MATH130068h, (Recitation) 2020 Spring

Contents

CHAPTER 1

Motives and Variations

1. Function on class

@Z.C: We can talk about functions form one set to another. Can we talk about functions from one class to another class?

1.1. Recall how a function is defined[1]:

- Let X, Y be sets.
- We define a relation from X to Y as a subset $R \subseteq X \times Y$.
- We say the correspondence is functional, if $(x, y) \in R$ and $(x, y') \in R$ implies that $y = y'$.
- A function is a functional relation from X to Y.

1.2. Thus we can define a function on a class as long as we can talk about product of classes, subclasses, object "belongs to" classes. Using this description, one shows that functions form an empty set to an arbitrary set is nonempty. As $\emptyset \times Y$ has precisely one subset the $\emptyset$, and it is a functional relation.

2. Yoneda embedding

@C.F: Explain more on Yoneda lemma?

2.1. Using Yoneda lemma (thus Yoneda embedding), we can talk about what is a function from one object A to another object B in an abstract category $\mathcal{C}$, even if $A \times B$ does not exist in $\mathcal{C}$. We just define a functions as before in the category of presheaves of sets $\mathrm{Psh}(\mathcal{C})$: A function from A to B is a sub-presheave R of $h_A \times h_B$ such that for any object T in $\mathcal{C}$, $R(T)$ is a functional relation of $h_A(T) \times h_B(T)$. As how we proved Yoneda lemma, one can show that functions from A to B are just morphisms from A to B: a function is determined by the element f in $h_B(A)$ such that $(\mathrm{id}_A, f) \in R(A)$.

2.2. We can define an equivalence relation on A and B as a sub-presheaf $F \subset h_A \times h_B$ such that for any object T in $\mathcal{C}$, we $F(T)$ is an equivalence relation in $h_A(T) \times h_B(T)$. We say a F is a $\mathcal{C}$-theoretic equivalence relation if F is represented by an object in $\mathcal{C}$. [2]

2.3. Similarly, we can define any notion in an abstract category, as long as we can define the notion in `Sets`. This is because

- Suppose a construction or property $\mathfrak{C}$ makes sense in `Sets`,
- Then the construction or property makes sense in $\mathrm{Psh}_{\texttt{Sets}}(\mathcal{C})$ for any category $\mathcal{C}$.
- The construction or property $\mathfrak{C}$ can be talked about in any subcategory of $\mathfrak{C}$.
- The category $\mathcal{C}$ embeds in $\mathrm{Psh}_{\texttt{Sets}}(\mathcal{C})$ via Yoneda embedding, the fully faithful functor
$$h_{-} \colon \mathcal{C} \to \mathrm{Psh}(\mathcal{C}).$$
- Thus we can talk about the construction $\mathfrak{C}$ in $\mathcal{C}$ via Yoneda embedding.

[1]See [**?**, 1.3.4].
[2]See [**?**, 5.6.3].

- The notion can be translated back to universal properties in $\mathfrak{C}$. To see this, one unwraps the Yoneda lemma, find where the id are mapped to.

3. Equalizer

`@Z.W`: What is an equalizer?

For example, we can define the notion of equalizer in the category of sets: an equalizer of two morphism of sets $f\colon A \to D$ and $g\colon B \to D$ as the subset $\mathrm{E}(f,g) := \{(a,d) \in A \times D\colon f(a) = g(b)\}$. Translating back, it coincides with usual notion defined by universal property[3]: A morphism $T \to \mathrm{E}(f,g)$ is a morphism $\alpha\colon T \to A$ and $\alpha\colon T \to D$ such that $f \circ \alpha = g \circ \beta$.

For another example, in order to verify the magic diagram,

$$\begin{tikzcd} X_1 \times_Y X_2 \arrow[r] \arrow[d] & X_1 \times_Z X_2 \arrow[d] \\ Y \arrow[r] & Y \times_Z Y \end{tikzcd}$$

it suffices to verify it in the category of sets. Once it is verified in `Sets`, it is automatically verified in $\mathrm{Psh}_{\mathtt{Sets}}(\mathcal{C})$ for whatever category $\mathcal{C}$. Then Yoneda's lemma tells us the diagram is verified in $\mathcal{C}$. If one does not believe this, let's unwrap and see what happens:

- This diagram is characterized by the property that, for any $T \to X_1 \times_Z X_2$ and $T \to Y$ such that the composed map $T \to X_1 \times_Z X_2 \to Y \times_Z Y$ coincides with $T \to Y \to Y \times_Z Y$, there exists a unique map $T \to X_1 \times_Y X_2$.
- Giving $T \to X_1 \times_Z X_2$ is the same as giving $T \to X_1$ and $T \to X_2$ such that the composition $T \to X_1 \to Z$ and $T \to X_2 \to Z$ coincides. Asking the maps $T \to X_1 \times_Z X_2 \to Y \times_Z Y$ coincides with $T \to Y \to Y \times_Z Y$ coincide is just asking $T \to X_1 \to Y$ and $T \to X_2 \to Y$ coincides.
- This is just the what the mapping $T \to X_1 \times_Y X_2$ says.

4. Isomorphism of functors

`@D.H`: Why do people study "isomorphism of functors"? Let X be a topological space. Let $\mathtt{Path}_X$ be the category, whose

- Objects consists of points in X.
- Morphisms $\mathrm{Mor}_{\mathtt{Path}_X}(x,y)$ consists image of $\gamma\colon [0,1] \to X$, such that $\gamma(0) = x, \gamma(1) = y$.
- Composition of morphism consists of taking union of subsets.

Let $f\colon X \to Y$ be a continuous map. On easily shows that f induces a functor

$$F\colon \mathtt{Path}_X \to \mathtt{Path}_Y.$$

The notion of isomorphic functors now enables us to pose the question:

Question 4.1. *If f, f' are homotopic maps, can one show that F', F' are naturally isomorphic. If so, is the converse true? At least, in the category of graphs?*

5. R-`Mod` and `Mod`-R

`@F.Z`: Are the categories R-`Mod` and `Mod`-R equivalent?

[3] [?, 1.3.14]

5.1. Maybe? Let M be an abelian group, let R be a commutative ring. Given a left R-module structure on M:

$$\texttt{left}\colon R \times M \to M,$$

can we simply define a right module structure

$$\texttt{right}\colon M \times R \to M,$$

by wall-crossing:

$$\texttt{right}(m, r) := \texttt{left}(r, m)?$$

5.2. Let's check associativity:

$$\texttt{right}(\texttt{right}(m, r), s) = \texttt{right}(\texttt{left}(r, m), s) = \texttt{left}(s, \texttt{left}(r, m))$$

It seems we've done, by formally moving a left module structure to the right hand side.

5.3. `@F.Z`: No. What we need to check is

$$\texttt{right}(\texttt{right}(m, r), s) = \texttt{right}(m, rs),$$

or equivalently,

$$\texttt{left}(s, \texttt{left}(r, m)) = \texttt{left}(rs, m),$$

which amounts to show

$$\texttt{left}(sr, m) = \texttt{left}(rs, m)$$

there's no reason why this is true, when R is non-commutative.

5.4. Good. And looking back, we have shown that R-`Mod` and `Mod`-R are equivalent, or even isomorphic, in the case when R is commutative.

5.5. `@F.Z`: Yes, and we've actually shown that

$$\texttt{Mod-}R \cong R^{opp}\texttt{-Mod}$$

so the question boils down to comparing

$$R\texttt{-Mod} \quad \text{and} \quad R^{opp}\texttt{-Mod}.$$

5.6. In order to approach the problems, let's think about some concrete non-commutative rings. For example, in our abstract algebra textbook, we have

- Matrix algebra $M_n(\mathbb{C})$.
- Group algebra $\mathbb{C}[G] := \bigoplus_{g\in G} \mathbb{C} \cdot g$.[4]
- Division algebras (or skew fields), e.g., quaternions[5].

Let's try them out:

5.7. Matrix algebras do not work, as we've shown in class[6] that $\mathbb{C}$-`Mod` and $M_n(\mathbb{C})$-`Mod` are equivalent, $\mathbb{C}$ is commutative, and equivalence of categories is an equivalence relation[7].

[4]See [**?**, 5.2].

[5]See [**?**, 2.1.6].

[6]See [**?**, 1.2.15]. This is the right module version, which can be adapted to our left module version by taking R^{opp}, as was remarked.

[7]On the *class* of objects in `Cat`. What is an equivalence relation on the class $\mathrm{Ob}(\texttt{Cat})$? A subclass of the product class $\mathrm{Ob}(\texttt{Cat}) \times \mathrm{Ob}(\texttt{Cat})$ such that...

5.8. Group rings are elaborations on a group: whenever $G \cong G'$, we have $\mathbb{C}[G] \cong \mathbb{C}[G']$. Then, is it true that

$$G \cong G^{opp}?$$

There is a notion of anti-homomorphism, which is not what we want, for otherwise R and R^{opp} are anti-isomorphic.[8] But luckily, we have the map $x \mapsto x^{-1}$, which indeed gives an isomorphism of G and G^{opp}. Thus the category of $\mathbb{C}[G]$-`Mod` and `Mod`-$\mathbb{C}[G]$ are isomorphic, although $\mathbb{C}[G]$ is non-commutative.

We remark that in the construction of $\mathbb{C}[G]$, we only asking G to be a semigroup, thus one may try to give examples of non-equivalent categories $\mathbb{C}[S]$-`Mod` and `Mod`-$\mathbb{C}[S]$ for non-commutative semigroup S. [9] We haven't studied semigroups a lot, so let's skip this and continue.

5.9. Now let's come to the skew fields. We have the following theorem[**?**, 4.2]:

THEOREM 5.1 (Wedderburn). *A left artinian ring R is simple if and only if $R \cong M_n(\Delta)$ for some unique skew field Δ.*

Artinian means ascending chain of submodules terminates, which is true if we consider finite dimension division algebras over $\mathbb{Q}$. This motivates the definition of the Brauer group: a group that classifies skew field extensions over a fixed base fields k, see [**?**, 4.7]. Here is the recipe:

- Let' fix a field k.
- An algebra A/k is called a central simple algebra if $A \otimes_k \overline{k} \cong M_r(\overline{k})$ for some r.[10].
- Tensor product of central simple algebras $A \otimes_k B$ is still a central simple algebra, because $M_n(\overline{k}) \otimes_k M_m(\overline{k}) \cong M_n(M_m(\overline{k})) \cong M_{mn}(\overline{k})$[11].
- We have $A \otimes A^{opp} \cong M_{\dim_k(A)}(k)$[12].
- Central simple algebras are left artinian and simple.[13]
- Thus the isomorphism classes of Central Simple Algebras, modulo the equivalence relations that
$$A \sim B \;\Leftrightarrow\; M_n(A) \cong M_m(B) \text{ ,for some } m, n$$
form an abelian group, where the identity is $[k]$, addition is given by $[A]+[B] = [A \otimes_k B]$, and inverse given by $-[A] = [A^{opp}]$.
- We define the Brauer group[14] of k, denoted by $\mathrm{Br}(k)$, as the set $\mathrm{CSA}_k/\sim$ equipped with the aforementioned group law.

The theory of Morita context[15] asserts that:

THEOREM 5.2. *Let k be a field. Let A, B be skew fields over a k, then A-`Mod` and B-`Mod` are equivalent if and only if $[A] = [B]$ in $\mathrm{Br}(k)$.*

We know from local class field theory [**?**, IV.4.3] that,

$$\mathrm{Br}(\mathbb{Q}_p) \cong \mathbb{Q}/\mathbb{Z}.$$

We may pick a skew field $F/\mathbb{Q}_p$ corresponding to $\frac{1}{3}$ in $\mathrm{Br}(\mathbb{Q}_p)$. Then

[8] it's not wise to cover your ears and steal the bell.

[9] There might be some combinatoric to play with monoids! Or, consider the semigroup of operators in functional analysis! Functional analysts study Hilbert modules, which are Hilbert spaces equipped with $\mathbb{C}[T_{z_1}, \ldots, T_{z_n}]$-actions, e.g., the Hardy space, Drury-Arveson spaces, etc, which leads to non-commutative geometry, see [**?**],[**?**].

[10] Say, for the quaternions algebra $\mathbb{H}/\mathbb{R}$, we have $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong M_2(\mathbb{C})$

[11] Taking tensor is the same as doing scalar change, say, $M_n(k) \otimes_k R \cong M_n(R)$

[12] See [**?**, 4.6], or find your own proof.

[13] Exercise in abstract algebra. Just check the definitions.

[14] See [**?**].

[15] See [**?**, 3.15]

PROPOSITION 5.3. *The categories F-**Mod** and **Mod**-F are not equivalent.*

PROOF. If F-Mod is equivalent to Mod-F, then F-Mod is equivalent to F^{opp}-Mod. Thus by Theorem 5.2, $[F] = [F^{opp}] = -[F]$ in $\mathrm{Br}(\mathbb{Q}_p)$. However, $2[F] = \frac{2}{3} \neq 0$, contradiction. Furthermore, this F can be explicitly contracted by assigning the multiplication table to a k-basis of F[16]. □

5.10.

[16] See [**?**, 4.2]

CHAPTER 2

Motives and variations

1. Unique up to a unique isomorphism

`@Z.C`: We say a functor $F\colon \mathcal{C}^{opp} \to \texttt{Sets}$ is representable if F is isomorphic to h_U for some U, furthermore U is unique up to a unique isomorphism. What does "unique up to a unique isomorphism" mean?

1.1. Let's clarify our situation: Let $\mathcal{C}$ be a category. We say a functor $F\colon \mathcal{C}^{opp} \to \texttt{Sets}$ is representable if

- There exists an object $U \in \mathrm{Ob}(\mathcal{C})$
- There exists a natural isomorphism $\eta\colon F \to h_U$

By the Yoneda lemma, we know

$$\begin{aligned}\eta\colon F(U) &\to \mathrm{Mor}_{\texttt{Psh}(\mathcal{C})}(h_U, F),\\ u &\mapsto (\eta_u\colon f \mapsto (F(f))u)\end{aligned}$$

is a bijection, thus the second condition can be rephrased as:

- There exists an element $u \in F(U)$, such that $\eta_u\colon h_U \to F$ is a natural isomorphism.

Thus, we make the following definition:

1.2. Let $\mathcal{C}$ be a category. We say a functor $F\colon \mathcal{C}^{opp} \to \texttt{Sets}$ is representable, if there exists a pair (U, u_F), where

- U is an object in $\mathcal{C}$,
- u_F is an element in $F(U)$;

such that

- $\eta_{u_F}\colon h_U \to F$ is an isomorphism.

In this situation, we say (U, u_F) is a representative of F.

1.3. Two representatives $(U, u_F), (U', u'_F)$ are said to be naturally isomorphic, if there exists an isomorphism $\phi\colon h_U \to h_{U'}$ such that one of the following three equivalent conditions hold:

- $\phi(\mathrm{id}_U)(u_F) = u'_F$
- the following diagram is commutative

$$\begin{tikzcd} h_U \arrow[rr, "\phi"] \arrow[dr, "\eta_{u_F}"'] & & h_{U'} \arrow[dl, "\eta_{u'_F}"] \\ & F & \end{tikzcd}$$

- $(h_U, \eta_{u_F}), (h_{U'}, \eta_{u'_F})$ are isomorphic in the category $\mathrm{Psh}(\mathcal{C})/F$.

1.4. After the clarification, one realizes that ϕ has no other choice but $\eta_{u'_F}^{-1} \circ \eta_{u_F}$, as $\eta_{u_F}, \eta_{u'_F}$ are isomorphisms.

2. Sheaves on "small" topological spaces

2.1. Recall in class, we saw the functor

$$\begin{aligned} \mathrm{Fgt}\colon \mathtt{Top}^{opp} &\to \mathrm{Sets} \\ (X, \mathcal{T}_X) &\mapsto \mathcal{T}_X \end{aligned}$$

is represented by the Sierpiński space[1]

$$S := (\{0,1\}, \{\emptyset, \{1\}, \{0,1\}\}).$$

Unwrapping the sheaf axioms, one observes that every presheaf on S is a sheaf.

We remark that, this is also true on the discrete space with one element

$$\{*\} := (\{0\}, \{\emptyset, \{0\}\}).$$

However, this is not true on a discrete space with two elements

$$D := (\{0,1\}, \{\emptyset, \{0\}, \{1\}, \{0,1\}\})$$

2.2. Let D be the discrete space with two points. By definition, a presheaf $\mathcal{F}$ on D is given by a commutative diagram

$$\begin{tikzcd}
& \{0,1\} & & & \mathcal{F}(\{01\}) \arrow[dl, "a"'] \arrow[dr, "b"] & \\
\{0\} \arrow[ur] & & \{1\} \arrow[ul] \arrow[r, squiggly, "\mathcal{F}"] & \mathcal{F}(\{0\}) \arrow[dr, "c"'] & & \mathcal{F}(\{1\}) \arrow[dl, "d"] \\
& \{\emptyset\} \arrow[ul] \arrow[ur] & & & \mathcal{F}(\emptyset) &
\end{tikzcd}$$

The sheafification $\mathcal{F}^\sharp$ of $\mathcal{F}$ is nothing but replacing $\mathcal{F}(\emptyset)$ by the final object, and replacing $\mathcal{F}(\{0,1\})$ by $\mathcal{F}^\sharp(\{0,1\}) = c \prod d$. Let;s take the simple example where $\mathcal{F}(\{0,1\}) = \{x, y\}, \mathcal{F}(\{0\}) = \{x, y, z\}, \mathcal{F}(\{1\}) = \{w\}, \mathcal{F}(\emptyset) = (v, w, z)$. It is happy to observe that:

- This is a working example of the sheafification $\mathfrak{F} := \mathrm{colim}_{h_U/\mathcal{F}} U$ via assembling lemma.
- This is a working example illustrating the usual sheafification [2] $\mathcal{F} \to \mathcal{F}^+ \to \mathcal{F}^{++} =: \mathcal{F}^\sharp$
- The two sheafifications coincide: $\mathrm{\acute{e}t}(\mathcal{F}^\sharp) = \mathfrak{F}$

For details, see the appendix 2.a.

2.3. Let's consider discrete topological space with three points

$$X := (\{0,1,2\}, \{\emptyset, \{0\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\})$$

[1] [**?**, 1.2.19]

[2] See [**?**, 3.1.2], ok there they used $\mathcal{F}^\dagger$, then see [**?**, 00WB] for $\mathcal{F}^+$.

One checks that this is indeed a topology, as $\mathcal{T}_X$ is closed under union and finite intersection. By sheaf axioms, a sheaf of abelian group $\mathcal{F}$ on X consists of an commutative diagram

$$\begin{tikzcd}
 & & \{0,1,2\} & & & & & & \mathcal{F}(\{0,1,2\}) \arrow[dll] \arrow[d] \arrow[drr] & & \\
\{0,1\} \arrow[urr] & & \{0,2\} \arrow[u] & & \{1,2\} \arrow[ull] \arrow[rr, squiggly, "\mathcal{F}"] & & \mathcal{F}(\{0,1\}) \arrow[d] \arrow[drr] & & \mathcal{F}(\{0,2\}) \arrow[dll] \arrow[drr] & & \mathcal{F}(\{1,2\}) \arrow[dll] \arrow[d] \\
\{0\} \arrow[u] \arrow[urr] & & \{1\} \arrow[ull] \arrow[urr] & & \{2\} \arrow[ull] \arrow[u] & & \mathcal{F}(\{0\}) \arrow[drr] & & \mathcal{F}(\{1\}) \arrow[d] & & \mathcal{F}(\{2\}) \arrow[dll] \\
 & & \emptyset \arrow[ull] \arrow[u] \arrow[urr] & & & & & & \mathcal{F}(\emptyset) & &
\end{tikzcd}$$

satisfying certain rules, which follows from the local nature of sheaves.

- Each of the three lower parallelograms
$$\mathcal{F}|_{\{0,1\}}, \mathcal{F}|_{\{0,2\}}, \mathcal{F}|_{\{1,2\}}$$
satisfy the sheaf axioms.
- The $\mathcal{F}(\{0,1,2\})$ is then pinned down by the sheaf axioms.

In the following examples:

$$\begin{tikzcd}
 & ? \arrow[dl] \arrow[d] \arrow[dr] & \\
M \arrow[d, "\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & M \arrow[dl, "\mathrm{id}"] \arrow[dr, "\mathrm{id}"] & M \arrow[dl, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] \\
M \arrow[dr] & M \arrow[d] & M \arrow[dl] \\
 & 0 &
\end{tikzcd}
\quad
\begin{tikzcd}
 & ? \arrow[dl] \arrow[d] \arrow[dr] & \\
M \arrow[d, "-\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & M \arrow[dl, "\mathrm{id}"] \arrow[dr, "\mathrm{id}"] & M \arrow[dl, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] \\
M \arrow[dr] & M \arrow[d] & M \arrow[dl] \\
 & 0 &
\end{tikzcd}
\quad
\begin{tikzcd}
 & ? \arrow[dl] \arrow[d] \arrow[dr] & \\
M \arrow[d, "-\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & M \arrow[dl, "\mathrm{id}"] \arrow[dr, "-\mathrm{id}"] & M \arrow[dl, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] \\
M \arrow[dr] & M \arrow[d] & M \arrow[dl] \\
 & 0 &
\end{tikzcd}
\quad
\begin{tikzcd}
 & ? \arrow[dl] \arrow[d] \arrow[dr] & \\
M \arrow[d, "-\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & M \arrow[dl, "\mathrm{id}"] \arrow[dr, "-\mathrm{id}"] & M \arrow[dl, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] \\
M \arrow[dr] & M \arrow[d] & M \arrow[dl] \\
 & 0 &
\end{tikzcd}$$

one easily calculates the global sections are

$$M, 0, M, 0$$

which only depend on the parity of the number of $-\mathrm{id}$s. This is exactly what happens when we construct a Mobius band. Thus, one vaguely observes that the existence of global section has some “topological” or “homotopical” interpretation. Let’s denote the diagram corresponding to the first two diagrams by $\mathcal{E}_0(M), \mathcal{E}_1(M)$. Then we have exact sequence of sheaves

$$0 \longrightarrow \mathcal{E}_1(\{\pm 1\}) \longrightarrow \mathcal{E}_1(\mathbb{R}^*) \longrightarrow \mathcal{E}_0(\mathbb{R}^+) \longrightarrow 0\ ,^3$$

---

[3]We remark that the 0 appears here because the $-\mathrm{id}$ cancells. Namely, we have the commutative diagram

$$\begin{tikzcd}
\{\pm 1\} \arrow[r] \arrow[d, "-\mathrm{id}"] & \mathbb{R}^* \arrow[r] \arrow[d, "-\mathrm{id}"] & \mathbb{R}^+ \arrow[d, "-\mathrm{id}"] \\
\{\pm 1\} \arrow[r] & \mathbb{R}^* \arrow[r] & \mathbb{R}^+
\end{tikzcd}$$

because they restrict to exact sequence of presheaves on an open cover[4] $\{0,1\},\{0,2\},\{1,2\}$. However, their group of global sections fit into

$$0 \longrightarrow 0 \longrightarrow \mathbb{R}^+$$

which is not exact.[5]

3. More on fiber products

Fiber product is a very flexible notion, we gave a few example to show its versatility:

3.1. Let $E \to B$, $F \to B$ be vector bundles, then $G := E \times_B F \to B$ is a vector bundle, where each fiber G_b is canonically isomorphic to $E_b \times F_b$. This is true in general, fiber product is what it literally means: taking product fiberwise.

3.2. Let X be set, let $f\colon X \to X$ be a map, then the set of fixed points can be written as

$$\mathrm{Fix}(f) := \Gamma_f \times_{X\times X} \Delta_X.$$

Recall in algebraic topology, the Lefschetz theorem was proved when the fixed point is a finite set, otherwise we cannot properly count. To name a few natural situations

- The number of fixed points of the identity map should recover the Euler number: the alternating sum of the Betti numbers.
- The number of fixed points of the sphere S^2 by reflection, is it zero?

Of course, the questions can be resolved by replacing $\#|\mathrm{Fix}|$ by $\chi(X)$. However, with the expression of fixed points as fiber products, we may do the same construction in a "homotopy category". where the "homotopy-fixed-points" can be made finite after some admissible replacement.

We remark that expression already yields interesting results. Let X be a differentiable manifold. Let TX be its tangent bundle. Given a generic section s, we can count it number of zeros and poles. We define the degree $\deg(TX) = \#|s^{-1}(0)| - \#|s^{-1}(\infty)|$. One can show $\deg(TX) = \chi(X)$. This is done by taking $f = \mathrm{id}$, recognizing the self-intersection of the diagonal as the degree of the normal bundle of $\Delta_X \subset X \times X$, and the fun observation that the normal bundle of the diagonal is the tangent bundle[6].[7]

3.3. Let $f\colon S^1 \to S^1, z \mapsto z^n$ be the n-to-1 covering, then

$$S^1 \times_{f,S^1,f} S^1 = S^1 \times \mathbb{Z}/n\mathbb{Z}.$$

This follows from three observations:

- Note that the fiber product is a n-to-1 cover to S^1 via the second projection.[8]
- The diagonal is a section to the second projection.
- The fibers are transitive by $\langle \zeta_n \rangle = \mathbb{Z}/n\mathbb{Z}$ action.

[4] We may regard this as our definition of exactness. We are in a finite topological space, presheaf-exactness on an open cover is equivalent to stalk exactness: our usual notion of exactness. Recall stalk is defined as colimit over a neighborhood basis, and each colimit runs over finitely many opens. We can show "stalk-exactness is equivalent to cover-exactness" in general, if the sheaves have finite cohomologies.

[5] This example can be viewed as a special case of Galois cohomology: Taking $\mathbb{Z}/2\mathbb{Z}$-invariants in the sequence $0 \to \{\pm 1\} \to \mathbb{R}^* \to \mathbb{R}^+ \to 0$ is not exact, we get the same sequence of invariants $0 \to 0 \to \mathbb{R}^+$. Note that the pointed discrete topological space $\{*,0,1,2\} = \mathrm{Spec}(\mathbb{C}\times_{\mathbb{R}}\mathbb{C})$, and that $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$.

[6] Which is reasonable, since the tangent bundle should consists of directions not in x, which is the complement of T_{Δ_X}

[7] See [**?**, 11.24], [**?**, 11.5].

[8] Check this locally on S^1 by hand.

We may replace S^1 by $\mathbb{F}_{q^n}$ and f by the Frobenius $\mathrm{Frob}_{\mathbb{F}_{q^n}/\mathbb{F}_q}$, we get the similar result in the category of commutative rings. Note that our previous argument can be applied to any Galois covering.[9]

4. R-Mod and $M_n(R)$-Mod

@F.Z: We know R-Mod and $M_2(R)$-Mod are equivalent categories[10], are they isomorphic?[11]

4.1. By definition, we are asked to construct functors

$$F\colon R\text{-}\mathtt{Mod} \to M_2(R)\text{-}\mathtt{Mod}$$

$$G\colon M_2(R)\text{-}\mathtt{Mod} \to R\text{-}\mathtt{Mod}$$

such that $GF = \mathrm{id}_{R\text{-}\mathtt{Mod}}$ and $FG = \mathrm{id}_{M_2(R)\text{-}\mathtt{Mod}}$

4.2. It feels like we are asking if homotopy equivalent topological spaces are necessarily homeomorphic. This is definitely incorrect. To be precise, let $\mathcal{C}$ and $\mathcal{D}$ be small categories, then by functoriality of simplicial nerve and geometric realization, we observe that:

- If $\mathcal{C}$ and $\mathcal{D}$ are equivalent, then $|N_{\mathcal{C}}|$ and $|N_{\mathcal{D}}|$ are homotopy equivalent in Top.
- If $\mathcal{C}$ and $\mathcal{D}$ are isomorphic, then $|N_{\mathcal{C}}|$ and $|N_{\mathcal{D}}|$ are isomorphic in Top.

For example, let $\mathcal{C} = \{*, 1_*\}$, let $\mathcal{D} = \{{\color{blue}*}, {\color{red}*} : 1_{\color{blue}*}, f\colon {\color{blue}*} \to {\color{red}*}, g\colon {\color{red}*} \to {\color{blue}*}, 1_{\color{red}*}\}$, then

$$|N_{\mathcal{C}}| = \bigcup_n \Delta^n,$$

$$|N_{\mathcal{D}}| = \bigcup_n S^n.$$

For details, see appendix 2.b.As a corollary, we show that the infinite sphere is contractible.[12] Here is an interesting question

Question 4.1. *Are $S^\infty := \bigcup_n S^n$ and $\Delta^\infty := \bigcup_n \Delta^n$ homeomorphic?*

Although this is unclear, it is pretty clear that $\mathcal{C}$ and $\mathcal{D}$ are not isomorphic, since $\#|\mathrm{Ob}(\mathcal{C}) = 1 \neq 2 = \#|\mathrm{Ob}(\mathcal{D})|$.

4.3. Let's sit down and think about the question in an algebraic way. There are several issues we need to deal with

- The bigness of the category R-Mod and $M_2(R)$-Mod, if we are not happy with proper classes[13], we may need some "universe" to discuss this.
- We may want to classify the R-modules and $M_2(R)$-modules.
- For each isomorphic class, we may want to discuss the groupoids of isomorphic objects.
- We need to play around and write a proof.

4.4. In order to simplify the bullets mentioned, we may assume $R = \mathbb{F}_p$ is a finite field, and consider the category

$$\mathtt{Fin}\text{-}R\text{-}\mathtt{Mod} \quad \text{and} \quad \mathtt{Fin}\text{-}M_2(R)\text{-}\mathtt{Mod}$$

of modules whose underlying abelian groups are finite. Still, these are big categories. But let's continue.

[9] This is why categorical arguments are sometimes preferred, say using "diagonal", etc.

[10] See [?, Prop 1.4]

[11] See [?, Ex 1.4.2].

[12] If you are not convinced, check the explicit construction [?, Example 1B.3].

[13] which we really aren't

4.5. An R-module is noting but an $\mathbb{F}_p$-vector space. What is an $M_2(R)$-module A? By [**?**, Proposition 4.4], we know $A = \bigoplus M_i$, where each M_i is an irreducible representation of $M_2(\mathbb{F}_p)$. Where each M_i is isomorphic to the column representation R^2. Also note that $\mathrm{End}_{M_2(R)}(R^2, R^2) = R^\times$, thus

- $M_2(R)$-modules behave like R-vector spaces, where the "atoms" are R^2 instead of R.

4.6. Thus R-$\mathtt{Mod}$ and $M_2(R)$-$\mathtt{Mod}$ are equivalent, if we can work in some "universe":

- If one can work in some "universe", where the corresponding class of objects are sets, then by axiom of choice, let's fix for any R-module B an isomorphism $B \cong \bigoplus_{\mu(B)} R$, and fix for any $M_2(R)$-module A an isomorphism $A \cong \bigoplus_{\nu(A)} R^2$. If the sets $\mathrm{Ob}(R\text{-}\mathtt{Mod})$ and $\mathrm{Ob}(M_2(R)\text{-}\mathtt{Mod})$ are closed under the association
$$\phi\colon B \to \bigoplus_{j(B)} R \quad \text{and} \quad \psi\colon A \to \bigoplus_{j(A)} R^2,$$
then by Schröder-Bernstein theorem, there exists a bijection
$$\mathfrak{F}\colon \mathrm{Ob}(\mathtt{Fin}\text{-}R\text{-}\mathtt{Mod}) \to \mathrm{Ob}(\mathtt{Fin}\text{-}M_2(R)\text{-}\mathtt{Mod}),$$
this can be upgraded to an isomorphism of categories, where morphism are assigned by matrices: The composition laws are satisfied, as we have fixed basis for each object, by the axiom of choice.
- Or equivalently, it suffices to give an isomorphism of the groupoid $\mathtt{Gpd}_1$ of free rank one R-modules and the groupoid $\mathtt{Gpd}_2$ of irreducible $M_2(R)$-modules. These categories are both large. Thus, if we can work in some universe, where the sets of objects have equal cardinality, we can construct the isomorphism of categories $\mathtt{Gpd}_1$ and $\mathtt{Gpd}_2$, and upgrade this to an isomorphism of categories $\mathfrak{F}\colon \mathtt{Fin}\text{-}R\text{-}\mathtt{Mod} \to \mathtt{Fin}\text{-}M_2(R)\text{-}\mathtt{Mod}$ by axiom of choice.

4.7. Let's currently not worry about the precise mathematical statements about "universe" and summarize what we have done.

- Recall a category consists of objects and morphisms.
- Let's assume we are working in a "universe" U, a set which is closed under mildly reasonable mathematical operations, then every category $\mathcal{C}$ is small, as the class $\mathrm{Ob}(\mathcal{C}) \in U$.
- Thus a category consists of a set of objects and a set of morphisms.
- The set of morphisms describes the relations.
- The relations of R-modules and $M_2(R)$-modules are "the same", by (1) the complete reducibility of $M_2(R)$-modules, (2) the classification of irreducible $M_2(R)$-modules, and (3) the fact that $\mathrm{Mor}_{R\text{-}\mathtt{Mod}}(R) = R^\times = \mathrm{Mor}_{M_2(R)\mathtt{Mod}}(R^2)$. Or, the groupoid of "building blocks" of the categories are equivalent.
- Thus the categories R-$\mathtt{Mod}$ and $M_2(R)$-$\mathtt{Mod}$ only differ in the set of objects.
- Two sets are identified if and only if they admit a bijection (or mutual injection, then we get bijection by Schröder-Bernstein).
- Finally, we need a precise definition of "universe" in order to write down a mathematically meaningful proof.

5. More on Yoneda

5.1. Is every sheaf representable? No: look at the sheaf of empty sets: we should at least have id_U over h_U. Thus the Yoneda embedding is never an equivalence, it is never essentially surjective. Another fun fact: let $\mathcal{C}$ be a category, then the coproduct of representable functors are never representable: test the coproduct on the functor T that maps every object to the constant set $\{0,1\}$ and every arrow to the identity.[14] Thus Yoneda's lemma does not necessarily preserved coproducts. In particular, the initial object is never mapped to an initial object: the sheaf of empty sets. However, Yoneda lemma preserves connected limits and colimits in certain cases, the verifications are just unwrapping definitions, one needs to take a grain of salt when using Yoneda "embedding": only the relations within the source category are preserved, we don't know much about how the relations of the essential image and other presheaf of sets. So our construction of étale space of sheaves is a quite fishy, maybe it can be fixed.

5.2. Yoneda lemma can be used to check monic morphisms, but it does not work for epic morphisms. The epic can be tested by co-Yoneda lemma. Actually, there are four notions: injective, surjective, co-injective, co-surjective. We defined injective and co-injective as monic and epic.

[14] See here for details.

CHAPTER 3

Motives and variations

1. Exact categories

1.1. Recall a pre-additive category is a category enriched over `Ab`: the sets of morphisms are equipped with structure of abelian groups, such that composition of morphisms are bilinear. Pre-additive categories are often called `Ab`-categories. An additive category is a pre-additive category equipped with finite products, coproduct and zero objects.[1] We call an additive category pre-abelian, if kernel and cokernel for any morphism exist.

In [**?**, 1.2.2], an abelian category is defined as a pre-abelian category such that

- Every monic is kernel of its cokernel, every epi is cokernel of its kernel.

It is remarked in class [**?**, 2.1.4] that the axiom is equivalent to

- Every cokernel of kernel is isomorphic to kernel of cokernel.

In [**?**, 6.2], there is an extra axiom (AC7):

- Every morphism has an epi-monic factorization.

1.2. `@F.Z`: (1) Why are the first two axioms equivalent? (2) Is (AC7) necessary?

1.3. First of all, one needs to figure out what the word "is" means in the axioms. Given a monic $f\colon A \to B$, what does "A is kernel of its cokernel mean? Let (Coker_f, p) be cokernel of f. Since $p \circ f = 0$, by the universal property of kernel, there exists a morphism $s\colon A \to \mathrm{Ker}_p$. The claim A is cokernel of p just means s is an isomorphism: There exists a morphism $s'\colon \mathrm{Ker}_p \to A$ such that $s \circ s' = 1_{\mathrm{Ker}_p}$ and $s' \circ s = 1_A$. The case of cokernel of kernel is similarly explained.

The next question is, what does Coker_i "is" Ker_p mean, in the second bullet? Let's draw the diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ker}_f \arrow[r, "i"] & A \arrow[r, "f"] \arrow[d, "p_i"'] & B \arrow[r, "p"] & \mathrm{Coker}_f \arrow[r] & 0 \\ & & \mathrm{Coker}_i \arrow[r, dotted, "\alpha"] & \mathrm{Ker}_p \arrow[u, "i_p"'] & & \end{tikzcd}$$

If there is a natural map α filling the dotted arrow, then we may pin down the "is" as

- There exist a morphism $\alpha^{-1}\colon \mathrm{Coker}_i \to \mathrm{Ker}_p$ such that $\alpha \circ \alpha' = 1_{\mathrm{Ker}_p}$ and $\alpha' \circ \alpha = 1_{\mathrm{Coker}_i}$.

Does such α really exist? Yes. Note that $p_i \circ i = 0$ by definition, by the universal property of Coker_{p_i}, there exists a unique $t\colon \mathrm{Coker}_i \to B$ making the diagram commute

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ker}_f \arrow[r, "i"] & A \arrow[r, "f"] \arrow[d, "p_i"'] & B \arrow[r, "p"] & \mathrm{Coker}_f \arrow[r] & 0 \, . \\ & & \mathrm{Coker}_i \arrow[ur, dotted, "t"] & \mathrm{Ker}_p \arrow[u, "i_p"'] & & \end{tikzcd}$$

[1] In class we shown that in this setting, the notion of finite products and finite coproducts coincide.

Then note that $0 = p \circ f = p \circ (t \circ p_i) = (p \circ t) \circ p_i$, and that p_i is epic[2], we know $p \circ t = 0$. By the universal property for Ker_p, the morphism t factors as $i_p \circ \alpha$ for some $\alpha \colon \mathrm{Coker}_i \to \mathrm{Ker}_p$.

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}_f \arrow[r, "i"] & A \arrow[r, "f"] \arrow[d, "p_i"'] & B \arrow[r, "p"] & \mathrm{Coker}_f \arrow[r] & 0 . \\
& & \mathrm{Coker}_i \arrow[ur, "t"] \arrow[r, dotted, "\alpha"] & \mathrm{Ker}_p \arrow[u, "i_p"'] & &
\end{tikzcd}
$$

1.4. It suffice to show $\mathrm{Ker}_\alpha = 0$ and $\mathrm{Coker}_\alpha = 0$:

- $\mathrm{Coker}_\alpha = 0$ is equivalent to α being epic,
- kernel of $\mathrm{Ker}_p \to 0$ is Ker_p,
- thus the axiom for the monic α implies that $\mathrm{Coker}_i \to \mathrm{Ker}_p$ is an isomorphism.

1.5. Since the dual of an pre-abelian category is still pre-abelian, it suffice to check α is monic. Given $w_1, w_2 \colon W \to \mathrm{Coker}_i$ such that $w_1 \circ \alpha = w_2 \circ \alpha$, then $t \circ w_1 = t \circ w_2$, here we use the fact that i_p is monic. It suffices to check that t is monic.

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}_f \arrow[r, "i"] & A \arrow[r, "f"] \arrow[d, "p_i"'] & B \arrow[r, "p"] & \mathrm{Coker}_f \arrow[r] & 0 . \\
& W \arrow[r, shift left, "w_1"] \arrow[r, shift right, "w_2"'] & \mathrm{Coker}_i \arrow[ur, "t"] \arrow[r, "\alpha"] & \mathrm{Ker}_p \arrow[u, "i_p"'] & &
\end{tikzcd}
$$

1.6. First we prove a useful filling lemma: For any factorization $f = t' \circ p'$, where p' is epic, there exists an induced map β with target $\mathrm{Coim}_f = \mathrm{Coker}_i$ making the triangles commute:

$$
\begin{tikzcd}
& \mathrm{Coim}_f \arrow[dr, "t"] & \\
A \arrow[ur, "p_i"] \arrow[dr, "p'"'] & & B \\
& \mathrm{Coker}_{i'} \arrow[ur, "t'"'] \arrow[uu, dotted, "\beta"] & .
\end{tikzcd}
$$

Let's write out kernels of the epics. As $t' \circ p' \circ i' = f \circ i' = 0$, there is an induced map $\gamma \colon \mathrm{Ker}_{p'} \to \mathrm{Ker}_f$ such that $i' = i \circ \gamma$

$$
\begin{tikzcd}
\mathrm{Ker}_{p'} \arrow[dr, "i'"] \arrow[dd, dotted, "\gamma"'] & & \mathrm{Coim}_f \arrow[dr, "t"] & \\
& A \arrow[ur, "p_i"] \arrow[rr, "f"] \arrow[dr, "p'"'] & & B \\
\mathrm{Ker}_f \arrow[ur, "i"'] & & \mathrm{Coker}_{i'} \arrow[ur, "t'"'] & .
\end{tikzcd}
$$

As $p_i \circ i' = p_i \circ i \circ \gamma$, the morphism p_i factors as $p_i = \beta \circ p'$. In order to show $t \circ \beta = t'$, it suffices to note that they equal when pre-composed with p', which is right-effaceable.

[2]Definition of cokernels (or co-equalizers) imply that they are necessarily epic. Epics are right-effaceable.

1.7. Then we show t is monic. It suffices to show $\mathrm{Ker}_t = 0$

$$
\begin{tikzcd}
\mathrm{Ker}_t \arrow[dr, "i_t"] & & \mathrm{Coker}_{i_t} \arrow[dl, dotted, "\beta'"] \arrow[dd, "t''"] \\
& \mathrm{Coim}_f \arrow[ur, "p_{i_t}"] \arrow[dr, "t"] & \\
A \arrow[ur, "p_i"] \arrow[dr, "p'"'] & & B \\
& \mathrm{Coker}_{i'} \arrow[uu, "\beta"] \arrow[ur, "t'"'] &
\end{tikzcd}
.
$$

Viewing $p_{i_t} \circ p_i$ as p', by the filling lemma, there exists a β' such that $t \circ \beta' = t''$ and $\beta' \circ p_{i_t} \circ p_i = p_i$. As p_i is epic, we know $\beta' \circ p_{i_t} = 1_{\mathrm{Coim}_f}$. Thus p_{i_t} is monic. Thus $p_{i_t} \circ i_t = 0$ is monic, thus Ker_t is forced to be zero.

1.8. We remark in the proof, we only need to assume every epimorphism is a cokernel of some map. We may simply replace $\mathrm{Ker}(p')$ by the presentation of cokernel, then all the steps follow. Let's sum up what we have done.

1.9. Let's call a category pre-exact, if zero object exists, and any morphism has kernel, cokernel. We have shown the following conditions are equivalent on an pre-exact category[3]:

- Every monic is a kernel, every epic is a cokernel.
- Every monic is a kernel of its cokernel, every epic is cokernel of its kernel
- Every morphism has a unique epi-monic decomposition.
- Kernel of cokernel equals cokernel of kernel.

We call a pre-exact category an exact category if one of the four equivalent conditions hold. An example of exact category is the category of pointed sets, which is obviously not an additive category, but the notion of exact sequence well makes sense. [4]

1.10. Now we can answer the question at the beginning: the axiom (AC7) is unnecessary, and it is strictly weaker than the previous axioms. For example, in the category of topological abelian groups the morphism $(\mathbb{Q}, \text{discrete}) \to (\mathbb{Q}, |\cdot|_v)$ is both monic and epic, so there exist non-unique epi-monic factorization. However, in an abelian category, a monic-epic morphism is isomorphism. We remark that Peter Scholze recently developed a theory called condensed mathematics, which enabled us to make the category of topological abelian groups into an abelian category, see [**?**] and [**?**].

2. Still more on Yoneda

2.1. Can we study abelian categories using Yoneda lemma? Yes, Yoneda embedding embeds an abelian category $\mathcal{A}$ as a full subcategory in $\mathtt{Psh}_{\mathtt{Ab}}(\mathcal{A})$. However, it does not preserve kernel and cokernel, so half of the structure of the abelian category goes away. This is the reason why we do not use Yoneda embedding to study abelian categories. For example, take the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$, then $\eta \colon X \mapsto \mathrm{Mor}_{\mathcal{A}}(-, X)$ is not an exact sequence of presheaves. However, Yoneda lemma has certain faithfulness:

[3] We proved this for additive category with kernels and cokernels, but actually we did not use the additive structure. We merely used the existence of zero object, kernels and cokernels.

[4] The name "exact category" follows [**?**, I], the proof is shamelessly copied from [**?**, VIII].

Lemma 2.1 (Yoneda Preservation). *The Yoneda functor preserves left exactness. Namely, given short exact sequence in an abelian category $\mathcal{A}$:*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0\ ,$$

For any $T \in \mathrm{Ob}(\mathcal{A})$, we have the following short exact sequence of abelian groups [5]

$$0 \longrightarrow \mathrm{Hom}(T, M') \longrightarrow \mathrm{Hom}(T, M) \longrightarrow \mathrm{Hom}(T, M'')$$

$$0 \longrightarrow \mathrm{Hom}(M'', T) \longrightarrow \mathrm{Hom}(M, T) \longrightarrow \mathrm{Hom}(M', T)$$

Lemma 2.2 (Yoneda Reflection). *The Yoneda functor reflects exactness. Namely, let given a pair of composable arrows in an abelian category $\mathcal{A}$:*

$$M' \longrightarrow M \longrightarrow M''\ ,$$

- *If for any $T \in \mathrm{Ob}(\mathcal{A})$,*

$$0 \longrightarrow \mathrm{Hom}(T, M') \longrightarrow \mathrm{Hom}(T, M) \longrightarrow \mathrm{Hom}(T, M'')$$

  is exact, then we have exactness

$$0 \to M' \to M \to M''$$

- *If for any $T \in \mathrm{Ob}(\mathcal{A})$,*

$$0 \longrightarrow \mathrm{Hom}(M'', T) \longrightarrow \mathrm{Hom}(M, T) \longrightarrow \mathrm{Hom}(M', T)$$

  is exact, then we have exactness

$$M' \to M \to M'' \to 0$$

Proof. For example, let's prove the last bullet: (1) We check $M \to M''$ is exact: Let $N := \mathrm{Coker}(M \to M'')$, then by Yoneda preservation, we have $0 \to \mathrm{Hom}(N, N) \to \mathrm{Hom}(M'', N) \to \mathrm{Hom}(M, N)$, since the second one is injection, we know $\mathrm{Hom}(N, N) = 0$, hence $1_N = 0$, thus $N = 0$. (2) Let $K = \mathrm{Ker}(\phi\colon M \to M'')$, $C = \mathrm{Coker}(\psi\colon M' \to M)$, it suffice to show $H = 0$ in either of the diagrams[6]

$$\begin{tikzcd}
 & M' \arrow[d] \arrow[dr] & & & \\
0 \arrow[r] & K \arrow[r] \arrow[d] & M \arrow[r] & M'' \arrow[r] & 0 \\
 & H_1 \arrow[d] & & & \\
 & 0 & & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & M' \arrow[d] & & \\
 & & M \arrow[d] \arrow[dr] & & \\
0 \arrow[r] & H_2 \arrow[r] & C \arrow[r] \arrow[d] & M'' \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd}$$

[7]

[5]Let's denote Mor by Hom, with the vision in mind that we are taking homomorphisms of abelian groups as morphisms

[6]Here H is the cohomology of the complex at M, which measures the failure of exactness at M.

[7]In order to show $H = 0$, we want to mimic the previous argument, and show $\mathrm{Hom}(H, H) = 0$. In order to fully exploit the condition, we take $\mathrm{Hom}(-, H)$ in the diagrams. However, it seems that naively playing with the diagram cannot fully exploit the power of abelian categories. We need some analysis of the concrete situation.

The key observation is $H_1 \cong H_2$, which follows from the snake lemma[8]

$$\begin{tikzcd}
 & & & H_2 \arrow[d, "i"] & \\
 & M' \arrow[r, "\psi"] \arrow[d, "f"] & M \arrow[r, "p_\psi"] \arrow[d, "1_M"] & \mathrm{Im}(\psi) \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & \mathrm{Ker}(\phi) \arrow[r, "i_\phi"] \arrow[d, "p"] & M \arrow[r, "\phi"] & M'' & \\
 & H_1 & & &
\end{tikzcd}$$

Let's finish the proof. Consider the diagram:

$$\begin{tikzcd}
M' \arrow[rr, "\phi"] \arrow[dr, "f"] & & M \arrow[rr, "\psi"] \arrow[dr, "p_\phi"] & & M'' \arrow[rr] \arrow[dl, dotted, bend left, "h"] & & 0 \\
 & K \arrow[ur, "i_\psi"] \arrow[rr, "\alpha"] \arrow[dr, "p"] & & C \arrow[ur, "g"] & & & \\
 & & H \arrow[ur, "i"] \arrow[dr] & & & & \\
 & 0 \arrow[ur] & & 0 & & &
\end{tikzcd}$$

Note that $H = 0$ if and only if $\alpha = i \circ 1_H \circ p = 0$, because p and i are effaceable. Let's take the testing object $T = C$, our assumption gives the exact sequence

$$0 \longrightarrow [M'', C] \xrightarrow{\psi^*} [M, C] \xrightarrow{\phi^*} [M', C] \ .$$

Note that $\phi^*(p_\phi) = p_\phi \circ \phi = 0$, the sequence asserts there exists a section $h\colon M'' \to C$ to g. Then $\alpha = \circ h\psi \circ i_\psi = 0$, thus $H = 0$, hence the exactness: the coimage coincides with image. $\square$

2.2. One may feel life being too hard in an abelian category. Can we work out everything as modules? A wishful thinking is that every abelian category is equivalent to R-`Mod` for some ring R. However this is too wishful, say, we may just delete any infinite product or certain infinite colimit in an R-`Mod`, the resulting thing is an abelian category, but not R-`Mod`, where infinite products exist. However, the guess is not so far away from being true.

THEOREM 2.3 (Freyd-Mitchell). *Let $\mathcal{A}$ be any small abelian category. Then there exists a ring R and an exact, faithfully full functor* A *into R-`Mod`, which embeds $\mathcal{A}$ as a full subcategory.*

The key feature here is exactness of the embedding: kernels and cokernels are preserved.[9] This is the theorem that enables us to chase diagram as if we are in the category `Ab`. Note that the ring R here is just used to put restriction on the set of morphisms, so that not every homomorphism $\mathrm{Hom}_{\mathbb{Z}}(M, N)$ is allowed. In chasing diagrams, we do not care about what this R actually is.

[8]See [**?**, 010H].

[9]And also finite limits and colimits.

3. Monoidal categories

3.1. Recall we defined a pre-abelian category as a category enriched over `Ab`, such that the composition are bilinear with respect to the group law. It seems that there are some requirements on the "enricher category". Usually, the enricher category should be a monoidal category[10], so that the "bilinearity" of the composition makes sense.

3.2. A monoidal category is a category $\mathcal{C}$ equipped with a monoidal structure consisting of the following:

- A bifunctor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the monoidal product, or tensor product
- An object $\mathbf{1}$ called the identity object.
- Unital constraints
$$\lambda_x \colon \mathbf{1} \otimes x \to x, \quad \rho \colon x \otimes \mathbf{1} \to x$$
- Associative constraints
$$\alpha_{x,y,z} \colon x \otimes (y \otimes z) \to (x \otimes y) \otimes z$$
- Subject to coherence conditions
$$(\alpha_{x,y,z} \otimes 1_z) \circ (\alpha_{x,y\otimes z,w}) \circ (1_x \otimes \alpha_{y,z,w}) = \alpha_{x\otimes y,z,w} \circ \alpha_{x,y,z\otimes w},$$
$$(\rho_x \otimes 1_y) \circ \alpha_{x,\mathbf{1},y} = 1_x \otimes \lambda_y.$$

A functor of monoidal categories is a functor $\mathcal{C} \to \mathcal{C}'$ plus a natural transformation $F(x) \otimes F(y) \to F(x \otimes y)$ satisfying some hexagon axiom.[11]

3.3. We call an object x in $\mathcal{C}$ is invertible, if there exists an object y such that $x \otimes y \cong y \otimes x \cong \mathbf{1}$, or equivalently $- \otimes x$ is an equivalence. We call an object y a left dual of x, it there are morphisms $\eta \colon \mathbf{1} \to x \otimes y$ and $\epsilon \colon y \otimes x \to \mathbf{1}$ so that $(1 \otimes \epsilon) \circ (\eta \otimes 1) = 1_x$ and $(\epsilon \otimes 1) \circ (1 \otimes \eta) = 1_y$, we call x the right dual of y. Here the left-right may be remembered with respect to η. In this case, we have functorial isomorphism $[a \otimes x, b] = [a, b \otimes y]$ and $[y \otimes a, b] = [a, x \otimes b]$. A symmetric monoidal category is a monoidal category plus the commutative constraint. A functor or symmetric monoidal categories asks the compatibility with symmetric constraints. A strict monoidal category is one which natural transformation α, λ, ρ are identities, for example `Sets` with cartesian product. However R-`Mod` is not strict. Every monoidal category is equivalent to a strict one, by axiom of choice.

3.4. Let $(\mathtt{M}, \otimes)$ be a monoidal category, then an `M`-category $\mathcal{C}$, consists of

- A assignment $C \colon \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{C}) \to \mathtt{M}$.
- For each object $a \in \mathrm{Ob}(\mathcal{C})$, a morphism $I \to C(a, a)$.
- For each ordered triple (a, b, c), a morphism $\circ_{a,b,c} \colon C(b, c) \otimes C(a, b) \to C(c, a)$ in `M`,
- Such that the composition is associative and unital.

4. Pseudo-abelian categories

Finally we remark that there is an interesting notion between pre-additive categories and pre-abelian categories.

4.1. Let $\mathcal{C}$ be an `Ab`-category. We say $\mathcal{C}$ is Karoubian (or pseudo-abelian), if every idempotent endomorphism of $\mathcal{C}$ has a kernel. Here are some lemma.

[10] Or tensor category

[11] See [**?**, 0FFL].

4.2. Karoubian lemma: In a pre-additive category, the following are equivalent

- Kernel of idempotents exist
- Cokernel of idempotents exist.
- Any idempotent $p\colon z \to z$ decomposes into direct sum, and p is a projection.

PROOF. We deduce the third bullet from the first bullet. Let p be an idempotent, then $1-p$ is automatically an idempotent. Let's take their kernels i_p and i_{1-p}. Note that $p(p-1)=0$, we have factorizations

$$\begin{tikzcd} & z \arrow[dl, "f"'] \arrow[d, "p"] & \\ x \arrow[r, "i_x"'] & z \arrow[r, "1-p"'] & z \end{tikzcd} \qquad \begin{tikzcd} & z \arrow[dl, "g"'] \arrow[d, "1-p"] & \\ y \arrow[r, "i_y"'] & z \arrow[r, "p"'] & z \end{tikzcd}$$

Then we can check the biproduct axiom for $(z; x, y; i_x, i_y, f, g)$

- $i_x \circ f + i_y \circ g = (1-p) + p = 1_z$,
- $f \circ i_y = 0$, because $i_x \circ f \circ i_y = (1 - i_y \circ g) \circ i_y = i_y - i_y(1-p) = 0$. Similarly $g \circ i_x = 0$.
- $f \circ i_x = 1_x$, because i_x is monic and $i_x \circ f \circ i_x = (1 - i_y \circ g) \circ i_x = i_x$. Similarly $g \circ i_y = 1_y$.

□

4.3. Karoubian criterion[12]: An `Ab` category is Karoubian, if either one of the following holds

- The category has countable products, and kernels have right inverse.
- The category has countable coproducts, and cokernels have right inverse.

4.4. There is a canonical way to upgrade a pre-additive category to a Karoubian category. The Karoubinization $\mathcal{C} \to \mathrm{Kar}(\mathcal{C})$ called the Karoubian envelope, or the pseduo-abelian completion of $\mathcal{C}$, is a fully faithful functor. It can be realized by explicitly taking pairs (X, p), where p is an idempotent as objects, and morphism $(X,p) \to (Y,q)$ are given by $f\colon X \to Y$ such that $f = q \circ f \circ p$.

4.5. Here are some examples why Karoubian envelopes are interesting

- The category of projective modules over any ring is the Karoubi envelope of its full subcategory of free modules.
- The category of vector bundles over any paracompact space is the Karoubi envelope of its full subcategory of trivial bundles. [13]
- Let's fix a base field k. The category of chow motives M_k is Karoubian, see [**?**, 0FGB].

4.6. We summarize the notions we encountered as the following diagram

$$\begin{tikzcd} \texttt{Cat} \arrow[rr] \arrow[d] & & \texttt{Pre-exact} \arrow[r] \arrow[dd] & \texttt{Exact} \arrow[dd] \\ \texttt{Pre-additive} \arrow[r] \arrow[d] & \texttt{Karoubian} \arrow[r, dash] & {} & \\ \texttt{Additive} \arrow[rr] & & \texttt{Pre-abelian} \arrow[r] & \texttt{Abelian} \end{tikzcd}$$

[12]See the fabulously clever trick [**?**, 05QV].

[13]This is in fact a special case of the previous example by the Serre-Swan theorem and conversely this theorem can be proved by first proving both these facts, the observation that the global sections functor is an equivalence between trivial vector bundles over X and free modules over $C(X)$ and then using the universal property of the Karoubi envelope.–Wikipedia

5. Ends and Coends

5.1. `@F.Z` Does it make sense to talk about natural transformations form a contravariant functor to a covariant functor?

5.2. Yes, one can define "antinatural-transformations". The interesting thing is: some similar considerations motivate notions called end and coend, which are special cases of limit and colimit. Let's look at how they work, just for fun.

5.3. Let $S, T\colon \mathcal{C}^{opp} \times \mathcal{C} \to \mathcal{X}$ be functors. A dinatural transformation α between S, T, associates each c in $\mathcal{C}$ an arrow $S(c,c) \to T(c,c)$ such that the sexagon coherence diagram is satisfied

$$\begin{tikzcd} & S(c,c) \arrow[r, "\alpha_c"] & T(c,c) \arrow[dr, "T(1,f)"] & \\ S(c',c) \arrow[ur, "S(f,1)"] \arrow[dr, "S(1,f)"'] & & & T(c,c') \\ & S(c',c') \arrow[r, "\alpha_{c'}"] & T(c',c') \arrow[ur, "T(f,1)"'] & \end{tikzcd}$$

5.4. We define the end of a functor $S\colon \mathcal{C}^{opp} \times \mathcal{C} \to \mathcal{X}$ to be the universal dinatural transformation form an object $e \in \mathrm{Ob}(\mathcal{X})$ to a cone of wedges. [14] Namely, an end of the functor S is a pair $(e, \{\omega_i\})$, such that for any dinatural transformation $\beta\colon x \in S$, there exists a unique $x \to e$, such that $\beta_i = \omega_i \circ h$. We call e the end of the functor S, and denote it by

$$e = \int_c S(c,c)$$

It can be characterized by equalizer

$$\int_c S(c,c) \longrightarrow \prod_{\mathcal{C}} S(c,c) \rightrightarrows \prod_{c \to c'} S(c,c')$$

5.5. Similarly we define the coend of a functor $S\colon \mathcal{C}^{opp} \times \mathcal{C} \to \mathcal{X}$ to be the universal dinatural transformation form a cone of wedges to an object $e \in \mathrm{Ob}(\mathcal{X})$. We denote the coend by

$$d = \int^c S(c,c)$$

It can be characterized by coequalizer

$$\int^c S(c,c) \longleftarrow \prod_{\mathcal{C}} S(c,c) \leftleftarrows \prod_{c \to c'} S(c,c')$$

5.6. Let $X\colon \mathbf{\Delta}^{opp} \to \mathtt{Sets}$ be a simplicial set. The discrete topology gives a functor $\mathtt{Sets} \to \mathtt{Top}$. There is a functor $\Delta\colon \mathbf{\Delta} \to \mathtt{Top}$ sending $\mathbf{n}$ to the standard n-simplex $\Delta^n \subset \mathbb{R}^{n+1}$. We define functor $X \cdot \Delta\colon \mathbf{\Delta}^{opp} \times \mathbf{\Delta} \to \mathtt{Top}$ by taking product. Then the geometric realization of S_X can be written as

$$|X| = \int^c X \cdot \Delta(c,c)$$

[14]Recall the limit is the universal transformation form an object to a cone of points.

CHAPTER 4

Motives and variations

1. Nilpotent deformation of vector bundles

1.1. Let R be a ring[1], let I be an ideal of R, let M be a module over R such that

- I is nilpotent, namely, $I^k = 0$ for some $k \in \mathbb{N}$.
- M is flat over R, namely, $M \otimes_R -$ is exact.
- M/I is projective over R/I, namely $\mathrm{Hom}_{R/I}(M/I, -)$ is exact.

We show the three conditions imply that M is projective.

1.2. First let's remark that none of these conditions can be dropped. For example, dropping each of the bullets, we have corresponding counterexamples

- Take $R = \mathbb{Z} \oplus \mathbb{Z}$, $M = \mathbb{Z} \oplus \mathbb{Q}$ and $I = 0 \oplus \mathbb{Z}$
- Take $R = \mathbb{Z}/4\mathbb{Z}$, $I = (2)$ and $M = \mathbb{Z}/2\mathbb{Z}$
- Take $R = \mathbb{Z}[x]/(x^2)$, $I = (x)$ and $M = \mathbb{Q}[x]/(x^2)$.

All these examples come from the geometric intuitions we explained last time.

1.3. We remark that the first and third condition can be dropped if M is finitely presented, for example when M is a finitely generated module over a noetherian ring. As "flat + f.p." implies projectivity.

1.4. Let's think about this question. First of all, one note that by induction, we may assume $I^2 = 0$. This may or may not be helpful, anyway, the problem looks simplified a little bit.

1.5. In the same spirit, we note that in order to show M is projective:

$$\text{“ } \mathrm{Ext}^1_R(M, N) = 0 \text{ for any } N \in \mathrm{Ob}(\mathtt{Mod}_R) \text{ ”}$$

it suffice to show

$$\text{“ } \mathrm{Ext}^1_R(M, N) = 0 \text{ for any } N \in \mathrm{Ob}(\mathtt{Mod}_{R/I}) \text{ ”}.$$

This is because, we always have short exact sequence

$$0 \longrightarrow N \longrightarrow N \longrightarrow N/IN \longrightarrow 0 \,,$$

whose the long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^1(M, IN) \longrightarrow \mathrm{Ext}^1(M, N) \longrightarrow \mathrm{Ext}^1(M, N/IN) \longrightarrow \cdots$$

would force $\mathrm{Ext}^1_R(M, N) = 0$.

[1]Let's discuss commutative rings, to make life easier.

1.6. We just decomposed N, why don't we do the same thing for M? The short exact sequence

$$0 \longrightarrow M \longrightarrow M \longrightarrow M/IM \longrightarrow 0\ ,$$

give us exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(M/IM, N) \longrightarrow \mathrm{Hom}_R(M, N) \longrightarrow \mathrm{Hom}_R(IM, N) \longrightarrow \mathrm{Ext}^1_R(M/IM, N)$$

In order to compute the $\mathrm{Ext}^1(M, N)$, we take a projective resolution

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{\mathrm{d}_0} M \longrightarrow 0$$

Take the kernel $K := \mathrm{Ker}(\mathrm{d}_0)$, the long exact sequence associated with the short exact sequence

$$0 \longrightarrow K \longrightarrow P_0 \longrightarrow M \longrightarrow 0\ ,$$

extends the previous diagram to

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
0 \arrow[r] & \mathrm{Hom}_R(M/IM, N) \arrow[r] & \mathrm{Hom}_R(M, N) \arrow[r] \arrow[d] & \mathrm{Hom}_R(IM, N) \arrow[r] & \mathrm{Ext}^1_R(M/IM, N) \\
 & & \mathrm{Hom}_R(P, N) \arrow[d] & & \\
 & & \mathrm{Hom}_R(K, N) \arrow[d] & & \\
 & & \mathrm{Ext}^1_R(M, N) \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

1.7. We fill in the missing terms in the cross

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
0 \arrow[r] & \mathrm{Hom}_R(M/IM, N) \arrow[r] & \mathrm{Hom}_R(M, N) \arrow[r] \arrow[d] & \mathrm{Hom}_R(IM, N) \arrow[r] & \mathrm{Ext}^1_R(M/IM, N) \\
0 \arrow[r] & \mathrm{Hom}_R(P/IP, N) \arrow[r] & \mathrm{Hom}_R(P, N) \arrow[r] \arrow[d] & \mathrm{Hom}_R(IP, N) \arrow[r] & \mathrm{Ext}^1_R(P/IP, N) \\
0 \arrow[r] & \mathrm{Hom}_R(K/IK, N) \arrow[r] & \mathrm{Hom}_R(K, N) \arrow[r] \arrow[d] & \mathrm{Hom}_R(IK, N) \arrow[r] & \mathrm{Ext}^1_R(K/IK, N) \\
 & & \mathrm{Ext}^1_R(M, N) \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

1.8. One observes that the first column form a short exact sequence, because

- We have natural isomorphism
$$\mathrm{Hom}_R(X/IX, N) = \mathrm{Hom}_{R/I}(X/IX, N)$$
- Projectivity of M/I tells us
$$\mathrm{Ext}^1_{R/I}(M/IM, N) = 0.$$

1.9. For the same reason, the third column form a short exact sequence

- Note that P_0, M being flat implies that K is flat, thus for $X = P_0, M, K$, we have
$$I \otimes_R X \cong IX$$
- Hom-tensor adjunction tells us
$$\mathrm{Hom}_R(IX, N) = \mathrm{Hom}_R(I \otimes_R X, N) = \mathrm{Hom}_R(X, \mathrm{Hom}_R(I, N))$$
- Since I is square zero, we know
$$\mathrm{Hom}_R(X, \mathrm{Hom}_R(I, N)) = \mathrm{Hom}_R(X/IX, \mathrm{Hom}_R(I, N))$$
- We have natural isomorphism
$$\mathrm{Hom}_R(X/IX, \mathrm{Hom}_R(I, N)) = \mathrm{Hom}_{R/I}(X/IX, \mathrm{Hom}_R(I, N)).$$
- Projectivity of M/I tells us
$$\mathrm{Ext}^1_{R/I}(M/IM, \mathrm{Hom}_R(I, N)) = 0.$$

1.10. One observes that the third column form a short exact sequence, because
$$0 \longrightarrow K/IK \longrightarrow P/IP \longrightarrow M/IM \longrightarrow 0$$
is split exact, as M/IM is projective.

1.11. To sum up, now we have the following diagram which is exact everywhere
$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}_R(M/IM, N) \arrow[r] \arrow[d] & \mathrm{Hom}_R(M, N) \arrow[r] \arrow[d] & \mathrm{Hom}_R(IM, N) \arrow[r] \arrow[d] & \mathrm{Ext}^1_R(M/IM, N) \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}_R(P/IP, N) \arrow[r] \arrow[d] & \mathrm{Hom}_R(P, N) \arrow[r] \arrow[d] & \mathrm{Hom}_R(IP, N) \arrow[r] \arrow[d] & \mathrm{Ext}^1_R(P/IP, N) \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}_R(K/IK, N) \arrow[r] \arrow[d] & \mathrm{Hom}_R(K, N) \arrow[r] \arrow[d] & \mathrm{Hom}_R(IK, N) \arrow[r] \arrow[d] & \mathrm{Ext}^1_R(K/IK, N) \arrow[d] \\
 & 0 & \mathrm{Ext}^1_R(M, N) \arrow[d] & 0 & 0 \\
 & & 0 & &
\end{tikzcd}$$

1.12. Finally we quotient the second row by the first and apply the five lemma.

2. Tensor and Hom of complexes

2.1. Let $A_\bullet, B_\bullet$ be chain complexes, let $C^\bullet$ be cochain complex. In this section, we define tensor $(A_\bullet \otimes B_\bullet)_\bullet$ and Hom $\mathrm{Hom}^\bullet(A_\bullet, C^\bullet)$ of such complexes, which are themselves complexes.

2.2. The first step is to form the double complexes,

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
A_0 \otimes B_3 & \longleftarrow & A_1 \otimes B_3 & \longleftarrow & A_2 \otimes B_3 & \longleftarrow & A_3 \otimes B_3 & \longleftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
A_0 \otimes B_2 & \longleftarrow & A_1 \otimes B_2 & \longleftarrow & A_2 \otimes B_2 & \longleftarrow & A_3 \otimes B_2 & \longleftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
A_0 \otimes B_1 & \longleftarrow & A_1 \otimes B_1 & \longleftarrow & A_2 \otimes B_1 & \longleftarrow & A_3 \otimes B_1 & \longleftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
A_0 \otimes B_0 & \longleftarrow & A_1 \otimes B_0 & \longleftarrow & A_2 \otimes B_0 & \longleftarrow & A_3 \otimes B_0 & \longleftarrow & \cdots
\end{array}$$

Then we simply define

$$(A_\bullet \otimes B_\bullet)_\bullet = \mathrm{Tot}_\bullet(A_\bullet \otimes B_\bullet),$$

where $A_\bullet \otimes B_\bullet$ is the double complex given by termwise tensor product.

2.3. Similarly, we form the double complex

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\mathrm{Hom}(A_0, B^3) & \longrightarrow & \mathrm{Hom}(A_1, B_3) & \longrightarrow & \mathrm{Hom}(A_2, B^3) & \longrightarrow & \mathrm{Hom}(A_3, B^3) & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\mathrm{Hom}(A_0, B^2) & \longrightarrow & \mathrm{Hom}(A_1, B_2) & \longrightarrow & \mathrm{Hom}(A_2, B^2) & \longrightarrow & \mathrm{Hom}(A_3, B^2) & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\mathrm{Hom}(A_0, B^1) & \longrightarrow & \mathrm{Hom}(A_1, B_1) & \longrightarrow & \mathrm{Hom}(A_2, B^1) & \longrightarrow & \mathrm{Hom}(A_3, B^1) & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\mathrm{Hom}(A_0, B^0) & \longrightarrow & \mathrm{Hom}(A_1, B^0) & \longrightarrow & \mathrm{Hom}(A_2, B^0) & \longrightarrow & \mathrm{Hom}(A_3, B^0) & \longrightarrow & \cdots
\end{array}$$

Let's carefully look at the components. The n-th component $\mathrm{Tot}^n(\mathrm{Hom}(A_\bullet, C^\bullet))$ consists of the degree preserving maps from $\bigoplus A^{-\bullet}$ to $\bigoplus B^\bullet[n]$ that does not necessarily commute with differentials. It is very interesting to note that the cohomology group $[A_\bullet, C^\bullet] := \mathrm{H}^n(\mathrm{Tot}^\bullet(A_\bullet, C^\bullet))$ is exactly the group of chain maps from $A^{-\bullet}$ to $C^\bullet[n]$ modulo homotopy equivalence. One simply checks this by hand. Or, better still, one simply define chain homotopy as the cocycle that realizes the connect the chain maps and build up the whole theory!

2.4. One may ask if it is possible, to define some generalized chain complexes by replacing the index $\mathbb{Z}$ by general directed sets. One can think about this. This is like we replace sequence convergence by net convergence. Can we define web complexes, web homotopy? Or they just give the same theory?

3. Application: Yoneda pairing

3.1. The aforementioned result easily extends to general complexes[2]: Let $X^\bullet, Y^\bullet$ be chain complexes, let $\mathrm{Hom}^\bullet(X^\bullet, Y^\bullet)$ denote the complex of

- abelian groups
$$\mathrm{Hom}^n(X^\bullet, Y^\bullet) = \prod_{p\in\mathbb{Z}} \mathrm{Hom}(X^p, Y^{p+n}),$$
- with boundary operator
$$(\mathrm{d}^n f)^p = \mathrm{d}f + (-1)^{n+1} f\mathrm{d},$$

then we have
$$[X^\bullet, Y^{\bullet+n}] = \mathrm{H}^n\mathrm{Hom}(X^\bullet, Y^\bullet).$$

3.2. This description is very interesting, for example, when $X^\bullet = M[0]$ is concentrated in degree zero and $Y^\bullet$ is an injective resolution of N, then $[X^\bullet, Y^{\bullet+n}] = \mathrm{Ext}^n(M, N)$, in particular, in the derived category $\mathtt{D}^b(\mathtt{Ab})$ of chain complexes[3], we can express the Ext group as the group of morphisms!
$$\mathrm{Ext}^n(M, N) = \mathrm{Mor}_{\mathtt{D}^b(\mathrm{Ab})}(M, N[n])$$
An interesting corollary is

PROPOSITION 3.1. *The elements in* $\mathrm{Ext}^n(M, N)$ *are uniquely (up to homotopy) represented by exact sequences*
$$0 \longrightarrow N \longrightarrow E_0 \longrightarrow \cdots \longrightarrow E_{n-1} \longrightarrow M \longrightarrow 0$$

This is noting but a generalization of the description of Ext^1.

3.3. By the description above, one observes that there is a multiplicative structure on the ext groups, called the Yoneda pairing.

PROPOSITION 3.2 (Yoneda). *There is a functorial bilinear map between ext groups of modules*
$$\mathrm{Ext}^m(M, N) \times \mathrm{Ext}^m(L, M) \to \mathrm{Ext}^{m+n}(L, N)$$
which are explicitly given by
$$[0 \to N \to \mathbf{E} \to M \to 0] \smile [0 \to M \to \mathbf{F} \to L \to 0] := [0 \to N \to \mathbf{E} \to \mathbf{F} \to L \to 0]$$

This will be upgraded to Serre duality in algebraic duality, where the pairing is perfect[4] under some compactness and smoothness assumptions.

4. Application: Koszul resolution

Let's also point out that the tensor product of complexes is very useful. For example, it can be used to construct the Koszul resolution of a regular sequence. The so called Hilbert Syzygy theorem is simple corollary of "Existence of Koszul resolution"+"Nakayama lemma".

[2]See [**?**, I.4] for details.
[3]a category where quasi-isomorphisms are formally made invertible
[4]Some fancier version of the pairing $V \times V^* \to \mathbb{R}$ between dual vector spaces.

4.1. Let R be a ring, let $x \in R$ be an element. We say x is regular[5] if the multiplication-by-x map is injective

$$0 \longrightarrow R \xrightarrow{x} R \longrightarrow R/(x) \longrightarrow 0$$

4.2. Let $x_1, \dots, x_n$ be elements in R. The Koszul complex $\mathrm{Kos}(x_1, \dots, x_n)_\bullet$ is inductively define by

- $\mathrm{Kos}_\bullet(x_1) = \{ R \xrightarrow{x} R \}$ is the two term complex concentrated in degree $1, 0$.
- $\mathrm{Kos}_\bullet(x_1, \dots, x_r)_\bullet := \mathrm{Tot}_\bullet(\mathrm{Kos}_\bullet(x_1, \dots, x_r) \otimes \mathrm{Kos}_\bullet(x_r))$.

One observes that the Koszul complex $\mathrm{Kos}_\bullet(x_1, \dots, x_r)$ is concentrated in degree $[0, r]$. This is a locally free resolution, where the degree-r component $\mathrm{Kos}_r(x_1, \dots, x_n)$ is given by $\binom{n}{r}$ copies of R.

4.3. For example, what is the complex $\mathrm{Kos}_\bullet(x, y)$? Take the total complex of:

$$\begin{tikzcd} R \arrow[d, "y"'] & R \arrow[l, "x"] \arrow[d, "y"] \\ R & R \arrow[l, "x"'] \end{tikzcd}$$

It writes out as a three term complex, whose degree 0 cohomology is $R/(x, y)$:

$$0 \longrightarrow R \xrightarrow{\binom{x}{-y}} R^{\oplus 2} \xrightarrow{(x,y)} R \longrightarrow R/(x, y) \longrightarrow 0$$

4.4. Is the complex exact?

- Not necessarily, for example: $x = y = 0$.
- How about the case x, y are nonzero-divisors? In this case, the degree-2 homology is exact. However, if $x = y$, the degree-1 homology contains $(-1, 1)^T$, which in general does not come from image of $(x, -x)$, unless x is a unit.
- When x, y are units, it is easy to see the complex is exact.
- When x is a unit, the complex is exact if and only if x is regular.
- When x is a nonzero divisor, then $\mathrm{Kos}_\bullet(x, y)$ is exact if and only if y is regular in $R/(x)$.

4.5. The observations in the last three bullets are crucial. It tells us that, the sequence should either behave like transversal slices or units, or: multiplication by x_r should always be an injection on $R/(x_1, \dots, x_{r-1})$. To see why this is true, one writes out $\mathrm{Kos}_\bullet(x_1, \dots, x_r)$, by definition, it is the total complex of the following double complex

$$\begin{tikzcd} \vdots \arrow[d] & \vdots \arrow[d] \\ K_2 \arrow[d, "\mathrm{d}_2"'] & K_1 \arrow[l, "x_r"'] \arrow[d] \\ K_1 \arrow[d, "\mathrm{d}_1"'] & K_1 \arrow[l, "x_r"'] \arrow[d] \\ K_0 \arrow[d, "\mathrm{d}_0"'] & K_0 \arrow[l, "x_r"'] \\ R/(x_1, \dots, x_{r-1}) & \end{tikzcd}$$

[5] In literature, people would ask x to be non-unit. But here we don't care.

For example, take $(a,b) \in K_1 \oplus K_0 = \mathrm{Kos}_1(x_1, \cdots, x_r)$, if $\mathrm{d}_1 a + x_r b = 0$, then b is a zero-divisor of x_r in $R/(x_1, \cdots, x_{r-1})$, thus $b = 0$. Hence $\mathrm{d}_1 x = 0$, the exactness shows $a = \mathrm{d}_1(c)$, thus we shown the exactness at $\mathrm{Kos}_1(x_1, \cdots, x_r)$, etc.

4.6. In particular, let $R = k[x_1, \ldots, x_n]$ be the polynomial ring, the Koszul complex is a free resolution of k in degree $[0, n]$. Thus $\mathrm{Tor}_{R,n+1}(M, k) = 0$ for any R-modules M. The Koszul complex also shows $\mathrm{Tor}_{R,n}(k, k) = k \neq 0$. Were there a complex of projective modules $P_\bullet \to k$, let $K = \mathrm{Ker}(\mathrm{d}_n)$, then $0 = \mathrm{Tor}_{R,n+1}(k, k) = \mathrm{Tor}_{R,0}(K, k) = K \otimes k$, by Nakayama's lemma, homogeneous version or local version, we know $K = 0$. This result is called Hilbert's Syzygy's theorem. The word syzygy comes from Greek, it was originally used to describe straight-line configuration of three or more celestial bodies in a gravitational system[6], say the system

$$\texttt{the Sun} \longleftarrow \texttt{the Earth} \longleftarrow \texttt{the Moon} \longleftarrow \texttt{M.J.}$$

5. Künneth formula

5.1. The previous discussion may motivate us to ask: what's the relation of the homology of $P_\bullet, Q_\bullet$ and $(P_\bullet \otimes Q_\bullet)_\bullet$ in general? In the category of vector spaces, the dimensions multiply by taking convolution, in general, we need to take Tor into consideration.

5.2. This is our homework problem:

THEOREM 5.1. *Let $P_\bullet, Q_\bullet$ be chain complexes. Let's assume $Q_n, \mathrm{d}Q_n$ are all projective. Then we have a split short exact sequence*

$$0 \longrightarrow \bigoplus \mathrm{H}_p(P_\bullet) \otimes_R \mathrm{H}_{n-p}(Q_\bullet) \longrightarrow \mathrm{H}_n(P_\bullet \otimes Q_\bullet) \longrightarrow \bigoplus \mathrm{Tor}_{R,1}(\mathrm{H}_p(P_\bullet), \mathrm{H}_{n-1-p}(Q_\bullet)) \longrightarrow 0$$

5.3. First consider the simple question where P is a single term concentrated in degree 0. Thus the problem reduces to

$$0 \longrightarrow P \otimes_R \mathrm{H}_n(Q_\bullet) \longrightarrow \mathrm{H}_n(P \otimes Q_\bullet) \longrightarrow \mathrm{Tor}_{R,1}(P, \mathrm{H}_{n-1}(Q_\bullet)) \longrightarrow 0$$

Then we may want to make $Q_\bullet$ homologically better. For example we take the Cartan-Eilenberg resolution $A_{\bullet,j} \to Q_\bullet$. Tensoring with P, we get double complex

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
P \otimes A_{0,2} \arrow[d] & P \otimes A_{1,2} \arrow[l] \arrow[d] & P \otimes A_{2,2} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
P \otimes A_{0,1} \arrow[d] & P \otimes A_{1,1} \arrow[l] \arrow[d] & P \otimes A_{2,1} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
P \otimes A_{0,0} \arrow[d] & P \otimes A_{1,0} \arrow[l] \arrow[d] & P \otimes A_{2,0} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
P \otimes Q_0 & P \otimes Q_1 \arrow[l] & P \otimes Q_2 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

[6]Homological algebraists seem to a fun preference for astronomic jargons, while algebraic geometers love agriculture words. Will there be a branch of mathematics using aquaculture terms to name things in the future?

5.4. Let's first take the vertical differential, since P is projective, we get $E^v_{a,b} = 0$ for $a \neq 0$ and $E^v_{0,b} = P \otimes Q_b$. Then taking the horizontal differential, we get $E^{hv}_{0,b} = \mathrm{H}_b(P \otimes Q_\bullet)$ at $(0, b)$ and $E^{hv}_{a,b} = 0$ for $a \neq 0$. On the other hand, if we first take the horizontal differential, we get $E^h_{a,b} = P \otimes A_a^{\mathrm{H}_b(Q_\bullet)}$, where $A_a^{\mathrm{H}_b(Q_\bullet)}$ is the projective resolution of $\mathrm{H}_b(Q_\bullet)$. Then taking vertical differential we get $E^{vh}_{0,b} = P \otimes \mathrm{H}_n(Q_\bullet)$ and $E^{vh}_{1,b} = \mathrm{Tor}_1(P, \mathrm{H}_{n-1}(Q_\bullet))$. The other terms $E^{hv}_{k,b} = \mathrm{Tor}_k(P, \mathrm{H}_{n-1}(Q_\bullet))$ all vanish as $\mathrm{H}^{n-1}(Q_\bullet)$ has a projective resolution of length 1. Now we are in such a situation:

- We start from a double complex in first quadrant.
- By first apply vertical differential, then horizontal differential, we get a page $E^{hv}_{a,b}$ where all differentials are used up. It reads

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \cdots \\
\mathrm{H}_0(P \otimes Q_0) & \mathrm{H}_0(P \otimes Q_1) & \mathrm{H}_0(P \otimes Q_1) & \cdots
\end{array}$$

- Similarly, first applying horizontal differential, then the vertical one, we get the page $E^{vh}_{a,b}$:

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
0 & 0 & 0 & \cdots \\
\mathrm{Tor}_1(P, \mathrm{H}_0(Q_\bullet)) & \mathrm{Tor}_1(P, \mathrm{H}_1(Q_\bullet)) & \mathrm{Tor}_1(P, \mathrm{H}_2(Q_\bullet)) & \cdots \\
P \otimes \mathrm{H}_0(Q_\bullet) & P \otimes \mathrm{H}_1(Q_\bullet) & P \otimes \mathrm{H}_2(Q_\bullet) & \cdots
\end{array}$$

- What approximately calculated homology of a double complex in two ways, they should give "the same result". Intuitively, we think of these two results converges to the homology of the double complex by $E_k = \mathrm{H}_k(\mathrm{Tot}_\bullet)$. And formally write
$$E^{hv}_{a,b} \Rightarrow E_{a+b}, \quad E^{vh}_{a,b} \Rightarrow E_{a+b}.$$
- The desired short exact sequence comes from the general machine: spectral sequence.

5.5. Let's end the discussion by two remarks

- Note that the projectivity (flatness is enough) assumption on Q allows us to reduce $P_\bullet$ to a the cohomologies $P := \mathrm{H}^i(P_\bullet)$.[7]
- The splitness follows from projectivity of $\mathrm{d}Q_n$. Splitness is very useful in calculation. For example, when both sides are $\mathbb{Z}/2\mathbb{Z}$, splitness tells us the extension cannot be $\mathbb{Z}/4\mathbb{Z}$.[8]

[7] For details, see [**?**, 3.6]

[8] An element or order 4 may have interesting geometric and arithmetic implications.

- Finally, if we take $P_\bullet, Q_\bullet$ as the singular chain complex, $C_\bullet(X), C_\bullet(Y)$, note that $C_\bullet(X \times Y) = C_\bullet(X) \otimes C_\bullet(Y)$, we see that homology of $X \times Y$ is determined by homology of X and Y, at least in $\mathbb{Z}$ coefficients, so that the assumptions on Q are met.

6. Universal coefficient theorems

6.1. The Künneth formula can also be viewed as a way to change coefficients of homology. Taking $Q = C_\bullet(X)$, and M any $\mathbb{Z}$-module, we have

$$\mathrm{H}_n(X, M) \cong (\mathrm{H}_n(X) \otimes M) \oplus \mathrm{Tor}_{\mathbb{Z},1}(\mathrm{H}_{n-1}(X), M).$$

One naturally wonders if we can express $\mathrm{H}^n(X, M)$ in a similar manner?

6.2. Recall that $\mathrm{H}^n(X, M) = \mathrm{H}^n(C_\bullet(X), M)$. Taking Cartan-Eilenberg resolution and run the spectral sequence, similarly, we get

$$\mathrm{H}^n(X, M) \cong \mathrm{Hom}(\mathrm{H}_n(X), M) \oplus \mathrm{Ext}^1_{\mathbb{Z}(\mathrm{H}_{n-1}(X,M))}.$$