Sample Spaces with Equally Likely Outcomes.

$S = \{1, 2, \dots, N\}$

$P(\{1\}) = P(\{2\}) = \dots = P(\{N\}) = \frac{1}{N}$.

hence

$$P(E) = \sum_{i \in E} P(\{i\})$$

$$= \frac{\#\text{ outcomes in } E}{\#\text{ outcomes in } S}.$$

example 2 fair dice rolled, sum of upturned faces.

(i, j) , $1 \le i \le 6$, $1 \le j \le 6$

36 outcomes, equally likely

$P(i+j=7) = \frac{6}{36} = \frac{1}{6}$

$\{1,6\}$, $\{2,5\}$, $\{3,4\}$, $\{4,3\}$, $\{5,2\}$, $\{6,1\}$ //

Counting Methods

Multiplication Rule.

k experimental stages s.t.

- 1^{st} stage results in n_1 possible outcomes
- 2^{nd} stage results in n_2 possible outcomes for each outcome of stage 1;
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⋮

- k^{th} stage results in n_k possible outcomes for each outcome of the first $(k-1)$ stages

Then $n_1 \times n_2 \times \dots \times n_k$ = # of possible outcomes of this k-stage experiment.

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examples

1) roll 1^{st} die $\longrightarrow n_1 = 6$
roll 2^{nd} die $\longrightarrow n_2 = 6$

$\Rightarrow$ # outcomes $= n_1 \times n_2 = 36$.

k dice $\longrightarrow 6^k$ outcomes.

2) a) How many different license plates with 7 places are possible if the first 3 places are letters and the last 4 places are digits?

$$\underbrace{\overset{*}{\_\_}\ \_\_\ \_\_}_{\text{letters}}\quad \underbrace{\_\_\ \_\_\ \_\_\ \_\_}_{\text{digits}}$$

$k = 7$ $\quad n_1 = 26,\ n_2 = 26,\ n_3 = 26$

(reuse letters, repetitions are allowed)

$n_4 = 10 = n_5 = n_6 = n_7$

(reuse numbers, repetitions are allowed).

$n_1 \times n_2 \times \dots \times n_7 = 26^3 \times 10^4$ = total number of such license plates.

b) Still 7-place license plate, still 3 letters first, 4 digits next.

Now, repetitions are $\underline{\text{not}}$ allowed.

$$\underbrace{\_\ \_\ \_}_{\text{letters}}\ \underbrace{\_\ \_\ \_\ \_}_{\text{digits}}$$

$n_1 = 26$, $n_2 = 25$, $n_3 = 24$

$n_4 = 10$, $n_5 = 9$, $n_6 = 8$, $n_7 = 7$

such license plates $= 26 \times 25 \times 24 \times 10 \times 9 \times 8 \times 7$

Permutations

of ways to arrange N numbers in order? (numbers are distinct)

$\{1, 2, \ldots, N\}$

①	②	③	$\cdots$	ⓝ
N	$N-1$	$N-2$		1
$\parallel$	$\parallel$	$\parallel$		$\parallel$
n_1	n_2	n_3		n_N

$$N \times (N-1) \times (N-2) \cdots \times 1 \equiv N!$$
$$n_1 \times n_2 \times n_3 \cdots \times n_N \quad (N \text{ factorial})$$

# of ways to arrange N distinct objects in order.

examples: 10 runners, 4 are women, 6 are men

1) # rankings without regard to gender

$10!$

2) # rankings with regard to gender

$6! \times 4!$ possibilities.

Arrange $k < N$ objects out of N distinct objects

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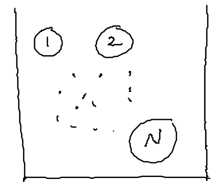

Select k out N, put them in order, without replacement (no repetitions).

possible outcomes?

1	2	......	k
N	$N-1$	......	$(N-k+1)$
$\parallel$	$\parallel$		$\parallel$
n_1	n_2		n_k

# of such outcomes $= \underbrace{N \times (N-1) \times \dots \times (N-k+1)}$

$$= \frac{N!}{(N-k)!} \equiv P_{N,k}$$

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examples Club of 25 members

Need to choose a president (P) and a vice president (VP)

P VP

$k = 2$

$$P_{25,2} = \frac{25!}{(25-2)!} = \frac{25!}{23!} = 25 \times 24$$

$\underset{\{7\}}{\underline{P}}$ $\underset{\{5\}}{\underline{VP}}$ is not the same as $\underset{\{5\}}{\underline{P}}$ $\underset{\{7\}}{\underline{VP}}$

$\Rightarrow$ order matters!

Now, imagine we are choosing 2 VPs.

VP VP $\{5, 7\}$ $\{7, 5\}$

$\Rightarrow$ order no longer matters!

$\frac{24 \times 25}{2}$ — # of outcomes w/o order.

More generally, choosing k objects out of N distinct objects, w/o replacement and not paying attention to order

$$\frac{N!}{(N-k)!\,k!} \equiv \binom{N}{k} \equiv C_N^k$$

("N choose k").

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example Group of 5 men and 7 women is selecting a commitee of 2 women and 3 men. How many commitees can be formed?

$$\binom{7}{2} \times \binom{5}{3}$$

Choosing k out of N distinct objects.

Situation 1 Choose k out N with replacement (allow repetition) and order matters.

$$\underset{\text{①}}{\_\_} \quad \underset{\text{②}}{\_\_} \quad \cdots\cdots \quad \underset{\text{ⓚ}}{\_\_}$$

$$N \times N \times \cdots \times N$$

N^k = # possible such arrangements.

Situation 2 Choose k out of N without replacement (no repetition) & order matters.

$$\underset{N}{\underset{\sqcup}{①}} \quad \cdots\cdots \quad \underset{(N-k+1)}{\underset{\sqcup}{ⓚ}}$$

$P_{N,k}$ = # outcomes.

Situation 3 Choose k out of N without replacement (no repetition) and order doesn't matter.

$$\frac{P_{N,k}}{k!} = \binom{N}{k} = C_N^k .$$

Situation 4 Choose k out of N distinct objects with replacement and order doesn't matter.

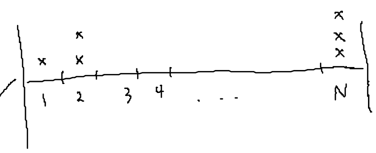
example $k = 3$ out of $N = 6$

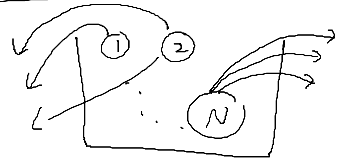
(4 5 6) ← 3! = 6 orderings

$\left\{\begin{array}{l} (4\,5\,5) \leftarrow \text{ 3 orderings} \\ (5\,4\,5) \\ (5\,5\,4) \end{array}\right.$

(5 5 5) ← 1 order

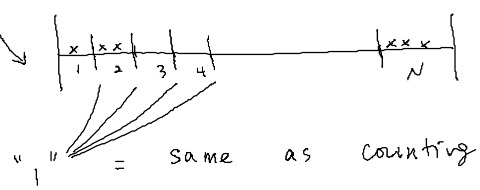
⇒ can not simply divide by k! like before!





Arrange k of "x" in N bins =

= counting selections of k with replacement & no order.



"|" = same as counting # of arrangements of "x" and "|".

How many "x"? k

How many "|"? $N-1$

$k+N-1$ $\qquad \binom{k+N-1}{k}$ − # of way of placing "x" in the spots.

$$\| $$

$$\binom{k+N-1}{N-1}$$

Summary (# ways of choosing k objects out of N distinct objects)

	order matters	order doesn't matter
with replacement (allow repetition)	N^k	$\binom{N+k-1}{k}$
without replacement (no repetition)	$P_{N,k} = \frac{N!}{(N-k)!}$	$C_N^k = \binom{N}{k} = \frac{N!}{(N-k)!k!}$

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examples (equally likely outcomes)

1) a bowl contains 6 black balls

& 5 white balls.

Choose 3 without replacement.

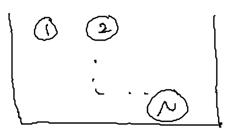
What is the probability that 2 are white and 1 is black?

$$P(E) = \frac{\#\text{ in } E}{\#\text{ in } S} =$$

$$= \frac{\binom{5}{2} \times \binom{6}{1}}{\binom{11}{3}} \quad //$$

2) k balls drawn out N without replacement. 1 out N is "special"

(1 ball is red, rest are black).



What is the chance of getting the special ball among k selected?

$\#$ of outcomes in $S = \binom{N}{k}$

$\#$ of outcomes in $E = \binom{N-1}{k-1}$

$$P(E) = \frac{\binom{N-1}{k-1}}{\binom{N}{k}} = \frac{\frac{(N-1)!}{\cancel{(N-k)!}\,(k-1)!}}{\frac{N!}{\cancel{(N-k)!}\,k!}} =$$

$$= \frac{k}{N}.$$

<u>Another</u> <u>solution</u> <u>to</u> <u>the</u> <u>same</u> <u>problem</u>

(using indicator events).

A i^{th} [illegible] results in the special ball (using indicator events).

A_i = i^{th} draw results in the special ball

$i = 1, 2, \ldots, k$

$A_i \cap A_j = \emptyset$ for $i \neq j$

($\Rightarrow A_j$ are mutually exclusive)

$$E = \bigcup_{i=1}^{k} A_i$$

$$P(E) = P\left(\bigcup_{i=1}^{k} A_i\right) = \sum_{i=1}^{k} P(A_i) \stackrel{}{=}$$

$P(A_1) = \frac{1}{N}$; by symmetry $P(A_i) = \frac{1}{N}$

$$= \frac{k}{N} \quad //$$

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examples A poker hand consists of 5 cards;

assume playing with a standard 52 card deck.

$52 = 13 \times 4$

values: $2, 3, \ldots 10, J, Q, K, A$

suits: ♣ clubs, ♦ diamonds, ♥ hearts, ♠ spades

① What's the chance of getting a straight?

↑ i.e. values line up in order

{A 2 3 4 5} is a possible straight, as long as not all suits are the same, b/c then it becomes "straight flush"

$\{10\ J\ Q\ K\ A\}$ – ok

$\{Q\ K\ A\ 2\ 3\}$ – is not

Dealing 5 cards out of a well-shuffled deck.

$\#$ in $S = \binom{52}{5}$

$\underbrace{A\,2\,3\,4\,5}_{10 \text{ choices } \leftarrow n_1}$, $2\,3\,4\,5\,6$, $3\,4\,5\,6\,7$, ..., $10\,J\,Q\,K\,A$

$4 \times \dots \times 4 - 4$

$$P(E) = \frac{10 \times (4^5 - 4)}{\binom{52}{5}}$$ //

② What is the probability of getting a full house?

3+2 combination : $\begin{cases} Q\,Q \quad K\,K\,K \\ K\,K \quad Q\,Q\,Q \end{cases}$

(order of choosing the value matters)

$$P(E) = \frac{13 \times 12 \times \binom{4}{2} \times \binom{4}{3}}{\binom{52}{5}}$$

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example Bridge

52 standard card deck

4 players, 13 cards are dealt to each player

S <= total number of outcomes?

$$\overset{1}{\binom{52}{13}} \times \overset{2}{\binom{39}{13}} \times \overset{3}{\binom{26}{13}} \times \overset{4}{\binom{13}{13}} =$$

$$\overset{1}{\binom{52}{13}} \times \overset{2}{\binom{39}{13}} \times \overset{3}{\binom{26}{13}} \times \overset{4}{\binom{13}{13}} =$$

$$= \frac{52!}{\cancel{39!}\,13!} \times \frac{\cancel{39!}}{\cancel{26!}\,13!} \times \frac{\cancel{26!}}{13!\,13!} \times 1$$

$$= \frac{52!}{(13!)^4} \quad \text{in } S$$

<u>Note:</u> (show at your own time)

More generally, there are

$$\frac{(n_1+n_2+\dots+n_r)!}{n_1!\cdots\cdots n_r!} \quad \text{ways of arranging}$$

$(n_1+n_2+\dots+n_r)$ distinct objects without replacement without order within groups into r groups of sizes $n_1, n_2, \dots, n_r$.

$$\binom{n_1+n_2+\dots+n_r}{n_1, n_2, \dots, n_r} = \frac{(n_1+\dots+n_r)!}{n_1!\,n_2!\,\cdots\,n_r!}$$

(side: $\underbrace{\binom{N}{k}}_{\text{short hand for } r=2.} \equiv \frac{N!}{(N-k)!\,k!} = \binom{N}{k, N-k} \quad r=2$)

The chance of one player getting all 13 spades?

$$P(E) = \frac{4 \times \binom{39}{13,13,13}}{\binom{52}{13,13,13,13}} \quad //$$