Expected Value (Expectation or Mean) of Random Variable.

Motivating Example

fair coin

I offer you to play a game

throw the fair coin

HEADS $\Rightarrow$ I pay you \$1

TAILS $\Rightarrow$ you pay me \$2.

How much can I offer you so you would agree to play?

Play 100 times

≈ 50 times is HEADS $\longrightarrow +\$1$

≈ 50 times is TAILS $\longrightarrow -\$2$

Your "average" winnings?

$$\approx \frac{50 \times 1 + 50 \times (-2)}{100} \$ = -\frac{1}{2} \$$$

Economists agree that both sides enter into an agreement only if both believe that game is fair.

$\Rightarrow$ 50 ¢ is the fair price for this game.

Dutch mathematician Christian Huygens (1657). In a box

$$\begin{cases} p \text{ prizes, each worth } \$a \\ q \text{ prizes, each worth } \$b \end{cases}$$

What should you pay to play, i.e. to draw a prize (blindly & randomly) from this box?

$$\frac{p \times a + q \times b}{p+q} \text{ — "average win"}$$

$$= a \times \frac{p}{p+q} + b \times \frac{q}{p+q}$$

$$= a \times p(a) + b \times p(b)$$ //

Def Expected value (expectation or mean) of a r.v. (discrete) X is

$$EX = E(X) = \sum_x x \cdot f_X(x),$$

f_X is the pmf of X.

= a weighted average of all possible values of X weighted by their probabilities.

<u>example</u> 1) X = outcome of a roll of a fair die.

$$EX = \sum_x x f_X(x) = \sum_{x=1}^{6} x \cdot \frac{1}{6}$$

$$\boxed{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}}$$

$$= \frac{1}{6} \cdot \frac{6 \cdot 7}{2} = 3.5 \leftarrow \text{is not a possible value of } X.$$

2) Let $Y = \begin{cases} 1 & \text{if } X \le 4 \\ 0 & \text{if } X > 4 \end{cases}$ | X as above

Indicator r.v. $I\{A\} = \mathbb{1}_A = \begin{cases} 1, & \text{if } A \\ 0, & \text{if } A^c. \end{cases}$

$A = \{X \le 4\}$

$$EY = E(\mathbb{1}_A) = 1 \cdot P(X \le 4) + 0 \cdot P(X > 4) =$$

$$= P(X \le 4) = 2/3$$

Generally, $E\mathbb{1}_A = P(A)$. //

<u>example</u> St Petersburg Paradox

(D. Bernoulli was is St. Petersburg when he wrote the paper.)

Toss a fair coin until H (heads)

Let X = # of tosses until H.

Possible Outcomes	# of tosses until H, x	$P(X=x)$	Payoff
H	1	$\frac{1}{2}$	\$1
TH	2	$\frac{1}{4}$	\$2
⋮	⋮	⋮	⋮
$\underbrace{TTT\ldots T}_{n}H$	$n+1$	$\left(\frac{1}{2}\right)^{n+1}$	$\$2^n$
⋮	⋮	⋮	⋮

What is the fair price for this game?

Y = payoff random variable

$$E(Y) = \sum_y y \cdot f_Y(y) =$$

$$= 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + \ldots + 2^n \cdot \left(\frac{1}{2}\right)^{n+1} + \ldots$$

$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \ldots + \frac{1}{2} + \ldots$$

$$= +\infty \; !$$

In practice, this game would be played against a bank (casino?) with limited resources.

The bank has $\$2^{27}$

If $X \geq 28 \Rightarrow$ payoff is fixed at $\$2^{27}$.

Then $\tilde{Y}$ = new payoff capped at 2^{27}.

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$= \min(Y, 2^{27})$

$$E\tilde{Y} = \underbrace{1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + \ldots + 2^{27} \cdot \left(\frac{1}{2}\right)^{28} + 2^{27} \cdot \left(\frac{1}{2}\right)^{29} + \ldots}_{} =$$

$$= \frac{1}{2} \cdot 28 + \frac{1}{4} \cdot \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = 14 + \frac{1}{4} \cdot \frac{1}{1-\frac{1}{2}}$$

$= \underline{14.5}$ \$. //

Expectations of Continuous r.v.'s

Def The mean of a continuous r.v. X

is

$$E(X) = \int_{-\infty}^{\infty} x \, f_X(x) \, dx \quad \text{, if exists}$$

where $f_X(x)$ is the pdf of X.

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example X has pdf

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0 \qquad [\lambda > 0 \text{ is known parameter}]$$

$$E(X) = \int_{-\infty}^{\infty} x \, f_X(x) \, dx =$$

$$= \int_0^{\infty} x \cdot \lambda e^{-\lambda x} \, dx = -\int_0^{\infty} x \, d(e^{-\lambda x})$$

$$= \cancel{-x e^{-\lambda x} \Big|_0^{\infty}}^{\to 0} + \int_0^{\infty} e^{-\lambda x} \, dx$$

$$= \cancel{-x e} \Big|_0 + \int_0 e \cdots$$

$$= -\frac{1}{\lambda} e^{-\lambda x} \Big|_0^{\infty} = \frac{1}{\lambda}. \quad //$$

<u>Note:</u> EX will almost always exist but

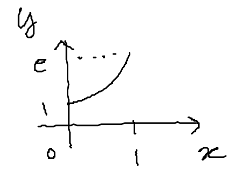
$$f(x) = \frac{1}{\pi(1+x^2)}, \quad x \in \mathbb{R},$$

$EX = ?$ diverges.

<u>example</u> $X \sim \text{Unif } [0,1]$

$$Y = e^X, \quad EY = ?$$

$$X = \ln Y, \quad 1 \le y \le e.$$



(one-to-one !)

$$f_Y(y) = f_X(x(y)) \left| \frac{\partial x}{\partial y} \right|$$

$$= 1 \cdot \left| \frac{1}{y} \right| = \frac{1}{y}, \quad 1 \le y \le e.$$

$$EY = \int_{-\infty}^{\infty} y f_Y(y)\, dy = \int_1^e y \cdot \frac{1}{y}\, dy = e - 1. \quad //$$

<u>Expectation of a Function of r.v.'s</u>

It turns out that it's not necessary to compute f_Y (pmf or pdf) of $Y = g(X)$.

In fact, for continuous X

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$$EY = E(g(X)) = \int_{-\infty}^{\infty} \underbrace{g(x)\, f_X(x)}\, dx$$

no need to compute $f_Y(y)$!

In discrete case,

$$EY = \sum_x g(x)\, f_X(x)$$

<u>example</u> $X \sim \text{Unif}\,[0,1]$

$$Y = e^X$$

$$EY = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx =$$

$$= \int_0^1 e^x \cdot 1\, dx = e^x \Big|_0^1 = e - 1. \quad //$$

<u>Properties of Expectations</u>

Let X be a r.v., s.t. EX exists.

<u>Theorem</u> If $Y = aX + b$, $a, b \in \mathbb{R}$,

then $EY = E(aX+b) = aEX + b$.

Caution! In general, X, $Y = g(X)$ then

$$Eg(X) \neq g(EX) \quad (\text{in linear case yes!})$$

$$\left\{ \int g(x) f(x)\, dx \neq g\left(\int x f(x)\, dx \right) \right\}$$

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▷ Suppose X is discrete then

$$E(aX+b) = \sum_x (ax+b) f_X(x)$$

$$= \sum_x ax f_X(x) + \sum_x b f_X(x)$$

$$= a \underbrace{\sum_x x f_X(x)} + b \underbrace{\sum_x f_X(x)}$$

$$= a \cdot EX + b \qquad \square$$

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example $\quad X \sim Bin(n,p), \quad Y = \frac{1}{2} X - 5$

$$EY = ? \qquad EY = \frac{1}{2} EX - 5$$

$$EX = \sum_{x=0}^{n} x f_X(x) = \sum_{x=0}^{n} x \cdot \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} \frac{x \cdot n!}{(n-x)!\, x!} p^x (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} \frac{\cancel{x} \cdot n!}{(n-x)!\, \cancel{x}!} p^x (1-p)^{n-x} = \sum_{x=1}^{n} \frac{n!}{(n-x)!\,(x-1)!} p^x (1-p)^{n-x}$$

$$= n \cdot p \cdot \sum_{x=1}^{n} \frac{(n-1)!}{(n-1-(x-1))!\,(x-1)!} p^{x-1} (1-p)^{(n-1)-(x-1)}$$

$u = x-1$

$$= np \cdot \underbrace{\sum_{u=0}^{n-1} \frac{(n-1)!}{(n-1-u)!\, u!} p^u (1-p)^{n-1-u}}$$

binomial formula

$$u=0\ (n-1-u)!\ u!$$ binomial formula

$$= np \cdot \left(p+(1-p)\right)^{n-1} = np.$$

For any $X \sim Bin(n,p)$, $EX = np$.

$$EY = \frac{1}{2}EX - 5 = \frac{1}{2}np - 5. \quad //$$

Functions of Several Random Variables

Let $X_1, X_2, \dots, X_n$ have joint density

$f_X(x_1, \dots, x_n)$.

Let $Y = r(\underset{\sim}{X}) = r(X_1, \dots, X_n)$, $r: \mathbb{R}^n \to \mathbb{R}$

(e.g. $Y = \sum_{i=1}^{n} X_i$). Then

$$EY = \int_{\mathbb{R}^n} \cdots \int r(x_1, \dots, x_n) f_X(x_1, \dots, x_n)\, dx_1 \dots dx_n$$

$$= \int_{\mathbb{R}^n} r(\underset{\sim}{x}) f_X(\underset{\sim}{x})\, d\underset{\sim}{x}.$$

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example Suppose X and Y have joint pdf

$$f(x,y) = 12y^2, \quad 0 \le y \le x \le 1.$$

$$E(XY) = \iint_{0 \le y \le x \le 1} xy \cdot 12y^2\, dx\, dy =$$

$$= \int_0^1 \int_0^x 12xy^3\, dy\, dx = \int_0^1 3x \cdot y^4 \Big|_0^x dx$$

$$= \int_0 \int_0 12xy\, dy\, dx = \int_0 \ldots |_0$$

$$= \int_0^1 3x^5\, dx = \frac{x^6}{2}\Big|_0^1 = \frac{1}{2}. \quad //$$

Similar statement is true for discrete r.v.

If $X_1, \ldots, X_n \sim$ pmf $f_X(x_1, \ldots, x_n)$

then $Y = u(X_1, \ldots, X_n)$ $(u : \mathbb{R}^n \to \mathbb{R})$

$$EY = \sum_{x_1} \cdots \sum_{x_n} u(x_1, \ldots, x_n) f_X(x_1, \ldots, x_n)$$

<u>Theorem</u> If exists a constant $a \in \mathbb{R}$ s.t.

$P(X \geq a) = 1$ then $EX \geq a$.

D $\quad EX = \int_{-\infty}^{\infty} x f_X(x)\, dx =$

$$= \int_a^{\infty} x f_X(x)\, dx \geq \int_a^{\infty} a \cdot f_X(x)\, dx$$

$$= a \int_a^{\infty} f_X(x)\, dx = a \quad //$$

If $P(X \leq b) = 1 \Rightarrow EX \leq b$.

& $P(a \leq X \leq b) = 1 \Rightarrow a \leq EX \leq b$.

<u>Theorem</u> $X_1, \ldots, X_n$ are r.v. s.t. EX_i exist

then $\quad E(X_1 + X_2 + .. + X_n) = \sum_{i=1}^{n} EX_i$

$\triangleright$ Let $n = 2$, X_1, X_2 discrete.

$$E(X_1 + X_2) = \sum_{x_1} \sum_{x_2} (x_1 + x_2) f_{12}(x_1, x_2)$$

$$= \sum_{x_1} x_1 \sum_{x_2} f_{12}(x_1, x_2) +$$

$$+ \sum_{x_2} \sum_{x_1} x_2 f_{12}(x_1, x_2)$$

$$= \sum_{x_1} x_1 f_1(x_1) + \sum_{x_2} x_2 f_2(x_2)$$

$$= EX_1 + EX_2. \quad //$$

Use induction for $n > 2$.

And if $a_1, a_2, .., a_n, b \in \mathbb{R}$

$$E(a_1 X_1 + a_2 X_2 + \dots + a_n X_n + b) =$$
$$= a_1 EX_1 + a_2 EX_2 + \dots + a_n EX_n + b. \quad \Big| \; EX_i < \infty$$

example $\quad X \sim Bin(n, p)$

X = # of successes in n trials

$$Y_i = \begin{cases} 1, & \text{if trial } i \text{ is a success} \\ 0, & \text{otherwise (o.w.)} \end{cases}$$

$$X = \sum_{i=1}^{n} Y_i, \qquad EY_i = p$$

$$EX = \sum_{i=1}^{n} EY_i = np. \qquad //$$

$$EX = \sum_{i=1}^{n} EY_i = np. \quad //$$

Theorem Let $X_1, \dots, X_n$ be <u>independent</u> r.v. such that $EX_i < \infty$ then

$$E\left(\prod_{i=1}^{n} X_i\right) = \prod_{i=1}^{n} EX_i.$$

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example Sps X_1, X_2, X_3 are independent s.t.

$$EX_i = 0, \quad EX_i^2 = 1, \quad i = 1, 2, 3.$$

$$E\left[X_1^2 (X_2 - 4X_3^2)\right] =$$

$$\overset{\text{linearity}}{=} EX_1^2 X_2 - 4 E X_1^2 X_3^2 \overset{\text{independence}}{=} EX_1^2 EX_2 - 4EX_1^2 EX_3^2$$

$$= -4 \quad //$$

If $P(X \geq 0) = 1$ and

1) X is discrete, taking values on $0, 1, 2, 3, \dots$ $(\mathbb{Z}_+)$

$$EX = \sum_{n=1}^{\infty} n P(X = n)$$

$P(X \geq 1) = P(X=1) + P(X=2) + P(X=3) + \dots$

$P(X \geq 2) = + P(X=2) + P(X=3) + \dots$

$P(X \geq 3) = + P(X=3) + \dots$

$$\boxed{EX = \sum_{n=1}^{\infty} P(X \geq n)} \quad \text{if } X \in \mathbb{Z}_+.$$

2) If X is continuous $P(X \geq 0) = 1$.

$$EX = \int_0^{\infty} P(X > x)\,dx = \int_0^{\infty} (1 - F_X(x))\,dx$$

(convince yourself)

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example We flip a coin until the first success. $P(H) = p$.

X = # trials until first success.

$$P(X = x) = (1-p)^{x-1} \cdot p, \quad x = 1, 2, \ldots$$

$$EX = \sum_{x=1}^{\infty} x \cdot p(1-p)^{x-1} \quad \text{(try this!)}$$

$$EX = \sum_{x=1}^{\infty} P(X \geq x) = \sum_{x=1}^{\infty} P(X > x-1)$$

$$= \sum_{x=1}^{\infty} (1-p)^{x-1} = \frac{1}{1-(1-p)} = \frac{1}{p}.$$

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example 1) N people throw their hats into the center of a room. Hats are mixed up then each person takes one hat (blindly & randomly).

Y = # of people that selects their own hat.

$$Y_i = \begin{cases} 1, & \text{person } i \text{ gets his hat} \\ 0, & \text{o.w.} \end{cases}$$

$$Y = \sum_{i=1}^{N} Y_i$$

$$EY_i = P(Y_i = 1) = \frac{1}{N}$$

$$EY = \sum_{i=1}^{N} EY_i = 1 \qquad //$$

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2) N different types of coupons.

On each trial, a person obtains one of the types with equal probability.

Expected # of different types of coupons in a set of n coupons.

$$X_i = \begin{cases} 1, & \text{if coupon } i \text{ was observed in at least one of } n \text{ trials} \\ 0, & \text{o.w.} \end{cases}$$

$$X = \sum_{i=1}^{N} X_i$$

$$EX_i = P(X_i = 1) = 1 - \left(\frac{N-1}{N}\right)^n$$

$$EX = N \cdot \left\{ 1 - \left(\frac{N-1}{N}\right)^n \right\}$$

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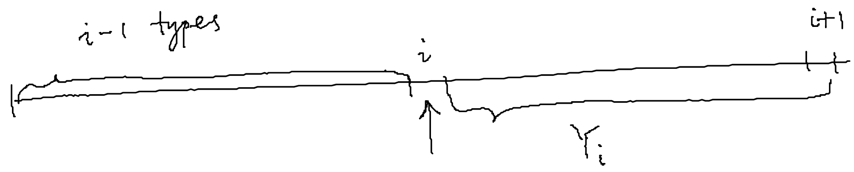
3) Expected # draws (trials) needed to get all kinds of coupons?



$$Y_0 = 1 \quad , \quad Y = \sum_{i=0}^{N-1} Y_i$$

$$p = \frac{N-i}{N} \quad , \quad P(Y_i = k) = p \cdot (1-p)^{k-1}$$

$$EY_i = \frac{1}{p} = \frac{N}{N-i}$$

$$EY = \sum_{i=0}^{N-1} EY_i = \sum_{i=0}^{N-1} \frac{N}{N-i} \quad //$$

(renewal theory from stochastic processes)

Variance of a Random Variable

Motivation

2 games

① win \$1 w/prob. $\frac{1}{2}$
loose \$1 w/prob. $\frac{1}{2}$

② win \$100 w/prob. $\frac{1}{2}$
loose \$100 w/prob $\frac{1}{2}$

$1 \cdot \frac{1}{2} - 1 \cdot \frac{1}{2} = 0 = 100 \cdot \frac{1}{2} - 100 \cdot \frac{1}{2}$ expected winnings.

To describe the difference in these two games need another measure (summary of the dist'ns) besides the men.

Def. If X is a r.v. with mean $\mu = EX$ then the variance of X

$$\mathrm{Var}(X) = E\left[(X-\mu)^2\right]$$

<u>Note.</u> ① $(X-\mu)^2 \geq 0 \Rightarrow \mathrm{Var}(X) \geq 0.$

② $$\mathrm{Var}\,X = E\left[(X-\mu)^2\right]$$
$$= E\left[X^2 - 2\mu X + \mu^2\right]$$
$$= EX^2 - 2\mu EX + \mu^2 =$$
$$= EX^2 - \mu^2 = EX^2 - (EX)^2.$$

<u>example</u> X = outcome of a roll of a fair die.

$$EX = 3.5 = \mu$$

$$EX^2 = \sum_{x=1}^{6} x^2 \cdot \frac{1}{6} =$$

$$\boxed{1^2+2^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}}$$

$$= \frac{6\cdot 7\cdot 13}{6}\cdot\frac{1}{6} = \frac{91}{6}$$

$$\mathrm{Var}\,X = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = 35/12. \quad //$$

If $Y = aX + b$, $a, b \in \mathbb{R}$

$$\mathrm{Var}(Y) = \mathrm{Var}(aX+b) =$$
$$= E\left[\left(aX+b - E(aX+b)\right)^2\right]$$
$$= E\left[\left(aX+b - aEX - b\right)^2\right] = E\left[a^2(X-EX)^2\right]$$

$$= a^2 E\left((X - EX)^2\right) = a^2 \cdot \mathrm{Var}\, X.$$

Def. The square root of $\mathrm{Var}(X)$ is called the standard deviation,

$$SD(X) = \sqrt{\mathrm{Var}(X)}.$$

Remark

1. $SD(aX + b) = |a|\, SD(X)$.

2. $\mu = EX$, $\sigma^2 = \mathrm{Var}\, X$,
   $\sigma = SD(X)$.

3. $\mathrm{Var}(X) = SD(X) = 0$
   $\Leftrightarrow P(X = c) = 1$, $c \in \mathbb{R}$.
