SPECIALIZING BRAUER CLASSES IN PICARD SCHEMES

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Abstract. Let X be a smooth projective curve defined over a field k. The existence of tautological line bundles on $X \times \mathrm{Pic}_{X/k}$ is obstructed by a Brauer class $\alpha \in \mathrm{Br}(\mathrm{Pic}_{X/k})$. We show that α splits at the generic point of various naturally defined loci in $\mathrm{Pic}_{X/k}$—the theta divisor and the generalized theta divisors associated with degree $2g-2$, semi-stable rank 2 vector bundles on X. We study the rank 2 analogue of Franchetta's theorem [Sch03], and give explicit constructions of degree $2g-2$, semi-stable rank 2 vector bundles on the generic genus g curve.

Contents

1. Introduction

1.1. The Brauer group of algebraic varieties is a central object of studies in algebraic geometry: the Tate conjecture can be rephrased by finiteness of Brauer groups [Mil75], Brauer-Manin obstruction measures the failure to local-global principle [Bor96], Brauer classes and their further generalization, the unramified cohomology classes, have been powerful tools in the study of the rationality problems [AM72] [HPT18] [Sch19].

1.2. There is a natural source of Brauer classes that come from geometry: the obstruction to the Zariski-representability of various moduli functors gives rise to Brauer classes over the moduli spaces. To be precise: Let k be a field, let X be a projective variety defined over k. Let M be the moduli space of stable sheaves on X with fixed rank and chern classes. Then the existence of tautological sheaves on $X \times_k M$ is obstructed by a Brauer class $\alpha \in \mathrm{Br}(M)$, see [Cal00, I.3.3].

1.3. Degeneration techniques have been powerful methods in the current study of rationality problems [Sch19], [NS19]. Brauer groups are stable birational invariants, and it is useful to understand how Brauer classes behave in the degeneracy loci. In this paper, we study specializations of the Brauer obstruction class to various loci in the moduli space, and focus on the simplest situation: moduli space of line bundles on curves.

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1.4. Interesting loci in the moduli space can usually be characterized by vanishing of sheaf cohomologies. When we base change such locus, twisted sheaves of interesting ranks will pop out and give us information about the specialized Brauer classes.

Let us focus on the case of curves. Let X be a smooth projective curve defined over a field k. Let d be an integer, let $M = \mathrm{Pic}^d_{X/k}$ be the moduli space of degree d line bundles on X, and let α_d be the obstruction class. While this class is nonzero in general, we show that it restricts to zero at the generic point of the naturally defined locus—theta divisor $\Theta = \{L\colon h^0(X, L) \neq 0\} \subset \mathrm{Pic}^{g-1}_{X/k}$. Let $k(\Theta)$ be its function field, we prove that:

Theorem. *The Brauer class $\alpha_{g-1} \in \mathrm{Br}(\mathrm{Pic}^{g-1}_{X/k})$ restricts to zero in $\mathrm{Br}(k(\Theta))$.*

The proof goes by explicitly finding a section in the Brauer-Severi variety that represents $\alpha_{g-1}|_{k(\Theta)}$ (essentially, showing the existence of an $\alpha_{g-1}|_{k(\Theta)}$-twisted sheaf of rank 1), which rises from the cohomological characterization of the theta divisor. One key fact we use in the proof is the reducedness of Θ.

1.5. In [Ray82], Raynaud defined generalized theta divisors $\Theta_E = \{L\colon h^0(X, L\otimes E) \neq 0\} \subset \mathrm{Pic}^0_{X/k}$ associated with degree $2g-2$, semi-stable rank 2 vector bundles E on X. With the previous theorem in mind, one naturally expects that the Brauer class α_0 splits at the generic point of Θ_E. Unlike the usual theta divisor Θ, the generalized theta divisor Θ_E may not be reduced in general. Moreover, it depends on the vector bundle E, which lacks a canonical choice. Let us recall Franchetta's theorem:

Theorem ([Sch03, 5.1])**.** *Let $g \geq 3$ be an integer, let C_g be the generic genus g curve, i.e., the generic fiber of the forgetful map $\mathcal{M}_{g,1} \to \mathcal{M}_g$. Then the canonical line bundle is the only degree $2g-2$ line bundle on C_g.*

One naturally asks the rank 2 analogue: Does there exist a "canonical" degree $2g-2$, semi-stable rank 2 vector bundle on C_g? Working over any base field, we prove that

Theorem. *Let $g \geq 3$ be an integer, let C_g/k be the generic genus g curve. Then there exist degree $2g-2$, semi-stable rank 2 vector bundles on C_g. For any such vector bundle E, the generalized theta divisor $\Theta_E \subset \mathrm{Pic}^0_{C_g/k}$ is reduced and irreducible. The Brauer class $\alpha_0 \in \mathrm{Br}(\mathrm{Pic}^0_{C_g/k})$ restricts to zero at the generic point of Θ_E.*

Such vector bundles are not unique. They can be obtained by extension, or by explicitly modifying the rank 2 vector bundle $\omega_{C_g}^{\oplus 2}$ along an effective canonical divisor $D \in |\omega_{C_g}|$. To show that E is semi-stable, we use the Franchetta theorem for line bundles, which allows us to check stability condition for a very limited class of subsheaves of E.

1.6. Our method of detecting the Brauer class in the cohomologically defined locus, of determining the irreducibility of the generalized theta divisors, and our constructions of semistable vector bundles on the generic curve maybe helpful to further studies of arithmetic properties of the generic objects over various moduli spaces.

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2. Preliminaries

2.1. **The Picard scheme.** We collect some facts from [Bos90, 8]. Let k be a field, let X be a smooth proper curve defined over k. Let us consider the relative Picard functor

$$P'_{X/k}\colon \mathrm{Sch}/k \to \mathrm{Sets},\ T \mapsto \mathrm{Pic}(X \times_k T)/\mathrm{pr}_2^*\mathrm{Pic}(T).$$

Let $P_{X/k}$ be the étale sheafification of $P'_{X/k}$. The functor $P_{X/k}\colon (\mathrm{Sch}/k)_{\acute{e}t} \to \mathrm{Sets}$ is represented by a group scheme

$$\mathrm{Pic}_{X/k} = \bigsqcup_{d\in\mathbb{Z}} \mathrm{Pic}^d_{X/k}.$$

The identity component $\mathrm{Pic}^0_{X/k}$ is an abelian variety, the components $\mathrm{Pic}^d_{X/k}$ are torsors of $\mathrm{Pic}^0_{X/k}$. By the representability of $P_{X/k}$, we mean that there exists a tautological line bundle on $X \times_k U$, where U is some étale cover of $\mathrm{Pic}_{X/k}$. By the representability of $P'_{X/k}$, we mean that there exists a tautological line bundle on $X \times_k \mathrm{Pic}_{X/k}$. The functor $P'_{X/k}$ is representable when X has a k-rational point.

2.2. **The Brauer class.** Let us consider the Leray spectral sequence of the étale sheaf $\mathbb{G}_m$, along the projection $\pi\colon X \times \mathrm{Pic}_{X/k} \to \mathrm{Pic}_{X/k}$. The low-degree terms fit into an exact sequence

$$0 \longrightarrow \mathrm{Pic}(\mathrm{Pic}_{X/k}) \xrightarrow{\pi^*} \mathrm{Pic}(X \times \mathrm{Pic}_{X/k}) \xrightarrow{e} \mathrm{Mor}(\mathrm{Pic}_{X/k}, \mathrm{Pic}_{X/k}) \xrightarrow{\mathrm{d}_2^{0,1}} \mathrm{Br}(\mathrm{Pic}_{X/k})\ ,$$

where π^* is the pullback, and the middle morphism e sends a line bundle L on $X \times \mathrm{Pic}_{X/k}$ to the morphism $e(L)\colon \mathrm{Pic}_{X/k} \to \mathrm{Pic}_{X/k}, b \mapsto L|_{X\times b}$. The obstruction to the existence of tautological line bundles on $X \times \mathrm{Pic}_{X/k}$ is given by the class

$$\alpha := \mathrm{d}_2^{0,1}(\mathrm{id}_{\mathrm{Pic}_{X/k}}) \in \mathrm{Br}(\mathrm{Pic}_{X/k}).$$

Let d be an integer. Let us denote the restriction of α to $\mathrm{Pic}^d_{X/k}$ by α_d. This class is in general nontrivial when k is not algebraically closed. For example, let k be the function field of $\mathcal{M}_{g,\mathbb{C}}$ and let C_g/k be the generic genus g curve, then the period of α_d equals to $\gcd(2g-2, d-g+1)$, see [MV14, Theorem 6.4].

3. Specializing to Θ

Let k be a field, let X be a smooth projective genus g curve defined over k. Let $\mathrm{Pic}^{g-1}_{X/k}$ be the Picard scheme of degree $g-1$ line bundles. The theta divisor consists of the subset of line bundles that admit nontrivial global sections:

$$\Theta := \{L | h^0(X, L) \neq 0\} \subseteq \mathrm{Pic}^{g-1}_{X/k}.$$

We show that the class $\alpha \in \mathrm{Br}(\mathrm{Pic}_{X/k})$ restricts to zero at generic point of the theta divisor $\Theta \subset \mathrm{Pic}^{g-1}_{X/k}$.

3.1. **The theta divisor.** The theta divisor Θ can be characterized as the image of the Abel-Jacobi morphism $\mathrm{Sym}^{g-1}(X) \to \mathrm{Pic}^{g-1}_{X/k}$. It can also be constructed schematically from tautological line bundles [ACGH85, IV.3], let us recall the construction:

Let U be an étale cover of $\mathrm{Pic}^{g-1}_{X/k}$, so that a tautological line bundle L exist on $X \times_k U$. Let $\pi\colon X \times_k U \to U$ be the projection. Let us choose an effective canonical divisor $Z \in |\omega_X|$. Let $Z_U = Z \times_k U \subset X \times_k U$. They fit in the diagram:

$$\begin{tikzcd} & Z_U \arrow[r, hook] & X \times_k U \arrow[dl] \arrow[dr, "\pi"] & \\ Z \arrow[r, hook] & X & & U. \end{tikzcd}$$

Let us consider the short exact sequence on $X \times_k U$:

$$0 \longrightarrow L(-Z_U) \longrightarrow L \longrightarrow L|_{Z_U} \longrightarrow 0\ .$$

Taking direct images along π_*, we get an exact sequence of sheaves on U:

$$0 \longrightarrow \pi_* L \longrightarrow \pi_*(L|_{Z_U}) \xrightarrow{\delta_U} R^1\pi_*(L(-Z_U)) \longrightarrow R^1\pi_* L \longrightarrow 0. \qquad (*)$$

Let us denote $\pi_*(L|_{Z_U})$ and $R^1\pi_*(L(-Z_U))$ by F_U and G_U. By cohomology and base change, we know that F_U, G_U are both locally free of rank $2g-2$. The subsheaf $\pi_* L \subset F_U$ is torsion, so $\pi_* L = 0$. Let $\delta_U\colon F_U \to G_U$ be the connecting homomorphism, then $\det(\delta_U)$ induces a nonzero section $s_U\colon \mathcal{O}_U \to \det(F_U)^\vee \otimes \det(G_U)$. The line bundle $\det(F_U)^\vee \otimes \det(G_U)$ and section s_U descend along the cover $U \to \mathrm{Pic}^{g-1}_{X/k}$. The vanishing locus T_U of s_U descends to a closed subscheme $T \subset \mathrm{Pic}^{g-1}_{X/k}$. We call T the theta divisor, and denote it by Θ.

3.2. Let us denote the base change $\Theta \times_{\mathrm{Pic}_{X/k}} U$ by Θ_U. Let $\pi'\colon X \times_k \Theta_U \to \Theta_U$ and $L' = L|_{X \times \Theta_U}$ be the restrictions. Let $Z' = Z \times_k \Theta_U \subset X \times_k \Theta_U$. Similar to $(*)$, we have an exact sequence of sheaves on Θ_U:

$$0 \longrightarrow \pi'_* L' \longrightarrow \pi'_*(L'|_{Z'}) \xrightarrow{\delta'} R^1\pi'_*(L'(-Z')) \longrightarrow R^1\pi'_* L' \longrightarrow 0. \qquad (**)$$

The sequence $(**)$ is not the base change of $(*)$ to the closed subscheme $\Theta_U \subset U$, because L is not cohomologically flat. Let us observe that:

Lemma 1. *The coherent sheaf $\pi'_* L'$ is torsion free rank 1.*

Proof. The sheaf $\pi_* L'$ is torsion free as a subsheaf of the locally free sheaf $\pi'_*(L'|_{Z'})$. It suffices to check that the connecting homomorphism δ' in $(**)$ has corank 1: Given a point $p \in \Theta_U$, if the corank of $\delta'|_p$ is at least 2, then $\det(\delta'_p) \subset \mathfrak{m}_p^2$, thus p lies in the non-regular locus of Θ_U. Notice that Θ is reduced, as it is the birational image of $\mathrm{Sym}^{g-1}(X)$, therefore the étale base change Θ_U is reduced, so p cannot be the generic point. $\square$

Theorem 2. *The Brauer class $\alpha_{g-1} \in \mathrm{Br}(\mathrm{Pic}^{g-1}_{X/k})$ restricts to zero in $\mathrm{Br}(k(\Theta))$.*

Proof. The class α_{g-1} is represented by the Brauer-Severi scheme corresponding to $\mathbb{P}((\pi_* L|_{Z_U})^\vee)$ [Gir71, 6], and therefore the class $\alpha_{g-1}|_{\Theta_U}$ is represented by the Brauer-Severi scheme corresponding to its restriction $\mathbb{P}(\pi_*(L'|_{Z'})^\vee)$. Notice that the rank 1 subsheaf $\pi'_* L' \subset \pi'_*(L'|_{Z'})$ yields a section to the Brauer-Severi scheme $\mathbb{P}((\pi'_* L')^\vee)$ over the generic point of Θ, therefore the Brauer-Severi scheme is trivial and $\alpha_{g-1}|_{k(\Theta)} = 0$. $\square$

Remark 3. *It is not clear if the Brauer class α_{g-1} restricts to zero in $\mathrm{Br}(\Theta)$.*

4. Over the generic curve

Are there any special loci in other components of the Picard scheme? Let E be a degree $2g-2$, semi-stable rank 2 vector bundle on X, Raynaud constructed generalized theta divisors

$$\Theta_E := \{L | h^0(X, E \otimes L) \neq 0\} \subset \mathrm{Pic}^0_{X/k}.$$

These divisors are not always reduced or irreducible (see Section 4.4 for their scheme structure). However, we show that everything is good when we consider *the generic* curves.

4.1. **The numerical class of divisors.** Let k be a field. Let L be a divisor on a proper smooth k-variety Y. By the numerical class of L, we mean its class $[L] \in \mathrm{NS}(Y) := \mathrm{Pic}(Y_{k^{\mathrm{sep}}})/\mathrm{Pic}^0(Y_{k^{\mathrm{sep}}})$.

Let X be a curve over k. We denote the numerical class of $\Theta \subset \mathrm{Pic}^{g-1}_{X/k}$ by $\theta \in \mathrm{NS}(\mathrm{Pic}^{g-1}_{X/k})$. Let d be an integer, let x_0 be a k^{sep}-point of X. Translation by $(d+1-g)\mathcal{O}_{X_{k^{\mathrm{sep}}}}(x_0)$ yields an isomorphism $(\mathrm{Pic}^{g-1}_{X/k})_{k^{\mathrm{sep}}} \cong (\mathrm{Pic}^d_{X/k})_{k^{\mathrm{sep}}}$, such that the induced isomorphism $\mathrm{NS}(\mathrm{Pic}^{g-1}_{X/k}) \cong \mathrm{NS}(\mathrm{Pic}^d_{X/k})$ is independent of choice of x_0 [PS99, 2]. Therefore we also denote the numerical class of $\Theta_{k^{\mathrm{sep}}} + \mathcal{O}_{X_{k^{\mathrm{sep}}}}((d+1-g)x_0) \subset (\mathrm{Pic}^d_{X/k})_{k^{\mathrm{sep}}}$ by θ.

4.2. **The generic genus g curve.** Let k_0 be a fixed field. Let $g \geq 3$ be an integer. Let $\mathcal{M}_g$ be the moduli stack of families of smooth genus g curves over k_0. Let k be the function field of $\mathcal{M}_g$. Let X/k be the generic genus g curve. We collect some facts:

Lemma 4. *The group* $\mathrm{Pic}(X)$ *is generated by* ω_X.

Proof. This is Franchetta's theorem, see [Sch03]. □

Lemma 5. *For any line bundle on* $\mathrm{Pic}^d_{X/k}$*, the numerical class is always a multiple of*

$$\frac{2g-2}{\gcd(2g-2, d+1-g)}\theta.$$

Proof. Let $P = \mathrm{Pic}^d_{X/k}$, we use the long exact sequence of Galois cohomology

$$\cdots \to \mathrm{Pic}(P_{k^{\mathrm{sep}}})^{\mathrm{Gal}_k} \longrightarrow \mathrm{NS}(P)^{\mathrm{Gal}_k} \longrightarrow H^1(k, \mathrm{Pic}^0_{P/k}) \to \cdots$$

associated with the short exact sequence

$$0 \longrightarrow \mathrm{Pic}^0(P_{k^{\mathrm{sep}}}) \longrightarrow \mathrm{Pic}(P_{k^{\mathrm{sep}}}) \longrightarrow \mathrm{NS}(P) \longrightarrow 0\ .$$

Let us denote the connecting homomorphism $\mathrm{NS}(P)^{\mathrm{Gal}_k} \to H^1(k, \mathrm{Pic}^0_{X/k})$ by δ. By [HL18, 2], we know that $\mathrm{NS}(P) = \langle \theta \rangle$. Therefore, if $k\theta$ comes from a line bundle on P, then $\delta(k\theta) = 0$. It suffices to show that the order of $\delta(\theta)$ equals to $\frac{2g-2}{gcd(2g-2,d+1-g)}$.

Let us calculate the order of $\delta(\theta)$. We pick a point $x_0 \in X(k^{\mathrm{sep}})$. By definition, the class $\delta(\theta)$ is represented by a 1-cocycle in $\mathrm{Pic}^0((\mathrm{Pic}^d_{X/k})_{k^{\mathrm{sep}}})$, which sends $\sigma \in G$ to the divisor

$$c_\sigma := t_{(d+1-g)\mathcal{O}_{X_{k^{\mathrm{sep}}}}(x_0)}\Theta - t_{(d+1-g)\mathcal{O}_{X_{k^{\mathrm{sep}}}}(\sigma(x_0))}\Theta.$$

By the theorem of the square, this equals to $t_{(d+1-g)\mathcal{O}_{X_{k^{\mathrm{sep}}}}(x_0 - \sigma(x_0))}\Theta - \Theta$. Notice that

(1) Autoduality of Jacobian gives us a k-isomorphism $\mathrm{Pic}^0_{X/k} \to \mathrm{Pic}^0_{\mathrm{Pic}^0_{X/k}}, L \mapsto t_L\Theta - \Theta$.
(2) Pulling back line bundles along any k^{sep}-translation induces a natural k-isomorphism $\mathrm{Pic}^0_{\mathrm{Pic}^0_{X/k}} \cong \mathrm{Pic}^0_{\mathrm{Pic}^d_{X/k}}$ [PS99, 2].

(3) The class of the torsor $[\mathrm{Pic}^1_{X/k}] \in \mathrm{H}^1(k, \mathrm{Pic}^0_{X/k})$ is represented by the 1-cocycle $\sigma \mapsto \mathcal{O}_X(x_0 - \sigma(x_0))$.

Therefore, the class $\delta(\theta)$ is represented by the torsor $[\mathrm{Pic}^{d+1-g}_{X/k}] \in \mathrm{H}^1(k, \mathrm{Pic}^0_{X/k})$ under the identification $\mathrm{H}^1(k, \mathrm{Pic}^0_{X/k}) \cong \mathrm{H}^1(k, \mathrm{Pic}^0_{P/k})$. Then we conclude by the strong Franchetta theorem [Sch03, Theorem 5.1], which says the torsor $[\mathrm{Pic}^1_{X/k}]$ has order $2g-2$. $\square$

4.3. **Semi-stable vector bundles on universal curves.** Let $g \geq 3$ be an integer. We show that there exist degree $2g-2$, semi-stable rank-2 vector bundles on the generic genus g curve X. Our constructions is based on the following observation:

Lemma 6. *A degree $2g-2$, rank 2 vector bundle E on X is semi-stable if and only if $h^0(E \otimes \omega_X^{-1}) = 0$.*

Proof. A rank 2 vector bundle is semi-stable, if and only if for any line bundle $L \subset E$, we have $\deg(L) < g-1$. By Lemma 4, we can write such line bundle $L = \omega_X^{\otimes k}$ for some $k \in \mathbb{Z}$. If $k \geq 1$, then there would exist an inclusion $\omega_X \hookrightarrow \omega_X^{\otimes k} \hookrightarrow E$. $\square$

4.3.1. The first construction is given by taking extension, we show that:

Proposition 7. *Any non-split extension of ω_X by $\mathcal{O}_X$ is semi-stable.*

Proof. Let $0 \to \mathcal{O}_X \to E \xrightarrow{p} \omega_X \to 0$ be a non-split extension. By Lemma 6 it suffices to show there exists no inclusion $\omega_X \hookrightarrow E$. We argue by contradiction: Given an inclusion $i \colon \omega_X \hookrightarrow E$, then the composition $p \circ i \colon \omega_X \hookrightarrow E \to \omega_X$ is an isomorphism or zero. If $p \circ i = 0$, then i factors through $\mathcal{O}_X$, which is impossible for degree reason; if $p \circ i$ is identity, then E splits, contradiction our assumption. $\square$

Remark 8. *For general curves, such extensions are not always semi-stable. Let C be a hyperelliptic curve of genus 5, and let $p \colon C \to \mathbb{P}^1$ be the canonical map. Let us consider the sequence of sheaves $0 \to \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(3) \to \mathcal{O}_{\mathbb{P}^1}(4) \to 0$, where the inclusion is given by a pair of sections with no common zero. Pulling back the sequence along p, we get an unstable extension of ω_C by $\mathcal{O}_C$.*

4.3.2. The second construction is more explicit. We carefully modify the direct sum of the canonical line bundles along canonical divisors. Let $D \in |\omega_X|$ be a separable effective canonical divisor. Let $\rho \colon \Gamma(X, \omega_X) \to \Gamma(D, \omega_X|_D)$ be the restriction map. For any $u \in \Gamma(D, \mathcal{O}_D)$, let $u\rho$ be the multiplication of ρ by u. Let $(\rho, u\rho) : \omega_X \oplus \omega_X \to \omega_X|_D$ be the sum of ρ and $u\rho$. Let $E_u := \mathrm{Ker}(\rho, u\rho)$ be the kernel. It has rank 2 and degree $2g-2$. We show that:

Proposition 9. *For a general choice of $u \in \Gamma(D, \mathcal{O}_D)$, the kernel E_u is semi-stable.*

Proof. By Lemma 6, it suffices to find conditions on u, such that E_u does not contain ω_X as a subsheaf. We will look for u such that $h^0(X, E_u) < h^0(X, \omega_X) = g$.

Since $h^0(X, E_u) = 2h^0(X, \omega_X) - \dim \mathrm{Im}(\Gamma(\rho, u\rho))$, where Γ is the functor of taking global section on X, it suffices to find u such that $\dim \mathrm{Im}(\Gamma(\rho, u\rho)) > g$. Let us consider the family of coherent sheaves E_u parameterized by $u \in \Gamma(D, \mathcal{O}_D)$. Notice that the rank of $\Gamma(u, \rho u)$ is a lower semi-continuous function on $\Gamma(D, \mathcal{O}_D)$ and that k is infinite, it suffices to find such u after base change to the algebraic closure $\overline{k}$.

Without loss of generality, let us assume that k is algebraically closed. Let us write $D = \sum_{i=1}^{2g-2} P_i$, where P_i are distinct points. Let us identify $\Gamma(D, \mathcal{O}_D)$ with $k^{\oplus 2g-2}$, so that a section $u \in \Gamma(D, \mathcal{O}_D)$ has coordinates $u = (\lambda_i)_{i=1}^{2g-2}$, given by its value of at P_i. Let us fix non-vanishing local sections of ω_X at P_i, denoted by ω_i, so that we identify $\omega_X|_D \cong \mathcal{O}_D$ by $\Omega \mapsto \Omega/\omega_i$ at p_i. Then

$$\Gamma(\rho, u\rho)\colon \Gamma(X, \omega_X)^{\oplus 2} \to \Gamma(D, \omega_X|_D) \cong \oplus_{i=1}^{2g-2}\Gamma(P_i, \mathcal{O}_{P_i})$$

is given by $(\Omega_a, \Omega_b) \mapsto (f_a(P_i) + \lambda_i f_b(P_i))_i$, where $f_a(P_i) = \frac{\Omega_a(P_i)}{\omega_i(P_i)}, f_b(P_i) = \frac{\Omega_b(P_i)}{\omega_i(P_i)}$.

Let us pick a basis $\{\Omega_i\}_{i=1}^{g}$ of $\Gamma(X, \omega_X)$. The map $\Gamma(\rho, u\rho)$ can be expressed by the $(2g-2) \times 2g$ matrix $M = [M_L|M_R]$, where $(M_L)_{i,j} = \frac{\Omega_j(P_i)}{\omega_i(P_i)}$ and $(M_R)_{i,j} = \frac{\lambda_i\Omega_j(P_i)}{\omega_i(P_i)}$ for $0 \le i \le 2g-2, 0 \le j \le g$.

Notice that M_L is the matrix for $\Gamma(\rho)$, which fits into the exact sequence

$$0 \longrightarrow k \cong \Gamma(X, \omega_X - D) \longrightarrow \Gamma(X, \omega_X) \xrightarrow{\Gamma(\rho)} \Gamma(D, \mathcal{O}_D)\ ,$$

therefore $\operatorname{rank}(M_L) = \dim \operatorname{Im}(\Gamma(\rho)) = g-1$. After rearranging the order of $\{P_i\}$ and replacing Ω_i by suitable linear combinations, we may write

$$M_L = \begin{bmatrix} 0_{(g-1)\times 1} & I_{g-1} \\ 0_{(g-1)\times 1} & A \end{bmatrix}$$

Let us denote the diagonal matrices $\operatorname{diag}_{i=1}^{g-1}(\lambda_i)$, $\operatorname{diag}_{i=g}^{2g-2}(\lambda_i)$ by L_1, L_2, then

$$M = \begin{bmatrix} 0_{(g-1)\times 1} & I_{g-1} & 0_{(g-1)\times 1} & L_1 \\ 0_{(g-1)\times 1} & A & 0_{(g-1)\times 1} & L_2 A \end{bmatrix}$$

Let us choose λ_i with $\lambda_i\lambda_{g-1+i} \neq 1$ for all i, so that $L_2^{-1}L_1 - I$ is invertible. Notice that $\dim \operatorname{Im}(\Gamma(\rho, u\rho)) = \operatorname{rank}(M) = g-1+\operatorname{rank}((L_2^{-1}L_1 - I)A) = g-1+\operatorname{rank}(A)$, it suffice to show that $\operatorname{rank}(A) \ge 2$.

Let $\psi\colon \omega_X \to \omega_X|_{\sum_{i=1}^{g-1} P_i}$ be the restriction, then we have $h^0(\omega_X(-\sum_{i=1}^{g-1} P_i)) = g - \dim \operatorname{Im}(\Gamma(\psi)) = 1$, because $\dim \operatorname{Im}(\Gamma(\psi)) = \operatorname{rank}(I_{g-1}) = g-1$. By Riemann-Roch, we know that $h^0(\omega_X(-\sum_{i=g}^{2g-2} P_i)) = h^0(\omega_X(-\sum_{i=1}^{g-1} P_i)) = 1$. Now consider the restriction $\phi\colon \omega_X \to \omega_X|_{\sum_{i=g}^{2g-2} P_i}$, we have $\operatorname{rank}(A) = \dim \operatorname{Im}(\Gamma(\phi)) = h^0(X, \omega_X) - h^0(\omega_X(-\sum_{i=g}^{2g-2} P_i)) = g-1 \ge 2$. □

Remark 10. *One can show that a general extension of ω_X by $\mathcal{O}_X$ cannot be embedded in $\omega_X \oplus \omega_X$, so that the constructions produce different semi-stable vector bundles, see* [ah] *for details.*

4.3.3. Notice that every degree $2g-2$, semi-stable rank 2 vector bundle on X can be written as an extension $0 \to \omega_X^{\otimes -k} \to E \to \omega_X^{\otimes k+1} \to 0$. When $k = 0$, we saw that every non-split extension is semi-stable. This is not true for $k > 0$.

Proposition 11. *For any $k > 0$, there exists non-split unstable extensions of $\omega_X^{\otimes k+1}$ by $\omega_X^{\otimes -k}$.*

Proof. Let E be an extension of $\omega_X^{\otimes k+1}$ by $\omega_X^{\otimes k}$. Then we have short exact sequence

$$0 \longrightarrow \omega_X^{\otimes -k-1} \longrightarrow E \otimes \omega_X^{-1} \longrightarrow \omega_X^{\otimes k} \longrightarrow 0\ .$$

By Lemma 6, we know that E is semi-stable if and only if the connecting homomorphism

$$\delta_E = \cup[E]\colon \mathrm{H}^0(X, \omega_X^{\otimes k}) \to \mathrm{H}^1(X, \omega_X^{\otimes -k-1})$$

is injective. Here $[E] \in \mathrm{Ext}^1_X(\omega_X^{\otimes k}, \omega_X^{\otimes -k-1}) \cong \mathrm{H}^0(X, \omega_X^{\otimes 2k+2})$ is the extension class of E. For any $s \neq 0 \in \mathrm{H}^0(X, \omega_X^{\otimes -k-1})$, we have linear map

$$\cup s \colon \mathrm{Ext}^1_X(\omega_X^{\otimes -k-1}, \omega_X^{\otimes k}) \to \mathrm{H}^1(X, \omega_X^{\otimes -k-1}).$$

Then $\ker(\cup s) \neq 0$ for dimension reason. Any element $[E]$ in $\ker(\cup s)\backslash\{0\}$ is a required non-split unstable extension. □

4.4. **The generalized theta divisor.** Let us spell out the specialized theta divisor in terms of tautological line bundles. Let X/k be a smooth projective curve. Let E be a degree $2g-2$, semi-stable rank 2 vector bundle on X. Let $U \to \mathrm{Pic}^0_{X/k}$ be an étale cover, such that a tautological line bundle L exist on $X \times_k U$. Let m be a positive integer, and $Z \in |m\omega_X|$ be a reduced effective divisor. Let $\pi \colon X \times_k U \to U$ be the projection, let E_U be the constant family. Let $Z_U = Z \times_k U$. Let us take the short exact sequence

$$0 \longrightarrow (E_U \otimes L)(-Z_U) \longrightarrow E_U \otimes L \longrightarrow (E_U \otimes L)|_{Z_U} \longrightarrow 0\ . \tag{†}$$

and consider the long exact sequence of direct images along π_*:

$$\pi_*(E_U \otimes L) \hookrightarrow \pi_*(E_U \otimes L)|_{Z_U} \xrightarrow{\delta_U} R^1\pi_*\left((E_U \otimes L)(-Z_U)\right) \twoheadrightarrow R^1\pi_*(E_U \otimes L).\ .$$

Let η be the generic point of U. Since E is semi-stable and $\chi(E) = 0$, we know by [Ray82, 1.6.2] that $h^0(X_\eta, E_\eta \otimes L_\eta) = 0$. By cohomology and base change, the middle two terms are locally free of the same rank for m large enough. Therefore, the torsion sheaf $\pi_*(E_U \otimes L) = 0$ and δ_U is an injection. The determinant of δ_U cuts out a divisor in $T_U \subset U$, which descends to the generalized theta divisor $\Theta_E \subset P$.

4.5. **Restriction to $k(\Theta_E)$.**

Lemma 12. *The subscheme Θ_E is reduced and irreducible.*

Proof. By [Ray82, 1.8.1], the divisor Θ_E is numerically equivalent to 2Θ. If Θ_E was reducible or non-reduced, its reduced component would have numerical class Θ, contradiction with Lemma 5. □

Theorem 13. *The class $\alpha_0 \in \mathrm{Br}(\mathrm{Pic}^0_{X/k})$ restricts to zero in $\mathrm{Br}(k(\Theta_E))$.*

Proof. Let us restrict the exact sequence (†) to $X \times (\Theta_E)_U$ and denote the restriction of E_U, L and Z_U by E', L' and Z'. Let $\pi' \colon X \times (\Theta_E)_U \to (\Theta_E)_U$ be the projection. We have exact sequence

$$\pi'_*(E' \otimes L') \hookrightarrow \pi'_*(E' \otimes L')|_{Z'} \xrightarrow{\delta_U} R^1\pi'_*\left((E' \otimes L')(-Z')\right) \twoheadrightarrow R^1\pi'_*(E' \otimes L').$$

By the same argument in Lemma 1, the reducedness of Θ_E implies that the generic rank of $\pi'_*(E' \otimes L')$ is 1, then we conclude by the same argument as in Theorem 2. □

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