Note For continuous distributions

cdf $F(x) = P(X \le x) = \int_{-\infty}^{x} f(y)\,dy$ pdf

$\Rightarrow$ can differentiate F, so it is continuous, i.e.

$$\lim_{y \to x+} F(y) = \lim_{y \to x-} F(y) = \lim_{y \to x} F(y) = F(x).$$

true for all r.v.

$$\Rightarrow P(X = \{x\}) = F(x) - \lim_{y \to x-} F(y)$$

$$= F(x) - F(x) = 0.$$

$$\frac{\partial F}{\partial x}(x) = \frac{\partial}{\partial x}\left(\int_{-\infty}^{x} f(y)\,dy\right) = f(x).$$

Discrete & continuous random variables.

cdf for both cases: $F(x) = P(X \le x),\ x \in \mathbb{R}$

pmf for discrete r.v.'s: $f(x) = P(X = x),\ x \in \mathbb{R}$

pdf for continuous r.v.'s: $P(X \in A) = \int_A f(x)\,dx$

Discrete: $f(x) = F(x) - \lim_{y \to x-} F(y)$

Continuous: $f(x) = \frac{\partial F}{\partial x}(x)$

example Cdf $\underbrace{F(x) = x^2, \quad 0 \le x \le 1}$

Shorthand for

$$F(x) = \begin{cases} 0, & x < 0 \\ x^2, & 0 \le x \le 1 \\ 1, & x > 1. \end{cases}$$

omit to save space

What's the pdf?

$$f(x) = 2x, \quad \underbrace{0 \le x \le 1}_{\text{"domain", or "support"}}$$

i.e. $\{x : f(x) > 0\}$ //

---

example Consider a candidate for a pdf

$$f(x) = c x e^{-\lambda x}, \quad x \ge 0.$$

Assume that $\lambda > 0$ is known.

1) $f \ge 0 \Rightarrow c > 0.$

$$P(X \in \mathbb{R}) = \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$1 = \int_0^{\infty} c x e^{-\lambda x}\,dx =$$

$$= -\frac{1}{\lambda} \int_0^{\infty} c x \, d\left(e^{-\lambda x}\right) =$$

$$= -\frac{1}{\lambda} c x e^{-\lambda x} \Big|_0^{\infty} + \frac{c}{\lambda} \int_0^{\infty} e^{-\lambda x}\,dx$$

λ

$$= 0 - \frac{c}{\lambda^2} e^{-\lambda x} \Big|_0^\infty = \frac{c}{\lambda^2}$$

$$\Rightarrow \quad c = \lambda^2 > 0.$$

$$f(x) = \lambda^2 x e^{-\lambda x}, \quad x \geq 0 \qquad //$$

So far we looked at one random variable at a time.

Bivariate Distributions

The joint probability distribution of two random variables is called a bivariate distribution.

The pair (X, Y) takes on some values.

If there are at most countably many possible values $\Rightarrow$ discrete bivariate dist'n.

In this case, use joint probability mass function to describe the dist'n of (X, Y)

$$f(x, y) = P(\{X = x\} \cap \{Y = y\})$$

$$= P(X = x, Y = y), \quad \text{so}$$

"and"

"and"

1) $0 \le f(x,y) \le 1$, for all $(x,y) \in \mathbb{R}^2$;

2) $$\sum_{\text{all } \{x,y\} \in \mathbb{R}^2} f(x,y) = 1;$$

3) $$P((X,Y) \in A) = \sum_{\{x,y\} \in A} f(x,y), \quad A \subset \mathbb{R}^2.$$

<u>example</u> X can take on values $1, 2, 3, 4$

Y can take on values $0, 5, 8$

joint pmf is given by the following table

$X \ge 3$

$Y \backslash X$	1	2	3	4	
0	.1	0	.01	.05	.16
5	.3	.2	.15	0	.65
$Y > 6$ { 8	.05	.08	.05	.01	.19
	.45	.28	.21	.06	1

$$P(X=1) = P(X=1, Y=0) + P(X=1, Y=5) +$$
$$+ P(X=1, Y=8) = .1 + .3 + .05 = .45$$

Row totals give us the dist'n Y

Column totals give us the dist'n X.

$$P(X \ge 3, Y > 6) = .06$$

//

<u>Def</u> (X, Y) are jointly continuous r.v.'s

Def (X, Y) are jointly continuous r.v.s (jointly continuous dist'n) if there exists a nonnegative fn $f : \mathbb{R}^2 \to [0, \infty)$ s.t. for all $A \subset \mathbb{R}^2$ (A ⓜ)

$$P\big((X, Y) \in A\big) = \iint_A f(x, y)\, dx\, dy$$

Such f is called the <u>joint</u> probability density function (pdf) of X, Y

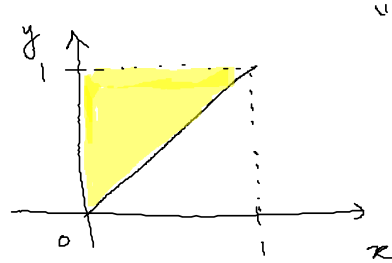
1) $f(x, y) \geq 0$, all $(x, y) \in \mathbb{R}^2$

2) $\displaystyle\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$

3) $P\big((X, Y) = \{x, y\}\big) = 0.$

<u>example</u> Suppose $f(x, y) = Cxy$, $\underbrace{0 < x \leq y < 1}_{\text{"support" or "domain"}}$

$\Rightarrow C > 0.$



$$1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy =$$

$\int^{1} \int^{1}$ …

$$= \int_0^1 \int_x^1 C\, x y \, dy\, dx =$$

$$= \int_0^1 c x \left(\int_x^1 y \, dy \right) dx = \int_0^1 c x \cdot \frac{y^2}{2} \Big|_x^1 dx$$

$$= \frac{1}{2} \int_0^1 c x \,(1 - x^2)\, dx = \frac{c}{2} \left(\frac{x^2}{2} - \frac{x^4}{4} \right) \Big|_0^1$$

$$= \frac{C}{2} \cdot \frac{1}{4} = C/8 \quad \Rightarrow \quad C = 8.$$

$$\Rightarrow \quad f(x,y) = 8xy, \quad 0 < x \leq y \leq 1.$$

Fix $a > 0$.

$$P(X < aY) = \iint_{\{x < ay\}} f(x,y)\, dx\, dy =$$

$$= \iint_{\substack{\{0 < x \leq y < 1\} \\ \cap \{x < ay\}}} 8xy \, dx\, dy =$$

$$= \begin{cases} \displaystyle\iint_{\{0 < x \leq y < 1\}} 8xy\, dx\, dy, & a > 1 \\ \displaystyle\iint_{\left\{\substack{0 < x < ay \\ 0 \leq y \leq 1}\right\}} 8xy\, dx\, dy, & 0 < a < 1 \end{cases}$$

$$= \begin{cases} 1, & a > 1 \\ \int_0^1 \int_0^{ay} \ldots & 0 < a < 1 \end{cases}$$

$$= \begin{cases} \quad , \\ \int_0^1 \int_0^{ay} 8xy \, dx \, dy, & 0 < a < 1 \end{cases}$$

compute on your own

$$= \begin{cases} 1, & a > 1 \\ a^2, & 0 < a < 1 \end{cases} //$$

We define the joint cumulative distribution function by

$$F(x, y) = P(X \le x, Y \le y)$$

"and" or $\cap$

$\Rightarrow$ for jointly continuous r.v.'s (X, Y)

$$F(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(u, v) \, du \, dv$$

and
$$f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y)$$

---

think on your own

$\underline{Q}$: Can you find a r.v. X that is neither discrete nor continuous?

(yes)

<u>Remark</u> Joint dist'ns are not always jointly discrete or jointly continuous.

Let $X \sim \text{Unif } [a, b]$

Let $X \sim \text{Unif}\,[a, b]$

$\Rightarrow$ pdf $f(x) = \frac{1}{b-a}$, $a \le x \le b$

$X \Rightarrow$ continuous.

Take $Y = -X$.

$$F_{Y}(y) = P(Y \le y) = P(-X \le y)$$

$$= P(-y \geq X) = 1 - P(X \le -y)$$

$$= 1 - F_{X}(-y)$$, so differentiating

we get $f_Y(y) = f_X(-y) = \frac{1}{b-a}$, $a \le -y \le b$

$$= \frac{1}{b-a}, \quad -b \le y \le -a$$

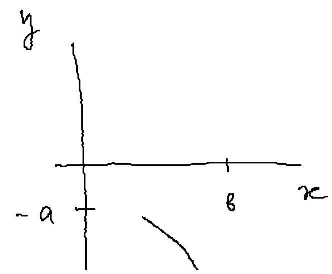
$\Rightarrow Y \sim \text{Unif}\,[-b, -a]$,

so Y is continuous.

$(X, Y) = (X, -X)$ not discrete.

But not jointly continuous!

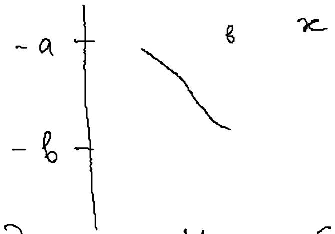
Does pdf f exist?



i.e. so

Does pdf f exist,

i.e. so



$P(X, Y \in A) =$

$= \int_A f \, dx dy > 0$?   No. Show on your own.

Brief overview of distributions & their summaries.

One variable case: X

Continuous	Discrete
1) pdf, $f \geq 0$ $P(X \in A) = \int_A f(x) \, dx$	1) pmf, $0 \leq f \leq 1$ $P(X \in A) = \sum_{x \in A} f(x)$
2) $\int_{-\infty}^{\infty} f(x) \, dx = 1$	2) $\sum_x f(x) = 1$
3) cdf $F(x) = P(X \leq x)$	
4) $P(X = x) = 0$, for all $x \in \mathbb{R}$	4) not true! there exist x s.t. $P(X = x) > 0$.
5) $F(x) = \int_{-\infty}^{x} f(y) \, dy$ $f(x) = \frac{\partial}{\partial x} F(x)$	5) $F(x) = \sum_{y \leq x} f(y)$ $f(x) = F(x) - \lim_{y \to x-} F(y)$

For Bivariate Distributions:

jointly continuous	jointly discrete
1) $P((X,Y) \in A) = \iint_A \underbrace{f(x,y)}_{\text{joint pdf}} \, dx\, dy$	1) $P((X,Y) \in A) = \sum\sum_{(x,y)\in A} \underbrace{f(x,y)}_{\text{joint pmf}}$
2) $f \geq 0$, $\iint_{\mathbb{R}^2} f(x,y)\, dx\, dy = 1$	2) $0 \leq f \leq 1$, $\sum_x \sum_y f(x,y) = 1$
3) $F(x,y) = P(X \leq x \underbrace{,}_{\text{"and"}} Y \leq y)$	
4) $P((X,Y) = (x,y)) = 0$ for all $(x,y) \in \mathbb{R}^2$	4) not true!
5) $F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)\, dv\, du$; $f(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y)$	5) $F(x,y) = \sum_{v \leq y}\sum_{u \leq x} f(u,v)$

<u>example</u> Bowl contains 10 chips, each marked

$$\underbrace{(1,1)}_{1} \quad \underbrace{(2,1)}_{1} \quad \underbrace{(3,1)}_{2} \quad \underbrace{(1,2)}_{1} \quad \underbrace{(2,2)}_{2} \quad \underbrace{(3,2)}_{3}$$

Draw one chip @ random.

Note the pair on that chip.

(X_1, X_2).

$X_2 \backslash X_1$	1	2	3

$X_2 \backslash X_1$	1	2	3	
1	.1	.1	.2	.4
2	.1	.2	.3	.6
	.2	.3	.5	1

} dist'n of X_2

dist'n of X_1.

Marginal Distributions

– (X, Y) discrete, joint pmf $f_{XY}(x, y)$

then what's the pmf of X?

pmf : $$f_X(x) = P(X = x) = \sum_y P(X = x, Y = y)$$

$$= \sum_y f_{XY}(x, y)$$

Similarly, $$f_Y(y) = \sum_x f_{XY}(x, y)$$

(same as taking row/column totals)

Suppose (X, Y) are jointly continuous with joint pdf $f(x, y)$

How to get $f_X(x)$, the pdf of X?
(marginal density of X).

Need $f_X(x)$ s.t.

$$P(X \in A) = \int_A f_X(x)\,dx$$

$$P(X \in A) = P(X \in A, Y \in \mathbb{R})$$

$$= \int_A \underbrace{\left(\int_{\mathbb{R}} f(x,y)\,dy\right)}_{\equiv f_X(x)} dx$$

$$\Rightarrow \quad f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy$$

$$f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx$$

---

example $\quad f(x_1, x_2) = 6x_1^2 x_2, \quad 0 < x_1 < 1, \quad 0 < x_2 < 1$

To get marginal density of X_1

$$f_1(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2)\,dx_2$$

$$= \int_0^1 6x_1^2 x_2\,dx = 3x_1^2, \quad \underbrace{0 \le x_1 \le 1}_{\text{support !}}$$

Check your computation?

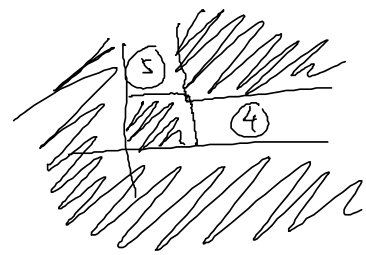
$$\int_{\mathbb{R}} f_1(x_1)\,dx_1 = \int_0^1 3x_1^2\,dx_1 = x_1^3\Big|_0^1 = 1.$$

Find $F(x_1, x_2)$ joint cdf.

$$P(X_1 \le x_1, X_2 \le x_2) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f(a_1, a_2)\, da_2\, da_1$$

① $\boxed{0 \le x_1 \le 1, \quad 0 \le x_2 \le 1}$

$$F(x_1, x_2) = \int_0^{x_1} \int_0^{x_2} 6a_1^2 a_2\, da_2\, da_1 = x_1^3 x_2^2$$



② $\boxed{x_1 > 1, \quad x_2 > 1}$

$$F(x_1, x_2) = 1$$

③ $\boxed{x_1 < 0 \quad \underline{\underline{or}} \quad x_2 < 0}$

$$F(x_1, x_2) = 0$$

④ $\boxed{x_1 > 1, \quad x_2 \in (0, 1)}$

$$F(x_1, x_2) = \int_0^1 \int_0^{x_2} 6a_1^2 a_2\, da_2\, da_1 = x_2^2$$

⑤ $\boxed{x_2 > 1, \quad x_1 \in (0, 1)}$

$$F(x_1, x_2) = \int_0^{x_1} \int_0^1 6a_1^2\, da_2\, da_1 = x_1^3$$

Summarizing, we get

$$F(x_1, x_2) = \begin{cases} 0, & x_1 \le 0 \text{ or } x_2 \le 0 \\ x_1^3 x_2^2, & 0 \le x_1 \le 1 \;\&\; 0 \le x_2 \le 1 \\ x_2^2, & x_1 > 1 \;\&\; 0 \le x_2 \le 1 \\ 3 & \end{cases}$$

marginals

$$\begin{cases} x_2^2, & x_1 > 1 \ \& \ 0 \le x_2 \le 1 \\ x_1^3, & 0 \le x_1 \le 1 \ \& \ x_2 > 1 \\ 1, & x_1 \ge 1 \ \& \ x_2 \ge 1. \end{cases}$$

//

Independent Random Variables.

Def The random variables X and Y are independent if for any $A, B \subset \mathbb{R}$ (Borel)

$$P(X \in A, Y \in B) = P(X \in A)\, P(Y \in B)$$

"and"

i.e. $\{X \in A\} \perp\!\!\!\perp \{Y \in B\}$

We write $X \perp\!\!\!\perp Y$, similarly as for events.

One can show (not in this class) that X and Y are independent iff $F(x, y) = F_X(x)\, F_Y(y)$ for all $(x, y) \in \mathbb{R}^2$.

When X and Y are jointly discrete

the independence is equivalent to

$$p(x,y) = p_X(x)\, p_Y(y) \quad \text{for all } (x,y) \in \mathbb{R}^2.$$

When X and Y are jointly continuous the independence is equivalent to

$$f(x,y) = f_X(x)\, f_Y(y),$$

for all $(x,y) \in \mathbb{R}^2$.

(a.e. in Lebesgue measure)

Intuitively, random variables X and Y are independent when knowing the value of one doesn't give us information about the other.

example We perform $n+m$ independent trials, each resulting in success with probability p.

Let X = # of successes in the first n trials

Y = # of successes in the last

Y = # of successes in the last m trials.

integers
$0 \le x \le n,\ 0 \le y \le m$

$$P(X = x, Y = y) = \binom{n}{x} p^x (1-p)^{n-x} \binom{m}{y} p^y (1-p)^{m-y}$$

$$= P(X = x)\, P(Y = y)$$

$$0 \le x \le n, \quad 0 \le y \le m$$

Z = # of successes in <u>all</u> $n+m$ trials.

X and Z ? Intuition suggests that

$$X \not\perp\!\!\!\perp Z.$$

$$0 = P(X = 1, Z = 0) \neq$$

$$P(X=1) \cdot P(Z=0) > 0$$

$$\binom{n}{1} p (1-p)^{n-1} \cdot \binom{n+m}{0} p^0 (1-p)^{n+m}$$

$$\Rightarrow \quad X \not\perp\!\!\!\perp Z. \qquad //$$

<u>example</u> A man and a woman decide to meet at a certain location.

If each person independently arrives

at a time uniformly dist'd between noon and 1pm. Find the chance that the first to arrive has to wait longer than 10 min.

M = man's arrival time $\sim$ Unif $[0, 60]$

W = woman's arrival time $\sim$ Unif $[0, 60]$

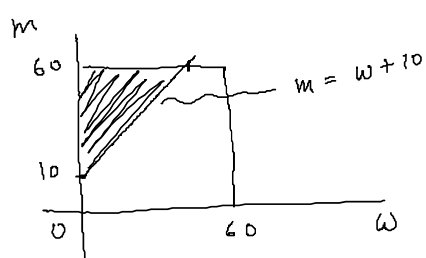
$M \perp\!\!\!\perp W$

$$\{ |W - M| > 10 \} = \{ W + 10 < M \} \cup \{ M + 10 < W \}$$

disjoint

By symmetry,

$$P(|W - M| > 10) = 2\, P(W + 10 < M)$$



$$= 2 \int_{10}^{60} \int_{0}^{m-10} \left(\frac{1}{60}\right)^2 dw\, dm =$$

$$= 25/36$$

//

Quiz next time! Cheat sheet!

Prepare (look for assignment) if

you are a hybrid student.