DATA130004: Homework 1

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1. Suppose that X and Y are continuous random variables with density f and g respectively, and a and b are constants. Prove the following arguments:

   (a) $E(aX + b) = aE(X) + b$

   Proof: Since X is a continuous random variable with density $f(x)$, denote $Z = X + c$ which is also a continuous variable with density $f_z(x)$.

$$E(aX+b) = E(Z) = \int_{-\infty}^{\infty} x f_z(x)\mathrm{d}x$$

   If $a = 0$, then $E(b) = b$ will be undeniable. So assume $a \neq 0$.

$$\begin{aligned}
\because F_z(x) &= P\{Z \le x\} = P\{aX + b \le x\} = P\{aX \le x - b\} \\
&= \begin{cases} P\{X \le \dfrac{x-b}{a}\}(a > 0) \\ P\{X \ge \dfrac{x-b}{a}\}(a < 0) \end{cases} \\
&= \begin{cases} F_x\Big(\dfrac{x-b}{a}\Big)(a > 0) \\ 1 - F_x\Big(\dfrac{x-b}{a}\Big)(a < 0) \end{cases}
\end{aligned}$$

$$\begin{aligned}
\therefore f_z(x) &= \begin{cases} \frac{1}{a} f_x\Big(\frac{x-b}{a}\Big)(a > 0) \\ -\frac{1}{a} f_x\Big(\frac{x-b}{a}\Big)(a < 0) \end{cases} \\
&= \frac{1}{|a|} f_x\Big(\frac{x-b}{a}\Big)(a \neq 0)
\end{aligned}$$

$$E(aX+b) = \int_{-\infty}^{\infty} \frac{x}{|a|} f_x\Big(\frac{x-b}{a}\Big)\mathrm{d}x$$

   Denote $t = \frac{x-b}{a}$, which means $x = at + b$

$$\begin{aligned}
\Rightarrow \; E(aX+b) &= \int_{-\infty}^{\infty} \frac{at+b}{|a|} f_x(t) a \mathrm{d}x \\
&= a\int_{-\infty}^{\infty} t f_x(t)\mathrm{d}x + b\int_{-\infty}^{\infty} f_x(t)\mathrm{d}x \\
&= aE(X) + b
\end{aligned}$$

(b) $E(X+Y) = E(X) + E(Y)$

Proof: Assume the joint density of X and Y as $\varphi(x,y)$.

$$\begin{aligned}
E(X+Y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x+y)\varphi(x,y)\mathrm{d}x\mathrm{d}y \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x\varphi(x,y)\mathrm{d}x\mathrm{d}y + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}y\varphi(x,y)\mathrm{d}x\mathrm{d}y \\
&= \int_{-\infty}^{\infty}x\mathrm{d}x\int_{-\infty}^{\infty}\varphi(x,y)\mathrm{d}y + \int_{-\infty}^{\infty}y\mathrm{d}y\int_{-\infty}^{\infty}\varphi(x,y)\mathrm{d}x \\
&= \int_{-\infty}^{\infty}xf(x)\mathrm{d}x + \int_{-\infty}^{\infty}yg(y)\mathrm{d}y \\
&= E(X) + E(Y)
\end{aligned}$$

(c) if X and Y are independent, then $E(XY) = E(X)E(Y)$

Proof: Assume the joint density of X and Y as $\varphi(x,y)$. And since X and Y are independent continuous variables, $\varphi(x,y) = f(x)g(y)$.

$$\begin{aligned}
E(XY) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}xy\varphi(x,y)\mathrm{d}x\mathrm{d}y \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}xyf(x)g(y)\mathrm{d}x\mathrm{d}y \\
&= \int_{-\infty}^{\infty}xf(x)\mathrm{d}x\int_{-\infty}^{\infty}yg(y)\mathrm{d}y \\
&= E(X)E(Y)
\end{aligned}$$

(d) $Var(a) = 0$

Proof:

$$Var(a) = E\big(a - E(a)\big)^2 = E(a-a) = 0$$

(e) $Var(aX+b) = a^2Var(X)$

Proof:

$$\begin{aligned}
Var(aX+b) &= E\big[aX+b-E(aX+b)\big]^2 = E\big[aX+b-aE(X)-b\big]^2 \\
&= E\big[a\big(X-E(X)\big)\big]^2 = a^2E\big(X-E(X)\big)^2 \\
&= a^2Var(X)
\end{aligned}$$

(f) $Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$

Proof:

$$\begin{aligned}
Var(X+Y) &= E\big[X+Y-E(X+Y)\big]^2 = E\big[X-E(X)+Y-E(Y)\big]^2 \\
&= E\Big[\big(X-E(X)\big)^2 + \big(Y-E(Y)\big)^2 + 2\big(X-E(X)\big)\big(Y-E(Y)\big)\Big] \\
&= E\big[X-E(X)\big]^2 + E\big[Y-E(Y)\big]^2 + 2E\big[\big(X-E(X)\big)\big(Y-E(Y)\big)\big] \\
&= Var(X) + Var(Y) + 2Cov(X,Y)
\end{aligned}$$

(g) if X and Y are independent, then $Var(X+Y) = Var(X) + Var(Y)$

Proof: If X and Y are independent, then according to argument (c), E(XY)=E(X)E(Y). Therefore,

$$Cov(X,Y) = E\big[\big(X-E(X)\big)\big(Y-E(Y)\big)\big] = E(XY) - E(X)E(Y) = 0$$

$$\Rightarrow \ Var(X+Y) = Var(X) + Var(Y)$$

2. Rizzo book Exercises 3 (starting from Page 94): 3, 5.

   (a) The Pareto(a, b) distribution has cdf

$$F(x) = 1 - \left(\frac{b}{x}\right)^a, \ x \geq b > 0, a > 0$$

Derive the probability inverse transformation $F^{-1}(U)$ and use the inverse transform method to simulate a random sample from the Pareto(2, 2) distribution. Graph the density histogram of the sample with the Pareto(2, 2) density superimposed for comparison.

$$F^{-1}(U) = \frac{b}{(1-U)^{\frac{1}{a}}}$$
$$\Rightarrow \ F^{-1}_{(2,2)}(U) = \frac{2}{\sqrt{1-U}}$$

To simulate a random sample from the Pareto(2,2) distribution using inverse transform method, the R script are as follows:

```
n <- 10000
u <- runif(n)
x <- 2/((1-u)^(1/2))
hist(x, prob = TRUE, main = expression(f(x) == (4*x-8)/(x^3)))
y <- seq(0,120,0.01)
lines(y, (4*y-8)/(y^3))
```

By running the script, I got Figure 1.

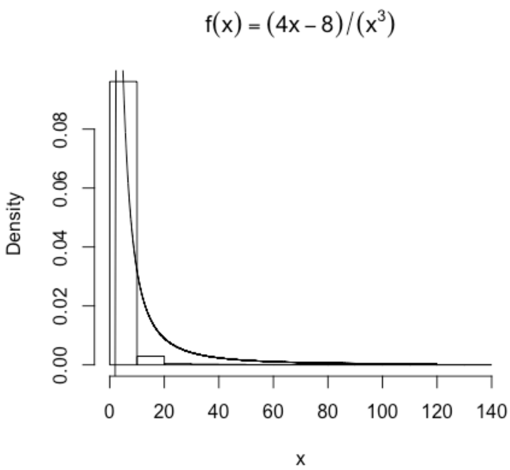

Figure 1: density histogram of sample and density function

(b) A discrete random variable X has probability mass function

x	0	1	2	3	4
$p(x)$	0.1	0.2	0.2	0.2	0.3

Use the inverse transform method to generate a random sample of size 1000 from the distribution of X. Construct a relative frequency table and compare the empirical with the theoretical probabilities. Repeat using the R sample function.

$$x = F_X^{-1}(u) = \begin{cases} 0, u \leq 0.1 \\ 1, 0.1 < u \leq 0.3 \\ 2, 0.3 < u \leq 0.5 \\ 3, 0.5 < u \leq 0.7 \\ 4, 0.7 < u \leq 1 \end{cases}$$

Here is the frequency table of the sample (sample size = 1000):

x	0	1	2	3	4
$p(x)$	0.103	0.222	0.198	0.185	0.292

Repeat using the R sample function:

```
x <- sample(c(0,1,2,3,4),1000,replace = TRUE, prob = c(0.1,0.2,0.2,0.2,0.3))
```

Here's the frequency table of the sample:

x	0	1	2	3	4
$p(x)$	0.093	0.208	0.198	0.197	0.304

Comparing theempirical with the theoretical probabilities, the two sample distributions I generated both quite close to the theoretical one, and one using the R sample function seems closer to the theoretical distribution. But more experiments needs to be done to examine this argument.