Conditional Distributions in Continuous Case.

Suppose Y is a continuous r.v., s.t.
$f_Y(y) > 0$. (i.e. $y \in$ domain (support) of Y.)

Assume that (X, Y) are jointly continuous. Then the conditional density of $X \mid Y = y$ is

joint density

$$f_{X|Y}(x \mid y) = \frac{f(x,y)}{f_Y(y)}$$

marginal density

So

$$P(X \in A \mid Y = y) = \int_A f_{X|Y}(x \mid y)\,dx$$

(the cond'l distribution of $X \mid Y = y$ is a dist'n in x; y acts as a parameter)

<u>example</u> $f(x,y) = \frac{15}{2} x(2-x-y)$, $0 < x < 1$, $0 < y < 1$.

$\Rightarrow X \not\perp\!\!\!\perp Y$ (f does not factor into a product $h(x) g(y)$)

∞

into a product $h(x)g(y)$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_0^1 \frac{15}{2} x(2-x-y)\,dx$$

$$= \frac{15x^2}{2} - \frac{15x^3}{6} - \frac{15yx^2}{4} \Big|_0^1$$

$$= 5 - \frac{15y}{4}, \quad 0 < y < 1.$$

$$f_{X|Y}(x|y) = \frac{\frac{15}{2}x(2-x-y)}{5-\frac{15y}{4}} = \frac{6x(2-x-y)}{4-3y},$$

$$0 < x < 1$$

$$(y \in (0,1) \text{ fixed}). \quad //$$

<u>example</u> $$f(x,y) = \frac{e^{-x/y}e^{-y}}{y}, \quad x > 0,\ y > 0$$

$$\Rightarrow X \not\perp\!\!\!\perp Y.$$

Want to compute $P(X > 1 \mid Y = y)$, $y > 0$.

$$f_Y(y) = \int_0^{\infty} e^{-x/y} e^{-y}/y\,dx$$

$$= e^{-y} \cdot \left(-e^{-x/y}\Big|_0^{\infty}\right)$$

$$= e^{-y}, \quad y > 0.$$

$$f_{X|Y}(x|y) = \frac{e^{-x/y}e^{-y}}{y} \Big/ e^{-y}$$

$$= \frac{1}{y} e^{-x/y}, \quad x > 0.$$

($y>0$ acts as a parameter for this distribution).

$$P(X > 1 \mid Y = y) = \int_1^\infty \frac{1}{y} e^{-x/y}\, dx$$

$$= -e^{-x/y}\Big|_1^\infty = e^{-1/y}, \quad \underline{\underline{y>0}}$$

fixed. //

Note If $X \perp\!\!\!\perp Y$, jointly continuous

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{f_X(x)\, f_Y(y)}{f_Y(y)}$$

$$= f_X(x).$$

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example Suppose $P \sim \text{Unif}\,[0,1]$ and $X \mid P = p \sim \text{Bin}(n, p)$.

Think of first selecting a number p for prob. of success uniformly over $[0,1]$.

Throw a coin n times with this p = prob. of success.

Then X = # of successes among n trials.

(X, P) neither jointly discrete nor jointly continuous.

Marginal dist'n of X?

$0, \dots, n$ are values for X.

Fix $x \in \{0, \dots, n\}$

$$P(X = x) = \int_{-\infty}^{\infty} \underbrace{P(X = x \mid P = p) f_P(p)}_{\text{anologue of joint}} \, dp$$

$$= \int_0^1 \binom{n}{x} p^x (1-p)^{n-x} \cdot 1 \, dp$$

$$= \binom{n}{x} \int_0^1 p^x (1-p)^{n-x} \, dp \ (=)$$

Reminder:

$a > -1,\ b > -1$

$$\int_0^1 x^a (1-x)^b \, dx = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)};$$

If $a \in \mathbb{Z}_+$, $\Gamma(a+1) = a!$

If $x > 0$: $\Gamma(x+1) = x\Gamma(x)$

$$(=) \binom{n}{x} \frac{\Gamma(x+1)\Gamma(n-x+1)}{\Gamma(n+2)}$$

$$= \binom{n}{x} \frac{x!\,(n-x)!}{(n+1)!}$$

$$= \frac{n!}{\cancel{x!\,(n-x)!}} \frac{\cancel{x!\,(n-x)!}}{(n+1)!} =$$

$$= \frac{1}{n+1}, \quad 0 \le x \le n, \text{ integer}$$

$\Rightarrow X \sim$ Uniform on $\{0, 1, \dots, n\}$ //

Extending all notions to 3 or more random variables.

Def The joint cdf of n random variab-

Def The joint cdf of n random variables $X_1, X_2, \dots, X_n$ is defined as a function $F : \mathbb{R}^n \to [0,1]$ s.t.

$$F(x_1, \dots, x_n) = P(X_1 \le x_1, \dots, X_n \le x_n).$$

"and"

Def A collection of random variables $X_1, X_2, \dots, X_n$ has jointly discrete distribution if $(X_1, \dots, X_n)$ can take on at most countably many values in $\mathbb{R}^n$.

Def The joint pmf of jointly discrete r.v.'s $X_1, X_2, \dots, X_n$ is $f : \mathbb{R}^n \to [0,1]$

s.t. $f(x_1, \dots, x_n) = P(X_1 = x_1, \dots, X_n = x_n)$

$A \subset \mathbb{R}^n$

"n summations"

$$P((X_1, \dots, X_n) \in A) = \sum_{(x_1, x_2, \dots, x_n) \in A} f(x_1, \dots, x_n)$$

In vector notation

$\underset{\sim}{X} = (X_1, \dots, X_n)$, $\underset{\sim}{x} = (x_1, \dots, x_n)$

$$f(\underset{\sim}{x}) = P(\underset{\sim}{X} = \underset{\sim}{x}),$$

$$P(\underset{\sim}{X} \in A) = \sum_{\underset{\sim}{x} \in A} f(\underset{\sim}{x}), \quad A \subset \mathbb{R}^n.$$

<u>Def</u> A collection of random variables $X_1, \ldots, X_n$ has <u>jointly continuous dist'n</u> if there exists a non-negative function $f : \mathbb{R}^n \to \mathbb{R}_+$ s.t.

$$P((X_1, \ldots, X_n) \in A) = \int \cdots \int_A f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n$$

for all $A \subset \mathbb{R}^n$; say that f is the joint pdf (density) of $X_1, X_2, \ldots, X_n$.

In vector notation,

$$P(\underset{\sim}{X} \in A) = \int_A f(\underset{\sim}{x})\, d\underset{\sim}{x}, \quad A \subset \mathbb{R}^n$$

$$f(x_1, \ldots, x_n) = \frac{\partial^n F(x_1, \ldots, x_n)}{\partial x_1 \partial x_2 \ldots \partial x_n}$$

– jointly continuous $X_1, X_2, \ldots X_n$.

Also, $X_1, X_2, \ldots, X_n$ are <u>independent</u> iff

1) $f(x_1, \ldots, x_n) = \prod_{i=1}^{n} f_i(x_i)$ $\left.\begin{matrix} \text{pmf (discrete)} \\ \text{pdf (continuous)} \end{matrix}\right.$

marginal pmf(pdf) for X_i.

marginal pmf(pdf) for X_i.

2) $F(x_1, \ldots, x_n) = \prod_{i=1}^{n} F_i(x_i)$ } for both cases.

3) Same principle applies when recognizing independence by looking at the joint pdf (pmf) as for 2 r.v.'s. Need to <u>factor</u> joint fns into fns that involve one variable at a time.

$\left(f(x_1, \ldots, x_n) = \prod_{i=1}^{n} h_i(x_i) \right.$, $h_i \geq 0$ and rectangular support $\Big)$

<u>Discrete case</u> (jointly)

$X_1, \ldots, X_n$

$$f_1(x_1) = \sum_{x_2} \sum_{x_3} \cdots \sum_{x_n} f(x_1, \ldots, x_n)$$

$$f_{12}(x_1, x_2) = \sum_{x_3} \cdots \sum_{x_n} f(x_1, \ldots, x_n)$$

Generally, $j \neq k$

summation over $n-2$ variables

$$f_{jk}(x_j, x_k) = \sum_{\substack{x_i :\ i \neq j \\ \&\ i \neq k}} f(x_1, \ldots, x_n).$$

More generally, $1 \leq r \leq n-1$,

$$f_{i_1,\dots,i_r}(x_{i_1},\dots,x_{i_r}) = \overbrace{\sum_{x_i \,:\, i \notin \{i_1,\dots,i_r\}}}^{\text{summation over } n-r \text{ variables}} f(x_1,\dots,x_n)$$

Here $\{i_1,\dots,i_r\} \subset \{1,2,\dots,n\}$.

<u>Continuous case, jointly</u> : $X_1,\dots,X_n$.

$$f_1(x_1) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1,x_2,\dots,x_n)\,dx_2 \dots dx_n$$

Moreover, if $\{i_1,\dots,i_r\} \subset \{1,2,\dots,n\}$

$$f_{i_1 \; i_r}(x_{i_1},\dots,x_{i_r}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1,\dots,x_n)\,dx_{j_1} \dots dx_{j_{n-r}}$$

$$j_k \notin \{i_1,\dots,i_r\}$$

In both jointly discrete & jointly continuous cases

$$F_1(x_1) = P(X_1 \le x_1) =$$

$$= P(X_1 \le x_1, X_2 < \infty, \dots, X_n < \infty)$$

$$= \lim_{\substack{x_2 \to \infty \\ \vdots \\ x_n \to \infty}} F(x_1,x_2,\dots,x_n)$$

More generally,

$$F_{\cdot \quad \cdot}(x_{i_1},\dots,x_{i_r}) = \lim_{x \to \infty} F(x_1,\dots,x_n)$$

$$F_{i_1,\dots,i_r}(x_{i_1},\dots,x_{i_r}) = \lim_{\substack{x_j \to \infty \\ \underline{\text{all}}\ j \notin \{i_1,\dots,i_r\}}} F(x_1,\dots,x_n)$$

Def n random variables are said to be i.i.d. (independent identically distributed) if they are independent and all X_i's have the same marginal distribution.

If $X_1, X_2, \dots, X_n$ are jointly discrete (continuous) and iid then

$$\left.\begin{aligned} f(x_1,\dots,x_n) &= f_1(x_1)\cdots f_n(x_n) \\ &= f(x_1)\cdots f(x_n) \end{aligned}\right\} \begin{array}{l} \text{pmfs} \\ \text{(pdfs)} \end{array}$$

here $f_1 \equiv f_2 \equiv \dots \equiv f_n \equiv f$.
(same marginals).

Def $X_1,\dots,X_n$ is a random sample from a distribution with pdf (pmf) $f(x)$ if $X_1,\dots,X_n$ are iid and have the common marginal with $f(x)$ as a pdf (pmf).

Continuous Case

$X_1 \mid X_2 = x_2, \dots, X_n = x_n$

pdf

$$f(x_1 \mid x_2, \dots, x_n) = \frac{f(x_1, \dots, x_n)}{f(x_2, \dots, x_n)} .$$

More generally, if

$\underset{\sim}{X}$ is divided into 2 vectors

$\underset{\sim}{Y}$ and $\underset{\sim}{Z}$, $\underset{\sim}{Y}$ – k-dim, $\underset{\sim}{Z}$ – $(n-k)$-dim

$$f(\underset{\sim}{y} \mid \underset{\sim}{z}) = \frac{f(\underset{\sim}{y}, \underset{\sim}{z})}{f_z(\underset{\sim}{z})}$$

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example Let X, Y, Z with joint pdf

$$f(x,y,z) = e^{-(x+y+z)}, \quad x \geq 0,\ y \geq 0,\ z \geq 0$$

$$= e^{-x} \cdot \mathbb{1}\{x \geq 0\} \cdot e^{-y} \cdot \mathbb{1}\{y \geq 0\} \cdot$$
$$\cdot e^{-z} \cdot \mathbb{1}\{z \geq 0\}$$

$$F(x,y,z) = \int_{-\infty}^{x} \int_{-\infty}^{y} \int_{-\infty}^{z} f(a,b,c)\, dc\, db\, da$$

$$= \int_0^x \int_0^y \int_0^z e^{-(a+b+c)}\, dc\, db\, da$$

$$= (1 - e^{-x})(1 - e^{-y})(1 - e^{-z}),$$

$x, y, z \geq 0.$

Distribution of $U = X + Y$?

$$F_{X,Y}(x,y) = \lim_{z \to \infty} F(x,y,z) =$$

$$= (1 - e^{-x})(1 - e^{-y}), \ \underbrace{x, y \geq 0.}$$

$$U = X + Y, \quad u \geq 0.$$

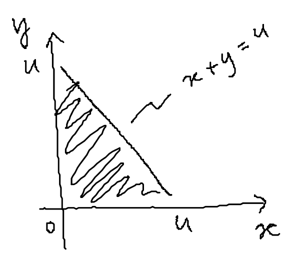
Fix $u \geq 0.$

cdf $F_U(u) = P(U \leq u) =$

$$= P(X + Y \leq u) =$$

$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y} = e^{-x-y}, \ x, y \geq 0.$$

$$= \iint_{x+y \leq u} e^{-x-y} \, dx \, dy =$$



$$= \int_0^u \int_0^{u-x} e^{-x} e^{-y} \, dy \, dx$$

$$= \int_0^u e^{-x} \left(-e^{-y} \Big|_0^{u-x} \right) dx =$$

$$= \int_0^u e^{-x} \left(1 - e^{-(u-x)} \right) dx$$

$$= -e^{-x} \Big|_0^u - e^{-u} \cdot u$$

$$= 1 - e^{-u} - u e^{-u}, \quad u \geq 0.$$

pdf of $U = X + Y$

$$\frac{\partial F_u(u)}{\partial u} = e^{-u} - e^{-u} + ue^{-u}, \quad u \geq 0$$

$$= ue^{-u}, \quad u \geq 0.$$

Dist'n of $X \mid (Y, Z) = ?$

same as marginal X :

$$F_X(x) = \lim_{y \to \infty} F_{XY}(x, y) = 1 - e^{-x}, \quad x \geq 0.$$

Dist'n of $X \mid U = u$? $U = X + Y$

$$f_{X|u}(x|u) = \frac{f_{X,u}(x,u)}{f_u(u)}$$

Can we get $f_{X,u}$ from $f_{X,Y}$? $\underline{\underline{\text{No}}}$

$$F_{X,u}(x,u) = P(X \leq x, U \leq u)$$

$$= P(X \leq x, X + Y \leq u) =$$

$$= \iint_{\substack{a \leq x \\ a+b \leq u}} f_{XY}(a, b)\, da\, db$$

$$= \dots \quad (\text{calculus exercize})$$

$$= 1 - e^{-x} - xe^{-u}, \quad 0 \leq x \leq u$$

$$= \begin{cases} 1 - e^{-x} - x e^{-u}, & 0 \le x \le u \\ 1 - e^{-u} - u e^{-u}, & x > u \end{cases}$$

$$f_{XU}(x,u) = \frac{\partial^2}{\partial x \partial u} F_{XU}(x,u) =$$

$$= e^{-u}, \quad 0 \le x \le u$$

$$f_{X|U}(x|u) = \frac{f_{XU}(x,u)}{f_U(u)}$$

$$= \frac{e^{-u}}{u e^{-u}} = \frac{1}{u}, \quad 0 \le x \le u$$

We can write

$$X \mid U = u \sim \mathrm{Unif}\,[0,u]. \quad //$$

Functions of Jointly Distributed Random Variables

r.v. $X : S \to \mathbb{R}$ & $f : \mathbb{R} \to \mathbb{R}$

$\Rightarrow \quad Y = f(X) : S \to \mathbb{R}$ is also a r.v.

Sums and convolutions.

Let X, Y jointly discrete

$Z = X + Y$ and know $f(x,y)$

pmf Z?

$$f_Z(z) = P(X+Y=z)$$
$$= \sum\sum_{x+y=z} f(x,y)$$
$$= \sum_x \sum_{y=z-x} f(x,y)$$
$$= \underbrace{\sum_x f(x, z-x)}_{\text{pmf of } Z}.$$

Suppose X, Y are jointly continuous and $Z = X+Y$.

$$F_Z(z) = P(X+Y \le z)$$
$$= \iint_{x+y\le z} f(x,y)\,dx\,dy$$
$$= \int_{-\infty}^{\infty}\int_{y\le z-x} f(x,y)\,dy\,dx \;\circledcirc$$

$$\boxed{\begin{array}{l}\text{Change of variables,}\\ \text{let } v = y+x\\ \quad y = v-x\\ \quad dy = dv\\ \{y \le z-x\} = \{v \le z\}\end{array}}$$

$$\circledcirc \int_{-\infty}^{\infty}\int_{v\le z} f(x, v-x)\,dv\,dx$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{z} f(x, v-x)\,dv\,dx$$
$$\overset{f\ge 0}{=} \int_{-\infty}^{z}\int_{-\infty}^{\infty} f(x, v-x)\,dx\,dv.$$

cdf of Z

$\Rightarrow$ pdf is $f_Z(z) = \frac{\partial}{\partial z} F_Z(z)$

$$= \int_{-\infty}^{\infty} f(x, z-x)\, dx$$

$Z = X + Y$

If $X \perp\!\!\!\perp Y$ and still jointly continuous

$$f(x,y) = f_X(x)\, f_Y(y)$$

then for $Z = X + Y$ the density

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx$$

f_Z is the convolution of f_X and f_Y.

General Case of Functions of Random Variables.

Suppose we know the dist'n of X.

We want to find the dist'n of

$Y = g(X)$, a r.v. itself.

Then $F_Y(y) = P(Y \leq y) =$

$$= P(g(X) \leq y).$$

example $X \sim Unif\ [-1, 1]$

$Y = X^2$, $y \in [0, 1]$

$$P(Y \le y) = P(X^2 \le y) =$$

$$= P(-\sqrt{y} \le X \le \sqrt{y})$$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2}\, dx = \sqrt{y}, \quad 0 \le y \le 1.$$

$$f_Y(y) = \frac{\partial F_Y(y)}{\partial y} = \frac{1}{2\sqrt{y}}, \quad 0 < y \le 1.$$