# MATH130068h Recitation Classes, 2020 Spring

# Contents

CHAPTER 1

# Categories and functors

## 1. Basic concepts

**1.1.** A category $\mathcal{C}$ consists of the following data:

- A class of objects
$$\mathrm{Ob}(\mathcal{C}).$$
- For each pair $x, y \in \mathcal{C}$, a set of morphisms
$$\mathrm{Mor}_{\mathcal{C}}(x, y).$$
- For each triple $x, y, z \in \mathcal{C}$, a composition map
$$\mathrm{Mor}_{\mathcal{C}}(y, z) \times \mathrm{Mor}_{\mathcal{C}}(x, y) \to \mathrm{Mor}_{\mathcal{C}}(x, z),$$
denoted by
$$(f, g) \mapsto f \circ g.$$

The data are to satisfy the following rules:

- For every element $x \in \mathrm{Ob}(\mathcal{C})$, there exists a morphism
$$\mathrm{id}_x \in \mathrm{Mor}_{\mathcal{C}}(x, x),$$
such that
$$\mathrm{id}_x \circ f = f \quad \text{and} \quad g \circ \mathrm{id}_x = g$$
whenever these compositions make sense.
- Composition is associative, i.e.,
$$(f \circ g) \circ h = f \circ (g \circ h)$$
whenever compositions make sense.

**1.2.** A functor $F \colon \mathcal{A} \to \mathcal{B}$ between two categories $\mathcal{A}, \mathcal{B}$ is given by the following data:

- A map
$$F \colon \mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(\mathcal{B}),$$
- For every $x, y \in \mathrm{Ob}(\mathcal{A})$, a map
$$F \colon \mathrm{Mor}_{\mathcal{A}}(x, y) \to \mathrm{Mor}_{\mathcal{B}}(F(x), F(y)),$$
denoted by $g \mapsto F(g)$.

The data are to be:

- Compatible with composition
$$F(g \circ h) = F(g) \circ F(h),$$
- Compatible with identity morphisms
$$F(\mathrm{id}_x) = \mathrm{id}_{F(x)}.$$

**1.3.** Let $F, G\colon \mathcal{A} \to \mathcal{B}$ be functors. A natural transformation, or morphism of functors $t\colon F \to G$, is a collection $\{t_x\}_{x\in \mathrm{Ob}(\mathcal{A})}$ such that

- $t_x\colon F(x) \to G(x)$ is a morphism in the category $\mathcal{B}$.
- for every morphism $f\colon x \to y$ of $\mathcal{A}$, the following diagram in $\mathcal{B}$ commutes

$$\begin{tikzcd} F(x) \arrow[r, "t_x"] \arrow[d, "F(f)"'] & G(x) \arrow[d, "G(f)"] \\ F(y) \arrow[r, "t_y"] & G(y) \end{tikzcd}$$

**1.4.** Let $\mathcal{C}$ be a category. A *presheaf of sets* on $\mathcal{C}$ is a contravariant functor $F$ from $\mathcal{C}$ to the category of sets, denoted by $\mathtt{Sets}$. The category of presheaves[1], where we take natural transformations as morphisms, is denoted by $\mathtt{Psh}(\mathcal{C})$.

**1.5.** Let $\mathcal{C}$ be a category. For any object $U \in \mathrm{Ob}(\mathcal{C})$, there is a contravariant functor,

$$\begin{aligned} h_U\colon \mathcal{C} &\to \mathtt{Sets} \\ X &\mapsto \mathrm{Mor}_{\mathcal{C}}(X, U) \end{aligned}$$

In other words $h_U$ is a presheaf. Given $f\colon X \to Y$, the corresponding map $h_U(f)\colon \mathrm{Mor}_{\mathcal{C}}(Y,U) \to \mathrm{Mor}_{\mathcal{C}}(X,U)$ takes $a$ to $a \circ f$. We call this presheaf the representable presheaf associated with $U$, or *the functor of points of $U$*. Note that the assignment $h\colon \mathcal{C} \to \mathtt{Psh}(\mathcal{C})$ is also a functor. We call it the *Yoneda embedding functor*. We call a contravariant functor (presheaf) on $\mathcal{C}$ is *representable* if it is isomorphic to $h_U$ for some $U \in \mathrm{Ob}(\mathcal{C})$, or equivalently, lies in the essential image of $h$.

**1.6.** (Yoneda Lemma) Given any presheaf $F$ on $\mathcal{C}$, we have a natural bijection

$$t_U\colon \mathrm{Mor}_{\mathtt{Psh}(\mathcal{C})}(h_U, F) \to F(U), \quad s \mapsto s_U(\mathrm{id}_U).$$

Here "natural" means, $t_U$ is a natural transformation of functors from $\mathrm{Mor}_{\mathtt{Psh}(\mathcal{C})}(h_-, F)$ to $F-$.

## 2. The category of finite ordered sets

**2.1.** The category of finite ordered sets, denoted by $\boldsymbol{\Delta}$, is the category consisting of the following data:

- Objects
$$\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots$$
where $\mathbf{n}$ is the set $\{0, 1, \ldots, n\}$[2].
- A morphism $\mathbf{n} \to \mathbf{m}$ is a nondecreasing map
$$\{0, 1, \ldots, n\} \to \{0, 1, \ldots, m\}$$
between the corresponding ordered sets.
- The composition of morphisms are given by composition of maps.

[1] Let's agree that we are talking about presheaf of sets, if we do not specify. In general, we can talk about presheaves of abelian groups, rings, etc, which are just functors $\mathcal{C}^{opp} \to \mathtt{Ab}$, $\mathcal{C}^{opp} \to \mathtt{Rings}$, etc.

[2] We may replace $\mathbf{n}$ by $[n]$, if the bold font is not preferred

**2.2.** The morphisms look a mess[3], but they can be clarified once we pick a "basis". Among all the functors $\mathbf{m} \to \mathbf{n}$, there are special ones[4]

$$\delta_j^n \colon \mathbf{n-1} \to \mathbf{n}, \quad \text{(coface)}$$
$$\sigma_j^n \colon \mathbf{n+1} \to \mathbf{n}. \quad \text{(codegeneracy)}$$

They denote[5] the injective order preserving map that skips $j$, and the surjective order preserving map that collapses $\{j, j+1\}$ to $j$, respectively. We have the generation lemma[6]:

Lemma 2.1. *Coface and codegeneracy maps generate all the morphisms in* $\Delta$.

**2.3.** The coface and codegeneracy morphisms satisfy the "belt axioms"[7]:

- If $0 \le i < j \le n+1$, then $\delta_j \circ \delta_i = \delta_i \circ \delta_{j-1} \colon \mathbf{n-1} \to \mathbf{n+1}$.
- If $0 \le i < j \le n-1$, then $\sigma_j \circ \delta_i = \delta_i \circ \sigma_{j-1} \colon \mathbf{n-1} \to \mathbf{n-1}$
- If $0 \le j \le n-1$, then $\sigma_j \circ \delta_j = \mathrm{id}_{\mathbf{n-1}}$, $\sigma_j \circ \delta_{j+1} = \mathrm{id}_{\mathbf{n-1}}$.
- If $0 < j+1 < i \le n$, then $\sigma_j \circ \delta_i = \delta_{i-1} \circ \sigma_j \colon \mathbf{n-1} \to \mathbf{n-1}$.
- If $0 \le i \le j \le n-1$, then $\sigma_j \circ \sigma_i = \sigma_i \circ \sigma_{j+1} \colon \mathbf{n+1} \to \mathbf{n-1}$.

**2.4.** The category $\mathbf{\Delta}$ is characterized by the universal property:

Proposition 2.2. *The category* $\mathbf{\Delta}$ *is the universal category with objects* $\mathbf{n}$ *for* $n \ge 0$*, and morphisms*

$$\delta_j^n \colon \mathbf{n-1} \to \mathbf{n}, \quad \sigma_j^n \colon \mathbf{n+1} \to \mathbf{n}, \quad 0 \le j \le n$$

*such that*

- *Every morphism is a composition of these morphisms,*
- *The relations in the "belt axioms" are satisfied,*
- *Any relation among the morphisms is a consequence of those relations.*

Can you prove it? (What are the precise statements you're going to prove?)

## 3. Simplicial sets

**3.1.** A simplicial set $X$ is a contravariant functor[8]

$$X \colon \mathbf{\Delta}^{opp} \to \mathtt{Sets}.$$

The simplicial sets form a category `sSets`, where a morphism $f \colon X \to Y$ is a *natural transformation* of functors.

**3.2. Nerve.** Given a small category $\mathcal{C}$, there is a naturally assigned simplicial set $N_{\mathcal{C}}$, called the *nerve* of the category. In order to define the functor $N_{\mathcal{C}} \colon \mathbf{\Delta}^{opp} \to \mathtt{Sets}$, we need to assign a set $N_{\mathcal{C}}(\mathbf{n})$ for each $\mathbf{n}$, and assign all the morphisms. Fortunately, by the previous discussion, it suffices to assign morphisms for $\delta, \sigma$ and verify the belt axioms.

Let $N_{\mathcal{C}}(\mathbf{n})$ be the set consisting of all $n$-tuple of composable morphisms

$$A_0 \to A_1 \to \cdots \to A_{n-1} \to A_n$$

We assign the face maps

$$d_i^n := N_{\mathcal{C}}(\delta_i^n) \colon N_{\mathcal{C}}(\mathbf{n}) \to N_{\mathcal{C}}(\mathbf{n-1})$$

[3]What is the size $\#|\mathrm{Mor}_{\mathbf{\Delta}}(\mathbf{m}, \mathbf{n})|$? For example, $m = 1$?

[4]Sometimes we omit $n$ in $\delta_j^n$, when the source and target are clear.

[5]We reserve the term "face map $d^j$", "degeneracy maps $s^j$" for the simplicial objects, to be defined later.

[6]Can you prove it in five lines?

[7]Draw the commutative diagrams and see how the stacked axioms looks like a belt [**Sta20**, 0167]

[8]In this section, in order not to cause confusion, we stick to the notion of contravariant functors. You may replace "contravariant functor" by "presheaf" everywhere, the category `sSets` (to be defined) will just be `Psh(`$\mathbf{\Delta}$`)`.

which send the $n$-tuple

$$A_0 \to \cdots \to A_{i-1} \to A_i \to A_{i+1} \cdots \to A_n,$$

by composing $A_{i-1} \to A_i \to A_{i+1}$, to

$$A_0 \to \cdots \to A_{i-1} \to A_{i+1} \cdots \to A_n,$$

and assign the degeneration maps

$$s_j^n := N_{\mathcal{C}}(\sigma_j^n)\colon N_{\mathcal{C}}(\mathbf{n}) \to N_{\mathcal{C}}(\mathbf{n}+\mathbf{1})$$

which send the $n$-tuple

$$A_0 \to \cdots \to A_{j-1} \to A_j \to A_{j+1} \cdots \to A_n,$$

by inserting $\mathrm{id}\colon A_j \to A_j$, to

$$A_0 \to \cdots \to A_{j-1} \to A_j \to A_j \to A_{j+1} \cdots \to A_n.$$

One verify the axioms and feel ensured that the assignment is a functor, hence $N_{\mathcal{C}}\colon \Delta^{opp} \to \mathtt{Sets}$ is a simplicial set. To sum up, we naturally assigned $\mathcal{C}$ with a simplicial set: the nerve $N_{\mathcal{C}}$.

**3.3. Singular Set.** Consider the standard $n$-simplex $|\Delta^n| \subset \mathbb{R}^{n+1}$ given by $\{(t_i) | \sum t_i = 1, t_i \geq 0\}$, endowed with subspace topology.

Given $\theta\colon \mathbf{n} \to \mathbf{m}$, we define an induced map[9]

$$\theta_*\colon |\Delta^n| \to |\Delta^m|,$$

given by

$$\theta_*(t_0, \ldots, t_m) = (s_0, \ldots, s_n),$$

where

$$s_i = \sum_{j \in \theta^{-1}(i)} t_j.$$

Let $T$ be a topological space, we define its *singular set* $S_T$ to be the simplicial set given by

$$\mathbf{n} \mapsto \mathrm{Mor}_{\mathtt{Top}}(|\Delta^n|, T)$$

The morphism $S_T(\theta)\colon S_T(\mathbf{m}) \to S_T(\mathbf{n})$ is given by $f \mapsto f \circ \theta_*$. To sum up[10], we naturally assigned a topological space $T$ with a simplicial set: the singular set $S_T$.

**3.4.** We can associate the topological space $T$ with a simplicial abelian group $A_T\colon \mathbf{\Delta}^{opp} \to \mathtt{Sets}$, $\mathbf{n} \mapsto \mathbb{Z}^{\oplus S_T(\mathbf{n})}$. The simplicial abelian group can be made into a chain complex by the Dold-Kan correspondence[11]. The chain complex $\Gamma(A_T)$ calculates the singular homology of $T$.

## 4. Fully faithfulness

**4.1.** Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor, we say it is faithful (full), if for any objects $x, y \in \mathrm{Ob}(\mathcal{A})$, the map

$$F\colon \mathrm{Mor}_{\mathcal{A}}(x,y) \to \mathrm{Mor}_{\mathcal{B}}(F(x), F(y))$$

is injective (surjective). If $F$ is fully faithful, $\mathcal{A}$ is a full subcategory of $\mathcal{B}$[12].

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[9]How did we use the non-decreasing of $\theta$?

[10]How can we check that $S_T(\theta)$ is a functor?

[11]How does the Dold-Kan functor $\Gamma\colon \mathrm{Ab}^{\mathbf{\Delta}^{opp}} \to \mathrm{Ch}_\bullet^+(\mathrm{Ab})$ work? Google or look at [**Sta20**, 0194].

[12]Here is a geometric version of full faithfulness: Whenever we embed a manifold in $\mathbb{R}^n$, we want the underlying topological space is an injection, and the tangent space do not degenerate.

**4.2.** The study of a category $\mathcal{C}$ can be carried out by embedding it in a well-understood category, for example, the category of sets[13]. We have Yoneda embedding theorem

THEOREM 4.1. *The Yoneda functor $U \mapsto h_U$ is fully faithful.*

PROOF. This follows from Yoneda's lemma $\mathrm{Mor}_{\mathtt{Psh}(\mathcal{C})}(h_X, F) = F(X)$, take $F = h_Y$. $\square$

**4.3.** It is noticeable that the nerve functor

$$N\colon \mathtt{Cat} \to \mathtt{sSets},$$
$$\mathcal{C} \mapsto N_{\mathcal{C}}$$

is fully faithful[14], hence we see that "*simplicial sets*" is a genuine generalization of "*categories*".

However, the singular set functor

$$S\colon \mathtt{Top} \to \mathtt{sSets},$$
$$T \mapsto S_T$$

is not faithfully full. Can you explain what fails? (Hint: Take $T$ to be a totally disconnected space with non-discrete topology.)

## 5. Intermission ♪

### 5.1. What we have done.

$$\begin{tikzcd} & \mathtt{sSets} & \\ \mathtt{Cat} \arrow[ur, "\text{Nerve}", "\mathcal{C}\mapsto N_{\mathcal{C}}"'] & & \mathtt{Top} \arrow[ul, "\text{Sing.Set}"', "T\mapsto S_T"] \end{tikzcd}$$

### 5.2. Wishful thinking.

$$\begin{tikzcd} & \mathtt{sSets} \arrow[dl, dotted, "\text{Categorize}"', "?"] \arrow[dr, dotted, "\text{Realize}", "?"'] & \\ \mathtt{Cat} & & \mathtt{Top} \end{tikzcd}$$

Can we, using simplicial sets as a hub, unify categories and topological spaces in a bigger framework? First of all, do we have the dotted arrows going backwards?

**5.3.** Presheaves in general. Let $\mathcal{C}, \mathcal{D}$ be a categories, we call a functor $F\colon \mathcal{C}^{opp} \to \mathcal{D}$ a presheaf of $D$ on $\mathcal{C}$. The category of such presheaves are denoted by $\mathtt{Psh}_{\mathcal{D}}(\mathcal{C})$. We'll omit subscript $\mathcal{D}$ if it is clear from context.

**5.4.** Simplicial sets. A simplicial set is just a presheaf of sets on the category of finite ordered sets $\mathbf{\Delta}$, namely $\mathtt{sSets} = \mathtt{Psh}(\mathbf{\Delta})$.

**5.5.** Yoneda embedding enables us to study a category by embedding in a well understood category, which is "close to" $\mathtt{Sets}$. In an abstract category, we cannot talk about elements of an object[15]. Yoneda embedding enables us to do so, using the functor of points[16].

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[13]More precisely, the category of "presheaf of sets on $\mathcal{C}$", denoted by $\mathtt{Psh}(\mathcal{C})$, or $\mathtt{Psh}_{\mathtt{Sets}}(\mathcal{C})$ or $\mathtt{Sets}^{\mathcal{C}^{opp}}$

[14]Challenge yourself to prove it? Or cowardly look at [**Lur20**, 002Y].

[15]Then it is very annoying to check if a morphism is epic/surjection, as we do not have elements to perform diagram chasing.

[16]See Freyd-Mitchell embedding theorem for a continuation of this philosophy with more structures involved. The embedding theorem enables one to prove the five-lemma in the abstract setting, pretending objects in the abelian category are actually modules [**Wei94**, 1.3.2,1.6.1].

## 6. Realization

**6.1.** Is there a realization functor

$$| \, | \colon \texttt{sSets} \to \texttt{Top}?$$

Let $X$ be a simplicial set. The idea is to assign each "simplex" in $X(\mathbf{n})$, a geometric simplex $|\Delta^n|$, glue them via the face and degeneration maps[17].

**6.2.** First we need to clarify what is an "$n$-simplex" in the category `sSets`? Were such a thing exist,

- First of all, as a simplicial set, it is a contravariant functor $\mathbf{\Delta}^{opp} \to \texttt{Sets}$.
- Furthermore, the $n$-simplex should contain an "identity" when acted on $\mathbf{n}$.

A natural choice of an $n$-simplex in `sSets` is

$$\mathbf{\Delta}^n := \mathrm{Mor}_{\mathbf{\Delta}}(-, \mathbf{n}),$$

this is a toy example of a representable functor.

**6.3.** Thinking simplices in $X$ as "maps $\Delta^n \to X$"[18], the collection of all the simplices in $X$ can be integrated into a category $\mathbf{\Delta}/X$, the simplex category over $X$ [19]. This is a subcategory of $\texttt{sSets} = \texttt{Psh}(\mathbf{\Delta})$, is it full?

Lemma 6.1 (Assembling lemma). *There is an isomorphism in* `sSets`*:*

$$X \cong \operatorname*{colim}_{f \in \mathbf{\Delta}/X} \mathbf{\Delta}^{n(f)}$$

**6.4.** Recall what is a "colimit". Let $\mathcal{I}$ be a small category, let $\mathcal{C}$ be a category. A diagram is simply a functor $M \colon \mathcal{I} \to \mathcal{C}$. The colimit of the $\mathcal{I}$-diagram $M$ in the category $\mathcal{C}$ is given by an object $\mathrm{colim}_{\mathcal{I}} M$ in $\mathcal{C}$, endowed with morphisms $s_i \colon M_i \to \mathrm{colim}_{\mathcal{I}} M$ which best approximates the family of arrows in $M$. More precisely, given $W \in \mathcal{C}$ and any compatible family of morphisms $M_i \to W$, there exists a unique morphism $\mathrm{colim}_{\mathcal{I}} M \to W$ that makes the "umbrella" diagram commute:

$$\begin{tikzcd}
M_{i_1} \arrow[dr] \arrow[drr] & & \\
M_{i_2} \arrow[r] \arrow[rr, bend left] & \mathrm{colim}_I M \arrow[r, dotted] & W \\
M_{i_3} \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

Let $U, V$ be the closed upper and lower hemisphere of $S^2$, let $E$ be the equator, consider the small category $\mathcal{I} = \{a \leftarrow c \to b\}$, and the functor $M \colon \mathcal{I} \to \texttt{Top}$, $a \mapsto U, b \mapsto V, c \mapsto E$, the arrows are mapped to inclusions, can you unwrap the definitions and justify $\mathrm{colim}_{\mathcal{I}} M = S^2$? [20] Checking this example make us feel safe that colimit is a kind of gluing.

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[17] One may look into any algebraic topology textbook for intuition.

[18] Let's think in the category of topological spaces (which is actually an natural transformation of functors)

[19] The set of objects $\mathrm{Ob}(\mathbf{\Delta}/X)$ consists of morphisms $f \colon \mathbf{\Delta}^{n(f)} \to X$, the set of morphisms $\mathrm{Mor}_{\mathbf{\Delta}/X}(f, g)$ consists of morphisms $h \colon \Delta^{n(f)} \to \Delta^{n(g)}$ such that $f = g \circ h$, see [**GJ99**, I.2].

[20] When does colimit over a small category exist? [**Sta20**, 002D]

**6.5.** With the assembling lemma[21] in mind, we define the realization of the simplicial set:

$$|X| := \operatorname*{colim}_{f \in \boldsymbol{\Delta}/X} |\Delta^{n(f)}|$$

**6.6.** By construction, the realization $|X|$ is always a CW complex.

**6.7.** The realization functor is a left adjoint to the singular set functor:

$$\mathrm{Mor}_{\mathtt{Top}}(|X|, Y) = \mathrm{Mor}_{\mathrm{sSet}}(X, S_Y).$$

Proof. The proof gives us a flavor how abstract nonsense is spoken:

$$\begin{aligned}
\mathrm{Mor}_{\mathtt{Top}}(|X|, Y) =& \mathrm{Mor}_{\mathtt{Top}}(\mathrm{colim}|\Delta^{n(f)}|, Y) \\
=& \lim \mathrm{Mor}_{\mathtt{Top}}(|\Delta^{n(f)}|, Y) \\
=& \lim \mathrm{Mor}_{\mathtt{sSets}}(\boldsymbol{\Delta}^{n(f)}, S_Y) \\
=& \mathrm{Mor}_{\mathtt{sSets}}(\mathrm{colim}\boldsymbol{\Delta}^{n(f)}, S_Y) \\
=& \mathrm{Mor}_{\mathtt{sSets}}(X, S_Y)
\end{aligned}$$

The commutation of (co)limit and Mor is just a restatement of the(ir) universal properties. □

**6.8.** We prove the assembling lemma. By (co)-Yoneda embedding theorem, it suffice give natural identification of the following sets for all $Y$ in $\mathtt{sSets} = \mathtt{Psh}(\boldsymbol{\Delta})$:

$$\mathrm{Mor}_{\mathtt{Psh}(\boldsymbol{\Delta})}(X, Y) = \mathrm{Mor}_{\mathtt{Psh}(\boldsymbol{\Delta})}(\operatorname*{colim}_{f \in \boldsymbol{\Delta}/X} \boldsymbol{\Delta}^{n(f)}, Y).$$

By the universal property of colimit and Yoneda lemma, the right hand side equals to

$$\lim_{f \in \boldsymbol{\Delta}/X} \mathrm{Mor}_{\mathtt{Psh}(\boldsymbol{\Delta})}(\boldsymbol{\Delta}^{n(f)}, Y) = \mathrm{Mor}_{\mathtt{Psh}(\boldsymbol{\Delta}/X)}(h_X, h_Y) = \mathrm{Mor}_{\mathtt{Psh}(\boldsymbol{\Delta})}(X, Y)$$

How do we get the equalities?

- The index category $\boldsymbol{\Delta}/X$ consists of elements of the form $\boldsymbol{\Delta}^m \xrightarrow{f} X$, whose morphisms are maps $h$ that make the following triangles commute:

$$\begin{tikzcd}
\color{red}{\boldsymbol{\Delta}^m} \arrow[rr, "h"] \arrow[dr, "f"'] & & \color{blue}{\boldsymbol{\Delta}^n} \arrow[dl, "g"] \arrow[r, dotted, "\text{Yoneda}"] & X(\color{red}{\mathbf{m}}) & X(\color{blue}{\mathbf{n}}) \arrow[l] \\
& X & & &
\end{tikzcd}$$

- The diagram[22] we consider is the functor $\boldsymbol{\Delta}/X \to \mathtt{Sets}, a \mapsto \mathrm{Mor}(a, Y)$. The compatibility conditions in the limit are given by commutativity of lower triangles:

$$\begin{tikzcd}
\mathrm{Mor}(\color{red}{\boldsymbol{\Delta}^m}, Y) & \mathrm{Mor}(\color{blue}{\boldsymbol{\Delta}^n}, Y) \arrow[l, "\widetilde{f} = \widetilde{g} \circ h \leftarrow \widetilde{g}"'] \arrow[r, dotted, "\text{Yoneda}"] & Y(\color{red}{\mathbf{m}}) & Y(\color{blue}{\mathbf{n}}) \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
\boldsymbol{\Delta}^m \arrow[rr, "h"] \arrow[dr, "\widetilde{f}"'] & & \boldsymbol{\Delta}^n \arrow[dl, "\widetilde{g}"] \\
& Y &
\end{tikzcd}$$

- By the previous interpretation, we see an element in the limit, is nothing but a natural transformation of functors from $h_X$ to $h_Y$ in $\mathtt{Psh}(\boldsymbol{\Delta}/X)$.

[21] which tells us how the abstract simplices $\boldsymbol{\Delta}^n$ are assembled, we want the geometric simplices $\Delta^n$ be assembled in exactly the same way

[22] of the limit

- By the interpretation of Yoneda lemma, such natural transformations are exactly the natural transformations from $X$ to $Y$ in $\mathtt{Psh}(\mathbf{\Delta}) = \mathtt{sSets}$.

**6.9.** The assembling lemma is our nickname for a general theorem [**ML98**, III.7.1]:

THEOREM 6.2 (the density theorem). *Let $\mathcal{C}$ be a small category. Then every presheaf is a colimit of representable sheaves:*

$$F \cong \operatorname*{colim}_{V \to F} V.$$

*Can you make this statement precise and prove it? How is the smallness*[23] *of $\mathcal{C}$ used?*

**6.10.** It is not true that every sheaf is a colimit of representable sheaves, unless every representable presheaf is a sheaf[24].

## 7. Questions

Here are some questions[25] one can think about when feeling bored. You are encouraged to use your imagination, pose your own questions, and share the joy with others.

**7.1.** Given a group $G$, let's consider $BG$: the category with a single object $e$, and $\mathrm{Mor}(e, e) = G$.

- Recall we defined the nerve functor

$$N\colon \mathtt{Cat} \to \mathtt{sSets}$$

  and the realization functor

$$|\ |\colon \mathtt{sSets} \to \mathtt{Top}.$$

  Applying these functors to $G$ produces a topological space $|N(BG)|$.
- Let $G = \mathbb{Z}/2\mathbb{Z}$, what do we get? Do we get $\mathbb{RP}^\infty$?
- What is the relation between $BG$ and $B(G^{opp})$?

**7.2.** During the intermission, we wondered if there is way to "reverse" the construction of taking nerve: Is there a way to categorize a simplicial set? Viewing a graph as a simplicial set(how?), can you categorize(what does this word mean) it?

**7.3.** Let $X$ be a topological space. Consider the category $\mathcal{T}_X$, whose objects are open subsets of $X$, whose morphisms are inclusions. For the following $X$, is it true that "every representable presheaf of sets on $\mathcal{T}_X$ is a sheaf"[26]?

- $X = (\{0, 1\}, \text{discrete topology})$
- $X = (\mathbb{C}^1, \text{Euclidean topology})$
- $X = (\mathbb{C}^1, \text{cofinite topology})$

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[23] namely $\mathrm{Ob}(\mathcal{C})$ is a set, not a proper class

[24] Can you prove this in one line?

[25] Ok, these questions are here just for fun, they are definitely not homework questions.

[26] One may want to look up the definition of a sheaf of sets, find out what to prove/disprove

**7.4.** Let $T$ be your favorite topological space. Recall we defined the singular set functor

$$S\colon \texttt{Top} \to \texttt{sSets}$$

and the realization functor

$$|\ |\colon \texttt{sSets} \to \texttt{Top}.$$

Composing these two functors on $T$, we get a topological space $|S_T|$.

- What is the relation between $T$ and $|S_T|$? Are they homotopic? Can they be homeomorphic? Does the association $T \mapsto |S_T|$ preserve homotopy equivalence?
- Let $T = \text{point}, \mathbb{R}^n, S^1, S^3, S^1 \times S^1$, what do we get?
- Do they have same homology groups?
- Do they have isomorphic fundamental groups?

**7.5.** Are there some special properties that simplicial sets in

- the image of the nerve functor $N\colon \texttt{Cat} \to \texttt{sSets}$
- the image of singular set functor $S\colon \texttt{Top} \to \texttt{sSets}$

have in common? Can find one such property and axiomatize it? (Hint: In a category, a key structure we have is composition of arrows. Thus given a horn $\wedge$ in $N_{\mathcal{C}}(\mathbf{1})$, there is a unique extension to the a simplex $\triangle$ in $N_{\mathcal{C}}(\mathbf{2})$. Similarly, given a geometrical horn $\wedge$ mapping to a topological space $T$, we can extend the map $\wedge \to T$ to $\triangle \to T$, by squeezing the interior.)

**7.6.** In the definition of categories, we may impose extra structure on the set of morphisms. For example, instead of merely ask $\mathrm{Mor}_{\mathcal{C}}(x, y)$ to be a set, we may ask $\mathrm{Mor}_{\mathcal{C}}(x, y)$ to have extra structure of

- an abelian group,
- a small category,
- a topological space.

The enriched category is correspondingly called

- A pre-additive category, it will be upgraded to additive categories and abelian categories in the future.
- A 2-category, by induction, one may naively enrich category with 2-categories and produce "3-categories", etc, up to "$\infty$-categories", whatever that means.
- A topologically enriched category, it is also called quasi-category, weak-Kan category, inner Kan complex, Boardman complex, quategory. Does it have another name?

**7.7.** Let's admit that the notion of *topologically enriched categories*[27] is the correct generality that well accommodates both `Top` and `Cat`, but what is the point of such generalization?

**7.8.** An answer: This allows us to justify certain absurd operations, with fully developed abstract theories[28]. Our situation is like justifying $1 + 2 + \cdots + n + \cdots = -\frac{1}{12}$ as $\zeta(-1)$.

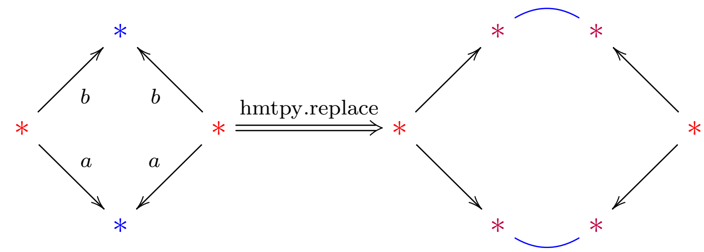
[27]which, by the name weak-Kan category, is a subcategory of `sSets` satisfying ceratin axiom on extension of horns.

[28]Grothendieck developed the general language schemes, which accommodate all commutative rings. This looked useless when people only care about varieties. However, the language of schemes facilitates the study of deformation and degeneration problems, which are posed for varieties.

**7.9.** Here is a toy example[29]. Let $A$ be a commutative ring, we "show" the Hochschild homology $\mathrm{H}(A \otimes_{A\otimes A} A)$ has an "$S^1$-action". We view

$$A \otimes_{A\otimes A} A = A \cdot (* \xleftarrow{a} ** \xrightarrow{b} *).$$

The push-out diagram is already seen to be a circle, if we read the arrows as edges:




If we work in the category of topological spaces, the push-out diagram collapses to a point. However, if we allow some homotopy flexibility, we may replace $**$ with arcs, the push-out[30] glues the arcs along the end points yield a circle $S^1$.

**7.10.** Let $\mathcal{I} = (S, \prec)$ be a directed set: a set $S$ equipped with a (1) reflexive and (2) transitive binary relations $\prec$, such that (3) every pair of elements has an upper bound. Let $\mathcal{I}' \subset \mathcal{I}$ be a sub-directed-set, we say $\mathcal{I}'$ is cofinal in $\mathcal{I}$ is every element in $\mathcal{I}$ is upper bounded by an element in $\mathcal{I}'$. Let $M\colon \mathcal{I} \to \mathcal{C}$ be a diagram, and let $M'\colon \mathcal{I}' \to \mathcal{C}$ be the induced diagram. Can you show $\mathrm{colim}_{\mathcal{I}} M \cong \mathrm{colim}_{\mathcal{I}'} M'$? Does the isomorphism hold if we drop condition (3) from $\mathcal{I}$? (Hint:$\mathcal{I} = \{* \leftarrow * \to *\}$, $\mathcal{I}' = \{* \quad *\}$)

**7.11.** In the assembling lemma, does it suffice to run colimit over all non-degenerate simplices[31]? If so, how does the notion of "non-degenerate simplex" generalize to the density theorem?

**7.12.** Let $\mathcal{C}$ be a category. What are the functors from $BG$ to $\mathcal{C}$? What are the functors from $\mathcal{C}$ to $BG$? Let $G, H$ be groups, what are the functors from $BG$ to $BH$?

**7.13.** In linear algebra, we studied the category $\mathtt{Vec}_{\mathbb{R}}$, whose objects are finite dimensional $\mathbb{R}$-vector spaces, whose morphisms are linear transformation. We know vector spaces are determined by their dimensions, so it seems that the category can be simplified. Let's consider the category $\mathtt{Mat}$:

- Whose objects are finite ordered sets $\mathbf{0}, \mathbf{1}, \ldots, \mathbf{n}, \ldots$ where $\mathbf{n} = \{0, 1, \ldots, n\}$
- A morphism $\mathbf{n} \to \mathbf{m}$ is a $(m+1) \times (n+1)$ matrix.
- Law of composition is given by matrix multiplication.

There is a functor $F\colon \mathtt{Mat} \to \mathtt{Vec}$, given by

- $F(\mathbf{n}) = \mathbb{R}^{n+1} = \bigoplus_0^n \mathbb{R}e_i$, where $e_i$ is the standard column matrix.
- For $A$ in $\mathrm{Mor}_{\mathtt{Mat}}(\mathbf{n}, \mathbf{m})$, we assign $F(A) \in \mathrm{Mor}_{\mathtt{Vec}_{\mathbb{R}}}(F(\mathbf{n}), F(\mathbf{m}))$ which sends $e_i$ to $Ae_i$.

Can you check the functor is faithfully full and essentially surjective, hence $F$ is an equivalence of categories? [32] Is the category $\mathtt{Mat}$ equivalent to the category of finite ordered sets $\mathbf{\Delta}$? Why?

**7.14.** Let $G, H$ be finite groups, when is $BG$ equivalent to $BH$?

---

[29]Caution: The following reasoning is as absurd as calculating sum of natural numbers by elementary operation on equalities of divergent sums.

[30]We did the push-out for $S^2$ when we discuss colimits

[31]namely, we run the colimit in the subcategory of $\mathbf{\Delta}/X$, where only monic morphisms $f\colon \mathbf{\Delta}^{n(f)} \to X$ are allowed. In $\mathtt{sSets}$, does every morphism have a unique epi-monic factorization?

[32]Hint: One can pick an "midterm exam" functor $M\colon \mathtt{Vec}_{\mathbb{R}} \to \mathtt{Mat}$ as the quasi-inverse of $F$, by fixing a basis whenever he see a vector space.

**7.15.** Consider the pair of adjoint functors $|\ | \dashv S$. Does $S$ has a right adjoint? Does $|\ |$ has a left adjoint? Do $S$, $|\ |$ preserve fiber products? Can you give an example of two good topological spaces over a third, whose fiber product is miserable? Is it true that fiber product of CW complexes remain a CW complex?

*There is a story about two Hungarian aristocrats who decided to play a game in which the one who calls the largest number wins:*

*"Well," said one of them, "you name your number first."*

*After a few minutes of hard mental work the second aristocrat finally named the largest number he could think of.*

*"Three," he said.*

*Now it was the turn of the first one to do the thinking, but after a quarter of an hour he finally gave up.*

*"You've won," he agreed.*

—George Gamow, *One, two, three..., infinity, 1946*

CHAPTER 2

# Sheaves-I

Let $\texttt{Top}$ be the category of topological spaces.[1]

## 1. $\texttt{Top}(X)$, $\texttt{Top}_X$ and $\texttt{Top}_{/X}$

Let $X$ be a topological space.

**1.1.** Let $\texttt{Top}_{/X}$ be the category of objects of $\texttt{Top}$ over $X$.[2]

**1.2.** Let $\texttt{Top}_X$ be the category of local homeomorphisms to $X$.

**1.3.** Let $\texttt{Top}(X)$ be category of open inclusions to $X$.[3]

**1.4.** Can you check that:

- Morphisms in $\texttt{Top}_X$ are automatically local homeomorphisms?
- $\texttt{Top}(X)$ is not a groupoid?

**1.5.** In the chain

$$\texttt{Top}(X) \subset \texttt{Top}_X \subset \texttt{Top}_{/X}$$

of nested subcategories,

- Is every subcategory is a full-subcategory?
- Does there exist quasi-inverses[4] to the forgetful[5] functors?

## 2. Sheaves: Definition

Let $X$ be a topological space.

**2.1.** A sheaf of sets on $X$ is a functor

$$\mathcal{F}\colon \texttt{Top}(X)^{opp} \to \texttt{Sets}$$

such that the diagram of pullback (where $p_i^*$ is induced from inclusion $p_i\colon U_i \cap U_j \to U_i$[6])

$$\mathcal{F}(U) \xrightarrow{p^*} \prod_{i\in I} \mathcal{F}(U_i) \underset{p_j^*}{\overset{p_i^*}{\rightrightarrows}} \prod_{i,j\in I} \mathcal{F}(U_i \cap U_j),$$

is an equalizer diagram[7] the category of sets, whenever $\mathcal{U} := \{U_i\}_{i\in I}$ is a covering of $U$.

[1] See [**Li20**, 1.1.2.7].

[2] See [**Jac80**, 1.1.Ex.3]. We also call $\texttt{Top}_{/X}$ the category of continuous maps to $X$.

[3] See [**Li20**, 1.2.4].

[4] See [**Li20**, 1.2.13].

[5] See [**Li20**, 1.2.2]. For an object in the subcategory, let it forget the fact that it lies in the subcategory.

[6] The letter $p$ comes from the word "projection": the intersection can be read as the fiber product $U_i \times_X U_j$.

[7] See [**Li20**, 1.3.14].

**2.2.** In words, the axiom means:

- "Local sections uniquely glue."
- Glueable local sections can be uniquely glued along the gluing datum.
- Local sections that match on overlaps, are restricted form a unique global section.
- A global section is uniquely determined by its local sections. Conversely, given a family of local sections, they glue to a global section as long as they match on overlaps.
- Sections of a sheave behave like continuous maps between topological spaces: it is uniquely determined by its restriction to an open cover; or,
- given a family of continuous maps over an open covering, as long as they coincide on overlaps, they define a unique global continuous map.
- Sections of a sheaf behave like solutions to differential equations. Recall we studied local uniqueness and existence of solutions, this is exactly what we ask of sheaves.

**2.3.** When we consider sheaves of abelian groups, the pullback diagram being equalizer diagram translates to: $\ker(p^*) = 0$ and $\mathrm{im}(p^*) = \ker(p_i^* - p_j^*)$. Or equivalently, we are asking the exactness of the sequence

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\mathrm{d}^0} \prod_{i \in I} \mathcal{F}(U_i) \xrightarrow{\mathrm{d}^1} \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j) \ ,$$

where $\mathrm{d}^0 = p^*$ and $\mathrm{d}^1 = p_1^* - p_2^*$ is taking difference.

**2.4.** Let's try to understand the axiom from simple examples. Let's work on the sheaves of abelian groups, since we are not very familiar with the universal property of equalizers, plus, in an equalizer diagram, there are more arrows — *"Three."*

- Let $I = \emptyset$. Note that the product over empty index set is the final object in the category,[8] then the sequence reads

$$\begin{tikzcd} & & 0 \arrow[d, equal] & 0 \arrow[d, equal] \\ 0 \arrow[r] & \mathcal{F}(U) \arrow[r, "\mathrm{d}^0"] & \prod_{i \in \emptyset} \mathcal{F}(U_i) \arrow[r, "\mathrm{d}^1"] & \prod_{i,j \in \emptyset} \mathcal{F}(U_i \cap U_j) \end{tikzcd} .$$

  In particular, we can prove *the theorem of empty-section* for sheaves of abelian groups:

  THEOREM 2.1. *Let $\mathcal{F}$ be a sheaf of abelian groups on a topological space $X$, then*
  $$\mathcal{F}(\emptyset) = 0.$$
  *In particular, there is no reason why a presheaf should satisfy such a condition*[9].[10]

- Let $I = \{0\}$, the exact sequence reads:

$$\begin{tikzcd} & & \mathcal{F}(U) \times \mathcal{F}(U) \arrow[dr, equal, "1 \times i"] & \\ 0 \arrow[r] & \mathcal{F}(U) \arrow[r, "\mathrm{id}_U^*"] & \mathcal{F}(U) \arrow[u, "\Delta"] \arrow[r, "\mathrm{d}^1"] & \mathcal{F}(U) \times \mathcal{F}(U) \end{tikzcd} .$$

[8] The universal property reads: For any empty set of objects, and any morphism from a testing object $T$ to the empty set of objects, (which can't be a more vacuous condition), there exists a unique morphism from $T$ to the product. This simply means any object has a unique morphism to the empty product. Hence the empty product is the final object in the category. Which is a point in `Sets` and 0 in the category of abelian groups.

[9] Thus the definition of presheaves in [**Har77**, II.1] is stronger than usual. The extra axiom $\mathcal{F}(\emptyset) = 0$ for presheaves is unnecessary, and is never used anywhere.

[10] Can you give a meaningful example of a presheaf of abelian groups, such that $\mathcal{F}(\emptyset) \neq 0$?

One just note that identity is the equalizer of diagonal.[11] This is indeed uninteresting, it holds even if $\mathcal{F}$ is a presheaf.

- Let $I = \{0, 1\}$, the exact sequence reads:

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\mathrm{d}^0} \mathcal{F}(U_0) \times \mathcal{F}(U_1) \xrightarrow{\mathrm{d}^1} \mathcal{F}(U_{00}) \times \mathcal{F}(U_{01}) \times \mathcal{F}(U_{10}) \times \mathcal{F}(U_{11}) \ ,$$

where

$$\mathrm{d}^1 \colon (a, b) \mapsto (a - a, a - b, b - a, b - b).$$

There seems to be some redundancy we could work on. Taking difference with oneself does not yield anything interesting: sections are always compatible with themselves.[12] Also, one need to check compatibility *only once* on each overlap. Thinking this through, one decide to give the set $I$ a total order, and only take product with ordered pairs. Note that

(1) $$(a - a, a - b, b - a, b - b) = 0 \Leftrightarrow (0, a - b, b - a, 0) \Leftrightarrow (a - b) = 0,$$

the exactness of our original sequence reduces to the exactness of

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\mathrm{d}^0} \mathcal{F}(U_0) \times \mathcal{F}(U_1) \xrightarrow{\mathrm{d}^1} \mathcal{F}(U_{01}) \ ,$$

where

$$\mathrm{d}_1 \colon (a, b) \mapsto (a - b).$$

Now it is very clear what the sheaf axiom asks of a presheaf: A pair of sections $(a, b)$ take the form $(s|_{U_0}, s|_{U_1})$ for some global section $s$, if and only if $a = b$ on $U_0 \cap U_1$. And the restriction map $s \mapsto (s|_{U_0}, s|_{U_1})$ is unique.

- Let $I = \{0, 1, 2, \cdots, n\}$, then the third term has $n^2$ components, and there are lots of redundant terms. The equality (1) can be generalized to an arbitrary set of coverings.[13] Let's denote the last product (unordered and ordered) by

$$\widetilde{C}^1(\mathcal{U}, \mathcal{F}) := \prod_{i_0, i_1 \in I \times I} \mathcal{F}(U_{i_0} \cap U_{i_1}),$$
$$C^1(\mathcal{U}, \mathcal{F}) := \prod_{i_0 \prec i_1 \in I \times I} \mathcal{F}(U_{i_0} \cap U_{i_1}).$$

Then the axiom, which was originally stated as:

$$\mathcal{F}(U) = \ker(\widetilde{C}^0(\mathcal{U}, \mathcal{F}) \to \widetilde{C}^1(\mathcal{U}, \mathcal{F})), \text{ for all } U \in \mathtt{Top}(X), \text{ and } \mathcal{U} \text{ covering } U.$$

can be restated as

$$\mathcal{F}(U) = \ker(C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F})), \text{ for all } U \in \mathtt{Top}(X), \text{ and } \mathcal{U} \text{ covering } U.$$

- Let's stop for a while and stare at the notations. Do they remind you of something familiar from the last recitation class? *There's a shadow hanging over them, the category* $\mathbf{\Delta}$ *came suddenly.* We'll come back to this in exercise 7.3.

---

[11] By abstract nonsense, or *it suffice to* check in the category of sets, by the Yoneda embedding lemma. Can you write down the precise statements: how the Yoneda embedding lemma is applied, and what is shown?

[12] Unlike human beings.

[13] Using the fact that "diagonal is injective"

## 3. Sheaves: Extension to $\mathtt{Top}_X$

The slogan "local sections uniquely glue" is vague. Being vague has many advantages, to name a few:

- It motivates one to think and work: carefully specify what is talked about, carefully *choose the definitions* so that the mathematics works.[14]
- It makes no mistakes, since it is vague!
- Poetic.

In our slogan, we didn't specify what "local section" means. We vaguely know that it is something local. It doesn't even ask the local section be taken over an open *subset* of $X$: the notion "subset of $X$" is already global. After a few minutes of thinking, the question pops out: can the sections be taken over "local homeomorphisms"?

**3.1.** Let's rephrasing our question: Let $X$ be a topological space, we have categories over $X$: $\mathtt{Top}(X)$ and $\mathtt{Top}_X$. Let $\mathcal{F}$ be a sheaf on $X$. We are asking if the functor $\mathcal{F}$ extends?

$$\begin{tikzcd}
\mathtt{Top}(X)^{opp} \arrow[r, "\mathcal{F}"] \arrow[d, "\mathrm{Fgt}"'] & \mathtt{Ab} \\
\mathtt{Top}_X^{opp} \arrow[ur, dotted, "??"'] &
\end{tikzcd}$$

**3.2.** It's a problem we've never dealt with.[15] We don't know an answer. But what would happen, *if all was settled*?

**3.3.** Let's take a $(w\colon W \to X) \in \mathrm{Ob}(\mathtt{Top}_X)$. And try to figure out what the group of sections $\mathcal{F}(W)$ should be.

**3.4.** Since $w$ is a local homeomorphism, we may choose an open covering $\{W_i\}_I$ of $W$, such that each $W_i \to X$ is an injective local homeomorphism, or, a homeomorphism onto its image in $X$. Can we identify $W$ as an open subset of $X$? This question sounds like: can we identify all vector spaces with $\mathbb{R}^n$? We can, but we prefer not to. We really want to *distinguish the isomorphic objects*.[16]

[14]Hopefully one understands the concepts and ideas better in this process.

[15]Isn't it, we just learnt what is a sheaf 5 minutes ago.

[16]Let us, using axiom of choice, identify every vector space with a standard $\mathbb{R}^n_{\mathtt{God}}$. Assume it happened that $\mathbb{R}^n_{\mathtt{Lihua}}$ is identified with the $\mathbb{R}^n_{\mathtt{God}}$ via matrix $(\zeta^n_m)$, and that $\mathbb{R}^n_{\mathtt{Hanmeimei}}$ is identified with $\mathbb{R}^n_{\mathtt{God}}$ via matrix $(n^m)$, then the map $\iota\delta\colon \mathbb{R}^n_{\mathtt{Hanmeimei}} \to \mathbb{R}^n_{\mathtt{Lihua}}, (a_1, \ldots, a_n) \mapsto (a_1, \ldots, a_n)$ would be $(\zeta^n_m)(n^m)^{-1}$, this make things extremely difficult: Are you still in sane after verifying surjectivity of $\iota\delta\colon (a_1, \ldots, a_n) \mapsto (a_1, \ldots, a_n)$? This might be a post-modern reason why Newton's idea of taking inertia frame does not work. Nowadays, there are mathematicians applying category theory to physics. For example, what is an "observer", precisely? One may read for fun: *The Categorical Language of Quantum Physics* by Eugene Rabinovich.

**3.5.** The question of extending $\mathcal{F}$ now decomposes into two steps:

$$\begin{tikzcd}
\mathtt{Top}(X)^{opp} \arrow[r, "\mathcal{F}"] \arrow[d, "\mathrm{Fgt}"'] & \mathrm{Ab} \\
\mathtt{Top}\{X\}^{opp} \arrow[d, "\mathrm{Fgt}"'] \arrow[ur, dotted, "?"] & \\
\mathtt{Top}_X^{opp} \arrow[uur, dotted, "??"'] &
\end{tikzcd}$$

Here the category $\mathtt{Top}\{X\}^{opp}$ consists of all local homeomorphisms onto $X$. In this category, we may extend the functor by assigning $w\colon W \to X$ to ${}^{w}\mathcal{F}(w(W))$, this is a group isomorphic to $\mathcal{F}(w(W))$, but *not equal* to $\mathcal{F}(w(W)) =: {}^{w(W)}\mathcal{F}(w(W))$! It sounds like a stupid "idealisticism" to distinguish the two, but actually not. We haven't finished the functor, morphisms! Now one see how *realistic* it is to bookkeep $w$ with the object.

**3.6.** Now we have the last arrow to conquer. Again, what would happen if *all was settled*?

**3.7.** Let $(u\colon U \to X)$ be an object in $\mathtt{Top}_X$. The assumption of local homeomorphism allows us to take an open covering $\mathcal{W}$ of $U$ in $\mathtt{Top}\{X\}$. Since everything was settled, the sheaf $\mathcal{F}$ has to satisfy the axiom

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \prod_{i_0 \in \mathcal{W}} \mathcal{F}(W_{i_0}) \longrightarrow \prod_{i_0, i_1 \in \mathcal{W}\times\mathcal{W}} \mathcal{F}(W_{i_0} \times_U W_{i_1})$$

This can be viewed as the definition of $\mathcal{F}(U)$!

**3.8.** Let's define $U$ as above, by picking any covering $\mathcal{W}$ of $U$ in $\mathtt{Top}\{X\}$. First, we need to check that this map is well defined. This should follow from the sheaf axiom on $\mathtt{Top}\{X\}$.[17] We leave this as an exercise, see 7.4.

**3.9.** Now we need to verify the sheaf axiom: let $\{\mathcal{U}_\alpha\}_{\mathcal{A}}$ be an open covering of $U$ in $\mathtt{Top}_X$. We need to check the exactness of the following sequence:

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\mathrm{d}^0} \prod_{\alpha_0 \in \mathcal{A}} \mathcal{F}(U_{\alpha_0}) \xrightarrow{\mathrm{d}^1} \prod_{\alpha_0, \alpha_1 \in \mathcal{A}\times\mathcal{A}} \mathcal{F}(U_{\alpha_0} \times_U U_{\alpha_1})$$

- Injectivity of $\mathrm{d}^0$.
- Exactness at the middle term: $\ker(\mathrm{d}^1) \subset \mathrm{im}(\mathrm{d}^0)$.

**3.10.** Let's translate the verifications step by step, first look at $\mathrm{d}^1$. The first term can be resolved by some covering from $\mathtt{Top}\{X\}$. Good. But the cheerful optimism would not propagate. The second and third term "*do not look like feul-efficient lamps*": each $U_\alpha$ needs to be again resolved by a cover in $\mathtt{Top}\{X\}$, unfortunately, by their definition. Let's be brave, take a cover $\mathcal{W}_{I_\alpha}$ of each $U_\alpha$, and see what happens.

[17] Which we haven't checked yet, right? However, since they are authentic homeomorphisms, it is reasonable to believed that the sheafiness is inherited form sheaf axiom for $\mathtt{Top}(X)$.

**3.11.** We have a beautiful diagram

$$
\begin{tikzcd}
\textbf{(0,0)} \arrow[dr, squiggly] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \mathcal{F}(U) \arrow[r, "\mathrm{d}^0"] \arrow[d] & \prod\limits_{\mathcal{A}} \mathcal{F}(U_{\alpha_0}) \arrow[r, "\mathrm{d}^1"] \arrow[d] & \prod\limits_{\mathcal{A}^2} \mathcal{F}(U_{\alpha_0\alpha_1}) \arrow[d] \\
0 \arrow[r] & \prod\limits_{I} \mathcal{F}(W_{i_0}) \arrow[r] \arrow[d] & \prod\limits_{I}\prod\limits_{\mathcal{A}}\prod\limits_{I_{\alpha_0}} \mathcal{F}(W_{i_0} \underset{U}{\times} W_{i_{\alpha_0}}) \arrow[r] \arrow[d] & \prod\limits_{I}\prod\limits_{\mathcal{A}^2}\prod\limits_{I_{\alpha_{01}}} \mathcal{F}(W_{i_0} \underset{U}{\times} W_{i_{\alpha_0\alpha_1}}) \arrow[d] \\
0 \arrow[r] & \prod\limits_{I^2} \mathcal{F}(W_{i_0i_1}) \arrow[r] & \prod\limits_{I^2}\prod\limits_{\mathcal{A}}\prod\limits_{I^2_{\alpha_0}} \mathcal{F}(W_{i_0i_1} \underset{W}{\times} W_{i_{\alpha_0,\alpha'_0}}) \arrow[r] & \prod\limits_{I^2}\prod\limits_{\mathcal{A}^2}\prod\limits_{I^2_{\alpha_{01}}} \mathcal{F}(W_{i_0i_1} \underset{U}{\times} W_{i_{\alpha_0}i_{\alpha_1},i'_{\alpha_0}i'_{\alpha_1}})
\end{tikzcd}
$$

where

- Columns are exact by definition of $\mathcal{F}$ on $\mathtt{Top}_X$.
- The second and third rows are exact, by sheaf axiom on $\mathtt{Top}\{X\}$ and *Fubini's theorem*: limits commute.
- The map $\mathrm{d}^0$ is injective.[18]

To sum up, we have a diagram where we have exactness everywhere except at $(1, 0)$.

**3.12.** It's time to learn some *Parseltongue*:

LEMMA 3.1 (Python). *In such a diagram, exactness at $(1, 0)$ is automatic.*

The Python lemma is also called the nine-lemma: in the $3 \times 3$ diagram, exactness at the rest of nodes implies the ninth.

PROOF. Recall how we proved five-lemma, then use the equality $9 < 5 + 5$. $\square$

**3.13.** Let's crystalize what we have done

THEOREM 3.2. *The forgetful functor*

$$\mathrm{Sh}(\mathit{Top}_X) \to \mathrm{Sh}(\mathit{Top}(X))$$

*is an equivalence of categories. In particular, we've just constructed its quasi-inverse.*[19]

[18] This actually follows from the previous bullets. Can you prove it?

[19] The category of sheaves of sets on a *site* is called a topos. A site is also called a grothendieck (pre)-topology, over which sheaves are naturally defined. See [**Sta20**, 00VG, 00X9]. So we can talk about "non point-set" topology. This reasonable, what really matters is the relation between the open subset, not the points. This is very important in the algebraic or rigid analytic world, where points are usually not enough. Instead of using the fancy language, people may also just add more points, see Brian Conrad's notes on *adic spaces*Here is an interesting example on "topology without enough points": [**Con14**, 1.3].

## 4. Sheaves: Representability

**4.1.** Let $X$ be a topological space, let $\mathcal{F}$ be a sheaf of sets on $\mathtt{Top}(X)$. By the previous discussion, it can be naturally extend to a sheaf on $\mathtt{Top}_X$. Here is our fancy dream:

- Any sheaf $\mathcal{F}\colon \mathtt{Top}_X \to \mathtt{Sets}$ is representable.

To be precise: we dream that for any functor

$$\mathcal{F}\colon \mathtt{Top}(X)^{opp} \to \mathtt{Sets},$$

there exists a continuous map of topological spaces $\pi_{\mathfrak{F}}\colon \mathfrak{F} \to X$, such that

- the structure map $\pi_{\mathfrak{F}}\colon \mathfrak{F} \to X$ is a local homeomorphism
- the functors $\mathcal{F}$ and $h_{\mathfrak{F}}$ are naturally isomorphic in $\mathtt{Top}_X$.

If such $\mathfrak{F}$ exist, we call it: the étale space associated with the sheaf $\mathcal{F}$. Why is this a fancy dream? If this is true, then the sections of $\mathcal{F}$ are actually realized as continuous maps $U \to \mathfrak{F}$[20]. So the poetic explanation bullet 5 in Subsection 2.2 can be made rigorous:

- We can redefine sheaf of sets on $X$ as objects in $\mathtt{Top}_X$
- Or, given an object $\mathfrak{F} \in \mathtt{Top}_X$, the functor $U \mapsto \mathrm{Mor}_{\mathtt{Top}_X}(U, \mathfrak{F})$ is a sheaf, and every sheaf arise in this way for a unique local homeomorphism of topological spaces $\pi_{\mathfrak{F}}\colon \mathfrak{F} \to X$.

**4.2.** Recall in the last recitation class, we proved the assembling lemma, which literally gives the proof of following fact: on a small category $\mathcal{C}$, any presheaf $\mathcal{F}$ is a colimit of representable presheaves:

$$\mathcal{F} = \operatorname*{colim}_{h_U/\mathcal{F}} h_U.$$

Note that Yoneda lemma preserves colimit[21], and that $\mathtt{Top}_X$ admit colimits, we know

$$\mathcal{F} = h_{\operatorname*{colim}_{h_U/\mathcal{F}} U}$$

we may simply define the étale space of $\mathcal{F}$ by $\mathfrak{F} := \operatorname*{colim}_{h_U/\mathcal{F}} U$.

**4.3.** This is not quite a proof, as the category $\mathtt{Top}_X$ is not small. In the proof, we need the category to be small, because limits and colimit are only defined on small diagrams. For example, we can take product for "a set of sets", but not "a class of sets". Why is the category $\mathtt{Top}_X$ big? Just take $X = \{*\}$, then $\mathtt{Top}_X = \mathtt{Top}$, which contains $\mathtt{Sets}$ as a full subcategory, but $\mathrm{Ob}(\mathtt{Sets})$ is not a set: *Who shaves the Barber*? However, the problem can be fixed: we may replace the colimit diagram by a small diagram see this link. Note that $\mathtt{Top}(X)$ is a small category, as the class of objects come is a subset of $2^X$, hence $\mathrm{Sh}(\mathtt{Top}(X))$ is a category. This illustrates how the notion of topos is preferred to site. In this case, the topos $\mathrm{Sh}(\mathtt{Top}_X)$ is equivalent to $\mathrm{Sh}(\mathtt{Top}(X))$: it automatically takes care of the bigness of $\mathtt{Top}_X$, which is mostly irrelevant for our discussion.

**4.4.** Now let's define the stalk of a sheaf at $x \in X$ to be the inverse image $\mathfrak{F}|_x = \pi_{\mathfrak{F}}^{-1}(x)$. It coincides what we see elsewhere

$$\mathcal{F}_x := \operatorname*{colim}_{x \in U} \mathcal{F}(U).$$

**4.5.** Psychologically, if one are still not convinced, one could read [**Har77**, II.Ex.1.13], where the étale space is constructed as the union of all stalks, equipping with certain topology, so that the representability of $\mathtt{Top}_X$ is satisfied.

[20] Of course, in as morphisms $\mathtt{Top}_X$, so they have to be compatible with the structure, which just mean $U \to \mathfrak{F} \to U$ is identity, which is what section literally means! See [**Li20**, 1.3.5]

[21] what faithful literally means, otherwise, what's the point of the so called "embedding".

**4.6.** We remark that local homeomorphism can be are very different from $n$-sheet covering in topology. For a local homeomorphism to $\mathbb{R}^1$, the source space can be highly non-separated, say, $\mathbb{C}$-copies of $\mathbb{R}^1$ identified on $\mathbb{R}^1 - \{0\}$. One gets a skyscraper sheaf $i_*\mathbb{C}$, where $i\colon \mathbb{R}^1 - \{0\}$ is the open inclusion.

**4.7.** Up to now, we only assumed knowledge of sets, topological spaces and abelian groups. Using categories and functors, we have developed such a bunch of things, isn't it cool? Of course, there are more to say in the future, when we discuss sheaves of modules.[22]

**4.8.** Lastly, we remark that, in the construction

$$\mathcal{F} = \underset{h_U/\mathcal{F}}{\operatorname{colim}} h_U,$$

we didn't actually ask $\mathcal{F}$ to be a sheaf. The construction goes word by word for a presheaf

THEOREM 4.1 (Sheafification). *There is a left adjoint functor*

$$\sharp\colon \mathrm{Psh}(\mathit{Top}_X) \to \mathrm{Sh}(\mathit{Top}_X)$$

*to the forgetful functor. Namely:*

- *Let $\mathcal{F} \in \mathrm{Ob}(\mathrm{Psh}(Top_X))$, then*
$$\mathcal{F}^\sharp := \underset{h_U/\mathcal{F}}{\operatorname{colim}} h_U \in \mathrm{Ob}(\mathrm{Sh}(\mathit{Top}_X)).$$
- *For any $\mathcal{G} \in \mathrm{Ob}(\mathrm{Sh}(\mathit{Top}_X))$, there is a natural identification:*
$$\mathrm{Mor}_{\mathrm{Psh}(\mathit{Top}_X)}(\mathcal{F}, \mathcal{G}) \cong \mathrm{Mor}_{\mathrm{Sh}(\mathit{Top}_X)}(\mathcal{F}^\sharp, \mathcal{G})$$

**4.9.** To apply the theorem, we remark that any presheaf $\mathtt{Top}(X)$ can be naturally viewed as a presheaf on $\mathtt{Top}_X$ by "zero extension".

## 5. Sheaves: Applications

**5.1.** Recall in complex analysis, given a multi-valued analytic function[23] $f$ on an open domain $\Omega \subset \mathbb{C}$, say

- $w = \mathrm{Log}(z)$, $w = z^{3/4}$, $w = z^{1/2} + z^{-1/2}$ on $\Omega = \mathbb{C} - \{0\}$,
- $w = \sqrt{z(z-1)(z-t)}$ on $\Omega = \mathbb{C} - [0,1] \bigcup [t, \infty)$,
- implicit function defined as the root of an irreducible polynomial $g(z, w) \in \mathbb{C}[z, w]$. Its maximal domain of definition
$$\Omega = \{z_1 \in \mathbb{C}\colon g(z_1)(w) \text{ has no multiple roots.}\}^{24}$$
Equivalently, by the fundamental theorem of algebra:
$$\Omega = \{z_1 \in \mathbb{C}\colon g(z_1)(w) \text{ has } \deg_w(g) \text{ distinct roots.}\}$$
- solution to an ordinary differential equation, say $w'' = p(z)w' + q(z)$[25],

[22] A module is a representation of a ring on an abelian group. A sheaf of modules can be thought as representation of a "topological space" on an abelian group.

[23] Multi-valued functions are not functions, as red pandas are not pandas. See [**Zor15**, 1.3.4] for the formal theoretic definition of a function (maybe you learned it in [**Lou09**]). We remark that using Yoneda embedding, we can define the notion of a function in an arbitrary category, even if the objects do not have underlying sets.

[24] Exercise: show this by inverse function theorem.

[25] See [**Ahl66**, 8.4.4], it is called *Hypergeometric Differential Equation*. It naturally appears in algebraic geometry.

there is a notion called *the Riemann surface* associated with $f$. A Riemann surface is a topological space, serving as the correct "domain of definition" of the multi-valued function. In our textbook[26], the Riemann surface $\mathcal{L}$ for $\mathrm{Log}(z)$ is constructed by patching domains of analytic continuation. The Riemann surface comes naturally equipped with a continuous map $\pi\colon \mathcal{L} \to \Omega \subset \mathbb{C}$.

On the Riemann surface, the multi-valued function $f$ can be well defined, more precisely: there is an analytic function $\widetilde{f}$ defined on $\mathcal{L}$, such that germs of $\widetilde{f}$ are carried to germs of $f$ via projection. We call $(\widetilde{f}, \mathcal{L})$ a uniformization of the multi-valued function $f$ on $\Omega$. Many things need to be justified: For example, how to check the analytic continuation is independent of the path: why is $\widetilde{f}$ well-defined on $\mathcal{L}$?[27] We show these verifications are automatic with the help of "étale space of sheaves".

**5.2.** Let $f$ be a multi-valued analytic function defined on $\Omega \subset \mathbb{C}$. Let $\mathcal{F}$ be the presheaf of single valued branches of $f$ on $\Omega$. The sheaf axioms are obviously satisfied[28]. We may simply define the étale space of $\mathcal{F}$ as our Riemann surface. To see that $f$ can be defined on $\mathfrak{F}$, it suffice to show the sheaf $\mathcal{F}$ of sections has a global section when pulled back to $\mathfrak{F}$. One observes that $\mathcal{F}$ pulls back to the sheaf of sections to the projection $\mathrm{pr}_2\colon \mathfrak{F} \times_X \mathfrak{F} \to \mathfrak{F}$, and that the diagonal $\Delta$ gives a global section.

**5.3.** We remark that when the section of $\mathcal{F}$ is equipped with a transitive group action, then $\mathfrak{F} \times_X \mathfrak{F} \cong \mathfrak{F} \times G/H$ [29]. Let's consider the analytic function in bullet 3, then the Galois group $\mathrm{Gal}_g$ acts on the branches of local sections of $g$ (which are roots of $g(z, w) \in \mathbb{C}(z)[w]$). Recall by our discussion there, the stalks of $g$ are of constant size, thus $\pi_{\mathfrak{F}}\colon \mathfrak{F} \to X$ is a $\deg_w(g)$-sheet covering space. Let's summarize what we have done:

**5.4.** Recall in topology and Galois theory, we have isomorphism of the lattices:

$$\begin{tikzcd}
\mathtt{Top}_\Omega[\Omega, \widetilde{\Omega}] \arrow[rr, leftrightarrow, "\widetilde{\Omega}/H \mapsfrom H", "\mathcal{F} \mapsto \mathrm{Stab}_{\pi_1(\Omega, \pi_{\mathfrak{F}}(p))}(p)"'] \arrow[d, leftrightarrow, dotted] & & \mathtt{Grp}[\pi_1(\Omega), \{\mathrm{id}\}] \arrow[d, leftrightsquigarrow, "\text{``equivalent''}", "almost"'] \\
\mathtt{Flds}_K[K, \overline{K}] \arrow[rr, leftrightarrow, "L \mapsto \mathrm{Gal}(\overline{K}/L)", "\overline{K}^H \mapsfrom H"'] & & \mathtt{Grp}[\mathrm{Gal}_K, \{\mathrm{id}\}]
\end{tikzcd}$$

Let $K = \mathbb{C}(z)$, let $g \in K[z]$ be an irreducible polynomial, let $\mathcal{F}$ be the sheaf of analytic germs on $\Omega$, then the left upwards arrow can be constructed by

$$g(z) \mapsto \mathfrak{F}.$$

We remark that the left downwards arrow can be constructed by taking the field of meromorphic functions:

$$\Omega' \mapsto \mathcal{M}(\Omega').$$

[26]See [**ZQ01**, 7.1].

[27]See [**Aut82**, 2.6.3*].

[28]An analytic function is a solution to the Cauchy-Riemann equation. The local existence and uniqueness of solutions are standard material in ODE or PDE courses. For ODE, see [**Aut**, 4.1.1], where the local existence and uniqueness is proved under the assumption that coefficients are Lipschitz. The local existence or uniqueness are not always guaranteed, for example, $y' = \sqrt{t}y$ or $x'^2 = 4x$, see [**Aut**, 1.3.1]. We may fix these problems by shrinking our domain of discussion. However, the study of the behavior at these singular points are very important. See Deligne's work [**Del70**] and some recent development in $p$-adic Hodge theory.

[29]We get all the sections by translating the diagonal section.

**5.5.** This presentation looks nice, but it still has some drawbacks:

- The bijection is stronger than just being a functor, it preserves the "Galois structure": Galois extension are sent to normal coverings, vice versa.
- The right hand side is not well explained.
- The infinite Galois correspondence only matches subfields with closed subgroups of $\mathrm{Gal}_K$. This is not taken care of in the diagram.

We remark that all these can be fully clarified with the notion of Galois categories.[30]

## 6. Sheaves: A brief history

This section follows from Wikipedia:

**6.1.** The first origins of sheaf theory are hard to pin down – they may be co-extensive with the idea of analytic continuation[clarification needed]. It took about 15 years for a recognisable, free-standing theory of sheaves to emerge from the foundational work on cohomology.

- 1936 Eduard Čech introduces the nerve construction, for associating a simplicial complex to an open covering.
- 1938 Hassler Whitney gives a 'modern' definition of cohomology, summarizing the work since J. W. Alexander and Kolmogorov first defined cochains.
- 1943 Norman Steenrod publishes on homology with local coefficients.
- 1945 Jean Leray publishes work carried out as a prisoner of war, motivated by proving fixed-point theorems for application to PDE theory; it is the start of sheaf theory and spectral sequences.
- 1947 Henri Cartan reproves the de Rham theorem by sheaf methods, in correspondence with André Weil (see De Rham–Weil theorem). Leray gives a sheaf definition in his courses via closed sets (the later carapaces).
- 1948 The Cartan seminar writes up sheaf theory for the first time.
- 1950 The "second edition" sheaf theory from the Cartan seminar: the sheaf space (espace étalé) definition is used, with stalkwise structure. Supports are introduced, and cohomology with supports. Continuous mappings give rise to spectral sequences. At the same time Kiyoshi Oka introduces an idea (adjacent to that) of a sheaf of ideals, in several complex variables.
- 1951 The Cartan seminar proves theorems A and B, based on Oka's work.
- 1953 The finiteness theorem for coherent sheaves in the analytic theory is proved by Cartan and Jean-Pierre Serre, as is Serre duality.
- 1954 Serre's paper Faisceaux algébriques cohérents (published in 1955) introduces sheaves into algebraic geometry. These ideas are immediately exploited by Friedrich Hirzebruch, who writes a major 1956 book on topological methods.
- 1955 Alexander Grothendieck in lectures in Kansas defines abelian category and presheaf, and by using injective resolutions allows direct use of sheaf cohomology on all topological spaces, as derived functors.
- 1956 Oscar Zariski's report Algebraic sheaf theory
- 1957 Grothendieck's Tohoku paper rewrites homological algebra; he proves Grothendieck duality (i.e., Serre duality for possibly singular algebraic varieties).
- 1957 onwards: Grothendieck extends sheaf theory in line with the needs of algebraic geometry, introducing: schemes and general sheaves on them, local cohomology, derived categories (with Verdier), and Grothendieck topologies. There emerges also his influential schematic idea of 'six operations' in homological algebra.

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[30] See [**Sta20**, 0BMQ].

- 1958 Roger Godement's book on sheaf theory is published. At around this time Mikio Sato proposes his hyperfunctions, which will turn out to have sheaf-theoretic nature.

At this point sheaves had become a mainstream part of mathematics, with use by no means restricted to algebraic topology. It was later discovered that the logic in categories of sheaves is intuitionistic logic (this observation is now often referred to as Kripke–Joyal semantics, but probably should be attributed to a number of authors). This shows that some of the facets of sheaf theory can also be traced back as far as Leibniz.

**6.2.** stalk and fiber

## 7. Questions

**7.1.** State and prove "*the theorem of empty-section*"[31] for sheaves of sets.

**7.2.** Let $\overline{\mathbf{\Delta}}$ be the "conjugate category" of finite ordered sets, whose

- set of objects are $\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots$, where $\mathbf{n} = \{0, 1, \ldots, n\}$,
- a morphism $\mathbf{n} \to \mathbf{m}$ is a non-increasing map of ordered sets.

Is the "conjugate category" isomorphic to $\mathbf{\Delta}^{opp}$?[32]

**7.3.** Let $\mathbf{\Delta}$ be the category of finite ordered sets. The discussion in 2.4 vaguely remind us of the category $\mathbf{\Delta}$. There we can still talk about non-decreasing maps, but we do not have $X(\mathbf{n})$ for $n \geq 1$. The morphisms are there, but we don't have all the objects—"Aha! Subcategories."

Let $\mathbf{\Delta}_{\leq n}$ be the full subcategory of $\mathbf{\Delta}$, whose objects are $\mathbf{0}, \ldots, \mathbf{n}$. The forgetful functor $\mathbf{\Delta}_{\leq n} \to \mathbf{\Delta}$ induces a *skeleton functor*

$$\mathrm{sk}_n \colon \mathrm{Psh}(\mathbf{\Delta}) \to \mathrm{Psh}(\mathbf{\Delta}_{\leq n})$$

Since are only allowed to take simplices up to $\mathbf{n}$, by the assembling lemma in $\mathrm{Psh}(\mathbf{\Delta}_{\leq n})$,

$$\mathrm{sk}_n(X) = \varinjlim_{f \in \mathbf{\Delta}_{\leq n}/\mathrm{sk}_n(X)} \mathbf{\Delta}^f,$$

the $n$-skeleton $\mathrm{sk}_n(X)$ is assembled form simplices of dimension no more than $n$, this coincides with our usual notion of skeleton: the thing left when the higher dimensional "flesh" is removed.[33]

As a common feature of many forgetful functors, the left adjoint functor (if it exits), would give the universal way to take closure, saturation, or extension.[34] Luckily, in our case, a left adjoint exists[35], called the $n$-coskeleton functor

$$\mathrm{cosk}_n \colon \mathrm{Psh}(\mathbf{\Delta}_{\leq n}) \to \mathrm{Psh}(\mathbf{\Delta}),$$

which can be thought as *the* way to assign the functor for $\mathbf{n+1}, \mathbf{n+2}$ and so on.

This being said, let's review what we did in 2.4.

[31] See Theorem 2.1

[32] : Use the *belt lemma*?

[33] This is creepy...

[34] It is usually called the blahfication functor, see [**Li20**, 1.4.3,1.4.5], where blah="sheafi", "groupi". Is literally taking topological closure a left adjoint to certain forgetful functor?

[35] See [**Sta20**, 0183].

- By take an ordering on the index set $I$ of the family of open covering, we gave a presheaf $C_{\mathcal{U},\mathcal{F}} \in \mathrm{Psh}(\mathbf{\Delta}_{\leq 1}^{opp})$:

$$\begin{aligned} C_{\mathcal{U},\mathcal{F}} \colon \mathbf{\Delta}_{\leq 1} &\to \mathtt{Ab} \\ \mathbf{k} &\mapsto C^k(\mathcal{U},\mathcal{F}) := \prod_{i_0 \prec \cdots \prec i_k} \mathcal{F}(U_{i_0} \times_U \cdots \times_U U_{i_k}) \\ (f \colon \mathbf{k} \to \mathbf{k}') &\mapsto C^k(\mathcal{U},\mathcal{F}) \overset{Cf^*}{\to} C^{k'}(\mathcal{U},\mathcal{F}) \end{aligned}$$

The homomorphism $Cf^*$ on component[36]

$$\prod_{j_0 \prec \cdots \prec j_k} \mathcal{F}(U_{j_0,\dots,j_k}) \to \mathcal{F}(U_{i_0,\dots,i_{k'}})$$

reads as the composition of the projection

$$\prod_{j_0 \prec \cdots \prec j_k} \mathcal{F}(U_{j_0,\dots,j_k}) \to \mathcal{F}(U_{i_{f(0)},\dots,i_{f(k)}})$$

and map induced from projection

$$\pi^* \colon \mathcal{F}(U_{i_{f(0)},\dots,i_{f(k)}}) \to \mathcal{F}(U_{i_0,\dots,i_{k'}}).$$

- Be careful, we only have a covariant functor emitting form $\mathbf{\Delta}_{\leq 1}$, this is not what we want. However, it can be fixed, by changing the target category to $\mathtt{Ab}^{opp}$.[37] Namely, viewing $C_{\mathcal{U},\mathcal{F}} \colon \mathbf{\Delta}_{\leq \mathbf{1}} \to \mathtt{Ab}$ as a presheaf in $\mathtt{Ab}^{opp}$, $\mathrm{Psh}_{\mathtt{Ab}^{opp}}(\mathbf{\Delta}_{\leq 1})$. Take 1-coskeleton, we get a simplicial opposite-abelian-group

$$\mathrm{cosk_n}(C_{\mathcal{U},\mathcal{F}}) \in \mathrm{Psh}_{\mathtt{Ab}^{opp}}(\mathbf{\Delta}).$$

- By the Dold-Kan correspondence, the simplicial opposite-abelian-group can be turned into a chain complex in $\mathrm{Ch}_{\geq 0}(\mathtt{Ab}^{opp})$. Reversing the arrows, we get a cochain complex

$$C^\bullet(\mathcal{U},\mathcal{F}) := \mathrm{DK}(\mathrm{cosk}_1(C_{\mathcal{U},\mathcal{F}})) \in \mathrm{Ch}^{\geq 0}(\mathrm{Ab}).$$

- Since coskeleton is an extension of the presheaf in the truncated category, it is reasonable to believe $\mathrm{sk}_n(cosk_n(X)) = X$. As Dold-Kan correspondence preserves the grading, one see the complex $C^\bullet(\mathcal{U},\mathcal{F})$ is a natural extension of the two term complex in the sheaf axiom.

$$\begin{tikzcd} & & C^0(\mathcal{U},\mathcal{F}) \arrow[r, "\mathrm{d}^0"] \arrow[d, equal] & C^1(\mathcal{U},\mathcal{F}) \arrow[r, "\mathrm{d}^1"] \arrow[d, equal] & \cdots \arrow[r, "\mathrm{d}^{n-1}"] & C^n(\mathcal{U},\mathcal{F}) \arrow[r, "\mathrm{d}^n"] & \cdots \\ 0 \arrow[r] & \mathcal{F} \arrow[r] & \prod_I \mathcal{F}(U_i) \arrow[r] & \prod_{i \prec j} \mathcal{F}(U_i \times_U U_j) & & \mathrm{H}^n(\mathcal{U},\mathcal{F}) \arrow[u, squiggly] & \end{tikzcd}$$

We call the complex $C^\bullet(\mathcal{U},\mathcal{F})$ the Čech complex associated to $\mathcal{F}$ with respect to the covering $\mathcal{U}$.

- The sub-quotient $\mathrm{H}^i(\mathcal{U},\mathcal{F}) := \ker(\mathrm{d}^i)/\mathrm{im}(\mathrm{d}^{i-1})$ is called the $i$-th Čech cohomology of $\mathcal{F}$ with respect to the covering $\mathcal{U}$. As Betti number counts "holes"[38] of a real manifold, Čech

[36] in $\mathrm{Hom}(\prod A, \prod B) = \prod \mathrm{Hom}(\prod A, B)$

[37] It is currently hard to understand read this post. However, we know it is still an abelian category, the axioms are symmetric! By the Freyd-Mitchell embedding theorem[**Wei94**, 1.6.1], we may view it as a category of $R$-modules, for some ring $R$. But the opposite category of sets is fuzzy.

[38] an $n$-hole yields an obstruction to contract $n$-spheres, and the Betti(=Singular)[39] cohomology $\mathrm{H}^n(X,\mathbb{Z})$ are generated by the $n$-spheres.

cohomology measures the "homological holes"[40] of a presheaf. The sheaf axiom thus, just asks that the degree-zero invariants to be geometrically meaningful: they are interpreted by sections.

**7.4.** Check the well-definedness of $\mathcal{F}$ on $\mathtt{Top}_X$, see Subsection 3.8. What is the statement you are going to check, precisely? [41]

**7.5.** In 3.11, we used Fubini's theorem to reinterpret the triple products. State and prove *the* Fubini's theorem we used there.

**7.6.** Can you reformulate the usual Fubini's theorem in calculus: "limit commutes" or "integration commutes", in a categorical language? Build *your category*.[42]

**7.7.** State and prove *the* Python lemma we used in 3.11.

**7.8.** Show that " $W_i \times_U W_j = W_i \times_X W_j$ when $U \to X$" is an injection: State and prove your favorite version.

*"The crucial thing here, from the viewpoint of the Weil conjectures, is that the new notion [of space] is vast enough, that we can associate to each scheme a "generalized space" or "topos" (called the "étale topos" of the scheme in question). Certain "cohomology invariants" of this topos ("childish" in their simplicity!) seemed to have a good chance of offering "what it takes" to give the conjectures their full meaning, and (who knows!) perhaps to give the means of proving them."*

—A. Grothendieck, *Récoltes et Semailles,1986*

[40]an $n$-cohomology class describes an obstruction to glue "$n$-sections", and sheaf cohomology are generated by the "$n$-sections"—$n$-cocycles. Sheaf with no such obstruction is called acyclic, the prefix "$a$-" means "antonym".

[41]Hint: Connect the two by a common refinement, then do some diagram chasing.

[42]— *"Let there be light."*

CHAPTER 3

# Motives and Variations

## 1. Function on class

`@Z.C`: We can talk about functions form one set to another. Can we talk about functions from one class to another class?

**1.1.** Recall how a function is defined[1]:

- Let $X, Y$ be sets.
- We define a relation from $X$ to $Y$ as a subset $R \subseteq X \times Y$.
- We say the correspondence is functional, if $(x, y) \in R$ and $(x, y') \in R$ implies that $y = y'$.
- A function is a functional relation from $X$ to $Y$.

**1.2.** Thus we can define a function on a class as long as we can talk about product of classes, subclasses, object "belongs to" classes. Using this description, one shows that functions form an empty set to an arbitrary set is nonempty. As $\emptyset \times Y$ has precisely one subset the $\emptyset$, and it is a functional relation.

## 2. Yoneda embedding

`@C.F`: Explain more on Yoneda lemma?

**2.1.** Using Yoneda lemma (thus Yoneda embedding), we can talk about what is a function from one object $A$ to another object $B$ in an abstract category $\mathcal{C}$, even if $A \times B$ does not exist in $\mathcal{C}$. We just define a functions as before in the category of presheaves of sets $\mathrm{Psh}(\mathcal{C})$: A function from $A$ to $B$ is a sub-presheave $R$ of $h_A \times h_B$ such that for any object $T$ in $\mathcal{C}$, $R(T)$ is a functional relation of $h_A(T) \times h_B(T)$. As how we proved Yoneda lemma, one can show that functions from $A$ to $B$ are just morphisms from $A$ to $B$: a function is determined by the element $f$ in $h_B(A)$ such that $(\mathrm{id}_A, f) \in R(A)$.

**2.2.** We can define an equivalence relation on $A$ and $B$ as a sub-presheaf $F \subset h_A \times h_B$ such that for any object $T$ in $\mathcal{C}$, we $F(T)$ is an equivalence relation in $h_A(T) \times h_B(T)$. We say a $F$ is a $\mathcal{C}$-theoretic equivalence relation if $F$ is represented by an object in $\mathcal{C}$. [2]

**2.3.** Similarly, we can define any notion in an abstract category, as long as we can define the notion in `Sets`. This is because

- Suppose a construction or property $\mathfrak{C}$ makes sense in `Sets`,
- Then the construction or property makes sense in $\mathrm{Psh}_{\texttt{Sets}}(\mathcal{C})$ for any category $\mathcal{C}$.
- The construction or property $\mathfrak{C}$ can be talked about in any subcategory of $\mathfrak{C}$.
- The category $\mathcal{C}$ embeds in $\mathrm{Psh}_{\texttt{Sets}}(\mathcal{C})$ via Yoneda embedding, the fully faithful functor

$$h_- \colon \mathcal{C} \to \mathrm{Psh}(\mathcal{C}).$$

- Thus we can talk about the construction $\mathfrak{C}$ in $\mathcal{C}$ via Yoneda embedding.

[1] See [**Zor15**, 1.3.4].
[2] See [**FGI**$^+$**05**, 5.6.3].

- The notion can be translated back to universal properties in $\mathfrak{C}$. To see this, one unwraps the Yoneda lemma, find where the id are mapped to.

## 3. Equalizer

`@Z.W`: What is an equalizer?

For example, we can define the notion of equalizer in the category of sets: an equalizer of two morphism of sets $f\colon A \to D$ and $g\colon B \to D$ as the subset $\mathrm{E}(f,g) := \{(a,d) \in A \times D\colon f(a) = g(b)\}$. Translating back, it coincides with usual notion defined by universal property[3]: A morphism $T \to \mathrm{E}(f,g)$ is a morphism $\alpha\colon T \to A$ and $\alpha\colon T \to D$ such that $f \circ \alpha = g \circ \beta$.

For another example, in order to verify the magic diagram,

$$\begin{tikzcd} X_1 \times_Y X_2 \arrow[r] \arrow[d] & X_1 \times_Z X_2 \arrow[d] \\ Y \arrow[r] & Y \times_Z Y \end{tikzcd}$$

it suffices to verify it in the category of sets. Once it is verified in `Sets`, it is automatically verified in $\mathrm{Psh}_{\texttt{Sets}}(\mathcal{C})$ for whatever category $\mathcal{C}$. Then Yoneda's lemma tells us the diagram is verified in $\mathcal{C}$. If one does not believe this, let's unwrap and see what happens:

- This diagram is characterized by the property that, for any $T \to X_1 \times_Z X_2$ and $T \to Y$ such that the composed map $T \to X_1 \times_Z X_2 \to Y \times_Z Y$ coincides with $T \to Y \to Y \times_Z Y$, there exists a unique map $T \to X_1 \times_Y X_2$.
- Giving $T \to X_1 \times_Z X_2$ is the same as giving $T \to X_1$ and $T \to X_2$ such that the composition $T \to X_1 \to Z$ and $T \to X_2 \to Z$ coincides. Asking the maps $T \to X_1 \times_Z X_2 \to Y \times_Z Y$ coincides with $T \to Y \to Y \times_Z Y$ coincide is just asking $T \to X_1 \to Y$ and $T \to X_2 \to Y$ coincides.
- This is just the what the mapping $T \to X_1 \times_Y X_2$ says.

## 4. Isomorphism of functors

`@D.H`: Why do people study "isomorphism of functors"? Let $X$ be a topological space. Let $\texttt{Path}_X$ be the category, whose

- Objects consists of points in $X$.
- Morphisms $\mathrm{Mor}_{\texttt{Path}_X}(x,y)$ consists image of $\gamma\colon [0,1] \to X$, such that $\gamma(0) = x, \gamma(1) = y$.
- Composition of morphism consists of taking union of subsets.

Let $f\colon X \to Y$ be a continuous map. On easily shows that $f$ induces a functor

$$F\colon \texttt{Path}_X \to \texttt{Path}_Y.$$

The notion of isomorphic functors now enables us to pose the question:

QUESTION 4.1. *If $f, f'$ are homotopic maps, can one show that $F', F'$ are naturally isomorphic. If so, is the converse true? At least, in the category of graphs?*

## 5. $R$-`Mod` and `Mod`-$R$

`@F.Z`: Are the categories $R$-`Mod` and `Mod`-$R$ equivalent?

---

[3][**Li20**, 1.3.14]

**5.1.** Maybe? Let $M$ be an abelian group, let $R$ be a commutative ring. Given a left $R$-module structure on $M$:

$$\mathtt{left}\colon R \times M \to M,$$

can we simply define a right module structure

$$\mathtt{right}\colon M \times R \to M,$$

by wall-crossing:

$$\mathtt{right}(m, r) := \mathtt{left}(r, m)?$$

**5.2.** Let's check associativity:

$$\mathtt{right}(\mathtt{right}(m,r),s) = \mathtt{right}(\mathtt{left}(r,m),s) = \mathtt{left}(s,\mathtt{left}(r,m))$$

It seems we've done, by formally moving a left module structure to the right hand side.

**5.3.** `@F.Z`: No. What we need to check is

$$\mathtt{right}(\mathtt{right}(m,r),s) = \mathtt{right}(m,rs),$$

or equivalently,

$$\mathtt{left}(s,\mathtt{left}(r,m)) = \mathtt{left}(rs,m),$$

which amounts to show

$$\mathtt{left}(sr,m) = \mathtt{left}(rs,m)$$

there's no reason why this is true, when $R$ is non-commutative.

**5.4.** Good. And looking back, we have shown that $R$-`Mod` and `Mod`-$R$ are equivalent, or even isomorphic, in the case when $R$ is commutative.

**5.5.** `@F.Z`: Yes, and we've actually shown that

$$\mathtt{Mod}\text{-}R \cong R^{opp}\text{-}\mathtt{Mod}$$

so the question boils down to comparing

$$R\text{-}\mathtt{Mod} \quad \text{and} \quad R^{opp}\text{-}\mathtt{Mod}.$$

**5.6.** In order to approach the problems, let's think about some concrete non-commutative rings. For example, in our abstract algebra textbook, we have

- Matrix algebra $M_n(\mathbb{C})$.
- Group algebra $\mathbb{C}[G] := \bigoplus_{g \in G} \mathbb{C} \cdot g$.[4]
- Division algebras (or skew fields), e.g., quaternions[5].

Let's try them out:

**5.7.** Matrix algebras do not work, as we've shown in class[6] that $\mathbb{C}$-`Mod` and $M_n(\mathbb{C})$-`Mod` are equivalent, $\mathbb{C}$ is commutative, and equivalence of categories is an equivalence relation[7].

[4]See [**Jac80**, 5.2].

[5]See [**Yan11**, 2.1.6].

[6]See [**Li20**, 1.2.15]. This is the right module version, which can be adapted to our left module version by taking $R^{opp}$, as was remarked.

[7]On the *class* of objects in `Cat`. What is an equivalence relation on the class $\mathrm{Ob}(\mathtt{Cat})$? A subclass of the product class $\mathrm{Ob}(\mathtt{Cat}) \times \mathrm{Ob}(\mathtt{Cat})$ such that...

**5.8.** Group rings are elaborations on a group: whenever $G \cong G'$, we have $\mathbb{C}[G] \cong \mathbb{C}[G']$. Then, is it true that

$$G \cong G^{opp}?$$

There is a notion of anti-homomorphism, which is not what we want, for otherwise $R$ and $R^{opp}$ are anti-isomorphic.[8] But luckily, we have the map $x \mapsto x^{-1}$, which indeed gives an isomorphism of $G$ and $G^{opp}$. Thus the category of $\mathbb{C}[G]$-`Mod` and `Mod`-$\mathbb{C}[G]$ are isomorphic, although $\mathbb{C}[G]$ is non-commutative.

We remark that in the construction of $\mathbb{C}[G]$, we only asking $G$ to be a semigroup, thus one may try to give examples of non-equivalent categories $\mathbb{C}[S]$-`Mod` and `Mod`-$\mathbb{C}[S]$ for non-commutative semigroup $S$. [9] We haven't studied semigroups a lot, so let's skip this and continue.

**5.9.** Now let's come to the skew fields. We have the following theorem[**Jac80**, 4.2]:

Theorem 5.1 (Wedderburn). *A left artinian ring $R$ is simple if and only if $R \cong M_n(\Delta)$ for some unique skew field $\Delta$.*

Artinian means ascending chain of submodules terminates, which is true if we consider finite dimension division algebras over $\mathbb{Q}$. This motivates the definition of the Brauer group: a group that classifies skew field extensions over a fixed base fields $k$, see [**Jac80**, 4.7]. Here is the recipe:

- Let' fix a field $k$.
- An algebra $A/k$ is called a central simple algebra if $A \otimes_k \overline{k} \cong M_r(\overline{k})$ for some $r$.[10].
- Tensor product of central simple algebras $A \otimes_k B$ is still a central simple algebra, because $M_n(\overline{k}) \otimes_k M_m(\overline{k}) \cong M_n(M_m(\overline{k})) \cong M_{mn}(\overline{k})$[11].
- We have $A \otimes A^{opp} \cong M_{\dim_k(A)}(k)$[12].
- Central simple algebras are left artinian and simple.[13]
- Thus the isomorphism classes of Central Simple Algebras, modulo the equivalence relations that
$$A \sim B \iff M_n(A) \cong M_m(B) \text{ ,for some } m, n$$
form an abelian group, where the identity is $[k]$, addition is given by $[A] + [B] = [A \otimes_k B]$, and inverse given by $-[A] = [A^{opp}]$.
- We define the Brauer group[14] of $k$, denoted by $\mathrm{Br}(k)$, as the set $\mathrm{CSA}_k/\sim$ equipped with the aforementioned group law.

The theory of Morita context[15] asserts that:

Theorem 5.2. *Let $k$ be a field. Let $A, B$ be skew fields over a $k$, then $A$-`Mod` and $B$-`Mod` are equivalent if and only if $[A] = [B]$ in $\mathrm{Br}(k)$.*

We know from local class field theory [**Mil13**, IV.4.3] that,

$$\mathrm{Br}(\mathbb{Q}_p) \cong \mathbb{Q}/\mathbb{Z}.$$

We may pick a skew field $F/\mathbb{Q}_p$ corresponding to $\frac{1}{3}$ in $\mathrm{Br}(\mathbb{Q}_p)$. Then

---

[8] it's not wise to cover your ears and steal the bell.

[9] There might be some combinatoric to play with monoids! Or, consider the semigroup of operators in functional analysis! Functional analysts study Hilbert modules, which are Hilbert spaces equipped with $\mathbb{C}[T_{z_1}, \dots, T_{z_n}]$-actions, e.g., the Hardy space, Drury-Arveson spaces, etc, which leads to non-commutative geometry, see [**DTY16**],[**Kha13**].

[10] Say, for the quaternions algebra $\mathbb{H}/\mathbb{R}$, we have $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong M_2(\mathbb{C})$

[11] Taking tensor is the same as doing scalar change, say, $M_n(k) \otimes_k R \cong M_n(R)$

[12] See [**Jac80**, 4.6], or find your own proof.

[13] Exercise in abstract algebra. Just check the definitions.

[14] See [**Jac80**].

[15] See [**Jac80**, 3.15]

PROPOSITION 5.3. *The categories $F$-`Mod` and `Mod`-$F$ are not equivalent.*

PROOF. If $F$-`Mod` is equivalent to `Mod`-$F$, then $F$-`Mod` is equivalent to $F^{opp}$-`Mod`. Thus by Theorem 5.2, $[F] = [F^{opp}] = -[F]$ in $\mathrm{Br}(\mathbb{Q}_p)$. However, $2[F] = \frac{2}{3} \neq 0$, contradiction. Furthermore, this $F$ can be explicitly contracted by assigning the multiplication table to a $k$-basis of $F$[16]. □

**5.10.**

[16]See [**Mil13**, 4.2]

# Bibliography


[Ahl66] Lars V. Ahlfors. *Complex analysis: An introduction of the theory of analytic functions of one complex variable*. Second edition. McGraw-Hill Book Co., New York-Toronto-London, 1966.

[Aut] The ODE Textbook Authors. *Ordinary differential Equation*. Department of Mathematics, Fudan University.

[Aut82] The Topology Textbook Authors. *Topology*. Department of Mathematics, Fudan University, 1982.

[Con14] Brian Conrad. *Motivation for adic spces*. `http://math.stanford.edu/~conrad/Perfseminar/Notes/L1.pdf`, 2014.

[Del70] Pierre Deligne. *Équations différentielles à points singuliers réguliers*. Lecture Notes in Mathematics, Vol. 163. Springer-Verlag, Berlin-New York, 1970.

[DTY16] Ronald G. Douglas, Xiang Tang, and Guoliang Yu. An analytic Grothendieck Riemann Roch theorem. *Adv. Math.*, 294:307–331, 2016.

[FGI+05] Barbara Fantechi, Lothar Göttsche, Luc Illusie, Steven L. Kleiman, Nitin Nitsure, and Angelo Vistoli. *Fundamental algebraic geometry*, volume 123 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005. Grothendieck's FGA explained.

[GJ99] Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory*, volume 174 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1999.

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[Jac80] Nathan Jacobson. *Basic algebra. II*. W. H. Freeman and Co., San Francisco, Calif., 1980.

[Kha13] Masoud Khalkhali. *Basic noncommutative geometry*. EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, second edition, 2013.

[Li20] Zhiyuan Li. *Abstract algebra III*. `https://elearning.fudan.edu.cn/courses/25538`, 2020.

[Lou09] Hongwei Lou. *Advanced calculus*. Science Press, 2009.

[Lur20] Jacob Lurie. Kerodon. `https://kerodon.net`, 2020.

[Mil13] James Milne. *Class field theory*. `https://www.jmilne.org/math/CourseNotes/CFT.pdf`, 2013.

[ML98] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[Sta20] The Stacks Project Authors. *Stacks Project*. `https://stacks.math.columbia.edu`, 2020.

[Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

[Yan11] Jin-gen Yang. *Lectures on Modern Algebra*. `http://homepage.fudan.edu.cn/jgyang/files/2012/02/algebra.pdf`, 2011.

[Zor15] Vladimir A. Zorich. *Mathematical analysis. I*. Universitext. Springer-Verlag, Berlin, second edition, 2015. Translated from the 6th corrected Russian edition, Part I, 2012 by Roger Cooke, With Appendices A–F and new problems translated by Octavio Paniagua T.

[ZQ01] Jinhao Zhang and Weiyuan Qiu. *Functions of a complex variable*. China Higher Education Press Beijing and Springer-Verlag Berlin Heidelberg, 2001.