Partially-Ordered Set

- A partially-ordered set (or poset) is a set S paired with a binary relation $\sqsubseteq$ that is
  - *reflexive*: $\forall x \in S: x \sqsubseteq x$.
  - *anti-symmetric*: $\forall x, y \in S: x \sqsubseteq y \land y \sqsubseteq x \Longrightarrow x = y$.
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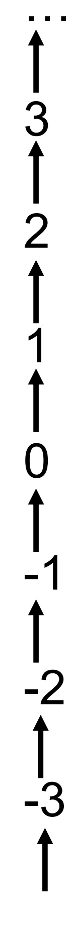
  - Poset generalizes this notion.
- Graphical representation of poset.
  - Nodes are elements of S. The relation $\sqsubseteq$ is shown by directed edges (i.e., $(x, y) \in E \equiv x \sqsubseteq y$).
  - We omit self-loops and transitive arrows to simplify the picture.



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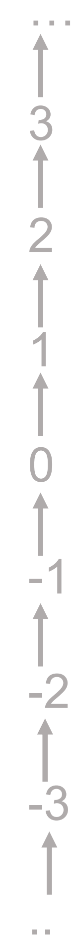
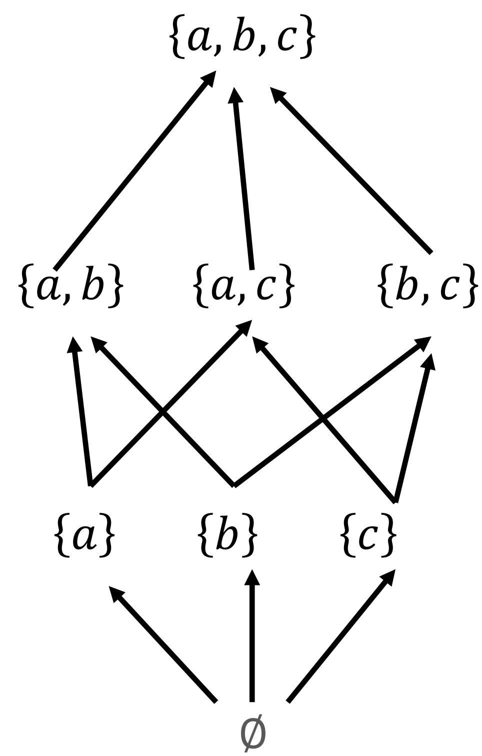
- Not a poset:
  - $S = \{a, b\}, \{a \sqsubseteq a, b \sqsubseteq b, a \sqsubseteq b, b \sqsubseteq a\}$.



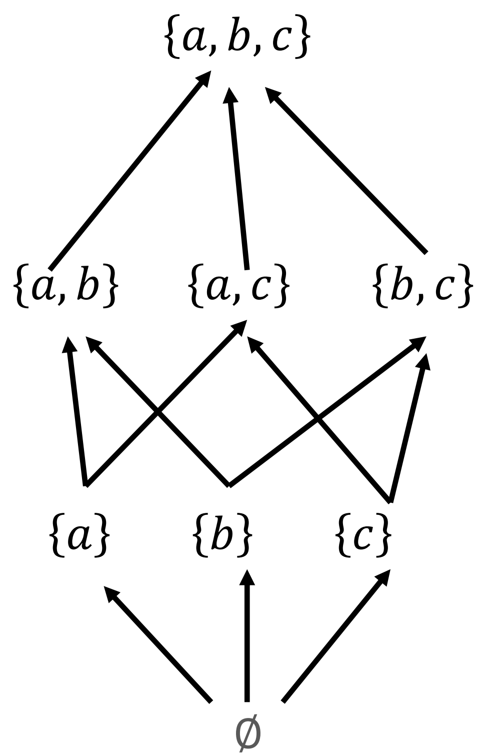
Another Example of A Poset

- The powerset of any set ordered by set containment is a poset.



Another Example of A Poset

- The powerset of any set ordered by set containment is a poset.
- In this example
  - The poset elements are $\emptyset$, $\{a\}$, $\{a, b\}$, $\{a, b, c\}$, etc.
  - $x \sqsubseteq y \equiv x \subseteq y$.



Domain

- A domain is a poset in which
  - the set S is finite; and
  - there is an element of S (denoted $\perp$) that is "below" all elements in the poset (i.e., $\forall x \in S: \perp \sqsubseteq x$).

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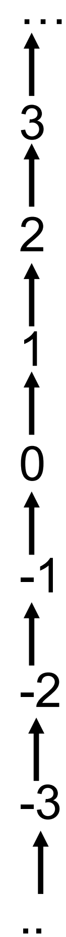
  - The set of prime numbers ordered by natural ordering is a poset but is not a domain, because the set is not finite.



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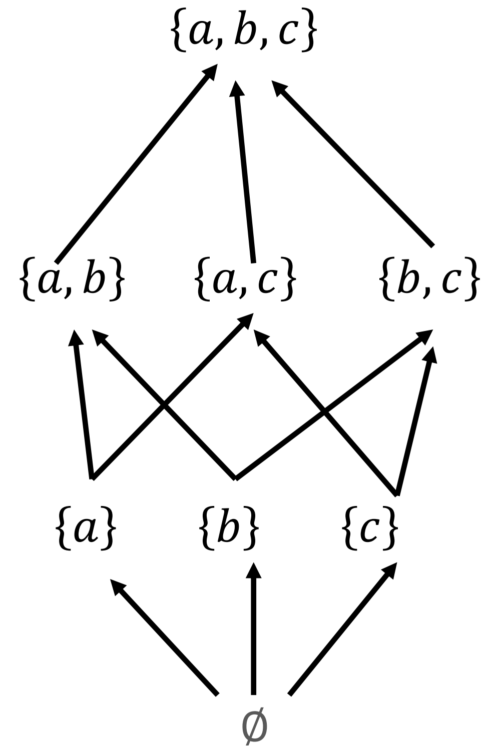

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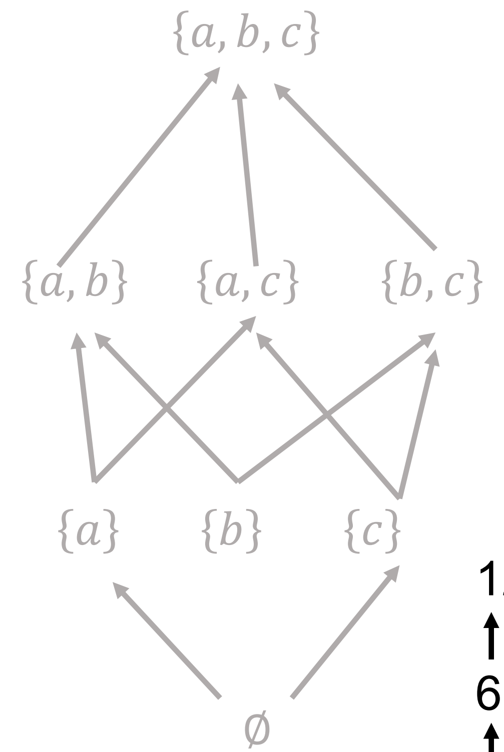
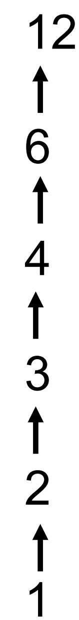
  - The factors of 12 ordered by natural ordering on integers is a poset and a domain (with $\bot = 1$).





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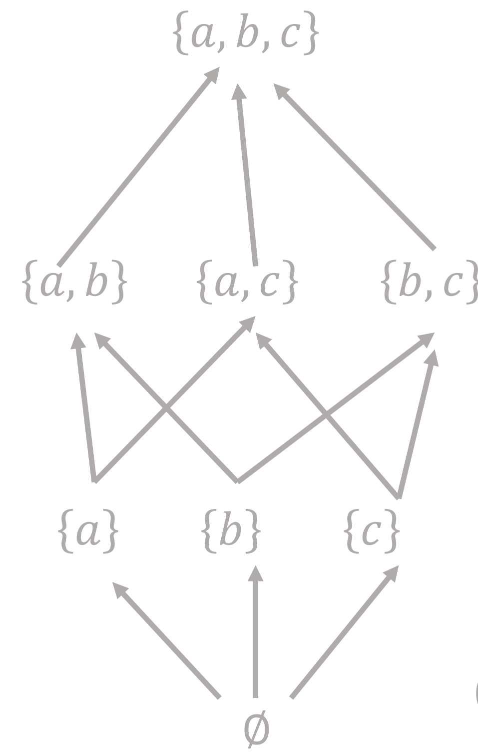
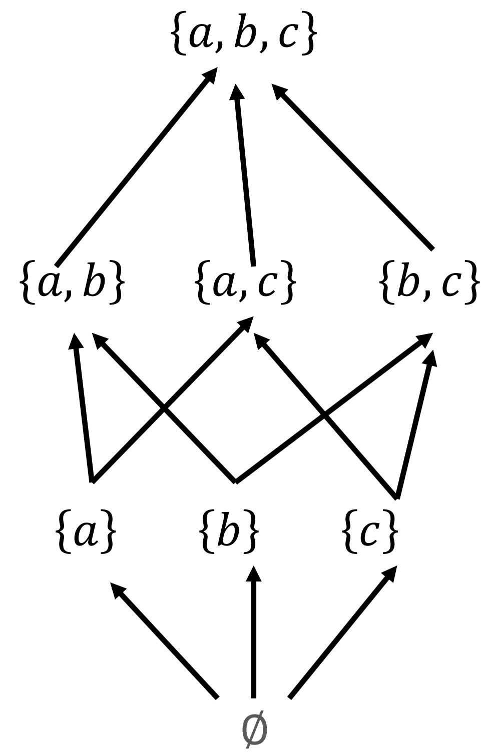
- We will write $D = (S, \sqsubseteq)$.



Functions on Domains

- If $D = (S, \sqsubseteq)$ is a domain, let $f: S \rightarrow S$ be a function that maps each element of S to some element of S itself.



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- For D = powerset of $\{a, b, c\}$ ordered by set containment:
  - $f_1(x) = x \cup \{a\}$.
    - So, $f_1(\emptyset) = \{a\}$, $f_2(b) = \{a, b\}$, etc.
  - $f_2(x) = x \setminus \{a\}$.
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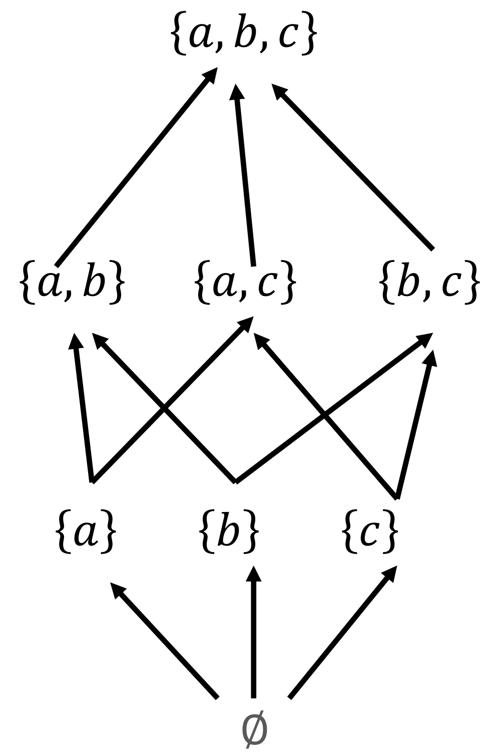

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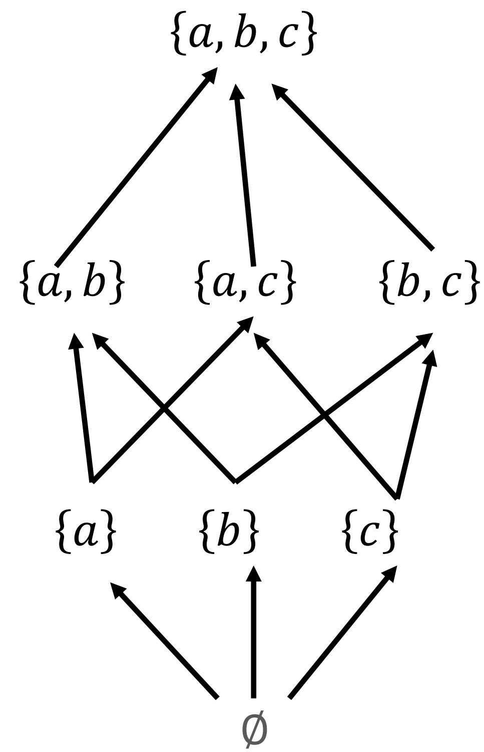
- Such a function f is said to be:
  - monotonic if $\forall x, y \in S: x \sqsubseteq y \rightarrow f(x) \sqsubseteq f(y)$.
  - extensive if $\forall x \in S: x \sqsubseteq f(x)$.



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- Examples
  - Function f_1 is extensive and monotonic.
  - Function f_2 is not extensive but monotonic.
  - Function f_3 is neither extensive nor monotonic.