

# SPECIALIZING BRAUER CLASSES IN MODULI SPACES

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ABSTRACT. Let  $X$  be a smooth projective curve defined over a field  $k$ . The existence of tautological line bundles on  $X \times \mathrm{Pic}_{X/k}$  is obstructed by a Brauer class  $\alpha \in \mathrm{Br}(\mathrm{Pic}_{X/k})$ . We show that  $\alpha$  splits at the generic point of various naturally defined loci in  $\mathrm{Pic}_{X/k}$ —the theta divisor and the generalized theta divisors associated with degree  $2g - 2$ , semi-stable rank 2 vector bundles on  $X$ . Motivated by Franchetta’s theorem [Sch03], we ask if such vector bundle (uniquely) exists on the generic genus  $g$  curve. We show by explicit construction that degree  $2g - 2$ , semi-stable rank 2 vector bundles on the generic genus  $g$  curve abounds.

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## 1. INTRODUCTION

1.1. The Brauer group of algebraic varieties is a central object of study in arithmetic and algebraic geometry. To name a few, the Tate conjecture can be rephrased in terms of finiteness of Brauer groups [Tat68]; for certain classes of varieties over global fields, the Brauer-Manin obstruction is shown to be the only obstruction to the local-global principle [Bor96], and therefore can be used to study the existence of rational points; Brauer classes and their further generalization, the unramified cohomology classes, have been powerful tools in the study of the rationality problems [AM72] and algebraic cycles [Sch20]. Despite the usefulness, Brauer classes are in general hard to explicitly write down and work with.

1.2. There is a natural source of Brauer classes that come from geometry: the obstruction to the Zariski-representability of various moduli functors gives rise to Brauer classes over the moduli spaces. To be concrete: Let  $k$  be a field, let  $X$  be a projective variety defined over  $k$ . Let  $M$  be the moduli space of stable sheaves on  $X$  with fixed rank and chern classes. Then the existence of tautological sheaves on  $X \times_k M$  is obstructed by a Brauer class  $\alpha \in \mathrm{Br}(M)$ , see [Cal00, I.3.3]. The Fourier-Mukai equivalences between twisted derived categories has been a powerful tool for studying the geometry of  $M$  ever since [Cal00]. On the other hand, the Brauer classes provide a rich source of candidates for various conjectures on the structure of division algebras, most notably the cyclicity problem [ABGV11, 1.1].

1.3. We study the specializations of the Brauer obstruction class to various loci in the moduli space. In many natural cases, the locus can be characterized by failure to cohomological flatness of twisted sheaves. We base change to the locus, and analyze the exact sequences of cohomologies. Failure of cohomological flatness indicates that some (higher) direct images of interesting rank will pop out, giving us information about the specialized Brauer class. Understanding the specialized Brauer class should be useful in a wide range of contexts, where the Brauer-class-related properties are preserved by specialization: e.g., stable rationality [NS19], existence of rational points, cyclicity, etc.

Let us illustrate our idea in the case of curves. Let  $X$  be a smooth projective curve defined over a field  $k$ . Let  $d$  be an integer, let  $M = \text{Pic}_{X/k}^d$  be the moduli space of degree  $d$  line bundles on  $X$ , and let  $\alpha_d$  be the obstruction class. While this class is nonzero in general [MV14, Theorem 6.4], we show that it restricts to zero at the generic point of the naturally defined locus—theta divisor. Let  $\Theta \subset \text{Pic}_{X/k}^{g-1}$  be the theta divisor, let  $k(\Theta)$  be its function field, we prove that:

**Theorem.** *The Brauer class  $\alpha_{g-1} \in \text{Br}(\text{Pic}_{X/k}^{g-1})$  restricts to zero in  $\text{Br}(k(\Theta))$ .*

The proof goes by explicitly finding a section in the Brauer-Severi variety that represents  $\alpha_{g-1}$  (essentially, showing the existence of an  $\alpha_{g-1}$ -twisted sheaf of rank 1), which rises from the cohomological characterization of the theta divisor.

1.4. In [Ray82], Raynaud defined generalized theta divisors  $\Theta_E \subset \text{Pic}_{X/k}^0$  associated with degree  $2g-2$ , semi-stable rank 2 vector bundles  $E$  on  $X$  (see Section 4.4). With the previous theorem in mind, one naturally expects that the Brauer class  $\alpha_0$  splits at the generic point of  $\Theta_E$ . Unlike the usual theta divisor  $\Theta$ , the generalized theta divisor  $\Theta_E$  depends on the vector bundle  $E$ , which lacks a canonical choice. Is it possible to choose one, generically? Let us recall Franchetta's theorem:

**Theorem** ([Sch03, 5.1]). *Let  $g \geq 3$  be an integer, let  $C_g$  be the generic genus  $g$  curve, i.e., the generic fiber of the forgetful map  $\mathcal{M}_{g,1} \rightarrow \mathcal{M}_g$ . Then the canonical bundle  $\omega_{C_g}$  is the only line bundle of degree  $2g-2$  on  $C_g$ .*

One naturally asks if the theorem has a rank 2 analogue: Does there exist a degree  $2g-2$ , semi-stable rank 2 vector bundle on the generic genus  $g$  curve? If so, is it unique? Working over any base field, we prove that

**Theorem.** *Let  $g \geq 3$  be an integer. Let  $k$  be the function field of  $\mathcal{M}_g$  and let  $X/k$  be the generic genus  $g$  curve. Then there exist degree  $2g-2$ , semi-stable rank 2 vector bundles on  $X$ . For any such vector bundle  $E$ , the generalized theta divisor  $\Theta_E \subset \text{Pic}_{X/k}^0$  is reduced and irreducible. The Brauer class  $\alpha_0 \in \text{Pic}_{X/k}^0$  restricts to zero at the generic point of  $\Theta_E$ .*

The construction if  $E$  is given by carefully modifying the rank 2 vector bundle  $\omega_X^{\oplus 2}$  along an effective canonical divisor  $D \in |\omega_X|$ . To show that  $E$  is semi-stable, we use the Franchetta theorem for line bundles, which allows us to check stability condition for very limited cases of subsheaves of  $E$ .

1.5. Our method of detecting the Brauer class in the cohomologically defined locus, of determining the irreducibility of the generalized theta divisors, and our construction of semi-stable vector bundles on the generic curve maybe helpful to further studies of arithmetic properties of the generic objects over various moduli spaces.

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## 2. PRELIMINARIES

**2.1. The Picard scheme.** We collect some facts from [Bos90, 8]. Let  $k$  be a field, let  $X$  be a smooth proper curve defined over  $k$ . Let us consider the relative Picard functor

$$P'_{X/k}: \text{Sch}/k \rightarrow \text{Sets}, T \mapsto \text{Pic}(X \times_k T)/\text{pr}_2^* \text{Pic}(T).$$

Let  $P_{X/k}$  be the étale sheafification of  $P'_{X/k}$ . The functor  $P_{X/k}: (\text{Sch}/k)_{\text{ét}} \rightarrow \text{Sets}$  is represented by a group scheme

$$\text{Pic}_{X/k} = \bigsqcup_{d \in \mathbb{Z}} \text{Pic}_{X/k}^d.$$

The identity component  $\text{Pic}_{X/k}^0$  is an abelian variety, the components  $\text{Pic}_{X/k}^d$  are torsors of  $\text{Pic}_{X/k}^0$ . By the representability of  $P_{X/k}$ , we mean that there exists a tautological line bundle on  $X \times_k U$ , where  $U$  is some étale cover of  $\text{Pic}_{X/k}$ . By the representability of  $P'_{X/k}$ , we mean that there exists a tautological line bundle on  $X \times_k \text{Pic}_{X/k}$ . The functor  $P'_{X/k}$  is representable when  $X$  has a  $k$ -rational point.

**2.2. The Brauer class.** Let us consider the Leray spectral sequence of the étale sheaf  $\mathbb{G}_m$ , along the projection  $\pi: X \times \text{Pic}_{X/k} \rightarrow \text{Pic}_{X/k}$ . The low-degree terms fit into an exact sequence

$$0 \longrightarrow \text{Pic}(\text{Pic}_{X/k}) \xrightarrow{\pi^*} \text{Pic}(X \times \text{Pic}_{X/k}) \xrightarrow{e} \text{Mor}(\text{Pic}_{X/k}, \text{Pic}_{X/k}) \xrightarrow{d_2^{0,1}} \text{Br}(\text{Pic}_{X/k}),$$

where  $\pi^*$  is the pullback, and the middle morphism  $e$  sends a line bundle  $L$  on  $X \times \text{Pic}_{X/k}$  to the morphism  $e(L): \text{Pic}_{X/k} \rightarrow \text{Pic}_{X/k}, b \mapsto L|_{X \times b}$ . The obstruction to the existence of tautological line bundles on  $X \times \text{Pic}_{X/k}$  is given by the class

$$\alpha := d_2^{0,1}(\text{id}_{\text{Pic}_{X/k}}) \in \text{Br}(\text{Pic}_{X/k}).$$

Let  $d$  be an integer. Let us denote the restriction of  $\alpha$  to  $\text{Pic}_{X/k}^d$  by  $\alpha_d$ . This class is in general nontrivial when  $k$  is not algebraically closed. For example, let  $k$  be the function field of  $\mathcal{M}_{g,\mathbb{C}}$  and let  $X/k$  be the generic genus  $g$  curve, then the period of  $\alpha_d$  equals to  $\gcd(2g-2, d-g+1)$ , see [MV14, Theorem 6.4].

## 3. SPECIALIZING TO $\Theta$

**3.1.** Let  $f: Y \rightarrow S$  be a dominant morphism of algebraic varieties over an algebraically closed field. It is in general hard to tell if rational sections to  $f$  exist. However, we have:

**Proposition 1** ([hM]). *There always exist codimension-one points  $\eta \in S$  such that the restriction  $f_\eta: Y_\eta \rightarrow \eta$  admits a section.*

*Proof.* Since the generic fiber of  $f$  admits separable multi-sections, taking closure of the multi-section and shrinking  $S$ , we may assume that  $S$  is smooth and that  $f$  is a finite étale morphism of degree  $d$ . Let  $p \in S$  be a closed point and let  $f^{-1}(p) = \{q_1, \dots, q_d\}$  be the inverse image. We take an irreducible divisor  $D \subset Y$  that passes through  $q_1$ , smooth at  $q_1$ , but does not pass through  $\{q_2, \dots, q_d\}$ , then  $f|_D: D \rightarrow S$  is generically of degree one, and

therefore  $f|_D: D \rightarrow f(D)$  is birational. Let  $\eta$  be the generic point of  $f(D)$ , then  $D_\eta$  is a section to  $f_\eta$ .  $\square$

3.2. The general discussion does not provide an effective way to check if a variety admits rational sections over an explicitly given codimension-one point. We will explicitly check it in the following specific case of our interest:

Let  $k$  be a field, let  $X$  be a smooth projective genus  $g$  curve defined over  $k$ . Let  $f: \mathcal{P} \rightarrow \text{Pic}_{X/k}^{g-1}$  be a Brauer-Severi scheme that represents the class  $\alpha_{g-1}$ . We show that the family  $f$ , when specialized to the generic point of the theta divisor, admit rational sections. Equivalently, we show that the class  $\alpha \in \text{Br}(\text{Pic}_{X/k})$  restricts to zero at generic point of the theta divisor  $\Theta \subset \text{Pic}_{X/k}^{g-1}$ .

3.3. **The theta divisor.** The divisor  $\Theta$  can be defined as the image of the Abel-Jacobi morphism  $\text{Sym}^{g-1}(X) \rightarrow \text{Pic}_{X/k}^{g-1}$ . Let us recall another construction via tautological line bundles, see [ACGH85, IV.3] for details.

Let  $U$  be an étale cover of  $\text{Pic}_{X/k}^{g-1}$ , so that a tautological line bundle  $L$  exist on  $X \times_k U$ . Let  $\pi: X \times_k U \rightarrow U$  be the projection. Let us choose an effective canonical divisor  $Z \in |\omega_X|$ . Let  $Z_U = Z \times_k U \subset X \times_k U$ . They fit in the diagram:

$$\begin{array}{ccc} & Z_U \hookrightarrow & X \times_k U \\ & \swarrow & \searrow \pi \\ Z \hookrightarrow & X & U. \end{array}$$

Let us consider the short exact sequence on  $X \times_k U$ :

$$0 \longrightarrow L(-Z_U) \longrightarrow L \longrightarrow L|_{Z_U} \longrightarrow 0.$$

Taking direct images along  $\pi_*$ , we get an exact sequence of sheaves on  $U$ :

$$0 \longrightarrow \pi_* L \longrightarrow \pi_*(L|_{Z_U}) \xrightarrow{\delta_U} R^1 \pi_*(L(-Z_U)) \longrightarrow R^1 \pi_* L \longrightarrow 0. \quad (*)$$

Let us denote  $\pi_*(L|_{Z_U})$  and  $R^1 \pi_*(L(-Z_U))$  by  $F_U$  and  $G_U$ . By cohomology and base change, we know that  $F_U, G_U$  are both locally free of rank  $2g-2$ . The subsheaf  $\pi_* L \subset F_U$  is torsion, so  $\pi_* L = 0$ . Let  $\delta_U: F_U \rightarrow G_U$  be the connecting homomorphism, then  $\det(\delta_U)$  induces a nonzero section  $s_U: \mathcal{O}_U \rightarrow \det(F_U)^\vee \otimes \det(G_U)$ . The line bundle  $\det(F_U)^\vee \otimes \det(G_U)$  and section  $s_U$  descend along the cover  $U \rightarrow \text{Pic}_{X/k}^{g-1}$ . The vanishing locus  $T_U$  of  $s_U$  descends to a closed subscheme  $T \subset \text{Pic}_{X/k}^{g-1}$ . We call  $T$  the theta divisor, and denote it by  $\Theta$ .

3.4. Let us denote the base change  $\Theta \times_{\text{Pic}_{X/k}} U$  by  $\Theta_U$ . Let  $\pi': X \times_k \Theta_U \rightarrow \Theta_U$  and  $L' = L|_{X \times_k \Theta_U}$  be the restrictions. Let  $Z' = Z \times_k \Theta_U \subset X \times_k \Theta_U$ . Similar to (\*), we have an exact sequence of sheaves on  $\Theta_U$ :

$$0 \longrightarrow \pi'_* L' \longrightarrow \pi'_*(L'|_{Z'}) \xrightarrow{\delta'} R^1 \pi'_*(L'(-Z')) \longrightarrow R^1 \pi'_* L' \longrightarrow 0. \quad (**)$$

The sequence (\*\*) is not the base change of (\*) to the closed subscheme  $\Theta_U \subset U$ , because  $L$  is not cohomologically flat. Here is a classical result:

**Lemma 2.** *The coherent sheaf  $\pi'_* L'$  is torsion free rank 1.*

*Proof.* The sheaf  $\pi_* L'$  is torsion free as a subsheaf of the locally free sheaf  $\pi'_*(L'|_{Z'})$ . We check that the connecting homomorphism  $\delta'$  in  $(**)$  has corank 1: Given a point  $t \in \Theta_U$ , if the corank of  $\delta'|_t$  is at least 2, then  $\det(\delta'_t) \in \mathfrak{m}_t^2$ , thus  $t$  lies in the non-regular locus of  $\Theta_U$ . To show that  $\Theta_U$  is regular at the generic point, or equivalently that  $\Theta_U$  is reduced, it suffices to notice that  $\Theta$  is reduced, which follows because the theta divisor  $\Theta$  is the birational image of  $\text{Sym}^{g-1}(X)$ .  $\square$

**Theorem 3.** *The Brauer class  $\alpha_{g-1} \in \text{Br}(\text{Pic}_{X/k}^{g-1})$  restricts to zero in  $\text{Br}(k(\Theta))$ .*

*Proof.* The class  $\alpha_{g-1}$  is represented by the Brauer-Severi scheme corresponding to  $\mathbb{P}((\pi_* L|_{Z_U})^\vee)$  [Gir71, 6], and therefore the class  $\alpha_{g-1}|_{\Theta_U}$  is represented by the Brauer-Severi scheme corresponding to its restriction  $\mathbb{P}(\pi_*(L'|_{Z'})^\vee)$ . Notice that the rank 1 subsheaf  $\pi'_* L' \subset \pi'_*(L'|_{Z'})$  yields a section to the Brauer-Severi scheme  $\mathbb{P}((\pi'_* L')^\vee)$  over the generic point of  $\Theta$ , and therefore the Brauer-Severi scheme is trivial, thus  $\alpha_{g-1}|_{k(\Theta)} = 0$ .  $\square$

**Remark 4.** *It is not clear if the Brauer class  $\alpha_{g-1}$  restricts to zero in  $\text{Br}(\Theta)$ .*

#### 4. OVER THE GENERIC CURVE

Are there any special loci in other components of the Picard scheme? Let  $E$  be a degree  $2g-2$ , semi-stable rank 2 vector bundle on  $X$ , Raynaud constructed generalized theta divisors  $\Theta_E \subset \text{Pic}_{X/k}^0$  (see Section 4.4). These divisor are in general neither reduced nor irreducible. However, we show that everything is good when we consider *the generic curves*.

**4.1. The numerical class of  $\Theta$ .** Let  $k$  be a field. Let  $L$  be a divisor on a  $k$ -variety  $Y$ . By the numerical class of  $L$ , we mean its class  $[L] \in \text{NS}(Y) := \text{Pic}(Y_{k^{\text{sep}}})/\text{Pic}^0(Y_{k^{\text{sep}}})$ . Let  $X$  be a curve over  $k$ . We denote the numerical class of  $\Theta \subset \text{Pic}_{X/k}^{g-1}$  by  $\theta \in \text{NS}(\text{Pic}_{X/k}^{g-1})$ . Let  $d$  be an integer, let  $x_0$  be a  $k^{\text{sep}}$ -point of  $X$ . Notice that translation by  $(d+1-g)\mathcal{O}_{X_{k^{\text{sep}}}}(x)$  yields an isomorphism  $(\text{Pic}_{X/k}^{g-1})_{k^{\text{sep}}} \cong (\text{Pic}_{X/k}^d)_{k^{\text{sep}}}$ , we also denote the numerical class of  $\Theta_{k^{\text{sep}}} + \mathcal{O}_{X_{k^{\text{sep}}}}((d+1-g)x) \subset (\text{Pic}_{X/k}^d)_{k^{\text{sep}}}$  by  $\theta$ . The numerical class is independent of the choice of  $x_0$ .

**4.2. The generic genus  $g$  curve.** Let  $k_0$  be a fixed field. Let  $g \geq 3$  be an integer. Let  $\mathcal{M}_g$  be the moduli stack of families of smooth genus  $g$  curves over  $k_0$ . Let  $k$  be the function field of  $\mathcal{M}_g$ . Let  $X/k$  be the generic genus  $g$  curve. We collect some facts:

**Lemma 5.** *The group  $\text{Pic}(X)$  is generated by  $\omega_X$ .*

*Proof.* This is Franchetta's theorem, see [Sch03].  $\square$

**Lemma 6.** *For any line bundle on  $\text{Pic}_{X/k}^d$ , the numerical class is always a multiple of*

$$\frac{2g-2}{\gcd(2g-2, d+1-g)} \theta.$$

*Proof.* Let  $P = \text{Pic}_{X/k}^d$ , we use the long exact sequence of Galois cohomology

$$\cdots \rightarrow \text{Pic}(P_{k^{\text{sep}}})^{\text{Gal}_k} \longrightarrow \text{NS}(P)^{\text{Gal}_k} \longrightarrow H^1(k, \text{Pic}_{P/k}^0) \rightarrow \cdots$$

associated with the short exact sequence

$$0 \longrightarrow \text{Pic}^0(P_{k^{\text{sep}}}) \longrightarrow \text{Pic}(P_{k^{\text{sep}}}) \longrightarrow \text{NS}(P) \longrightarrow 0.$$

Let us denote the connecting homomorphism  $\mathrm{NS}(P)^{\mathrm{Gal}_k} \rightarrow H^1(k, \mathrm{Pic}_{X/k}^0)$  by  $\delta$ . By [HL18, 2], we know that  $\mathrm{NS}(P) = \langle \theta \rangle$ . Therefore, if  $k\theta$  comes from a line bundle on  $P$ , then  $\delta(k\theta) = 0$ .

Let us calculate the order of  $\delta(\theta)$ . We pick a point  $x_0 \in X(k^{\mathrm{sep}})$ . By definition, the class  $\delta(\theta)$  is represented by a 1-cocycle in  $\mathrm{Pic}^0((\mathrm{Pic}_{X/k}^d)_{k^{\mathrm{sep}}})$ , which sends  $\sigma \in G$  to the divisor

$$c_\sigma := t_{(d+1-g)\mathcal{O}_{X_{k^{\mathrm{sep}}}(x_0)}}\Theta - t_{(d+1-g)\mathcal{O}_{X_{k^{\mathrm{sep}}}(\sigma(x_0))}}\Theta.$$

By the theorem of the square, this equals to  $t_{(d+1-g)\mathcal{O}_{X_{k^{\mathrm{sep}}}(x_0-\sigma(x_0))}}\Theta - \Theta$ . Notice that

- (1) Autoduality of Jacobian gives us a  $k$ -isomorphism  $\mathrm{Pic}_{X/k}^0 \rightarrow \mathrm{Pic}_{\mathrm{Pic}_{X/k}^0}^0, L \mapsto t_L\Theta - \Theta$ .
- (2) By the theorem of the square, pulling back line bundles along any  $k^{\mathrm{sep}}$ -translation induces a natural  $k$ -isomorphism  $\mathrm{Pic}_{\mathrm{Pic}_{X/k}^0}^0 \cong \mathrm{Pic}_{\mathrm{Pic}_{X/k}^d}^0$ .
- (3) The class of the torsor  $[\mathrm{Pic}_{X/k}^1] \in H^1(k, \mathrm{Pic}_{X/k}^0)$  is represented by the 1-cocycle  $\sigma \mapsto \mathcal{O}_X(x_0 - \sigma(x_0))$ .

Therefore, the class  $\delta(\theta)$  is represented by the torsor  $[\mathrm{Pic}_{X/k}^{d+1-g}] \in H^1(k, \mathrm{Pic}_{X/k}^0)$  under the identification  $H^1(k, \mathrm{Pic}_{X/k}^0) \cong H^1(k, \mathrm{Pic}_{P/k}^0)$ . Then we conclude by the strong Franchetta theorem [Sch03, Theorem 5.1], which says the torsor  $[\mathrm{Pic}_{X/k}^1]$  has order  $2g - 2$ .  $\square$

**4.3. Semi-stable vector bundles on universal curves.** Let  $g \geq 3$  be an integer. We show that there exist degree  $2g - 2$ , semi-stable rank-2 vector bundles on the generic genus  $g$  curve. We give explicit construction of such bundles by carefully modifying the direct sum of the canonical line bundles along canonical divisors.

**Proposition 7.** *Let  $g \geq 3$  be an integer, let  $X$  be the generic genus  $g$  curve. Then there exist degree  $2g - 2$ , semi-stable rank 2 vector bundles on  $X$ .*

*Proof.* Let  $D \in |\omega_X|$  be an effective canonical divisor. Let  $\rho: \Gamma(X, \omega_X) \rightarrow \Gamma(D, \omega_X|_D) = \Gamma(D, \mathcal{O}_D)$  be the restriction map. For any  $u \in \Gamma(D, \mathcal{O}_D)$ , let  $u\rho$  be the multiplication of  $\rho$  by  $u$ . Let  $(\rho, u\rho): \omega_X \oplus \omega_X \rightarrow \mathcal{O}_D$  be the sum of  $\rho$  and  $u\rho$ . Let  $E_u := \mathrm{Ker}(\rho, u\rho)$  be the kernel, then  $E_u$  is a locally free rank 2 sheaf on  $X$ . It has degree  $2g - 2$  and slope  $\mu(E_u) = g - 1$ . We show that for suitable choices of  $u$ , the sheaf  $E_u$  is semi-stable.

Let  $L$  be a subsheaf of  $E_u$ . As  $X$  is a smooth curve and  $E_u$  is locally free, so is  $L$ . We want to find conditions for  $u$ , such that the slope inequality  $\mu(L) \leq \mu(E_u)$  always holds:

- (1) If  $\mathrm{rank}(L) = 2$ , then  $\deg(L) = \deg(E_u) - \deg(E_u/L) \leq \deg(E_u)$ , so  $\mu(L) \leq \mu(E_u)$ .
- (2) If  $\mathrm{rank}(L) = 1$ , since  $\mathrm{Pic}(X) = \mathbb{Z} \cdot \omega_X$ , we may write  $L = \omega_X^{\otimes k}$ . If  $k \leq 0$ , then  $\mu(L) \leq 0 \leq \mu(E_u)$ . If  $k \geq 1$ , we may pick a nonzero section  $t$  of  $\omega_X^{\otimes(k-1)}$ . The section gives an embedding  $\omega_X \xrightarrow{t} \omega_X^{\otimes k} \hookrightarrow E_u$ , so  $h^0(E_u) \geq h^0(\omega_X) = g$ . Notice that  $E_u = \mathrm{Ker}(\rho, u\rho)$ , so  $h^0(E_u) = 2h^0(\omega_X) - \dim \mathrm{Im}(\Gamma(\rho, u\rho))$ , where  $\Gamma$  is the functor of taking global section on  $X$ . In order that  $E_u$  is semi-stable, it suffices to find  $u$  such that  $\dim \mathrm{Im}(\Gamma(\rho, u\rho)) \geq g + 1$ .

Let us consider the family of coherent sheaves  $E_u$  parameterized by  $u \in \Gamma(D, \mathcal{O}_D)$ . Since the rank of the linear transform  $\Gamma(\rho, u\rho)$  is a lower semi-continuous function on  $\Gamma(D, \mathcal{O}_D)$  and that  $k$  is infinite, it suffices to show that geometrically

$$\{u \in \Gamma(D, \mathcal{O}_D)_{\bar{k}} \cong \bar{k}^{2g-2} \mid \dim_{\bar{k}}(\mathrm{Im}(\Gamma(\rho, u\rho))) \geq g + 1\} \neq \emptyset.$$

From now on, let us assume that  $k$  is algebraically closed.

Without loss of generality, we may assume that  $D$  is a separable, i.e.,  $D = \sum_{i=1}^{2g-2} P_i$ , where  $P_i$  are distinct points. Let us identify  $\Gamma(D, \mathcal{O}_D)$  with  $k^{2g-2}$ , so that a section  $u \in \Gamma(D, \mathcal{O}_D)$

has coordinates  $u = (\lambda_i)_{i=1}^{2g-2}$ , given by its value of at  $P_i$ . Let us fix non-vanishing local sections of  $\omega_X$  at  $P_i$ , denoted by  $\omega_i$ . Then

$$\Gamma(\rho, u\rho): \Gamma(X, \omega_X)^{\oplus 2} \rightarrow \bigoplus_{i=1}^{2g-2} \Gamma(P_i, \mathcal{O}_{P_i})$$

is given by  $(\Omega_a, \Omega_b) \mapsto (f_a(P_i) + \lambda_i f_b(P_i))_i$ , where  $f_a(P_i) = \frac{\Omega_a(P_i)}{\omega_i(P_i)}$ ,  $f_b(P_i) = \frac{\Omega_b(P_i)}{\omega_i(P_i)}$ .

Let us pick a basis  $\{\Omega_i\}_{i=1}^g$  of  $\Gamma(X, \omega_X)$ . The map  $\Gamma(\rho, u\rho): \Gamma(X, \omega_X)^{\oplus 2} \rightarrow \bigoplus_{i=1}^{2g-2} \Gamma(P_i, \mathcal{O}_{P_i})$  can be expressed by the  $(2g-2) \times 2g$  matrix  $M = [M_L | M_R]$ , where  $(M_L)_{i,j} = \frac{\Omega_j(P_i)}{\omega_i(P_i)}$  and  $(M_R)_{i,j} = \frac{\lambda_i \Omega_j(P_i)}{\omega_i(P_i)}$  for  $0 \leq i \leq 2g-2, 0 \leq j \leq g$ .

Notice that  $M_L$  is the matrix for  $\Gamma(\rho)$ , which fits into the exact sequence

$$0 \longrightarrow k \cong \Gamma(X, \omega_X - D) \longrightarrow \Gamma(X, \omega_X) \xrightarrow{\Gamma(\rho)} \Gamma(D, \mathcal{O}_D),$$

therefore  $\text{rank}(M_L) = \dim \text{Im}(\Gamma(\rho)) = g-1$ . After rearranging the order of  $\{P_i\}$  and replacing  $\Omega_i$  by suitable linear combinations, we may write

$$M_L = \begin{bmatrix} 0_{(g-1) \times 1} & I_{g-1} \\ 0_{(g-1) \times 1} & A \end{bmatrix}$$

Let  $L_1 = \text{diag}_{i=1}^{g-1}(\lambda_i)$  and  $L_2 = \text{diag}_{i=g}^{2g-2}(\lambda_i)$  be diagonal matrices, we may write:

$$M = \begin{bmatrix} 0_{(g-1) \times 1} & I_{g-1} & 0_{(g-1) \times 1} & L_1 \\ 0_{(g-1) \times 1} & A & 0_{(g-1) \times 1} & L_2 A \end{bmatrix}$$

Let us choose  $\lambda_i$  such that  $\lambda_i \lambda_{g-1+i} \neq 1$  for all  $i$ , so that  $L_2^{-1} L_1 - I$  is invertible. Notice that  $\dim \text{Im}(\Gamma(\rho, u\rho)) = \text{rank}(M) = g-1 + \text{rank}((L_2^{-1} L_1 - I)A) = g-1 + \text{rank}(A)$ , it suffice to show that  $\text{rank}(A) \geq 2$ .

Let us consider the restriction maps:

$$\phi: \omega_X \rightarrow \omega_X|_{\sum_{i=g}^{2g-2} P_i} \quad \text{and} \quad \psi: \omega_X \rightarrow \omega_X|_{\sum_{i=1}^{g-1} P_i},$$

we have  $g = \dim \text{Im}(\Gamma(\phi)) + h^0(\omega_X(-\sum_{i=g}^{2g-2} P_i))$  and  $g = \dim \text{Im}(\Gamma(\psi)) + h^0(\omega_X(-\sum_{i=1}^{g-1} P_i))$ . Notice that with our arrangement, we have

$$\dim \text{Im}(\Gamma(\psi)) = \text{rank}(I_{g-1}) = g-1, \quad \dim \text{Im}(\Gamma(\phi)) = \text{rank}(A).$$

By Riemann-Roch, we know that  $h^0(\omega_X(-\sum_{i=1}^{g-1} P_i)) = h^0(\omega_X(-\sum_{i=g}^{2g-2} P_i))$  and therefore

$$\text{rank}(A) = \dim \text{Im}(\Gamma(\phi)) = \dim \text{Im}(\Gamma(\psi)) = g-1 \geq 2.$$

□

**4.4. The generalized theta divisor.** Let  $E$  be a semi-stable vector bundle on  $X$  obtained as in the last section. Let  $U \rightarrow \text{Pic}_{X/k}^0$  be an étale cover, such that a tautological line bundle  $L$  exist on  $X \times_k U$ . Let  $m$  be a positive integer, and  $Z \in |m\omega_X|$  be a reduced effective divisor. Let  $\pi: X \times_k U \rightarrow U$  be the projection, let  $E_U$  be the constant family. Let  $Z_U = Z \times_k U$ . Let us take the short exact sequence

$$0 \longrightarrow (E_U \otimes L)(-Z_U) \longrightarrow E_U \otimes L \longrightarrow (E_U \otimes L)|_{Z_U} \longrightarrow 0. \quad (\dagger)$$

and consider the long exact sequence of direct images along  $\pi_*$ :

$$\pi_*(E_U \otimes L) \hookrightarrow \pi_*(E_U \otimes L)|_{Z_U} \xrightarrow{\delta_U} R^1 \pi_*((E_U \otimes L)(-Z_U)) \twoheadrightarrow R^1 \pi_*(E_U \otimes L).$$

Let  $\eta$  be the generic point of  $U$ . Since  $E$  is semi-stable and  $\chi(E) = 0$ , we know by [Ray82, 1.6.2] that  $h^0(X_\eta, E_\eta \otimes L_\eta) = 0$ . By cohomology and base change, the middle two terms are locally free of the same rank for  $m$  large enough. Therefore, the torsion sheaf  $\pi_*(E_U \otimes L) = 0$  and  $\delta_U$  is an injection. The determinant of  $\delta_U$  cuts out a divisor in  $T_U \subset U$ , which descends to the generalized theta divisor  $\Theta_E \subset P$ .

#### 4.5. Restriction to $k(\Theta_E)$ .

**Lemma 8.** *The subscheme  $\Theta_E$  is reduced and irreducible.*

*Proof.* By [Ray82, 1.8.1], the divisor  $\Theta_E$  is numerically equivalent to  $2\Theta$ . If  $\Theta_E$  is reducible or non-reduced, its reduced component will have numerical class  $\Theta$ , which is not possible by Lemma 6.  $\square$

**Theorem 9.** *The class  $\alpha_0 \in \text{Br}(\text{Pic}_{X/k}^0)$  restricts to zero in  $\text{Br}(k(\Theta_E))$ .*

*Proof.* Let us restrict the exact sequence  $(\dagger)$  to  $X \times (\Theta_E)_U$  and denote the restriction of  $E_U, L$  and  $Z_U$  by  $E', L'$  and  $Z'$ . Let  $\pi': X \times (\Theta_E) \rightarrow (\Theta_E)$  be the projection. We have exact sequence

$$\pi'_*(E' \otimes L') \hookrightarrow \pi'_*(E' \otimes L')|_{Z'} \xrightarrow{\delta_U} R^1\pi'_*((E' \otimes L')(-Z')) \twoheadrightarrow R^1\pi'_*(E' \otimes L').$$

By the same argument in Lemma 2, the reducedness of  $\Theta_E$  implies that the generic rank of  $\pi'_*(E' \otimes L')$  is 1, then we conclude by the same argument as in Theorem 3.  $\square$

## REFERENCES

- [ABGV11] Asher Auel, Eric Brussel, Skip Garibaldi, and Uzi Vishne. Open problems on central simple algebras. *Transform. Groups*, 16(1):219–264, 2011.
- [ACGH85] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris. *Geometry of algebraic curves. Vol. I*, volume 267 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1985.
- [AM72] M. Artin and D. Mumford. Some elementary examples of unirational varieties which are not rational. *Proc. London Math. Soc. (3)*, 25:75–95, 1972.
- [Bor96] Mikhail Borovoi. The Brauer-Manin obstructions for homogeneous spaces with connected or abelian stabilizer. *J. Reine Angew. Math.*, 473:181–194, 1996.
- [Bos90] Siegfried Bosch. *Néron Models*. Springer Berlin Heidelberg, Berlin, Heidelberg, 1990.
- [Cal00] Andrei Horia Caldararu. *Derived categories of twisted sheaves on Calabi-Yau manifolds*. ProQuest LLC, Ann Arbor, MI, 2000. Thesis (Ph.D.)—Cornell University.
- [Gir71] Jean Giraud. *Cohomologie non abélienne*. Springer-Verlag, Berlin-New York, 1971. Die Grundlehren der mathematischen Wissenschaften, Band 179.
- [HL18] Wei Ho and Max Lieblich. Splitting Brauer classes using the universal Albanese. *arXiv e-prints*, page arXiv:1805.12566, May 2018.
- [hM] <https://mathoverflow.net/users/9502/mohan> (Mohan). Does a finite morphism to  $\mathbb{P}^n$  necessarily split at some height one point in the étale locus? MathOverflow. URL:<https://mathoverflow.net/q/359344> (version: 2020-05-04).
- [MV14] Margarida Melo and Filippo Viviani. The Picard group of the compactified universal Jacobian. *Doc. Math.*, 19:457–507, 2014.
- [NS19] Johannes Nicaise and Evgeny Shinder. The motivic nearby fiber and degeneration of stable rationality. *Invent. Math.*, 217(2):377–413, 2019.
- [Ray82] Michel Raynaud. Sections des fibrés vectoriels sur une courbe. *Bull. Soc. Math. France*, 110(1):103–125, 1982.
- [Sch03] Stefan Schröer. The strong Franchetta conjecture in arbitrary characteristics. *Internat. J. Math.*, 14(4):371–396, 2003.



- [Sch20] Stefan Schreieder. Algebraic cycles and refined unramified cohomology, 2020.
- [Tat68] John Tate. On the conjectures of Birch and Swinnerton-Dyer and a geometric analog [see MR1610977]. In *Dix exposés sur la cohomologie des schémas*, volume 3 of *Adv. Stud. Pure Math.*, pages 189–214. North-Holland, Amsterdam, 1968.

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