DATA130004: Homework 4

董晴园 14300680173

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1. Exercise 5.12: Let $\hat{\theta}_f^{IS}$ be an importance sampling estimator of $\theta = \int g(x)dx$, where the importance function f is a density. Prove that if $g(x)/f(x)$ is bounded, then the variance of the importance sampling estimator $\hat{\theta}_f^{IS}$ is finite.

   The variance of the importance sampling estimator is

$$Var(\hat{\theta}_f^{IS}) = Var\Big(\frac{1}{m}\sum_{i=1}^{m}\frac{g(X_i)}{f(X_i)}\Big) = \frac{1}{m}Var\Big(\frac{g(X)}{f(X)}\Big)$$

$$\begin{aligned}\therefore Var\Big(\frac{g(X)}{f(X)}\Big) &= E\Big(\frac{g^2(X)}{f^2(X)}\Big) - \Big[E\Big(\frac{g(X)}{f(X)}\Big)\Big]^2 \\ &= \int_{-\infty}^{\infty}\frac{g^2(x)}{f^2(x)}f(x)dx - \theta^2\end{aligned}$$

   Since $g(x)/f(x)$ is bounded, $\frac{g^2(x)}{f^2(x)}$ should also be bounded, which means the expectation of $\frac{g^2(x)}{f^2(x)}$ is finite. And also θ is finite, so the variance of $\hat{\theta}_f^{IS}$ is finite.

2. Exercise 5.14: Obtain a Monte Carlo estimate of

$$\int_1^{\infty}\frac{x^2}{\sqrt{2\pi}}e^{-x^2/2}dx$$

   by importance sampling.

   As can be indicated from the question,

$$g(x) = \frac{x^2}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}, \quad x \geq 1.$$

   Generate $X_I, ..., X_m$ from Truncated Standard Normal Distribution with density f:

$$f(x) = \frac{\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}}{1-\Phi(1)}, \quad x \geq 1.$$

   Thus, we can view the integration as the expectation of X:

$$\theta = \int_1^{\infty} g(x)dx = \int_1^{\infty}\frac{g(x)}{f(x)}f(x)dx = \big(1-\Phi(1)\big)\int_1^{\infty}x^2 f(x)dx$$

   Then, estimate it with

$$\hat{\theta}_f^{IS} = \frac{1}{m}\sum_{i=1}^{m}\frac{g(X_i)}{f(X_i)} = \frac{1-\Phi(1)}{m}\sum_{i=1}^{m}{X_i}^2, \quad X_i \sim f(x).$$

Here is the R code to realize such a estimate (using adverse transformation to generate X_i from $f(x)$):

```
rm(list=ls())
m <- 10000
u <- runif(m)
x <- qnorm(u*(1-pnorm(1))+pnorm(1))
fg <- (1-pnorm(1))*(x^2)
theta.hat <- mean(fg)
theta.hat
```

And the estimate result is 0.3963846.

3. Given two random variables X and Y , prove the law of total variance

$$Var(Y) = E\{Var(Y|X)\} + Var\{E(Y|X)\}.$$

Proof:

$$\begin{aligned}
Var(Y) &= E\Big\{E(Y^2|X) - \big[E(Y|X)\big]^2\Big\} + E\Big\{\big[E(Y|X)\big]^2\Big\} - \big[E(Y)\big]^2 \\
&= E(Y^2) - E\Big\{\big[E(Y|X)\big]^2\Big\} + E\Big\{\big[E(Y|X)\big]^2\Big\} - \big[E(Y)\big]^2 \\
&= E(Y^2) - \big[E(Y)\big]^2 = Var(Y)
\end{aligned}$$

4. (Importance sampling) Define $\theta = \int_A g(x)dx$, where A is a bounded set and $g \in \mathcal{L}_2(A)$. Let f be an importance function which is also a density function on the set A.

   (a) Describe the steps to obtain the importance sampling estimator $\hat{\theta}_n$, where n is the number of random samples generated during the process.

   Steps to obtain the importance sampling estimator $\hat{\theta}_n$:

   i. Generate $X_1, ..., X_n \sim^{i.i.d.} f$.

   ii. Estimator:

$$\hat{\theta}_n = \frac{1}{n}\sum_{i=1}^{n}\frac{g(X_i)}{f(X_i)}$$

   (b) Show that the Monte Carlo variance of $\hat{\theta}_n$ is

$$Var(\hat{\theta}_n) = \frac{1}{n}\left\{\int_A \frac{g^2(x)}{f(x)}dx - \theta^2\right\}$$

$$
\begin{aligned}
Var(\hat{\theta}_n) &= \frac{1}{n} Var\Big(\frac{g(X)}{f(X)}\Big) = \frac{1}{n}\Bigg\{ E\Big(\frac{g^2(X)}{f^2(X)}\Big) - \Big[E\big(\frac{g(X)}{f(X)}\big)\Big]^2 \Bigg\} \\
&= \frac{1}{n}\Bigg\{ \int_A \frac{g^2(X)}{f^2(X)} f(X)dx - \Big(\int_A \frac{g(X)}{f(X)} f(X)dx\Big)^2 \Bigg\} \\
&= \frac{1}{n}\Bigg\{ \int_A \frac{g^2(X)}{f(X)} dx - \theta^2 \Bigg\}
\end{aligned}
$$

(c) Show that the optimal importance function f^*, i.e., the minimizer of $Var(\hat{\theta}_n)$, is

$$
f^*(x) = \frac{|g(x)|}{\int_A |g(x)|dx},
$$

and derive the theoretical lower bound of $Var(\hat{\theta}_n)$.

Since θ is independent from $f(X)$, to minimize $Var(\hat{\theta}_n)$, we only need to minimize $\int_A \frac{g^2(X)}{f(X)}dx$.

$$
\int_A \frac{g^2(X)}{f(X)}dx = \int_A |g(X)| \frac{|g(X)|}{f(X)} dx
$$

Denote

$$
\int_A \phi(X)dx = \frac{|g(X)|}{f(X)}.
$$

Then according to Cauchy-Schwars Inequality,

$$
\int_A \frac{g^2(X)}{f(X)}dx = \int_A |g(X)|dx \int_A \phi(X)dx \geq \Bigg(\int_A \sqrt{|g(X)|}\sqrt{\phi(X)} \Bigg)^2,
$$

and the two sides are equal if and only if $\phi(X) = |g(X)|$. Therefore, the minimizer of $Var(\hat{\theta}_n)$ is

$$
f^*(x) = \frac{|g(x)|}{\int_A \phi(x)dx} = \frac{|g(x)|}{\int_A |g(x)|dx},
$$

and the lower bound of $Var(\hat{\theta}_n)$ is

$$
\frac{1}{n}\Bigg\{ \Big(\int_A |g(X)|dx \Big)^2 - \theta^2 \Bigg\}.
$$