SPECIALIZING BRAUER CLASSES IN PICARD SCHEMES

QIXIAO MA

Abstract. Let X be a smooth projective curve defined over a field k. The existence of tautological line bundles on $X \times \mathrm{Pic}_{X/k}$ is obstructed by a Brauer class $\alpha \in \mathrm{Br}(\mathrm{Pic}_{X/k})$. We show that α splits at the generic point of various naturally defined loci in $\mathrm{Pic}_{X/k}$—the theta divisor and the generalized theta divisors associated with degree $2g-2$, semi-stable rank 2 vector bundles on X. We show by explicit construction that degree $2g-2$, semi-stable rank 2 vector bundles exist on the generic genus g curve.

Contents

1. Introduction

1.1. The Brauer group of algebraic varieties is a central object of studies in arithmetic and algebraic geometry: the Tate conjecture can be rephrased in terms of finiteness of Brauer groups [Tat68]; for certain classes of varieties over global fields, the Brauer-Manin obstruction is shown to be the only obstruction to the local-global principle [Bor96], and therefore can be used to study the existence of rational points; Brauer classes and their further generalization, the unramified cohomology classes, have been powerful tools in the study of the rationality problems [AM72] and algebraic cycles [Sch20]. Despite the usefulness, Brauer classes are in general hard to explicitly write down and work with.

1.2. There is a natural source of Brauer classes that come from geometry: the obstruction to the Zariski-representability of various moduli functors gives rise to Brauer classes over the moduli spaces. To be precise: Let k be a field, let X be a projective variety defined over k. Let M be the moduli space of stable sheaves on X with fixed rank and chern classes. Then the existence of tautological sheaves on $X \times_k M$ is obstructed by a Brauer class $\alpha \in \mathrm{Br}(M)$, see [Cal00, I.3.3].

1.3. Degeneration techniques have been powerful methods in the current study of rationality problems [NS19],[Sch19]. Brauer groups are stable birational invariants, it is useful to understand how Brauer classes behave in the degeneracy loci. In this paper, we study the specializations of the Brauer obstruction class to various loci in the moduli space, and focus on the simplest situation: moduli space of line bundles on curves.

Date: February 3, 2021.

1.4. Interesting loci in the moduli space can usually be characterized by vanishing of sheaf cohomologies. When base change to the locus, twisted sheaves of interesting ranks will pop out and give us information about the specialized Brauer classes.

Let us illustrate the discussion with the case of curves. Let X be a smooth projective curve defined over a field k. Let d be an integer, let $M = \mathrm{Pic}^d_{X/k}$ be the moduli space of degree d line bundles on X, and let α_d be the obstruction class. While this class is nonzero in general, we show that it restricts to zero at the generic point of the naturally defined locus—theta divisor $\Theta = \{L\colon h^0(X.L) \neq 0\} \subset \mathrm{Pic}^{g-1}_{X/k}$. Let $k(\Theta)$ be its function field, we prove that:

Theorem. *The Brauer class $\alpha_{g-1} \in \mathrm{Br}(\mathrm{Pic}^{g-1}_{X/k})$ restricts to zero in $\mathrm{Br}(k(\Theta))$.*

The proof goes by explicitly finding a section in the Brauer-Severi variety that represents $\alpha_{g-1}|_{k(\Theta)}$ (essentially, showing the existence of an $\alpha_{g-1}|_{k(\Theta)}$-twisted sheaf of rank 1), which rises from the cohomological characterization of the theta divisor.

1.5. In [Ray82], Raynaud defined generalized theta divisors $\Theta_E = \{L\colon h^0(X, L\otimes E) \neq 0\} \subset \mathrm{Pic}^0_{X/k}$ associated with degree $2g-2$, semi-stable rank 2 vector bundles E on X. With the previous theorem in mind, one naturally expects that the Brauer class α_0 splits at the generic point of Θ_E. Unlike the usual theta divisor Θ, the generalized theta divisor Θ_E depends on the vector bundle E, which lacks a canonical choice. Let us recall Franchetta's theorem:

Theorem ([Sch03, 5.1])**.** *Let $g \geq 3$ be an integer, let C_g be the generic genus g curve, i.e., the generic fiber of the forgetful map $\mathcal{M}_{g,1} \to \mathcal{M}_g$. Then every line bundle on C_g is a tensor power of the canonical bundle.*

One naturally asks the rank 2 analogue: Does there exist a degree $2g-2$, semi-stable rank 2 vector bundle on C_g? Working over any base field, we prove that

Theorem. *Let $g \geq 3$ be an integer. Let k be the function field of $\mathcal{M}_g$ and let X/k be the generic genus g curve. Then there exist degree $2g-2$, semi-stable rank 2 vector bundles on X. For any such vector bundle E, the generalized theta divisor $\Theta_E \subset \mathrm{Pic}^0_{X/k}$ is reduced and irreducible. The Brauer class $\alpha_0 \in \mathrm{Pic}^0_{X/k}$ restricts to zero at the generic point of Θ_E.*

The vector bundle E is constructed by carefully modifying the rank 2 vector bundle $\omega_X^{\oplus 2}$ along an effective canonical divisor $D \in |\omega_X|$. To show that E is semi-stable, we use the Franchetta theorem for line bundles, which allows us to check stability condition for a very limited class of subsheaves of E.

1.6. The existence of such vector bundles on the generic curve should be new in the literature. The result may be helpful to arithmetic aspects of rationality problems. For example, it is asked:

Question 1 ([Kol01, Problem 60])**.** *Find a smooth geometrically unirational k-variety V, which has k-points, but not unirational over k.*

The moduli space of $M_X(r, L)$ rank r vector bundles on a smooth curve X with fixed determinant L are known to be smooth and geometrically unirational in many cases [Ses82, Theorem I.49], e.g., $L = \omega_X, r = 2g-3$. When $X = C_g$, it is interesting to know if the corresponding vector bundles on X are rare. If the rational points representing the vector bundles lie in a Zariki closed subset, then the space $V := M_X(r, L)$ answers Question 1.

Acknowledgements. The author heartily thank Professor Aise Johan de Jong and Daniel Krashen for generously sharing their insights, and for many helpful comments on improving the presentation.

2. Preliminaries

2.1. **The Picard scheme.** We collect some facts from [Bos90, 8]. Let k be a field, let X be a smooth proper curve defined over k. Let us consider the relative Picard functor

$$P'_{X/k}\colon \mathrm{Sch}/k \to \mathrm{Sets},\ T \mapsto \mathrm{Pic}(X \times_k T)/\mathrm{pr}_2^*\mathrm{Pic}(T).$$

Let $P_{X/k}$ be the étale sheafification of $P'_{X/k}$. The functor $P_{X/k}\colon (\mathrm{Sch}/k)_{\acute{e}t} \to \mathrm{Sets}$ is represented by a group scheme

$$\mathrm{Pic}_{X/k} = \bigsqcup_{d\in\mathbb{Z}} \mathrm{Pic}^d_{X/k}.$$

The identity component $\mathrm{Pic}^0_{X/k}$ is an abelian variety, the components $\mathrm{Pic}^d_{X/k}$ are torsors of $\mathrm{Pic}^0_{X/k}$. By the representability of $P_{X/k}$, we mean that there exists a tautological line bundle on $X \times_k U$, where U is some étale cover of $\mathrm{Pic}_{X/k}$. By the representability of $P'_{X/k}$, we mean that there exists a tautological line bundle on $X \times_k \mathrm{Pic}_{X/k}$. The functor $P'_{X/k}$ is representable when X has a k-rational point.

2.2. **The Brauer class.** Let us consider the Leray spectral sequence of the étale sheaf $\mathbb{G}_m$, along the projection $\pi\colon X \times \mathrm{Pic}_{X/k} \to \mathrm{Pic}_{X/k}$. The low-degree terms fit into an exact sequence

$$0 \longrightarrow \mathrm{Pic}(\mathrm{Pic}_{X/k}) \xrightarrow{\pi^*} \mathrm{Pic}(X \times \mathrm{Pic}_{X/k}) \xrightarrow{e} \mathrm{Mor}(\mathrm{Pic}_{X/k}, \mathrm{Pic}_{X/k}) \xrightarrow{\mathrm{d}_2^{0,1}} \mathrm{Br}(\mathrm{Pic}_{X/k}),$$

where π^* is the pullback, and the middle morphism e sends a line bundle L on $X \times \mathrm{Pic}_{X/k}$ to the morphism $e(L)\colon \mathrm{Pic}_{X/k} \to \mathrm{Pic}_{X/k}, b \mapsto L|_{X\times b}$. The obstruction to the existence of tautological line bundles on $X \times \mathrm{Pic}_{X/k}$ is given by the class

$$\alpha := \mathrm{d}_2^{0,1}(\mathrm{id}_{\mathrm{Pic}_{X/k}}) \in \mathrm{Br}(\mathrm{Pic}_{X/k}).$$

Let d be an integer. Let us denote the restriction of α to $\mathrm{Pic}^d_{X/k}$ by α_d. This class is in general nontrivial when k is not algebraically closed. For example, let k be the function field of $\mathcal{M}_{g,\mathbb{C}}$ and let X/k be the generic genus g curve, then the period of α_d equals to $\gcd(2g-2, d-g+1)$, see [MV14, Theorem 6.4].

3. Specializing to Θ

Let k be a field, let X be a smooth projective genus g curve defined over k. Let $\mathrm{Pic}^{g-1}_{X/k}$ be the Picard scheme of degree $g-1$ line bundles. The theta divisor consists of the subset of line bundles that admit nontrivial global sections:

$$\Theta := \{L | h^0(X, L) \neq 0\} \subseteq \mathrm{Pic}^{g-1}_{X/k}.$$

We show that the class $\alpha \in \mathrm{Br}(\mathrm{Pic}_{X/k})$ restricts to zero at generic point of the theta divisor $\Theta \subset \mathrm{Pic}^{g-1}_{X/k}$.

3.1. **The theta divisor.** The theta divisor Θ can be characterized as the image of the Abel-Jacobi morphism $\mathrm{Sym}^{g-1}(X) \to \mathrm{Pic}^{g-1}_{X/k}$. Let us recall a more useful construction via tautological line bundles, see [ACGH85, IV.3] for details.

Let U be an étale cover of $\mathrm{Pic}^{g-1}_{X/k}$, so that a tautological line bundle L exist on $X \times_k U$. Let $\pi\colon X \times_k U \to U$ be the projection. Let us choose an effective canonical divisor $Z \in |\omega_X|$. Let $Z_U = Z \times_k U \subset X \times_k U$. They fit in the diagram:

$$\begin{tikzcd} & Z_U \arrow[r, hook] & X \times_k U \arrow[dl] \arrow[dr, "\pi"] & \\ Z \arrow[r, hook] & X & & U. \end{tikzcd}$$

Let us consider the short exact sequence on $X \times_k U$:

$$0 \longrightarrow L(-Z_U) \longrightarrow L \longrightarrow L|_{Z_U} \longrightarrow 0\ .$$

Taking direct images along π_*, we get an exact sequence of sheaves on U:

$$0 \longrightarrow \pi_*L \longrightarrow \pi_*(L|_{Z_U}) \xrightarrow{\delta_U} R^1\pi_*(L(-Z_U)) \longrightarrow R^1\pi_*L \longrightarrow 0. \qquad (*)$$

Let us denote $\pi_*(L|_{Z_U})$ and $R^1\pi_*(L(-Z_U))$ by F_U and G_U. By cohomology and base change, we know that F_U, G_U are both locally free of rank $2g-2$. The subsheaf $\pi_*L \subset F_U$ is torsion, so $\pi_*L = 0$. Let $\delta_U\colon F_U \to G_U$ be the connecting homomorphism, then $\det(\delta_U)$ induces a nonzero section $s_U\colon \mathcal{O}_U \to \det(F_U)^\vee \otimes \det(G_U)$. The line bundle $\det(F_U)^\vee \otimes \det(G_U)$ and section s_U descend along the cover $U \to \mathrm{Pic}^{g-1}_{X/k}$. The vanishing locus T_U of s_U descends to a closed subscheme $T \subset \mathrm{Pic}^{g-1}_{X/k}$. We call T the theta divisor, and denote it by Θ.

3.2. Let us denote the base change $\Theta \times_{\mathrm{Pic}_{X/k}} U$ by Θ_U. Let $\pi'\colon X \times_k \Theta_U \to \Theta_U$ and $L' = L|_{X\times\Theta_U}$ be the restrictions. Let $Z' = Z \times_k \Theta_U \subset X \times_k \Theta_U$. Similar to $(*)$, we have an exact sequence of sheaves on Θ_U:

$$0 \longrightarrow \pi'_*L' \longrightarrow \pi'_*(L'|_{Z'}) \xrightarrow{\delta'} R^1\pi'_*(L'(-Z')) \longrightarrow R^1\pi'_*L' \longrightarrow 0. \qquad (**)$$

The sequence $(**)$ is not the base change of $(*)$ to the closed subscheme $\Theta_U \subset U$, because L is not cohomologically flat. Here is a classical result:

Lemma 2. *The coherent sheaf π'_*L' is torsion free rank* 1*.*

Proof. The sheaf π_*L' is torsion free as a subsheaf of the locally free sheaf $\pi'_*(L'|_{Z'})$. It suffices to check that the connecting homomorphism δ' in $(**)$ has corank 1: Given a point $p \in \Theta_U$, if the corank of $\delta'|_p$ is at least 2, then $\det(\delta'_p) \subset \mathfrak{m}_p^2$, thus p lies in the non-regular locus of Θ_U. Notice that Θ is reduced, as it is the birational image of $\mathrm{Sym}^{g-1}(X)$, therefore the étale base change Θ_U is reduced, so p cannot be the generic point. $\square$

Theorem 3. *The Brauer class $\alpha_{g-1} \in \mathrm{Br}(\mathrm{Pic}^{g-1}_{X/k})$ restricts to zero in $\mathrm{Br}(k(\Theta))$.*

Proof. The class α_{g-1} is represented by the Brauer-Severi scheme corresponding to $\mathbb{P}((\pi_*L|_{Z_U})^\vee)$ [Gir71, 6], and therefore the class $\alpha_{g-1}|_{\Theta_U}$ is represented by the Brauer-Severi scheme corresponding to its restriction $\mathbb{P}(\pi_*(L'|_{Z'})^\vee)$. Notice that the rank 1 subsheaf $\pi'_*L' \subset \pi'_*(L'|_{Z'})$ yields a section to the Brauer-Severi scheme $\mathbb{P}((\pi'_*L')^\vee)$ over the generic point of Θ, therefore the Brauer-Severi scheme is trivial and $\alpha_{g-1}|_{k(\Theta)} = 0$. $\square$

Remark 4. *It is not clear if the Brauer class α_{g-1} restricts to zero in $\mathrm{Br}(\Theta)$.*

4. Over the generic curve

Are there any special loci in other components of the Picard scheme? Let E be a degree $2g-2$, semi-stable rank 2 vector bundle on X, Raynaud constructed generalized theta divisors

$$\Theta_E := \{L | h^0(X, E \otimes L) \neq 0\} \subset \mathrm{Pic}^0_{X/k}.$$

These divisor are in general neither reduced nor irreducible (see Section 4.4 for their scheme structure). However, we show that everything is good when we consider *the generic* curves.

4.1. **The numerical class of divisors.** Let k be a field. Let L be a divisor on a k-variety Y. By the numerical class of L, we mean its class $[L] \in \mathrm{NS}(Y) := \mathrm{Pic}(Y_{k^{\mathrm{sep}}})/\mathrm{Pic}^0(Y_{k^{\mathrm{sep}}})$.

Let X be a curve over k. We denote the numerical class of $\Theta \subset \mathrm{Pic}^{g-1}_{X/k}$ by $\theta \in \mathrm{NS}(\mathrm{Pic}^{g-1}_{X/k})$. Let d be an integer, let x_0 be a k^{sep}-point of X. Notice that translation by $(d+1-g)\mathcal{O}_{X_{k^{\mathrm{sep}}}}(x)$ yields an isomorphism $(\mathrm{Pic}^{g-1}_{X/k})_{k^{\mathrm{sep}}} \cong (\mathrm{Pic}^d_{X/k})_{k^{\mathrm{sep}}}$, we also denote the numerical class of $\Theta_{k^{\mathrm{sep}}} + \mathcal{O}_{X_{k^{\mathrm{sep}}}}((d+1-g)x) \subset (\mathrm{Pic}^d_{X/k})_{k^{\mathrm{sep}}}$ by θ. The class is independent of the choice of x_0.

4.2. **The generic genus g curve.** Let k_0 be a fixed field. Let $g \geq 3$ be an integer. Let $\mathcal{M}_g$ be the moduli stack of families of smooth genus g curves over k_0. Let k be the function field of $\mathcal{M}_g$. Let X/k be the generic genus g curve. We collect some facts:

Lemma 5. *The group* $\mathrm{Pic}(X)$ *is generated by* ω_X.

Proof. This is Franchetta's theorem, see [Sch03]. ☐

Lemma 6. *For any line bundle on* $\mathrm{Pic}^d_{X/k}$*, the numerical class is always a multiple of*

$$\frac{2g-2}{\gcd(2g-2, d+1-g)}\theta.$$

Proof. Let $P = \mathrm{Pic}^d_{X/k}$, we use the long exact sequence of Galois cohomology

$$\cdots \to \mathrm{Pic}(P_{k^{\mathrm{sep}}})^{\mathrm{Gal}_k} \longrightarrow \mathrm{NS}(P)^{\mathrm{Gal}_k} \longrightarrow H^1(k, \mathrm{Pic}^0_{P/k}) \to \cdots$$

associated with the short exact sequence

$$0 \longrightarrow \mathrm{Pic}^0(P_{k^{\mathrm{sep}}}) \longrightarrow \mathrm{Pic}(P_{k^{\mathrm{sep}}}) \longrightarrow \mathrm{NS}(P) \longrightarrow 0\ .$$

Let us denote the connecting homomorphism $\mathrm{NS}(P)^{\mathrm{Gal}_k} \to H^1(k, \mathrm{Pic}^0_{X/k})$ by δ. By [HL18, 2], we know that $\mathrm{NS}(P) = \langle\theta\rangle$. Therefore, if $k\theta$ comes from a line bundle on P, then $\delta(k\theta) = 0$.

Let us calculate the order of $\delta(\theta)$. We pick a point $x_0 \in X(k^{\mathrm{sep}})$. By definition, the class $\delta(\theta)$ is represented by a 1-cocycle in $\mathrm{Pic}^0((\mathrm{Pic}^d_{X/k})_{k^{\mathrm{sep}}})$, which sends $\sigma \in G$ to the divisor

$$c_\sigma := t_{(d+1-g)\mathcal{O}_{X_{k^{\mathrm{sep}}}}(x_0)}\Theta - t_{(d+1-g)\mathcal{O}_{X_{k^{\mathrm{sep}}}}(\sigma(x_0))}\Theta.$$

By the theorem of the square, this equals to $t_{(d+1-g)\mathcal{O}_{X_{k^{\mathrm{sep}}}}(x_0 - \sigma(x_0))}\Theta - \Theta$. Notice that

(1) Autoduality of Jacobian gives us a k-isomorphism $\mathrm{Pic}^0_{X/k} \to \mathrm{Pic}^0_{\mathrm{Pic}^0_{X/k}}$, $L \mapsto t_L\Theta - \Theta$.
(2) By the theorem of the square, pulling back line bundles along any k^{sep}-translation induces a natural k-isomorphism $\mathrm{Pic}^0_{\mathrm{Pic}^0_{X/k}} \cong \mathrm{Pic}^0_{\mathrm{Pic}^d_{X/k}}$.
(3) The class of the torsor $[\mathrm{Pic}^1_{X/k}] \in \mathrm{H}^1(k, \mathrm{Pic}^0_{X/k})$ is represented by the 1-cocycle $\sigma \mapsto \mathcal{O}_X(x_0 - \sigma(x_0))$.

Therefore, the class $\delta(\theta)$ is represented by the torsor $[\mathrm{Pic}^{d+1-g}_{X/k}] \in \mathrm{H}^1(k, \mathrm{Pic}^0_{X/k})$ under the identification $\mathrm{H}^1(k, \mathrm{Pic}^0_{X/k}) \cong \mathrm{H}^1(k, \mathrm{Pic}^0_{P/k})$. Then we conclude by the strong Franchetta theorem [Sch03, Theorem 5.1], which says the torsor $[\mathrm{Pic}^1_{X/k}]$ has order $2g-2$. ☐

4.3. **Semi-stable vector bundles on universal curves.** Let $g \geq 3$ be an integer. We show that there exist degree $2g-2$, semi-stable rank-2 vector bundles on the generic genus g curve. We give explicit construction of such bundles by carefully modifying the direct sum of the canonical line bundles along canonical divisors.

Proposition 7. *Let $g \geq 3$ be an integer, let X be the generic genus g curve. Then there exist degree $2g-2$, semi-stable rank 2 vector bundles on X.*

Proof. Let $D \in |\omega_X|$ be an effective canonical divisor. Let $\rho\colon \Gamma(X, \omega_X) \to \Gamma(D, \omega_X|_D) = \Gamma(D, \mathcal{O}_D)$ be the restriction map. For any $u \in \Gamma(D, \mathcal{O}_D)$, let $u\rho$ be the multiplication of ρ by u. Let $(\rho, u\rho) : \omega_X \oplus \omega_X \to \mathcal{O}_D$ be the sum of ρ and $u\rho$. Let $E_u := \mathrm{Ker}(\rho, u\rho)$ be the kernel, then E_u is a locally free rank 2 sheaf on X. It has degree $2g-2$ and slope $\mu(E_u) = g-1$. We show that for suitable choices of u, the sheaf E_u is semi-stable.

Let L be a subsheaf of E_u. As X is a smooth curve and E_u is locally free, so is L. We want to find conditions for u, such that the slope inequality $\mu(L) \leq \mu(E_u)$ always holds:

(1) If $\mathrm{rank}(L) = 2$, then $\deg(L) = \deg(E_u) - \deg(E_u/L) \leq \deg(E_u)$, so $\mu(L) \leq \mu(E_u)$.
(2) If $\mathrm{rank}(L) = 1$, since $\mathrm{Pic}(X) = \mathbb{Z} \cdot \omega_X$, we may write $L = \omega_X^{\otimes k}$. If $k \leq 0$, then $\mu(L) \leq 0 \leq \mu(E_u)$. If $k \geq 1$, we may pick a nonzero section t of $\omega_X^{\otimes(k-1)}$. The section gives an embedding $\omega_X \overset{t}{\hookrightarrow} \omega_X^{\otimes k} \hookrightarrow E_u$, so $h^0(E_u) \geq h^0(\omega_X) = g$. Notice that $E_u = \mathrm{Ker}(\rho, u\rho)$, so $h^0(E_u) = 2h^0(\omega_X) - \dim \mathrm{Im}(\Gamma(\rho, u\rho))$, where Γ is the functor of taking global section on X. In order that E_u is semi-stable, it suffices to find u such that $\dim \mathrm{Im}(\Gamma(\rho, u\rho)) \geq g+1$.

Let us consider the family of coherent sheaves E_u parameterized by $u \in \Gamma(D, \mathcal{O}_D)$. Since the rank of the linear transform $\Gamma(\rho, u\rho)$ is a lower semi-continuous function on $\Gamma(D, \mathcal{O}_D)$ and that k is infinite, it suffices to show that geometrically

$$\{u \in \Gamma(D, \mathcal{O}_D)_{\overline{k}} \cong \overline{k}^{2g-2} \,|\, \dim_{\overline{k}}(\mathrm{Im}(\Gamma(\rho, u\rho))) \geq g+1\} \neq \emptyset.$$

From now on, let us assume that k is algebraically closed.

Without loss of generality, we may assume that D is a separable, i.e., $D = \sum_{i=1}^{2g-2} P_i$, where P_i are distinct points. Let us identify $\Gamma(D, \mathcal{O}_D)$ with k^{2g-2}, so that a section $u \in \Gamma(D, \mathcal{O}_D)$ has coordinates $u = (\lambda_i)_{i=1}^{2g-2}$, given by its value of at P_i. Let us fix non-vanishing local sections of ω_X at P_i, denoted by ω_i. Then

$$\Gamma(\rho, u\rho)\colon \Gamma(X, \omega_X)^{\oplus 2} \to \oplus_{i=1}^{2g-2} \Gamma(P_i, \mathcal{O}_{P_i})$$

is given by $(\Omega_a, \Omega_b) \mapsto (f_a(P_i) + \lambda_i f_b(P_i))_i$, where $f_a(P_i) = \frac{\Omega_a(P_i)}{\omega_i(P_i)}, f_b(P_i) = \frac{\Omega_b(P_i)}{\omega_i(P_i)}$.

Let us pick a basis $\{\Omega_i\}_{i=1}^g$ of $\Gamma(X, \omega_X)$. The map $\Gamma(\rho, u\rho)\colon \Gamma(X, \Omega_X)^{\oplus 2} \to \oplus_{i=1}^{2g-2} \Gamma(P_i, \mathcal{O}_{P_i})$ can be expressed by the $(2g-2) \times 2g$ matrix $M = [M_L | M_R]$, where $(M_L)_{i,j} = \frac{\Omega_j(P_i)}{\omega_i(P_i)}$ and $(M_R)_{i,j} = \frac{\lambda_i \Omega_j(P_i)}{\omega_i(P_i)}$ for $0 \leq i \leq 2g-2, 0 \leq j \leq g$.

Notice that M_L is the matrix for $\Gamma(\rho)$, which fits into the exact sequence

$$0 \longrightarrow k \cong \Gamma(X, \omega_X - D) \longrightarrow \Gamma(X, \omega_X) \xrightarrow{\Gamma(\rho)} \Gamma(D, \mathcal{O}_D)\ ,$$

therefore $\operatorname{rank}(M_L) = \dim \operatorname{Im}(\Gamma(\rho)) = g-1$. After rearranging the order of $\{P_i\}$ and replacing Ω_i by suitable linear combinations, we may write

$$M_L = \begin{bmatrix} 0_{(g-1)\times 1} & I_{g-1} \\ 0_{(g-1)\times 1} & A \end{bmatrix}$$

Let $L_1 = \operatorname{diag}_{i=1}^{g-1}(\lambda_i)$ and $L_2 = \operatorname{diag}_{i=g}^{2g-2}(\lambda_i)$ be diagonal matrices, we may write:

$$M = \begin{bmatrix} 0_{(g-1)\times 1} & I_{g-1} & 0_{(g-1)\times 1} & L_1 \\ 0_{(g-1)\times 1} & A & 0_{(g-1)\times 1} & L_2 A \end{bmatrix}$$

Let us choose λ_i such that $\lambda_i\lambda_{g-1+i} \neq 1$ for all i, so that $L_2^{-1}L_1 - I$ is invertible. Notice that $\dim \operatorname{Im}(\Gamma(\rho, u\rho)) = \operatorname{rank}(M) = g-1+\operatorname{rank}((L_2^{-1}L_1 - I)A) = g-1+\operatorname{rank}(A)$, it suffice to show that $\operatorname{rank}(A) \geq 2$.

Let us consider the restriction maps:

$$\phi\colon \omega_X \to \omega_X|_{\sum_{i=g}^{2g-2} P_i} \quad\text{and}\quad \psi\colon \omega_X \to \omega_X|_{\sum_{i=1}^{g-1} P_i},$$

we have $g = \dim\operatorname{Im}(\Gamma(\phi)) + h^0(\omega_X(-\sum_{i=g}^{2g-2} P_i))$ and $g = \dim\operatorname{Im}(\Gamma(\psi)) + h^0(\omega_X(-\sum_{i=1}^{g-1} P_i))$. Notice that with our arrangement, we have

$$\dim \operatorname{Im}(\Gamma(\psi)) = \operatorname{rank}(I_{g-1}) = g-1,\ \dim \operatorname{Im}(\Gamma(\phi)) = \operatorname{rank}(A).$$

By Riemann-Roch, we know that $h^0(\omega_X(-\sum_{i=1}^{g-1} P_i)) = h^0(\omega_X(-\sum_{i=g}^{2g-2} P_i))$ and therefore

$$\operatorname{rank}(A) = \dim \operatorname{Im}(\Gamma(\phi)) = \dim \operatorname{Im}(\Gamma(\psi)) = g-1 \geq 2.$$

□

4.4. **The generalized theta divisor.** Let E be a degree $2g-2$, semi-stable rank 2 vector bundle on X. Let $U \to \operatorname{Pic}^0_{X/k}$ be an étale cover, such that a tautological line bundle L exist on $X \times_k U$. Let m be a positive integer, and $Z \in |m\omega_X|$ be a reduced effective divisor. Let $\pi\colon X \times_k U \to U$ be the projection, let E_U be the constant family. Let $Z_U = Z \times_k U$. Let us take the short exact sequence

$$0 \longrightarrow (E_U \otimes L)(-Z_U) \longrightarrow E_U \otimes L \longrightarrow (E_U \otimes L)|_{Z_U} \longrightarrow 0\,. \qquad (\dagger)$$

and consider the long exact sequence of direct images along π_*:

$$\pi_*(E_U \otimes L) \hookrightarrow \pi_*(E_U \otimes L)|_{Z_U} \xrightarrow{\delta_U} R^1\pi_*\left((E_U \otimes L)(-Z_U)\right) \twoheadrightarrow R^1\pi_*(E_U \otimes L).\,.$$

Let η be the generic point of U. Since E is semi-stable and $\chi(E) = 0$, we know by [Ray82, 1.6.2] that $h^0(X_\eta, E_\eta \otimes L_\eta) = 0$. By cohomology and base change, the middle two terms are locally free of the same rank for m large enough. Therefore, the torsion sheaf $\pi_*(E_U \otimes L) = 0$ and δ_U is an injection. The determinant of δ_U cuts out a divisor in $T_U \subset U$, which descends to the generalized theta divisor $\Theta_E \subset P$.

4.5. **Restriction to $k(\Theta_E)$.**

Lemma 8. *The subscheme Θ_E is reduced and irreducible.*

Proof. By [Ray82, 1.8.1], the divisor Θ_E is numerically equivalent to 2Θ. If Θ_E is reducible or non-reduced, its reduced component will have numerical class Θ, which is not possible by Lemma 6. □

Theorem 9. *The class* $\alpha_0 \in \mathrm{Br}(\mathrm{Pic}^0_{X/k})$ *restricts to zero in* $\mathrm{Br}(k(\Theta_E))$.

Proof. Let us restrict the exact sequence (†) to $X \times (\Theta_E)_U$ and denote the restriction of E_U, L and Z_U by E', L' and Z'. Let $\pi' \colon X \times (\Theta_E) \to (\Theta_E)$ be the projection. We have exact sequence

$$\pi'_*(E' \otimes L') \hookrightarrow \pi'_*(E' \otimes L')|_{Z'} \xrightarrow{\delta_U} R^1\pi'_*\left((E' \otimes L')(-Z')\right) \twoheadrightarrow R^1\pi'_*(E' \otimes L').$$

By the same argument in Lemma 2, the reducedness of Θ_E implies that the generic rank of $\pi'_*(E' \otimes L')$ is 1, then we conclude by the same argument as in Theorem 3. $\square$

References

[ACGH85] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris. *Geometry of algebraic curves. Vol. I*, volume 267 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1985.

[AM72] M. Artin and D. Mumford. Some elementary examples of unirational varieties which are not rational. *Proc. London Math. Soc. (3)*, 25:75–95, 1972.

[Bor96] Mikhail Borovoi. The Brauer-Manin obstructions for homogeneous spaces with connected or abelian stabilizer. *J. Reine Angew. Math.*, 473:181–194, 1996.

[Bos90] Siegfried Bosch. *Néron Models*. Springer Berlin Heidelberg, Berlin, Heidelberg, 1990.

[Cal00] Andrei Horia Caldararu. *Derived categories of twisted sheaves on Calabi-Yau manifolds*. ProQuest LLC, Ann Arbor, MI, 2000. Thesis (Ph.D.)–Cornell University.

[Gir71] Jean Giraud. *Cohomologie non abélienne*. Springer-Verlag, Berlin-New York, 1971. Die Grundlehren der mathematischen Wissenschaften, Band 179.

[HL18] Wei Ho and Max Lieblich. Splitting Brauer classes using the universal Albanese. *arXiv e-prints*, page arXiv:1805.12566, May 2018.

[Kol01] János Kollár. Which are the simplest algebraic varieties? *Bull. Amer. Math. Soc. (N.S.)*, 38(4):409–433, 2001.

[MV14] Margarida Melo and Filippo Viviani. The Picard group of the compactified universal Jacobian. *Doc. Math.*, 19:457–507, 2014.

[NS19] Johannes Nicaise and Evgeny Shinder. The motivic nearby fiber and degeneration of stable rationality. *Invent. Math.*, 217(2):377–413, 2019.

[Ray82] Michel Raynaud. Sections des fibrés vectoriels sur une courbe. *Bull. Soc. Math. France*, 110(1):103–125, 1982.

[Sch03] Stefan Schröer. The strong Franchetta conjecture in arbitrary characteristics. *Internat. J. Math.*, 14(4):371–396, 2003.

[Sch19] Stefan Schreieder. Stably irrational hypersurfaces of small slopes. *J. Amer. Math. Soc.*, 32(4):1171–1199, 2019.

[Sch20] Stefan Schreieder. Algebraic cycles and refined unramified cohomology, 2020.

[Ses82] C. S. Seshadri. *Fibrés vectoriels sur les courbes algébriques*. Number 96 in Astérisque. Société mathématique de France, 1982.

[Tat68] John Tate. On the conjectures of Birch and Swinnerton-Dyer and a geometric analog [see MR1610977]. In *Dix exposés sur la cohomologie des schémas*, volume 3 of *Adv. Stud. Pure Math.*, pages 189–214. North-Holland, Amsterdam, 1968.

Shanghai Center for Mathematical Sciences, Fudan University,
Email address: `qxma10@fudan.edu.cn`