MATH130068h, 2020 Spring

Contents

CHAPTER 1

Sheaves-I

Let $\mathtt{Top}$ be the category of topological spaces.[1]

1. $\mathtt{Top}(X)$, $\mathtt{Top}_X$ and $\mathtt{Top}_{/X}$

Let X be a topological space.

1.1. Let $\mathtt{Top}_{/X}$ be the category of objects of $\mathtt{Top}$ over X.[2]

1.2. Let $\mathtt{Top}_X$ be the category of local homeomorphisms to X.

1.3. Let $\mathtt{Top}(X)$ be category of open inclusions to X.[3]

1.4. Can you check that:

- Morphisms in $\mathtt{Top}_X$ are automatically local homeomorphisms?
- $\mathtt{Top}(X)$ is not a groupoid?

1.5. In the chain

$$\mathtt{Top}(X) \subset \mathtt{Top}_X \subset \mathtt{Top}_{/X}$$

of nested subcategories,

- Is every subcategory is a full-subcategory?
- Does there exist quasi-inverses[4] to the forgetful[5] functors?

2. Sheaves: Definition

Let X be a topological space.

2.1. A sheaf of sets on X is a functor

$$\mathcal{F} \colon \mathtt{Top}(X)^{opp} \to \mathtt{Sets}$$

such that the diagram of pullback (where p_i^* is induced from inclusion $p_i \colon U_i \cap U_j \to U_i$[6])

$$\mathcal{F}(U) \xrightarrow{p^*} \prod_{i\in I} \mathcal{F}(U_i) \overset{p_i^*}{\underset{p_j^*}{\rightrightarrows}} \prod_{i,j\in I} \mathcal{F}(U_i \cap U_j),$$

is an equalizer diagram[7] the category of sets, whenever $\mathcal{U} := \{U_i\}_{i\in I}$ is a covering of U.

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[1] See [**?**, 1.1.2.7].

[2] See [**?**, 1.1.Ex.3]. We also call $\mathtt{Top}_{/X}$ the category of continuous maps to X.

[3] See [**?**, 1.2.4].

[4] See [**?**, 1.2.13].

[5] See [**?**, 1.2.2]. For an object in the subcategory, let it forget the fact that it lies in the subcategory.

[6] The letter p comes from the word "projection": the intersection can be read as the fiber product $U_i \times_X U_j$.

[7] See [**?**, 1.3.14].

2.2. In words, the axiom means:

- "Local sections uniquely glue."
- Glueable local sections can be uniquely glued along the gluing datum.
- Local sections that match on overlaps, are restricted form a unique global section.
- A global section is uniquely determined by its local sections. Conversely, given a family of local sections, they glue to a global section as long as they match on overlaps.
- Sections of a sheave behave like continuous maps between topological spaces: it is uniquely determined by its restriction to an open cover; or,
- given a family of continuous maps over an open covering, as long as they coincide on overlaps, they define a unique global continuous map.
- Sections of a sheaf behave like solutions to differential equations. Recall we studied local uniqueness and existence of solutions, this is exactly what we ask of sheaves.

2.3. When we consider sheaves of abelian groups, the pullback diagram being equalizer diagram translates to: $\ker(p^*) = 0$ and $\operatorname{im}(p^*) = \ker(p_i^* - p_j^*)$. Or equivalently, we are asking the exactness of the sequence

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\mathrm{d}^0} \prod_{i\in I} \mathcal{F}(U_i) \xrightarrow{\mathrm{d}^1} \prod_{i,j\in I} \mathcal{F}(U_i \cap U_j)\ ,$$

where $\mathrm{d}^0 = p^*$ and $\mathrm{d}^1 = p_1^* - p_2^*$ is taking difference.

2.4. Let's try to understand the axiom from simple examples. Let's work on the sheaves of abelian groups, since we are not very familiar with the universal property of equalizers, plus, in an equalizer diagram, there are more arrows — *"Three."*

- Let $I = \emptyset$. Note that the product over empty index set is the final object in the category,[8] then the sequence reads

$$\begin{tikzcd}
 & & 0 \arrow[d, equal] & 0 \arrow[d, equal] & . \\
0 \arrow[r] & \mathcal{F}(U) \arrow[r, "\mathrm{d}^0"] & \prod_{i\in\emptyset} \mathcal{F}(U_i) \arrow[r, "\mathrm{d}^1"] & \prod_{i,j\in\emptyset} \mathcal{F}(U_i \cap U_j) &
\end{tikzcd}$$

In particular, we can prove *the theorem of empty-section* for sheaves of abelian groups:

THEOREM 2.1. *Let $\mathcal{F}$ be a sheaf of abelian groups on a topological space X, then*

$$\mathcal{F}(\emptyset) = 0.$$

In particular, there is no reason why a presheaf should satisfy such a condition[9].[10]

- Let $I = \{0\}$, the exact sequence reads:

$$\begin{tikzcd}
 & & & \mathcal{F}(U) \times \mathcal{F}(U) \arrow[dr, equal, "1\times i"] & & . \\
0 \arrow[r] & \mathcal{F}(U) \arrow[r, "\mathrm{id}_U^*"] & \mathcal{F}(U) \arrow[ur, "\Delta"] \arrow[rr, "\mathrm{d}^1"] & & \mathcal{F}(U) \times \mathcal{F}(U) &
\end{tikzcd}$$

[8]The universal property reads: For any empty set of objects, and any morphism from a testing object T to the empty set of objects, (which can't be a more vacuous condition), there exists a unique morphism from T to the product. This simply means any object has a unique morphism to the empty product. Hence the empty product is the final object in the category. Which is a point in `Sets` and 0 in the category of abelian groups.

[9]Thus the definition of presheaves in [?, II.1] is stronger than usual. The extra axiom $\mathcal{F}(\emptyset) = 0$ for presheaves is unnecessary, and is never used anywhere.

[10]Can you give a meaningful example of a presheaf of abelian groups, such that $\mathcal{F}(\emptyset) \neq 0$?

One just note that identity is the equalizer of diagonal.[11] This is indeed uninteresting, it holds even if $\mathcal{F}$ is a presheaf.

- Let $I = \{0, 1\}$, the exact sequence reads:

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\mathrm{d}^0} \mathcal{F}(U_0) \times \mathcal{F}(U_1) \xrightarrow{\mathrm{d}^1} \mathcal{F}(U_{00}) \times \mathcal{F}(U_{01}) \times \mathcal{F}(U_{10}) \times \mathcal{F}(U_{11}) \ ,$$

where

$$\mathrm{d}^1 \colon (a, b) \mapsto (a - a, a - b, b - a, b - b).$$

There seems to be some redundancy we could work on. Taking difference with oneself does not yield anything interesting: sections are always compatible with themselves.[12] Also, one need to check compatibility *only once* on each overlap. Thinking this through, one decide to give the set I a total order, and only take product with ordered pairs. Note that

$$(a - a, a - b, b - a, b - b) = 0 \Leftrightarrow (0, a - b, b - a, 0) \Leftrightarrow (a - b) = 0, \tag{1}$$

the exactness of our original sequence reduces to the exactness of

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\mathrm{d}^0} \mathcal{F}(U_0) \times \mathcal{F}(U_1) \xrightarrow{\mathrm{d}^1} \mathcal{F}(U_{01}) \ ,$$

where

$$\mathrm{d}_1 \colon (a, b) \mapsto (a - b).$$

Now it is very clear what the sheaf axiom asks of a presheaf: A pair of sections (a, b) take the form $(s|_{U_0}, s|_{U_1})$ for some global section s, if and only if $a = b$ on $U_0 \cap U_1$. And the restriction map $s \mapsto (s|_{U_0}, s|_{U_1})$ is unique.

- Let $I = \{0, 1, 2, \cdots, n\}$, then the third term has n^2 components, and there are lots of redundant terms. The equality (1) can be generalized to an arbitrary set of coverings.[13] Let's denote the last product (unordered and ordered) by

$$\widetilde{C}^1(\mathcal{U}, \mathcal{F}) := \prod_{i_0, i_1 \in I \times I} \mathcal{F}(U_{i_0} \cap U_{i_1}),$$
$$C^1(\mathcal{U}, \mathcal{F}) := \prod_{i_0 \prec i_1 \in I \times I} \mathcal{F}(U_{i_0} \cap U_{i_1}).$$

Then the axiom, which was originally stated as:

$$\mathcal{F}(U) = \ker(\widetilde{C}^0(\mathcal{U}, \mathcal{F}) \to \widetilde{C}^1(\mathcal{U}, \mathcal{F})), \text{ for all } U \in \mathtt{Top}(X), \text{ and } \mathcal{U} \text{ covering } U.$$

can be restated as

$$\mathcal{F}(U) = \ker(C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F})), \text{ for all } U \in \mathtt{Top}(X), \text{ and } \mathcal{U} \text{ covering } U.$$

- Let's stop for a while and stare at the notations. Do they remind you of something familiar from the last recitation class? *There's a shadow hanging over them, the category* $\boldsymbol{\Delta}$ *came suddenly.* We'll come back to this in exercise 7.3.

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[11]By abstract nonsense, or *it suffice to* check in the category of sets, by the Yoneda embedding lemma. Can you write down the precise statements: how the Yoneda embedding lemma is applied, and what is shown?

[12]Unlike human beings.

[13]Using the fact that "diagonal is injective"

3. Sheaves: Extension to $\mathtt{Top}_X$

The slogan "local sections uniquely glue" is vague. Being vague has many advantages, to name a few:

- It motivates one to think and work: carefully specify what is talked about, carefully *choose the definitions* so that the mathematics works.[14]
- It makes no mistakes, since it is vague!
- Poetic.

In our slogan, we didn't specify what "local section" means. We vaguely know that it is something local. It doesn't even ask the local section be taken over an open *subset* of X: the notion "subset of X" is already global. After a few minutes of thinking, the question pops out: can the sections be taken over "local homeomorphisms"?

3.1. Let's rephrasing our question: Let X be a topological space, we have categories over X: $\mathtt{Top}(X)$ and $\mathtt{Top}_X$. Let $\mathcal{F}$ be a sheaf on X. We are asking if the functor $\mathcal{F}$ extends?

$$\begin{tikzcd}
\mathtt{Top}(X)^{opp} \arrow[r, "\mathcal{F}"] \arrow[d, "\mathrm{Fgt}"'] & \mathtt{Ab} \\
\mathtt{Top}_X^{opp} \arrow[ur, dotted, "??"'] &
\end{tikzcd}$$

3.2. It's a problem we've never dealt with.[15] We don't know an answer. But what would happen, *if all was settled*?

3.3. Let's take a $(w\colon W \to X) \in \mathrm{Ob}(\mathtt{Top}_X)$. And try to figure out what the group of sections $\mathcal{F}(W)$ should be.

3.4. Since w is a local homeomorphism, we may choose an open covering $\{W_i\}_I$ of W, such that each $W_i \to X$ is an injective local homeomorphism, or, a homeomorphism onto its image in X. Can we identify W as an open subset of X? This question sounds like: can we identify all vector spaces with $\mathbb{R}^n$? We can, but we prefer not to. We really want to *distinguish the isomorphic objects*.[16]

[14] Hopefully one understands the concepts and ideas better in this process.

[15] Isn't it, we just learnt what is a sheaf 5 minutes ago.

[16] Let us, using axiom of choice, identify every vector space with a standard $\mathbb{R}^n_{\mathtt{God}}$. Assume it happened that $\mathbb{R}^n_{\mathtt{Lihua}}$ is identified with the $\mathbb{R}^n_{\mathtt{God}}$ via matrix (ζ^n_m), and that $\mathbb{R}^n_{\mathtt{Hanmeimei}}$ is identified with $\mathbb{R}^n_{\mathtt{God}}$ via matrix (n^m), then the map $\iota\delta\colon \mathbb{R}^n_{\mathtt{Hanmeimei}} \to \mathbb{R}^n_{\mathtt{Lihua}}, (a_1, \ldots, a_n) \mapsto (a_1, \ldots, a_n)$ would be $(\zeta^n_m)(n^m)^{-1}$, this make things extremely difficult: Are you still in sane after verifying surjectivity of $\iota\delta\colon (a_1, \ldots, a_n) \mapsto (a_1, \ldots, a_n)$? This might be a post-modern reason why Newton's idea of taking inertia frame does not work. Nowadays, there are mathematicians applying category theory to physics. For example, what is an "observer", precisely? One may read for fun: *The Categorical Language of Quantum Physics* by Eugene Rabinovich.

3.5. The question of extending $\mathcal{F}$ now decomposes into two steps:

$$\begin{tikzcd}
\mathtt{Top}(X)^{opp} \arrow[r, "\mathcal{F}"] \arrow[d, "\mathrm{Fgt}"'] & \mathrm{Ab} \\
\mathtt{Top}\{X\}^{opp} \arrow[ur, dotted, "?"] \arrow[d, "\mathrm{Fgt}"'] & \\
\mathtt{Top}_X^{opp} \arrow[uur, dotted, "??"'] &
\end{tikzcd}$$

Here the category $\mathtt{Top}\{X\}^{opp}$ consists of all local homeomorphisms onto X. In this category, we may extend the functor by assigning $w\colon W \to X$ to ${}^{w}\mathcal{F}(w(W))$, this is a group isomorphic to $\mathcal{F}(w(W))$, but *not equal* to $\mathcal{F}(w(W)) =: {}^{w(W)}\mathcal{F}(w(W))$! It sounds like a stupid "idealisticism" to distinguish the two, but actually not. We haven't finished the functor, morphisms! Now one see how *realistic* it is to bookkeep w with the object.

3.6. Now we have the last arrow to conquer. Again, what would happen if *all was settled*?

3.7. Let $(u\colon U \to X)$ be an object in $\mathtt{Top}_X$. The assumption of local homeomorphism allows us to take an open covering $\mathcal{W}$ of U in $\mathtt{Top}\{X\}$. Since everything was settled, the sheaf $\mathcal{F}$ has to satisfy the axiom

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \prod_{i_0 \in \mathcal{W}} \mathcal{F}(W_{i_0}) \longrightarrow \prod_{i_0, i_1 \in \mathcal{W}\times\mathcal{W}} \mathcal{F}(W_{i_0} \times_U W_{i_1})$$

This can be viewed as the definition of $\mathcal{F}(U)$!

3.8. Let's define U as above, by picking any covering $\mathcal{W}$ of U in $\mathtt{Top}\{X\}$. First, we need to check that this map is well defined. This should follow from the sheaf axiom on $\mathtt{Top}\{X\}$.[17] We leave this as an exercise, see 7.4.

3.9. Now we need to verify the sheaf axiom: let $\{\mathcal{U}_\alpha\}_\mathcal{A}$ be an open covering of U in $\mathtt{Top}_X$. We need to check the exactness of the following sequence:

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\mathrm{d}^0} \prod_{\alpha_0 \in \mathcal{A}} \mathcal{F}(U_{\alpha_0}) \xrightarrow{\mathrm{d}^1} \prod_{\alpha_0, \alpha_1 \in \mathcal{A}\times\mathcal{A}} \mathcal{F}(U_{\alpha_0} \times_U U_{\alpha_1})$$

- Injectivity of d^0.
- Exactness at the middle term: $\ker(\mathrm{d}^1) \subset \mathrm{im}(\mathrm{d}^0)$.

3.10. Let's translate the verifications step by step, first look at d^1. The first term can be resolved by some covering from $\mathtt{Top}\{X\}$. Good. But the cheerful optimism would not propagate. The second and third term "*do not look like feul-efficient lamps*": each U_α needs to be again resolved by a cover in $\mathtt{Top}\{X\}$, unfortunately, by their definition. Let's be brave, take a cover $\mathcal{W}_{I_\alpha}$ of each U_α, and see what happens.

[17] Which we haven't checked yet, right? However, since they are authentic homeomorphisms, it is reasonable to believed that the sheafiness is inherited form sheaf axiom for $\mathtt{Top}(X)$.

3.11. We have a beautiful diagram

$$
\begin{tikzcd}
\mathbf{(0,0)} \arrow[dr, squiggly] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \mathcal{F}(U) \arrow[r, "\mathrm{d}^0"] \arrow[d] & \prod_{\mathcal{A}} \mathcal{F}(U_{\alpha_0}) \arrow[r, "\mathrm{d}^1"] \arrow[d] & \prod_{\mathcal{A}^2} \mathcal{F}(U_{\alpha_0\alpha_1}) \arrow[d] \\
0 \arrow[r] & \prod_{I} \mathcal{F}(W_{i_0}) \arrow[r] \arrow[d] & \prod_{I}\prod_{\mathcal{A}}\prod_{I_{\alpha_0}} \mathcal{F}(W_{i_0} \underset{U}{\times} W_{i_{\alpha_0}}) \arrow[r] \arrow[d] & \prod_{I}\prod_{\mathcal{A}^2}\prod_{I_{\alpha_{01}}} \mathcal{F}(W_{i_0} \underset{U}{\times} W_{i_{\alpha_0\alpha_1}}) \arrow[d] \\
0 \arrow[r] & \prod_{I^2} \mathcal{F}(W_{i_0i_1}) \arrow[r] & \prod_{I^2}\prod_{\mathcal{A}}\prod_{I^2_{\alpha_0}} \mathcal{F}(W_{i_0i_1} \underset{W}{\times} W_{i_{\alpha_0,\alpha'_0}}) \arrow[r] & \prod_{I^2}\prod_{\mathcal{A}^2}\prod_{I^2_{\alpha_{01}}} \mathcal{F}(W_{i_0i_1} \underset{U}{\times} W_{i_{\alpha_0}i_{\alpha_1},i'_{\alpha_0}i'_{\alpha_1}})
\end{tikzcd}
$$

where

- Columns are exact by definition of $\mathcal{F}$ on $\texttt{Top}_X$.
- The second and third rows are exact, by sheaf axiom on $\texttt{Top}\{X\}$ and *Fubini's theorem*: limits commute.
- The map d^0 is injective.[18]

To sum up, we have a diagram where we have exactness everywhere except at $(1,0)$.

3.12. It's time to learn some *Parseltongue*:

LEMMA 3.1 (Python). *In such a diagram, exactness at $(1,0)$ is automatic.*

The Python lemma is also called the nine-lemma: in the 3×3 diagram, exactness at the rest of nodes implies the ninth.

PROOF. Recall how we proved five-lemma, then use the equality $9 < 5 + 5$. $\square$

3.13. Let's crystalize what we have done

THEOREM 3.2. *The forgetful functor*

$$\mathrm{Sh}(\mathit{Top}_X) \to \mathrm{Sh}(\mathit{Top}(X))$$

is an equivalence of categories. In particular, we've just constructed its quasi-inverse.[19]

[18]This actually follows from the previous bullets. Can you prove it?

[19]The category of sheaves of sets on a *site* is called a topos. A site is also called a grothendieck (pre)-topology, over which sheaves are naturally defined. See [**?**, 00VG, 00X9]. So we can talk about "non point-set" topology. This reasonable, what really matters is the relation between the open subset, not the points. This is very important in the algebraic or rigid analytic world, where points are usually not enough. Instead of using the fancy language, people may also just add more points, see Brian Conrad's notes on *adic spaces*Here is an interesting example on "topology without enough points": [**?**, 1.3].

4. Sheaves: Representability

4.1. Let X be a topological space, let $\mathcal{F}$ be a sheaf of sets on $\mathtt{Top}(X)$. By the previous discussion, it can be naturally extend to a sheaf on $\mathtt{Top}_X$. Here is our fancy dream:

- Any sheaf $\mathcal{F}\colon \mathtt{Top}_X \to \mathtt{Sets}$ is representable.

To be precise: we dream that for any functor

$$\mathcal{F}\colon \mathtt{Top}(X)^{opp} \to \mathtt{Sets},$$

there exists a continuous map of topological spaces $\pi_{\mathfrak{F}}\colon \mathfrak{F} \to X$, such that

- the structure map $\pi_{\mathfrak{F}}\colon \mathfrak{F} \to X$ is a local homeomorphism
- the functors $\mathcal{F}$ and $h_{\mathfrak{F}}$ are naturally isomorphic in $\mathtt{Top}_X$.

If such $\mathfrak{F}$ exist, we call it: the étale space associated with the sheaf $\mathcal{F}$. Why is this a fancy dream? If this is true, then the sections of $\mathcal{F}$ are actually realized as continuous maps $U \to \mathfrak{F}$[20]. So the poetic explanation bullet 5 in Subsection 2.2 can be made rigorous:

- We can redefine sheaf of sets on X as objects in $\mathtt{Top}_X$
- Or, given an object $\mathfrak{F} \in \mathtt{Top}_X$, the functor $U \mapsto \mathrm{Mor}_{\mathtt{Top}_X}(U, \mathfrak{F})$ is a sheaf, and every sheaf arise in this way for a unique local homeomorphism of topological spaces $\pi_{\mathfrak{F}}\colon \mathfrak{F} \to X$.

4.2. Recall in the last recitation class, we proved the assembling lemma, which literally gives the proof of following fact: on a small category $\mathcal{C}$, any presheaf $\mathcal{F}$ is a colimit of representable presheaves:

$$\mathcal{F} = \operatorname*{colim}_{h_U/\mathcal{F}} h_U.$$

Note that Yoneda lemma preserves colimit[21], and that $\mathtt{Top}_X$ admit colimits, we know

$$\mathcal{F} = h_{\operatorname*{colim}_{h_U/\mathcal{F}} U}$$

we may simply define the étale space of $\mathcal{F}$ by $\mathfrak{F} := \operatorname*{colim}_{h_U/\mathcal{F}} U$.

4.3. This is not quite a proof, as the category $\mathtt{Top}_X$ is not small. In the proof, we need the category to be small, because limits and colimit are only defined on small diagrams. For example, we can take product for "a set of sets", but not "a class of sets". Why is the category $\mathtt{Top}_X$ big? Just take $X = \{*\}$, then $\mathtt{Top}_X = \mathtt{Top}$, which contains $\mathtt{Sets}$ as a full subcategory, but $\mathrm{Ob}(\mathtt{Sets})$ is not a set: *Who shaves the Barber*? However, the problem can be fixed: we may replace the colimit diagram by a small diagram see this link. Note that $\mathtt{Top}(X)$ is a small category, as the class of objects come is a subset of 2^X, hence $\mathrm{Sh}(\mathtt{Top}(X))$ is a category. This illustrates how the notion of topos is preferred to site. In this case, the topos $\mathrm{Sh}(\mathtt{Top}_X)$ is equivalent to $\mathrm{Sh}(\mathtt{Top}(X))$: it automatically takes care of the bigness of $\mathtt{Top}_X$, which is mostly irrelevant for our discussion.

4.4. Now let's define the stalk of a sheaf at $x \in X$ to be the inverse image $\mathfrak{F}|_x = \pi_{\mathfrak{F}}^{-1}(x)$. It coincides what we see elsewhere

$$\mathcal{F}_x := \operatorname*{colim}_{x \in U} \mathcal{F}(U).$$

4.5. Psychologically, if one are still not convinced, one could read [?, II.Ex.1.13], where the étale space is constructed as the union of all stalks, equipping with certain topology, so that the representability of $\mathtt{Top}_X$ is satisfied.

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[20] Of course, in as morphisms $\mathtt{Top}_X$, so they have to be compatible with the structure, which just mean $U \to \mathfrak{F} \to U$ is identity, which is what section literally means! See [?, 1.3.5]

[21] what faithful literally means, otherwise, what's the point of the so called "embedding".

4.6. We remark that local homeomorphism can be are very different from n-sheet covering in topology. For a local homeomorphism to $\mathbb{R}^1$, the source space can be highly non-separated, say, $\mathbb{C}$-copies of $\mathbb{R}^1$ identified on $\mathbb{R}^1 - \{0\}$. One gets a skyscraper sheaf $i_*\mathbb{C}$, where $i\colon \mathbb{R}^1 - \{0\}$ is the open inclusion.

4.7. Up to now, we only assumed knowledge of sets, topological spaces and abelian groups. Using categories and functors, we have developed such a bunch of things, isn't it cool? Of course, there are more to say in the future, when we discuss sheaves of modules.[22]

4.8. Lastly, we remark that, in the construction

$$\mathcal{F} = \operatorname*{colim}_{h_U/\mathcal{F}} h_U,$$

we didn't actually ask $\mathcal{F}$ to be a sheaf. The construction goes word by word for a presheaf

THEOREM 4.1 (Sheafification). *There is a left adjoint functor*

$$\sharp\colon \mathrm{Psh}(\mathit{Top}_X) \to \mathrm{Sh}(\mathit{Top}_X)$$

to the forgetful functor. Namely:

- *Let* $\mathcal{F} \in \mathrm{Ob}(\mathrm{Psh}(Top_X))$*, then*

$$\mathcal{F}^\sharp := \operatorname*{colim}_{h_U/\mathcal{F}} h_U \in \mathrm{Ob}(\mathrm{Sh}(\mathit{Top}_X)).$$

- *For any* $\mathcal{G} \in \mathrm{Ob}(\mathrm{Sh}(\mathit{Top}_X))$*, there is a natural identification:*

$$\mathrm{Mor}_{\mathrm{Psh}(\mathit{Top}_X)}(\mathcal{F}, \mathcal{G}) \cong \mathrm{Mor}_{\mathrm{Sh}(\mathit{Top}_X)}(\mathcal{F}^\sharp, \mathcal{G})$$

4.9. To apply the theorem, we remark that any presheaf $\mathtt{Top}(X)$ can be naturally viewed as a presheaf on $\mathtt{Top}_X$ by "zero extension".

5. Sheaves: Applications

5.1. Recall in complex analysis, given a multi-valued analytic function[23] f on an open domain $\Omega \subset \mathbb{C}$, say

- $w = \mathrm{Log}(z)$, $w = z^{3/4}$, $w = z^{1/2} + z^{-1/2}$ on $\Omega = \mathbb{C} - \{0\}$,
- $w = \sqrt{z(z-1)(z-t)}$ on $\Omega = \mathbb{C} - [0,1] \bigcup [t, \infty)$,
- implicit function defined as the root of an irreducible polynomial $g(z,w) \in \mathbb{C}[z,w]$. Its maximal domain of definition

  $$\Omega = \{z_1 \in \mathbb{C}\colon g(z_1)(w) \text{ has no multiple roots.}\}^{24}$$

  Equivalently, by the fundamental theorem of algebra:

  $$\Omega = \{z_1 \in \mathbb{C}\colon g(z_1)(w) \text{ has } \deg_w(g) \text{ distinct roots.}\}$$

- solution to an ordinary differential equation, say $w'' = p(z)w' + q(z)$[25],

[22] A module is a representation of a ring on an abelian group. A sheaf of modules can be thought as representation of a "topological space" on an abelian group.

[23] Multi-valued functions are not functions, as red pandas are not pandas. See [?, 1.3.4] for the formal theoretic definition of a function (maybe you learned it in [?]). We remark that using Yoneda embedding, we can define the notion of a function in an arbitrary category, even if the objects do not have underlying sets.

[24] Exercise: show this by inverse function theorem.

[25] See [?, 8.4.4], it is called *Hypergeometric Differential Equation*. It naturally appears in algebraic geometry.

there is a notion called *the Riemann surface* associated with f. A Riemann surface is a topological space, serving as the correct "domain of definition" of the multi-valued function. In our textbook[26], the Riemann surface $\mathcal{L}$ for $\mathrm{Log}(z)$ is constructed by patching domains of analytic continuation. The Riemann surface comes naturally equipped with a continuous map $\pi\colon \mathcal{L} \to \Omega \subset \mathbb{C}$.

On the Riemann surface, the multi-valued function f can be well defined, more precisely: there is an analytic function $\widetilde{f}$ defined on $\mathcal{L}$, such that germs of $\widetilde{f}$ are carried to germs of f via projection. We call $(\widetilde{f}, \mathcal{L})$ a uniformization of the multi-valued function f on Ω. Many things need to be justified: For example, how to check the analytic continuation is independent of the path: why is $\widetilde{f}$ well-defined on $\mathcal{L}$?[27] We show these verifications are automatic with the help of "étale space of sheaves".

5.2. Let f be a multi-valued analytic function defined on $\Omega \subset \mathbb{C}$. Let $\mathcal{F}$ be the presheaf of single valued branches of f on Ω. The sheaf axioms are obviously satisfied[28]. We may simply define the étale space of $\mathcal{F}$ as our Riemann surface. To see that f can be defined on $\mathfrak{F}$, it suffice to show the sheaf $\mathcal{F}$ of sections has a global section when pulled back to $\mathfrak{F}$. One observes that $\mathcal{F}$ pulls back to the sheaf of sections to the projection $\mathrm{pr}_2\colon \mathfrak{F} \times_X \mathfrak{F} \to \mathfrak{F}$, and that the diagonal Δ gives a global section.

5.3. We remark that when the section of $\mathcal{F}$ is equipped with a transitive group action, then $\mathfrak{F} \times_X \mathfrak{F} \cong \mathfrak{F} \times G/H$ [29]. Let's consider the analytic function in bullet 3, then the Galois group Gal_g acts on the branches of local sections of g (which are roots of $g(z,w) \in \mathbb{C}(z)[w]$). Recall by our discussion there, the stalks of g are of constant size, thus $\pi_{\mathfrak{F}}\colon \mathfrak{F} \to X$ is a $\deg_w(g)$-sheet covering space. Let's summarize what we have done:

5.4. Recall in topology and Galois theory, we have isomorphism of the lattices:

$$\begin{tikzcd}
\mathtt{Top}_{\Omega}[\Omega, \widetilde{\Omega}] \arrow[rrr, leftrightarrow, "\widetilde{\Omega}/H \leftarrowtail H", "\mathcal{F} \mapsto \mathrm{Stab}_{\pi_1(\Omega, \pi_{\mathfrak{F}}(p))}(p)"'] \arrow[d, dotted, leftrightarrow] & & & \mathtt{Grp}[\pi_1(\Omega), \{\mathrm{id}\}] \arrow[d, squiggly, dash, "\text{``equivalent''}", "\textit{almost}"'] \\
\mathtt{Flds}_K[K, \overline{K}] \arrow[rrr, leftrightarrow, "L \mapsto \mathrm{Gal}(\overline{K}/L)", "\overline{K}^H \leftarrowtail H"'] & & & \mathtt{Grp}[\mathrm{Gal}_K, \{\mathrm{id}\}]
\end{tikzcd}$$

Let $K = \mathbb{C}(z)$, let $g \in K[z]$ be an irreducible polynomial, let $\mathcal{F}$ be the sheaf of analytic germs on Ω, then the left upwards arrow can be constructed by

$$g(z) \mapsto \mathfrak{F}.$$

We remark that the left downwards arrow can be constructed by taking the field of meromorphic functions:

$$\Omega' \mapsto \mathcal{M}(\Omega').$$

[26] See [**?**, 7.1].

[27] See [**?**, 2.6.3*].

[28] An analytic function is a solution to the Cauchy-Riemann equation. The local existence and uniqueness of solutions are standard material in ODE or PDE courses. For ODE, see [**?**, 4.1.1], where the local existence and uniqueness is proved under the assumption that coefficients are Lipschitz. The local existence or uniqueness are not always guaranteed, for example, $y' = \sqrt{t}y$ or $x'^2 = 4x$, see [**?**, 1.3.1]. We may fix these problems by shrinking our domain of discussion. However, the study of the behavior at these singular points are very important. See Deligne's work [**?**] and some recent development in p-adic Hodge theory.

[29] We get all the sections by translating the diagonal section.

5.5. This presentation looks nice, but it still has some drawbacks:

- The bijection is stronger than just being a functor, it preserves the "Galois structure": Galois extension are sent to normal coverings, vice versa.
- The right hand side is not well explained.
- The infinite Galois correspondence only matches subfields with closed subgroups of Gal_K. This is not taken care of in the diagram.

We remark that all these can be fully clarified with the notion of Galois categories.[30]

6. Sheaves: A brief history

This section follows from Wikipedia:

6.1. The first origins of sheaf theory are hard to pin down – they may be co-extensive with the idea of analytic continuation[clarification needed]. It took about 15 years for a recognisable, free-standing theory of sheaves to emerge from the foundational work on cohomology.

- 1936 Eduard Čech introduces the nerve construction, for associating a simplicial complex to an open covering.
- 1938 Hassler Whitney gives a 'modern' definition of cohomology, summarizing the work since J. W. Alexander and Kolmogorov first defined cochains.
- 1943 Norman Steenrod publishes on homology with local coefficients.
- 1945 Jean Leray publishes work carried out as a prisoner of war, motivated by proving fixed-point theorems for application to PDE theory; it is the start of sheaf theory and spectral sequences.
- 1947 Henri Cartan reproves the de Rham theorem by sheaf methods, in correspondence with André Weil (see De Rham–Weil theorem). Leray gives a sheaf definition in his courses via closed sets (the later carapaces).
- 1948 The Cartan seminar writes up sheaf theory for the first time.
- 1950 The "second edition" sheaf theory from the Cartan seminar: the sheaf space (espace étalé) definition is used, with stalkwise structure. Supports are introduced, and cohomology with supports. Continuous mappings give rise to spectral sequences. At the same time Kiyoshi Oka introduces an idea (adjacent to that) of a sheaf of ideals, in several complex variables.
- 1951 The Cartan seminar proves theorems A and B, based on Oka's work.
- 1953 The finiteness theorem for coherent sheaves in the analytic theory is proved by Cartan and Jean-Pierre Serre, as is Serre duality.
- 1954 Serre's paper Faisceaux algébriques cohérents (published in 1955) introduces sheaves into algebraic geometry. These ideas are immediately exploited by Friedrich Hirzebruch, who writes a major 1956 book on topological methods.
- 1955 Alexander Grothendieck in lectures in Kansas defines abelian category and presheaf, and by using injective resolutions allows direct use of sheaf cohomology on all topological spaces, as derived functors.
- 1956 Oscar Zariski's report Algebraic sheaf theory
- 1957 Grothendieck's Tohoku paper rewrites homological algebra; he proves Grothendieck duality (i.e., Serre duality for possibly singular algebraic varieties).
- 1957 onwards: Grothendieck extends sheaf theory in line with the needs of algebraic geometry, introducing: schemes and general sheaves on them, local cohomology, derived categories (with Verdier), and Grothendieck topologies. There emerges also his influential schematic idea of 'six operations' in homological algebra.

[30] See [**?**, 0BMQ].

- 1958 Roger Godement's book on sheaf theory is published. At around this time Mikio Sato proposes his hyperfunctions, which will turn out to have sheaf-theoretic nature.

At this point sheaves had become a mainstream part of mathematics, with use by no means restricted to algebraic topology. It was later discovered that the logic in categories of sheaves is intuitionistic logic (this observation is now often referred to as Kripke–Joyal semantics, but probably should be attributed to a number of authors). This shows that some of the facets of sheaf theory can also be traced back as far as Leibniz.

6.2. stalk and fiber

7. Questions

7.1. State and prove "*the theorem of empty-section*"[31] for sheaves of sets.

7.2. Let $\overline{\mathbf{\Delta}}$ be the "conjugate category" of finite ordered sets, whose

- set of objects are $\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots$, where $\mathbf{n} = \{0, 1, \ldots, n\}$,
- a morphism $\mathbf{n} \to \mathbf{m}$ is a non-increasing map of ordered sets.

Is the "conjugate category" isomorphic to $\mathbf{\Delta}^{opp}$?[32]

7.3. Let $\mathbf{\Delta}$ be the category of finite ordered sets. The discussion in 2.4 vaguely remind us of the category $\mathbf{\Delta}$. There we can still talk about non-decreasing maps, but we do not have $X(\mathbf{n})$ for $n \geq 1$. The morphisms are there, but we don't have all the objects—"Aha! Subcategories."

Let $\mathbf{\Delta}_{\leq n}$ be the full subcategory of $\mathbf{\Delta}$, whose objects are $\mathbf{0}, \ldots, \mathbf{n}$. The forgetful functor $\mathbf{\Delta}_{\leq n} \to \mathbf{\Delta}$ induces a *skeleton functor*

$$\mathrm{sk}_n \colon \mathrm{Psh}(\mathbf{\Delta}) \to \mathrm{Psh}(\mathbf{\Delta}_{\leq n})$$

Since are only allowed to take simplices up to $\mathbf{n}$, by the assembling lemma in $\mathrm{Psh}(\mathbf{\Delta}_{\leq n})$,

$$\mathrm{sk}_n(X) = \varinjlim_{f \in \mathbf{\Delta}_{\leq n}/\mathrm{sk}_n(X)} \mathbf{\Delta}^f,$$

the n-skeleton $\mathrm{sk}_n(X)$ is assembled form simplices of dimension no more than n, this coincides with our usual notion of skeleton: the thing left when the higher dimensional "flesh" is removed.[33]

As a common feature of many forgetful functors, the left adjoint functor (if it exits), would give the universal way to take closure, saturation, or extension.[34] Luckily, in our case, a left adjoint exists[35], called the n-coskeleton functor

$$\mathrm{cosk}_n \colon \mathrm{Psh}(\mathbf{\Delta}_{\leq n}) \to \mathrm{Psh}(\mathbf{\Delta}),$$

which can be thought as *the* way to assign the functor for $\mathbf{n+1}, \mathbf{n+2}$ and so on.

This being said, let's review what we did in 2.4.

---

[31]See Theorem 2.1

[32]◦◦⌣: Use the *belt lemma*?

[33]This is creepy...

[34]It is usually called the blahfication functor, see [**?**, 1.4.3,1.4.5], where blah="sheafi", "groupi". Is literally taking topological closure a left adjoint to certain forgetful functor?

[35]See [**?**, 0183].

- By take an ordering on the index set I of the family of open covering, we gave a presheaf $C_{\mathcal{U},\mathcal{F}} \in \mathrm{Psh}(\mathbf{\Delta}^{opp}_{\leq 1})$:

$$
\begin{aligned}
C_{\mathcal{U},\mathcal{F}} \colon \mathbf{\Delta}_{\leq 1} &\to \mathtt{Ab} \\
\mathbf{k} &\mapsto C^k(\mathcal{U},\mathcal{F}) := \prod_{i_0 \prec \cdots \prec i_k} \mathcal{F}(U_{i_0} \times_U \cdots \times_U U_{i_k}) \\
(f \colon \mathbf{k} \to \mathbf{k}') &\mapsto C^k(\mathcal{U},\mathcal{F}) \xrightarrow{Cf^*} C^{k'}(\mathcal{U},\mathcal{F})
\end{aligned}
$$

The homomorphism Cf^* on component[36]

$$\prod_{j_0 \prec \cdots \prec j_k} \mathcal{F}(U_{j_0,\dots,j_k}) \to \mathcal{F}(U_{i_0,\dots,i_{k'}})$$

reads as the composition of the projection

$$\prod_{j_0 \prec \cdots \prec j_k} \mathcal{F}(U_{j_0,\dots,j_k}) \to \mathcal{F}(U_{i_{f(0)},\dots,i_{f(k)}})$$

and map induced from projection

$$\pi^* \colon \mathcal{F}(U_{i_{f(0)},\dots,i_{f(k)}}) \to \mathcal{F}(U_{i_0,\dots,i_{k'}}).$$

- Be careful, we only have a covariant functor emitting form $\mathbf{\Delta}_{\leq 1}$, this is not what we want. However, it can be fixed, by changing the target category to $\mathtt{Ab}^{opp}$.[37] Namely, viewing $C_{\mathcal{U},\mathcal{F}} \colon \mathbf{\Delta}_{\leq \mathbf{1}} \to \mathtt{Ab}$ as a presheaf in $\mathtt{Ab}^{opp}$, $\mathrm{Psh}_{\mathtt{Ab}^{opp}}(\mathbf{\Delta}_{\leq 1})$. Take 1-coskeleton, we get a simplicial opposite-abelian-group

$$\mathrm{cosk_n}(C_{\mathcal{U},\mathcal{F}}) \in \mathrm{Psh}_{\mathtt{Ab}^{opp}}(\mathbf{\Delta}).$$

- By the Dold-Kan correspondence, the simplicial opposite-abelian-group can be turned into a chain complex in $\mathrm{Ch}_{\geq 0}(\mathtt{Ab}^{opp})$. Reversing the arrows, we get a cochain complex

$$C^\bullet(\mathcal{U},\mathcal{F}) := \mathrm{DK}(\mathrm{cosk}_1(C_{\mathcal{U},\mathcal{F}})) \in \mathrm{Ch}^{\geq 0}(\mathrm{Ab}).$$

- Since coskeleton is an extension of the presheaf in the truncated category, it is reasonable to believe $\mathrm{sk}_n(cosk_n(X)) = X$. As Dold-Kan correspondence preserves the grading, one see the complex $C^\bullet(\mathcal{U},\mathcal{F})$ is a natural extension of the two term complex in the sheaf axiom.

$$
\begin{tikzcd}
 & & C^0(\mathcal{U},\mathcal{F}) \arrow[r, "\mathrm{d}^0"] \arrow[d, equal] & C^1(\mathcal{U},\mathcal{F}) \arrow[r, "\mathrm{d}^1"] \arrow[d, equal] & \cdots \arrow[r, "\mathrm{d}^{n-1}"] & C^n(\mathcal{U},\mathcal{F}) \arrow[r, "\mathrm{d}^n"] & \cdots \\
0 \arrow[r] & \mathcal{F} \arrow[r] & \prod_I \mathcal{F}(U_i) \arrow[r] & \prod_{i \prec j} \mathcal{F}(U_i \times_U U_j) & & \mathrm{H}^n(\mathcal{U},\mathcal{F}) \arrow[u, squiggly] &
\end{tikzcd}
$$

We call the complex $C^\bullet(\mathcal{U},\mathcal{F})$ the Čech complex associated to $\mathcal{F}$ with respect to the covering $\mathcal{U}$.

- The sub-quotient $\mathrm{H}^i(\mathcal{U},\mathcal{F}) := \ker(\mathrm{d}^i)/\mathrm{im}(\mathrm{d}^{i-1})$ is called the i-th Čech cohomology of $\mathcal{F}$ with respect to the covering $\mathcal{U}$. As Betti number counts "holes"[38] of a real manifold, Čech

[36] in $\mathrm{Hom}(\prod A, \prod B) = \prod \mathrm{Hom}(\prod A, B)$

[37] It is currently hard to understand read this post. However, we know it is still an abelian category, the axioms are symmetric! By the Freyd-Mitchell embedding theorem[**?**, 1.6.1], we may view it as a category of R-modules, for some ring R. But the opposite category of sets is fuzzy.

[38] an n-hole yields an obstruction to contract n-spheres, and the Betti(=Singular)[39] cohomology $\mathrm{H}^n(X,\mathbb{Z})$ are generated by the n-spheres.

cohomology measures the "homological holes"[40] of a presheaf. The sheaf axiom thus, just asks that the degree-zero invariants to be geometrically meaningful: they are interpreted by sections.

7.4. Check the well-definedness of $\mathcal{F}$ on $\mathtt{Top}_X$, see Subsection 3.8. What is the statement you are going to check, precisely? [41]

7.5. In 3.11, we used Fubini's theorem to reinterpret the triple products. State and prove *the* Fubini's theorem we used there.

7.6. Can you reformulate the usual Fubini's theorem in calculus: "limit commutes" or "integration commutes", in a categorical language? Build *your category.*[42]

7.7. State and prove *the* Python lemma we used in 3.11.

7.8. Show that" $W_i \times_U W_j = W_i \times_X W_j$ when $U \to X$" is an injection: State and prove your favorite version.

"The crucial thing here, from the viewpoint of the Weil conjectures, is that the new notion [of space] is vast enough, that we can associate to each scheme a "generalized space" or "topos" (called the "étale topos" of the scheme in question). Certain "cohomology invariants" of this topos ("childish" in their simplicity!) seemed to have a good chance of offering "what it takes" to give the conjectures their full meaning, and (who knows!) perhaps to give the means of proving them."

—A. Grothendieck, *Récoltes et Semailles,1986*

[40]an n-cohomology class describes an obstruction to glue "n-sections", and sheaf cohomology are generated by the "n-sections"—n-cocycles. Sheaf with no such obstruction is called acyclic, the prefix "a-" means "antonym".

[41]Hint: Connect the two by a common refinement, then do some diagram chasing.

[42]— *"Let there be light."*

CHAPTER 2

Sheaves-II

1. A brief review of sheaves

1.1. Let $\mathcal{C}$ be a category. We call a functor $\mathcal{F}\colon \mathcal{C}^{opp} \to \mathtt{Sets}$ a presheaf on $\mathcal{C}$. In many interesting situations, it suffice to consider presheaves on small categories. For example, let X be a topological space, then $\mathtt{Top}(X)$ is small: the class of objects are open subsets of X, which are elements in the power set $\mathcal{P}(X)$, so $\mathrm{Ob}(\mathtt{Top}(X)) \in \mathcal{P}(\mathcal{P}(X))$. However, even if $\mathcal{C}$ is small, the category of presheaves $\mathtt{Psh}(\mathcal{C})$ is always big, as $\mathtt{Sets}$ is big.

1.2. Let X be a topological space. Let $\mathcal{F}$ be a presheaf on X. We say $\mathcal{F}$ is a sheaf, if for any covering $\{U_i \to U\}_{i\in I}$ in $\mathtt{Top}(X)$, the following commutative diagram is always an equalizer:

$$\mathcal{F}(U) \xrightarrow{p^*} \prod_{i\in I} \mathcal{F}(U_i) \underset{p_j^*}{\overset{p_i^*}{\rightrightarrows}} \prod_{i,j\in I} \mathcal{F}(U_i \cap U_j),$$

1.3. Let X be a topological space. Let $\mathcal{F}$ be a presheaf on $\mathtt{Top}(X)$. There is a canonical way to upgrade $\mathcal{F}$ to a sheaf. Here is the recipe:

- Let $\mathfrak{F} := \mathrm{colim}_{h_U/\mathcal{F}} U$[1] be the colimit in $\mathtt{Top}_X$, and $\pi_{\mathfrak{F}}\colon \mathfrak{F} \to X$ be the projection.
- For $V \in \mathtt{Top}(X)$, we take $\mathcal{F}^\sharp(V) = \mathrm{Mor}_{\mathtt{Top}_X}(V, \mathfrak{F})$.[2]

We call $\mathcal{F}^\sharp\colon \mathtt{Top}(X) \to \mathtt{Sets}$ the sheafification of $\mathcal{F}$. The sheafification is characterized as the left adjoint to the forgetful functor: For any sheaf $\mathcal{G}$ on X, we can prove the key fact[3]

$$\sharp \dashv \mathrm{Fgt}, \qquad \mathrm{Mor}_{\mathtt{Sh}(X)}(\mathcal{F}^\sharp, \mathcal{G}) = \mathrm{Mor}_{\mathtt{Psh}(X)}(\mathcal{F}, \mathrm{Fgt}(\mathcal{G})).$$

We define the stalk of $\mathcal{F}$ at x by $\mathcal{F}_x := \pi_{\mathfrak{F}}^{-1}(x)$. Let's write $x = \lim_{x\in V} V$[4], then by Fubini:

$$\pi_{\mathfrak{F}}^{-1}(x) = \mathrm{Mor}_{\mathtt{Top}}(x, \mathfrak{F}) = \mathrm{Mor}_{\mathtt{Top}_X}(\lim_{x\in V} V, \mathfrak{F}) = \mathrm{colim}_{x\in V} \mathrm{Mor}_{\mathtt{Top}_X}(V, \mathfrak{F}) = \mathrm{colim}_{x\in V} \mathcal{F}(V).$$

1.4. If one prefers a more concrete description, one may follow the usual recipe

- Given any presheaf $\mathcal{F}$, we define its stalk at $x \in X$
$$\mathcal{F}_x := \mathrm{colim}_{x\in V} \mathcal{F}(V).$$
- One defines the étale space of the sheaf by putting suitable topology on $\mathfrak{F} := \coprod_{x\in X} \mathcal{F}_x$, so that the sections of the presheaf are local homeomorphisms.
- One define the sheafification $\mathcal{F}^\sharp$ as the presheaf of local sections to $\pi_{\mathfrak{F}}\colon \mathfrak{F} \to X$.

[1] We call $\mathfrak{F}$ the étale space associated with $\mathcal{F}$

[2] By definition of $\mathtt{Top}_X$, this equals to $\{s_V | s_V \in \mathrm{Mor}_{\mathtt{Top}}(V, \mathfrak{F}), \pi_{\mathfrak{F}} \circ s_V = \mathrm{id}_V\}$

[3] See 3.3.

[4] At least when X is Hausdorff.

2. The category of abelian sheaves

Let $\mathcal{C}$ be a category. Let $\mathtt{Psh}_{\mathtt{Ab}}(\mathcal{C})$ be the category of presheaves of abelian groups on $\mathcal{C}$. For convenience, let's rename "presheaves of abelian groups" to "abelian presheaves". When the sheaf axioms are satisfied, we rename "sheaf of abelian groups" to "abelian sheaves".

We show that $\mathtt{Sh}_{\mathtt{Ab}}(X)$ is an abelian category.

2.1. Recall a category $\mathcal{C}$ is called an abelian category, if

- $\mathcal{C}$ is an additive: "pre-additive category" + "finite products and coproduct exists".
- $\mathcal{C}$ is an exact: "existence of zero object[5]"+ "existence of kernel and cokernel"+ "first isomorphism theorem"

Let's recall the definition of kernel and cokernel.

2.2. Let $f\colon \mathcal{F} \to \mathcal{G}$ be a morphism.

- The kernel for f is the final object $i\colon \mathrm{Ker}_f \to \mathcal{F}$ in
$$\mathcal{C}_{\mathrm{Ker}(f)} := \{g\colon X \to \mathcal{F} | f \circ g = 0\}$$
- The cokernel for f is the initial object $p\colon \mathcal{G} \to \mathrm{Coker}_f$ in
$$\mathcal{C}_{\mathrm{Coker}(f)} := \{h\colon \mathcal{G} \to Y | h \circ f = 0\}$$

2.3. It's easy to verify that $\mathtt{Psh}_{\mathtt{Ab}}(\mathcal{C})$ is an abelian category for any $\mathcal{C}$, as the verification can be done pointwise[6]. Let's check that $\mathtt{Psh}_{\mathtt{Ab}}(X) := \mathtt{Psh}_{\mathtt{Ab}}(\mathtt{Top}(X))$ is an abelian category for any topological space X. Let $f\colon \mathcal{F} \to \mathcal{G}$ be a morphism of abelian sheaves on X:

- We check that kernels exist. Let's define the presheaf $\mathrm{Ker}_f(U) := \mathrm{Ker}(f\colon \mathcal{F}(U) \to \mathcal{G}(U))$. As kernel and equalizer are both limits, by Fubini's lemma, we know Ker_f is a sheaf. The universal property can be verified pointwise.
- We check that cokernels exist. Let's define the presheaf $\mathrm{Coker}^\circ_f(U) := \mathrm{Coker}(f\colon \mathcal{F}(U) \to \mathcal{G}(U))$. This is not necessarily a sheaf, then let's take $\mathrm{Coker}_f := (\mathrm{Coker}^\circ_f)^\sharp$. Let $h\colon \mathcal{G} \to \mathcal{H}$ be a morphism of sheaves on X, such that $h \circ f = 0$. Viewing them as presheaves, there exists a unique morphism $t\colon \mathrm{Coker}^\circ_f \to \mathcal{H}$, such that $h = t^\circ \circ p^\circ$
$$\begin{tikzcd}
\mathcal{F} \arrow[r, "f"] & \mathcal{G} \arrow[r, "h"] \arrow[d, "p^\circ"'] & \mathcal{H} \\
& \mathrm{Coker}^\circ_f \arrow[ur, "t^\circ"] \arrow[d, "\sharp"'] & \\
& \mathrm{Coker}_f \arrow[uur, dotted, "t"'] &
\end{tikzcd}$$
Note that $\mathrm{Mor}_{\mathtt{Psh}}(\mathrm{Coker}^\circ_f, \mathcal{H}) = \mathrm{Mor}_{\mathtt{Sh}}(\mathrm{Coker}_f, \mathcal{H})$, so there exists a unique $t\colon \mathrm{Coker}_f \to \mathcal{H}$ making the diagram commute. Hence $(\mathrm{Coker}_f, p := \sharp \circ p^\circ)$ is a cokernel for f in $\mathtt{Sh}(X)$.
- We show $\mathrm{Coker}_i \to \mathrm{Ker}_p$ in
$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}_f \arrow[r, "i"] & \mathcal{F} \arrow[r, "f"] \arrow[d] & \mathcal{G} \arrow[r, "p"] & \mathrm{Coker}_f \arrow[r] & 0 \\
& & \mathrm{Coker}_i \arrow[r] & \mathrm{Ker}_p \arrow[u] & &
\end{tikzcd}$$

---

[5] object both initial and final

[6] One can show that $\mathtt{Psh}_{\mathcal{A}}(\mathcal{C})$ is an abelian category as long as $\mathcal{A}$ is.

is an isomorphism. Since $\texttt{Psh}_{\texttt{Ab}}(X)$ is abelian, we know $\mathrm{Coker}_i^\circ \to \mathrm{Ker}_{p^\circ}$ is an isomorphism. It suffice to show the natural inclusion are isomorphisms

$$\begin{tikzcd}
(\mathrm{Ker}_{p^\circ})^\sharp \arrow[r, "\alpha"] \arrow[d, equal] & \mathrm{Ker}_p \arrow[d, dotted, dash, "?"] \\
(\mathrm{Coker}_i^\circ)^\sharp \arrow[r, "\beta"] & \mathrm{Coker}_i
\end{tikzcd}$$

For example, if we want to show α is an isomorphism, the only thing we could do it to unwrap the definitions. First of all, p and p° are define as the presheaf-cokernel and sheaf-cokernel, they fit into the following diagram in $\texttt{Sh}(X)$:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "p^\circ"] \arrow[d, equal] & \mathrm{Coker}_f^\circ \arrow[r] \arrow[d] & 0 \\
A \arrow[r, "f"] & B \arrow[r, "p"] & \mathrm{Coker}_f &
\end{tikzcd}$$

Then $\mathrm{Ker}(p^\circ)$ and $\mathrm{Ker}(p)$ are defined as presheaf-kernel

$$\begin{tikzcd}
 & & & \mathcal{H} \arrow[d, hook] & \\
0 \arrow[r] & \mathrm{Ker}_{p^\circ} \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, equal] & \mathrm{Coker}_f^\circ \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Ker}_p \arrow[r] \arrow[d] & B \arrow[r] & \mathrm{Coker}_f & \\
 & \mathcal{H} & & &
\end{tikzcd}$$

By snake lemma[7] (or second isomorphism theorem of abelian groups), we have exact sequence of presheaves

$$0 \longrightarrow \mathrm{Ker}_{p^\circ} \longrightarrow \mathrm{Ker}_p \longrightarrow \mathrm{Coker}_f^\circ \longrightarrow \mathrm{Coker}_f$$

Namely, for any $U \subset X$, we have exact sequences

$$0 \longrightarrow \mathrm{Ker}_{p^\circ}(U) \longrightarrow \mathrm{Ker}_p(U) \longrightarrow \mathcal{H}(U) \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{H}(U) \longrightarrow \mathrm{Coker}_f^\circ(U) \longrightarrow \mathrm{Coker}_f(U)$$

Since taking filtered colimit preserves finite limits, taking colimit at $x \in X$, we have

$$0 \longrightarrow \mathrm{Ker}_{p^\circ,x} \longrightarrow \mathrm{Ker}_{p,x} \longrightarrow \mathcal{H}_x \longrightarrow 0$$

$$0 \longrightarrow \mathcal{H}_x \longrightarrow \mathrm{Coker}_{i,x}^\circ \xrightarrow{\sim} \mathrm{Coker}_{i,x} \ .$$

From the second sequence, we know $\mathcal{H}_x = 0$, hence by the first sequence, $\mathrm{Ker}_{p^\circ,x} = \mathrm{Ker}_{p,x}$ for each $x \in X$. Thus the induced map on étale space $\mathfrak{K}er_{p^\circ} \to \mathfrak{K}er_p$ is an homeomorphism: a bijective local homeomorphism, hence the presheaf of sections $(\mathrm{Ker}_{p^\circ})^\sharp \to \mathrm{Ker}_p$ is an isomorphism.

[7]See [**?**, 2.2.7].

3. Miscellaneous

3.1. Let X be a topological space, let $\mathcal{F}, \mathcal{G} \in \mathtt{Sh}(X)$. Then $f\colon \mathcal{F} \to \mathcal{G}$ is monic (epic) if and only if f is injective (surjective) on stalks. This follows from the observation

- In an abelian category, a morphism is monic (epic) iff the kernel (cokernel) is zero.
- A sheaf is zero if and only if all the stalks are zero.

3.2. Let $X \to Y$ be a continuous map of topological spaces. For any sheaf $\mathcal{F}$ on X, we define

- the direct image sheaf $f_*\mathcal{F}$ on Y by
$$f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V)).$$
- the inverse image sheaf $f^{-1}\mathcal{G}$ on X to be the sheaf associated with the presheaf
$$U \mapsto \lim_{V \supseteq f(U)} \mathcal{G}(V).$$

3.3. We show the assembling lemma for sheaves holds

Lemma 3.1. *Let $\mathcal{G}$ be a sheaf satisfying sheaf axioms. Then $\sharp\colon \mathcal{G} \to \mathcal{G}^\sharp$ is an isomorphism.*

Proof. We need to check that the following natural map is an isomorphism for any open $V \subset X$:
$$\mathcal{G}(V) \to \mathrm{Mor}_{\mathtt{Top}_X}(V, \mathrm{colim}_{h_U/\mathcal{G}} U).$$
It follows from two simple observations:

- If $f\colon X_1 \to X_2$ is a continuous map between topological spaces, and $f(X_1) \subset X_3$, then f is uniquely determined by $f\colon X_1 \to X_3$. Thus we have
$$\mathrm{Mor}_{\mathtt{Top}_X}(V, \mathrm{colim}_{h_U/\mathcal{G}} U) = \mathrm{Mor}_{\mathtt{Top}_X}(V, \mathrm{colim}_{h_U/\mathcal{G}, U \subseteq V} U)$$
- By Yoneda lemma, $\mathrm{Mor}_{\mathtt{Psh}}(h_U, \mathcal{G}) = \mathcal{G}(U)$. Thus by sheaf property of $\mathcal{G}$,
$$\mathrm{colim}_{h_U/\mathcal{G}, U \subseteq V} U = \mathrm{colim}_{h_V/\mathcal{G}} V = \coprod_{s \in \mathcal{G}(V)} V.$$

□

Then the bijection in the key fact is given by bijection $a\colon f \mapsto f \circ \sharp$ and $b\colon f \mapsto \sharp^{-1} \circ f^\sharp$
$$\mathrm{Mor}_{\mathtt{Sh}(X)}(\mathcal{F}^\sharp, \mathcal{G}) \underset{b}{\overset{a}{\rightleftarrows}} \mathrm{Mor}_{\mathtt{Psh}(X)}(\mathcal{F}, \mathrm{Fgt}(\mathcal{G}))$$

4. Homework

4.1. Let X be a topological space. Let $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$ be an exact sequence of abelian presheaves on X. Let $x \in X$ be a point. Show that the induced sequence on stalks is exact:
$$0 \longrightarrow \mathrm{colim}_{x \in U} \mathcal{F}'(U) \longrightarrow \mathrm{colim}_{x \in U} \mathcal{F}(U) \longrightarrow \mathrm{colim}_{x \in U} \mathcal{F}''(U) \longrightarrow 0\,.$$

4.2. (Choose one of the following) We resume the discussion in subsection 3.2,

- Show that the étale space of $f^{-1}\mathcal{G}$ is isomorphic to $X \times_Y \mathfrak{G}$.
- Show that the étale space of $f_*\mathcal{F}$ is isomorphic to $\mathfrak{F}$, when $X \to Y$ is a local homeomorphism.
- What is the étale space of $f_*\mathcal{F}$ in general?

CHAPTER 3

Baby-Tohoku

1. Abelian categories

1.1. Recall an abelian category is an additive category plus two axioms:

- (AB1) Every morphism has kernel and cokernel.
- (AB2) For any morphism f, the natural map $\mathrm{Coim}_f \to \mathrm{Im}_f$ is an isomorphism.

1.2. In order to discuss homological algebra, Grothendieck considered enhanced axioms for abelian categories:[1]

- (AB3) Coproducts exist.
- (AB4) Direct sums are exact+(AB3).
- (AB5) Filtered colimits are exact+(AB3).
- (AB3*) Products exist.
- (AB4*) Products are exact+(AB3*).
- (AB5*) Filtered limits are exact+(AB3*).

1.3. In general, a nonzero abelian category can never satisfy both (AB5) and (AB5*). Consider $\bigoplus_I A \to \prod_I A$ for a nonzero object A: Taking filtered colimit in the source, (AB5) implies that α is monic. taking finite factors in the target, (AB5*) implies that α is epic. However, α is never an isomorphisms when I is infinite.

1.4. In the category of locally compact abelian groups, (AB5*) fails: the limit of $p^n\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ goes to $0 \to \mathbb{Z} \to \mathbb{Z}_p$. By Pontryagin duality, (AB5) also fails.[2]

1.5. Let $\mathcal{C}$ be a category, and $(U_i)_{i\in I}$ be a set of objects. We say (U_i) is a family of generators if for any object A in $\mathcal{C}$, and any proper subobject $B \to A$, there exists an $U_i \to A$ that does not factor through B. When $\{U\}$ is a family of generators, we say U is a generator. If a category has a family of generators, then the category of subobjects of any fixed object is small, as any object is determined by the family of morphisms that does not factor through it. Similarly, if a category has a family of cogenerators, then the category of quotients of any object is small. The situation is particularly nice with (AB3):

1.6. In an abelian category satisfying (AB3), the following are equivalent:

- $(U_i)_{i\in I}$ is a family of generators of $\mathcal{C}$.
- $\bigoplus_{i\in I} U_i$ is a generator of $\mathcal{C}$.
- Any object is a quotient of direct summands of U.

The third bullet follows from the second, as one take the direct sum of U indexed by $\mathrm{Mor}_{\mathcal{C}}(U, A)$, then the image is a sub of A, which has to be A, so the cokernel is zero, hence $U^{(I)} \to A$ is epic. From the third bullet, take any sub-object $W \to A$, if all $U^{(I)}$ factors through W, then $q \circ f \colon U^{(I)} \to A \to A/W$ is zero. Since f is epic, we know $q = 0$, hence $W = A$. For example, $\{\mathbb{Z}|_U\}_{\mathrm{Ob}(\mathtt{Top}(X))}$ is a family of generators in $\mathtt{Sh}(X)$ and $\mathbb{Z}_U(V) = \bigoplus_{[V,U]} \mathbb{Z}$ form a family of generators for $\mathtt{Psh}(X)$, although the category of presheaves is big.[3]

[1] Grothendieck's (AB5) is stated as "filtered union of subobjects is distributive over intersection with another subject". Is it equivalent to the exactness of filtered colimit. See [**?**, 079A].

[2] Explicitly, the colimit of S^1 in $\mathtt{LCAb}$ is a point, note that the Hausdorff condition forces the solenoid to collapse. Then note that when $\mathbb{Z}_p$ as a limit in $\mathtt{LCAb}$ is equipped with profinite topology, then $[\mathbb{Z}_p, S^1] = \mathbb{Q}_p/\mathbb{Z}_p$.

[3] Are there abelian category without generators?

1.7. A Grothendieck abelian category is an abelian category which satisfies (AB5) and has generators. We remark that (AB4) is implied by (AB5), by taking colimit of finite summands. [4]

2. Cardinality

2.1. Let $\mathcal{A}$ be an abelian category with a generator U and M an object of $\mathcal{A}$. We define the size $|M|$ as the cardinality of subobjects of M. For example, one can show in a short exact sequence $0 \to M' \to M \to M'' \to 0$, we have $|M'|, |M''| \leq |M|$. One can show if $|M| \leq \kappa$, then M is a quotient of a direct sum of at most κ copies of U, then there is a set of isomorphic classes of objects[5] with $|M| \leq \kappa$.

2.2. Let $\{B_\beta\}_{\beta\in\alpha}$ be an inductive indexed by some ordinal. Assume $\mathrm{colim}_{\beta\in\alpha} B_\beta$ exists, then there is a natural map

$$\operatorname*{colim}_{\beta\in\alpha}\mathrm{Mor}(A, B_\beta) \to \mathrm{Mor}(A, \operatorname*{colim}_{\beta\in\alpha} B_\beta)$$

We say A is α-small with respect to a set of arrows I, if for all direct systems in I, the map is an isomorphism. [6] Here is a reason why we study cardinality:

Lemma 2.1 (smallness). *If $|M| \leq \kappa$, then for $\mathrm{cf}(\alpha) > \kappa$, M is α-small with respect to injections.*

3. Injectives

Let $\mathcal{A}$ be a Grothendieck category with a generator U.

3.1. We call an object I injective, if $\mathrm{Mor}_{\mathcal{A}}(-, I)$ is right exact. Every injection to I extends from the source to the target. By abstract non-sentence we have the useful lemma:

Lemma 3.1. *Given $v \dashv u$, if v preserves monics, then u preserves injectives.*

Here is a simple application:

- Any abelian group D is injective $\mathbb{Z}$-modules iff it is divisible.
- Thus $\mathrm{Hom}_{\mathbb{Z}}(R, D)$ is an injective R-module iff D is divisible abelian group
- Thus R-`Mod` has enough injectives:

$$M \to \prod_{\mathrm{Hom}_R(M, \mathrm{Hom}(R, \mathbb{Q}/\mathbb{Z}))} \mathrm{Hom}(R, \mathbb{Q}/\mathbb{Z})$$

3.2. One can show that an object I of $\mathcal{A}$ is injective if any subobject $M \subset U$, $M \to I$ extends.[7] Furthermore, as Godement resolutions, objects in $\mathcal{A}$ has functorial injective embeddings, defined by first doing the push-forward diagram

$$\begin{tikzcd}
\bigoplus_{N\subset U}\bigoplus_{f\in[N,M]} N \arrow[r] \arrow[d] & M \arrow[d] \\
\bigoplus_{N\subset U}\bigoplus_{f\in[N,M]} U \arrow[r] & \mathbf{M}_0(M)
\end{tikzcd}$$

then use transfinite induction to define $\mathbf{M}_\alpha$ for any ordinal, until some ordinal $\mathrm{cf}(\alpha) > |U|$.[8] Then note that $M \to \mathbf{M}_\alpha(M)$ factors through some $\mathbf{M}_{\alpha'}(M)$, with $\alpha' < \alpha$. By construction ϕ extends to $\mathbf{M}_{\alpha'+1}(M)$ hence into $\mathbf{M}_\alpha(M)$, we win by Baer.

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[4] Discrete diagrams, (including empty diagram) are not filtered, filtered categories are connected.

[5] The objects themselves may not be a set, though.

[6] See [?, 05AB] for where the "smallness" comes from.

[7] The proof is similar to Baer criterion for injective modules, see [?, 079G], where Zorn's lemma is used.

[8] Here we do not have to deal with proper classes, we only need to do the transfinite induction up to $\alpha = |U|+1$. Note that $|U|$ may depend on our choice of U.

3.3. Let $\mathcal{A}$ be an abelian category. A monic $A \to B$ of $\mathcal{A}$ is called an essential injection, if for every quotient $B \to B'$, the composition $A \to B \to B'$ is not monic. A epic $f\colon A \to B$ is called essential surjection, if for every proper subobject $A' \subset A$, the composition $A' \to A \to B$ is not epic. One easily observes that composition of essential injections (surjections) are essential, and are preserved by pullback by injections (pushout by surjections).[9]

$$\begin{tikzcd}
A \bigcap C \arrow[r, hook, "\text{ess}"] \arrow[d, hook] & C \arrow[r, dotted, two heads] \arrow[d, hook] \arrow[rr, bend right] & Q \arrow[r, dotted] \arrow[d, dotted, equal] & T \\
A \arrow[r, hook, "\text{ess}"] \arrow[d] & B \arrow[r, dotted, two heads] \arrow[d] & Q & \\
B/C \arrow[r, equal] & B/C & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & C \arrow[r, equal] \arrow[d, hook] & C \arrow[d] \\
 & K \arrow[r, dotted, hook] \arrow[d, dotted, equal] & A \arrow[r, two heads, "\text{ess}"] \arrow[d] & B \arrow[d] \\
T \arrow[r, dotted] \arrow[rr, bend left] & K \arrow[r, dotted, hook] & A/C \arrow[r, two heads, "\text{ess}"] & C/f(B)
\end{tikzcd}$$

Note that by (AB5), filtered colimits are exact, similar arguments show that filtered colimit of essential injections is essential injection, this is true for example in R-$\mathtt{Mod}$. [10]

3.4. We say an injection $M \to I$ is an injective hull, if I is injective and $M \to I$ is an essential injection. In a Grothendieck abelian category, every object has an injective hull[11]. A basic example of injective hull is the field of fractions K of a domain R: K is injective, as Baer criterion for principal ideals holds, and $K \to R$ is essential. Here are some properties of injection hulls

- Injection induce injection on injective hulls.
- Essential injection induces isomorphism on injective hulls.
- Injective hull splits from larger injective embeddings.

The last bullet motivates to describe the injective modules which cannot be properly written as direct sums

- An indecomposable injective module is the hull of any of its submodules.
- The endomorphism of an injective hull is a local ring[12].
- The set of zerodivisors for an indecomposable E is a prime in R, and E is $R_{\mathfrak{p}}$-injective.

The discussion cumulates to the following classification theorem [**?**, 08YA]:

Theorem 3.2 (Classification). *On a Noetherian ring:*

- *Every injective module is direct sum of indecomposable injectives.*
- *Every indecomposable injective is a hull of some residue field.*

Injective hulls produce a beautiful duality theorem

Theorem 3.3 (Matlis duality). *Let R be a complete local Noetherian ring. Let E be an injective hull of κ over R, then* $\mathrm{Hom}_R(-, E)$ *induces an anti-equivalence of categories*

$$\mathtt{a.c.c.Mod}_R \Longleftrightarrow \mathtt{d.c.c.Mod}_R$$

[9] See [**?**, 08XK].

[10] Note that Freyd-Mitchell does not preserve colimits, this does not imply arbitrary abelian category satisfy (AB5), of course.

[11] See [**?**, 08Y3].

[12] Possibly noncommutative

4. Localization

4.1. In the study a categories, people may want to forget some unimportant information, for example, one may want to

- regard quasi-isomorphism as isomorphisms or
- identify the homotopic maps

Thus one is led to think about "localization" and "quotient" of categories. Let's consider the pre-additive category $\mathtt{B}R$, where R is a commutative ring. Inverting maps in $R\backslash\mathfrak{p}$ yields the category $\mathtt{B}R_{\mathfrak{p}}$, while forcing maps in $\mathfrak{p}$ to be zero yields the category $\mathtt{B}R/\mathfrak{p}$. Currently let's focus on localizations, as it preserves a little bit more information.[13]

4.2. Let $\mathcal{C}$ be a small category. A set of arrows S of $\mathcal{C}$ is called a left multiplicative system if

- (LM1) S preserves all compositions (1-ary and 2-ary)
- (LM2) Pushout diagrams with opposite edges in S exist.

$$\begin{tikzcd} X \arrow[r, "g"'] \arrow[d, "t"'] & Y \arrow[d, dotted, "s"] \\ Z \arrow[r, dotted, "f"] & W \end{tikzcd}$$

- (LM3) Coequal arrows can be equalized.

$$X \xrightarrow{s} Y \underset{g}{\overset{f}{\rightrightarrows}} Z \overset{t}{\dashrightarrow} W$$

We define a new category $S^{-1}\mathcal{C}$ by doing left calculus of fractions on $\mathrm{Mor}_{\mathcal{C}}$.

- A pre-morphism from X to Y in $S^{-1}\mathcal{C}$ is a pair $X \longrightarrow Z \xleftarrow{s} Y$
- We define two pre-morphisms to be S-equivalent if they dominate a third.
- We define the set of morphisms in $S^{-1}\mathcal{C}$ as equivalent classes of pre-arrows.
- Composable morphisms compose by pushing out the corner square.

An equivalent way to define the set of morphisms in

$$\mathrm{Mor}_{S^{-1}\mathcal{C}}(X,Y) := \mathrm{colim}_{(s\colon Y\to Y')\in Y/S}\mathrm{Mor}_{\mathcal{C}}(X,Y')$$

Here we note that the left multiplicative system has to be a set. Is this a problem in the previous setup? Yes, for otherwise the set of pre-morphisms are is not necessarily a set.

4.3. One checks that every morphism in the left multiplicative system becomes an isomorphism via the localization functor $Q\colon \mathcal{C} \to S^{-1}\mathcal{C}$, sending $X \to Y$ to $X \to Y \xleftarrow{1} Y$. The localization category enjoys the similar universal property as localization of rings: For any functor $F\colon \mathcal{C} \to \mathcal{D}$ such that $F(S)$ are invertible, the functor uniquely factors as $F' \circ Q$. By filtered Fubini:

LEMMA 4.1 (localization). *The localization functor commutes with finite colimits.*

PROOF. The key point here is that by (LM3), the category Y/S is filtered, and filtered limits commutes with finite colimits in $\mathtt{Sets}$. See [?, 04VG]. $\square$

4.4. There is more to say about abelian categories:

- If S is left multiplicative system, then $S^{-1}\mathcal{A}$ has cokernels, and Q commutes with cokernels.
- If S is right multiplicative system, then $S^{-1}\mathcal{A}$ has kernels, and Q commutes with cokernels.
- If S is multiplicative system, then $S^{-1}\mathcal{A}$ is abelian and Q is exact.

[13] Quotients can be obtained form localizations but not otherwise.

4.5. Here is a useful special case of localization. Let $\mathcal{A}$ be an abelian category, a proper subcategory $\mathcal{C}$ is called a Serre subcategory, if $\mathcal{C}$ is closed under zero, subobjects, quotients, and extension. For example, let $F\colon \mathcal{A} \to \mathcal{B}$ be an exact functor, then the subcategory of objects C such that $F(C) = 0$ is a Serre subcategory. Conversely,

THEOREM 4.2. *Let $\mathcal{A}$ be an abelian category, and $\mathcal{C}$ be a Serre subcategory. There exists an abelian category $\mathcal{A}/\mathcal{C}$ and an exact functor $F\colon \mathcal{A} \to \mathcal{A}/\mathcal{C}$ which is essentially surjective and kernel is $\mathcal{C}$. We define the set of arrows $S = \{f \in \mathrm{Arr}(\mathcal{A})\colon \mathrm{Ker}_f, \mathrm{Coker}_f \in \mathrm{Ob}(\mathcal{C})\}$. and define $\mathcal{A}/\mathcal{C} := S^{-1}\mathcal{A}$. In this category, an object is zero iff it lies in $\mathcal{C}$.*

Let $S = R\backslash\mathfrak{p}$. Let Σ be the class of morphisms $f\colon M \to N$ such that $S^{-1}f$ is an isomorphism, and $R\text{-}_\Sigma\mathtt{Mod}$ be the category of objects M such that $S^{-1}M = 0$. This is not $R/\mathfrak{p}\text{-}\mathtt{Mod}$[14]. We have equivalence

$$\Sigma^{-1}(R\text{-}\mathtt{Mod}) \longleftrightarrow S^{-1}R\text{-}\mathtt{Mod} \longleftrightarrow R\text{-}\mathtt{Mod}/R\text{-}_\Sigma\mathtt{Mod}.$$

5. Delta-functors

5.1. Let $\mathcal{A}, \mathcal{B}$ be abelian categories. A delta functor in degree (a, b) is given by the following data:

- a collection $T^n\colon \mathcal{A} \to \mathcal{B}$ of additive functors for $a < n < b$
- for every short exact sequence, we are given connecting morphisms

$$\delta_{A' \to A \to A''}\colon T^n(A'') \to T^{n+1}(A')$$

such that

- For every short exact sequence, the sequence
$$\cdots \to T^0(A') \to T^0(A) \to T^0(A'') \to T^1(A') \to T^1(A) \to T^1(A'') \to T^2(A') \to \cdots$$
is a complex[15].
- The connecting homomorphism are natural, namely, the following square commutes:

$$\begin{tikzcd} F^n(A'') \arrow[r, "\delta"] \arrow[d] & F^{n+1}(A') \arrow[d] \\ F^n(B'') \arrow[r, "\delta"] & F^{n+1}(B') \end{tikzcd}$$

We say a delta is an exact delta-functor, if the first bullet is exact. We say an exact delta-functor is a cohomological functor, if it is defined for all degrees.

5.2. We say an additive functor $F\colon \mathcal{A} \to \mathcal{B}$ is effaceable, if for any $A \in \mathcal{A}$, we can find a monic $u\colon A \to M$, such that $F(u) = 0$. Note that this is weaker than asking $F(A') = 0$. We say a delta functor is a universal delta-functor, if it is initial. One can show

LEMMA 5.1 (Key). *A termwise effaceable delta functor is universal.*

PROOF. Stare at the diagram

$$\begin{tikzcd} T^0(M) \arrow[r] \arrow[d] & T^0(A') \arrow[r] \arrow[d] & T^1(A) \arrow[r, "0"] \arrow[d] & T^1(M) \\ T'^0(M) \arrow[r] & T'^0(A') \arrow[r] & T'^0(A) & \end{tikzcd}$$

∎

[14]For example, it is not closed with respect to extensions.

[15]Note necessarily exact

Given a functor $F\colon \mathcal{A} \to \mathcal{B}$, there exists at most one universal delta functor with F in degree 0, denote by $F = S^0F, S^1F, \cdots$, we call S^iF the i-th right satellite functor of F, if it exists.

THEOREM 5.2. *Let $\mathcal{A}$ be an abelian category such that any object A admits an effacement, then S^iF exists and are effaceable functors for $i > 0$. In order that S^iF be exact, it is necessary and sufficient that F is half-exact*[16] *: For $P \subseteq Q \subseteq R$, we have*

$$\mathrm{Im}(F(Q) \to F(Q/P)) \supseteq \mathrm{Ker}(F(Q/P) \to F(R/P)).$$

5.3. Here is a very different situation where satellites also exist. Let $F\colon \mathcal{A} \to \mathcal{A}'$ be a functor. Suppose the isomorphism classes of objects in $\mathcal{A}$ is a set and $\mathcal{A}'$ is cocomplete (Coc), namely finite direct sums exist, then any additive functor has satellites. If $\mathcal{A}'$ is moreover (AB5), then the satellites are exact. Here we do not require $\mathcal{A}$ to be cocomplete. This will be useful, for example, when we work with the category of abelian algebraic groups over $\mathbb{C}$[17]. The functor $S^iF(A)$ is constructed as colimit of objects $F(M/A)/\mathrm{Im}(F(M))$ for all monics $A \to M$ in $\mathcal{A}$.[18]

5.4. We summarize the concepts into the following diagram

$$\begin{tikzcd}
 & \texttt{F:(AB2)}\to\texttt{(Coc)} \arrow[r] \arrow[d] & \texttt{F:(AB2)}\to\texttt{(AB5)+half-exact} \arrow[d] \\
\texttt{universal-delta} \arrow[r] & \texttt{Satellite} \arrow[r] & \texttt{Exact Satellites} \\
 & \texttt{effaceable } F \arrow[u] \arrow[r] & \texttt{half-exact } F \arrow[u] \\
\texttt{delta-functor} \arrow[uu] \arrow[ur, bend left] \arrow[uur, dotted, bend left] \arrow[rr] & & \texttt{exact delta functors} \arrow[d] \arrow[uu, bend right] \\
 & & \texttt{cohomological functor} \arrow[uuu, bend right=80]
\end{tikzcd}$$

6. Derived functors

6.1. In a category with enough injectives, by the homotopy triviality of injectives, we can show

- The functors $\mathrm{R}^iF(A) := \mathrm{H}^i(I_A^\bullet)$ are universal delta functors.
- Derived functors can be calculated by acyclic resolutions.[19]

Sometimes one may want to deal with multiple functors simultaneously, say $f_*(- \otimes \mathcal{L})$: what is its "derived functor"?. One may want to first think about the domain and codomain of derived functors: Derived categories. They are the homotopical incarnations of the original abelian category.

[16]Note that this is different form $F(Q) \to F(Q/P) \to F(R/Q)$ being exact! Left and right exactness both imply half-exactness.

[17]In order to be an abelian category, the inseparable isogenies are not in consideration

[18]See [**?**, 2.2], [**?**, 2.2].

[19]This motivates a question: in an abelian category without enough injectives, but with enough "F-acyclic objects", can we define derived functors of F using acyclics? Let's defer the discussion to the future.

CHAPTER 4

Sheaves-III

We give some examples of acyclic sheaves.

1. Flasque sheaves

Let X be a topological space. Let $\mathcal{F}$ be a sheaf on X. We say $\mathcal{F}$ is flasque[1] if for every inclusion $V \subseteq U$, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective.

1.1. Here are some trivial examples:

- Any constant presheaf on an irreducible topological space is flasque sheaf. Recall a topological space is irreducible if any two open subsets has nonempty intersection. Thus the sheaf condition is obviously verified.
- Any surjection $f\colon A \to B$ of sets gives a flasque sheaf $\mathcal{F}_f$ on the Sierpińsky space
$$(\{0,1\}; \{\emptyset, \{0\}, \{0,1\}\})$$
by taking f to be $\rho^{\{0,1\}}_{\{0\}}\colon \mathcal{F}_f(\{0,1\}) \to \mathcal{F}_f(\{0\})$.

1.2. Here are some further properties. For any short exact sequence in $\mathtt{Sh}(X)$,
$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

- If $\mathcal{F}', \mathcal{F}''$ are flasque, then so is $\mathcal{F}$.
$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$
- If $\mathcal{F}', \mathcal{F}$ are flasque, then so is $\mathcal{F}''$
$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$
- If $\mathcal{F}, \mathcal{F}''$ are flasque, then $\mathcal{F}'$ isn't necessarily flasque.
$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$
- If $\mathcal{F}'$ is flasque, then the sequence is also exact in $\mathtt{Psh}(X)$.[2] Namely, the presheaf-quotient of flasque sheaves is automatically a sheaf.[3]
- If $f\colon X \to Y$ is a continuous map, and $\mathcal{F} \in \mathtt{Sh}(X)$ is flasque, then $f_*\mathcal{F}$ is flasque.

1.3. Any sheaf $\mathcal{F}$ can be embedded into a flasque sheaf: the sheaf of discontinuous sections: for any $U \in \mathtt{Top}(X)$, we take
$$\mathcal{F}^\wedge(U) := \mathrm{Mor}_{\mathtt{Sets}_{/X}}(U, \mathfrak{F}).$$
This is a sheaf as local sections unique glue. The sheaf is flasque by axiom of choice. Further, the morphism of sheaves of sets $\mathcal{F} \to \mathcal{F}^\wedge$ is an injection. We can embed $\mathcal{F}^\wedge/\mathcal{F}$ into $(\mathcal{F}^\wedge/\mathcal{F})^\wedge =: \mathcal{F}^{\wedge 2}$, etc.

[1] Flasque is a French word that has sometimes been translated into English as flabby.

[2] Hint: What is the global section of a sheafification? If the global section of the quotient sheaf is represented on two pieces, can one fix it? In general, use Zorn's lemma.

[3] In general, flasque presheaves are not necessarily sheaves, say, adding whatever to $\mathcal{F}(X)$.

1.4. Iterating the process, for any $\mathcal{F}$, we have an exact sequence of sheaves,

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^{\wedge} \longrightarrow \mathcal{F}^{\wedge 2} \longrightarrow \cdots$$

We call $\{\mathrm{Gd}^{\bullet}_{\mathcal{F}}\}_{\mathbb{N}} := \{\mathcal{F}^{\wedge(\bullet-1)}\}_{\mathbb{N}^{+}}$ the Godement resolution of $\mathcal{F}$. It is an exact sequence of flasque sheaves, except in degree 0, where the kernel recovers $\mathcal{F}$.

1.5. It is noticeable that the Godement resolution is canonical: Given $\mathcal{F} \to \mathcal{G}$, there is a canonical extension $\mathrm{Gd}^{\bullet}_{\mathcal{F}} \to \mathrm{Gd}^{\bullet}_{\mathcal{G}}$. The situation is like, for any topological space X, there is a canonically associated singular chain complex $\mathrm{DK}(S_X)$.

Note that singular cohomology $\mathrm{H}^{*}_{\mathrm{Sing}}(X, \mathbb{R})$ is defined as the cohomology of free dual vector space generated by the complex. We may define flasque cohomology

$$\mathrm{H}^{i}_{\mathrm{Fl}}(X, \mathcal{F}) := \mathrm{Ker}(\mathrm{d}^{i})/\mathrm{Im}(\mathrm{d}^{i-1})$$

as the cohomology of the complex generated by global sections $\Gamma(X, \mathrm{Gd}^{\bullet}_{\mathcal{F}})$:

$$0 \longrightarrow \Gamma(X, \mathrm{Gd}^{0}_{\mathcal{F}}) \xrightarrow{\mathrm{d}^{0}} \Gamma(X, \mathrm{Gd}^{1}_{\mathcal{F}}) \xrightarrow{\mathrm{d}^{1}} \cdots \xrightarrow{\mathrm{d}^{n-1}} \Gamma(X, \mathrm{Gd}^{n}_{\mathcal{F}}) \xrightarrow{\mathrm{d}^{n}} \cdots$$

2. Fine sheaves

2.1. Let's introduce a notion of support. Let $\mathcal{F}$ be a sheaf on X, let $s \in \mathcal{F}(U)$ be a section over an open set U. The support of section s, or the non-vanishing locus of s, denote by $\mathrm{supp}(s)$, is defined to be the complement of the equalizer of the section with 0, namely

$$\mathrm{supp}(s) := \mathrm{Eq}(s, 0).$$

The support of sheaf $\mathcal{F}$, is define as the subset of points of X, over which the stalks are nontrivial

$$\mathrm{Supp}(\mathcal{F}) := \{x \in X | \mathcal{F}_x \neq 0\}$$

2.2. Let X be a topological space. Let $\mathcal{A}$ be a sheaf of rings on X. We say $\mathcal{A}$ is a fine sheaf of rings, if for every open cover $\{U_i \to X\}_{i \in I}$, there exist a family of compactly supported sections $f_i \in \mathcal{A}(U_i)$[4], such that the sections f_i, viewed as sections on X by zero-extension, satisfy $\sum_I f_i = 1$. In general, people call $\{f_i\}_{i \in I}$ a partition of unity subordinate to the covering $\{U_i\}_{i \in I}$. In topology, we have the standard result[5]

THEOREM 2.1 (Existence of Partition of Unity). *Let M be a topological manifold, and $\{U_i\}_{i \in I}$ be an open cover of M, there always exist a partition of unity subordinate to $\{U_i\}_{i \in I}$. In other words, the sheaf of continuous function $\mathcal{C}_X$ is a fine sheaf of rings.*

However, we remark that partition of unity does not hold on non-separated topological space, say the non-separated line, as we need to use Tietze-Urysohn extension lemma in the proof.

2.3. We call a sheaf of modules over a fine sheaf of rings, a fine sheaf[6]. Fine sheaves are flasque-acyclic:

LEMMA 2.2 (Fine lemma). *If $\mathcal{F}$ is a fine sheaf, then $\mathrm{H}^{i}_{\mathrm{Fl}}(X, \mathcal{F}) = 0$ for all $i > 0$.*

PROOF. By definition, we are asked to check that $\Gamma(X, \mathrm{Gd}^{\bullet}_{\mathcal{F}})$ is exact for $n \geq 1$. We split the resolution into short exact sequences $0 \to \mathrm{Gd}'^{n}_{\mathcal{F}} \to \mathrm{Gd}^{n}_{\mathcal{F}} \to \mathrm{Gd}''^{n}_{\mathcal{F}} \to 0$. Since $\Gamma(X, -)$ is left exact, it suffices to show that it preserves surjection for each n. Given a section $s \in \Gamma(X, \mathrm{Gd}''^{n}_{\mathcal{F}})$, it is represented by a collection of sections of $s|_{U_i} = \overline{s_i}$, for some $s_i \in \mathrm{Gd}^{n}_{\mathcal{F}}(U_i)$. As $s = 1 \cdot s = (\sum f_i) \cdot s = \sum (f_i \cdot s) = \sum f_i \cdot s|_{u_i}$, we know $\sum f_i \cdot s_i$ is a global section in $\mathrm{Gd}^{n}_{\mathcal{F}}$ representing s. □

[4]One observes that the fine sheaves are designed for smooth manifolds, where bump functions naturally lives.

[5]See [**?**, 1.11].

[6]See [**?**, 4.35]

2.4. Fine sheaves are not necessarily flasque. Flasque sheaves are not necessarily fine, it even does not have a sheaf of ring to act on.[7] The only thing we used here is the functoriality of Godement resolution: An $\mathcal{A}$-module structure is an action $\mathcal{A} \times \mathcal{F} \to \mathcal{F}$ satisfying certain compatibility conditions, it naturally induces a compatible action $\mathcal{A} \times \mathcal{F}^\wedge \to \mathcal{F}^\wedge$.

2.5. Let X be an n-dimensional differentiable manifold. Let $\mathbb{R}_X$ be the constant sheaf, the sheafification of the constant presheaf valued in $\mathbb{R}$. Let Ω_X^i be the sheaf of smooth i-forms, this is a fine sheaf, with respect to $\mathcal{C}_X^\infty$. Exterior differentials assemble the sheaves into a complex

$$\mathrm{dR}_X^\bullet := \ 0 \longrightarrow \Omega_X^0 \xrightarrow{\mathrm{d}^0} \Omega_X^1 \xrightarrow{\mathrm{d}^1} \cdots \longrightarrow \Omega_X^{n-1} \xrightarrow{\mathrm{d}^{n-1}} \Omega_X^n \longrightarrow 0$$

We call the i-th cohomology group of complex of global sections $\Gamma(X, \mathrm{dR}^\bullet)$ the i-th de Rham cohomology of X, namely

$$\mathrm{H}_{\mathrm{dR}}^i(X, \mathbb{R}) := \mathrm{Ker}(\mathrm{d}^i)/\mathrm{Im}(\mathrm{d}^{i-1})$$

What's the relation between de Rham cohomology and sheaf cohomology? They are isomorphic. The fact follows from three lemmas:

- The Poincaré lemma: the complex $0 \to \mathbb{R}_X \to \mathrm{dR}_X^\bullet$ is exact.
- The acyclicty lemma: flasque-cohomology can be calculated by flasque-acyclic resolutions.
- The fine lemma: fine sheaves are flasque-acyclic.

The acyclicty lemmas is somehow mysterious, as we did not specify what "calculate" means. For example, do we have a "mysterious functor" $\rightsquigarrow$: $\mathrm{H}_{\mathrm{Fl}}^i(X, \mathbb{R}) \to \mathrm{H}_{\mathrm{dR}}^i(X, \mathbb{R})$ or the other way?

2.6. In class we also saw Čech cohomology $\check{\mathrm{H}}^i(X, \mathbb{R})$, which is defined via Čech complexes, and derived functor cohomology $\mathrm{H}^i(X, \mathbb{R})$, which is defined using injective sheaves. These five cohomology theories are isomorphic, via the "Čech-to-blah" spectral sequence $\check{\mathrm{H}}^i(X, \mathrm{H}_{\mathrm{blah}}^j(\mathbb{B}^n, \mathbb{R})) \Rightarrow \mathrm{H}_{\mathrm{blah}}^{i+j}(X, \mathbb{R})$. and the banal calculation for the cohomology of balls $\mathbb{B}^n \subset \mathbb{R}^n$.[8]

$$\begin{tikzcd}
 & \mathrm{H}^i(X, \mathbb{R}) & \\
\mathrm{H}_{\mathrm{Sing}}^i(X, \mathbb{R}) & \check{\mathrm{H}}^i(X, \mathbb{R}) \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \mathrm{H}_{\mathrm{dR}}^i(X, \mathbb{R}) \\
 & \mathrm{H}_{\mathrm{Fl}}^i(X, \mathbb{R}) &
\end{tikzcd}$$

3. Soft sheaves

3.1. We introduce a very useful kind of sheaves in the study of locally compact spaces[9]. In particular, it applies to topological manifolds.

3.2. A sheaf $\mathcal{F}$ on X is called soft, if for any compact open inclusion $K \subset U$ in X, and any $s \in \mathcal{F}(U)$, there exists a section $t \in \mathcal{F}(X)$, which restricts to s on some open neighborhood $K \subset V \subset U$. Or equivalently, $\mathcal{F}$ is soft, if for any compact inclusion $i\colon Z \to X$, the induced map $\Gamma(X, \mathcal{F}) \to \Gamma(K, i^*\mathcal{F})$ is always a surjection.

[7]One may even not equip $\mathcal{F}$ with a fine sheaf of rings, for example, when X is irreducible.

[8]A spectral sequence in this situation is nothing but upgrading the snake to *the Snake* by chasing on a larger diagram.

[9]Hausdorff spaces in which every point has a compact neighborhood, see [**?**, III.1]

3.3. The soft sheaves are designed to study cohomology with compact supports. Let's consider the functor of compactly supported global section $\Gamma_c(X, -)$.

- Restriction lemma The restriction of soft sheaves to locally closed subset is soft.
- Acyclicity lemma Soft sheaves are $\Gamma_c(X, -)$-acyclic.[10] ($\Gamma(X, -)$ acyclic if X is compact.)
- Softness lemma: a sheaf of modules over a soft sheaf of rings is soft.[11]
- Urysohn lemma: the sheaf $\mathcal{C}_X$ is soft for any topological manifold X.

3.4. We crystallize our discussion to the following theorem

THEOREM 3.1 (Proper Base Change). *Let $f\colon X \to Y$ be a proper morphism between locally compact spaces, and* F *be a sheaf on X, then for any $y \in Y$, the colimit*

$$(\mathrm{R}^n f_* \mathcal{F})_y \xrightarrow{\sim} \mathrm{H}^n(f^{-1}(y), \mathcal{F})$$

where $\mathrm{R}^n f_* \mathcal{F}$ is a presheaf on ***Top***(Y)*, where $V \mapsto \mathrm{H}^n(f^{-1}(V), \mathcal{F})$*

The proper base change follows from the following observation

- Properness implies that $\{U_y\} := \{f^{-1}(V)\}_{y \in V}$ is cofinal in neighborhoods of $f^{-1}(y)$.
- Injective resolutions are soft.
- Soft resolution restricts to soft resolution.
- Soft sheaves are acyclic, so can be used to calculate cohomology:
$$\mathrm{H}^n(f^{-1}(y), i^* \mathcal{F}) = \mathrm{H}^n \Gamma(f^{-1}(y), i^* I^\bullet).$$
- Softness implies that the following natural map is isomorphism:
$$\varinjlim \Gamma(U_y, I^\bullet) \to \Gamma(f^{-1}(y), i^* I^\bullet).$$
- Filtered colimit is exact:
$$\mathrm{H}^n(\varinjlim \Gamma(U_y, I^\bullet)) = \varinjlim \mathrm{H}^n \Gamma(U_y, I^\bullet).$$
- Summing up, the natural map induced by restriction is an isomorphism:
$$\varinjlim \mathrm{H}^n(U_y, \mathcal{F}) \to \mathrm{H}^n(f^{-1}(y), i^* \mathcal{F}).$$

4. Injective sheaves

4.1. Let X be a topological space, let $\mathcal{F}$ be a sheaf on X, we show there always exists an embedding of $\mathcal{F}$ into an injective sheaf $\mathcal{I}$. Namely, we show that the abelian category $\mathtt{Sh}(X)$ has enough injectives, so that we can do homological algebra, define sheaf cohomology as derived functor of global sections, etc. If we use the argument in Tohoku, it suffices to show $\mathtt{Sh}(X)$ is an (AB5) abelian category with enough injectives. This is easily verified, we get everything for free. However, we may prefer an argument based on the fact that the category of abelian groups $\mathtt{Ab}$ has enough injectives.

4.2. Recall an injective object $\mathcal{I}$ in $\mathtt{Sh}(X)$ is a sheaf such that $\mathrm{Mor}_{\mathtt{Sh}(X)}(-, \mathcal{I}) = \mathrm{Mor}_{\mathtt{PSh}(X)}(-, \mathcal{I})$ turns injections into surjections. We show every sheaf $\mathcal{F}$ can be embedded in an injective one. Here is the recipe:

- For each point $i_x\colon x \in X$, embed the stalk $\mathcal{F}_x$ in an injective module I_x.
- The $i_x^*, i_{x,*}$ adjunction gives a morphism of sheaves $\alpha_x\colon \mathcal{F} \to i_{x,*} I_x$.
- The direct product $\alpha\colon \mathcal{F} \to \prod_{x \in X} i_{x,*} I_x$ is the required injective imbedding.
- We remark that by Fubini, presheaf-direct-product of sheaves is a sheaf.

To show this, one checks two things:

[10] $\Gamma_c(X, \mathcal{F}) \to \Gamma_c(X, \mathcal{F}/\mathcal{G})$ is surjective if $\mathcal{G}$ is a soft subsheaf of $\mathcal{F}$.

[11] We do some Tietze extension

- α is an injection: if suffice to check stalk-wise, use the colimit description of stalks.
- We show $\mathrm{Mor}(-, \prod i_{x,*}I_x)$ is exact by abstract nonsense. (Hint: $i_x^* \dashv i_{x,*}$)

5. Exercises

5.1. Here are some fun questions to think about, they are **not** homework problems:

- Is the presheaf-direct-sum of sheaves necessarily a sheaf?
- Is an injective sheaf necessarily an injective presheaf?
- Is the sheafification of an injective presheaf necessarily an injective sheaf?

5.2. Here is the content for the second part of the recitation class on Mar 26.

- We prove the simultaneous resolution lemma for injective resolutions, we construct the middle resolution layer by layer, use snake lemma somewhere.[12]
- We show a module is torsion free iff $\mathrm{Tor}_1(M, N) = 0$ for any N. We use the simple observation that any module is colimit of its cyclic submodules, and the fact that left adjoint preserves colimit.
- We show an abelian group is injective $\mathbb{Z}$-modules iff it is divisible. Slightly modify the proof, we get Baer's criterion.
- We show the right adjoint of an monic-preserving functor preserves injectives. In particular, $\mathrm{Hom}_{\mathbb{Z}}(R, M)$ is an injective R-module, say $\mathbb{Z}/n\mathbb{Z}$ is injective $\mathbb{Z}/n\mathbb{Z}$-modules. It is both projective and injective! Fun question: However, does the category of finite abelian groups have enough injectives?
- We show R-`Mod` has enough injectives. But we cannot immediately say anything by the Freyd-Mitchell, the injectives may not lie in the subcategory!
- We show a finitely presented flat modules is projective. This is wrong when finitely generation condition is dropped, say $\mathbb{Q}$ is flat over $\mathbb{Z}$, thank everybody, $\mathbb{Z}[\mathbb{Q}] \to \mathbb{Q} \to 0$.
- We show the nilpotent-thickening of a projective modules is projective. We show the nilpotent-thickening of a flat modules is flat. The slogan: vector bundles/flat sheaves are stable under deformation.

[12] This is also called the Horseshoe lemma, see [**?**, 2.2.8]