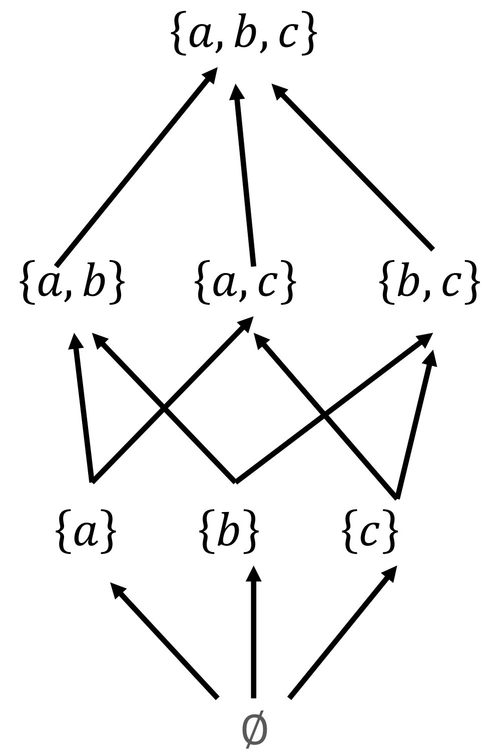
Power-Set Domains: $\cup$ and $\cap$

- Consider a power-set domain.
  - As set union and intersection are monotonic functions, we can use them in systems of fixpoint equations.



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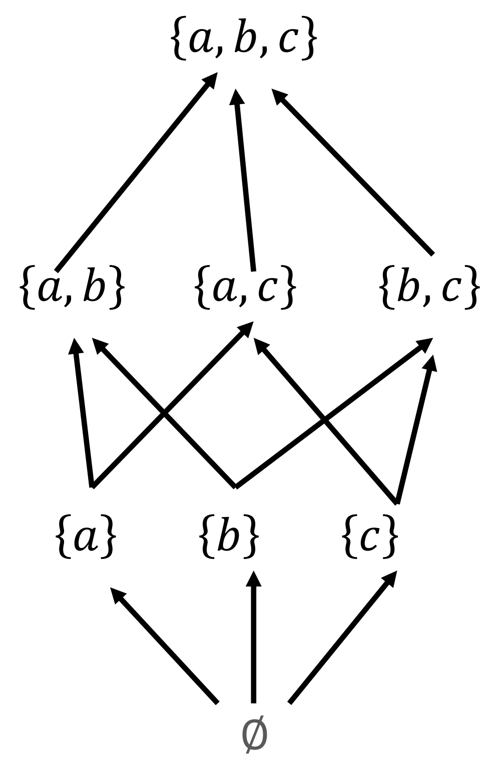
- Example:

$$f(x, y) = \{a\}$$

- Equations:

$$x = f(x, y)$$
$$y = x \cup y$$



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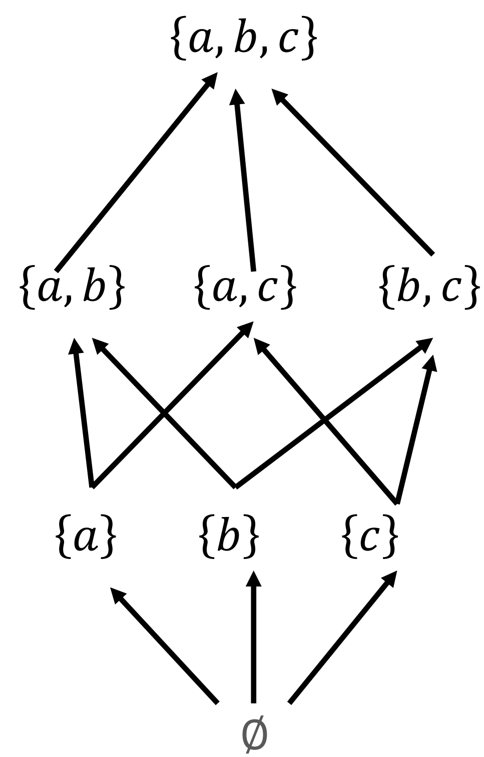
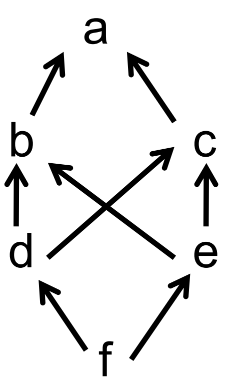
- Can we generalize this idea to domains that are not power-sets?



Meet and Join

- If $D = (S, \sqsubseteq)$ is a poset and $T \subseteq S$, then $\ell \in S$ is a *lower bound* of T if $\forall x \in T. \ell \sqsubseteq x$.
  - Example: lower bounds of $\{c, d\}$ are d and f.
- In general, a given T may have multiple lower bounds.



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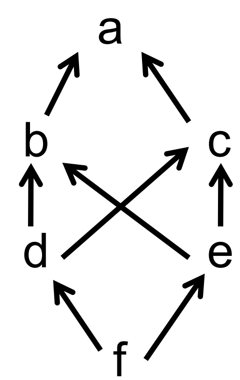
- The *greatest lower bound* (glb) of T is the greatest element of S that is a lower bound of T, if such an element exists.
  - Caveat: The glb may not always exist. E.g., d, e, and f are all lower bounds of $\{b, c\}$, but there is no glb.



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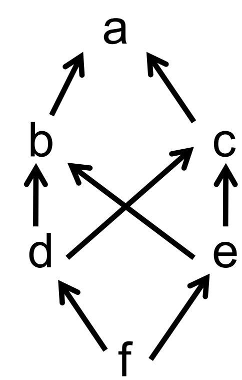
- If $\mathrm{glb}(\{x, y\})$ exists for every pair of elements $x, y \in S$, we can define a function called meet ($\wedge: D \times D \to D$) as $x \wedge y = \mathrm{glb}(\{x, y\})$.



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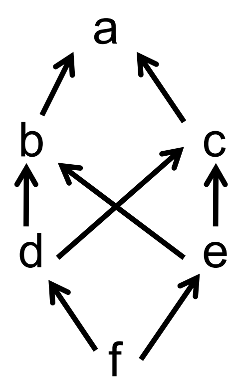
- A meet semilattice is a partially ordered set in which every pair of elements has a glb.



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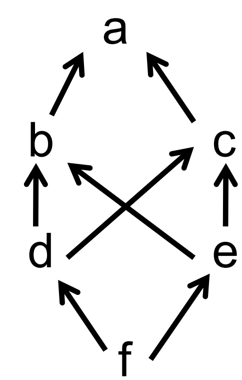
- Analogous notions:
  - upper bounds, least upper bounds, join ($\vee$), join semilattice.



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- A meet semilattice is a partially ordered set in which every pair of elements has a glb.
- Analogous notions:
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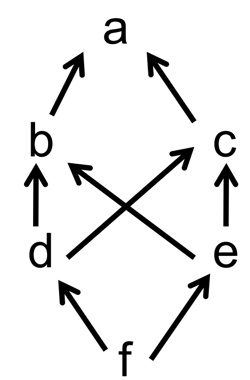
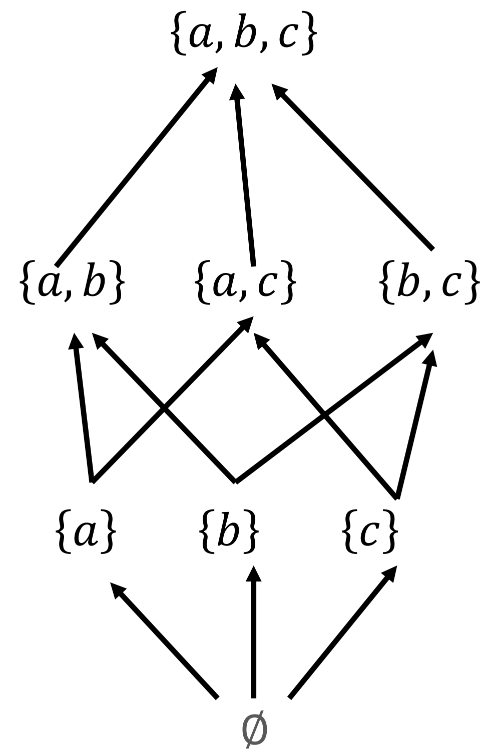
- A lattice is a poset that is both a meet and a join semilattice. We write $L = (D, \sqsubseteq, \wedge, \vee)$.



Back To Power-Set Domains

- The powerset of a finite set under subset ordering is a canonical example of a lattice, with $\perp = \emptyset$, $\wedge = \cap$, and $\vee = \cup$.



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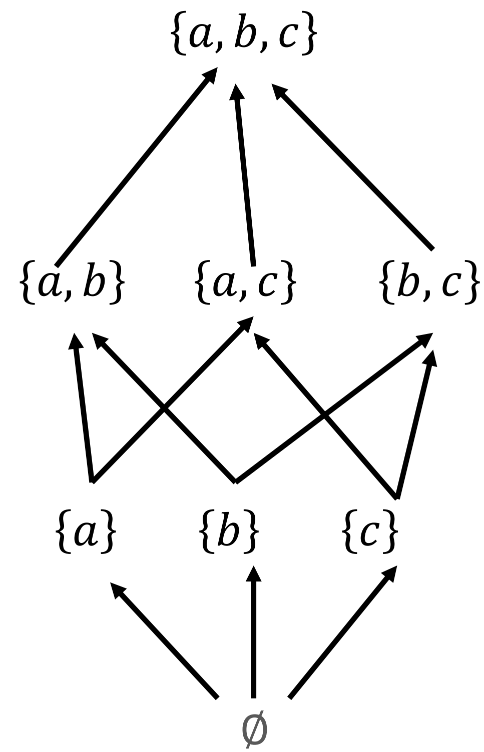
- The powerset of a finite set under subset ordering is a canonical example of a lattice, with $\perp = \emptyset$, $\wedge = \cap$, and $\vee = \cup$.
- If you "flip" this lattice over, you get another lattice, in which $\perp = \{a, b, c\}$, $\wedge = \cup$, and $\vee = \cap$.



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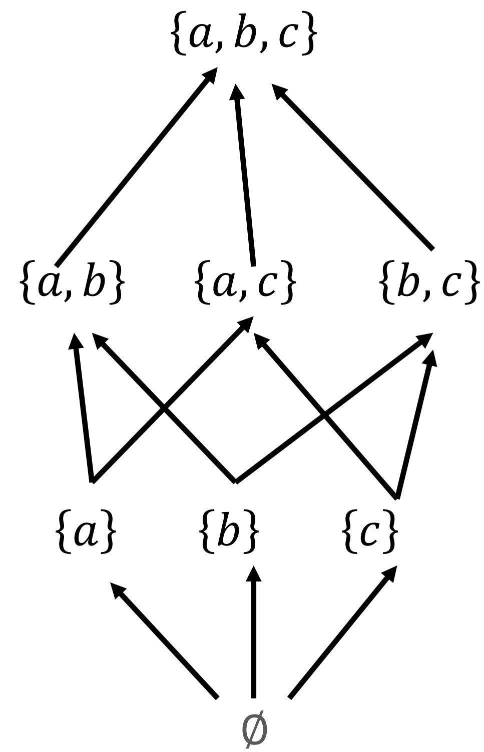
- The powerset of a finite set under subset ordering is a canonical example of a lattice, with $\bot = \emptyset$, $\wedge = \cap$, and $\vee = \cup$.
- If you "flip" this lattice over, you get another lattice, in which $\bot = \{a, b, c\}$, $\wedge = \cup$, and $\vee = \cap$.
- Another canonical example of a lattice is the set of divisors of a composite number under "is-a-divisor-of" ordering, with meet being GCD and join being LCM.



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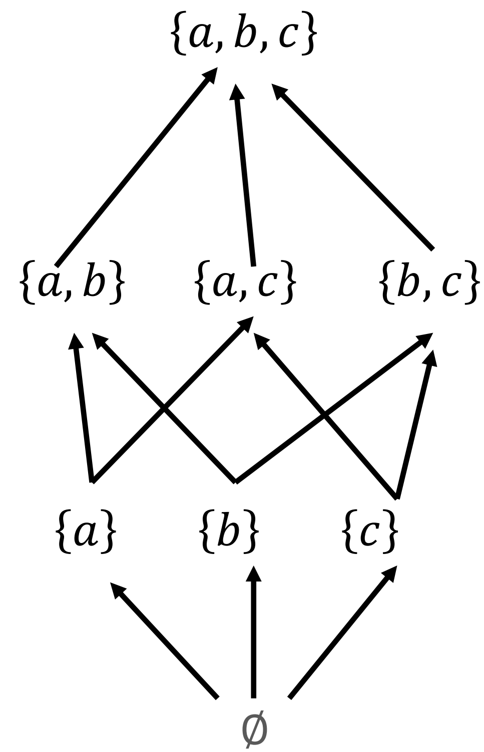

- Examples of posets that are not lattices
  - See previous slide.