Theoretical Computer Science

Winter semester 21/22

Prof. Dr. Georg Schied

Assignment 11

Deadline: Wednesday, 22 December 2021

10 out of 20 points have to be achieved in order to pass.

Exercise 11.1 - obligatory (4 points)

Which of the following propositional formulas are *satisfiable*, which are *valid* (*tautologies*), which are *unsatisfiable*?

(1) rich $\wedge$ sick $\rightarrow$ $\neg$ happy

(2) $\neg$ happy $\wedge$ $\neg$ sick $\wedge$ (happy $\vee$ sick)

(3) (sick $\rightarrow$ $\neg$ happy) $\vee$ sick

(4) (happy $\wedge$ $\neg$ rich) $\vee$ ($\neg$ happy $\wedge$ rich)

Exercise 11.2

Are both formulas equivalent? (give a short justification)

(1) $\neg$ (p $\wedge$ (q $\vee$ r)) and $\neg$p $\vee$ $\neg$q $\wedge$ $\neg$r

(2) a $\rightarrow$ b and $\neg$a $\rightarrow$ $\neg$b

(3) a $\rightarrow$ b and $\neg$b $\rightarrow$ $\neg$a

Exercise 11.3 - obligatory (6 points)

Let the following vocabulary for first-order logic formulas be given:

- Function symbols: f (unary)
  g (binary)
- Predicate symbols: P (unary)
  Q (binary)
- Constants: a, b
- Variables: x, y, z

Which of the following first-order formulas are **syntactically correct**? Indicate all errors.

(1) `Q(f(x), g(a))`

(2) $\exists$z f(z) $\rightarrow$ $\forall$`x Q(x, f(z))`

(3) $\exists$`z` $\forall$`x Q( f(x), z )` $\vee$ $\forall$`z P(g(z,z))`

(4) $\forall$ `y Q( f(y)` $\vee$ `g(a,b) )`

(5) $\exists\neg\forall$`y Q( y, f(a) )`

(6) $\exists$`y` $\neg$ `(` $\forall$`z P(f(x))` $\vee$ $\exists$`y Q(a,y) )`

Exercise 11.4

Let the following predicates on persons be given:

Male(x)	"x is male"
Brother(p_1, p_2)	"p_1 is brother of p_2"
Child(c, p)	"c is a child of p" (i.e. p is parent of c)
Aunt(a, p)	"a is aunt of p"

a) Explain in everyday language the meaning of the following formulas. Are the formulas true if they are considered as statements about all living persons?

(1) $\neg\forall p\ \text{Male}(p)$

(2) $\forall p\ \exists b\ \text{Brother}(b, p)$

(3) $\exists p\ \exists b\ \text{Brother}(b, p)$

(4) $\exists p\ \forall b\ \text{Brother}(b, p)$

(5) $\forall p\ \forall b\ \text{Brother}(b, p)$

b) Formulate as a first-order formula:

(1) "There is someone who is male."

(2) If a person p_1 is brother of p_2, then also p_2 is brother of p_1.

(3) "Not every person has a child."

(4) "A male person is never aunt of someone else."

(5) "If any two persons are brothers, then they have a common parent."

(6) "If a person is brother of someone else, then the person is male."

Exercise 11.5 - obligatory (10 points)

Let the following vocabulary for first-order formulas be given. We assume natural numbers $\mathbb{N}$ as domain of discourse.

Constants:	0, 1, 2	
Function symbols:	+	(addition, binary),
	$\cdot$	(multiplication, binary)
Predicate symbols:	=	(equality, binary),
	<	(less-than relation, binary),
	Prime	(is a prime number, unary)

As usual, +, $\cdot$, =, and < can be written in infix notation.

Formulate the following statements about natural numbers as first order formulas:

a) For any number n, $n+n$ is never a prime number.

b) If p is a prime number greater than 2, then $p+1$ is not prime.

c) There is no number $n > 1$, so that $n+n$ is a prime number.

d) The sum of two numbers is always less than the product of both numbers.

e) For every prime number, there is a larger number that is also prime.

Exercise 11.6

Let the following vocabulary for first-order formulas be given:

Constants:	a, b	
Variables:	x, y	
Function symbols:	f (unary),	g (binary)
Predicate symbols:	P (unary),	Q (binary)

Two interpretations *A* and B are defined as following:

	Interpretation *A*	**Interpretation B**
Universe	$\mathbb{N}$ (natural numbers)	$\{0,1\}^*$ (strings over alphabet $\{0,1\}$)
a	1	00
b	2	11
x	10	101
y	3	ε
f	$f_A(n) = 2n$	$f_B(w) = w^2$
g	$g_A(n,k) = n + k$ (sum)	$g_B(u,w) = uw$ (concatenation)
P(*x*)	*x* is odd	$\|x\|$ is even
Q(x,y)	$x \leq y$	$x = y$

a) What ist the result of interpretations ***A*** and **B** for the following terms?

(1) f(x)

(2) g(b, y)

(3) g(f(a), f(x))

b) What are the results of interpretations ***A*** and **B** for the following first-order formulas?

(1) Q(a, f(x))

(2) P(a) $\wedge$ Q(b,x)

(3) $\exists$x P(x)

(4) $\forall$x Q(x, f(x))

c) Is formula $\exists$x P(x) *satisfiable*?

d) Is formula $\forall$x Q(x, f(x)) *valid*?