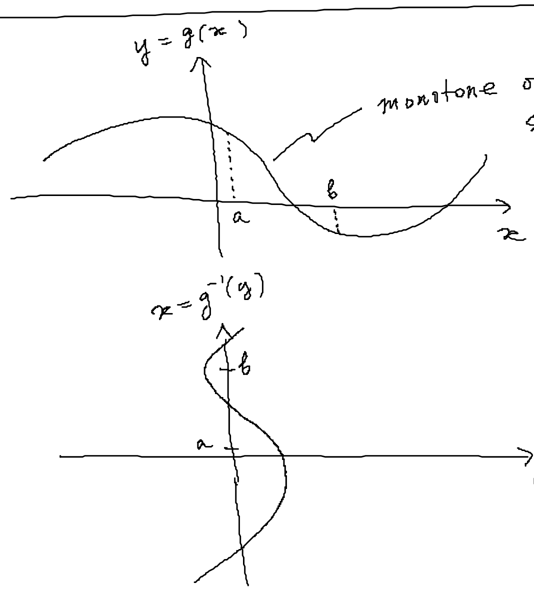
Theorem Suppose X is a random variable with density $f_X(x)$ and g is a **monotone** transformation on the domain of X (i.e. $\{x : f_X(x) > 0\}$).
Then if $Y = g(X)$ then

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \left| \frac{\partial g^{-1}(y)}{\partial y} \right| ,$$

$\{y : y = g(x)$ for some x s.t. s.t. $f_X(x) > 0\}$

---

example



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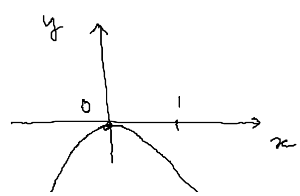
example X has density

$$f_X(x) = 2x, \quad 0 \le x \le 1.$$

$$Y = -X^2$$



$1 = -x$

$y = g(x) = -x^2$ is monotone on the support of $x : [0,1]$

Find the support of $Y : [-1, 0]$

and $g^{-1}(y) = \sqrt{-y}$ (solve for $x : y = g(x)$ to find g^{-1})

$$f_Y(y) = f_X\left(g^{-1}(y)\right) \left| \frac{\partial g^{-1}(y)}{\partial y} \right|$$

$$= 2\sqrt{-y} \cdot \left| \frac{\partial \sqrt{-y}}{\partial y} \right|$$

$$= 2\sqrt{-y} \cdot \left| \frac{-1}{2\sqrt{-y}} \right| = 2\sqrt{-y} \cdot \frac{1}{2\sqrt{-y}} =$$

$$= 1, \quad -1 \le y \le 0.$$

$Y \sim Unif\ [-1, 0]$. //

$$\int_A f_X(x)\, dx \xrightarrow[\text{to } y = g(x)]{\text{change } x} \int f_X(g^{-1}(y))\, d g^{-1}(y)$$

$$= \int f_X(g^{-1}(y)) \left| \frac{\partial g^{-1}(y)}{\partial y} \right| dy$$

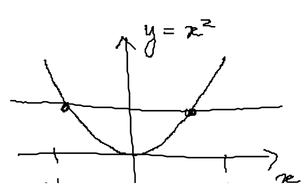
If g is not monotone then

$$f_Y(y) = \sum_{i : g(g_i^{-1}(y)) = y} f_X(g_i^{-1}(y)) \left| \frac{\partial g_i^{-1}(y)}{\partial y} \right|$$

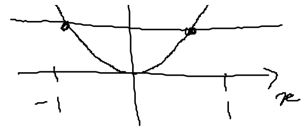
---

example $X \sim Unif\ [-1, 1]$

$Y = X^2 = g(X)$



$$Y = X^2 = g(X)$$



Solve for x : $y = x^2 \Rightarrow x = \pm\sqrt{y}$

i.e. $(\pm\sqrt{y})^2 = y$, so

$$g_1^{-1}(y) = \sqrt{y}, \quad g_2^{-1}(y) = -\sqrt{y}.$$

$$f_Y(y) = \sum_{i=1}^{2} f_X(g_i^{-1}(y)) \left| \frac{\partial g_i^{-1}(y)}{\partial y} \right|$$

$$= \frac{1}{2} \left| \frac{1}{2\sqrt{y}} \right| + \frac{1}{2} \left| \frac{-1}{2\sqrt{y}} \right|$$

$$= \frac{1}{2\sqrt{y}}, \quad 0 \le y \le 1. \quad //$$

In discrete case the situation is simpler. (but may be tedious)

<u>example</u> Let $Y \sim Unif\{10\}$

$$y \in \{1, 2, \dots, 10\}$$

Let $X = Y - 5$, $Z = X^2 = (Y-5)^2$

$x \in \{-4, \dots, 5\}$, $z \in \{0, 1, 4, 9, 16, 25\}$

$$P(X = i) = P(X = -i) = \frac{1}{10}, \quad i = 1, 2, 3, 4$$

$$P(X = 5) = P(X = 0) = \frac{1}{10}.$$

$$P(Z = z) = P(X = \sqrt{z}) + P(X = -\sqrt{z})$$

$$= \frac{1}{5}, \quad z \in \{1, 4, 9, 16\}$$

$$P(Z = 0) = P(Z = 25) = \frac{1}{10}.$$

z	0	1	4	9	16	25
$f_Z(z)$	.1	.2	.2	.2	.2	.1

(sums to 1 !)

In discrete Case:

$$Z = g(Y)$$

$$f_Z(z) = \sum_{y:\, z = g(y)} f_Y(y)$$

//

Special Transformation

PIT = probability integral transformation

Suppose X with cdf $F_X(x)$

assume F_X is strictly monotone on the domain of X.

Then for every $y \in (0,1)$ there exists exactly one x s.t. $y = F_X(x)$, so $G(x) = F_X^{-1}(x)$, so well-defined.

Let $Y = F_X(X)$. Then

$$F_Y(y) = P(Y \le y) = P(F_X(X) \le y)$$

$$= P(X \le F_X^{-1}(y)) =$$

$$= F_X(F_X^{-1}(y)) = y, \quad 0 \le y \le 1.$$

$$f_Y(y) = \frac{\partial F_Y(y)}{\partial y} = 1, \quad 0 \le y \le 1$$

$Y = F_X(X) \sim$ Unif $[0,1]$.

$Y = Y_X(X)$

Let $U \sim \mathcal{U}nif\,[0,1]$ and

$$Z = F_X^{-1}(U), \text{ then}$$

$$F_Z(z) = P(Z \le z) = P(F_X^{-1}(U) \le z)$$

$$= P\left(F_X(F_X^{-1}(U)) \le F_X(z)\right)$$

$$= \underbrace{P(\overset{\text{r.v.}}{U} \le \underbrace{\overset{\text{non-random}}{F_X(z)}})} = F_X(z)$$

$\left(U \sim Unif\,[0,1] \text{ then } \underbrace{F_U(u) = u}, \ 0 \le u \le 1 \right.$

$\left. F_X : \mathbb{R} \to [0,1] \implies F_X(z) \in [0,1] \right)$

$$\Rightarrow Z = F^{-1}(U) \sim X.$$

(same distribution)

Gives you a way to simulate from <u>any</u> distribution.

<u>Note</u>: I assumed strict monotonicity of F_X, the cdf, but it can be relaxed (easy but not here) to include <u>all</u> cdfs.

<u>Functions of 2 or more random variables.</u>

Suppose $X_1, \dots, X_n$ are discrete r.v.s with pmf $f_X(\underset{\sim}{x})$ and

Suppose $X_1, \ldots, X_n$ ...

pmf $\quad f_{\underset{\sim}{X}}(\underset{\sim}{x})$ and

$$Y_1 = r_1(\underset{\sim}{X}),\ Y_2 = r_2(\underset{\sim}{X}),\ \ldots,\ Y_m = r_m(\underset{\sim}{X})$$

$$r_i : \mathbb{R}^n \to \mathbb{R}, \quad \underset{\sim}{r} : \mathbb{R}^n \to \mathbb{R}^m \quad (\underset{\sim}{r} = (r_1, \ldots, r_m))$$

Then the joint pmf of $Y_1, \ldots, Y_m$

$$f_{\underset{\sim}{Y}}(y_1, \ldots, y_m) = f_{\underset{\sim}{Y}}(\underset{\sim}{y}) = \sum_{\underset{\sim}{x} : \underset{\sim}{r}(\underset{\sim}{x}) = \underset{\sim}{y}} f_{\underset{\sim}{X}}(\underset{\sim}{x})$$

(same as your intuition would suggest).

If $X_1, \ldots, X_n$ are jointly continuous with joint density $f_{\underset{\sim}{X}}$ and $\underset{\sim}{r} : \mathbb{R}^n \to \mathbb{R}^m$.

Then we let $\underset{\sim}{Y} = \underset{\sim}{r}(\underset{\sim}{X})$

$$\begin{aligned} F_{\underset{\sim}{Y}}(\underset{\sim}{y}) &= P(\underset{\sim}{Y} \le \underset{\sim}{y}) \\ &= P(\underset{\sim}{r}(\underset{\sim}{X}) \le \underset{\sim}{y}) \\ &= \int \cdots \int_{\underset{\sim}{x} : \underset{\sim}{r}(\underset{\sim}{x}) \le y} f_{\underset{\sim}{X}}(\underset{\sim}{x})\, d\underset{\sim}{x} \end{aligned}$$

(again, from basic definitions).

Important example: Order Statistics

Let $X_1, \ldots, X_n$ be iid (random sample) from the distribution with density $f_X(x)$ and marginal cdf $F_X(x)$.

Let $Y_1 = \min(X_1, \dots, X_n)$ (smallest X_i)

$Y_2 =$ second smallest X_i

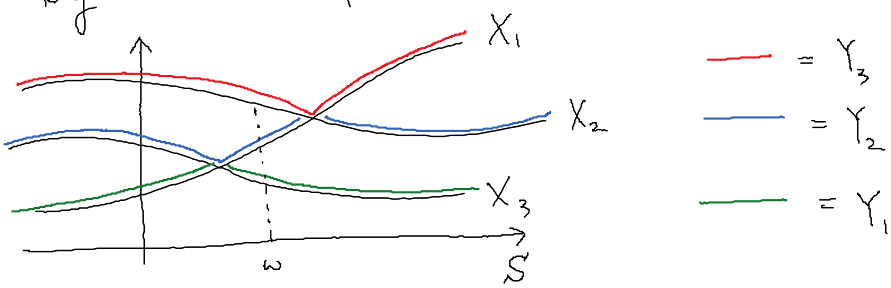
$\vdots$

$Y_r = r^{th}$ smallest X_i

$\vdots$

$Y_n = \max(X_1, \dots, X_n)$ (largest X_i)

By this definition $Y_1 \le Y_2 \le \dots \le Y_n$.



$Y_1(w) = x_3$, $Y_2(w) = x_1$, $Y_3(w) = x_2$

$(X_1(w) = x_1, X_2(w) = x_2, X_3(w) = x_3)$

Are Y_i's independent? NO

(even though X_i's were.)

<u>Notation</u>: $Y_i = X_{(i)}$ – order statistics.

It is not hard to show that

$$f_{\underset{\sim}{Y}}(y_1, \dots, y_n) = n!\, f_X(y_1) f_X(y_2) \dots f_X(y_n)$$

$$\underbrace{y_1 \le y_2 \le y_3 \le \dots \le y_n}_{\text{not a rectangular support!}}$$

Consider $Y_n = X_{(n)}$ – maximal order statistic

cdf of Y_n: $F_n(y) = P(Y_n \le y) =$

$\stackrel{\circ}{=}$ $P(X_1 \le y, \dots, X_n \le y) =$

$= P(X_1 \le y) \cdots P(X_n \le y) = F_X^n(y)$

pdf of Y_n : $f_n(y) = \dfrac{\partial F_n(y)}{\partial y} = \dfrac{\partial \left(F_X^n(y) \right)}{\partial y}$

$= n F_X^{(n-1)}(y) \, f_X(y).$

Consider $Y_1 = X_{(1)}$, the minimal order statistic.

cdf of Y_1 : $F_1(y) = P(Y_1 \le y) =$

$= 1 - P(Y_1 > y) = 1 - P(X_1 > y, \dots, X_n > y)$

$= 1 - \underbrace{P(X_1 > y)} \cdots P(X_n > y)$

$= 1 - \left(1 - F_X(y) \right)^n$

pdf of Y_1 : $f_1(y) = \dfrac{\partial F_1(y)}{\partial y} = \dfrac{\partial \{ 1 - (1 - F_X(y))^n \}}{\partial y}$

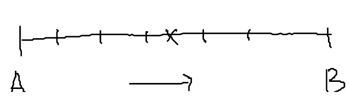
$= n \left(1 - F_X(y) \right)^{n-1} f_X(y).$

---

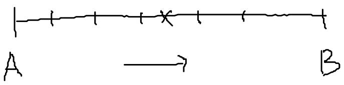
example Suppose the system consists of 100 components, each with failure time dist'n with density

$f(x) = \lambda e^{-\lambda x}, \quad x > 0 \quad (\lambda > 0$ known const$)$

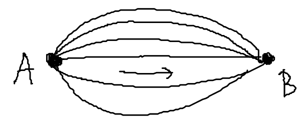
(a) If the system fails when at least one of of the components fails, what's the distribution of the system failure time?



failure time?



(b) If the system fails when $\underline{\underline{\text{all}}}$ of components fail, what's the distribution of the system failure time?



(a) $X_1, \dots, X_n$ be component failure times.

$Y_1 = X_{(1)}$ = the system failure time.

$$f_1(x) = n\,(1 - F_X(x))^{n-1} f_X(x)$$

$$F_X(x) = \int_{-\infty}^{x} f_x(u)\,du = \int_0^x \lambda e^{-\lambda u}\,du = $$

$$= 1 - e^{-\lambda x}, \quad x > 0.$$

$$f_1(x) = n e^{-(n-1)\lambda x} \cdot \lambda e^{-\lambda x} = \lambda n e^{-\lambda n x}, \quad x > 0.$$

(same form as for X_i's : $\lambda \to \lambda n$) //

---

b) $X_{(n)} = Y_n$ - system failure time,

$$f_n(x) = n F_X^{n-1}(x)\, f_X(x)$$

$$= n\,(1 - e^{-\lambda x})^{n-1} \cdot \lambda e^{-\lambda x}, \quad x > 0.$$

//

In general, for k^{th} order statistic

(statistic = any function of a random sample)

<u>cdf of $X_{(k)}$</u> :

$$F(x) = P(X_{(k)} \le x)$$

$$F_k(x) = P(X_{(k)} \le x)$$

$$= P(\text{at least } k \text{ of the } X_i\text{'s are} \le x)$$

$$= \sum_{\ell=k}^{n} P(\text{exactly } \ell \text{ of } X_i\text{'s} \le x)$$

$$= \sum_{\ell=k}^{n} \binom{n}{\ell} F_X^{\ell}(x)\,(1-F_X(x))^{n-\ell}$$

pdf of Y_k:

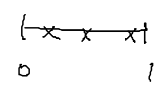
$$f_k(x) = \frac{\partial F_k(x)}{\partial x} = \underset{\text{(do it!)}}{\cdots\cdots}$$

$$= k\binom{n}{k} F_X^{k-1}(x)\,(1-F_X(x))^{n-k} f_X(x)$$

---

example $U_1, U_2, U_3 \sim \text{Unif } [0,1]$

- locations of 3 gas stations on a mile of a road



What's the probability that all 3 are located within 1/3 mile of each other?

$U_{(1)}, U_{(2)}, U_{(3)}$ order statistics?

$$P\left(U_{(3)} - U_{(1)} < \tfrac{1}{3}\right).$$

What's the joint density of $U_{(1)}, U_{(3)}$?

$$P(\underset{B}{U_{(3)} \le u_3},\ \underset{A}{U_{(1)} > u_1}) =$$

$$= P(U_{(3)} \le u_3) - \overbrace{P(U_{(1)} \le u_1, U_{(3)} \le u_3)}^{\text{cdf of } U_{(1)},\, U_{(3)}}$$

$$= P(\underbrace{\mathcal{U}_{(3)} \le u_3}_{B}) - P(\underbrace{\mathcal{U}_{(1)} \le u_1}_{A^c}, \underbrace{\mathcal{U}_{(3)} \le u_3}_{B})$$

$$= P(u_1 \le \mathcal{U}_i \le u_3, \; i = 1, 2, 3)$$

$$= \prod_{i=1}^{3} P(u_1 \le \mathcal{U}_i \le u_3) = \left(F_u(u_3) - F_u(u_1)\right)^3$$

So joint cdf of $\mathcal{U}_{(1)}, \mathcal{U}_{(3)}$ is

$$P(\mathcal{U}_{(1)} \le u_1, \mathcal{U}_{(3)} \le u_3) = \overbrace{P(\mathcal{U}_{(3)} \le u_3)} - \left(F_u(u_3) - F_u(u_1)\right)^3$$

$$\frac{\partial^2}{\partial u_1 \partial u_3}\left(F_{1,3}(u_1, u_3)\right) = -\frac{\partial^2}{\partial u_1 \partial u_3}\left\{\left(F_u(u_3) - F_u(u_1)\right)^3\right\}$$

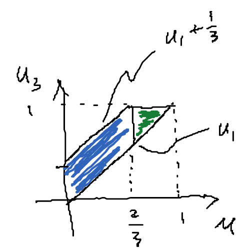
$$= \underset{\cdots\cdots}{\overset{\text{compute}}{}} = 6(u_3 - u_1),$$

$$\underbrace{0 \le u_1 \le u_3 \le 1.}_{\text{joint pdf of } \mathcal{U}_{(1)} \text{ \& } \mathcal{U}_{(3)}}$$

$$P\left(\mathcal{U}_{(3)} - \mathcal{U}_{(1)} < \tfrac{1}{3}\right) =$$

$$= \iint_{u_3 - u_1 < \frac{1}{3}} f_{13}(u_1, u_3)\, du_1\, du_3$$

$$= \iint_{\substack{0 \le u_1 \le u_3 \le 1 \\ u_3 - u_1 < \frac{1}{3}}} 6(u_3 - u_1)\, du_1\, du_3$$



$$= \int_0^{2/3}\int_{u_1}^{u_1 + \frac{1}{3}} 6(u_3 - u_1)\, du_3\, du_1 + \int_{2/3}^{1}\int_{u_1}^{1} 6(u_3 - u_1)\, du_3\, du_1$$

$$\underset{\cdots}{\overset{\text{calculus! fun!}}{=}} = 1/3. \qquad //$$

In general, joint density of 2 order statistics

$X_{(i)}$ and $X_{(j)}$, $\quad 1 \le i < j \le n$

$$f_{ij}(u, v) = \frac{n!}{(i-1)!\,(j-i-1)!\,(n-j)!} f(u) f(v) \cdot$$
$$\cdot F^{i-1}(u) \left[F(v) - F(u) \right]^{j-i-1} (1 - F(v))^{n-j},$$
$$u \le v$$

<u>example</u> (continued, gas stations)

$U_1, U_2, U_3 \sim$ Unif $[0,1]$, for $U_{(1)}, U_{(3)}$

$$f_{13}(u, v) = \frac{3!}{0!\,1!\,0!} \cdot 1 \cdot 1 \cdot u^0 (v-u)^1 (1-v)^0$$
$$= 6(v-u), \quad 0 \le u \le v \le 1.$$

(same as above).

Back to general transformation of 2 or more random variables.

Again, $\underset{\sim}{r} : \mathbb{R}^n \to \mathbb{R}^n$, so that

$$Y_1 = r_1(X_1, \dots, X_n)$$
$$\vdots$$
$$Y_n = r_n(X_1, \dots, X_n).$$

Assume that $\underset{\sim}{r}$ is one-to-one; so there exists an inverse tranformation, $\underset{\sim}{s} = \underset{\sim}{r}^{-1}$

s.t. $\underset{\sim}{X} = \underset{\sim}{s}(\underset{\sim}{Y})$.

Let

$$J = \det \begin{bmatrix} \frac{\partial s_1}{\partial y_1} & \cdots & \frac{\partial s_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial s_n}{\partial y_1} & \cdots & \frac{\partial s_n}{\partial y_n} \end{bmatrix}$$

(partial derivatives of old w/respect to new)

– jacobian of this tranformation.

If $f_{\underset{\sim}{X}}(\underset{\sim}{x})$ is the joint density of $\underset{\sim}{X}$

then

$$f_Y(y) = f_X(\underset{\sim}{s}(\underset{\sim}{y})) \cdot |J|$$

---

example Suppose X has density

$$f_X(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 \le x \le 1$$

$(\alpha, \beta > 0$ fixed$)$

and Y with density

$$f_Y(y) = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha+\beta)\Gamma(\gamma)} y^{\alpha+\beta-1}(1-y)^{\gamma-1}, \quad 0 \le y \le 1$$

$(\gamma > 0$ fixed$)$

and $X \perp\!\!\!\perp Y$.

Find distribution of $X \cdot Y$?

not one-to-one.

Let $U = XY$, $V = X$

$(X, Y) \rightsquigarrow (U, V)$ one-to-one now.

$X = V$, $Y = U/V$.

$$\{0 < x < 1,\ 0 < y < 1\} = \{0 < v < 1,\ 0 < \tfrac{u}{v} < 1\} =$$

$$= \{0 < v < 1,\ 0 < u < v\} = \{0 < u < v < 1\}.$$

$$J = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \det\begin{pmatrix} 0 & 1 \\ \frac{1}{v} & -\frac{u}{v^2} \end{pmatrix}$$

$$= -\frac{1}{v}, \text{ so}$$

$$f_{UV}(u,v) = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} v^{\alpha-1}(1-v)^{\beta-1} \cdot \left(\frac{u}{v}\right)^{\alpha+\beta-1}\left(1-\frac{u}{v}\right)^{\gamma-1} \cdot \frac{1}{v}$$

$$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} v^{-(\beta+\gamma)}(1-v)^{\beta-1} u^{\alpha+\beta-1}(v-u)^{\gamma-1}$$

$$0 < u < v < 1.$$

Distribution of XY?

Find marginal by integrating out V.

$$f_U(u) = \int_{-\infty}^{\infty} f_{UV}(u,v)\,dv$$

$$= \int_u^1 \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} v^{-(\beta+\gamma)}(1-v)^{\alpha+\beta-1} u^{\alpha+\beta-1}(v-u)^{\gamma-1}\,dv$$

change to a new variable $t = \dfrac{u/v - u}{1-u}$

$dt = \ldots$

$$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta+\gamma)}(1-u)^{\gamma+\beta-1} u^{\alpha-1}, \quad 0 < u < 1.$$