AN ELEMENTARY CALCULATION OF PERIOD OF THE GENERIC GENUS-g CURVE VIA DEGENERATION

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Abstract. Let X be a stable curve with geometrically rational components, we study conics obtained by taking Stein-factorization of the normalization. We show that the conics do not split in some generic cases. As an application, we show that for $g \geq 3$, the period and the index of the generic genus g curve both equal to $2g-2$.

1. Introduction

1.1. Let k be a field, let X be a smooth genus g curve defined over k. There are two basic invariants that measure the arithmetic complexity of X:

(1) the index of X: the greatest common divisor of the degrees of zero-cycles on X,
(2) the period of X: the order of the $\mathrm{Pic}^0_{X/k}$-torsor $[\mathrm{Pic}^1_{X/k}]$.

1.2. Let $g \geq 3$ be an integer. Let $\mathcal{M}_g$ ($\mathcal{M}_{g,1}$) be the moduli stack of smooth genus g curves (with a marked point). Forgetting the marked point induces a proper smooth morphism $\pi\colon \mathcal{M}_{g,1} \to \mathcal{M}_g$. Let us denote the generic fiber of π by X_g/k_g and call X_g the generic genus g curve. It is known that:

Theorem 1. *The period and the index of X_g both equal to $2g-2$.*

The result is a consequence of the strong Franchetta conjecture [Sch03], whose proof uses Moriwaki's calculation of the $\mathbb{Q}$-Picard group of the moduli space of pointed curves. In this paper, we reprove the result by direct degeneration arguments.

1.3. We will degenerate the generic genus g curve to stable curves with geometrically rational components. For simplicity, let us call such curves totally degenerate stable curves. We recall some facts from [ACG11, XII.12]:

Let us work over a fixed field k. Let $g \geq 3$ be an integer. Let $\Gamma = (V, E)$ be a graph of genus g, with degree at least four at all the vertices. Let $\overline{\mathcal{M}}_g$ be the moduli stack of genus g stable curves.

(1) The stack $\overline{\mathcal{M}}_g$ has a locally closed substack $\mathcal{D}_\Gamma$ that parameterizes families of totally degenerate stable curves, whose geometric fibers all have dual graph Γ.
(2) For any vertex $v \in V$ in Γ, let $E^+(v)$ be the set of half-edges connected to v. Let $\mathcal{M}_{0,E^+(v)}$ be the moduli scheme of families of smooth genus-zero curves with distinct $E^+(v)$-marked-points. Let us denote the product scheme $\prod_{v\in V} \mathcal{M}_{0,E^+(v)}$ by $\mathcal{M}_\Gamma$. Then $\mathcal{D}_\Gamma$ is isomorphic to the quotient stack $[\mathcal{M}_\Gamma/\mathrm{Aut}(\Gamma)]$.
(3) As $|E^+(v)| \geq 4$ for all $v \in V$, the $\mathrm{Aut}(\Gamma)$-action on $\mathcal{M}_\Gamma$ is generically free, and therefore $\mathcal{D}_\Gamma$ has an dense open integral subscheme. Let us denote the function field of $\mathcal{D}_\Gamma$ by k_Γ

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and denote the generic fiber of $\mathcal{X}_\Gamma := \overline{\mathcal{M}}_{g,1} \times_{\overline{\mathcal{M}}_g} \mathcal{D}_\Gamma \to \mathcal{D}_\Gamma$ by X_Γ/k_Γ. Let $\widetilde{k}_\Gamma$ be the function field of $\mathcal{M}_\Gamma$, then $(X_\Gamma)_{\widetilde{k}_\Gamma}$ is a union of rational curves.

(4) Let $v_0 \in V$ be a fixed vertex, let $\mathrm{Stab}(v_0)$ be the stabilizer of v_0 in $\mathrm{Aut}(\Gamma)$, then the quotient stack $\mathcal{D}_{\Gamma,v_0} := [\mathcal{M}_\Gamma/\mathrm{Stab}(v_0)]$ parameterizes families of totally degenerate stable curves with dual graph Γ and a distinguished component corresponding to v_0. Forgetting the distinguished component induces a covering morphism $[\mathcal{M}_\Gamma/\mathrm{Stab}(v_0)] \to [\mathcal{M}_\Gamma/\mathrm{Aut}(\Gamma)]$. Let k_{Γ,v_0} be the function field of $\mathcal{D}_{\Gamma,v_0}$, then the generic curve $X_{\Gamma,v_0}/k_{\Gamma,v_0}$ is isomorphic to the base change $(X_\Gamma)_{k_{\Gamma,v_0}}$.

1.4. Let us call a smooth curve of geometric genus zero a conic, and say a conic splits if it is birational to $\mathbb{P}^1$. Let us keep the notations in the last paragraph. Let X^ν_{Γ,v_0} be the normalization of X_{Γ,v_0}. Let $R_{\Gamma,v_0} = \mathrm{H}^0(X^\nu_{\Gamma,v_0}, \mathcal{O}_{X^\nu_{\Gamma,v_0}})$ be the ring of regular functions on X^ν_{Γ,v_0}. The Stein factorization of $X^\nu_{\Gamma,v_0}/k_{\Gamma,v_0}$ produces a conic C_{Γ,v_0} over $\mathrm{Spec}(R_{\Gamma,v_0})$. Our degeneration argument of Theorem 1 is base on the following observation:

Theorem 2. *If there exists a vertex of $v_0 \in \Gamma$ and an order two automorphism of Γ that acts freely on $E^+(v_0)$, then the conic C_{Γ,v_0} is does not split. In particular, the distinguished component $C \subset X_{\Gamma,v_0}$ corresponding to v_0 is a non-split conic.*

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2. The associated conic

Let us work over a fixed field k_0. Let Γ be a graph, with degree at least four at all the vertices. Given a family of totally degenerate stable curves that fiber-wise has dual graph Γ, we introduce the notion of its associated family of conics.

2.1. Let $\pi\colon \mathcal{X}_\Gamma \to \mathcal{D}_\Gamma$ be the universal family of curves with dual graph Γ. Since $\mathcal{D}_\Gamma$ is smooth, by local property of smooth morphism and normalization, we know that the usual normalization $\nu\colon (\mathcal{X}_\Gamma)^\nu \to \mathcal{X}_\Gamma$ is also fiber-wise normalization over $\mathcal{D}_\Gamma$. This operation extends to arbitrary families by base change.

Definition 1. *Let $f\colon X \to S$ be a family of totally degenerate curves with dual graph Γ. Let $\varphi\colon S \to \mathcal{D}_\Gamma$ be the classifying morphism induced by f. We define the normalization f^ν of f by the base change $(\pi \circ \nu)_\varphi\colon (\mathcal{X}_\Gamma)^\nu \times_{\mathcal{D}_\Gamma,\varphi} S \to S$.*

Let $f\colon X \to S$ be a family of totally degenerate curves with a fixed dual graph Γ. Let $f^\nu\colon X^\nu \to S$ be its normalization. The geometric components of fibers of f^ν are smooth rational curves, and therefore the Stein factorization $f^{\nu,st}\colon X^\nu \to \underline{\mathrm{Spec}}((f^\nu)_*\mathcal{O}_{X^\nu})$ is a conic bundle [Har77, 11.5].

Definition 2. *We call $f^{\nu,st}$ the conic associated with the family f.*

2.2. The construction commutes with arbitrary base change:

Lemma 3. *Let $f\colon X \to S$ be a family of totally degenerate curves with a fixed dual graph. Let $T \to S$ be a morphism, let $f_T\colon X_T \to T$ be the base change of f along T, then $(f^{\nu,st})_T = (f_T)^{\nu,st}$*

Proof. The construction of f^ν is defined via base change, so it commutes with base change. Notice that $h^i(X_s^\nu, \mathcal{O}_{X_s})$ is constant for $s \in S$, and therefore the construction of $(f^\nu)_*$ commutes with base change. $\square$

3. Non-splitness of the conic

Let k_0 be a fixed field. Let $\Gamma = (V, E)$ be a graph with degree at least four at each vertex. Let $v_0 \in V$ be a distinguished vertex. Let $f\colon X_{\Gamma,v_0} \to \operatorname{Spec}(k_{\Gamma,v_0})$ be the generic curve with dual graph Γ and a distinguished component corresponding to v_0. Let $f^{\nu,st}\colon C_{\Gamma,v_0} \to \operatorname{Spec}(R_{\Gamma,v_0})$ be the associated conic. We study when the conic splits.

In the following discussion we will consider the generic curve over different bases. For a commutative ring B, let us denote the generic "curve over $\operatorname{Spec}(B)$" with dual graph Γ and a distinguished component corresponding to v_0 by $X_{\Gamma,v_0,B}$ and denote the associated conic by $C_{\Gamma,v_0,B}$, etc.

3.1. First let us reduce the base field k_0 to global fields.

Proposition 4. *If the conic $C_{\Gamma,v_0,k_0}/R_{\Gamma,v_0,k_0}$ splits, then there exists a global field M such that the conic $C_{\Gamma,v_0,M}/R_{\Gamma,v_0,M}$ splits.*

Proof. Let $\mathbb{F}$ be the prime field of k_0 (i.e., $\mathbb{Q}$ or $\mathbb{F}_p$). If C_{Γ,v_0,k_0} splits, then we can pick a section $s\colon \operatorname{Spec}(R_{\Gamma,v_0,k_0}) \to C_{\Gamma,v_0,k_0}$. By Lemma 3, we know $C_{\Gamma,v_0,k_0} = (C_{\Gamma,v_0,\mathbb{F}})_{k_0}$, therefore s is defined over a finite type $\mathbb{F}$-subalgebra $A \subset k_0$. Namely, the section s is the base change of a section $s_A\colon \operatorname{Spec}(R_{\Gamma,v_0,A}) \to C_{\Gamma,v_0,A}$. If $\operatorname{char}(k_0) = 0$, we take a maximal ideal $\mathfrak{m}$ of A, then $M := A/\mathfrak{m}$ is a finite extension of $\mathbb{Q}$. The section s_A specializes to a section $s_{A/\mathfrak{m}}\colon \operatorname{Spec}(R_{\Gamma,v_0,M}) \to C_{\Gamma,v_0,M}$, and therefore $C_{\Gamma,v_0,M}$ splits. If $\operatorname{char}(k_0) > 0$, we take a dominant map from $\operatorname{Spec}(A)$ to a $\mathbb{A}^1_{\mathbb{F}}$, replace A by its localization at the generic point and argue as before. $\square$

3.2. We further reduce the problem to finding one stable curve with non-split associated conic.

Proposition 5. *If the conic C_{Γ,v_0,k_0} splits, then for any field extension k/k_0, and any totally degenerate stable curve $h\colon X \to k$ with dual graph Γ and a distinguished component corresponding to v_0, the associated conic of h, denoted by C_h, splits.*

Proof. By Lemma 3, we may assume $k = k_0$. Let's work over k in the rest of the proof. Let $g\colon \mathcal{X}_{\Gamma,v_0} \to \mathcal{D}_{\Gamma,v_0}$ be the universal family of stable curves with dual graph Γ and a distinguished component corresponding to v_0. Let $[X]$ be the closed point in $\mathcal{D}_{\Gamma,v_0}$ corresponding to the curve X. Let $d = \dim(\mathcal{D}_{\Gamma,v_0})$. By [Sta19, 0DR0], there exists a flat morphism $\varphi\colon \operatorname{Spec}(k[[t_1,\ldots,t_d]]) \to \mathcal{D}_{\Gamma,v_0}$ centered at $[X]$. Let g_φ be the family of curves over $\operatorname{Spec}(k[[t_1,\ldots,t_d]])$ induced by φ. Let C' be the generic fiber of $g_\varphi^{\nu,st}$. Since φ is flat, the generic point of $\operatorname{Spec}(k[[t_1,\ldots,t_d]])$ maps to the generic point of $\mathcal{D}_{\Gamma,v_0}$. If $C_{\Gamma,v_0,k}$ splits, then its base change C' splits. Notice that any rational section of C' yields a rational section of C_h by successive specialization along the chain of discrete valuation rings: $k((t_1,\ldots,t_{d-1}))[[t_d]] \rightsquigarrow k((t_1,\ldots,t_{d-2}))[[t_{d-1}]] \rightsquigarrow \cdots \rightsquigarrow k[[t_1]]$, therefore C_h splits. $\square$

3.3. In the following discussion, we will consider graphs that have degree at least three at each vertex, let us call them stable graphs. Given the datum as in Theorem 2, we will construct the desired stable curve, whose associated conic does not split. Let us recall a useful lemma from [Pal04].

Lemma 6. *Let L/K be a Galois extension. Let Γ be a stable graph with an action by $\mathrm{Gal}(L/K)$. Assume K is infinite, then there exists a totally degenerate stable curve X, such that there is a Galois equivariant isomorphism between Γ and the dual graph of X.*

Proof. See [Pal04, Proposition 4.6]. □

Proposition 7. *Let $\Gamma = (V, E)$ be a stable graph. Let $v_0 \in V$ be a vertex, let σ be an order-2 element in $\mathrm{Stab}(v_0) \subset \Gamma$, such that σ acts freely on $E^+(v_0)$. Let K be a global field. Then there exists a totally degenerate stable curve with dual graph Γ defined over K, such that the associated conic does not split. In particular, the component corresponding to v_0 is a non-split conic.*

Proof. By global class field theory, there exists an element $\gamma \in \mathrm{Br}(K)$ of order 2 that is represented by a conic C. Let L/K be a quadratic extension over which C splits. Let $\tau \in \mathrm{Gal}(L/K)$ be the generator.

By Lemma 6, there exists a totally degenerate curve X over K such that τ acts as σ on the dual graph. Let $Y \subset X_L$ be the component corresponding to v_0, since v_0 is Galois invariant, we know Y descends to K. Therefore we may write $X = Y \coprod_{Y \cap Z} Z$, where $Z = \overline{X \backslash Y}$. Since σ acts freely on $E^+(v_0)$, the nodes $Y \cap Z \subset Y$ consists of a disjoint union of closed points with residue field L. Since $C_L \cong \mathbb{P}^1_L$, we may pick an embedding $Y \cap Z \hookrightarrow C$, then $C \coprod_{Y \cap Z} Z$ give the desired curve. □

3.4. The proof of Theorem 2 follows from Proposition 4, Proposition 5 and Proposition 7. One may also consider the conic C_Γ associated with the generic curve X_Γ with fixed dual graph (without singling out a distinguished component). Since X_{Γ, v_0} is the base change of X_Γ, with the same assumption, we know that the conic C_Γ is non-split.

4. Non-triviality of Picard torsors

Let k be a field. For any totally degenerate stable curve X defined over k, let $\mathrm{Pic}^{\mathbf{1}}_{X/k}$ be the component of the Picard scheme that parameterizes line bundles with degree one on each geometrical irreducible component. The scheme $\mathrm{Pic}^{\mathbf{1}}_{X/k}$ is a torsor of the Picard torus $\mathrm{Pic}^0_{X/k}$. We show that the torsor is non-trivial in certain cases.

4.1. Let us consider the generic genus g curve with the following dual graph $\Gamma = (V, E)$ (intuitively, the graph is a $(g-1)$-gon with a loop at each vertex):

- the set of vertices $V = \{v_i\}_{i \in \mathbb{Z}/(g-1)\mathbb{Z}}$,
- the set of edges $E = \{e_{\{i,j\}}\}_{i,j \in \mathbb{Z}/(g-1)\mathbb{Z}}$, where the edge $e_{\{i,j\}}$ connects v_i and v_j, the edge $e_{\{i\}}$ is a loop at v_i.

4.2. We keep the notations as in Section 1.3. Let $\widetilde{k}_\Gamma$ be the function field of $\mathcal{M}_\Gamma$, then $k_\Gamma = (\widetilde{k}_\Gamma)^{\mathrm{Aut}(\Gamma)}$ and $k_{\Gamma, v_0} = (\widetilde{k}_\Gamma)^{\mathrm{Stab}(v_0)}$. Let $C \subset (X_\Gamma)_{k_{\Gamma, v_0}}$ be the component that corresponds to v_0.

Lemma 8. *There exists a surjective morphism from $(\mathrm{Pic}^{\mathbf{1}}_{X_\Gamma/k_\Gamma})_{k_{\Gamma, v_0}}$ to $\mathrm{Pic}^1_{C/k_{\Gamma, v_0}}$.*

Proof. Restricting a line bundle on $(X_\Gamma)_{k_{\Gamma, v_0}}$ to the component C gives a morphism from $(\mathrm{Pic}^{\mathbf{1}}_{X_\Gamma/k_\Gamma})_{k_{\Gamma, v_0}}$ to $\mathrm{Pic}^1_{C/k_{\Gamma, v_0}}$, it suffices to check that the morphism is surjective. The surjectivity can be checked after base change to the algebraic closure of k_Γ, which is clear. □

Proposition 9. *Let us work over a field k_0. Let X_Γ/k_Γ be the generic genus g curve with dual graph Γ, then $\mathrm{Pic}^{\mathbf{1}}_{X_\Gamma/k_\Gamma}$ does not have k_Γ-rational points.*

Proof. It suffices to show that the base change $(\mathrm{Pic}^{\mathbf{1}}_{X_\Gamma/k_\Gamma})_{k_{\Gamma,v_0}}$ admits no k_{Γ,v_0}-rational points. By Lemma 8, there is a surjection of varieties $(\mathrm{Pic}^{\mathbf{1}}_{X_\Gamma/k_\Gamma})_{k_{\Gamma,v_0}} \to \mathrm{Pic}^1_{C/k_{\Gamma,v_0}}$. Therefore any k_Γ-rational point of $\mathrm{Pic}^{\mathbf{1}}_{X_\Gamma/k_\Gamma}$ would give rise to a k_{Γ,v_0} rational point of $\mathrm{Pic}^1_{C/k_{\Gamma,v_0}}$. Notice that C is a nodal curve, the Abel-Jacobi map induces an isomorphism from the smooth locus of C to its Picard scheme: $C^{\mathrm{sm}} \to \mathrm{Pic}^1_{C/k_{\Gamma,v_0}}, c \mapsto \mathcal{O}_C(c)$. Therefore, it suffices to show that C^ν admits no k_{Γ,v_0}-rational points. This follows from Theorem 2. □

5. Proof of Theorem 1

Let us work over a fixed base field k_0. Let $g \geq 3$ be an integer. Let k be the function field of $\mathcal{M}_g$. Let X/k be the generic genus-g curve. We show that the period of X and the index of X both equal to $2g-2$.

5.1. We show that the period of X equals to $2g-2$.

Proposition 10. *The torsor $[\mathrm{Pic}^1_{X/k}] \in \mathrm{H}^1(k, \mathrm{Pic}^0_{X/k})$ has order $2g-2$.*

Proof. Let $\alpha \in \mathrm{Br}(\mathrm{Pic}^0_{X/k})$ be the Brauer obstruction class to the existence of tautological line bundles on $X \times \mathrm{Pic}^0_{X/k}$. By degeneration methods, one can show that the period of α equals to $g-1$, see [Ma19, Remark 4.4.2] or [Mes87]. Moreover, there exists a group homomorphism from a subgroup of $\mathrm{H}^1(k, \mathrm{Pic}^0_{X/k})$ to $\mathrm{Br}(\mathrm{Pic}^0_{X/k})$, which sends the class $[\mathrm{Pic}^1_{X/k}]$ to α [Ma19, Proposition 2.5.3], and therefore the period of $[\mathrm{Pic}^1_{X/k}]$ is divisible by $g-1$. Notice that $[\mathrm{Pic}^{2g-2}_{X/k}]$ is a trivial $\mathrm{Pic}^0_{X/k}$-torsor, and therefore it suffices to show that the class $[\mathrm{Pic}^{g-1}_{X/k}]$ is non-trivial, or that $\mathrm{Pic}^{g-1}_{X/k}$ admits no k-rational points.

Let Γ be the graph as in Section 4.1. By Proposition 9, the torsor $\mathrm{Pic}^{\mathbf{1}}_{X_\Gamma/k_\Gamma}$ does not have a rational point. Let $R_1 = k_\Gamma[[t_1]]$. By [Bak08, B.2], we may find a regular surface $\mathcal{X}/R_1$, with special fiber X_Γ/k_Γ and smooth generic fiber X_1/k_1. It suffices to show that $\mathrm{Pic}^{g-1}_{X_1/k_1}$ admit no rational points (because rational points on $\mathrm{Pic}^{g-1}_{X/k}$ yield rational points on $\mathrm{Pic}^{g-1}_{X_1/k_1}$ by successive specialization as in Proposition 5).

Let R_1^{sh} be the strict henselization of R_1. Suppose that Q_1 is a rational point on $\mathrm{Pic}^{g-1}_{X_1/k_1}$, represented by a line bundle L on the generic fiber of $(\mathcal{X})_{R_1^{\mathrm{sh}}}$. Since $\mathcal{X}$ is regular, so is $(\mathcal{X})_{R_1^{\mathrm{sh}}}$, and therefore the line bundle L extends to $(\mathcal{X})_{R_1^{\mathrm{sh}}}$ by taking closure of the corresponding Weil divisor. The restriction of the line bundle to X_Γ/k_Γ yields a rational point Q_0 on $\mathrm{Pic}_{X_\Gamma/k_\Gamma}$. Notice that Q_0 is $\mathrm{Gal}(\widetilde{k_\Gamma}/k_\Gamma)$-invariant, and that all the components of $(X_\Gamma)_{\widetilde{k_\Gamma}}$ are Galois conjugate, thus Q_0 lands in $\mathrm{Pic}^{\mathbf{1}}_{X_\Gamma/k_\Gamma}$. This contradicts the non-triviality of $\mathrm{Pic}^{\mathbf{1}}_{X_\Gamma/k_\Gamma}$, see Proposition 9. □

5.2. The index can be easily calculated from the period.

Proposition 11. *The index of X equals to $2g-2$.*

Proof. Canonical divisors provide zero-cycles of degree $2g-2$ on X, and therefore the index of X divides $2g-2$. Notice that the period of X divides the index of X, see [LT58]. By Proposition 10, the period of X equals to $2g-2$, and therefore the index of X divides $2g-2$, hence equals to $2g-2$. □

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