RATIONAL POINTS ON UNIVERSAL HYPERSURFACES (DRAFT)

Abstract. We show that for $d \geq 2n+1$ and $r \leq d$, the generic "r-marked degree-d hypersurface" in $\mathbb{P}^n$ admits the marking as the only rational point.

1. Introduction

Let $\mathcal{M}_{g,n}$ be the moduli space of genus-g curves with n marked points. Let $f\colon \mathcal{M}_{g,n+1} \to \mathcal{M}_{g,n}$ be the tautological family. Let us denote the generic fiber of f by $C_{g,n}$. The marked points give rise to $k(\mathcal{M}_{g,n})$-rational sections of $C_{g,n}$.

In the study of Grothendieck's section conjecture for hyperbolic curves, Hain [Hai11] and Watanabe [Wat19] showed that the markings are all the $k(\mathcal{M}_{g,n})$-rational points of $C_{g,n}$ for $g \geq 3$.

In [BK20], it is shown that general high-degree hypersurfaces in projective spaces are hyperbolic. Does Hain and Watanabe's result hold for their higher dimensional analogues? We prove some positive results based on the work of Riedl-Woolf [RW18].

1.1. Let us work over a field k. Let n, d be positive integers. Let $\mathcal{X}_{n,d} \subset |\mathcal{O}_{\mathbb{P}^n}(d)| \times \mathbb{P}^n$ be the universal family of degree-d hypersurfaces in $\mathbb{P}^n$. Let us denote the projections from $\mathcal{X}_{n,d}$ to $|\mathcal{O}_{\mathbb{P}^n}(d)|, \mathbb{P}^n$ by π_1, π_2.

$$\begin{tikzcd}
X_{n,d} \arrow[r, hook] \arrow[d] & & \mathcal{X}_{n,d} \arrow[r, hook] \arrow[dl, "\pi_1"'] \arrow[d, "\pi_2"] & {|\mathcal{O}_{\mathbb{P}^n}(d)| \times \mathbb{P}^n} \\
\operatorname{Spec}(K_{n,d}) \arrow[r, hook] & {|\mathcal{O}_{\mathbb{P}^n}(d)|} & \mathbb{P}^n &
\end{tikzcd}$$

Let $K_{n,d}$ be the function field of $|\mathcal{O}_{\mathbb{P}^n}(d)|$, and let $X_{n,d} \subset \mathbb{P}^n_{K_{n,d}}$ be the generic fiber of π_1. We call $X_{n,d}$ *the* generic degree-d hypersurface in $\mathbb{P}^n$.

1.2. Let $\mathcal{X}^1_{n,d} := \mathcal{X}_{n,d} \times_{|\mathcal{O}_{\mathbb{P}^n}(d)|} \mathcal{X}_{n,d} \subset \mathcal{X}_{n,d} \times \mathbb{P}^n$ be the tautological family of "hypersurfaces with a marked point". Let us denote the projections to $\mathcal{X}_{n,d}$ and $\mathbb{P}^n$ by π^1_1, π^1_2. The diagonal embedding $\Delta\colon \mathcal{X}_{n,d} \to \mathcal{X}^1_{n,d}$ gives a regular

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section to π_1^1, we call it the marked point (tautological section) of π_1^1.

$$\begin{tikzcd}
X^1_{n,d} \arrow[rr, hook] \arrow[d, shift right] & & \mathcal{X}^1_{n,d} \arrow[rr, hook] \arrow[dl, "\pi_1^1"'] \arrow[dr, "\pi_2^1"] & & \mathcal{X}_{n,d} \times \mathbb{P}^n \\
\mathrm{Spec}(K^1_{n,d}) \arrow[r, hook] \arrow[u, bend right, "\delta"'] & \mathcal{X}_{n,d} \arrow[ur, bend right, "\Delta"'] & & \mathbb{P}^n &
\end{tikzcd}$$

Let $K^1_{n,d}$ be the function field of $\mathcal{X}_{n,d}$, and let $X^1_{n,d} \subset \mathbb{P}^n_{K^1_{n,d}}$ be the generic fiber of π_1^1. We call $X^1_{n,d}$ *the* generic pointed degree-d hypersurface in $\mathbb{P}^n$. The restriction of Δ yields a marked point $\delta \in X^1_{n,d}(K^1_{n,d})$. We show that δ is the only rational point of the generic hypersurface:

Theorem 1. *For $d \geq 2n+1$, we have $X^1_{n,d}(K^1_{n,d}) = \{\delta\}$.*

It is necessary to put some restrictions on the degree and the dimension: For $d = 2$ or 3, one easily constructs other rational points by taking residue intersection with lines and tangent planes through the marked point. However, for $d \geq 4$, it is unclear how to find a second rational point using projective geometry. We pose the following question:

Question. *Does $X^1_{n,d}(K^1_{n,d}) = \{\delta\}$ hold for all $d \geq 4$?*

1.3. In [Kol01, Problem 60], Kollár raised the open question of finding a smooth geometrically rationally connected variety X, which carries a k-rational point, but X is not k-unirational. Notice that Fano hypersurfaces are rationally connected, an affirmative answer to our Question for any $d \leq n$ would also answer Kollár's question.

2. Proof of Theorem 1

2.1. We show that every $K^1_{n,d}$-rational point in $X^1_{n,d}$ necessarily extends to a regular section of $\pi_{1,1} \colon \mathcal{X}^1_{n,d} \to \mathcal{X}_{n,d}$. Then Theorem 1 follows from the simple observation:

Proposition 2. *For $d > 1$, the marked point Δ is the only regular section to π_1^1.*

Proof. Let $s \colon \mathcal{X}_{n,d} \to \mathcal{X}^1_{n,d} \subset \mathcal{X}_{n,d} \times \mathbb{P}^n$ be a regular section to π_1^1, it suffices to show that $\pi_2^1 \circ s = \pi_2^1 \circ \Delta$. Notice that $\mathcal{X}_{n,d} = \bigcup_{p \in \mathbb{P}^n} \pi_2^{-1}(p)$, and that $(\pi_2^1 \circ \Delta)(\pi_2^{-1}(p)) = p$, it suffices to that $(\pi_2^1 \circ s)(\pi_2^{-1}(p)) = p$ for every $p \in \mathbb{P}^n$. The inverse image $\pi_2^{-1}(p)$ is a projective space of dimension $\binom{n+d}{d} - 1 > n$, therefore the morphism $(\pi_2^1 \circ s)|_{\pi_2^{-1}(p)} \colon \pi_2^{-1}(p) \to \mathbb{P}^n$ is constant. Then s being a section to π_1^1 implies that $\mathrm{Im}(\pi_2^1 \circ s|_{\pi_2^{-1}(p)}) \subseteq \bigcap_{[X] \in \pi_2^{-1}(p)} X = p$. $\square$

2.2. Let $R_{n,d} \subset \mathcal{X}_{n,d}$ be the subset consisting of pairs $([X], p \in X)$ such that there exist a rational curve $C \subset X$ passing through p. In other words, the subset $R_{n,d}$ is the union of rational curves in the fibers of the universal family. We define the dimension of $R_{n,d}$ to be the maximal dimension of its irreducible components.

Proposition 3. *If* $\dim |\mathcal{O}_{\mathbb{P}^n}(d)| - \dim R_{n,d} \geq 2$, *then the closure of every rational point of* $X^1_{n,d}$ *is a regular section to* $\pi^1_1 \colon \mathcal{X}^1_{n,d} \to \mathcal{X}_{n,d}$.

Proof. Let us take a point $z \in X^1_{n,d}(K^1_{n,d})$, and let $Z \subset \mathcal{X}^1_{n,d}$ be its closure. The projection $\pi^1_1|_Z \colon Z \to \mathcal{X}_{n,d}$ is a birational morphism. Let $W \subseteq Z$ be the locus where $(\pi^1_1|_Z)^{-1}$ is not defined. By Ahbyankar's lemma [KSC04, Theorem 4.26], for every $w \in W$, the fiber $(\pi^1_1|_Z)^{-1}(w)$ is uniruled, and therefore

$$W \subset R_{n,d}.$$

Since $\mathcal{X}_{n,d}$ is smooth, by [KM98, Corollary 2.63], we know that $\pi^1_1|_Z \colon Z \to \mathcal{X}_{n,d}$ is an isomorphism in codimension at least two, hence an isomorphism. Therefore Z is a regular section. □

2.3. We assemble the previous discussion:

Proof of Theorem 1. Let z be a rational point of $X^1_{n,d}$. By Proposition **??** and 3, we know that the closure $\overline{\{z\}} \subset \mathcal{X}^1_{n,d}$ is a regular section to π^1_1. Then we conclude from Proposition 2. □

2.4. The dimension estimate is not optimal. For example, when $n = 2$, a plane degree-4 curve which contains a rational component is either reducible or irreducible with δ-invariant 3 (3-nodes), so $\operatorname{codim}_{|\mathcal{O}_{\mathbb{P}^2}(4)|}(W_1) \geq 3$. By the same proof of Theorem 1, we know that $X^1_{2,4}(K^1_{2,4}) = \{\delta\}$.

3. More marked points

3.1. Let us define the tautological family

$$\pi^r_1 \colon \mathcal{X}^r_{n,d} \to \mathcal{X}^{r-1}_{n,d}$$

of degree-d hypersurface in $\mathbb{P}^n$ with r marked points recursively by

$$\mathcal{X}^0_{n,d} := \mathcal{X}_{n,d},$$

$$\mathcal{X}^r_{n,d} := \mathcal{X}^{r-1}_{n,d} \times_{|\mathcal{O}_{\mathbb{P}^n}(d)|} \mathcal{X}_{n,d} \subset \mathcal{X}_{n,d} \times \mathbb{P}^n,$$

with projections $\mathcal{X}^r_{n,d} \to \mathcal{X}^{r-1}_{n,d}$ and $\mathcal{X}^r_{n,d} \to \mathbb{P}^n$ denoted by π^r_1, π^r_2.

Let $\mathrm{p}_1, \mathrm{p}_2, \cdots, \mathrm{p}_r \colon \mathcal{X}^{r-1}_{n,d} = \mathcal{X}_{n,d} \times_{|\mathcal{O}_{\mathbb{P}^n}(d)|} \times \cdots \times_{|\mathcal{O}_{\mathbb{P}^n}(d)|} \mathcal{X}_{n,d} \to \mathcal{X}_{n,d}$ be the projections onto factors.

Lemma 4. *Let n, d be integers. Let $S = \{x_1, \cdots, x_d\}$ be any d distinct points in $\mathbb{P}^n$. Let $\Lambda = |\mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathcal{I}_S|$ be the linear system of degree-d hypersurfaces passing through the points, then $\bigcap_{[X]\in\Lambda} X = S$.*

Proof. Notice that Λ contains the subset Λ_S of hypersurfaces which consists of union of d planes passing through the points. Clearly we have

$$S \subseteq \bigcap_{[X]\in\Lambda} X \subseteq \bigcap_{[X]\in\Lambda_S} X = S$$

so equality holds everywhere. □

Proposition 5. *For every $r \leq d$, the regular sections to*

$$\pi_1^r \colon \mathcal{X}_{n,d}^r = \mathcal{X}_{n,d}^{r-1} \times_{|\mathcal{O}_{\mathbb{P}^n}(d)|} \mathcal{X}_{n,d} \to \mathcal{X}_{n,d}^{r-1}$$

are given by graphs of $\mathrm{p}_1, \mathrm{p}_2, \cdots, \mathrm{p}_r$.

Proof. Let s be a section to π_1^r, let $f \colon \mathcal{X}_{n,d}^r \to \mathbb{P}^r$ be the r-th self-product of π_2. We need to show that $\mathrm{Im}(f \circ s) \subset \bigcup_{i=1}^r \mathrm{Im}(f \circ \Gamma_{\mathrm{p}_i})$. Since $\mathrm{Im}(s) \cong \mathcal{X}_{n,d}^{r-1}$ is irreducible, it suffices to verify the inclusion over a Zariski dense subset in $|\mathcal{O}_{\mathbb{P}^n}(d)|$. Let $\{q_1, \cdots, q_s\}$ be distinct points in $\mathbb{P}^n$. By the previous lemma, we know that $\{q_i\}_{i=1}^r = \bigcap_{[X]\in f^{-1}(\{q_i\}_{i=1}^r)} X$.

The inverse image $f^{-1}(\{q_i\}_{i=1}^r)$ is a linear subspace in $|\mathcal{O}_{\mathbb{P}^n}(d)|$ of dimension $\binom{n+d}{d} - 1 - r > n$, so every morphism from $f^{-1}(\{q_i\}_{i=1}^r)$ to $\mathbb{P}^n$ is constant, and therefore $f \circ s|_{f^{-1}(\{q_i\}_{i=1}^r)} \subseteq (\bigcap_{[X]\in f^{-1}(\{q_i\}_{i=1}^r} X) = \{q_i\}_{i=1}^r$. Since every hypersurface of dimension at least 1 contains r distinct points, the union of $f^{-1}(\{q_i\}_{i=1}^r)$ for $\{q_i\}_1^r$ in $\mathbb{P}^n$ is Zariski dense in $|\mathcal{O}_{\mathbb{P}^n}(d)|$. Indeed, the union is the whole space. □

Let $X_{n,d}^r$ be the generic fiber of the projection $\pi_1^r \colon \mathcal{X}_{n,d}^r \to \mathcal{X}_{n,d}^{r-1}$, then we have:

Theorem 6. *Let $n \geq 3$, let $r \leq d$ be an integer, then the generic r-marked hypersurface $X_{n,d}^r$ admits the r marked points as the only rational points.*

Proof. We use the same proof as in Theorem 1, where Proposition 2 is replaced by Proposition 5. There is problem applying Ahbyankar's lemma: the product space

$$\mathcal{X}_{n,d} \times_{|\mathcal{O}_{\mathbb{P}^n}(d)|} \cdots \times_{|\mathcal{O}_{\mathbb{P}^n}(d)|} \mathcal{X}_{n,d}$$

is not necessarily smooth. We only know that it is $\mathbb{Q}$-factorial, since it is complete intersection, and is smooth in codimension at least 3. The remedy is, we replace the product by a resolution—the Fulton-Macpherson compactification, which compactified the space of distinct n-ordered points on a smooth projective variety. The compactification can be carried out by explicit blow-ups.

In the case of twice self-product, the Fulton-Macpherson compactification is just blowing up the product along the diagonal. Let us consider the diagram

$$\begin{tikzcd}
\mathcal{X}_{n,d} \times_{|\mathcal{O}_{\mathbb{P}^n}(d)|} \mathcal{X}_{n,d} \arrow[r, hook] & |\mathcal{O}_{\mathbb{P}^n}(d)| \times \mathbb{P}^n \times \mathbb{P}^n \\
\mathrm{Bl}_\Delta(\mathcal{X}_{n,d} \times_{|\mathcal{O}_{\mathbb{P}^n}(d)|} \mathcal{X}_{n,d}) \arrow[u] \arrow[r, hook, "\pi_2'"] & |\mathcal{O}_{\mathbb{P}^n}(d)| \times \mathrm{Bl}_\Delta(\mathbb{P}^n \times \mathbb{P}^n) \arrow[u]
\end{tikzcd}$$

The key observation is that the morphism π_2' is a projective bundle, the smoothness of the base implies the smoothness of Fulton-Macpherson compactificaiton of the universal family. (More details to be added.) $\square$

4. Bertini for generic hypersufaces

4.1. Recall that the classical Bertini theorem asserts that a *general* hyperplane section of a smooth projective variety is smooth. We show that

Theorem 7. *Let $n \geq 3, d \geq n+2$, then for any linear subspace H in $\mathbb{P}^n_{K_{n,d}}$, the intersection $H \cap X_{n,d}$ is smooth.*

4.2. Let us begin with a simple observation on the generic hypersurface $X_{n,d}$.

Proposition 8. *The degree of any closed point on $X_{n,d}$ is a multiple of d.*

Proof. Notice that $\mathcal{X}_{n,d}$ is a projective bundle over $\mathbb{P}^n$ via π_2. Therefore, the Chow ring $A^*(\mathcal{X}_{n,d})$ is generated by L_1 and L_2, where L_i is the pullback of hyperplane class via π_i, see [Ful84, Theorem 3.3]. Given any cycle z on $X_{n,d}$, the closure $\overline{\{z\}} \subset \mathcal{X}_{n,d}$ is a codimension-i cycle on $\mathcal{X}_{n,d}$, and therefore we can write it as a $\mathbb{Z}$-linear combination of $\{L_1^k \cdot L_2^{n-1-k}\}_{0 \leq k \leq i}$. Restricting to the generic fiber, the class L_1 becomes zero, so $\deg(z)$ is a multiple of $d = \deg(L_2^{n-2}|_{X_{n,d}})$. $\square$

4.3. The proof of Theorem 7 is based on the following result on generic hypersurfaces.

Proposition 9 ([Xu94, Prop.3, Prop.4])**.** *Let M be a hyperplane section of a generic hypersurface of degree $d \geq n+2$ in $\mathbb{P}^n$, $n \geq 3$, then M has at most n singularities.*

Proof of Theorem 7. When $d \geq n+2$, let us consider the subscheme of singularities

$$S := \mathrm{Sing}(H \cap X_{n,d}).$$

The closure of H in $\mathcal{X}_{n,d}$ is a family of hyperplanes defined outside a codimension 2 subset in $|\mathcal{O}_{\mathbb{P}^n}(d)|$. For a general geometric point $\overline{a}$ in $|\mathcal{O}_{\mathbb{P}^n}(d)|$, by Proposition 9 we know that $\deg(S_{\mathrm{red}}|_{\overline{a}}) \leq n$. If $S \neq \emptyset$, then Proposition 8 implies that $n+2 \leq d \leq \deg(S_{\mathrm{red}})$, contradiction. $\square$

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