ADDITION OF BRAUER CLASSES VIA PICARD SCHEMES

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Abstract. We give an explicit way to construct a Brauer-Severi variety corresponding to the sum of the two Brauer classes that split by a common Galois extension.

Contents

1. Introduction

1.1. Let K be a field, let L/K be a Galois extension of degree n. It is well-known that elements in the Brauer group $\mathrm{Br}(L/K)$ can be represented by Brauer-Severi varieties of dimension $n-1$, see [GS17, Proposition 4.5.6]. Addition of Brauer classes can be carried out via taking tensor product of the corresponding Azumaya algebras. The construction yields a Brauer-Severi variety of dimension n^2-1. We give an alternative construction of the sum of Brauer-Severi classes via Picard schemes, whose outcome remains a Brauer-Severi variety of dimension $n-1$.

1.2. Let L/K be a Galois extension of degree n. Let X_1, X_2 be Brauer-Severi varieties of dimension $n-1$ that split over L. We construct a singular K-variety W by clutching X_1, X_2 along certain closed points. The Picard scheme of W has a component $\mathrm{Pic}^{-1,-1}_{W/K}$ that parameterizes line bundles of multi-degree $(-1,-1)$. We show that $\mathrm{Pic}^{-1,-1}_{W/K}$ admits a natural compactification by some Brauer-Severi variety X_3, furthermore:

Theorem. *The classes* $[X_1], [X_2], [X_3] \in \mathrm{Br}(L/K)$ *satisfy* $[X_1]+[X_2]=[X_3]$.

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2. Closed points on Brauer-Severi varieties

2.1. Let L/K be a finite Galois extension of degree n. Let $\alpha \in \mathrm{Br}(L/K)$ be a Brauer class of K that splits over L, represented by a $(n-1)$-dimensional Brauer-Severi X variety defined over K. We show that

Lemma 1. *There exists a closed point P on X with residue field L, such that the geometric points of P span $X_L \cong \mathbb{P}_L^{n-1}$ linearly.*

Proof. Let us view $\mathrm{H}^0(X, T_X)$ as an affine space over K. Let $U \subset \mathrm{H}^0(X, T_X)$ be the Zariski open subset that parameterizes global sections of T_X whose zero-locus consists of n distinct points in linearly general position. One checks that U is nonempty after base change to L (e.g. pick distinct $\lambda_i \in L$, then the section $\sum \lambda_i X_i \partial_{X_i}$ has coordinate vertices as vanishing locus). Therefore, the vanishing locus of a general section of T_X gives a separable subscheme Q whose geometric points span $\mathbb{P}_L^{n-1}$.

If Q is irreducible, then it is a desired closed point. Otherwise, let $Q_1 \subset Q$ be an irreducible component, it spans a twisted linear subvariety $V_1 \subset X$ of dimension $m_1 := [K(Q_1) : K] - 1$. By [Kol18, Theorem 1], the Grassmannian $\mathbb{G}(m_1, X)$ is rational, and therefore we may pick $n_1 = [L : K(Q_1)]$ distinct twisted-linear subvarieties $\{V_j\}_{j=1}^{n_1}$ that span X_L linearly. Let us fix a choice of coordinates $\{X_{i,j}\}_{i=0}^{m_1}$ on $V_{j,L} \cong \mathbb{P}_L^{m_1}$ for each $1 \leq j \leq n_1$, and choose coordinates $\{X_{i,j}\}_{0 \leq i \leq m_1, 1 \leq j \leq n_1}$ of $X_L \cong \mathbb{P}_L^{n-1}$ so that $\{X_{i,j}\}_{i=0}^{m_1}$ restricts to our chosen coordinates on $V_{j,L}$. Let us pick representatives for the homogeneous coordinates of the L-points of Q_1, and denote them by $\mathbf{a}_i = [a_{i,0} \colon \cdots : a_{i,m_1}]$, $0 \leq i \leq m_1$ with entries $a_{i,j} \in L$. Let us choose a 1-cycle $\sigma \to A_\sigma$ that represents the class $\alpha \in \mathrm{H}^1(L/K, \mathrm{PGL}_{m_1+1}(L))$, then the Galois group $\mathrm{Gal}(L/K)$ acts on the set of L-points of X via $\mathbf{a} \mapsto A_\sigma(\sigma(\mathbf{a}))$. Let us take a primitive element β of $[L : K(Q_1)]$ and consider the following set of L-points on $X_L \cong \mathbb{P}_L^{n-1}$:

$$\{[\mathbf{a}_i : \sigma(\beta)\mathbf{a}_i : \cdots : \sigma(\beta^{n_1})\mathbf{a}_i]\}_{0 \leq i \leq m_1, \sigma \in \mathrm{Gal}(L/K)}.$$

By properties of Vandermonde matrix, we know that these points span $\mathbb{P}_L^{n-1}$ linearly. Furthermore, these n points form a $\mathrm{Gal}(L/K)$-orbit, and therefore descends to a closed points of residue field L on X. $\square$

3. Brauer-Severi varieties and torus-torsors

3.1. Let L/K be a finite Galois extension of degree n. Let X be a Brauer-Severi variety of dimension $n-1$, representing a class $\alpha \in \mathrm{Br}(L/K)$. By Lemma 1, we may find a separable point P of degree n, whose geometric points span X_L linearly. Let us denote the L-points of P by $\{P_\sigma\}_{\sigma \in G}$, in a way that $P_{\tau\sigma} = \tau(P_\sigma)$ for all $\sigma, \tau \in G$. For each $\sigma \in G$, the set of L-points $\{P_g\}_{g \in G \setminus \{\sigma\}}$ spans a hyperplane $H_\sigma \subset X_L$. The complement $X_L - \bigcup_{\sigma \in G} H_\sigma$

is a torus. Notice that $\bigcup_{\sigma\in G} H_\sigma$ is $\mathrm{Gal}(L/K)$-invariant, so it descends to a closed subscheme $B \subset X$. Let $A := X\backslash B$ be the complement of B in X. We show that A is naturally a torus-torsor.

3.2. Let $T \subset \mathrm{Aut}(X)$ be the subgroup-scheme of automorphism of X that leaves $P \subset X$ fixed. Since automorphisms of X are geometrically projective-linear, the union of hyperplanes B is stable under the T-action. Therefore, we have a natural T-action on $X\backslash B = A$, denoted by $a\colon T \times A \to A$. Let us denote the Weil restriction $\mathrm{Res}_{K(P)/P}\mathbb{G}_m$ by G_P.

Lemma 2. *The group scheme T is a torus, isomorphic to $G_P/\mathbb{G}_m$. The natural action $a\colon T \times A \to A$ realizes A as a T-torsor.*

Proof. One checks that after base change to L, we have $T_L \cong (\prod_{\sigma\in G} \mathbb{G}_{m,P_\sigma})/\mathbb{G}_{m,L}$, which descends to $G_P/\mathbb{G}_m$. One checks that the T_L action on $A_L \cong \mathbb{G}_m^{\oplus(n-1)}$ is free and transitive, so A is a T-torsor under the a-action. □

3.3. The short exact sequence

$$0 \longrightarrow \mathbb{G}_m \longrightarrow G_P \longrightarrow T \longrightarrow 0$$

gives a natural connecting homomorphism

$$\delta\colon \mathrm{H}^1(L/K, T) \to \mathrm{H}^2(L/K, \mathbb{G}_m).$$

This is an isomorphism by Hilbert 90. Let $[A]$ be the class of A as a T-torsor as in the previous lemma, we show that

Proposition 3. $\delta([A]) = \alpha$.

Proof. By Lemma 1, we may pick a closed point $C \subset \mathbb{P}^{n-1}_K$ with residue field L, whose geometric points span $\mathbb{P}^{n-1}_L$ linearly. By Lemma 2, we know that after deleting the union of hyperplanes $Z \subset \mathbb{P}^{n-1}$ determined by C, the open subvariety $\mathbb{P}^{n-1}\backslash Z$ is a T-torsor. This is a trivial torsor as it admits K-sections, so $\mathbb{P}^{n-1}\backslash Z$ is isomorphic to T.

The class α of the Brauer-Severi variety X is represented by the PGL_n-torsor

$$\mathrm{Isom}_K(\mathbb{P}^{n-1}, X),$$

where PGL_n-acts on the source $\mathbb{P}^{n-1}$. It contains a subvariety

$$\mathrm{Isom}_K(C \hookrightarrow \mathbb{P}^{n-1}, P \hookrightarrow X) \subset \mathrm{Isom}_K(\mathbb{P}^{n-1}, X)$$

of "isomorphisms that preserves a length-n subscheme". Notice that $\mathrm{Aut}(C \hookrightarrow \mathbb{P}^{n-1}) \cong T$, the subvariety admits a T-action through the source.

Let us consider the commutative diagram induced by inclusions:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{G}_{m,L} \arrow[r] \arrow[d, equal] & G_{P,L} \arrow[r] \arrow[d, hook] & T_L \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathbb{G}_{m,L} \arrow[r] & \mathrm{GL}(X_L, \mathcal{O}_{X_L}(1)) \arrow[r] & \mathrm{PGL}(X_L, \mathcal{O}_{X_L}(1)) \arrow[r] & 0
\end{tikzcd}$$

By naturality of the connecting homomorphisms, it suffices to show that the T-torsor A is isomorphic to the T-torsor $\mathrm{Isom}_K(C \hookrightarrow \mathbb{P}^{n-1}, P \hookrightarrow X)$.

Notice that an isomorphism of $\mathbb{P}^{n-1}$ that fixes the n points also fixes the union of n hyperplanes generated by the $(n-1)$-tuples of points. Deleting the union of hyperplanes, we get an induced morphism of schemes with T-actions

$$\rho\colon \mathrm{Isom}_K(C \hookrightarrow \mathbb{P}^{n-1}, P \hookrightarrow X) \to \mathrm{Isom}_K(T, A).$$

Let $\mathrm{Isom}_T(T, A) \subset \mathrm{Isom}_K(T, A)$ be the scheme of isomorphism of T-torsors. By passing to L, one checks that $\mathrm{Im}(\rho)$ lands in $\mathrm{Isom}_T(T, A)$. Notice that $\mathrm{Isom}_T(T, A) \cong A$, so ρ gives the desired isomorphism. □

3.4. We showed that the Brauer class of a Brauer-Severi variety can be recovered from its open subscheme as torus torsor. Conversely, given a torus torsor, it can be compactified into a Brauer-Severi variety.

Lemma 4. *Let $P, P' \subset X$ be closed points as in Lemma 1, then there exist an automorphism $\gamma\colon X \to X$ such that $\gamma(P) = P'$.*

Proof. Let us consider the scheme of isomorphisms $\mathrm{Isom}_K(P \hookrightarrow X, P' \hookrightarrow X)$, it is a torsor under $T := G_P/\mathbb{G}_m \cong \mathrm{Aut}_K(P \hookrightarrow X)$. It suffices to show that this is a trivial T-torsor. Let A, A' be the complement of X by the union of hypersurfaces determined by P, P', then we have an isomorphism $\mathrm{Isom}_K(P \hookrightarrow X, P' \hookrightarrow X) \cong \mathrm{Isom}_T(A, A')$. Notice that $[A] = \delta^{-1}([X]) = [A']$, and therefore $\mathrm{Isom}_T(A, A')$ represents the class $[A] - [A'] = 0$ in $\mathrm{H}^1(L/K, T)$. □

Proposition 5. *Let L/K be a Galois extension of degree n. Let A be a torsor under $T := (\mathrm{Res}_{L/K}\mathbb{G}_m)/\mathbb{G}_m$, then there exists a unique compactification of A by a Brauer-Severi variety X, whose complement is a union of hyperplanes in one $\mathrm{Gal}(L/K)$-orbit (or equivalently, the vertices of intersections of the hyperplanes are in one $\mathrm{Gal}(L/K)$-orbit). The class of A naturally matches the class of X as in Proposition 3.*

Proof. Let $C := \mathrm{Spec}(L) \hookrightarrow \mathbb{P}^{n-1}_K$ be a closed embedding, whose geometric points linearly spans $\mathbb{P}^{n-1}_L$. By Hilbert 90 and [GS17, Proposition 4.5.6], we have a natural identification $\delta\colon \mathrm{H}^1(L/K, T) \cong \mathrm{Br}(L/K) \cong \mathrm{H}^1(L/K, \mathrm{PGL}_n)$, or $\delta\colon \mathrm{H}^1(L/K, \mathrm{Aut}_L(C_L \hookrightarrow \mathbb{P}^{n-1}_L)) \cong \mathrm{H}^1(L/K, \mathrm{Aut}_L(\mathbb{P}^n_L))$. Given a T-torsor A, the class $\delta([A])$ is a Brauer-Severi variety with desired property. The uniqueness of such X is promised by Lemma 4. □

Since Brauer-Severi varieties of the same dimension are uniquely determined by the Brauer class, and therefore we may call X *the* Brauer-Severi compactification of the torus torsor A.

4. Picard schemes of clutched Brauer-Severi varieties

4.1. Let L/K be a Galois extension of degree n, let $\alpha \in \mathrm{Br}(L/K)$ be a Brauer class. Let X be a $(n-1)$-dimensional Brauer-Severi variety that represents α. Let $P \subset X$ be a closed point with residue L, whose geometric points linearly spans $X_L \cong \mathbb{P}_L^{n-1}$. Let us denote the Weil restriction $\mathrm{Res}_{K(P)/K}\mathbb{G}_m$ by G_P.

4.2. Let us consider the coproduct $Y := X \coprod_P \mathrm{Spec}(K)$. It is a singular variety obtained from X by collapsing the n geometric points of P to one point.

Lemma 6. *The connected component of the Picard scheme of Y, denoted by $\mathrm{Pic}^0_{Y/K}$, is isomorphic to the torus $G_P/\mathbb{G}_m$. The Picard scheme $\mathrm{Pic}_{Y/K}$ is a disjoint union of $\mathrm{Pic}^0_{Y/K}$-torsors indexed by $\mathbb{Z}$, via the degree of the pullback line bundle on $X_{\overline{K}}$.*

Proof. Let us consider the normalization map $\nu\colon X \to Y$. Comparing the étale sheaves $\mathbb{G}_m$ on X and Y, we get the following short exact sequence

$$0 \longrightarrow \mathbb{G}_{m,Y} \longrightarrow \nu_*\mathbb{G}_{m,X} \longrightarrow \nu_*\mathbb{G}_{m,X}/\mathbb{G}_{m,Y} \longrightarrow 0\ .$$

The quotient sheaf $\nu_*\mathbb{G}_{m,X}/\mathbb{G}_{m,Y}$ is supported on $\mathrm{Spec}(K)$, and it is isomorphic to $G_P/\mathbb{G}_m$. We conclude from the corresponding long exact sequence, which gives:

$$0 \longrightarrow G_P/\mathbb{G}_m \longrightarrow \mathrm{Pic}_{Y/K} \longrightarrow \mathrm{Pic}_{X/K} \cong \mathbb{Z} \longrightarrow 0\ .$$

The last arrow is surjective, because geometrically, we may always twist a line bundle with a hyperplane divisor disjoint from the closed point P. $\square$

4.3. Let X' be the dual Brauer-Severi variety of X. The n geometric points of P correspond to n hyperplanes in X'. Let $B' \subset X'$ be the union of the hyperplanes in X' determined by P, let $A' \subset X$ be the complement of B. We show that

Proposition 7. *There is an isomorphism of $G_P/\mathbb{G}_m$-torsors $A' \to \mathrm{Pic}^1_{Y/K}$.*

Proof. A geometric point $x \in A' \subset X'$ represents a hyperplane $H_x \subset X$ that does not pass through any of geometric points over P'. Consider the "Abel-Jacobi" morphism $\varphi\colon A' \to \mathrm{Pic}^1_{Y/K}$, $x \mapsto \mathcal{O}_X(H_x)$, it suffices to checks that φ is a $G_P/\mathbb{G}_m$-equivariant isomorphism after base change to L. We omit the explicit verification. $\square$

Remark 8. *Let $\overline{\mathrm{Pic}}^1_{Y/K}$ be the Brauer-Severi compactification of the Picard torsor, as promised by Proposition 5. Then Proposition 3 and Proposition 7 shows that the Brauer classes satisfy $[\overline{\mathrm{Pic}}^1_{Y/K}] = -[X]$. Since the connecting map δ is a homomorphism, we see that $[\overline{\mathrm{Pic}}^d_{Y/K}] = -d[X]$ for any d.*

4.4. Let X_1, X_2 be Brauer-Severi varieties representing classes $\alpha_1, \alpha_2 \in \mathrm{Br}(L/K)$. Let $P := \mathrm{Spec}(L)$. We pick closed immersions $i_1 \colon P \hookrightarrow X_1, i_2 \colon P \hookrightarrow X_2$ as in Lemma 1. Let $W = X_1 \coprod_{i_1,i_2} X_2$ be the coproduct.

Lemma 9. *The connected component of the Picard scheme of W, denoted by $\mathrm{Pic}^0_{W/K}$, is isomorphic to the torus $G_P/\mathbb{G}_m$. The Picard scheme $\mathrm{Pic}_{W/K}$ is a disjoint union of $\mathrm{Pic}^0_{W/K}$-torsors indexed by $\mathbb{Z} \oplus \mathbb{Z}$, via the degrees of the pullback line bundle on $X_{1,\overline{K}}, X_{2,\overline{K}}$.*

Proof. Let $\nu \colon X_1 \coprod X_2 \to W$ be the normalization, comparing the étale sheaves $\mathbb{G}_m$, we get the short exact sequence

$$0 \longrightarrow \mathbb{G}_{m,W} \longrightarrow \nu_* \mathbb{G}_{m,X_1 \coprod X_2} \longrightarrow \nu_* \mathbb{G}_{m,X_1 \coprod X_2}/\mathbb{G}_{m,W} \longrightarrow 0 \ ,$$

we conclude from the corresponding long exact sequence, which gives

$$0 \longrightarrow G_P/\mathbb{G}_m \longrightarrow \mathrm{Pic}_{Y/K} \longrightarrow \mathrm{Pic}_{X_1 \coprod X_2/K} \cong \mathbb{Z} \oplus \mathbb{Z} \longrightarrow 0 \ .$$

□

Therefore, the Picard scheme $\mathrm{Pic}^{1,1}_{W/K}$ is a torsor under $G_P/\mathbb{G}_m$. Let $\overline{\mathrm{Pic}}^{1,1}_{W/K}$ be its compactification by Brauer-Severi variety, we show that

Theorem 10. $[X_1] + [X_2] = -[\overline{\mathrm{Pic}}^{1,1}_{W/K}]$

Proof. Let X_1', X_2' be the dual Brauer-Severi varieties of X_1, X_2. Let $A_1' \subset X_1', A_2' \subset X_2'$ be the torus-torsors defined as in Section 4.3. By Proposition 3 and naturality of the connecting homomorphism δ, it suffices to show the equality $[A_1'] + [A_2'] = [\mathrm{Pic}^{1,1}_{W/K}]$ in $\mathrm{H}^1(X, G_P/\mathbb{G}_m)$.

Notice that in general, for a group scheme G, given two G-torsors T_1, T_2, the sum of their class is given by the contracted product, i.e., the quotient $[T_1 \times T_2/G]$ with the diagonal G-action on the product. Therefore, it suffices to show that the torsor $\mathrm{Pic}^{1,1}_{W/K}$ can be identified with $[A_1' \times A_2'/(G_P/\mathbb{G}_m)]$. The identification can be made explicit: Consider the morphism $\varphi \colon A_1' \times A_2' \to \mathrm{Pic}^{1,1}_{W/K}$ defined by $(x_1, x_2) \mapsto \mathcal{O}_W(H_{x_1} + H_{x_2})$, where H_{x_i} is the hyperplane corresponding represented x_i via projective duality. One explicitly verifies that the morphism descends to an isomorphism of $G_P/\mathbb{G}_m$-torsors $[A_1' \times A_2'/(G_P/\mathbb{G}_m)] \cong \mathrm{Pic}^{1,1}_{W/K}$. □

Remark 11. *If X_1 and X_2 are conics, the Picard scheme $\mathrm{Pic}^{1,1}_W$ is one dimensional. Therefore, any smooth compactification is conic with Brauer class $[X_1] + [X_2]$.*

5. General cases

5.1. In the previous discussions, in order to perform explicit constructions in projective geometry, we asked the Brauer-Severi varieties to have certain fixed dimension. In this section, we generalize the previous discussion to sum of Brauer-Severi varieties of arbitrary (possibly different) dimensions.

5.2. Let L/K be a Galois extension, let X a Brauer-Severi variety over K representing classes α in $\mathrm{Br}(K)$. Let $P \subset X$ be a separable closed point with function field N. By the same argument in Lemma 6, the Picard torus of the singular variety $Y := X \coprod_P \mathrm{Spec}(K)$ is isomorphic to the torus $T = (\mathrm{Res}_{N/K}\mathbb{G}_m)/\mathbb{G}_m$. The component $\mathrm{Pic}^1_{Y/K}$ can be uniquely compactified into a Brauer-Severi variety $\overline{\mathrm{Pic}}^1_{Y/K}$.

Proposition 12. *We have equality $[\overline{\mathrm{Pic}}^1_{Y/K}] = -[X]$ in $\mathrm{Br}(K)$.*

Proof. Let X' be the dual variety of X, and let $B \subset X'$ be the union of hyperplanes determined by P. Let $\varphi\colon X'\backslash B \to \mathrm{Pic}^1_{Y/K}, x \mapsto \mathcal{O}_Y(H_x)$ be the Abel-Jacobi morphism. Let $\overline{\mathrm{Pic}}^1_{Y/K}$ be the Brauer-Severi compactification as in Proposition 5.

(1) If the geometric points of P spans $X_{\overline{K}}$, we check that $X'\backslash B \to \overline{\mathrm{Pic}}^1_{Y/K}$ is a twisted-linear embedding: After base change to K^{sep}, we may separate the components of B. The morphism φ is given by $x \mapsto [\frac{l_{H_x}}{l}(p_0) : \cdots : \frac{l_{H_x}}{l}(p_r)]$, where l is some fixed linear form, l_{H_x} is a linear form corresponding to the hyperplane H_x and $p_0, \cdots, p_r$ are the geometric points of P. Clearly φ_L is linear. Our assumption that $p_0, \cdots, p_r$ spans $X_{\overline{K}}$ implies that φ_L is an embedding, so X' is a twisted linear subvariety of $\overline{\mathrm{Pic}}^1_{Y/K}$ and therefore $[\overline{\mathrm{Pic}}^1_{Y/K}] = [X'] = -[X]$.
(2) In general, let X_0 as a twisted linear subvariety of X spanned by the geometric points of P, then $[X_0] = [X]$. Let $Y_0 = X_0 \coprod_P \mathrm{Spec}(K)$. By the previous discussion, we have $[\overline{\mathrm{Pic}}^1_{Y_0/K}] = -[X_0] = -[X]$. Then we notice that "restriction to Y_0" gives an isomorphism $\mathrm{Pic}^1_{Y_0/K} \cong \mathrm{Pic}^1_{Y/K}$ of $(\mathrm{Res}_{N/K}\mathbb{G}_m)/\mathbb{G}_m$-torsors, which can be check after base change to L.

□

5.3. Let L/K be a Galois extension. Let X_1, X_2 be Brauer-Severi varieties over K representing classes $[X_1], [X_2]$ in $\mathrm{Br}(L/K)$. Let N/K be a separable field extension, and let $i_1\colon \mathrm{Spec}(N) \hookrightarrow X_1$, $i_2\colon \mathrm{Spec}(N) \to \mathrm{X}_2$ be immersion of closed points. Let $W = X_1 \coprod_{i_1,i_2} X_2$. Let $\mathrm{Pic}^{n_1,n_2}_{W/K}$ be the component that parameterizes line bundles of degree n_1 on X_1 and n_2 on X_2. Combing the arguments in Proposition 12 and Theorem 10, one can show that

Proposition 13. $n_1[X_1] + n_2[X_2] = -[\overline{\mathrm{Pic}}^{n_1,n_2}_{W/K}]$

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