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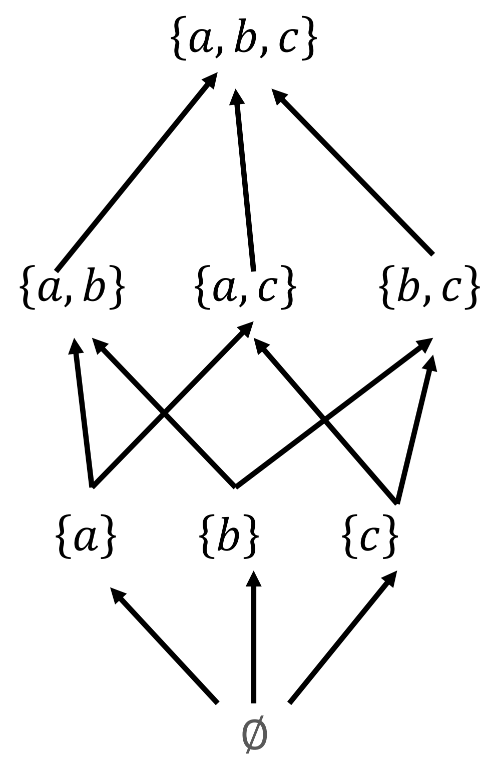

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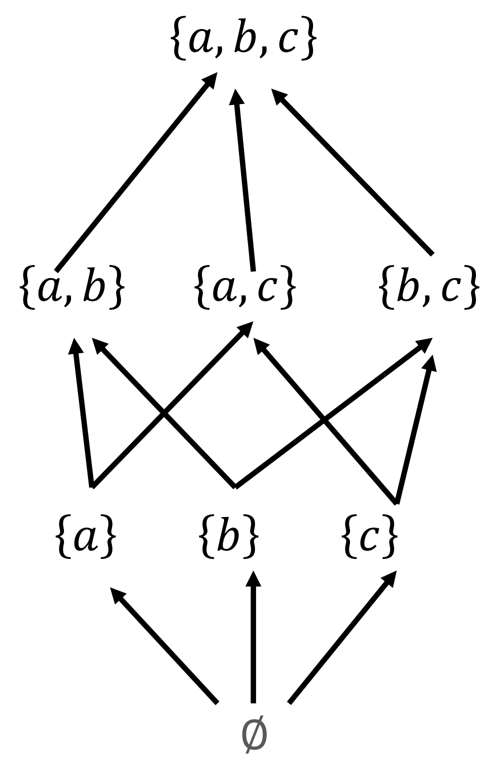

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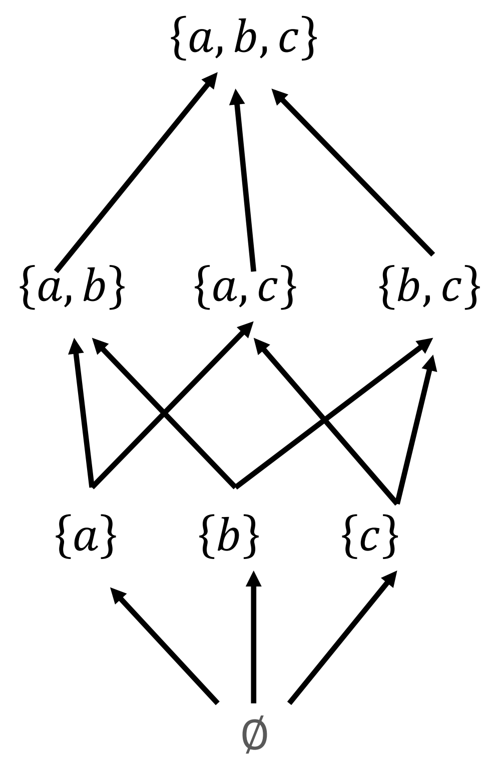

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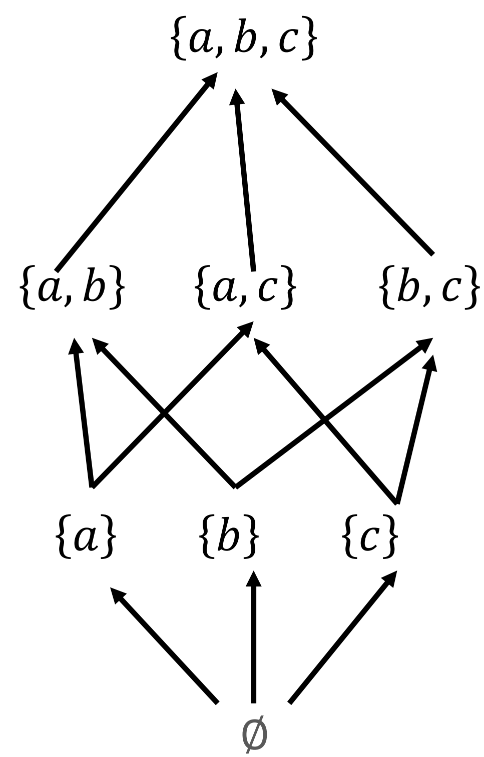
  - The function $H = \lambda x. \{a\} \setminus x$ has no fixpoints.



Fixpoint Theorem #1

- If $D = (S, \sqsubseteq)$ is a domain, $\perp$ is its least element, and $f: D \to D$ is monotonic, then f has a least fixpoint that is the largest element in the sequence (chain) $\left[\perp, f(\perp), f\big(f(\perp)\big), f\Big(f\big(f(\perp)\big)\Big), \ldots\right]$.

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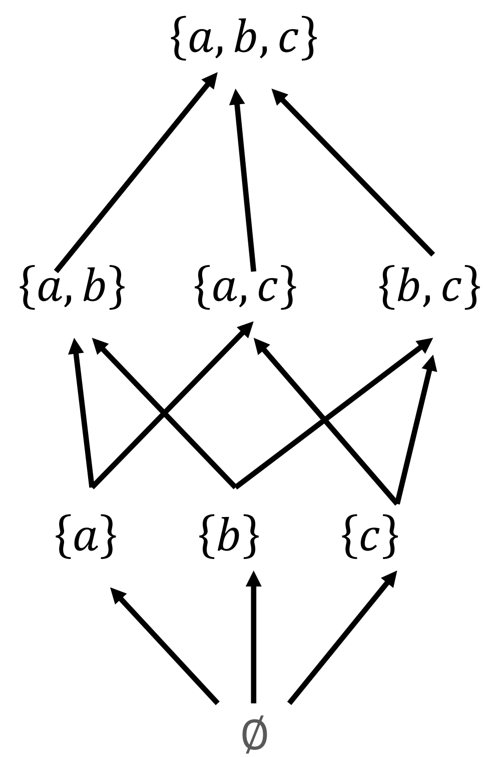

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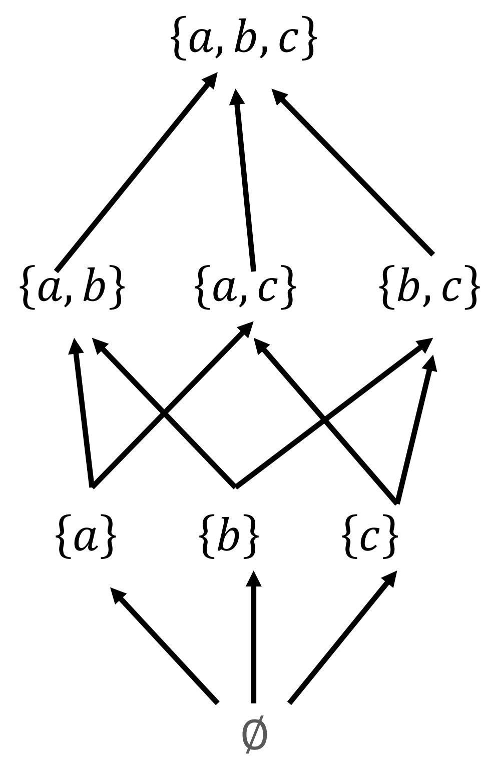

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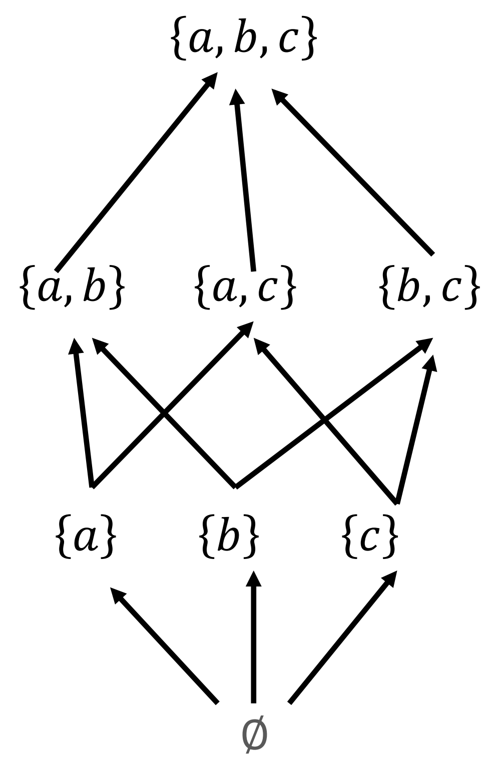

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- The largest element of the sequence is a fixpoint.
  - $\perp \sqsubseteq f(\perp)$ (by definition of $\perp$).
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  - $f\big(f(\perp)\big) \sqsubseteq f\Big(f\big(f(\perp)\big)\Big)$ (same argument).
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- This is the least fixpoint.
  - Let p be any other fixpoint of f.
  - $\perp \sqsubseteq p$ (from definition of $\perp$).
  - So $f(\perp) \sqsubseteq f(p) = p$ (by monotonicity of f).
  - Similarly, $f(f(\perp)) \sqsubseteq p$, etc.
  - Therefore, all elements of the chain are $\sqsubseteq p$, so largest element of chain must be $\sqsubseteq p$.
  - Therefore, the largest element of chain is the least fixpoint of f.

Solving A Fixpoint Equation

- If D is a domain and $f: D \to D$ is monotonic, then the equation $x = f(x)$ has a least solution given by the largest element in the sequence $[\perp, f(\perp), f\big(f(\perp)\big), f\Big(f\big(f(\perp)\big)\Big), \dots]$.
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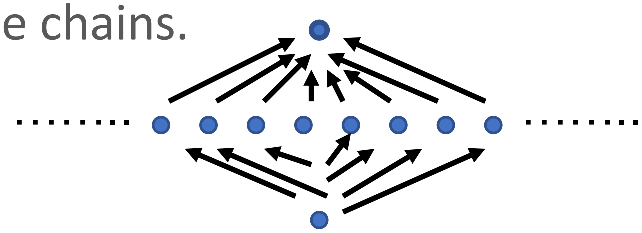

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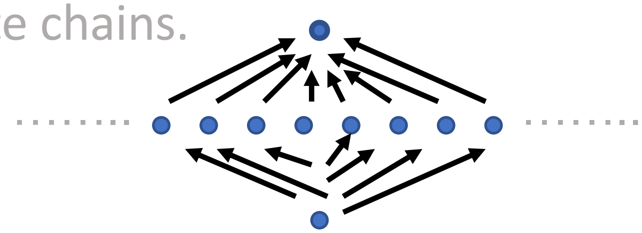

- Generalization #2: If D is a domain with a *greatest* element $\top$ and $f: D \to D$ is monotonic, then the equation $x = f(x)$ has a *greatest* solution given by the smallest element in the descending sequence $[\top, f(\top), f\big(f(\top)\big), f\Big(f\big(f(\top)\big)\Big), \ldots]$.

Functions with Multiple Arguments

- If D is a domain, a function $f: D \times D \rightarrow D$ that takes two arguments is said to be monotonic if it is monotonic in each argument when the other argument is held constant.
  - $\forall x_0, x_1, y \in D: x_0 \sqsubseteq x_1 \Longrightarrow f(x_0, y) \sqsubseteq f(x_1, y)$.
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- Fixpoint Theorem #2
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- This theorem generalizes to more than two equations and to the case when D has a greatest element T.

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- Check that D is a domain and that h is a monotonic function.