MONODROMY OF CLOSED POINTS ON GENERIC HYPERSURFACES

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ABSTRACT. We show that on the generic degree-d hypersurface in $\mathbb{P}^{n+1}$, when $d \geq n+2$, every closed point of degree d is separable, and has monodromy group S_d. We study related Franchetta-type questions on generic objects, and give affirmative answers in specific cases.

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1. INTRODUCTION

1.1. Let K be a field with algebraic closure $\overline{K}$. Let $L \subset \overline{K}$ be a finite separable extension of K, let N be the normal closure of L in $\overline{K}$. We call $\mathrm{Gal}(N/K)$ the monodromy group of L/K. In [Har79, 1], it is shown that: Given a finite dominant morphism of complex algebraic varieties $f\colon Y \to X$, the monodromy group of the function field extension $\mathbb{C}(Y)/\mathbb{C}(X)$ coincides with the usual topological monodromy group.

1.2. Here are some classical examples where the monodromy groups are determined:

(1) Let $K = \mathbb{C}(a_0, \cdots, a_5)$ be the function field of the parameter space of quintic equations in one variable, the closed point $\mathrm{Spec}(L) := V(\sum a_i T^{5-i}) \subset \mathbb{A}^1_K$ has monodromy group S_5. This implies Abel's theorem on the insolvability of general quintic equations by radicals.

(2) Let K be the function field of $|\mathcal{O}_{\mathbb{P}^2}(d)|$, the parameter space of degree-d plane curves. Let X/K be the generic degree-d plane curve, let $P \subset X$ be the closed point of flexes. When $d = 3$, the monodromy group is $\mathrm{ASL}_2(\mathbb{Z}/3\mathbb{Z})$; when $d \geq 4$, the monodromy group is $S_{3d(d-2)}$. Harris studied the monodromy groups of special geometric configurations extensively in [Har79], [EH87]. In most cases, the monodromy group are maximal (full permutation groups).

1.3. As we observed, special points on generic objects tend to have full monodromy groups. Closed points of minimal degree are special, and therefore we ask if the maximality of the monodromy group holds for *all closed points of minimal degree* on generic objects. While this is too optimistic, some mild assumptions make it work. Let $K_{n,d}$ be the function field of $|\mathcal{O}_{\mathbb{P}^{n+1}}(d)|$, the parameter space of degree-d hypersurfaces in $\mathbb{P}^{n+1}$. Let $X_{n,d} \subset \mathbb{P}^{n+1}_{K_{n,d}}$ be the generic degree d-hypersurface, we show that:

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Theorem. *Let n, d be positive integers. If $d \geq n+2$, then every length-d closed subscheme on $X_{n,d}$ is irreducible and separable, and has monodromy group S_d.*

One may view this as a "Franchetta-type" result. The Franchetta theorem [Sch03] asserts that every line bundle on the generic curve is a power of the canonical bundle. We regard this as an affirmative answer to the meta-question *whether all constructions on generic objects can be "canonically" explained.* Our theorem is an answer to the meta-question, if we view the full permutation group as the "canonical" choice of the monodromy group of the d geometric points.

1.4. A more interesting object to study is the generic "pointed hypersurface". Let $p \in \mathbb{P}^{n+1}_{\mathbb{C}}(\mathbb{C})$ be a fixed point, let $\mathcal{I}_p \subset \mathcal{O}_{\mathbb{P}^{n+1}}$ be the ideal sheaf of p. Let $K_{n,d,p}$ be the function field of $|\mathcal{O}_{\mathbb{P}^{n+1}}(d) \otimes \mathcal{I}_p|$, the parameter space of the degree-d hypersurfaces in $\mathbb{P}^{n+1}$ that passes through p. Let $X_{n,d,p} \subset \mathbb{P}^{n+1}_{K_{n,d,p}}$ be the generic hypersurface. We show that:

Theorem. *When $d \geq n+4$, for any line L tangent to $X_{n,d,p}$ at p, the residue intersection is a separable closed point and has monodromy group S_{d-2}. When $n = 1$ and $d \geq 4$, for any secant line L passing through p but not tangent to $X_{n,d,p}$ at p, the residue intersection is a separable closed point and has monodromy group S_{d-1}.*

We show that when $n \geq 2$ and $d \geq n+4$, given a secant line L passing through p but not tangent to $X_{n,d,p}$ at p, the monodromy group of the residue intersection is S_{d-1} if and only if the following Franchetta-type question has an affirmative answer:

Question 1. *For $d \geq 4$, is p the only $K_{n,d,p}$-rational point on $X_{n,d,p}$?*

1.5. The question is a natural analogue of Hain's theorem [Hai11, Theorem 1] in higher dimensions. Here we ask the degree to be at least 4, as the question has an negative answer for $d = 3$: Cubic hypersurfaces with a rational point are unirational [Kol02].

1.6. For a fixed n, when $d \geq 3n+2$, we show that Question 1 would have an affirmative answer if certain dimension estimate holds for singular hypersurfaces. While we can say little about the case when $d << n$, we have an affirmative answer to Question 1 if we allow the hypersurfaces to have mild singularities. Let $p \in \mathbb{P}^n_{\mathbb{C}}$ be a fixed point. Let X_p be the generic degree-d hypersurface that has a node at p, defined over the corresponding function field K_p. We show that:

Theorem. *If $d \geq 4$, then p is the only K_p-rational point on X_p.*

Finally, based on our previous discussions, we pose some candidates to a question of Kollár, which was the original motivation of this work.

2. Generic hypersurfaces

2.1. We study the generic fiber of the tautological family of degree-d hypersurfaces in $\mathbb{P}^{n+1}$. Let k be a field. Let n, d be positive integers. Let $P_{n,d} := |\mathcal{O}_{\mathbb{P}^{n+1}}(d)|$ be the complete linear system of degree-d hypersurfaces in $\mathbb{P}^{n+1}$. Let $\mathcal{X}_{n,d} \subset P_{n,d} \times \mathbb{P}^{n+1}$ be the tautological family. Let $K_{n,d}$ be the function field of $P_{n,d}$. Let $X_{n,d}$ be the generic fiber of the projection $\pi_1 \colon \mathcal{X}_{n,d} \to P_{n,d}$.

Lemma 1. *Let H be the hyperplane class in $\mathbb{P}^{n+1}_{K_{n,d}}$, then the Chow group $A^i(X_{n,d})$ is freely generated by $(H^i|_{X_{n,d}})$ for all $0 \leq i \leq n$. Therefore, the degree of any proper subvariety of $X_{n,d}$ is a multiple of d.*

Proof. Notice that $\mathcal{X}_{n,d}$ is a projective bundle over $\mathbb{P}^{n+1}$ via projection $\pi_2\colon P_{n,d}\times\mathbb{P}^{n+1}\to\mathbb{P}^{n+1}$. Therefore, the Chow ring $A^*(\mathcal{X}_{n,d})$ is generated by L and H, where L is the pullback of hyperplane class on $P_{n,d}$, see [Ful84, Theorem 3.3].

Given any cycle class $\alpha\in A^i(X_{n,d})$, we represent it by some cycle Z on $X_{n,d}$, the closure $\overline{Z}\subset\mathcal{X}_{n,d}$ is a codimension-i cycle on $\mathcal{X}_{n,d}$, and therefore the class $[\overline{Z}]$ is a $\mathbb{Z}$-linear combination of $\{L^k\cdot H^{i-k}\}_{0\le k\le i}$. Restricting to the generic fiber, the class L becomes zero, so α is a multiple of $(H^i|_{X_{n,d}})$. □

2.2. The aforementioned lemma shows that there are no rational points on the generic hypersurface, and that the minimal degree of closed points on $X_{n,d}$ is d. We show that:

Theorem 2. *Let $d\ge n+2$, then any closed point P in $X_{n,d}$ of degree d is separable, and the geometric points of P have monodromy group S_d.*

The idea is to specialize the closed point P to certain locus in $|\mathcal{O}_{\mathbb{P}^{n+1}}(d)|$ where the degenerated hypersurfaces have large monodromy action on the irreducible components, which forces the specialized point (hence P) to have large monodromy group.

The naive specialization of a subscheme is given by taking closure in the total space, then restrict to the desired locus. This works well when the base is a DVR, by valuative criteria. In general, problems rise when the closure has jumping dimension. We get around of this issue by the trick of "successive specialization". The specialized closed points may not be unique, and depends on the choice of the "path" of specialization. Nevertheless, we only need their existence for our purpose.

Lemma 3. *Let S be a regular integral scheme of dimension r, let η be the generic point and s be a closed point. Then there exists a sequence of discrete valuation rings $R_1,\dots,R_r$ and immersions $\mathrm{Spec}(R_i)\to S$, such that η is the generic point of $\mathrm{Spec}(R_1)$, that s is the closed point of $\mathrm{Spec}(R_r)$, and that the closed point of $\mathrm{Spec}(R_i)$ coincides with the generic point of $\mathrm{Spec}(R_{i+1})$ for all $1\le i\le r-1$.*

Proof. Let $R=\mathcal{O}_{S,s}$ be the local ring of S at s, we may replace S by $\mathrm{Spec}(R)$. We pick a system of parameters $(u_1,\dots,u_r)\in\mathfrak{m}_R$. By [Sta20, Tag 00NU] we know that $R/(u_1,\dots,u_k)$ are regular local rings for $1\le k\le r$. Then we pick the localization $(R/(u_1,\dots,u_{i-1}))_{(u_1,\dots,u_i)}$ as R_i and verify the properties by definition. □

Remark 4. *The previous lemma enables us to specialize rational sections of a proper family over a regular base, to any closed subfamily by iterating the process of taking closure over a chosen chain of DVRs.*

Proof of Theorem 2. We perform successive specialization of P in the Hilbert scheme of d points on $\mathcal{X}_{n,d}$, which is proper by [Sta20, Tag 0DM8]. Let $Z\subset P_{n,d}$ be the locus where the hypersurface degenerates to the union of d hyperplanes.

Notice that we may identify Z with an open part of the Hilbert scheme of d points in $(\mathbb{P}^{n+1})^\vee$. Therefore, Z is irreducible and it admits an S_d-Galois extension over which the components of hyperplanes are distinguishable. Let $k(Z)$ be the function field of Z. Let P' be a length-d closed subscheme on $(\mathcal{X}_{n,d})_{k(Z)}$ obtained by successive specialization.

We stratify the singular hypersurface $(\mathcal{X}_{n,d})_{k(Z)}$ by the singularity types. Geometrically, there are d disjoint open part of hyperplanes, $\binom{d}{2}$ disjoint open part of codimension two subspaces that rise from the intersection of two hyperplanes, etc. The geometric points of P' are Galois conjugate, and therefore they lie in the same type of singular stratum.

The number of geometric points of P' are no more than d. On the other hand, for each $1 \leq k \leq n+1$, there are $\binom{d}{k}$ pieces of disjoint codimension-k singular strata in the $(\mathcal{X}_{n,d})_{k(Z)}$, which are furthermore in one Galois orbit. Notice that $\binom{d}{k} \geq d$, the equality holds if and only if $k=1$ or $k=n+1=d-1$. Thus the specialized subscheme P' either lie in the smooth locus, or in the singular stratum of the d Galois conjugate points, both of which have monodromy group S_d. Therefore, the closed subscheme P' consists of d distinct geometric points, and has full monodromy group S_d. Notice that the size of monodromy group can only drop after specialization, so P is separable with monodromy group S_d. $\square$

Remark 5. *The theorem does not hold for all n, d. For example, when $n=2, d=2$, the generic plane conic $X_{1,2}$ admit degree 2 inseparable points: Let $k=\mathbb{F}_2$ and let us work in the affine setup, we write $X_{1,2}=V(ax^2+by^2+cxy+dx+ey+f)$, the intersection with line $L=V(y-mx-n)$ is a closed subscheme $V((a+bm^2+cm)x^2+(cn+d+em)x+bn^2+en+f) \subset \mathbb{A}^1$, take $n=(-d-em)/c$, the intersection $L \cap X_{1,2}$ is inseparable.*

3. Pointed hypersurfaces

3.1. We apply the previous discussion to the generic "pointed hypersurfaces". Let k be a field, e.g.,$\mathbb{C}$. Let n, d be positive integers. Let $p \in \mathbb{P}^{n+1}(k)$ be a k-rational point. Let $P_{n,d,p} := |\mathcal{O}_{\mathbb{P}^{n+1}}(d) \otimes \mathcal{I}_p|$ be the linear system of degree d-hypersurfaces in $\mathbb{P}^{n+1}$ that passes through p. Let $\mathcal{X}_{n,d,p} \subset P_{n,d,p} \times \mathbb{P}^{n+1}$ be the tautological family. Let $K_{n,d,p}$ be the function field of $P_{n,d,p}$. Let $X_{n,d,p}$ be the generic fiber of the projection $\pi_2 \colon \mathcal{X}_{n,d,p} \to P_{n,d,p}$.

Lemma 6. *Let H be the hyperplane class of $\mathbb{P}^{n+1}_{K_{n,d,p}}$, then the Chow group of $A^i(X_{n,d,p})$ is generated by $(H^i|_{X_{n,d,p}})$ for $0 \leq i \leq n-1$. The Chow group of zero-cycles $A^n(X_{n,d,p})$ is generated by $(H^n|_{X_{n,d,p}})$ and the class $[p]$ of the constant section; when $d=2$, we have $(H^n|_{X_{n,d,p}}) = 2[p]$.*

Proof. Notice that the projection $\pi_2 \colon \mathcal{X}_{n,d,p} \to \mathbb{P}^{n+1}$ realizes $\mathcal{X}_{n,d,p}$ as a projective bundle away from p, and therefore we have the excision sequence

$$A_{\dim(P_{n,k,p})+n-i}(P_{n,k,p}) \longrightarrow A^i(\mathcal{X}_{n,d,p}) \longrightarrow A^i(\pi_2^{-1}(\mathbb{P}^{n+1} - \{p\})) \longrightarrow 0\ ,$$

whose leftmost term is zero when $i \leq n-1$. In this range, we know that $A^i(X_{n,d,p})$ is generated by restriction of H^i as in Lemma 1. When $i=n$, the Chow group of codimension n cycles $A^n(\mathcal{X}_{n,d,p})$ is generated by the restriction of H^n and the class of $P_{n,d,p} \times p$, and therefore the Chow group of zero cycles $A^n(X_{n,d,p})$ is generated by restriction of H^n and class of $[p]$.

When $d=2$, we take any line L passing through p. The line intersects $X_{n,d,p}$ properly, as any positive dimensional cycle has degree a multiple of d. The residue intersection is a rational point p'. Let's take a general plane section with $X_{n,d,p}$ passing through L. The section is a smooth conic, isomorphic to $\mathbb{P}^1$, any two closed points of the same degree are linearly equivalent, so $(H^n|_{X_{n,d,p}}) = 2[p]$. $\square$

3.2. Let us take a look at monodromy of closed points on pointed hypersurfaces.

Lemma 7. *Let k be an infinite field. Let X be a variety over k with only finitely many k-rational points $\{P_1, \cdots, P_r\}$. Let K be a subextension of a purely transcendental extension of k, then the K-rational points on X_K are exactly $\{P_1, \cdots, P_r\}$.*

Proof. Notice that distinct rational points base change to distinct rational points, and therefore it suffices to prove the lemma when K is purely transcendental over k. Since X is locally of finite type, any rational section $K \to X$ extends to some Zariski open $U \subset \mathbb{A}^m_k \to X$, where $m = \mathrm{tr.deg}(K/k)$. There are infinitely many k-rational points in U, but X contains only finitely many k points, so the image of U in X has to collapse to a point in $X(k)$, and therefore the K-points of X_K corresponds to the k-points of X. □

Theorem 8. *Let L be any tangent line to $X_{n,d,p}$ at p, then the residue intersection Q is a closed point of degree $d-2$. If $d \geq n+4$, then the monodromy group of $Q/K_{n,d,p}$ equals to S_{d-2}.*

Proof. Let $Z \subset P_{n,d,p}$ be the locally closed locus where $X_{n,d,p}$ degenerates into a union of a smooth quadric passing through p and a degree-$(d-2)$ hypersurface. Then Z is birational to $P_{n,2,p} \times P_{n,d-2}$. The generic fiber $(\mathcal{X}_{n,d,p})_{k(Z)}$ is birational to the union of $X_{n,2,p} \times_k K_{n,d-2}$ and $X_{n,d-2} \times_k K_{n,2,p}$.

We show that neither of the component contain lines, and therefore any line in $\mathbb{P}^{n+1}_{k(Z)}$ intersects $(\mathcal{X}_{n,d,p})_{k(Z)}$ properly. Notice that the only lines on a quadric passing through the point p are contained in its tangent space T_p. By Lemma 6, we know the quadric $X_{n,2,p}$ does not contain lines, and therefore the intersection $T_p \cap X_{n,2,p}$ contains p as the only rational point. By Lemma 7, we know that the base change $(T_p \cap X_{n,2,p}) \times_k K_{n,d-2}$ contains no rational points other than p, and therefore $X_{n,2,p} \times_k K_{n,d-2}$ contain no lines. Similarly, Lemma 1 and Lemma 7 tell us that $X_{n,d-2} \times_k K_{n,2,p}$ does not contain lines.

We successively specialize the tangent line L to the generic point of Z. The residue intersection will be contained a subscheme of degree $(d-2)$ in $X_{n,d-2} \times_k K_{n,2,p}$: the base change of the generic degree-$(d-2)$ hypersurface. Notice that the Galois group of the irreducible components do not change after purely transcendental extension, we conclude that the monodromy group equals to S_{d-2} as in Theorem 2. □

Proposition 9. *Let L be any secant line of $X_{n,d,p}$ through p but not tangent to $X_{n,d,p}$ at p. If the residue intersection Q does not contain a rational point, then Q is a closed point with monodromy group S_{d-1}.*

Proof. Let $Z \subset P_{n,d,p}$ be the locus as in Theorem 8. Let L_1 be the specialization of L to $k(Z)$ along some chosen chain of successive specialization. Let us denote the geometric points of $(X_{n,d-2} \times_k K_{n,2,p}) \cap L$ by $p_1, \cdots, p_{d-2}$, let $q_1, \cdots, q_{d-2}$ be the corresponding geometric points of $X_{n,d,p} \cap L$ identified via the chosen chain of successive specialization. Let q_0 be the remaining geometric point in $X_{n,d,p} \cap L$.

By the argument in Theorem 8, we know that the monodromy group of $\{p_1, \cdots, p_{d-2}\}$ equals to S_{d-2}, and therefore the monodromy group of $\{q_0, \ldots, q_{d-2}\}$ contains S_{d-2}, which acts as permutation on $\{q_1, \cdots, q_{d-2}\}$ and fixes q_0. The assumption that Q does not contain a rational point means that q_0 is conjugate to other geometric points under monodromy. Therefore, after reordering $\{1, \cdots, d-2\}$, the monodromy action on (the indexes of) $\{q_0, \cdots, q_{d-2}\}$ contains a rotation $(0, 1, \cdots, r)$ for some r. Therefore, the monodromy group of Q contains the transposition $(0, r) = (r, \cdots, 2, 1)(0, 1, \cdots, r)$, which generates all transpositions by the S_{d-2} conjugation, hence the monodromy group of Q is S_{d-1}. □

3.3. We can say more about the generic pointed planes curves.

Lemma 10. *The classes $[p]$ and $H|_{X_{1,d,p}}$ are $\mathbb{Z}$-linearly independent when $d \geq 3$.*

Proof. Notice that $[p]$ and $H|_{X_{1,d,p}}$ represent rational sections of the Picard variety of $X_{1,d,p}$. If $a \cdot [p] + b \cdot (H|_{X_{1,d,p}})$ is the identity section, so would its specializations. However, we may always find a degree d-plane curve over $\overline{k}$ passing through p where p is disjoint from the multiples of $(H|_{X_{1,d,p}})$. □

Proposition 11. *When $d \geq 4$, the curve $X_{1,d,p}$ admits p as the unique rational section. When $d = 3$, the rational points on the generic curve $X_{1,3,p}$ can be identified with $\mathbb{Z} \cdot [\mathcal{O}_{\mathbb{P}^2}(1)|_{X_{1,3}} \otimes \mathcal{O}_{X_{1,3}}(-3p)]$.*

Proof. Let Z be a $K_{1,d,p}$-rational point of $X_{1,d,p}$. Let $\overline{Z}$ be its closure in $\mathcal{X}_{1,d,p}$. Notice that the projection $\pi_1 \colon \overline{Z} \to P_{1,d,p}$ is a birational morphism. Let $U \subset \mathbb{P}_{1,d,p}$ be the maximal Zariski open over which π_1 is an isomorphism. By Abhyankar's lemma [Kol96, Theorem VI.1.2], we know that the fibers over $P_{1,d,p} \backslash U$ are uniruled. Let $W \subset P_{n,d}$ be the locus of plane curves that contain a rational component, then $P_{1,d,p} \backslash U \subset W$. A plane curve with rational components is either is reducible, or consists of one rational component. Notice that:

(1) The locus of reducible curves in $P_{n,d}$ has codimension $\binom{d+2}{2} - 2 - \binom{d+1}{2} = d - 1 \geq 3$.
(2) The locus of degree-d irreducible rational curves in $P_{n,d}$ has codimension $\binom{d+2}{2} - (3(d+1) - 1 - 3) > 3$ (The curve can be viewed as a degree-d morphism $\mathbb{P}^1 \to \mathbb{P}^2$, which is determined by 3 polynomials of degree $d+1$, up to scaling and the 3-dimensional automorphism of $\mathbb{P}^1$.)

The family of hypersurfaces $P_{n,d,p} \subset P_{n,d}$ is base point free, therefore the codimension of $P_{1,d,p} \cap W$ in $P_{1,d,p}$ equals to the codimension of W in $W_{1,d}$. The fibers of the projection $\pi_1 \colon \overline{Z} \to P_{1,d,p}$ are at most 1-dimensional, so π_1 is an isomorphism in codimension at least $\operatorname{codim}_{P_{1,d}}(W) - 1 = 2$. By [KM98, Corollary 2.63], we know that $\pi_1 \colon \overline{Z} \to P_{1,d,p}$ is an isomorphism. Therefore, the section $\overline{Z}$ induces a morphism $P_{1,d,p} \to \mathbb{P}^2$. Notice that morphisms $\mathbb{P}^m \to \mathbb{P}^n$ for $m > n$ are constant, and that the only constant section of the family $\mathcal{X}_{1,d,p}$ is given by p, thus we prove the first assertion.

When $d = 3$, we may identify $X_{1,3,p}$ with the Picard scheme $\mathrm{Pic}^0_{X_{1,3,p}/K_{1,3,p}}$. Notice that $X_{1,3,p}$ admits a rational section, and therefore rational points are in bijection with degree-zero line bundles on $X_{1,3,p}$. Then by Proposition 6 and Lemma 10, we know the group $\mathrm{Pic}^0(X_{1,3,p})$ is freely generated by $\mathcal{O}(1)|_{X_{1,3,p}} \otimes \mathcal{O}_{X_{1,3,p}}(-3p)$. □

The proposition show that the generic pointed plane curves $X_{1,d,p}$ satisfies the assumption in Proposition 9. Therefore,

Theorem 12. *Let $d \geq 4$, let $X_{1,d,p}$ be the generic degree d plane curve that passes through a fixed point $p \in \mathbb{P}^2$. Then for any secant line passing through p but not tangent to $X_{1,d,p}$ at p, the residue intersection has full monodromy group S_{d-1}.*

3.4. Our proof of Proposition 11 generalizes to hypersurfaces of high degree, if certain dimension estimate holds. Let $W \subset P_{n,d}$ be the locus of hypersurfaces in $\mathbb{P}^{n+1}$ that contains a rational curve. The locus is stable under specialization, and therefore it is a countable union of closed subsets, so it has a well-defined codimension.

Proposition 13. *If $\operatorname{codim}_{P_{n,d}}(W) \geq n+2$, then $X_{n,d,p}(K_{n,d,p}) = \{p\}$.*

Proof. We apply the same argument as in Proposition 11: the codimension estimate implies that the projection to $P_{n,d,p}$ is an isomorphism away from a codimension 2 subset. □

Remark 14. *Let $P^{\circ}_{n,d} \subset P_{n,d}$ be the locus parameterizing smooth hypersurfaces. In* [Voi96, 1.4], *it is shown that $\operatorname{codim}_{P^{\circ}_{n,d}}(W \cap P^{\circ}_{n,d}) \geq d - 2n$. If the estimate extends to all hypersurfaces, namely $\operatorname{codim}_{P_{n,d}}(W) \geq d - 2n$, then $X_{n,d,p}(K_{n,d,p}) = \{p\}$ for $d \geq 3n+2$.*

4. Franchetta-type questions

4.1. We study the meta-question "whether constructions on generic objects can be canonically explained" in various specific cases. We show that:

(1) The generic "node-pointed hypersurfaces" admits the node as the only rational section.
(2) On the generic plane curve, every closed point of minimal degree is collinear.
(3) On the generic pointed curve $X_{1,d,p}$, the only effective zero-cycles of degree 2 is $2[p]$.

4.2. **Node-pointed hypersurfaces.** Let k be a field. Let n, d be positive integers. Let us consider the parameter space $|\mathcal{O}_{\mathbb{P}^n}(d)|$ of degree d hypersurfaces in $\mathbb{P}^n$. Let us fix coordinates $\{X_i\}_{i=0}^n$ of $\mathbb{P}^n$. If we write an general element F in $|\mathcal{O}_{\mathbb{P}^n}(d)|$ explicitly as $\sum a_{i_0,\cdots,i_n} X_0^{i_0}\cdots X_n^{i_n}$, then the coefficients serves as coordinates of $|\mathcal{O}_{\mathbb{P}^n}(d)|$.

Let us fix the point $p = [0 : \cdots : 0 : 1] \in \mathbb{P}^n_k$. We are interested in the family of hypersurfaces passing through p and singular at p. The condition that F passes through p translates to $a_{0,\cdots,0,d} = 0$, and F being singular at p, namely $\frac{\partial F}{\partial X_i}(p) = 0$, translates to $a_{1,0,\cdots,0,d-1} = 0, a_{0,1,\cdots,0,d-1} = 0, \cdots$ and $a_{0,0,\cdots,1,d-1} = 0$. Therefore, the family of hypersurfaces form a codimension-$(n+1)$ linear subspace $P_p \subset |\mathcal{O}_{\mathbb{P}^n}(d)|$. Since a general singular hypersurface has a node at p, let us call the generic fiber of the family the "generic node-pointed hypersurface".

Let $\mathcal{X}_p \subset P_p \times \mathbb{P}^n$ be the tautological family of hypersurfaces. Let K_p be the function field of P_p and $X_p \subset \mathbb{P}^n_{K_p}$ be the generic fiber of the projection $\pi_1\colon \mathcal{X}_p \to P_p$. Let H be the hyperplane class in $\mathbb{P}^n_{K_p}$, by excision sequence we know that the Chow group $A^i(X_p)$ is freely generated by restriction of H^i for $i \leq n-2$, and therefore X_p does not contain lines.

Theorem 15. *When $d \geq 4$, the generic node-pointed hypersurface $X_p \subset \mathbb{P}^n_{K_p}$ admits p as the only rational section. If $n \leq d-4$, the residue intersection of any line through p has monodromy group S_{d-2}.*

Proof. Let $Z \subset P_p$ be the locus parameterizing hypersurfaces which consists of a quadric cone with vertex p and a smooth hypersurface of degree $(d-2)$. One verifies that the cone does not contain rulings, for otherwise projecting form the vertex, we get a rational point on lower dimensional generic quadrics. Therefore, lines in the ambient space intersect the fiber of $\mathcal{X}_p$ over $k(Z)$ properly. Then we argue as in Theorem 8. □

4.3. **Generic plane curves.** On the generic plane curve $X_{1,d}$, we showed that degree-d closed point have always have full monodromy. Actually we can say more:

Theorem 16. *Every closed subscheme $Z \subset X_{1,d}$ of length d is collinear.*

Proof. Lemma 1 tells us Z is rationally equivalent to $H := \mathcal{O}_{\mathbb{P}^2}(1)|_{X_{2,d}}$. On a smooth curve, linearly equivalence is the same as rational equivalence, and therefore $Z \in |H|$. Notice that $X_{1,d} \subset \mathbb{P}^2_{K_{2,d}}$ is a complete intersection, Kodaira vanishing tells us the restriction $|\mathcal{O}_{\mathbb{P}^2}(1)| \to |H|$ is surjection, and therefore Z is the intersection of $X_{1,d}$ with a line. □

It seems that the only natural way to construct zero cycles of degree d on $X_{1,d}$ is to take intersection with a line in the ambient space. Therefore we ask in general, for $n \geq 2$ and all d:

Question 2. *Is every degree d closed subscheme in $X_{n,d}$ necessarily collinear?*

4.4. **Generic pointed curves.** On the generic plane curve, as we showed in Lemma 1, every closed point has degree divisible by d. There is no immediate obstruction to the existence of closed points of degree 2 on the generic pointed plane curve. However, we show such points do not exist, or equivalently:

Theorem 17. *When $d \geq 5$, on the generic pointed curve $X_{1,d,p}$, the cycle $2[p]$ is the only effective zero-cycle of degree 2 on $X_{1,d,p}$.*

Proof. We showed in Proposition 11 that $X_{1,d,p}$ contains no rational points other than p, and therefore it suffices to show that there exists no closed points of degree 2. Suppose that y is such a point, then $\mathcal{O}_{X_{1,d,p}}(y - 2 \cdot p)$ is a line bundle on $X_{1,d,p}$ of degree 0. Let H be the hyperplane class on $X_{1,d,p}$. Let us use the notation $l(D) := h^0(\mathcal{O}(D)) - 1$.

By Lemma 6 and 10, we know that $y - 2 \cdot p$ is linearly equivalent to $m(H - d \cdot p)$ for some integer m, and therefore $l(mH - (md - 2) \cdot p) \geq 0$. By upper semi-continuity of coherent cohomology, this would imply that for every degree d-plane curve $X \subset \mathbb{P}^2_{\mathbb{C}}$ and every point $q \in X(\mathbb{C})$, we have $l(mH - (md - 2) \cdot q) \geq 0$.

(1) If $m = 0$, since $y \neq 2 \cdot p$, we know that $l(2 \cdot p) > 1$. This is impossible as the generic plane curve is not hyperelliptic when $d \geq 4$.
(2) If $m \geq d - 2$, by Riemann-Roch we have $l(\mathcal{O}_X(m)) = md - g$. Notice that when $d \geq 4$, we have $g \geq 2$, and therefore $l(mH - (md - 2) \cdot q) = -1$ for general q on any X.
(3) If $1 < m \leq d - 3$, notice that $l(\mathcal{O}_X(m)) = \frac{(m+1)(m+2)}{2}$ and that $\frac{(m+1)(m+2)}{2} = m(\frac{m}{2} + \frac{3}{2} + \frac{3}{m}) - 2 \leq \max\{4m, (\frac{d}{2} + 1)m\} - 2 < md - 2$, we have $l(mH - (md - 2) \cdot q) = -1$ for general q on any X.
(4) If $m = 1$, we have $l(H - (md - 2) \cdot p) \leq l(H - 3p) = -1$, as there are only finitely many flexes on a general plane curve.
(5) If $m < 0$, let $m' = -m$, we show that $l((m'd + 2) \cdot q - m'H) = -1$. By Riemann-Roch, this equals to $l((d - 3 + m')H - (m'd + 2) \cdot q) + 3 - g$. By the calculation in (2), we have $l((d - 3 + m')H) = (d - 3 + m')d - g$. Finally, as q is general, we have $l((d - 3 + m')H - (m'd + 2) \cdot q) = l(\mathcal{O}_X(d - 3 + m')) - (m'd + 2) = g - 4$ (we use that $g = (d - 1)(d - 2)/2$).

Summing up, we know that the equality $l(mH - (md - 2) \cdot p) \geq 0$ does not hold for any m, and therefore the closed point y such that $y \in |mH - (md - 2) \cdot q|$ does not exist. □

Remark 18. *When $d = 4$, the residue intersection of tangent line at p gives a closed point of degree 2 on the generic degree 4 plane curve.*

Remark 19. *Let $X_{g,1}$ be the generic genus g curve, it is shown in* [Hai95, Theorem 12.3] *that* $\mathrm{Pic}^0(X_{g,1})$ *is freely generated by the class of $\omega_{X_g}(-(2g - 2) \cdot \xi)$, where ξ is the marked point. By the same argument as above, one can show that $m[\xi]$ the only degree-m effective zero-cycles on $X_{g,1}$ for $1 \leq m \leq g - 1$.*

5. Further discussions

5.1. Let k be a field, let $X \subset \mathbb{P}^{n+1}_k$ be a smooth cubic hypersurface. It is well known that $X_{\overline{k}}$ is unirational for $n \geq 2$. In [Kol02], Kollár shown that every smooth cubic hypersurface that admits a k-rational point is unirational over k. Kollár also shown that the smoothness assumption cannot be relaxed to regularity [Kol02, Example 17], [OS20]. Therefore, the following question is raised:

Question 3 (Kollár)**.** *Find a smooth geometrically unirational variety that admit rational points, but not unirational.*

5.2. We give some candidates to the questions based on our previous discussions:

(1) For fixed d, when n is sufficiently large, the smooth variety $X_{n,d,p}$ is geometrically unirational by [HMP98]. If Question 1 has an affirmative answer, then $X_{n,d,p}$ will not be unirational.
(2) Smooth cubic surfaces are geometrically rational, Hilbert scheme of points on smooth surfaces are smooth, and therefore $\mathrm{Hilb}^3_{X_{2,3}/K_{2,3}}$ is smooth and geometrically rational [Mat68].

If Question 2 has an affirmative answer, in particular, every degree 3 closed point on the generic cubic surface $X_{2,3}$ are collinear, then $\mathrm{Hilb}^3_{X_{2,3}/K_{2,3}}$ will not be unirational: the rational points will be contained in a codimension-2 closed subvariety $\mathrm{Gr}(2,4)$.

(3) If Question 2 has an affirmative answer for cubic surfaces, Colliot-Thélène [CT20, Theorem 7.1] showed that, projecting $X_{2,3}$ from any line in $\mathbb{P}^3$, taking minimal regular model of the relative Jacobian, then blow down the identity section, one obtains a del-Pezzo surface of degree 1 with a rational point, whose rational points are concentrated in the union of finitely many curves, and therefore not unirational. This would answer Kollár's question in the smallest possible dimension.

(4) We would also ask if the nodal hypersurface X_p in Section 4.2 admit no closed points of degree 2 when $d \geq 5$. If this is true, let Y be the blow-up of X_p at the node. Again by [HMP98], when $n >> d$, the variety X_p is geometrically unirational. Then any desingularization of $\mathrm{Sym}^2(Y)$ would serve as an example to Question 3, as the rational points will be always contained in the exceptional locus.

5.3. On the generic plane quartic curve $X_{1,4}$, the canonical divisor has degree 4, and therefore by Theorem 2, every effective canonical divisor is a separable closed point with monodromy group S_4. Let us observe that the moduli space of genus 4 curves $\mathcal{M}_4$ is dominated by the moduli of plane quartics $P_{1,4}$, and therefore every canonical divisor on the generic genus 4 curve is a separable closed point with monodromy group S_4. It is interesting to know if this holds in general:

Question 4. *Does every effective canonical divisor on the generic genus g curve have monodromy group S_{2g-2}?*

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