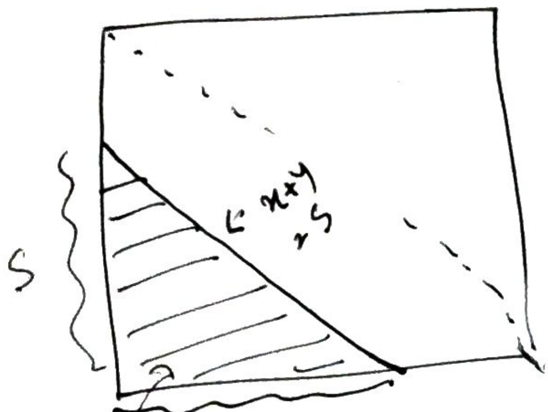
① $U_1, U_2 \sim U[0,1]$

$S = U_1 + U_2$

$F(s)\ P(S \leq s) = P(U_1 + U_2 \leq s)$

Fix $s \in [0,1]$ $= \frac{1}{2} s^2$



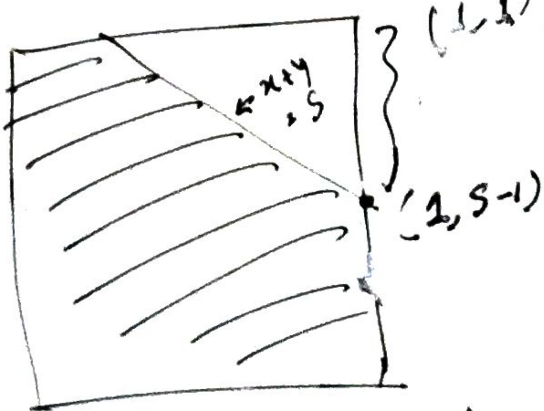
Area $= \frac{1}{2} s^2$

$F(s) =$

$P(S \leq s) = P(U_1 + U_2 \leq s)$

$= 1 - \frac{1}{2}(2-s)^2$

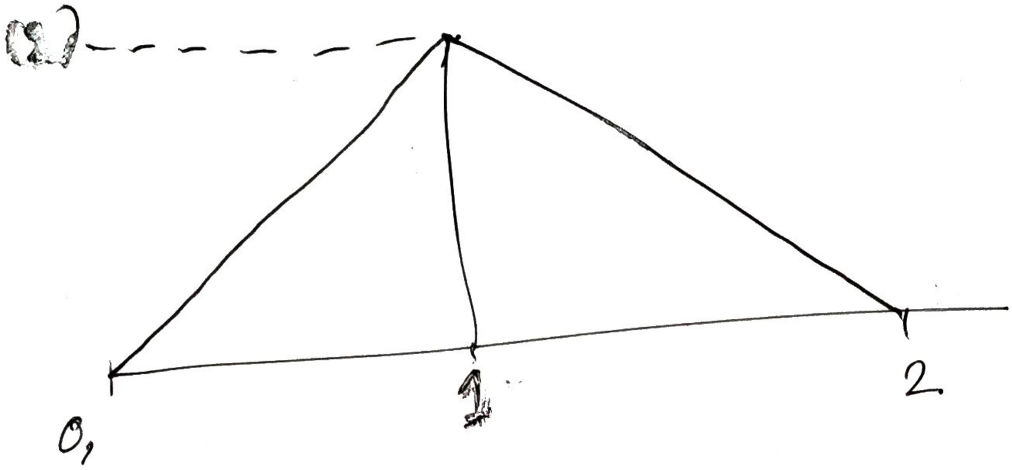
for $s \in [1,2]$



Hence, Density of S is.

$$f_S(s) = \frac{\partial}{\partial s} F(s) = \begin{cases} s & \text{if } s \in [0,1] \\ 2-s & \text{if } s \in [1,2] \end{cases}$$

Hence, the density is



② X has density

$$f(x) = \frac{1+\alpha x}{2} \qquad -1 \le x \le 1$$
$$-1 \le \alpha \le 1$$

$$E[X] = \int_{-1}^{1} x \cdot \frac{1+\alpha x}{2} \cdot dx$$

$$= \int_{-1}^{1} \frac{x}{2} + \alpha \cdot \int_{-1}^{1} \frac{x^2}{2} \cdot dx$$

$$= 0 + \frac{\alpha}{2}\left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{\alpha}{2} \times \frac{2}{3}$$
$$= \alpha/3$$

$$E[X^2] = \int_{-1}^{1} x^2 \cdot \frac{1+\alpha x}{2} \cdot dx$$

$$= \int_{-1}^{1} \frac{x^2}{2} + \alpha \int_{-1}^{1} \frac{x^3}{2} \cdot dx$$

$$= \frac{1}{2} \cdot \left[\frac{x^3}{3}\right]_{-1}^{1} + 0$$

$$= \frac{1}{2} \times 2/3 = \frac{1}{3}$$

$$\therefore Var(X) = E[X^2] - (E[X])^2$$

$$= \frac{1}{3} - \alpha^2/9$$

③ $U_1, U_2, \dots U_n \sim \text{Unif}[0,1]$.

$U_{(n)} = \max(U_1, U_2, \dots U_n)$

fix $s \in [0,1]$

$P(U_{(n)} \leq s) = P(U_1 < s, U_2 < s, \dots U_n < s)$

$= P(U_1 < s)\, P(U_2 < s) \dots P(U_n < s)$

$= s^n$

$\therefore$ density of $U_{(n)}$ is

$f_{(n)}(s) = \frac{\partial}{\partial s} s^n = n \cdot s^{n-1} \quad s \in [0,1]$

$\therefore E[U_{(n)}] = \int_0^1 s \cdot n \cdot s^{n-1} \cdot ds$

$= n \cdot \int_0^1 s^n \cdot ds$

$= \frac{n}{n+1}$

$P(U_{(1)} < s)$
$= 1 - P(U_{(1)} > s)$
$= 1 - P(U_1 > s, U_2 > s, \ldots U_n > s)$
$= 1 - (1-s)^n$

$\therefore$ Density of $U_{(1)}$ is

$$f_{(1)}(s) = \frac{\partial}{\partial s} \cdot P(U_{(1)} < s)$$
$$= \frac{\partial}{\partial s} \cdot \left[1 - (1-s)^n\right]$$
$$= n(1-s)^{n-1}$$

$$\therefore E[U_{(1)}] = \int_0^1 n(1-s)^{n-1} \cdot s \cdot ds$$
$$= n \int_0^1 s(1-s)^{n-1} \cdot ds$$

Let
$x = 1-s$
$\Rightarrow dx = -ds$

$$= n \int_1^0 (1-x) \cdot x^{n-1} \cdot (-dx)$$
$$= n \int_0^1 (1-x) x^{n-1} \cdot dx$$

$\therefore E\left[U_{(n)} - U_{(1)}\right]$

$$= \frac{n}{n+1} - \frac{1}{n+1} = \frac{n-1}{n+1}$$

---

(4) Let X be Unif $[0,1]$ the side length of the square.

The Area of the square is X^2

$\therefore$ Expected Area of square is

$$E[X^2] = \int_0^1 x^2 \cdot 1 \, dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 dx$$

$$= \frac{1}{3}$$

(5) $X \sim$ Poisson distribution with mean λ

$$\therefore P(X=x) = e^{-\lambda} \cdot \frac{\lambda^x}{x!} \qquad x = 0, 1, 2, \ldots$$

Hence,

$$E\left[\frac{1}{X+1}\right] = \sum_{K=0}^{\infty} \frac{1}{K+1} \cdot e^{-\lambda} \cdot \frac{\lambda^K}{K!}$$

$$= \sum_{K=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^K}{(K+1)!}$$

$$= \frac{e^{-\lambda}}{\lambda} \cdot \sum_{K=0}^{\infty} \frac{\lambda^{K+1}}{(K+1)!}$$

$$= \frac{e^{-\lambda}}{\lambda} \left[\sum_{K'=1}^{\infty} \frac{\lambda^{K'}}{K'!}\right]$$

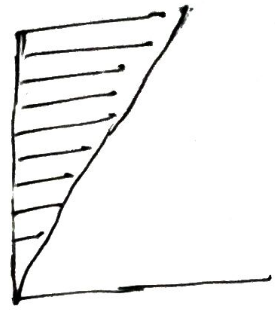
$$= \frac{e^{-\lambda}}{\lambda} \cdot \left[\underbrace{\sum_{K'=0}^{\infty} \frac{\lambda^{K'}}{K'!}}_{= e^{\lambda}} - 1\right]$$

$$= \frac{e^{-\lambda}}{\lambda} \left[e^{\lambda} - 1\right]$$

$$= \boxed{\frac{1 - e^{-\lambda}}{\lambda}}$$

⑥ X, Y have joint pdf

$f(x,y) = e^{-y} \quad 0 < x < y$



ⓐ $Cov(X,Y) = E[XY] - E[X]E[Y]$

$$E[XY] = \int_0^\infty \int_0^y xy\, e^{-y}\, dx\, dy$$

$$= \int_0^\infty y e^{-y} \int_0^y x\, dx\, dy$$

$$= \int_0^\infty y e^{-y} \left[\frac{x^2}{2}\right]_0^y dy$$

$$= \int_0^\infty y e^{-y} \frac{y^2}{2} \cdot dy$$

$$= \frac{1}{2} \int_0^\infty y^3 \cdot e^{-y}\, dy$$

$$= \frac{1}{2} \Gamma(4) = \frac{1}{2} \times 3! = 3$$

$$E[X] = \int_0^\infty \int_0^y x \cdot e^{-y} \cdot dx\, dy$$

$$= \int_0^\infty e^{-y} \int_0^y x \cdot dx \cdot dy$$

$$= \int_0^\infty e^{-y} \cdot \frac{y^2}{2} \cdot dy = \frac{1}{2} \Gamma(3)$$

$$= \frac{1}{2} \cdot 2! = 1$$

$$E[Y] = \int_0^\infty \int_0^y y \cdot e^{-y}\, dx\, dy$$

$$= \int_0^\infty y \cdot e^{-y} \int_0^y 1 \cdot dx\, dy$$

$$= \int_0^\infty y \cdot e^{-y} \cdot y \cdot dy$$

$$= \int_0^\infty y^2 \cdot e^{-y}\, dy = \Gamma(3) = 2!$$

$$= 2$$

$$\therefore \operatorname{Cov}(X, Y) = E[XY] - E[X] \cdot E[Y]$$

$$= 3 - 1 \cdot 2$$

$$= 1$$

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2$$

$$= E[X^2] - 1$$

$$= \int_0^\infty \int_0^y x^2 \cdot e^{-y} \, dx \cdot dy - 1$$

$$= \int_0^\infty e^{-y} \frac{y^3}{3} \, dy - 1$$

$$= \frac{1}{3} \Gamma(4) = \frac{3!}{3} - 1 = 1$$

$$\mathrm{Var}(Y) = E[Y^2] - (E[Y])^2$$

$$= \int_0^\infty \int_0^y y^2 \cdot e^{-y} \, dx \cdot dy - 4$$

$$= \int_0^\infty y^2 \cdot e^{-y} \, y \cdot dy - 4$$

$$= \int_0^\infty y^3 \cdot e^{-y} \, dy - 4$$

$$= \Gamma(3) - 4$$

$$= 6 - 4 = 2.$$

$$\therefore \ \mathrm{Cor}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\cdot \mathrm{Var}(Y)}}$$

$$= \frac{1}{\sqrt{2}}$$

(b) The conditional density of X given $Y=y$ is $\dfrac{f(x,y)}{f_Y(y)}$ where $f_Y(y)$ is marginal density of y.

Marginal Density of Y is given by

$$f_Y(y) = \int_0^y e^{-y}\cdot dx = y\cdot e^{-y} \quad y \in [0,\infty)$$

Hence conditional density of X given $Y=y$ is.

$$f_{X|Y}(x) = \frac{e^{-y}}{y\cdot e^{-y}} \quad \text{for} \quad 0 < x < y, \quad y \in \mathbb{R}^+ \text{ fixed}$$

$$= \frac{1}{y} \quad \text{for} \quad 0 < x < y$$

i.e. $f_{X|Y}(x) \sim \mathrm{Unif}[0, Y]$

Hence $E[X \mid Y=y]$ ... $y/$

Marginal of X is given by

$\frac{f(x,y)}{f_X(x)}$ where $f_X(x)$ is marginal density of X.

$$f_X(x) = \int_x^\infty e^{-y} \cdot dy$$

$$= \left[-e^{-y}\right]_x^\infty = e^{-x} \quad x \in [0, \infty)$$

Conditional density of Y given $X = x$ is given by

$$\frac{f(x,y)}{f_X(x)} = e^{-y+x} \quad \text{for } y > x$$

$x \in \mathbb{R}^+$ fixed

$$\Rightarrow E[Y \mid X = x] = \int_x^\infty y \cdot e^{-y+x} \cdot dy$$

$Z = y - x$

$dZ = dy$

$$= \int_0^\infty (Z + x)\, e^{-Z} \cdot dZ$$

$$= \int_0^\infty Z \cdot e^{-Z}\, dZ + x \cdot \int_0^\infty e^{-Z}\, dZ$$

(c) Note

$$E[X|Y] = \frac{Y}{2}$$

$$E[Y|X] = 1+X$$

Note that
Marginal density of Y is

$$f_Y(y) = y\, e^{-y} \cdot \quad \boxed{y>0}$$

Hence, Marginal density of $U = \frac{Y}{2}$ is
given by

Due to Jacobian

$$f_U(u) = 2u \cdot e^{-2u} \times 2 \qquad u>0$$

$$= 4u\, e^{-2u} \qquad u>0.$$

Marginal density of X is $f_X(x) = e^{-x}$, $x>0$
Marginal density of $V = 1+X$ is.

$$f_V(v) = e^{-(v-1)} \qquad \boxed{v>1}$$

(7) (a)

$$\operatorname{Cov}(X_1 + 2X_2,\ X_2 + X_3 + 1)$$

$$= \operatorname{Cov}(X_1, X_2) + \operatorname{Cov}(X_1, X_3) + \operatorname{Cov}(X_1, 1)$$
$$+ 2\operatorname{Cov}(X_2, X_2) + 2\operatorname{Cov}(X_2, X_3)$$
$$+ 2\operatorname{Cov}(X_2, 1)$$

$$= 0 + 0 + 0 + 2 + 0 + 0 = 2$$

$$\operatorname{Var}(X_1 + 2X_2)$$

$$= \operatorname{Var}(X_1) + 4 \cdot \operatorname{Var}(X_2) + 2\operatorname{Cov}(X_1, 2X_2)$$

$$= 1 + 4 + 0 = 5$$

$$\operatorname{Var}(X_2 + X_3 + 1)$$

$$= \operatorname{Var}(X_2 + X_3) = \operatorname{Var}(X_2) + \operatorname{Var}(X_3)$$
$$+ 2\operatorname{Cov}(X_2, X_3)$$

$$= 1 + 1 + 0 = 2$$

$$\therefore \operatorname{cor}(X_1 + 2X_2,\ X_2 + X_3 + 1)$$

$$= \frac{2}{\sqrt{2 \times 5}} = \frac{2}{\sqrt{10}}$$

7) (b) $\operatorname{Cov}(X_1+X_2+3,\ 5X_3+X_4)$

$= \operatorname{Cov}(X_1+X_2,\ 5X_3+X_4) = \operatorname{Cov}(X_1, 5X_3) + \operatorname{Cov}(X_1, X_4) + \operatorname{Cov}(X_2, 5X_3) + \operatorname{Cov}(X_2, X_4)$

$= 0$

$\therefore \operatorname{Cov}(X_1+X_2+3,\ 5X_3+X_4)$

$= 0$.

---

(8) The event that we need "n" tries to get both Heads and tails can be broken into two scenario's.

$\underbrace{HHH\cdots H}_{(n-1)\text{ times}}T$ and $\underbrace{TTT\cdots T}_{(n-1)\text{ times}}H$

where $n \geq 2$.

X be the "count" that number of tries needed.

$\therefore P(X=n) = P(\underbrace{HH\cdots H}_{n-1}\cdot T) + P(\underbrace{TT\cdots T}_{n-1}H)$

$= p^{n-1}\cdot(1-p) + (1-p)^{n-1}\cdot p$.

$= p(1-p)\left[p^{n-2} + (1-p)^{n-2}\right]$ $\boxed{n \geq 2}$

$$E[X] =$$

$$= \sum_{n=2}^{\infty} n P(1-P) \left[P^{n-2} + (1-P)^{n-2} \right]$$

$$= P(1-P) \cdot \sum_{n=2}^{\infty} \left[n P^{n-2} + n \cdot (1-P)^{n-2} \right]$$

$$= P(1-P) \left[\sum_{k=0}^{\infty} (k+2) \cdot P^{k} + (k+2) \cdot (1-P)^{k} \right]$$

$$= P(1-P) \left[2 \sum_{k=0}^{\infty} P^{k} + (1-P)^{k} + \sum_{k=0}^{\infty} k \cdot P^{k} + \sum_{k=0}^{\infty} k \cdot (1-P)^{k} \right]$$

$$= P(1-P) \left[\frac{2}{1-P} + \frac{2}{P} + \frac{1}{(1-P)^2} - \frac{1}{1-P} + \frac{1}{P^2} - \frac{1}{P} \right]$$

$$= P(1-P) \left[\frac{1}{P^2} + \frac{1}{(1-P)^2} + \frac{1}{1-P} + \frac{1}{P} \right]$$

$$= \frac{1-P}{P} + \frac{P}{1-P} + P + 1 - P$$

$$= 1 + \frac{1-P}{P} + \frac{P}{1-P}. \quad (Ans).$$

8) (b) $P(\text{Last flip is heads})$

$= P(\text{first flip is Tails})$

$= (1-p)$

---

(9) Let $X_i = 1$ if ith ball is Red

$= 0$ if " " " not Red

$Y_i = 1$ if ith ball is blue

$= 0$ " " " " not "

$\therefore X = \sum_{i=1}^{12} X_i \qquad Y = \sum_{i=1}^{12} Y_i$

$$Cov(X,Y) = Cov\left(\sum_{i=1}^{12} X_i, \sum_{i=1}^{12} Y_i\right)$$

$$= \sum_{i=1}^{12} Cov(X_i, Y_i) + 2\sum_{i<j} Cov(X_i, X_j)$$

$$Cov(X_i, Y_i) = E[X_i Y_i] - E[X_i]E[Y_i]$$

$$= 0 - \frac{10}{30} \times \frac{8}{30}$$

See that $X_iY_i = 0$

$$= -\frac{1}{3} \times \frac{4}{\ldots} = \boxed{-\frac{4}{\ldots}}$$

$Cov(X_i, Y_j)$

$= E[X_i Y_j] - E[X_i]E[Y_j]$

$= \frac{10}{30} \times \frac{8}{29} - \frac{10}{30} \times \frac{8}{30}$

$= \frac{1}{3}\left[\frac{8}{29} - \frac{8}{30}\right]$

$= \frac{8}{3}\left[\frac{30-1}{29 \times 30}\right] = \boxed{\frac{8}{3} \times \frac{1}{29 \times 30}.}$

$\therefore Cov(X, Y)$

$= -12 \times \frac{4}{15} + 2 \times \binom{12}{2} \times \frac{8}{3} \times \frac{1}{29 \times 30}$

$= -12 \times \frac{4}{15} + \frac{12 \times 11 \times 8}{3 \times 29 \times 30}.$

(Ans).

(10) conditional joint pdf of $Y_1, \ldots Y_n$ given $X = x$ is.

$$h(y_1, \ldots y_n | x) = \begin{cases} \frac{1}{x^n} & 0 < y_i < x \\ & \forall i \\ 0 & \text{o.w} \end{cases}$$

Joint p.d.f of X and $Y_1, \ldots Y_n$ is.

$$f(x) \cdot h(y_1, \ldots y_n | x) = \begin{cases} \frac{1}{n!} \exp(-x) & 0 < y_i < x \\ & \forall i \\ 0 & \text{o.w} \end{cases}$$

(a) For $y_i > 0$ $\forall i$, the marginal pdf of $Y_1, Y_2, \ldots Y_n$ is.

$$\int_{\max(y_1, \ldots y_n)}^{\infty} \frac{1}{n!} \exp(-x) \, dx = \frac{1}{n!} \exp\left(-\max_{(y_1, \ldots y_n)}\right)$$

(b) ∴ Conditional p.d.f of X given Y_i is.

$$\frac{\frac{1}{n!} \cdot \exp(-x)}{\frac{1}{n!} \exp(-\max(y_1, \ldots y_n))} = \exp\left(-\left(x - \max_{(y_1, \ldots y_n)}\right)\right)$$

where $x > \max(y_1, \ldots y_n)$

(17) We see that $X \in [0,2]$

This imply that $Y \in [0,1]$

$$P(Y \leq y)$$

$$= P(X^2 - 2X \geq -y)$$

$$= P((X-1)^2 \geq 1-y)$$

$$= P(X-1 \geq \sqrt{1-y}) + P(X-1 \leq -\sqrt{1-y})$$

$$= P(X \geq 1+\sqrt{1-y}) + P(X \leq 1-\sqrt{1-y})$$

$$= \int_{1+\sqrt{1-y}}^{2} \frac{1}{2}x\,dx + \int_{0}^{1-\sqrt{1-y}} \frac{1}{2}x\,dx$$

$$= 1 - \sqrt{1-y}$$

$\therefore$ Density of Y is $\frac{\partial}{\partial y} \cdot P(Y \leq y)$

$$= \frac{\partial}{\partial y} \cdot (1-\sqrt{1-y}) = \frac{1}{2(1-y)^{1/2}}$$

$y \in (0,1)$,

(12) Let $Z = X_1$.

Hence $(X_1, X_2) \longleftrightarrow (Y, Z)$ is one to one transform.

$X_1 = Z$

$X_2 = Y/Z$

Therefore

$$J = \det \begin{bmatrix} \frac{\partial x_1}{\partial y} & \frac{\partial x_1}{\partial z} \\ \frac{\partial x_2}{\partial y} & \frac{\partial x_2}{\partial z} \end{bmatrix}$$

$$= \det \begin{bmatrix} 0 & 1 \\ 1/z & -\frac{y}{z^2} \end{bmatrix} = -\frac{1}{z}$$

For $0 < y < z < 1$, the joint pdf of Y, Z is.

$$g(y,z) = f\left(z, \frac{y}{z}\right) \cdot |J| = \left(z + \frac{y}{z}\right) \cdot \frac{1}{z}.$$

$$= 1 + \frac{y}{z^2} \quad \boxed{0 < y < z < 1}$$

Hence, marginal of Y is.

$$g_Y(y) = \int_y^1 1 + \frac{y}{z^2} \cdot dz$$

$$= 2(1-y) \qquad y \in [0,1].$$

(13) mgf of Z is.

$$\psi_1(t) = E[\exp(tZ)]$$

$$= E[\exp(t(2X-3Y+4))]$$

$$= e^{4t}\cdot E[e^{2tX-3tY}]$$

$$= e^{4t}\cdot E[e^{2tX}]\cdot E[e^{-3tY}]$$

(As, X, Y indep)

$$= e^{4t}\,\psi(2t)\,\psi(-3t)$$

$$= e^{4t}\cdot e^{4t^2+6t}\, e^{9t^2-9t}$$

$$= e^{13t^2+t}.$$ (Ans).

(14) X can take finite no of values $x_1, x_2 .. x_k$ with prob $P_1, .. P_k$

then mgf is

$$\psi(t) = P_1 \exp(tx_1) + \cdots + P_k \exp(tx_k)$$

Hence by the expression of $\psi(t)$ we get X take values 1, 4, 8 with probabilities $\frac{1}{5}, \frac{2}{5}, 2/5$ respectively.

---

(15) P.d.f of X is $h(x) = \frac{1}{2} f(x) + \frac{1}{2} g(x)$

$$E[X] = \int_{-\infty}^{\infty} x \left[\frac{1}{2} f(x) + \frac{1}{2} g(x)\right] dx$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} x f(x) + \frac{1}{2} \int_{-\infty}^{\infty} x\, g(x)\, dx$$

$$= \frac{1}{2} \mu_f + \frac{1}{2} \mu_g$$ [where μ_f & μ_g are expections of f & g respectively]

Now, $\int_{-\infty}^{1} h(x)\, dx = \frac{1}{2} \int_{-\infty}^{1} f(x)\, dx = \frac{1}{2}$ &

$\int_{2}^{\infty} h(x)\, dx = \frac{1}{2} \int_{2}^{\infty} g(x)\, dx = \frac{1}{2}$

$\therefore$ Any value between $[1, 2]$ is a median.

(16) Simple calculation gives

$$E[X] = \int_0^1 \int_0^2 x \cdot \frac{1}{3}(x+y)\, dy\, dx = 5/9$$

$$E[Y] = \int_0^1 \int_0^2 y \, \frac{1}{3}(x+y)\, dy\, dx = \frac{11}{9}$$

$$E[X^2] = \int_0^1 \int_0^2 x^2 \frac{1}{3}(x+y)\, dy\, dx = 7/18$$

$$E[Y^2] = \int_0^1 \int_0^2 y^2 \frac{1}{3}(x+y)\, dy\, dx = 16/9$$

$$E[XY] = \int_0^1 \int_0^2 xy \, \frac{1}{3}(x+y)\, dy\, dx = 2/3.$$

$$\Rightarrow Var(X) = E[X^2] - (E[X])^2 = \frac{13}{162}$$

$$Var(Y) = E[Y^2] - (E(Y))^2 = \frac{23}{81}$$

$$cov(X, Y) = E[XY] - E(X)\, E(Y) = -\frac{1}{81}$$

$$Var(2X - 3Y + 8)$$

$$= 4\, Var(X) + 3^2 Var(Y) - 2\cdot 2\cdot 3 \; Cov(X, Y)$$

$$= 245/81$$