RESEARCH STATEMENT

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My field of interest is arithmetic geometry, focusing on problems around Brauer groups. Some other keywords: Franchetta-type question over moduli spaces, rationality problems.

The Brauer group of algebraic varieties plays a key role in many areas of algebraic geometry: it is related to the Tate conjecture, local-global principles over number fields, structure theory of division algebras, rationality problems, moduli problems, etc. Among all these, rationality problems and moduli problems are the main motivations of my research.

1. Past work

1.1. **Brauer class over the Picard scheme of curves.** In this work, we study the Brauer obstruction classes that come from moduli problems. This work is motivated by the cyclicity question [ABGV11, 1.1].

Let k be a field, let X be a genus g stable curve defined over k. Let $\mathrm{Pic}^0_{X/k}$ be the connected component of the Picard scheme of X. In general, tautological line bundles do not necessarily exist on $X \times \mathrm{Pic}^0_{X/k}$, the obstruction is given by a Brauer class $\alpha \in \mathrm{Br}(\mathrm{Pic}^0_{X/k})$. Our main results are the following:

- We identified some division algebras of prime degree, rising from unramified Brauer classes, they provide new candidates for the cyclicity problem. We give a new way to bound period of Brauer class from below. [Ma21b]
- We studied specialization of the Brauer class in the Picard scheme. We shown that the Brauer classes split at the generic point of generalized theta divisors. The method may apply to general moduli of sheaves. [Ma21g]
- In the study of generalized theta divisors, we constructed slope $g-1$, rank 2 semistable vector bundles on the generic genus g curve, answering a rank 2 analogue of the Franchetta's theorem (see Section 2) [Ma21g].

1.2. **Period of the generic curve via degeneration.** In this work, by degeneration method, we reprove the known result that the period of the generic genus g curve equals to $2g-2$.

Let $g \geq 3$ be an integer, let $\mathcal{M}_g$ be the moduli stack of smooth genus g curves. Let $\pi\colon \mathcal{M}_{g,1} \to \mathcal{M}_g$ be the universal family. Let C/k be the generic fiber of π. The period of C is defined as the order of the torsor $[\mathrm{Pic}^1_{C/k}] \in \mathrm{H}^1(k, \mathrm{Pic}^0_{C/k})$. It is a basic arithmetic invariant of C. By the strong Franchetta conjecture, see [Sch03], we know that the period of C equals to $2g-2$.

- We reprove the result by degenerating to stable curve whose geometrical irreducible components are all rational. The main difficulty in calculation of the period is that rational points on Picard schemes are not necessarily represented by line bundles. We overcome this difficulty by associating rational points on the Picard scheme to rational points on certain conic, which is non-split by global class field theory [Ma21e].
- A byproduct of the work: Brauer classes are represented by Brauer-Severi varieties. A natural question is how the classes sum up in terms of these varieties? We provide a geometric way to sum up Brauer-Severi varieties that split by a common Galois extension, via the Picard scheme of their clutching [Ma21a].

1.3. **Closed points on cubic hypersurfaces.** Cubic hypersurfaces are a natural class of object to study, many basic problems remain open, most notably their rationality. In this work, we generalize some results of Coray on closed points on cubic hypersurfaces. We showed that certain symmetric products of cubic hypersurfaces are stably birational.

Let k be a field, let X be a k-variety. The index of X, denoted by $\mathrm{ind}(X)$, is the greatest common divisor of degree of closed points on X. Cassels and Swinnerton-Dyer conjectured that if X is a cubic hypersurface, then $\mathrm{ind}(X) = 1$ implies that $X(k) \neq \emptyset$. The conjecture is largely open, but for cubic surface over perfect fields, it has been reduced to the case when X has a closed point of degree 4 or 10 [Cor76].

- We romove the assumption on perfectness of the base field in Coray's result, by a lifting trick. For cubic threefolds and fourfolds, we can also descend degree of closed points [Ma21c].
- We interpret this construction as rational maps between symmetric products of X. We show $\mathrm{Sym}^7(X)$ is stably birational to $\mathrm{Sym}^5(X)$ for a cubic threefold X; $\mathrm{Sym}^8(X)$ is stably birational to $\mathrm{Sym}^7(X)$ for a cubic fourfold X [Ma21c].

2. Current work

It is a general philosophy in mathematics that:

$$\texttt{On a universal object, the obvious classes are the only ones.} \qquad (\dagger)$$

For example, the cohomologies of the Grassmannian are generated by chern classes of the tautological bundle. Since Mumford's pioneering work [Mum83], many similar results are verified for moduli of curves.

Instead of working over the whole moduli, the philosophy can also be tested generically. Let C/k be the generic genus g curve (see section 1.2), Franchetta conjectured that all the line bundles on C are tensor powers of the canonical bundle ω_C. This has been enhanced to the strong Franchetta conjecture $\mathrm{Pic}_{C/k}(k) = \langle \omega_C \rangle$ and proved in [Sch03].

Let us call the problem of verifing philosophy (†) over *the whole* moduli space "questions of Mumford type", and call the problem of verifying (†) over the *generic point* of the moduli space "questions of Franchetta type". Unlike Mumford type questions, Franchetta type questions usually have negative answer. Take the simplest example, every homomorphic function on $\mathcal{M}_g$ is constant, while meromorphic functions abounds.

We are particularly interested in three kinds of Franchetta type questions:

- Closed points on generic hypersurfaces,
- Line bundles on plane slices of generic hypersurfaces.
- Picard torsors of the generic curves. (Future work)

2.1. Rational points on generic hypersurfaces.

The motivation is arithmetic aspects of rationality problems. Many rationality questions are posed and solved over an algebraically closed field, less is known on the arithmetic side. One basic open question on the arithmetic side is the following

Question 2.1 ([Kol01, Problem 60]). *Does there exist a smooth geometrically unirational k-variety, which carries a k-point, but not unirational over k?*

We pose several candidates to this question based on the philosophy (†):

- If *the* generic marked quartic hypersurface in $\mathbb{P}^6$ admits the marking as the only rational point, then it gives a desired example. The main evidence for such candidate is Hain's [Hai11] famous result on generic curves—the markings are the only rational points on the generic marked curves.

  Here is an evidence we found for hypersurfaces with mild singularities

  Theorem ([Ma21d]). *Let $Y_{n,d}$ be the generic degree-d nodal hypersurface in $\mathbb{P}^n$. If $d \geq 4$, then $Y_{n,d}$ admits the node as the only rational point.*

- If every closed point of degree 3 on *the* generic cubic surfaces X are collinear, then Hilb^3_X is a desired example. Colliot-Thélène shown that if collinearity holds, one can further give a del Pezzo surface example to Kollár's question.

  Here are some evidence we found for degree-d on generic degree d hypersurfaces

Theorem ([Ma21d])**.** *All closed points of degree d on the generic degree-d plane curve are collinear. All closed points of degree d on the generic degree-d hypersurface in $\mathbb{P}^n$ have the full permutation group S_d as monodromy group.*

Based on the previous work, we pose:

Conjecture 1. *Let $X^1_{n,d}/k$ the generic marked hypersurface of degree d in $\mathbb{P}^n$, let δ be the marked point. If $d \geq 4$, then $X^1_{n,d}(k) = \{\delta\}$.*

Here is our best result up to now:

Theorem ([Ma21f])**.** *Let $X^1_{n,d}/k$ be the generic marked hypersurface of degree d in $\mathbb{P}^n$. let δ be the marked point. If $d > 2n$, then $X^1_{n,d}(k) = \{\delta\}$. For $d > 2n$, consider the generic hypersurface with r marked points $X^r_{n,d}$. If $r \leq d$, then $X^r_{n,d}$ has exactly r rational points.*

The proof goes by analyzing the rational curves in the fibers. A more refined study of the rational curves in hypersurfaces may relax the degree to $d \geq 2n$. It would be very interesting result, if we could push the degree all way down to the Fano range $d < n$. For low degree hypersurfaces, we are studying their Fano variety of lines, which are more likely to be of general type—rational curves on general type varieties are more managable.

2.2. **Franchetta conjecture for plane slices of generic hypersurfaces.** The Franchetta theorem asserts that the generic genue g curve has only trivial degree-zero line bundles. We expect that arbitrary plane slices of generic hypersurfaces also have only trivial degree-zero line bundle. Such Franchetta type result would verify the aforementioned candidates in a unified way:

- Taking any plane slice of $X^1_{n,d}$ that passes through the marked point δ. If we can show that slice has only trivial degree-zero line bundles, then by the Abel-Jacobi embedding there exists no rational point on the slice. Hence $X^1_{n,d} = \{\delta\}$.
- If we can show that any slice of the generic cubic surface has only trivial degree-zero line bundle, then all the degree-3 points are collinear. (Otherwise they span a plane, the non-collinear points yields nontrivial line bundle on the plane cubic)

For vector bundle of higher rank, a natural Franchetta type question is whether the semistable line bundles of given rank and determinant exists on the generic genus g curve, and how they can be classified. Some results on rank 2 case

Theorem ([Ma21g])**.** *Degree $2g-2$, semi-stable rank 2 vector bundles on the generic genus g curve X exist and are not unique. Any extension of ω_X by $\mathcal{O}_X$ is semi-stable, but they do not give all semi-stable bundles on C.*

Characterizing such semi-stable vector bundles on generic objects is also related to Question 2.1: Notice that many moduli space of stable vector bundles on a variety X are smooth unirational, if their rational points are rare, then M would be an wanted example for Question 2.1.

3. Future work

3.1. **Rationality of hypersurfaces.** Determining if a hypersurface can be rationally parameterized is an old problem that kept motivating major developments of algebraic geometry: from abelian integrals, classification of algebraic surfaces to birational rigidity, intermediate Jacobians, Brauer group and unramified cohomologies, etc.

One open problem is whether a general hypersurface of degree $d \geq 4$ is rational. It is known that general hypersurface in $\mathbb{P}^{n+1}$ of degree $d \geq \log_2(n) + 3$ is irrational [Sch19], the method is given by degeneration to specific quadric-bundle hypersurfaces and finding a nontrivial unramified cohomology class. The unramified cohomology is explicitly constructed by pulling back carefully chosen symbol form étale cohomology of the base of the quadric bundle, showing it is unramified, and nonzero. Notice that general étale cohomology class is only a sum of symbols. One would like to ask

Question 3.1. *Can we possibly improve the log bound on d by constructing unramified cohomology classes from sum of symbols?*

3.2. **Franchetta type result for Picard torsors.** In [MM14], after determining the Picard group of universal compactified Jacobian $\overline{\mathcal{J}ac}$, the authors posed the following Mumford type question:

Question 3.2 (Melo-Viviani)**.** *Is the group of torsors* $\mathrm{H}^1(\mathcal{M}_g, \mathrm{Pic}_{\mathcal{M}_{g,1}/\mathcal{M}_g})$ *generated by the tautological torsor* $\mathrm{Pic}^1_{\mathcal{M}_{g,1}/\mathcal{M}_g}$*?*

The question is very interesting as it is the few known attempts to study Mumford type questions with arithmetic flavor. The question is answered in [FP19] using the fact that $\mathrm{Br}(\mathcal{M}_{g,n}) = 0$. Which is again based on the fact that every analytic line bundle on moduli of curves is algebraic [AC87], plus the deep result of Harer on group cohomology of mapping class groups [Har83]. On the other hand, the Franchetta type question remains mysterious:

Question 3.3. *Let C be the generic genus g curve, is the group* $\mathrm{H}^1(k, \mathrm{Pic}^0_{C/k})$ *generated by the torsor* $[\mathrm{Pic}^1_{C/k}]$*?*

One may think of the question as a "moduli version" of the Tate conjecture, focusing on the function field of $\mathcal{M}_{g,n}$ instead of global fields. Besides its own interest, the results are also interesting to researches on Grothendieck section conjecture over boundary of moduli space of curves [LLSS20]. A main difficulty is that abelian-variety-torsors are hard to extend over high codimensional locus by "taking closure", so Franchetta-type question cannot be directly related to the Mumford-type questions.

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