RESEARCH STATEMENT

QIXIAO MA

My field of interest is arithmetic geometry, focusing on problems around Brauer groups. Some other keywords: Franchetta-type question over moduli spaces, arithmetic aspect of rationality problems.

The Brauer group plays a crucial role in many areas of algebraic geometry: the Tate conjecture can be rephrased as finiteness of Brauer groups [Tat68], Brauer-Manin obstruction measures the failure to the local-global principle [Bor96], Brauer classes and their generalization, the unramified cohomology classes, has been important tools in studying rationality problems [AM72] and algebraic cycles [Sch20], etc.

1. Past work

1.1. **Brauer class over the Picard scheme of curves.** In this work, we study the Brauer obstruction classes that come from moduli problems. This work is motivated by the cyclicity question [ABGV11, 1.1]— an old problem in the theory of division algebras.: Is every central division algebra of prime degree cyclic?

Let k be a field, let X be a genus g stable curve defined over k. Let $\mathrm{Pic}^0_{X/k}$ be the connected component of the Picard scheme of X. In general, tautological line bundles do not necessarily exist on $X \times \mathrm{Pic}^0_{X/k}$, the obstruction is given by a Brauer class $\alpha \in \mathrm{Br}(\mathrm{Pic}^0_{X/k})$. Our main results are the following:

- We identified some division algebras of prime degree, rising from unramified Brauer classes, they provide new candidates for the cyclicity problem. We give a new way to bound period of Brauer class from below. [Ma21a]
- We studied specialization of the Brauer class in the Picard scheme. We shown that the Brauer classes split at the generic point of generalized theta divisors. The method may apply to general moduli of sheaves. [Ma21f]
- In the study of generalized theta divisors, we constructed slope $g-1$, rank 2 semistable vector bundles on the generic genus g curve, answering a rank 2 analogue of the Franchetta's theorem (see Section 2) [Ma21f].

1.2. **Period of the generic curve via degeneration.** In this work, by simple degeneration arguments, we reprove the known fact that the period and index of the generic genus g curve both equal to $2g-2$. [Ma21c]

Let $g \geq 3$ be an integer, let $\mathcal{M}_g$ be the moduli stack of smooth genus g curves. Let $\pi\colon \mathcal{M}_{g,1} \to \mathcal{M}_g$ be the universal family. Let C/k be the generic fiber of π. The period of C is defined as the order of the torsor $[\mathrm{Pic}^1_{C/k}] \in \mathrm{H}^1(k, \mathrm{Pic}^0_{C/k})$. By the strong Franchetta conjecture, see [Sch03], the period of C equals to $2g-2$.

We reprove the result in an elementary way: notice that "period drops after specialization", it suffices give lower bound on order of Picard torsors for certain degenerated curves.

The key observation is the following: For any given stable curve whose geometrical irreducible components are all rational, Stein factorization of its normalization gives us a conic. Using elementary class field theory, we show the conic is non-split in certain cases. By careful study of the degeneration, we show the Picard torsor $\mathrm{Pic}^{g-1}_{C/k}$ is nontrivial. Then we conclude from the lower bound in Section 1.1.

Class field theory is an instance of the Langlands program. One may wonder if the whole machinery can be exploited to solve some other classical moduli problems in algebraic geometry, especially, constructing objects with general moduli.

1.3. **Closed points on cubic hypersurfaces.** In this work, we generalize some results of Coray on closed points on cubic hypersurfaces. We showed that certain symmetric products of cubic hypersurfaces are stably birational [Ma21b].

Let k be a field, let X be a k-variety. The index of X, denoted by $\mathrm{ind}(X)$, is the greatest common divisor of degree of closed points on X. Cassels and Swinnerton-Dyer conjectured that if X is a cubic hypersurface, then $\mathrm{ind}(X) = 1$ implies that $X(k) \neq \emptyset$. The conjecture is largely open, but for cubic surface over perfect fields, it has been reduced to the case when X has a closed point of degree 4 or 10 [Cor76]. Coray's idea is to descend the degree of closed points on X, by taking residue intersection with carefully chosen space curves. Coray's method does not immediately work when the base field is non-perfect.

We romove the assumption on perfectness of the base field in Coray's result, by a lifting trick. For cubic threefolds and fourfolds, we can also descend degree of closed points by taking residue intersection with rational normal curves passing through the closed points. We interpret this construction as rational maps between symmetric products of X. Studying the fibers, we show some of these symmetric products are stably birational, e.g., $\mathrm{Sym}^7(X)$ is stably birational to $\mathrm{Sym}^5(X)$ for a cubic threefold X; $\mathrm{Sym}^8(X)$ is stably birational to $\mathrm{Sym}^7(X)$ for a cubic fourfold X, etc.

2. Current and Future work: Franchetta type questions

It is a general philosophy in mathematics that:

`On a universal object, the obvious classes are the only ones.` (†)

For example, the cohomologies of the Grassmannian are generated by chern classes of the tautological bundle. Since Mumford's pioneering work [Mum83], many similar results are verified for moduli of curves.

Instead of working over the whole moduli, the philosophy can also be tested generically. Let C/k be the generic genus g curve (see section 1.2), Franchetta conjectured that all the line bundles on C are tensor powers of the canonical bundle ω_C. This has later been enhanced to the strong Franchetta conjecture $\mathrm{Pic}_{C/k}(k) = \langle\omega_C\rangle$ and verified in [Sch03].

Let us call the problem of verifing philosophy (†) over *the whole* moduli space "questions of Mumford type", and call the problem of verifying (†) over the *generic point* of the moduli space "questions of Franchetta type". Unlike Mumford type questions, Franchetta type questions usually have negative answer. Take the simplest example, every homomorphic function on $\mathcal{M}_g$ is constant, while meromorphic functions abounds.

We are particularly interested in three kinds of Franchetta type questions:

- Closed points on generic hypersurfaces,
- Semi-stable vector bundles on the generic curve.
- Picard torsors of the generic curves. (Future work)

2.1. Generic Hypersurfaces.

The motivation is arithmetic aspects of rationality problems. Rationality problems have been of great importance in algebraic geometry and have motivated fundamental developments: Abelian integrals, Riemann surfaces; Lurōth problem motivated Enriques classification; irrationality of threefolds established the importance of Griffiths theory of periods and of the Brauer group, etc.

Most of the questions are solved and posed over an algebraically closed field, less is known on the arithmetic side. One the basic open question on the arithmetic side is the following

Question 2.1 ([Kol01, Problem 60])**.** *Does there exist a smooth geometrically unirational k-variety, which carries a k-point, but not unirational over k?*

We pose several candidates to this question based on the philosophy (†). Here are the two most fascinating examples:

- If *the* generic marked quartic hypersurface in $\mathbb{P}^6$ admits the marking as the only rational point, then it gives a desired example.
- If every closed point of degree 3 on *the* generic cubic surfaces X are collinear, then Hilb^3_X is a desired example. Colliot-Thélène shown that if collinearity holds, one can further give a del Pezzo surface example.

The evidence for the first candidate is Hain's [Hai11] famous result on generic curves—the markings are the only rational points on the generic marked curves. The evidence for the second candidate comes from plane curves. We established various results in low dimensions, with mild singularities, etc [Ma21d].

Currently we are working on

Conjecture 1. *Let $X^1_{n,d}/k$ the generic marked hypersurface of degree d in $\mathbb{P}^n$, let δ be the marked point. If $d \geq 4$, then $X^1_{n,d}(k) = \{\delta\}$.*

The condition $d \geq 4$ should not be neglected, as cubic hypersurfaces with a point are unirational [Kol02].

Here is our best result up to now.

Theorem. [Ma21e] *Let $X^1_{n,d}/k$ be the generic marked hypersurface of degree d and dimension $n-1$. let δ be the marked point. If $d > 2n$, then $X^1_{n,d}(k) = \{\delta\}$. For $d > 2n$, consider the generic hypersurface with r marked points $X^r_{n,d}$. If $r \leq d$, then $X^r_{n,d}$ has exactly r rational points.*

The proof goes by analyzing the rational curves in the fibers. A more refined study of the rational curves in hypersurfaces may push the condition on degree to $d \geq 2n$. It would be very interesting result, if we could push the degree all way down to the Fano range $d < n$.

2.2. Semi-stable vector bundles on the generic curve.

Shifting the point of view, the uniqueness of rational point on the $X^1_{n,d}$, and the collinearity question, can be unified in terms of line bundles on the curve slices of the hypersurfaces.

- Taking any plane slice of $X^1_{n,d}$ that passes through the marked point. If we can show that slice has only trivial line bundles, then by the Abel-Jacobi embedding there exists no rational point on the slice. Thus Conjecture 1 can be reduced to the study of line bundles on slices of the generic hypersurface.
- If we can show that any slice of the generic cubic surface has only trivial line bundle, then all the points are collinear. (non-collinear points span a plane, the non-collinear points yields nontrivial line bundle on the plane cubic)

For vector bundle of higher rank, a natural Franchetta type question is whether the semistable line bundles of given rank and determinant (necessarily power of canonical bundle) exists on the generic genus g curve. In a previous work, we have shown that semi-stable vector bundles of degree $2g-2$ and rank 2 exists on the generic curve [Ma21f]. The case remains unclear for higher rank vector bundles.

Characterizing such semi-stable vector bundles is also related to Question 2.1: Notice that moduli space $M_X(r, L)$ of semistable vector bundles with some fixed rank r and determinant ω_X are smooth unirational, if semi-stable bundles are rare (their corresponding rational points lie in a Zariski closed subset), then $M_X(r, \omega_X)$ would be an wanted example for Question 2.1.

2.3. **Picard torsors.** In [MM14], after determining the Picard group of universal compactified Jacobian $\overline{\mathcal{J}ac}$, the authors posed the following Mumford type question:

Question 2.2 (Melo-Viviani)**.** *Is the group of torsors* $\mathrm{H}^1(\mathcal{M}_g, \mathrm{Pic}_{\mathcal{M}_{g,1}/\mathcal{M}_g})$ *generated by the tautological torsor* $\mathrm{Pic}^1_{\mathcal{M}_{g,1}/\mathcal{M}_g}$*?*

The question is very interesting as it is the few known attempts to study Mumford type questions with arithmetic flavor. The question is answered in [FP19] using the fact that $\mathrm{Br}(\mathcal{M}_{g,n}) = 0$. Which is again based on the fact that every analytic line bundle on moduli of curves is algebraic [AC87], plus the deep result of Harer on group cohomology of mapping class groups [Har83]. On the other hand, the Franchetta type question remains mysterious:

Question 2.3. *Let C be the generic genus g curve, is the group* $\mathrm{H}^1(k, \mathrm{Pic}^0_{C/k})$ *generated by the torsor* $[\mathrm{Pic}^1_{C/k}]$*?*

The Tate conjecture for surfaces is equivalent to finiteness of the Brauer group, or finiteness of the Tate-Shafarevich group for Jacobian fibrations. Thus one may think of the question as a "moduli version" of the Tate conjecture, focusing on the function field of $\mathcal{M}_{g,n}$ instead of global fields. Besides its own interest, the results are also interesting to researches on Grothendieck section conjecture over boundary of moduli space of curves [LLSS20].

A main difficulty is that, for line bundles, we can associate Franchetta type questions with the Mumford type question by taking closure. But the abelian variety torsors are hard to extend over high codimensional locus by "taking closure". It is also possible that the Franchetta type Question 2.3 just have a negative answer.

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