

MAT1856/APM466 Assignment 1

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Fundamental Questions - 25 points

1.
 - (a) Governments issue bonds instead of just printing more money because in some countries the government is not allowed to print money (only the central bank can) and also to avoid inflation caused by increased money supply and limited goods since issuing bonds does not increase the supply of money.
 - (b) Regardless of market conditions, the yield curve tends to have a flatter tail because investors are more uncertain about the more distant future, but if the curve flattens (or even inverts), this could be because short term interest rates increase more than the long term interest rates due to increased volatility in the short term market or that investors expect long-term interest rates to drop due to slowed economic growth and could signal a recession [2].
 - (c) Quantitative Easing (QE) is a monetary policy where the central bank buys government bonds and other securities on the open market to increase the supply of money (more liquidity), lower interest rates, and increase economic activity; during the COVID-19 pandemic, the Fed used QE to increase its holdings to account for 56 percent of the Treasury issuance of securities through the first quarter of 2021 and eventually claimed that it would buy as much securities as needed to support smooth market functioning [6, 7].
2. The list of bonds I have chosen to construct the yield, spot, and forward curves are CAN 0.25 AUG 23, CAN 2.25 MAR 24, CAN 1.5 SEP 24, CAN 1.25 MAR 25, CAN 0.5 SEP 25, CAN 0.25 MAR 26, CAN 1.0 SEP 26, CAN 1.25 MAR 27, CAN 2.75 SEP 27, CAN 3.5 MAR 28. I chose these because their maturity dates are closest to 6 months apart so their cash flows match most closely. This way, I can approximately use the previous zeros (spots) calculated using bootstrapping of the bonds with earlier maturity dates to discount the cash flows of the bonds with later maturity dates. Notice that I skipped the CAN 1.75 MAR 23 bond since that is approximated to have a time to maturity of 0 years. I also picked the bonds with the most similar coupon rates (0.25-3.5 percent) and issuance dates (2018-2022) because it makes the bonds more comparable (more of the same "class" or "market" of bonds). Specifically, bonds with a much higher coupon payment has a larger reinvestment risk since the yield to maturity assumes that the investor reinvests the coupon payments back at a ytm rate. Very different issuance dates of the bonds may effect the liquidity of the bond and introduce a liquidity risk premium in the bond prices because investors are more willing to buy an on-the-run bond.
3. PCA decomposes the covariance matrix to a new set of independent vectors (eigenvectors) that capture the most variability in the data. Larger eigenvalues mean more variability along that direction (the eigenvector explains more variance in the data). In the case of the stochastic processes, the eigenvalues indicate how much the stochastic curve changes in the direction of each eigenvector. The eigenvectors in the case of the stochastic processes describe the directions that the stochastic curve changes the most (such as constant shifts in all the processes, or slopes and curves).

Empirical Questions - 75 points

4.

- (a) Using the bond data collected, we can construct the yield to maturity curve [5] (Figure 1) by solving for the IRR of the present value equation of the bonds using Python (see code in repository). The graph uses linear interpolation to interpolate the missing values since that is the easiest method, provides smooth estimates for values between observed points, and is robust to outliers. (We use the same interpolation method for the spot and forwards curve construction.) Notice that the yield curve is inverted and peaks at 1 year. This is because people expect the long term interest rate to be lower than the short term rate perhaps because short term interest rate may be increased further (within 1 year) to combat the current inflation and then later decreased as inflation and economic growth slows (this explains the peak at 1-yr). This signifies a recession. Over the 11 day period, there seems to be slight shifts vertically of the ytm curve.

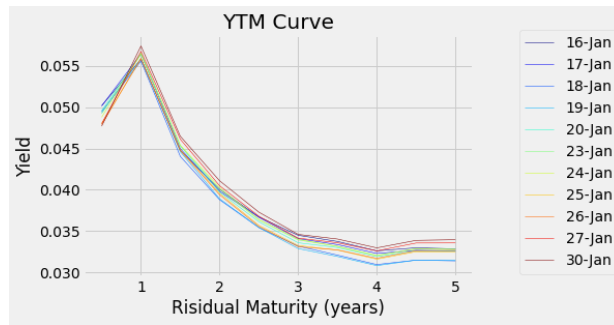


Figure 1: YTM Curve

- (b) Using the bond data collected, we can construct the spots curve [4] (Figure 2) through bootstrapping (Algorithm 1). Notice that the spot curve values are slightly lower than that of the ytm but follows the same general shape as the yield curve for the same reasons as explained in part (a). The spots being lower than the ytm is usually the case for inverted yield curves because in general, assuming annual coupon payments, the ym_n is the rate such that

$$Price = \frac{c}{1 + ytm_n} + \frac{c}{(1 + ytm_n)^2} + \dots + \frac{c + 100}{(1 + ytm_n)^n}$$

and $spot_n$ is the rate z_n such that

$$Price = \frac{c}{1 + z_1} + \frac{c}{(1 + z_2)^2} + \dots + \frac{c + 100}{(1 + z_n)^n}$$

so ym_n can be thought of as some sort of "average" of the z_1, z_2, \dots, z_n 's and since $z_1 > z_2 > \dots > z_n$ (because the yield curve is inverted), ym_n must be larger than z_n .

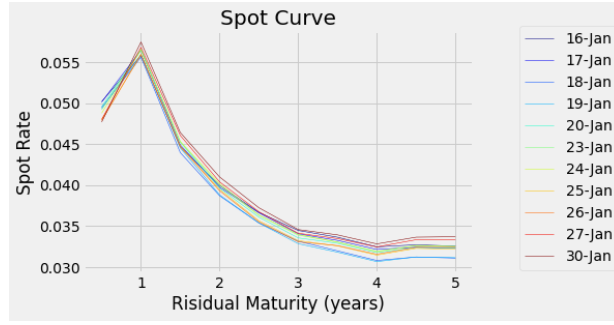


Figure 2: Spots Curve

Algorithm 1 Deriving Spot Curve

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1: function GETSPOTCURVE(prices, coupons)                                ▷ get the spot rates for one day of bond data
2:   zeros  $\leftarrow \square$                                               ▷ initialize empty zeros list
3:    $z_1 \leftarrow \left( \frac{\frac{coupons[1]}{2} + 100}{prices[1] - \frac{coupons[1]}{2}} - 1 \right) * 2$     ▷ calculate the first zero
4:   append  $z_1$  to zeros
5:   for i in [2, 3, ..., 10] do                                       ▷ for each of the maturity dates
6:     price  $\leftarrow prices[i]$ 
7:     coupon  $\leftarrow coupons[i]/2$ 
8:     frontPV  $\leftarrow$  GETPV(coupon, i - 1, zeros)                 ▷ get PV for all CF except for last
9:      $z \leftarrow \left( \sqrt[i]{\frac{coupon + 100}{price - frontPV}} - 1 \right) * 2$ 
10:    append z to zeros
11:   end for
12:   return zeros
13: end function

14: function GETPV(coupon, nperiods, zeros)                            ▷ get the present value of all cash flows up to nperiods
15:   pv  $\leftarrow 0$ 
16:   pv  $\leftarrow p_v + coupon$ 
17:   for i in [1, 2, ..., nperiods] do                                ▷ for each cashflow period
18:     pv  $\leftarrow p_v + \frac{coupon}{(1 + \frac{zeros[i]}{2})^i}$ 
19:   end for
20:   return pv
21: end function

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- (c) We can construct the forward zero curve [3] (Figure 3b) by using our zero (spot) rates calculated earlier to derive the forward rates (figure 3a) and use that to get the one-year forward zero rates (Algorithm 2). Notice the relationship where the $(1 + \frac{0zero_n}{2})$ equals the geometric mean of $(1 + \frac{0forward_{0.5}}{2})$, $(1 + \frac{0.5forward_1}{2})$, ..., $(1 + \frac{n-0.5forward_n}{2})$; and the forward zero $(1 + \frac{1zero_n}{2})$ equals the geometric mean of $(1 + \frac{1forward_{1.5}}{2})$, $(1 + \frac{1.5forward_1}{2})$, ..., $(1 + \frac{n-0.5forward_n}{2})$. We see that the 1-yr forward zero rates are upward sloping unlike the ytm. This could mean that the market expects the inversion of the yield curve to be temporary and expect the economic conditions to improve in one year, leading to the usual upward-sloping rates as we would normally expect where investors are compensated with a higher interest for locking in their money for a longer period. There also seems to be slight vertical shifts of the forward curves over the 11 day period.

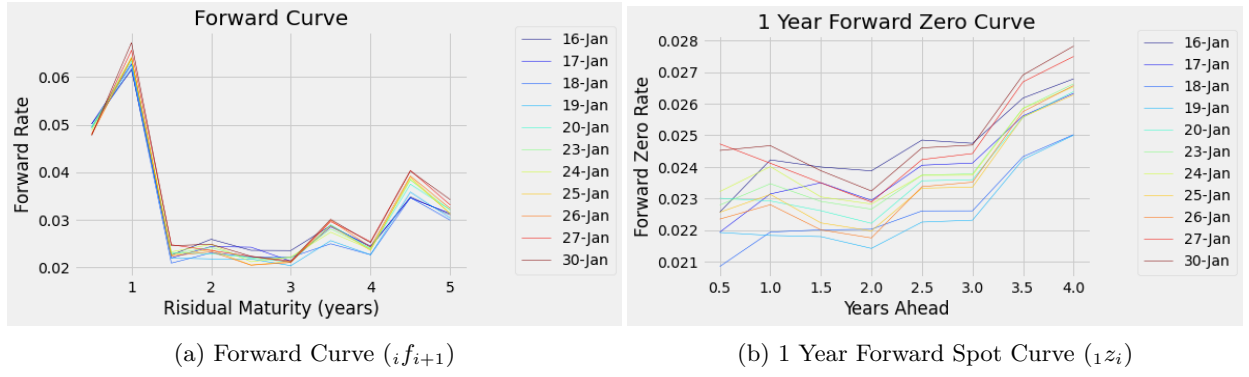


Figure 3: Forward Curves

Algorithm 2 Deriving Forward Spot Curve

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1: function GETONEYEARFORWARDZEROS(forwards)  ▷ get 1-year forward zeros given forward rates
2:   forwardzeros  $\leftarrow []$   ▷ initialize empty forward zeros list
3:   for i in [3, ..., 10] do  ▷ for each of the maturity dates starting 1 year from now
4:     forwardproduct  $\leftarrow 1$ 
5:     for j in [3, ..., i] do  ▷ for each of the maturity dates until i
6:       forwardproduct  $\leftarrow$  forwardproduct *  $(1 + \frac{\text{forwards}[j]}{2})$ 
7:     end for
8:     forwardzi  $\leftarrow (\sqrt[i]{\text{forwardproduct}} - 1) * 2$ 
9:     append forwardzi to forwardzeros
10:  end for
11:  return forwardzeros
12: end function

13: function GETFORWARDCURVE(zeros)  ▷ get the forward rates for one day of bond data
14:   forwards  $\leftarrow []$   ▷ initialize empty forwards list
15:   f1  $\leftarrow$  zeros[1]  ▷ first forward is same as first zero
16:   append f1 to forwards
17:   for i in [2, 3, ..., 10] do  ▷ for each of the maturity dates
18:     factor  $\leftarrow$  GETPREVIOUSFORWARDSPRODUCT(forwards, i - 1)
19:     f  $\leftarrow [((1 + \frac{\text{zeros}[i]}{2})^i * \text{factor}) - 1] * 2$ 
20:     append f to forwards
21:   end for
22:   return forwards
23: end function

24: function GETPREVIOUSFORWARDSPRODUCT(forwards, nperiods)  ▷ get the product of all the (1 +
   forward rates) for the first nperiods
25:   factor  $\leftarrow 1$ 
26:   for i in [1, 2, ..., nperiods] do  ▷ for each of the maturity dates
27:     factor  $\leftarrow$  factor /  $(1 + \frac{\text{forwards}[i]}{2})$ 
28:   end for
29:   return factor
30: end function

```

5. See below for the covariance matrices of the YTM log returns (Table 1) and the 1-YR forward zeros log returns (Table 2).

Table 1: Covariance Matrix of YTM Log Returns

1.1111	0.9362	0.8522	0.8762	0.8711
0.9362	1.1111	1.0845	1.0362	1.0184
0.8522	1.0845	1.1111	1.0231	0.9974
0.8762	1.0362	1.0231	1.1111	1.1067
0.8711	1.0184	0.9974	1.1067	1.1111

Table 2: Covariance Matrix of 1-YR Forward Zero Log Returns

1.1111	1.0750	0.9914	0.9600
1.0750	1.1111	0.9733	0.9306
0.9914	0.9733	1.1111	1.1040
0.9600	0.9306	1.1040	1.1111

6. The first eigenvalue of the ytm covariance matrix shows that the first principal component explains 91 percent of the total variance and its first eigenvector represents the situation that all rates in the yield curve move in the same direction by relatively the same amount (level shift) (Table 3), while the first eigenvalue of the 1-YR forward zeros covariance matrix shows that its first principal component explains 93 percent of total variance and its first eigenvector similarly represents the situation that all rates in the forward zero curve move in the same direction by relatively the same amount since all the values have the same sign and are around the same in magnitude (Table 4) [1].

Eigenvectors	-0.41040864	-0.89228493	0.10231514	0.15769443	0.0075125
	-0.46074175	0.02541976	-0.42663837	-0.77572515	-0.05745497
	-0.45077206	0.24491404	-0.60163085	0.5963938	0.13847894
	-0.45828112	0.27087663	0.39908526	0.11620089	-0.73745005
	-0.45394355	0.26424309	0.53505381	-0.06476692	0.65850855
Eigenvalue	5.04	0.33	0.17	0.02	0.003

Table 3: YTM Covariance Matrix Eigen Decomposition

Eigenvectors	-0.50101216	-0.44759861	-0.73930376	-0.04552194
	-0.49521041	-0.54839202	0.65893737	0.14083459
	-0.50632263	0.42516427	0.13120093	-0.73868739
	-0.49738402	0.56405409	-0.04491996	0.65759738
Eigenvalue	4.13	0.28	0.04	0.00

Table 4: 1YR Forward Zeros Covariance Matrix Eigen Decomposition

References

- [1] Pca for yield curve modelling. URL: <https://www.moodyanalytics.com/-/media/whitepaper/2014/2014-29-08-pca-for-yield-curve-modelling.pdf>.
- [2] Flat yield curve, 2023. URL: <https://corporatefinanceinstitute.com/resources/fixed-income/flat-yield-curve/>.
- [3] J. Chen. Forward rate: Definition, uses, and calculations, 2022. URL: <https://www.investopedia.com/terms/f/forwardrate.asp>.
- [4] A. Hayes. Spot rate treasury curve: Definition, uses, example, and formula, 2022. URL: https://www.investopedia.com/terms/s/spot_rate_yield_curve.asp.

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- [6] E. Milstein and D. Wessel. What did the fed do in response to the covid-19 crisis?, 2022. URL: <https://www.brookings.edu/research/fed-response-to-covid19/>.
- [7] T.I. Team. What is quantitative easing (qe), and how does it work?, 2023. URL: <https://www.investopedia.com/terms/q/quantitative-easing.asp>.

GitHub Link to Code

<https://github.com/qylisayu/Fixed-Income>