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Nonsingular fast terminal sliding-mode control for nonlinear dynamical systems

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SUMMARY

This paper investigates fast finite-time control of nonlinear dynamics using terminal sliding-mode (TSM) scheme. Some new norms of fast TSM strategies are proposed, and a faster convergence rate is established in comparison with the conventional fast TSM. A novel concept of nonsingular fast TSM, which is able to avoid the possible singularity during the control phase, is adopted in the robust high-precision control of uncertain nonlinear systems. Numerical simulation on a two-link rigid robot manipulator demonstrates the effectiveness of the proposed algorithm. Copyright © 2010 John Wiley & Sons, Ltd.

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KEY WORDS: terminal sliding mode; fast convergence rate; singularity; Lyapunov stability; nonlinear systems

1. INTRODUCTION

In recent decades, the sliding-mode control (SMC) technique has been widely used in plenty of uncertain systems due to its attractiveness such as strong robustness, order reduction, easier implementation and design simplification. To perform SMC design, one needs to first define a switching surface that prescribes the desired convergence property, and then design a sliding-mode controller to drive the system states to the chosen manifold, which is not affected by any uncertainties or disturbances [1–5]. One characteristic of the conventional SMC is that the plants usually converge to the equilibrium points asymptotically, and the reason is that the linear switching manifolds are commonly chosen with the asymptotical convergence property.

In terms of finite-time differential equations [6], terminal SMC has been developed in [7–12] to achieve finite-time convergence of the system dynamics in the switching surface. In the work [7–10], the first-order terminal sliding-mode (TSM) technique has been first developed for the control of second-order nonlinear systems. Yu and Man [11] have extended the TSM technique to high-order single-input and single-output (SISO) linear systems, where a hierarchical TSM structure has been proposed to establish the sequential convergence of the sliding variables. Then, a general TSM has been developed in [12] for multi-input and multi-output (MIMO) linear systems. Note that compared with the conventional sliding-mode scheme, TSM may not offer the same convergence performance when the system states are far away from the equilibrium. Yu and Man [13, 14] have adopted a fast terminal sliding-mode (FTSM) concept, which ensures fast transient

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convergence both at a distance from and at a close range of the equilibrium. However, through the analysis of control design, it can be found that the prior TSM and FTSM methods both suffer from a singularity problem. To overcome the difficulty, a few methods have been proposed. One approach is to switch the sliding mode between TSM and linear hyperplane-based sliding mode [12]. Another approach is to transfer the trajectory to a prescribed region in advance where no singularity occurs, which is the so-called two-phase control strategy [15]. It should be noted that these methods are adopting indirect approaches to avoid the singularity. In [16, 17], a new concept of nonsingular terminal sliding mode (NTSM) has been adopted to get rid of the singularity without adding any extra procedures. Based on the nonsingular strategy, Feng *et al.* [18] has put forward a second-order NTSM decomposed control algorithm to linear high-dimensional multivariable systems. Further, some more general forms of TSM, FTSM and NTSM have been advanced in [19]. Since the TSM strategies can speed up the convergence rate greatly, they are particularly useful in high-precision control cases. So far, they have been utilized into a wide range of applications, like the chaotic systems [20] and robotic manipulators [21, 22]. Nevertheless, it should be pointed out that the FTSM method still needs to be further investigated on better convergence rate and singularity avoidance.

In this paper, the FTSM technique is employed to study the finite-time stabilization problem of nonlinear dynamics. Some new models for FTSM are advanced to establish a faster rate of convergence in comparison with the traditional FTSM, especially when the system states are far away from the equilibrium. According to the new FTSM, an equivalent nonsingular fast terminal sliding mode (NFTSM) is put forward to get rid of the singularity completely. Then, to show the basic principle of NFTSM control technique, stabilization of a second-order nonlinear system is studied in detail. It is verified that the closed-loop system is provided with some superior properties such as fast finite-time convergence, strong robustness and singularity avoidance. Finally, the NFTSM strategy is proposed to control a class of practical systems including mechanical systems, and numerical simulation on rigid robotic manipulators is presented to validate the proposed algorithm.

The organization of this paper is given as follows. In Section 2, in virtue of the basic idea of TSM, we propose a few new forms of FTSM, and analyze their relevant properties. Also, some recent results are briefly reviewed. In Section 3, a second-order nonlinear system is stabilized to introduce the basic principle of NFTSM control technique, which is then utilized to fast finite-time control of a class of practical systems. In Section 4, a numerical example is performed in support of the proposed control scheme. Finally, concluding remarks in Section 5 close the paper.

2. BASIC CONCEPTS ON TSM

In this section, we review the results of TSM in recent studies, propose some new forms of FTSM, and analyze their properties for the application in control of nonlinear systems.

Before we proceed, we consider the following first-order nonlinear dynamics:

$$\dot{x} = -k\,\mathrm{sign}^a x, \quad a>0, \quad k>0, \tag{1}$$

where $x \in R$ and $\mathrm{sign}^a x := \mathrm{sign} x \cdot |x|^a$. It is clear to see that, given positive odd integers p, q such that $a = p/q$, (1) will reduce to $\dot{x} = -\mathrm{sign}^a x = -x^a$, which is the form of conventional TSM. It can be verified that the solution of (1) in the forward time direction is unique for any nonzero initial condition in spite of non-Liptchitz condition [23]. The properties of system (1) are listed: When $a = 1$, the system becomes $\dot{x} = -kx$, and it follows that the system is exponentially stable; When $0<a<1$, compared with the previous case, the system has a lower rate at a distance from zero and converges much more quickly near the origin, which results in finite-time convergence; When $a>1$, it offers a high convergence rate at a distance from zero, and it slows down as it comes close to the origin. Moreover, the more closer, the lower rate, and vice versa.

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Definition 1 (*Yu et al.* [*19*])
The TSM and NTSM are described by the following first-order nonlinear differential equations:

$$\sigma(t) = \dot{x} + k\,\mathrm{sign}^{a} x = 0, \quad k>0, \ 0<a<1, \tag{2a}$$

$$\sigma(t) = x + k'\,\mathrm{sign}^{a'} \dot{x} = 0, \quad k'>0, \ 1<a'<2. \tag{2b}$$

Remark 1
It is evident that TSM (2a) is equivalent to NTSM (2b) provided that $k'=k^{-a'}$ and $a'=a^{-1}$. Based on the finite-time differential equation [6], we conclude that the dynamics (2a) and (2b) are both globally finite-time stable, and they converge to the equilibrium point within the time

$$T_t = \int_0^{|x(0)|} \frac{1}{kx^a}\,\mathrm{d}x = \frac{|x(0)|^{1-a}}{k(1-a)} \quad \text{and} \quad T_n = \int_0^{|x(0)|} \frac{k'^{1/a'}}{x^{1/a'}}\,\mathrm{d}x = \frac{k'^{1/a'}\cdot|x(0)|^{1-1/a'}}{1-1/a'},$$

respectively. Note that TSM and NTSM converge in a relatively slow rate when far away from the equilibrium point. The following FTSM and NFTSM concepts are developed to remedy this drawback.

Definition 2
The FTSM is described by the following differential equation:

$$\sigma(t) = \dot{x} + k_1\,\mathrm{sign}^{a_1} x + k_2\,\mathrm{sign}^{a_2} x = 0, \quad k_1>0, \ k_2>0, \ a_1 \geqslant 1, \ 0<a_2<1. \tag{3}$$

Remark 2
It can be found that FTSM (3) has the form of FTSM [19] when $a_1=1$, and further, (3) coincides with FTSM in [13, 14] if there exists positive odd integers p, q satisfying $a_2 = p/q$ with $p<q$. By solving the differential equation (3), it is concluded that $x=0$ will be reached in a finite time determined by

$$\begin{aligned} T_{f1} = \int_0^{|x(0)|} \frac{1}{k_1 x^{a_1} + k_2 x^{a_2}}\,\mathrm{d}x &= \frac{|x(0)|^{1-a_1}}{(1-a_1)} \cdot k_1^{(1-a_1)/a_1} \\ &\cdot \mathrm{F}\left(1, \frac{a_1-1}{a_1-a_2}; \frac{2a_1-a_2-1}{a_1-a_2}; -k_2 k_1^{-1} |x(0)|^{a_2-a_1}\right), \end{aligned} \tag{4}$$

where $\mathrm{F}(\cdot)$ denotes Gauss' Hypergeometric function [24], and the conditions of a_1, a_2, k_1, k_2 induce that $\mathrm{F}(\cdot)$ will keep convergent. Usually, the exact form of $\mathrm{F}(\cdot)$ varies with the involved parameters. For example,

$$\mathrm{F}(a, b; b; x) = (1-x)^{-a}; \ \mathrm{F}(\tfrac{1}{2}, 1; \tfrac{3}{2}; -x^2) = x^{-1} \arctan x. \tag{5}$$

For more details on Gauss' Hypergeometric function, one can refer to the work [24].

Remark 3
It is noted that FTSM is the combination of cases mentioned at the beginning, which is interpreted in the following: When the system state stays at a distance from equilibrium, $k_1\mathrm{sign}^{a_1}x$ dominates over $k_2\mathrm{sign}^{a_2}x$. Thus (3) can be approximated by $\sigma(t) = \dot{x} + k_1\mathrm{sign}^{a_1}x = 0$, which guarantees a high convergence rate; When the system state is close to the origin, the dominant term $k_2\mathrm{sign}^{a_2}x$ determines finite-time convergence. Hereby, the dynamics converges very quickly in the whole FTSM (3). To demonstrate the fast convergence performance of FTSM, we consider the following sliding modes:

$$\sigma_1 = \dot{x} + \mathrm{sign}^{0.3} x = 0, \quad \sigma_2 = \dot{x} + \mathrm{sign}^{3} x + \mathrm{sign}^{0.3} x = 0, \quad \sigma_3 = \dot{x} + x + \mathrm{sign}^{0.3} x = 0 \tag{6}$$

with the initial state $x(0)=5$. It can be seen from Figure 1 that FTSMs have better convergence rate than the conventional TSM, and the convergence rate of FTSM at $a_1>1$ is higher than its counterpart at $a_1=1$.

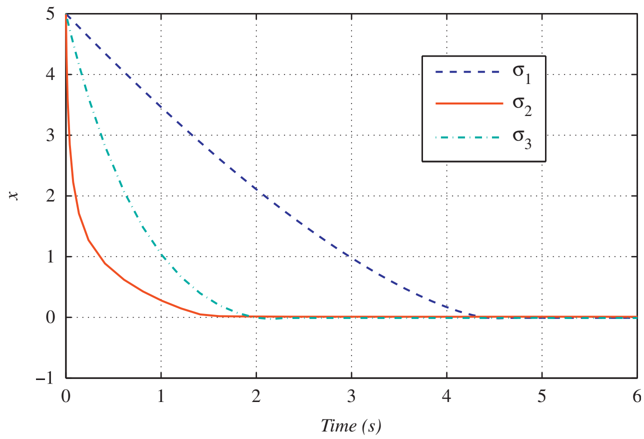

Figure 1. Comparison of TSM, FTSM ($a_1=1$) and FTSM ($a_1>1$).

Definition 2′
The FTSM can be also given by

$$\sigma(t)=\dot{x}+\bar{k}_1\,\mathrm{sign}^{1/\bar{a}_2}(x+\bar{k}_2\,\mathrm{sign}^{\bar{a}_1}x)=0,\quad \bar{k}_1>0,\ \bar{k}_2>0,\ \bar{a}_1>\bar{a}_2>1. \tag{7}$$

Remark 4
It is noted that $|x|\ll\bar{k}_2|x|^{\bar{a}_1}$ when the state x is far away from zero. At this time, (7) can be approximated by the following equation:

$$\sigma_1(t)=\dot{x}+\bar{k}_1\bar{k}_2^{1/\bar{a}_2}\,\mathrm{sign}^{\bar{a}_1/\bar{a}_2}x=0,\quad \bar{a}_1/\bar{a}_2>1, \tag{8}$$

which prescribes a fast convergence rate. As x closes to zero, x dominates in the term $x+\bar{k}_2\,\mathrm{sign}^{\bar{a}_1}x$. The approximated equation of (7) is given as

$$\sigma_2(t)=\dot{x}+\bar{k}_1\,\mathrm{sign}^{1/\bar{a}_2}x=0,\quad 1/\bar{a}_2<1, \tag{9}$$

which is the form of TSM (2a). In terms of finite-time differential equation, it can be easily verified that the dynamics (7) will reach the equilibrium in

$$T_{f2}=\int_0^{|x(0)|}\frac{1}{\bar{k}_1(x+\bar{k}_2x^{\bar{a}_1})^{1/\bar{a}_2}}\,\mathrm{d}x=\frac{\bar{a}_2|x(0)|^{1-1/\bar{a}_2}}{\bar{k}_1(\bar{a}_2-1)}$$
$$\cdot\mathrm{F}\left(\frac{1}{\bar{a}_2},\frac{\bar{a}_2-1}{(\bar{a}_1-1)\bar{a}_2};1+\frac{\bar{a}_2-1}{(\bar{a}_1-1)\bar{a}_2};-\bar{k}_2|x(0)|^{\bar{a}_1-1}\right). \tag{10}$$

During the control design of FTSMs, there often exists a singularity problem. The reason is that the designed controllers have nonlinear terms $x^{-\alpha}(\alpha>0)$, which results in the unboundness of the control input when x diminishes to zero. To overcome this difficulty, NFTSM is put forward in the following.

Definition 3
NFTSM is described as

$$\sigma(t)=x+k_1'\,\mathrm{sign}^{a_1'}x+k_2'\,\mathrm{sign}^{a_2'}\dot{x}=0,\quad k_1'>0,\ k_2'>0,\ 1<a_2'<2,\ a_1'>a_2'. \tag{11}$$

Remark 5
If $k_1'=\bar{k}_2, k_2'=\bar{k}_1^{-\bar{a}_2}, a_2'=\bar{a}_2, a_1'=\bar{a}_1$, NFTSM (11) is equivalent to FTSM (7). It can be simply verified in the following. It follows from NFTSM (11) that $\mathrm{sign}x=-\mathrm{sign}\dot{x}$ and $k_2'|\dot{x}|^{a_2'}=|x|+$

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$k_1'|x|^{a_1'}$. Owing to the relationship of the involved parameters, the latter equality can be rewritten as $|\dot{x}|=\bar{k}_1(|x|+\bar{k}_2|x|^{\bar{a}_1})^{1/\bar{a}_2}$. Also, it is apparent that $|x+\bar{k}_2x^{\bar{a}_1}|=|x|+\bar{k}_2|x|^{\bar{a}_1}$. Thus, we have

$$\dot{x}=|\dot{x}|\cdot\text{sign}\dot{x}=-\bar{k}_1(|x+\bar{k}_2x^{\bar{a}_1}|)^{1/\bar{a}_2}\,\text{sign}(x+\bar{k}_2x^{\bar{a}_1})=-\bar{k}_1\,\text{sign}^{1/a_2}(x+\bar{k}_2x^{\bar{a}_1}), \tag{12}$$

which implies the FTSM (7). Clearly, the sufficiency of (11) has been ensured. Indeed, the necessity of (11) can also be verified by using the similar process. In addition, it can be concluded that the settling time of the dynamics (11) is given by

$$T_{nf}=\int_0^{|x(0)|}\frac{k_2'^{1/a_2'}}{(x+k_1'x^{a_1'})^{1/a_2'}}\,\mathrm{d}x=\frac{a_2'|x(0)|^{1-1/a_2'}}{k_1'(a_2'-1)}$$
$$\cdot\mathrm{F}\left(\frac{1}{a_2'},\frac{a_2'-1}{(a_1'-1)a_2'};1+\frac{a_2'-1}{(a_1'-1)a_2'};-k_1'|x(0)|^{a_1'-1}\right). \tag{13}$$

Lemma 1 (*Jensens inequality* [25, 26])

$$\left(\sum_{i=1}^{m}\theta_i^{\lambda_2}\right)^{1/\lambda_2}\leqslant\left(\sum_{i=1}^{m}\theta_i^{\lambda_1}\right)^{1/\lambda_1},\quad 0<\lambda_1<\lambda_2 \tag{14}$$

with $\theta_i\geqslant 0$, $1\leqslant i\leqslant m$.

Proposition 1
Given the equalities

$$k'=(k_2'/k_1')^{a'/a_2'},\quad a'=a_2'/a_1',\quad k_1=k_2'^{-1/a_2'},\quad k_2=(k_1'/k_2')^{1/a_2'},\quad a_1=a_2'^{-1},\quad a_2=a_1'a_2'^{-1}, \tag{15}$$

we can derive that

$$T_n>T_{nf}\geqslant T_{f1}, \tag{16}$$

i.e. for any nonzero initial state $x(0)$, the convergence time of NFTSM (11) is shorter than that of NTSM (2b), and not shorter than that of FTSM (3).

Proof
It follows from Lemma 1 that

$$(x+k_1'x^{a_1'})^{1/a_2'}\leqslant x^{1/a_2'}+k_1'^{1/a_2'}x^{a_1'/a_2'},\quad x>0, \tag{17}$$

which implies

$$T_{nf}=\int_0^{|x(0)|}\frac{k_2'^{1/a_2'}}{(x+k_1'x^{a_1'})^{1/a_2'}}\,\mathrm{d}x\geqslant\int_0^{|x(0)|}\frac{1}{k_2'^{-1/a_2'}x^{1/a_2'}+(k_1'/k_2')^{1/a_2'}x^{a_1'/a_2'}}\,\mathrm{d}x=T_{f1}. \tag{18}$$

On the other hand, it is obvious that

$$(x+k_1'x^{a_1'})^{1/a_2'}>k_1'^{1/a_2'}x^{a_1'/a_2'},\quad x>0, \tag{19}$$

which follows that

$$T_{nf}=\int_0^{|x(0)|}\frac{k_2'^{1/a_2'}}{(x+k_1'x^{a_1'})^{1/a_2'}}\,\mathrm{d}x<\int_0^{|x(0)|}\frac{(k_2'/k_1')^{1/a_2'}}{x^{a_1'/a_2'}}\,\mathrm{d}x=T_n. \tag{20}$$

This completes the proof. □

3. CONTROL DESIGN OF THE NFTSM CONCEPT

To briefly introduce the basic principle of NFTSM control technique, stabilization of a simple second-order nonlinear system is investigated in Section 3.1. It is concluded that the closed-loop system converges fast to the equilibrium within finite time while no singularity occurs. Then, in Section 3.2, the nonsingular technique is adopted to control a class of uncertain MIMO nonlinear systems. Accordingly, the proposed algorithm can be utilized to high-precision control of many mechanical systems, like robotic systems in practical applications.

3.1. A simple analysis

Consider the second-order dynamical system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= f(x_1, x_2) + g(x_1, x_2)u, \end{aligned} \tag{21}$$

where x_1, x_2 are the system states, and $f(\cdot), g(\cdot)$ are nonlinear functions of x_1, x_2, and u is the control input.

According to the strategy of NFTSM (11), we define the following switching surface:

$$\sigma = x_1 + \alpha\,\mathrm{sign}^{\gamma_1} x_1 + \beta\,\mathrm{sign}^{\gamma_2} x_2, \tag{22}$$

where $\alpha>0, \beta>0,\ 1<\gamma_2<2,\ \gamma_1>\gamma_2$. It is easy to see that the sliding surface (22) establishes the fast finite-time convergence. To make (21) reach the equilibrium fast along the given manifold, the control u is chosen as

$$u = -1/g[1/(\beta\gamma_2)\,\mathrm{sign}^{2-\gamma_2} x_2 \cdot (1+\alpha\gamma_1 |x_1|^{\gamma_1 - 1}) + f + K_1\,\mathrm{sign}\sigma + K_2\sigma], \tag{23}$$

in which $K_1>0, K_2>0$. In the sequel, we will testify that the prescribed manifold (22) is reached fast within finite time.

Theorem 1
For the system (21), if the sliding surface is chosen as (22), and it is controlled by u expressed in (23), then the system trajectory will converge fast to zero within finite time. Moreover, no occurring of singularity is ensured during the whole process.

Proof
Consider the Lyapunov function candidate as

$$V_1 = \tfrac{1}{2}\sigma^2, \tag{24}$$

whose time derivative along (21) is

$$\dot{V}_1 = \sigma \cdot (x_2 + \alpha\gamma_1 |x_1|^{\gamma_1 - 1} x_2 + \beta\gamma_2 |x_2|^{\gamma_2 - 1}(f + g\cdot u)). \tag{25}$$

From (23), it follows that

$$\begin{aligned} \dot{V}_1 &= -K_1\beta\gamma_2 |x_2|^{\gamma_2-1}|\sigma| - K_2\beta\gamma_2|x_2|^{\gamma_2-1}\sigma^2 \\ &= -\sqrt{2}K_1\beta\gamma_2|x_2|^{\gamma_2-1}V_1^{1/2} - 2K_2\beta\gamma_2|x_2|^{\gamma_2-1}V_1, \end{aligned} \tag{26}$$

which can be rewritten as

$$\dot{V}_1 = -\rho_1(x_2)V_1^{1/2} - \rho_2(x_2)V_1, \tag{27}$$

where $\rho_1(x_2) = \sqrt{2}K_1\beta\gamma_2|x_2|^{\gamma_2-1}$ and $\rho_2(x_2) = 2K_2\beta\gamma_2|x_2|^{\gamma_2-1}$. It can be derived that when $x_2 \neq 0$, $\rho_1>0, \rho_2>0$ and thus (27) has the form of FTSM (3) since $V_1 \geqslant 0$. In the light of the analysis in Section 2, it is concluded that for the case $x_2 \neq 0$, the system states will move fast to the sliding

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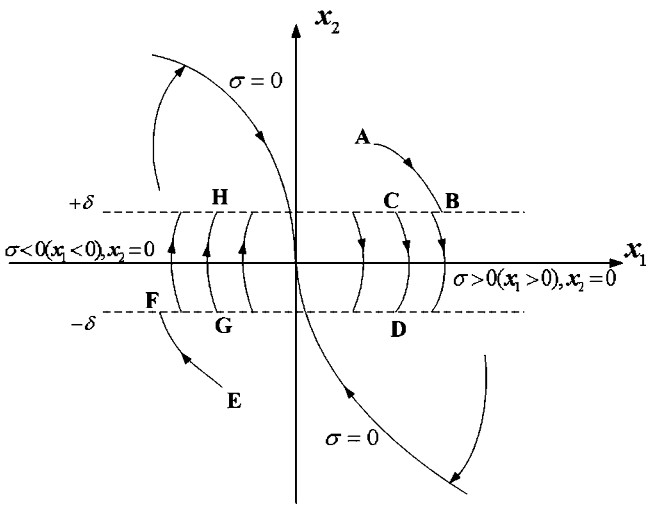

Figure 2. The phase-trajectory analysis of the system.

mode $\sigma=0$ within finite time. Then we consider the case $x_2=0$. By substituting (23) into the second equation of (21), one can obtain

$$\dot{x}_2=-1/\beta\,\mathrm{sign}^{2-\gamma_2}x_2\cdot(1+\alpha|x_1|^{\gamma_1-1})-K_1\,\mathrm{sign}\sigma-K_2\sigma. \tag{28}$$

From $x_2=0$, (28) can be rewritten as

$$\dot{x}_2=-K_1\,\mathrm{sign}\sigma-K_2\sigma, \tag{29}$$

which suggests that $\dot{x}_2<-K_1$ and $\dot{x}_2>K_1$ for the cases $\sigma>0$ and $\sigma<0$, respectively. It is easy to understand that the system will not always stay on the points ($x_1\neq 0$ and $x_2=0$). Moreover, it is reasonable to assume that there exists a vicinity of $x_2=0$, i.e. $|x_2|\leqslant\delta$ (δ is a small positive constant), satisfying $\dot{x}_2<-K_1$ and $\dot{x}_2>K_1$ for $\sigma>0$ and $\sigma<0$, respectively. Thus, it follows that the crossing of trajectories between two boundaries of $|x_2|\leqslant\delta$ is performed in finite time, and the dynamics from the region $|x_2|>\delta$ converge to the boundaries in finite time as well [16]. The analysis of phase trajectory is presented in Figure 2. Therefore, it is concluded that the sliding mode $\sigma=0$ can be reached from anywhere in the phase plane in finite time. Besides, in comparison with the form $\dot{x}_2=-K\,\mathrm{sign}\sigma(K>0)$ in [16], (29) ensures that the crossing of trajectories between $x_2=\delta$ and $x_2=-\delta$, and the convergence from the region $|x_2|>\delta$ to $\sigma=0$ occur at a faster rate due to the additional linear term $-K_2\sigma$.

It is noted that the control (23) does not contain any singularity since $\gamma_1>1$ and $\gamma_2<2$. Hence, we conclude that from any initial states, the closed-loop system may converge quickly to the origin along NFTSM (11) in finite time without any singularity. This completes the proof. □

Remark 6
From above, we can get the conclusion that u (23) is designed to achieve

$$\dot{\sigma}=-\rho_1^*(x_2)\,\mathrm{sign}\sigma-\rho_2^*(x_2)\sigma,\quad \rho_1^*=\rho_1/\sqrt{2}\geqslant 0,\ \ \rho_2^*=\rho_2/2\geqslant 0. \tag{30}$$

It is clear to see that (30) coincides with the constant plus proportional reaching law in [2], provided that $x_2\neq 0$ which often holds before $\sigma=0$ is arrived. As is known, such a reaching law can accelerate the convergence rate toward the manifold due to the proportional term $-\rho_2^*\sigma$, and thus it enhances the property of robustness. Moreover, it is concluded that by using this control law, the chattering phenomenon can be alleviated when system () operates on the sliding surface.

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However, in essence, chattering is caused by the discontinuity of the sign function. In order to totally eliminate the discontinuity, we propose

$$\psi(\sigma)=\frac{e^{b\sigma}-1}{e^{b\sigma}+1}, \quad b>0, \tag{31}$$

to replace the sign function in (23). It can be shown that the larger the parameter b, the better performance we will get. One can choose a suitable parameter in practical implementation.

3.2. Application of practical systems

Consider the uncertain nonlinear system

$$\begin{aligned}\dot{X}_1 &= F_1(X_1, X_2),\\ \dot{X}_2 &= F_2(X_1, X_2)+H(X_1, X_2)+B(X_1, X_2)U,\end{aligned} \tag{32}$$

where the states $X_1=[x_{11}, x_{12}, \ldots, x_{1n}]\in R^n$, $X_2=[x_{21}, x_{22}, \ldots, x_{2n}]\in R^n$, $F_1(\cdot), F_2(\cdot)$ are nonlinear smooth vector functions, $H(\cdot)$ represents the uncertainties and disturbances, $B(\cdot)$ is a nonsingular matrix and $u\in R^n$ is the control vector. For the convenience of study, we give the following assumptions.

Assumption 1
$[X_1, X_2]=0$ if and only if $[X_1, \dot{X}_1]=0$.

Assumption 2
$H(X_1, X_2)$ is a bounded function satisfying

$$\|H(X_1, X_2)\|\leqslant\rho(X_1, X_2), \tag{33}$$

where $\rho(X_1, X_2)$ denotes the upper bound of the system uncertainty..

Assumption 3
The matrix $(\partial F_1/\partial X_2)B(X_1, X_2)$ is nonsingular.

It should be pointed out that plenty of practical dynamical systems, especially the mechanical systems, can be expressed in (32) satisfying the above conditions. For example, robotic systems [8, 16, 19, 21, 22] are not exactly in the form of (32), but it can be transformed into such a form by the coordinates' change. Therefore, the proposed algorithm in the work can be applied to any plant, which can be transformed to (32).

Now, we construct the NFTSM for the system (32) as

$$\Sigma=X_1+\Lambda_1\text{sign}^{\Gamma_1}X_1+\Lambda_2\text{sign}^{\Gamma_2}\dot{X}_1, \tag{34}$$

in which Σ denotes n-dimensional switching manifold, and the involved matrices are represented by

$$\Lambda_1 = \text{diag}(\lambda_{11}, \lambda_{12}, \ldots, \lambda_{1n}), \quad \Lambda_2=\text{diag}(\lambda_{21}, \lambda_{22}, \ldots, \lambda_{2n}),$$

$$\Gamma_1 = \text{diag}(\gamma_{11}, \gamma_{12}, \ldots, \gamma_{1n}), \quad \Gamma_2=\text{diag}(\gamma_{21}, \gamma_{22}, \ldots, \gamma_{2n}),$$

with $\lambda_{1i}>0, \lambda_{2i}>0, \gamma_{1i}>\gamma_{2i}, 1<\gamma_{2i}<2$, for every $i=1, 2, \ldots, n$. Besides, $\text{sign}^{\Gamma_1}X_1$ is a vector defined as

$$\text{sign}^{\Gamma_1}X_1=\text{diag}(\text{sign}X_1)\cdot|X_1|^{\Gamma_1}=[\text{sign}^{\gamma_{11}}x_{11}, \text{sign}^{\gamma_{12}}x_{12}, \ldots, \text{sign}^{\gamma_{1n}}x_{1n}]^{\mathrm{T}}$$

and $\text{sign}^{\Gamma_2}\dot{X}_1$ can also be defined accordingly.

To facilitate the control design, we adopt the notion that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\text{sign}^{\Gamma_1}X_1)=\Gamma_1\text{diag}(|X_1|^{\Gamma_1-I_n})\cdot\dot{X}_1, \quad \frac{\mathrm{d}}{\mathrm{d}t}(\text{sign}^{\Gamma_2}\dot{X}_1)=\Gamma_2\,\text{diag}(|\dot{X}_1|^{\Gamma_2-I_n})\cdot\ddot{X}_1, \tag{35}$$

where I_n is the n-dimensional unit matrix, and it can be easily verified.

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In what follows, we will present a criterion for fast finite-time control of system (32).

Theorem 2
Under Assumption 1–3, the following control law:

$$U = -\left(\frac{\partial F_1}{\partial X_2}B\right)^{-1}\left[M_2\Sigma + \left(\left\|\frac{\partial F_1}{\partial X_2}\right\|\rho + M_1\right)\frac{\Sigma}{\|\Sigma\|} + \frac{\partial F_1}{\partial X_1}F_1 + \frac{\partial F_1}{\partial X_2}F_2 \right.$$
$$\left. + \Lambda_2^{-1}\Gamma_2^{-1}(I_n + \Lambda_1\Gamma_1 \operatorname{diag}(|X_1|^{\Gamma_1 - I_n}))\operatorname{sign}^{2I_n-\Gamma_2}(\dot{X}_1)\right], \quad M_1>0, \quad M_2>0, \tag{36}$$

along with the switching manifold (34) is capable of globally regulating the uncertain system (32) at the origin within finite time, while no singularity occurs.

Proof
The Lyapunov function is chosen as

$$V_2 = \tfrac{1}{2}\Sigma^{\mathrm{T}}\Sigma. \tag{37}$$

From (35), its time derivative can be given by

$$\dot{V}_2 = \Sigma^{\mathrm{T}}\dot{\Sigma} = \Sigma^{\mathrm{T}}\cdot[\dot{X}_1 + \Lambda_1\Gamma_1 \operatorname{diag}(|X_1|^{\Gamma_1-I_n})\cdot\dot{X}_1 + \Lambda_2\Gamma_2\operatorname{diag}(|\dot{X}_1|^{\Gamma_2-I_n})\cdot\ddot{X}_1. \tag{38}$$

Since

$$\ddot{X}_1 = \frac{\mathrm{d}}{\mathrm{d}t}F_1(X_1, X_2) = \frac{\partial F_1}{\partial X_1}F_1 + \frac{\partial F_1}{\partial X_2}(F_2 + H + BU), \tag{39}$$

it can be obtained that

$$\dot{V}_2 = \Sigma^{\mathrm{T}}\left[\dot{X}_1 + \Lambda_1\Gamma_1\operatorname{diag}(|X_1|^{\Gamma_1-I_n})\dot{X}_1 + \Lambda_2\Gamma_2\operatorname{diag}(|\dot{X}_1|^{\Gamma_2-I_n})\left(\frac{\partial F_1}{\partial X_1}F_1 + \frac{\partial F_1}{\partial X_2}(F_2+H+BU)\right)\right]. \tag{40}$$

Substituting (36) into (40), it follows that

$$\dot{V}_2 = -\Sigma^{\mathrm{T}}\cdot\Lambda_2\Gamma_2\cdot\operatorname{diag}(|\dot{X}_1|^{\Gamma_2-I_n})\cdot\left(M_1\frac{\Sigma}{\|\Sigma\|} + M_2\Sigma + \left\|\frac{\partial F_1}{\partial X_2}\right\|\cdot\frac{\rho\Sigma}{\|\Sigma\|} - \frac{\partial F_1}{\partial X_2}H\right)$$
$$\leqslant -\min_{i=1,\ldots,n}(\lambda_{2i}\gamma_{2i}|\dot{x}_i|^{\gamma_2-1})\cdot(M_1\|\Sigma\| + M_2\Sigma^{\mathrm{T}}\Sigma). \tag{41}$$

Define

$$\varrho_1(\dot{X}_1) = \min_{i=1,\ldots,n}(\lambda_{2i}\gamma_{2i}|\dot{x}_i|^{\gamma_2-1})\cdot\sqrt{2}M_1, \quad \varrho_2(\dot{X}_1) = \min_{i=1,\ldots,n}(\lambda_{2i}\gamma_{2i}|\dot{x}_i|^{\gamma_2-1})\cdot 2M_2. \tag{42}$$

Equation (41) can be rewritten as

$$\dot{V}_2 \leqslant -\varrho_1 V_2^{1/2} - \varrho_2 V_2. \tag{43}$$

By using Comparison Lemma [27] and the similar arguments in the proof of Theorem 1, it can be concluded that the fast finite-time convergence to $\Sigma = 0$ is established, and thus the uncertain MIMO systems will converge to $X_1 = X_2 = 0$ at a fast rate within finite time. Besides, one can easily see that the NTSM control (36) does not involve any terms that have negative powers.

Consequently, it is claimed that the NTSM control (36) provides the closed-loop system with fast finite-time convergence rate and singularity avoidance during the whole control process. This completes the proof. □

Remark 7

It can be found that the control (36) is designed to satisfy

$$\dot{\Sigma} = -\varrho_1^*(x_2)\frac{\Sigma}{\|\Sigma\|} - \varrho_2^*(x_2)\Sigma, \tag{44}$$

in which $\varrho_1^* = M_1\Lambda_2\Gamma_2 \cdot \mathrm{diag}(|\dot{X}_1|^{\Gamma_2 - I_n}) + \|\partial F_1/\partial X_2\| \cdot \rho - (\partial F_1/\partial X_2)(H \cdot \Sigma/\|\Sigma\|) \geqslant 0$, $\varrho_2^* = M_2\Lambda_2\Gamma_2 \cdot \mathrm{diag}(|\dot{X}_1|^{\Gamma_2 - I_n}) \geqslant 0$. It is obvious that (44) can be regarded as the multidimensional form of (31). Thus accordingly, such a reaching law may provide the closed-loop system with strong robustness and reduce the possible chattering phenomenon caused by imperfect switching. Likewise, in order to further get rid of the discontinuous property, the multidimensional switching function $\Sigma/\|\Sigma\|$ can be substituted by

$$\Psi(\sigma) = \frac{\Sigma}{\|\Sigma\| + \delta}, \tag{45}$$

where $\delta > 0$ is a small constant.

4. NUMERICAL SIMULATION

To illustrate the effectiveness and applicability of the proposed control algorithm, we design the NFTSM controller for high-precision tracking of a two-link rigid robot manipulator, whose simulation results are compared with the NTSM control scheme.

Consider that the dynamic equation of the manipulator model [16] is given by

$$M(q)\ddot{q} + C(q, \dot{q}) + G(q) = \tau(t) + \tau_d, \tag{46}$$

where $q, \dot{q}, \ddot{q} \in R^2$ are the vectors of joint angular position, velocity and acceleration, respectively. $M(q)$ is the 2×2 symmetric positive-definite inertia matrix, $C(q; \dot{q})$ is the 2×1 vector containing Coriolis and centrifugal forces, $G(q)$ is the 2×1 gravitational torque and $\tau(t)$ is 2×1 vector of applied joint torques that are actually the control inputs and τ_d denotes the 2×1 bounded input disturbance vector. Moreover, the involved matrices can be expressed as

$$M = \begin{bmatrix} (m_1 + m_2)r_1^2 + m_2 r_2^2 + 2m_2 r_1 r_2 \cos q_2 + J_1 & m_2 r_2^2 + m_2 r_1 r_2 \cos q_2 \\ m_2 r_2^2 + m_2 r_1 r_2 \cos q_2 & m_2 r_2^2 + J_2 \end{bmatrix},$$

$$C(q, \dot{q}) = \begin{bmatrix} -\beta_{12}(q_2)\dot{q_1}^2 - 2\beta_{12}(q_2)\dot{q_1}\dot{q_2} \\ \beta_{12}(q_2)\dot{q_2}^2 \end{bmatrix}, \quad \tau = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}, \quad \tau_d = \begin{bmatrix} \tau_{d_1} \\ \tau_{d_2} \end{bmatrix},$$

$$G(q) = \begin{bmatrix} (m_1 + m_2) g r_1 \cos q_2 + m_2 g r_2 \cos(q_1 + q_2) \\ m_2 g r_2 \cos(q_1 + q_2) \end{bmatrix}$$

with parameters given as $r_1 = 1\,\mathrm{m}$, $r_2 = 0.8\,\mathrm{m}$, $J_1 = 5\,\mathrm{kg\,m}$, $J_2 = 5\,\mathrm{kg\,m}$, $m_1 = 0.5\,\mathrm{kg}$, $m_2 = 1.5\,\mathrm{kg}$. It is assumed that rigid robotic manipulators have uncertainties, i.e.

$$M(q) = M_0(q) + \delta M(q), \quad C(q; \dot{q}) = C_0(q; \dot{q}) + \delta C(q; \dot{q}) \quad G(q) = G_0(q) + \delta G(q), \tag{47}$$

where $M_0(q)$, $C_0(q; \dot{q})$, $G_0(q)$ are the estimated terms, and $\delta M(q)$, $\delta C(q; \dot{q})$, $\delta G(q)$ are uncertain terms. Here we assume that the nominal values of m_1 and m_2 are $m_{10} = 0.4\,\mathrm{kg}$, $m_{20} = 1.2\,\mathrm{kg}$, and the external disturbances are $\tau_d = [2\sin(t) + 0.5\sin(200\pi t), \cos(2t) + 0.5\sin(200\pi t)]^{\mathrm{T}}$. The initial values of the system are selected as $q_1(0) = 3.0$, $q_2(0) = 2.5$, $\dot{q}_1(0) = 0$ and $\dot{q}_2(0) = 0$, and the desired reference signals are given by $q_d = [q_{d_1}, q_{d_2}]^{\mathrm{T}}$ with $q_{d_1} = 1.25 - 7/5\exp(-t) + 7/20\exp(-4t)$ and $q_{d_2} = 1.25 + \exp(-t) - (1/4)\exp(-4t)$.

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Define $\varepsilon_1(t)=q-q_d$ and $\varepsilon_2(t)=\dot{q}-\dot{q}_d$. The error equation of the rigid robotic manipulator can be obtained as follows:

$$\begin{aligned}\dot{\varepsilon}_1 &= \varepsilon_2,\\ \dot{\varepsilon}_2 &= F_2(\varepsilon_1,\varepsilon_2)+H(\varepsilon_1,\varepsilon_2)+B(\varepsilon_1,\varepsilon_2)\tau,\end{aligned} \tag{48}$$

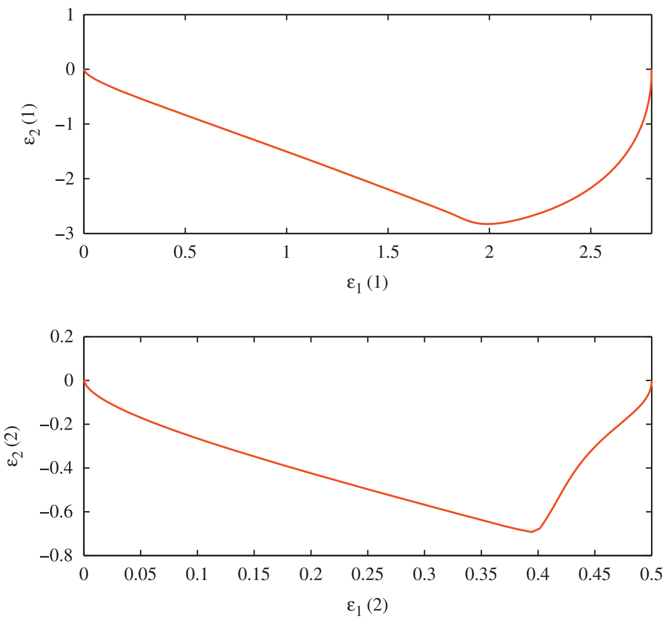

Figure 3. The phase plot of tracking error of joints 1 and 2.

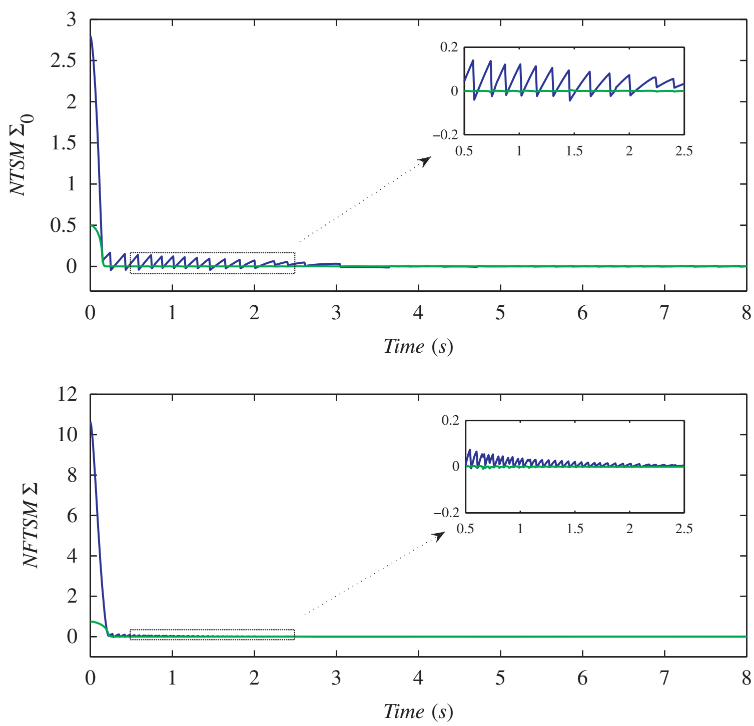

Figure 4. The time evolution of NTSM Σ and NFTSM Σ_0.

in which $F_2(\varepsilon_1,\varepsilon_2) = -M_0^{-1}C_0 - M_0^{-1}G - \ddot{q}_d$, $H(\varepsilon_1,\varepsilon_2) = -M_0^{-1}(\delta M\ddot{q} + \delta C + \delta G - \tau_d)$ and $B(\varepsilon_1,\varepsilon_2) = M_0^{-1}$. The uncertainty term $H(\varepsilon_1,\varepsilon_2)$ is assumed to satisfy the following:

$$\|H(\varepsilon_1,\varepsilon_2)\| \leqslant \|M_0^{-1}\| \cdot (b_0 + b_1\|q\| + b_2\|\dot{q}\|^2), \tag{49}$$

where $b_0 = 12, b_1 = 2.2, b_2 = 2.8$. It can be found that the error system will have the form of (32), and the upper-bounded function is $\rho(\varepsilon_1,\varepsilon_2) = \|M(0)^{-1}\| \cdot (b_0 + b_1\|q\| + b_2\|\dot{q}\|)$.

According to Theorem 2, we define the sliding surface as

$$\Sigma = \varepsilon_1 + \mathrm{sign}^{\Gamma_1}\varepsilon_1 + \mathrm{sign}^{\Gamma_2}\varepsilon_2, \tag{50}$$

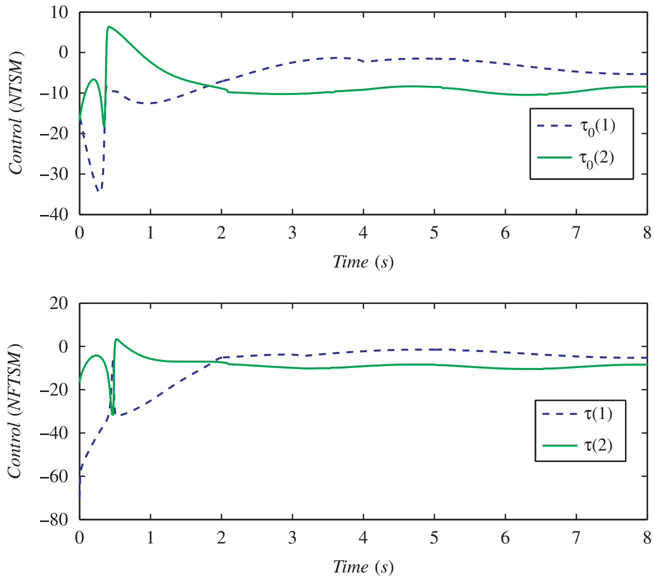

Figure 5. The time evolution of control signals τ_0 and τ.

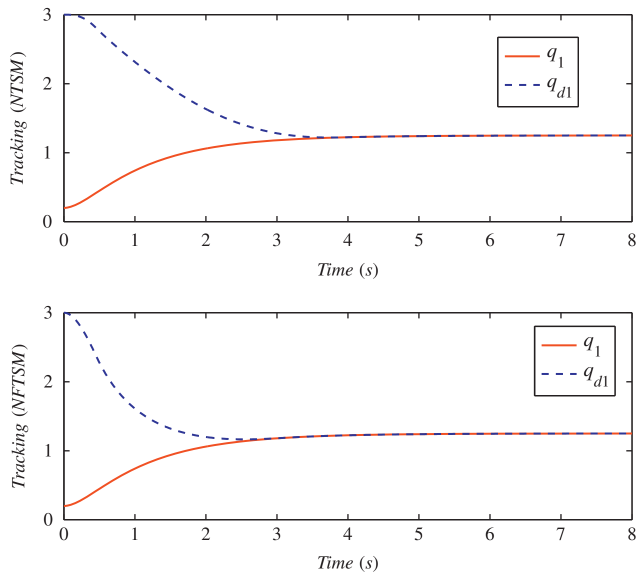

Figure 6. Output tracking of joint 1 in the case of NTSM and NFTSM.

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with

$$\Gamma_1=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix},\quad \Gamma_2=\begin{bmatrix}\frac{5}{3} & 0\\ 0 & \frac{5}{3}\end{bmatrix}.$$

Then, the following controller is designed:

$$\tau=-M_0[M_2\Sigma+(\rho+M_1)\frac{\Sigma}{\|\Sigma\|}+F_2+\Gamma_2^{-1}(I_2+\Gamma_1\,\mathrm{diag}(|\varepsilon_1|^{\Gamma_1-I_2}))\,\mathrm{sign}^{2I_2-\Gamma_2}(\varepsilon_2)], \tag{51}$$

in which $M_1=1$, $M_2=2$.

For comparison purposes, we design the following sliding surface and NTSM controller in the light of Theorem 2 [16] as

$$\Sigma_0=\varepsilon_1+\mathrm{sign}^{\Gamma}\varepsilon_2,\quad \tau_0=-M_0[(\rho+M_{10})\frac{\Sigma}{\|\Sigma\|}+F_2+\Gamma^{-1}\,\mathrm{sign}^{2I_2-\Gamma}(\varepsilon_2)], \tag{52}$$

where $M_{10}=4.3$ and

$$\Gamma=\begin{bmatrix}\frac{5}{3} & 0\\ 0 & \frac{5}{3}\end{bmatrix}$$

.

The simulation plots are presented in the following to demonstrate the superior performance of faster and high-precision tracking with the proposed NFTSM scheme. The sample time is given as $t_s=0.01\,$s. The phase plots of tracking errors of NFTSM system's joints 1 and 2 are reported in Figure 3. It can be seen from Figure 4 that the tracking errors reach NTSM $\Sigma_0=0$ and NFTSM $\Sigma=0$ within the finite time. Moreover, in comparison with the NTSM case, less chattering occurs during the NFTSM control process, which is by reason of a smaller gain for the switching term in (51).

In order to further eliminate the chattering, the approximated function (45) is adopted with $\delta=0.01$. Figure 5 depicts the control signals of two joints. It is noticed that neither singularity nor chattering occurs during the whole NFTSM control process. As shown in Figures 6 and 7, the system states track the desired reference signals quickly in the cases of NTSM and NFTSM, respectively. Moreover, since the concept of NFTSM and the constant plus proportional reaching

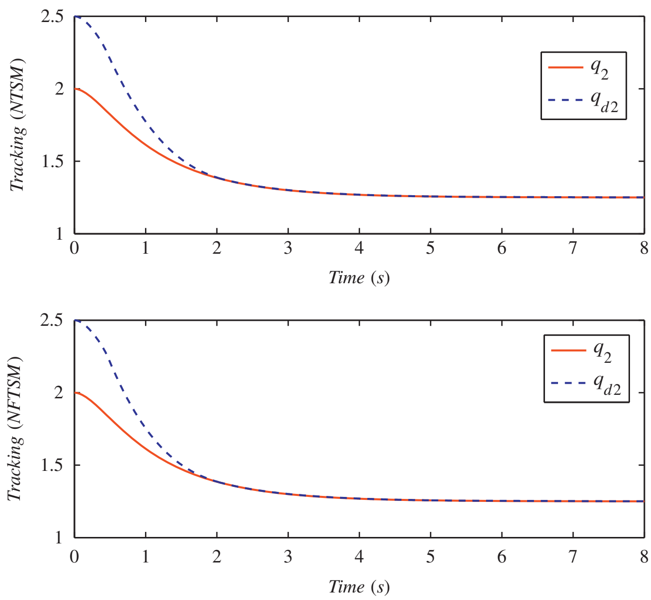

Figure 7. Output tracking of joint 2 in the case of NTSM and NFTSM.

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law decides a faster reaching, it is seen that the NFTSM system is steered to reference signals within a shorter time.

5. CONCLUSIONS

In this paper, we have studied high-precision control of nonlinear dynamics. Some new forms of FTSM have been proposed to establish fast finite-time convergence and strong robustness. It has been verified that new FTSMs have better performance than the conventional one, especially when the initial state is far away from the equilibrium. With the merits of FTSM, the concept of NFTSM has been advanced to avoid the singularity during the control process. The nonsingular scheme has been employed to control a class of uncertain MIMO nonlinear dynamics including mechanical systems. Through numerical simulation on robotic manipulators, the NFTSM control technique has demonstrated its superior performance. It should be noted that the proposed strategies could be conveniently extended in a wide range of applications such as chaotic suppression and synchronization, and filter designs.

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REFERENCES

1. Utkin VI. Variable structure systems with sliding modes. *IEEE Transactions on Automatic Control* 1977; **22**(2):212–222.
2. Gao W, Hung JC. Variable structure control of nonlinear systems: a new approach. *IEEE Transactions on Industrial Electronics* 1993; **40**(1):45–55.
3. Yu X, Kaynak O. Sliding-mode control with soft computing: a survey. *IEEE Transactions on Industrial Electronics* 2009; **56**(9):3275–3285.
4. Edwards C, Spurgeon SK. *Sliding Mode Control*: *Theory and Application*. Taylor and Francis: London, England, 1998.
5. Sabanovic A, Fridman L, Spurgeon SK. *Variable Structure Systems*: *from Principles to Implementation*. The Institution of Engineering and Technology: London, U.K., 2004.
6. Haimo VT. Finite time differential equations. *Proceedings of 24th Conference on Decision and Control*, Ft. Lauderdale, FL, 1985; 1729–1733.
7. Venkataraman ST, Gulati S. Control of nonlinear systems using terminal sliding modes. *Proceedings American Control Conference*, Chicago, IL, U.S.A., 1992; 891–893.
8. Man Z, Paplinski AP, Wu HR. A robust MIMO terminal sliding mode control scheme for rigid robotic manipulators. *IEEE Transactions on Automatic Control* 1994; **39**:2464–2469.
9. Man Z, Paplinski AP, Wu HR. A robust neural controller for a class of second-order nonlinear systems. In *Complex Systems*: *Mechanism of Adaptation*, Stonier RJ, Yu XH (eds). IOS: The Netherlands, Amsterdam, 1994; 26–267.
10. Park KB, Tsuiji T. Terminal sliding mode control of second-order nonlinear uncertain systems. *International Journal of Robust and Nonlinear Control* 1999; **9**(11):769–780.
11. Yu X, Man Z. Model reference adaptive control systems with terminal sliding modes. *International Journal of Control* 1996; **66**(6):1165–1176.
12. Man Z, Yu X. Terminal sliding mode control of MIMO linear systems. *IEEE Transactions on Circuits and Systems I* 1997; **44**(11):1065–1070.
13. Yu X, Man Z. Fast terminal sliding mode control for single input systems. *Proceedings of 2000 Asian Control Conference*, Shanghai, China, July 2000.
14. Yu X, Man Z. Fast terminal sliding-mode control design for nonlinear dynamical systems. *IEEE Transactions on Circuits and Systems I* 2002; **49**(2):261–264.
15. Wu YQ, Yu X, Man Z. Terminal sliding mode control design for uncertain dynamic systems. *Systems and Control Letters* 1998; **34**(5):281–288.
16. Feng Y, Yu X, Man Z. Non-singular adaptive terminal sliding mode control of rigid manipulators. *Automatica* 2002; **38**(12):2159–2167.
17. Yu X, Man Z, Feng Y, Guan Z. Nonsingular terminal sliding mode control of a class of nonlinear dynamical systems. *Proceedings of 2002*, *IFAC 15th Triennial World Congress*, Barcelona, Spain, 2002.

DOI: 10.1002/rnc

18. Feng Y, Zheng X, Yu X. Second-order nonsingular terminal sliding mode decomposed control of uncertain multivariable systems. *Asian Journal of Control* 2003; **5**(4):505–512.
19. Yu S, Yu X, Shirinzadeh B, Man Z. Continuous finite-time control for robotic manipulators with terminal sliding mode. *Automatica* 2005; **41**(11):1957–1964.
20. Wang H, Han ZZ, Xie QY *et al.* Finite-time chaos control via nonsingular terminal sliding mode control. *Communications in Nonlinear Science and Numerical Simulation* 2009; **14**:2728–2733.
21. Jin M, Lee J, Hun Chang P, Choi C. Practical nonsingular terminal sliding-mode control of robot manipulators for high-accuracy tracking control. *IEEE Transactions on Industrial Electronics* 2009; **56**(9):3593–3601.
22. Zhao D, Li S, Gao F. A new terminal sliding mode control for robotic manipulators. *Proceedings of the 17th World Congress The International Federation of Automatic Control*, Seoul, Korea, 2008; 9888–9893.
23. Nemitskii VV, Stepanov VV. *Qualitative Theory of Differential Equations*. Princeton University Press: Princeton, NJ, 1960.
24. Abramowitz M, Stegun IA. *Handbook of Mathematical Functions*: *with Formulas*, *Graphs*, *and Mathematical Tables*. Dover: New York, U.S.A., 1972.
25. Beckenbach E, Bellman R. *Inequalities*. Springer: Berlin, Germany, 1971.
26. Hong Y, Jiang Z. Finite-time stabilization of nonlinear systems with parametric and dynamic uncertainties. *IEEE Transactions on Automatic Control* 2006; **51**(12):1950–1956.
27. Arnold VI. *Ordinary Differential Equations*. MIT Press: Cambridge, MA, 1978.

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