algorithm will produce flow at least  $1/b$  times the maximum possible on that input.

Is your friend right? Provide a proof supporting your choice.

4. *Kleinberg, Jon. Algorithm Design (p. 418, q. 8)* Consider this problem faced by a hospital that is trying to evaluate whether its blood supply is sufficient:

In a (simplified) model, the patients each have blood of one of four types: A, B, AB, or O. Blood type A has the A antigen, type B has the B antigen, AB has both, and O has neither. Patients with blood type A can receive either A or O blood. Likewise patients with type B can receive either B or O type blood. Patients with type O can only receive type O blood, and patients with type AB can receive any of the four types.

- (a) Let integers  $s_O, s_A, s_B, s_{AB}$  denote the hospital's blood supply on hand, and let integers  $d_A, d_B, d_O, d_{AB}$  denote their projected demand for the coming week. Give a polynomial time algorithm to evaluate whether the blood supply is enough to cover the projected need.

- (b) Network flow is one of the most powerful and versatile tools in the algorithms toolbox, but it can be difficult to explain to people who don't know algorithms. Consider the following instance. Show that the supply is **insufficient** in this case, and provide an explanation for this fact that would be understandable to a non-computer scientist. (For example: to a hospital administrator.) Your explanation should not involve the words *flow*, *cut*, or *graph*.

blood type	supply	demand
O	50	45
A	36	42
B	11	8
AB	8	3

5. Kleinberg, Jon. *Algorithm Design* (p.416 q.6). Suppose you're a consultant for the Ergonomic Architecture Commission, and they come to you with the following problem.

They're really concerned about designing houses that are "user-friendly", and they've been having a lot of trouble with the setup of light fixtures and switches in newly designed houses. Consider, for example, a one-floor house with  $n$  light fixtures and  $n$  locations for light switches mounted in the wall. You'd like to be able to wire up one switch to control each light fixture, in such a way that a person at the switch can see the light fixture being controlled.

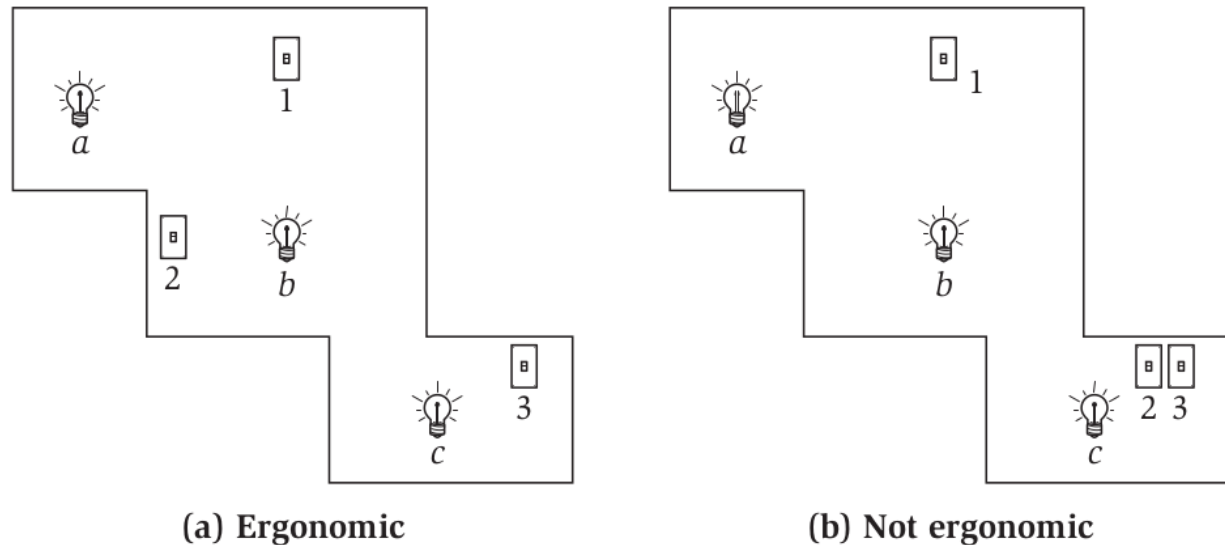
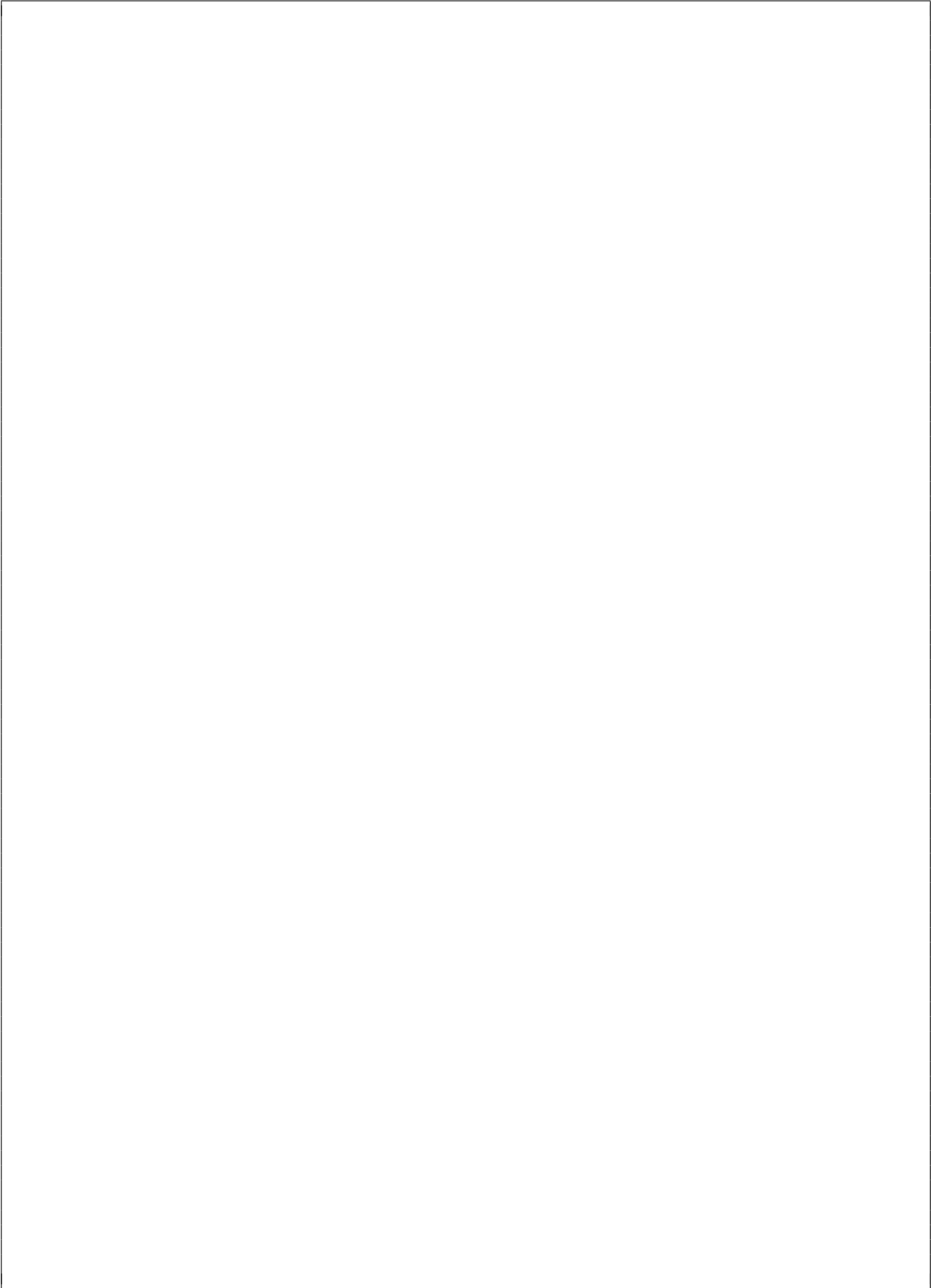


Figure 1: The floor plan in (a) is ergonomic, because we can wire switches to fixtures in such a way that each fixture is visible from the switch that controls it. (This can be done by wiring switch 1 to  $a$ , switch 2 to  $b$ , and switch 3 to  $c$ .) The floor plan in (b) is not ergonomic, because no such wiring is possible.

Sometimes this is possible and sometimes it isn't. Consider the two simple floor plans for houses in Figure 1. There are three light fixtures (labelled  $a, b, c$ ) and three switches (labelled 1, 2, 3). It is possible to wire switches to fixtures in Figure 1(a) so that every switch has a line of sight to the fixture, but this is not possible in Figure 1(b).

Let's call a floor plan, together with  $n$  light fixture locations and  $n$  switch locations, ergonomic if it's possible to wire one switch to each fixture so that every fixture is visible from the switch that controls it. A floor plan will be represented by a set of  $m$  horizontal or vertical line segments in the plane (the walls), where the  $i$ -th wall has endpoints  $(x_i, y_i), (x'_i, y'_i)$ . Each of the  $n$  switches and each of the  $n$  fixtures is given by its coordinates in the plane. A fixture is visible from a switch if the line segment joining them does not cross any of the walls.

Give an algorithm to decide if a given floor plan is ergonomic. The running time should be polynomial in  $m$  and  $n$ . You may assume that you have a subroutine with  $O(1)$  running time that takes two line segments as input and decides whether or not they cross in the plane.



6. Kleinberg, Jon. *Algorithm Design* (p.426 q.20).

Your friends are involved in a large-scale atmospheric science experiment. They need to get good measurements on a set  $S$  of  $n$  different conditions in the atmosphere (such as the ozone level at various places), and they have a set of  $m$  balloons that they plan to send up to make these measurements. Each balloon can make at most two measurements. Unfortunately, not all balloons are capable of measuring all conditions, so for each balloon  $i = 1, \dots, m$ , they have a set  $S_i$  of conditions that balloon  $i$  can measure. Finally, to make the results more reliable, they plan to take each measurement from at least  $k$  different balloons. (Note that a single balloon should not measure the same condition twice.) They are having trouble figuring out which conditions to measure on which balloon.

**Example.** Suppose that  $k = 2$ , there are  $n = 4$  conditions labelled  $c_1, c_2, c_3, c_4$ , and there are  $m = 4$  balloons that can measure conditions, subject to the limitation that  $S_1 = S_2 = c_1, c_2, c_3$ , and  $S_3 = S_4 = c_1, c_3, c_4$ . Then one possible way to make sure that each condition is measured at least  $k = 2$  times is to have

- balloon 1 measure conditions  $c_1, c_2$ ,
  - balloon 2 measure conditions  $c_2, c_3$ ,
  - balloon 3 measure conditions  $c_3, c_4$ , and
  - balloon 4 measure conditions  $c_1, c_4$ .
- (a) Give a polynomial-time algorithm that takes the input to an instance of this problem (the  $n$  conditions, the sets  $S_i$  for each of the  $m$  balloons, and the parameter  $k$ ) and decides whether there is a way to measure each condition by  $k$  different balloons, while each balloon only measures at most two conditions.

- (b) You show your friends a solution computed by your algorithm from (a), and to your surprise they reply, “This won’t do at all—one of the conditions is only being measured by balloons from a single subcontractor.” You hadn’t heard anything about subcontractors before; it turns out there’s an extra wrinkle they forgot to mention...

Each of the balloons is produced by one of three different subcontractors involved in the experiment. A requirement of the experiment is that there be no condition for which all  $k$  measurements come from balloons produced by a single subcontractor.

Explain how to modify your polynomial-time algorithm for part (a) into a new algorithm that decides whether there exists a solution satisfying all the conditions from (a), plus the new requirement about subcontractors.

7. Kleinberg, Jon. *Algorithm Design* (p.442, q.41).

Suppose you're managing a collection of  $k$  processors and must schedule a sequence of  $m$  jobs over  $n$  time steps.

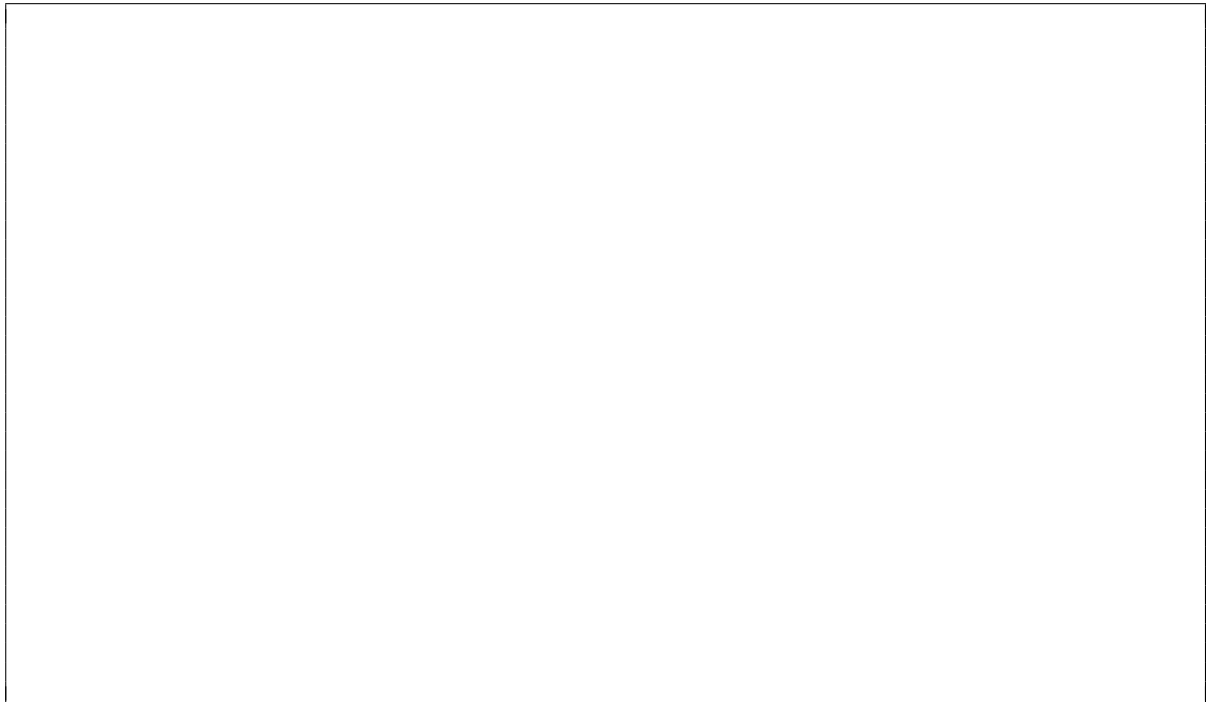
The jobs have the following characteristics. Each job  $j$  has an arrival time  $a_j$  when it is first available for processing, a length  $\ell_j$  which indicates how much processing time it needs, and a deadline  $d_j$  by which it must be finished. (We'll assume  $0 < \ell_j \leq d_j - a_j$ .) Each job can be run on any of the processors, but only on one at a time; it can also be preempted and resumed from where it left off (possibly after a delay) on another processor.

Moreover, the collection of processors is not entirely static either: You have an overall pool of  $k$  possible processors; but for each processor  $i$ , there is an interval of time  $[t_i, t'_i]$  during which it is available; it is unavailable at all other times.

Given all this data about job requirements and processor availability, you'd like to decide whether the jobs can all be completed or not. Give a polynomial-time (in  $k$ ,  $m$ , and  $n$ ) algorithm that either produces a schedule completing all jobs by their deadlines or reports (correctly) that no such schedule exists. You may assume that all the parameters associated with the problem are integers.

**Example.** Suppose we have two jobs  $J_1$  and  $J_2$ .  $J_1$  arrives at time 0, is due at time 4, and has length 3.  $J_2$  arrives at time 1, is due at time 3, and has length 2. We also have two processors  $P_1$  and  $P_2$ .  $P_1$  is available between times 0 and 4;  $P_2$  is available between times 2 and 3. In this case, there is a schedule that gets both jobs done.

- At time 0, we start job  $J_1$  on processor  $P_1$ .
- At time 1, we preempt  $J_1$  to start  $J_2$  on  $P_1$ .
- At time 2, we resume  $J_1$  on  $P_2$ . ( $J_2$  continues processing on  $P_1$ .)
- At time 3,  $J_2$  completes by its deadline.  $P_2$  ceases to be available, so we move  $J_1$  back to  $P_1$  to finish its remaining one unit of processing there.
- At time 4,  $J_1$  completes its processing on  $P_1$ . Notice that there is no solution that does not involve preemption and moving of jobs.





8. Kleinberg, Jon. *Algorithm Design* (p.444, q.45).

Consider the following definition. We are given a set of  $n$  countries that are engaged in trade with one another. For each country  $i$ , we have the value  $s_i$  of its budget surplus; this number may be positive or negative, with a negative number indicating a deficit. For each pair of countries  $i, j$ , we have the total value  $e_{ij}$  of all exports from  $i$  to  $j$ ; this number is always nonnegative. We say that a subset  $S$  of the countries is *free-standing* if the sum of the budget surpluses of the countries in  $S$ , minus the total value of all exports from countries in  $S$  to countries not in  $S$ , is nonnegative. Give a polynomial-time algorithm that takes this data for a set of  $n$  countries and decides whether it contains a nonempty free-standing subset.

### 9. Coding Question 1: Ford-Fulkerson Method

Implement the Ford-Fulkerson method for finding maximum flow in graphs with only integer edge capacities, in either C, C++, C#, Java, Python, or Rust. Be efficient and implement it in  $O(mF)$  time, where  $m$  is the number of edges in the graph and  $F$  is the value of the maximum flow in the graph. We suggest using BFS or DFS to find augmenting paths. (You may be able to do better than this.)

The input will start with a positive integer, giving the number of instances that follow. For each instance, there will be two positive integers, indicating the number of nodes  $n = |V|$  in the graph and the number of edges  $|E|$  in the graph. Following this, there will be  $|E|$  additional lines describing the edges. Each edge line consists of a number indicating the source node, a number indicating the destination node, and a capacity. The nodes are not listed separately, but are numbered  $\{1 \dots n\}$ .

Your program should compute the maximum flow value from node 1 to node  $n$  in each given graph.

A sample input is the following:

```
2
3 2
2 3 4
1 2 5
6 9
1 2 9
1 3 4
2 4 1
2 5 6
3 4 4
3 5 5
4 6 8
5 6 5
5 6 3
```

The sample input has two instances. For each instance, your program should output the maximum flow on a separate line. Each output line should be terminated by a newline. The correct output for the sample input would be:

```
4
11
```

**10. Coding Question 2: Perfect Matching**

Implement an algorithm to determine the maximum matching in a bipartite graph and if that matching is perfect (all nodes are matched) in either C, C++, C#, Java, Python, or Rust. Be efficient and use your max-flow implementation from the previous week.

The input will start with an positive integer, giving the number of instances that follow. For each instance, there will be 3 positive integers  $m$ ,  $n$ , and  $q$ . Numbers  $m$  and  $n$  are the number of nodes in node set  $A$  and node set  $B$ . Number  $q$  is the number of edges in the bipartite graph. For each edge, there will be 2 more positive integers  $i$ , and  $j$  representing an edge between node  $1 \leq i \leq m$  in  $A$  and node  $1 \leq j \leq n$  in  $B$ .

A sample input is the following:

```
3
2 2 4
1 1
1 2
2 1
2 2
2 3 4
2 3
2 1
1 2
2 2
5 5 10
1 1
1 3
2 1
2 2
2 3
2 4
3 4
4 4
5 4
5 5
```

The sample input has 3 instances.

For each instance, your program should output the size of the maximum matching, followed by a space, followed by an  $N$  if the matching is not perfect and a  $Y$  if the matching is perfect. Each output line should be terminated by a newline. The correct output to the sample input would be:

```
2 Y
2 N
4 N
```