PERIOD OF THE GENERIC GENUS-g CURVE VIA DEGENERATION

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Abstract. We show that the period of the generic genus g curve ($g \geq 3$) equals to $2g-2$. We prove this by degenerating the generic genus g curve to stable curves with rational components.

1. Introduction

1.1. Let k be a field, let X be a smooth genus g curve defined over k. There are two basic invariants that measure the arithmetic complexity of X:

(1) the index of X: the greatest common divisor of the degrees of zero-cycles on X,
(2) the period of X: the order of the $\mathrm{Pic}^0_{X/k}$-torsor $[\mathrm{Pic}^1_{X/k}]$.

1.2. Let $g \geq 3$ be an integer. Let $\mathcal{M}_g$ ($\mathcal{M}_{g,1}$) be the moduli stack of smooth genus g curves (with a marked point). Forgetting the marked point induces a proper smooth morphism $\pi\colon \mathcal{M}_{g,1} \to \mathcal{M}_g$. Let us denote the generic fiber of π by X_g/k_g and call X_g the generic genus g curve. It is known that:

Theorem 1. *The period and the index of X_g both equal to $2g-2$.*

The result is a consequence of the strong Franchetta conjecture [Sch03], whose proof uses Moriwaki's calculation of the $\mathbb{Q}$-Picard group of the moduli space of pointed curves. In this paper, we reprove the result by direct degeneration arguments.

1.3. We will degenerate the generic genus g curve to stable curves with geometrically rational components. For simplicity, let us call such curves totally degenerate stable curves. We recall some facts from [ACG11, XII.12]:

Let us work over a fixed field k. Let $g \geq 3$ be an integer. Let $\Gamma = (V, E)$ be a graph of genus g, with degree at least four at all the vertices. Let $\overline{\mathcal{M}}_g$ be the moduli stack of genus g stable curves.

(1) The stack $\overline{\mathcal{M}}_g$ has a locally closed substack $\mathcal{D}_\Gamma$ that parameterizes families of totally degenerate stable curves, whose geometric fibers all have dual graph Γ.
(2) For any vertex $v \in V$ in Γ, let $E^+(v)$ be the set of half-edges connected to v. Let $\mathcal{M}_{0,E^+(v)}$ be the moduli scheme of families of smooth genus-zero curves with distinct $E^+(v)$-marked-points. Let us denote the product scheme $\prod_{v\in V} \mathcal{M}_{0,E^+(v)}$ by $\mathcal{M}_\Gamma$. Then $\mathcal{D}_\Gamma$ is isomorphic to the quotient stack $[\mathcal{M}_\Gamma/\mathrm{Aut}(\Gamma)]$.
(3) As $|E^+(v)| \geq 4$ for all $v \in V$, the $\mathrm{Aut}(\Gamma)$-action on $\mathcal{M}_\Gamma$ is generically free, and therefore $\mathcal{D}_\Gamma$ has an dense open integral subscheme. Let us denote the function field of $\mathcal{D}_\Gamma$ by k_Γ and denote the generic fiber of $\mathcal{X}_\Gamma := \overline{\mathcal{M}}_{g,1} \times_{\overline{\mathcal{M}}_g} \mathcal{D}_\Gamma \to \mathcal{D}_\Gamma$ by X_Γ/k_Γ. It is the generic "totally degenerate stable curve with dual graph Γ". The geometric irreducible

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components of X_Γ are not necessarily defined over k_Γ. However, let $\widetilde{k}_\Gamma$ be the function field of $\mathcal{M}_\Gamma$, then the base change $X_\Gamma \times_{k_\Gamma} \widetilde{k}_\Gamma$ splits into a union of rational curves.

(4) Let $v_0 \in V$ be a fixed vertex, let $\mathrm{Stab}(v_0)$ be the stabilizer of v_0 in $\mathrm{Aut}(\Gamma)$, then the quotient stack $\mathcal{D}_{(\Gamma,v_0)} := [\mathcal{M}_\Gamma/\mathrm{Stab}(v_0)]$ parameterizes families of totally degenerate stable curves with labelled dual graph (Γ, v_0), i.e., nodal union of rational curves with dual graph Γ and a distinguished component corresponding to v_0. Forgetting the distinguished component induces a covering morphism $[\mathcal{M}_\Gamma/\mathrm{Stab}(v_0)] \to [\mathcal{M}_\Gamma/\mathrm{Aut}(\Gamma)]$. Let $\mathcal{X}_{(\Gamma,v_0)} = \overline{\mathcal{M}}_{g,1} \times_{\overline{\mathcal{M}}_g} \mathcal{D}_{(\Gamma,v_0)} \to \mathcal{D}_{(\Gamma,v_0)}$ be the universal family, then its generic fiber $X_{(\Gamma,v_0)}/k_{(\Gamma,v_0)}$ is isomorphic to the base change $X_\Gamma \times_{k_\Gamma} k_{(\Gamma,v_0)}$.

1.4. Let us call a smooth curve of geometric genus zero a conic, and say a conic splits if it admits a rational section. Let us keep the notations in the last paragraph. Let $X^\nu_{(\Gamma,v_0)}$ be the normalization of $X_{(\Gamma,v_0)}$. Let $R_{(\Gamma,v_0)} = \mathrm{H}^0(X^\nu_{(\Gamma,v_0)}, \mathcal{O}_{X^\nu_{(\Gamma,v_0)}})$ be the ring of regular functions on $X^\nu_{(\Gamma,v_0)}$. The Stein factorization of $X^\nu_{(\Gamma,v_0)}/\mathrm{Spec}(k_{(\Gamma,v_0)})$ produces a conic $C_{(\Gamma,v_0)}$ over $\mathrm{Spec}(R_{(\Gamma,v_0)})$. Our degeneration argument of Theorem 1 is base on the following observation:

Theorem 2. *If there exists a vertex of $v_0 \in \Gamma$ and an order two automorphism of Γ that acts freely on $E^+(v_0)$, then the conic $C_{(\Gamma,v_0)}$ does not split. In particular, the distinguished component $Y \subset X_{(\Gamma,v_0)}$ corresponding to v_0 is a non-split conic.*

1.5. Period of curves are in general hard to determine, as rational points on Picard torsors are not necessarily represented by line bundles. The main novelty of this paper is that we overcome this difficulty by associating rational points on Picard schemes to rational points on certain conics, which are known to be non-split by Theorem 2.

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2. The associated conic

Let us work over a fixed field k_0. Let Γ be a graph, with degree at least four at all the vertices. Given a family of totally degenerate stable curves with dual graph Γ, we introduce its associated family of conics.

2.1. Let $\pi\colon \mathcal{X}_\Gamma \to \mathcal{D}_\Gamma$ be the universal family of curves with dual graph Γ. Since $\mathcal{D}_\Gamma$ is smooth, by local property of smooth morphism and normalization, we know that the usual normalization $\nu\colon (\mathcal{X}_\Gamma)^\nu \to \mathcal{X}_\Gamma$ is also fiber-wise normalization over $\mathcal{D}_\Gamma$. This operation extends to arbitrary families by base change.

Definition 1. *Let $f\colon X \to S$ be a family of totally degenerate curves with dual graph Γ. Let $\varphi\colon S \to \mathcal{D}_\Gamma$ be the classifying morphism induced by f. We define the normalization f^ν of f to be the base change of $\pi \circ \nu$ along φ, namely $f^\nu := (\pi\circ\nu)_\varphi\colon (\mathcal{X}_\Gamma)^\nu \times_{\mathcal{D}_\Gamma,\varphi} S \to S$.*

Let $f\colon X \to S$ be a family of totally degenerate curves with a fixed dual graph Γ. Let $f^\nu\colon X^\nu \to S$ be its normalization. The geometric components of fibers of f^ν are smooth rational curves, and therefore the Stein factorization $f^{\nu,st}\colon X^\nu \to \underline{\mathrm{Spec}}((f^\nu)_*\mathcal{O}_{X^\nu})$ is a conic bundle [Har77, 11.5].

Definition 2. *We call $f^{\nu,st}$ the conic associated with the family f.*

2.2. The construction commutes with arbitrary base change:

Lemma 3. *Let $f\colon X \to S$ be a family of totally degenerate curves with a fixed dual graph. Let $T \to S$ be a morphism, let $f_T\colon X_T \to T$ be the base change of f along T, then $(f^{\nu,st})_T = (f_T)^{\nu,st}$*

Proof. The construction of f^ν is defined via base change, so it commutes with base change. Notice that $h^i(X_s^\nu, \mathcal{O}_{X_s})$ is constant for $s \in S$, and therefore the construction of $(f^\nu)_*$ commutes with base change. $\square$

3. Proof of Theorem 2

Let k_0 be a fixed field. Let $\Gamma = (V, E)$ be a graph with degree at least four at each vertex. Let $v_0 \in V$ be a distinguished vertex. Let $f\colon X_{(\Gamma,v_0)} \to \mathrm{Spec}(k_{(\Gamma,v_0)})$ be the generic curve with dual graph Γ and a distinguished component corresponding to v_0. Let $f^{\nu,st}\colon C_{(\Gamma,v_0)} \to \mathrm{Spec}(R_{(\Gamma,v_0)})$ be the associated conic. We show that the conic is non-split under the assumption of Theorem 2.

3.1. Let us reduce the problem to finding specific curves over global fields. In the following discussion we will consider the generic curve over different bases. For a commutative ring B, let us denote the generic "curve over $\mathrm{Spec}(B)$" with labelled dual graph (Γ, v_0) by $X_{(\Gamma,v_0),B}$, and denote the associated conic by $C_{(\Gamma,v_0),B}$, etc.

Proposition 4. *If the conic $C_{(\Gamma,v_0),k_0}/\mathrm{Spec}(R_{(\Gamma,v_0),k_0})$ splits, then there exists a global field M such that the conic $C_{(\Gamma,v_0),M}/\mathrm{Spec}(R_{(\Gamma,v_0),M})$ splits.*

Proof. Let $\mathbb{F}$ be the prime field of k_0 (i.e., $\mathbb{Q}$ or $\mathbb{F}_p$). If $C_{(\Gamma,v_0),k_0}$ splits, then we can pick a section $s\colon \mathrm{Spec}(R_{(\Gamma,v_0),k_0}) \to C_{(\Gamma,v_0),k_0}$. By Lemma 3, we know $C_{(\Gamma,v_0),k_0} = (C_{(\Gamma,v_0),\mathbb{F}})_{k_0}$, therefore s is defined over a finite type $\mathbb{F}$-subalgebra $A \subset k_0$. Namely, the section s is the base change of a section $s_A\colon \mathrm{Spec}(R_{(\Gamma,v_0),A}) \to C_{(\Gamma,v_0),A}$. If $\mathrm{char}(k_0) = 0$, we take a maximal ideal $\mathfrak{m}$ of A, then $M := A/\mathfrak{m}$ is a finite extension of $\mathbb{Q}$. The section s_A specializes to a section $s_{A/\mathfrak{m}}\colon \mathrm{Spec}(R_{(\Gamma,v_0),M}) \to C_{(\Gamma,v_0),M}$, and therefore $C_{(\Gamma,v_0),M}$ splits. If $\mathrm{char}(k_0) > 0$, we take a dominant map from $\mathrm{Spec}(A)$ to a $\mathbb{A}^1_{\mathbb{F}}$, replace A by its localization at the generic point of $\mathbb{A}^1_{\mathbb{F}}$ and argue as before. $\square$

Proposition 5. *If the conic $C_{(\Gamma,v_0),k_0}$ splits, then for any field extension k/k_0, and any totally degenerate stable curve $h\colon X \to k$ with labelled dual graph (Γ, v_0), the conic $h^{\nu,st}$ splits.*

Proof. By Lemma 3, we may assume $k = k_0$ and work over k_0 in the rest of the proof. Let $g\colon \mathcal{X}_{(\Gamma,v_0)} \to \mathcal{D}_{(\Gamma,v_0)}$ be the universal family of stable curves with labelled dual graph (Γ, v_0). Let $[X]$ be the closed point in $\mathcal{D}_{(\Gamma,v_0)}$ that corresponds to the curve X. Let $d = \dim \mathcal{D}_{(\Gamma,v_0)}$. By [Sta19, 0DR0], there exists a flat morphism $\varphi\colon \mathrm{Spec}(k[[t_1, \dots, t_d]]) \to \mathcal{D}_{(\Gamma,v_0)}$ centered at $[X]$. Let g_φ be the family of curves over $\mathrm{Spec}(k[[t_1, \dots, t_d]])$ induced by φ. Let C' be the generic fiber of $(g_\varphi)^{\nu,st}$. Since φ is flat, the generic point of $\mathrm{Spec}(k[[t_1, \dots, t_d]])$ maps to the generic point of $\mathcal{D}_{(\Gamma,v_0)}$. If $C_{(\Gamma,v_0)}$ splits, then its base change C' splits. Any rational section of C' yields a section of $h^{\nu,st}$ by successive specialization along the chain of discrete valuation rings: $k((t_1, \dots, t_{d-1}))[[t_d]] \rightsquigarrow k((t_1, \dots, t_{d-2}))[[t_{d-1}]] \rightsquigarrow \cdots \rightsquigarrow k[[t_1]]$. $\square$

3.2. Given the datum as in Theorem 2, we construct the desired stable curve, whose associated conic does not split. Let us call graphs that have degree at least three at each vertex stable graphs.

Lemma 6. *Let L/K be a Galois extension. Let Γ be a stable graph with an action by $\mathrm{Gal}(L/K)$. Assume K is infinite, then there exists a totally degenerate stable curve X, such that there is a Galois equivariant isomorphism between Γ and the dual graph of X.*

Proof. See [Pal04, Proposition 4.6]. □

Proposition 7. *Let $\Gamma = (V, E)$ be a stable graph. Let $v_0 \in V$ be a vertex, let σ be an order-2 element in $\mathrm{Stab}(v_0) \subset \Gamma$, such that σ acts freely on $E^+(v_0)$. Let K be a global field. Then there exists a totally degenerate stable curve with dual graph Γ defined over K, such that the associated conic does not split. In particular, the component corresponding to v_0 is a non-split conic.*

Proof. By global class field theory, there exists an element $\gamma \in \mathrm{Br}(K)$ of order 2 that is represented by a conic C. Let L/K be a quadratic extension over which C splits. Let $\tau \in \mathrm{Gal}(L/K)$ be the generator.

By Lemma 6, there exists a totally degenerate curve X over K such that τ acts as σ on the dual graph. Let $Y \subset X_L$ be the component corresponding to v_0. Since v_0 is Galois invariant, we know that the subscheme Y descends to K. Therefore we may write $X = Y \coprod_{Y\cap Z} Z$, where $Z = \overline{X\backslash Y}$. Since σ acts freely on $E^+(v_0)$, the nodes $Y \cap Z \subset Y$ is a disjoint union of closed points with residue field L. Since $C_L \cong \mathbb{P}^1_L$, we may pick an embedding $Y \cap Z \hookrightarrow C$, then $C \coprod_{Y\cap Z} Z$ give the desired curve. □

3.3. Now we assemble our discussion

Proof of Theorem 2. By Proposition 4, Proposition 5, we reduce the problem to finding a totally degenerate stable curve defined over a global field, with labelled dual graph (Γ, v_0) and non-split associated conic. This is constructed in Proposition 7. □

4. Non-triviality of Picard torsors

Let k be a field. For any totally degenerate stable curve X defined over k, let $\mathrm{Pic}^{\mathbf{1}}_{X/k}$ be the component of the Picard scheme that parameterizes line bundles with degree one on each geometric irreducible component. The scheme $\mathrm{Pic}^{\mathbf{1}}_{X/k}$ is a torsor of the Picard torus $\mathrm{Pic}^0_{X/k}$. We show that the torsor is non-trivial in certain cases.

4.1. Let us consider the generic genus g curve with the following dual graph $\Gamma = (V, E)$ (intuitively, the graph is a $(g-1)$-gon with an extra loop at each vertex):

- the set of vertices $V = \{v_i\}_{i\in\mathbb{Z}/(g-1)\mathbb{Z}}$,
- the set of edges $E = \{e_{\{i,i+1\}}, e_{\{i\}}\}_{i\in\mathbb{Z}/(g-1)\mathbb{Z}}$, where the edge $e_{\{i,i+1\}}$ connects v_i and v_{i+1}, the edge $e_{\{i\}}$ is a loop at v_i.

4.2. We keep the notations as in Section 1.3. Let $\widetilde{k}_\Gamma$ be the function field of $\mathcal{M}_\Gamma$, then $k_\Gamma = (\widetilde{k}_\Gamma)^{\mathrm{Aut}(\Gamma)}$ and $k_{(\Gamma,v_0)} = (\widetilde{k}_\Gamma)^{\mathrm{Stab}(v_0)}$. Let $Y \subset X_\Gamma \times_{k_\Gamma} k_{(\Gamma,v_0)}$ be the component that corresponds to v_0.

Proposition 8. *Let us work over a field k_0. Let X_Γ/k_Γ be the generic genus g curve with dual graph Γ, then $\mathrm{Pic}^{\mathbf{1}}_{X_\Gamma/k_\Gamma}$ does not have k_Γ-rational points.*

Proof. It suffices to show that the base change $(\mathrm{Pic}^{\mathbf{1}}_{X_\Gamma/k_\Gamma})_{k_{(\Gamma,v_0)}}$ admits no $k_{(\Gamma,v_0)}$-rational points. Restriction of line bundles gives us a morphism $(\mathrm{Pic}^{\mathbf{1}}_{X_\Gamma/k_\Gamma})_{k_{(\Gamma,v_0)}} \to \mathrm{Pic}^1_{Y/k_{(\Gamma,v_0)}}$, so any

k_Γ-rational point of $\mathrm{Pic}^{\mathbf{1}}_{X_\Gamma/k_\Gamma}$ would give rise to a $k_{(\Gamma,v_0)}$ rational point of $\mathrm{Pic}^1_{Y/k_{(\Gamma,v_0)}}$. Notice that Y is a nodal curve, the Abel-Jacobi map $Y^{\mathrm{sm}} \to \mathrm{Pic}^1_{Y/k_{(\Gamma,v_0)}}, y \mapsto \mathcal{O}_Y(y)$ induces an isomorphism from the smooth locus of Y to its Picard scheme. Therefore, it suffices to show that Y^ν admits no $k_{(\Gamma,v_0)}$-rational points. This follows from Theorem 2. $\square$

5. The period and the index

Let us work over a fixed base field k_0. Let $g \geq 3$ be an integer. Let k be the function field of $\mathcal{M}_g$. Let X/k be the generic genus-g curve. We show that the period of X and the index of X both equal to $2g-2$.

The period of the generic genus g curve is shown to be divisible by $g-1$ [Mes87], we give a different proof, by associating $[\mathrm{Pic}^1_{X/k}]$ with the Brauer obstruction class to tautological line bundles on $X \times \mathrm{Pic}^0_{X/k}$.

Lemma 9. *The order of the torsor* $[\mathrm{Pic}^1_{X/k}] \in \mathrm{H}^1(k, \mathrm{Pic}^0_{X/k})$ *is divisible by* $g-1$.

Proof. Let us denote $\mathrm{Pic}^0_{X/k}$ by P and consider the Hochschild-Serre spectral sequence

$$\mathrm{H}^i(k, \mathrm{H}^j(P_{k^{\mathrm{sep}}}, \mathbb{G}_m)) \Rightarrow \mathrm{H}^{i+j}(P, \mathbb{G}_m).$$

Let $0 \subset \mathrm{Fil}^0\mathrm{Br}(P) = \mathrm{Br}(P) \subset \mathrm{Fil}^1\mathrm{Br}(P) \subset \mathrm{Fil}^2\mathrm{Br}(P) = \mathrm{Br}(P)$ be the induced filtration on $\mathrm{Br}(P)$. Then we have $\frac{\mathrm{Fil}^1\mathrm{Br}(P)}{\mathrm{Br}(k)} \cong \ker(\mathrm{d}_2^{1,1}) \subseteq \mathrm{H}^1(k, \mathrm{Pic}_{P/k})$ and $\frac{\mathrm{Br}(P)}{\mathrm{Fil}^1\mathrm{Br}(P)} \subseteq \mathrm{Br}(P_{k^{\mathrm{sep}}})^G$.

Let $\alpha \in \mathrm{Br}(P)$ be the Brauer obstruction to the tautological line bundle on $X \times P$. The obstruction vanishes after base change to k^{sep} [Bos90, 8.2], thus $\alpha \in \mathrm{Fil}^1\mathrm{Br}(P)$. Let $\{e\} \subset P$ be the identity section. Since tautological line bundle exists on $X \times \{e\}$ (trivial line bundle), we have $\langle \alpha \rangle \cap \mathrm{Br}(k) = 0$. Therefore the order of α equals to the order of its image $\overline{\alpha}$ in $\mathrm{H}^1(k, \mathrm{Pic}_{P/k})$. By [Ma21, Example 5.2], we know that the order of α, hence the order of $\overline{\alpha}$ equals to $g-1$.

Let us fix a point $x_0 \in X(k^{\mathrm{sep}})$, and consider the morphism $f\colon X_{k^{\mathrm{sep}}} \to P_{k^{\mathrm{sep}}}, x \mapsto \mathcal{O}_{X_{k^{\mathrm{sep}}}}(x - x_0)$. The pullback induces an isomorphism $\mathrm{Pic}^0_{P_{k^{\mathrm{sep}}}/k^{\mathrm{sep}}} \to \mathrm{Pic}^0_{X_{k^{\mathrm{sep}}}/k^{\mathrm{sep}}}$ by auto-duality of Jacobian. By the theorem of the square, the isomorphism is independent of x_0, and its inverse descends to a well-defined k-isomorphism $J\colon P \to \mathrm{Pic}^0_{P/k}$. Let $\phi\colon \mathrm{H}^1(k, P) \cong \mathrm{H}^1(k, \mathrm{Pic}^0_{P/k}) \to \mathrm{H}^1(k, \mathrm{Pic}_{P/k})$ be the homomorphism induced by J and inclusion of identity component. One explicitly checks that $\phi([\mathrm{Pic}^1_{X/k}]) = -\overline{\alpha}$:

(1) The torsor $[\mathrm{Pic}^1_{X/k}]$ is represented by the 1-cocyle $c\colon G \to P(k^{\mathrm{sep}}), \sigma \mapsto \mathcal{O}_{X_{k^{\mathrm{sep}}}}(\sigma(x_0) - x_0)$, therefore $\phi([\mathrm{Pic}^1_{X/k}])$ is represented by $\sigma \mapsto J(\mathcal{O}_{X_{k^{\mathrm{sep}}}}(\sigma(x_0) - x_0))$.

(2) Let us also represent $\overline{\alpha}$ by a 1-cocycle. For any $x \in X(k^{\mathrm{sep}})$, let L_x be the tautological line bundle on $X \times P_{k^{\mathrm{sep}}}$ that is trivialized along $\{x\} \times P_{k^{\mathrm{sep}}}$. Applying the see-saw lemma to $\pi_2\colon X \times P_{k^{\mathrm{sep}}} \to P_{k^{\mathrm{sep}}}$, we may write the difference $L_{\sigma(x_0)} \otimes L_{x_0}^\vee \cong \pi_2^* N_\sigma$ for some $N_\sigma \in \mathrm{Pic}(P_{k^{\mathrm{sep}}})$. Then $\overline{\alpha}$ is represented by $\sigma \mapsto N_\sigma$.

(3) By auto-duality, the line bundle L_{x_0} is determined by its restriction to $X \times X_{k^{\mathrm{sep}}}$ via $1 \times f$. By our definition of f and L_{x_0}, we have $(1 \times f)^* L_{x_0} \cong \mathcal{O}_{X \times X_{k^{\mathrm{sep}}}}(\Delta - \{x_0\} \times X - X \times \{x_0\})$. Restricting to $\{\sigma(x_0)\} \times P_{k^{\mathrm{sep}}}$, we know that $L_{x_0}|_{\{\sigma(x_0)\} \times P_{k^{\mathrm{sep}}}} \cong J(\mathcal{O}_{X_{k^{\mathrm{sep}}}}(\sigma(x_0) - x_0))$.

(4) Restricting $L_{\sigma(x_0)} \otimes L_{x_0}^\vee \cong \pi_2^* N_\sigma$ to $\{\sigma(x_0)\} \times P_{k^{\mathrm{sep}}}$, we get $N_\sigma \cong L_{x_0}^\vee|_{\{\sigma(x_0)\} \times P_{k^{\mathrm{sep}}}}$. By (1)(2)(3), we see that the 1-cocyles representing $\phi([\mathrm{Pic}^1_{X/k}])$ and $-\overline{\alpha}$ coincide.

Therefore, the order of $[\mathrm{Pic}^1_{X/k}]$ is divisible by $g-1$, the order of $\overline{\alpha}$. $\square$

Proposition 10. *The torsor* $[\mathrm{Pic}^1_{X/k}] \in \mathrm{H}^1(k, \mathrm{Pic}^0_{X/k})$ *has order* $2g-2$.

Proof. Notice that $[\mathrm{Pic}^{2g-2}_{X/k}]$ is a trivial $\mathrm{Pic}^0_{X/k}$-torsor, by Lemma 9 it suffices to show that the class $[\mathrm{Pic}^{g-1}_{X/k}]$ is non-trivial, or that $\mathrm{Pic}^{g-1}_{X/k}$ admits no k-rational points.

Let Γ be the graph as in Section 4.1. By Proposition 8, the torsor $\mathrm{Pic}^{\mathbf{1}}_{X_\Gamma/k_\Gamma}$ does not have a rational point. Let $R_1 = k_\Gamma[[t_1]]$. By [Bak08, B.2], we may find a regular surface $\mathcal{X}$ over $\mathrm{Spec}(R_1)$, with special fiber X_Γ/k_Γ and smooth generic fiber X_1/k_1. It suffices to show that $\mathrm{Pic}^{g-1}_{X_1/k_1}$ admit no rational points, because rational points on $\mathrm{Pic}^{g-1}_{X/k}$ yield rational points on $\mathrm{Pic}^{g-1}_{X_1/k_1}$ by successive specialization as in Proposition 5.

Let R_1^{sh} be the strict henselization of R_1. Suppose that Q_1 is a rational point on $\mathrm{Pic}^{g-1}_{X_1/k_1}$, represented by a line bundle L on the generic fiber of $(\mathcal{X})_{R_1^{\mathrm{sh}}}$. Since $\mathcal{X}$ is regular, so is $(\mathcal{X})_{R_1^{\mathrm{sh}}}$, and therefore the line bundle L extends to $(\mathcal{X})_{R_1^{\mathrm{sh}}}$ by taking closure of the corresponding Weil divisor. The restriction of the line bundle to X_Γ/k_Γ yields a rational point Q_0 on $\mathrm{Pic}_{X_\Gamma/k_\Gamma}$. Notice that Q_0 is $\mathrm{Gal}(\widetilde{k_\Gamma}/k_\Gamma)$-invariant, and that all the components of $(X_\Gamma)_{\widetilde{k_\Gamma}}$ are Galois conjugate, thus Q_0 lands in $\mathrm{Pic}^{\mathbf{1}}_{X_\Gamma/k_\Gamma}$. This contradicts Proposition 8. ☐

The index can be easily calculated from the period.

Proposition 11. *The index of* X *equals to* $2g-2$.

Proof. Canonical divisors provide zero-cycles of degree $2g-2$ on X, thus the index divides $2g-2$. On the other hand, the period divides the index [LT58], so we conclude from Proposition 10. ☐

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