Math 126 Homework Assignment #1, Due Wed Feb 10

1 Required Problems

1. (a) Prove Nichomachus' theorem by induction, i.e., $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$ for all positive integers n.

   (b) Prove that the cube of any integer can written as the difference of two squares.

2. Derive the formula $\text{Te}_n = \dfrac{n(n+1)(n+2)}{6}$ for the nth tetrahedral number. To do this, I don't want you to use induction. Rather, I want you to use the properties of summation Σ and the fact that $\text{Te}_n = \displaystyle\sum_{k=1}^{n} \text{Tr}_k$ is a sum of triangular numbers $\text{Tr}_k = \dfrac{k(k+1)}{2}$.

3. Derive Newton's identity
$$\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}$$
for integers $n \geq k \geq r \geq 0$. (Hint: Start with the LHS, use the definition of binomial coefficients, and then cancel/shuffle some factors to make the expression look like the RHS.)

4. For integers $n \geq 2$, use induction and Pascal's rule to prove that
$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots \binom{n}{2} = \binom{n+1}{3}.$$

5. Show that the sum of any two consecutive triangular numbers $T_n + T_{n+1}$ is a perfect square. (Hint: You should not use induction, just write out the sum, expand the expressions, and then find a way of recollecting them as the square of an expression in n.)

6. Show that for any $n \in \mathbb{N}$ the sum of the reciprocals of the first n triangular numbers if less than 2... i.e., that
$$\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \cdots + \frac{1}{T_n} < 2$$
Hint: the sum "telescopes" to $2(1 - 1/(n+1))$.

2 Optional Challenge Problems

1. (a) Find the limit in problem 8... i.e., compute
$$\sum_{n=1}^{\infty} \frac{1}{T_n} = \text{ ?}$$

   (b) Now compute the sum of reciprocals of pentagonal numbers
$$\sum_{n=1}^{\infty} \frac{1}{P_n} = \text{ ?}$$

2. Use required problem 4 above and the identity $m^2 = 2\binom{m}{2} + m$ (for $m \geq 2$) to reprove the identity
$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

3. Write down the precise relationship we noticed mentioned between the Fibonacci numbers and the diagonals of Pascal's triangle. Prove this identity.