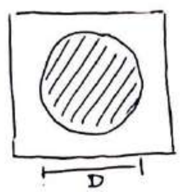
Report.pdf (A3)

In solving the questions in this assignment, I worked together with my classmate Addison Weatherhead, 1005691128. I confirm that I have written the solutions/- code/report in my own words. – Lisa Yu, 1005786366, Yulisa5

Q1)

I : 

$$LoG = \frac{1}{\pi\sigma^4}\left(\frac{x^2+y^2}{2\sigma^2}-1\right)e^{-\left(\frac{x^2+y^2}{2\sigma^2}\right)}$$

1.1) white pixel = 255
black pixel = 0

(normalized).

$$\sigma^2(LoG)*I = \sigma^2\int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty}\frac{1}{\pi\sigma^4}\left(\frac{x^2+y^2}{2\sigma^2}-1\right)e^{-\left(\frac{x^2+y^2}{2\sigma^2}\right)}\cdot\mathbb{1}\left(x^2+y^2>\left(\tfrac{D}{2}\right)^2\right)\cdot 255\,dx\,dy$$

in polar coordinates:

$$=\sigma^2\int_{r=\frac{D}{2}}^{\infty}\int_{\theta=0}^{2\pi}\frac{1}{\pi\sigma^4}\left(\frac{r^2}{2\sigma^2}-1\right)e^{\frac{-r^2}{2\sigma^2}}\cdot 255\, r\, d\theta\, dr$$

since inside circle is 0.

$$=\sigma^2\int_{r=\frac{D}{2}}^{\infty}\frac{510r}{\sigma^4}\left(\frac{r^2}{2\sigma^2}-1\right)e^{\frac{-r^2}{2\sigma^2}}\,dr$$

$$=\frac{255D^2e^{-\left(\frac{D^2}{8\sigma^2}\right)}}{4\sigma^4}$$

to maximize/minimize this,
take derivative w.r.t. σ:

$$\frac{d}{d\sigma}(LoG*I) = \frac{-(2040D^2\sigma^2-255D^4)e^{-\frac{D^2}{8\sigma^2}}}{16\sigma^5} \overset{set}{=} 0$$

$$\Rightarrow 2040D^2\sigma^2-255D^4=0$$

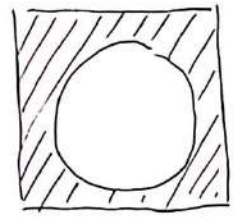
$$\Rightarrow \sigma^2=\frac{255D^4}{2040D^2}=\frac{D^2}{8}$$

$$\Rightarrow \boxed{\sigma=\sqrt{\frac{D^2}{8}}=\frac{D}{\sqrt{8}}}$$ since negative σ is not possible.

* note that $\frac{d^2}{d\sigma^2}(LoG*I)$ at $\sigma=\frac{D}{\sqrt{8}}$ is negative, so this is the local maximum. since this is the only first derivative equals zero, we can conclude that this is the global maximum.

* also note that this is when the zero of the LoG matches with the edge of the circle. (I.e. when $LoG(\sigma)_{x,y}=0 \Leftrightarrow x^2+y^2=\left(\frac{D}{2}\right)^2$).

1.2)

$I =$ 

D

again, $LoG = \frac{1}{\pi\sigma^4}\left(\frac{x^2+y^2}{2\sigma^2}-1\right)e^{-\left(\frac{x^2+y^2}{2\sigma^2}\right)}$

white pixel = 255
black " = 0.

$$\sigma^2(LoG) * I = \sigma^2\int_{r=0}^{\frac{D}{2}}\int_{\theta=0}^{2\pi} \frac{1}{\pi\sigma^4}\left(\frac{x^2+y^2}{2\sigma^2}-1\right)e^{-\frac{(x^2+y^2)}{2\sigma^2}}\cdot 255\, r\, d\theta\, dr$$

$$=\sigma^2\int_{r=\frac{D}{2}}^{\infty}\frac{510\, r}{\sigma^4}\left(\frac{r^2}{2\sigma^2}-1\right)e^{-\frac{r^2}{2\sigma^2}}\, dr$$

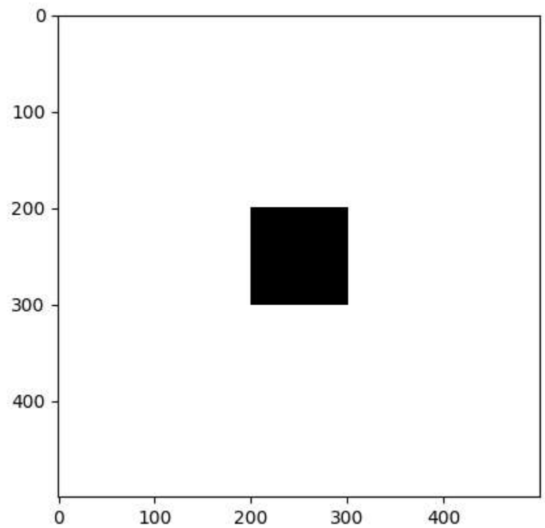
$$=\frac{-255\cdot D^2 e^{-\frac{D^2}{8\sigma^2}}}{4\sigma^4}$$

, which is the negative version of $\sigma^2(LoG)*I$ for I = (black circle on white) in previous part.

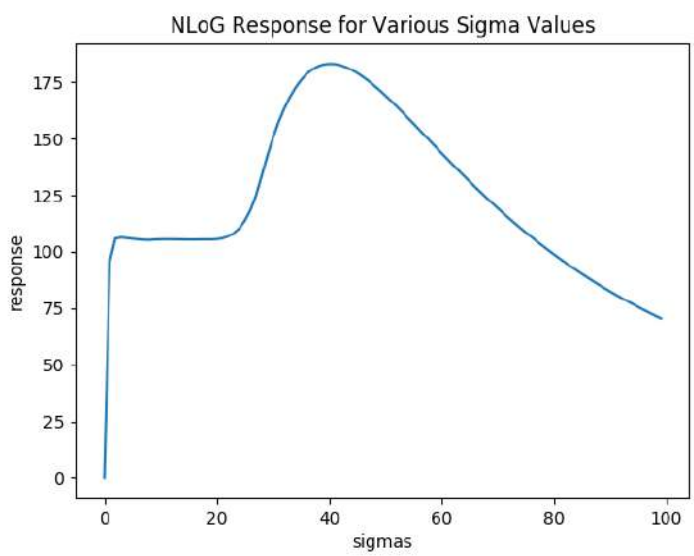
so ~~###~~ the global extrema for this as a function of σ is when $\boxed{\sigma = \sqrt{\frac{D^2}{8}} = \frac{D}{\sqrt{8}}}$, same as in 1.1.

except here, this would be the global minimum instead.

Q1.3)



The left shows the image constructed.



The optimal sigma value is 40

2) $M = \sum_x \sum_y w(x,y) \begin{pmatrix} I_x^2 & I_xI_y \\ I_xI_y & I_y^2 \end{pmatrix}$, $N = \begin{pmatrix} I_x^2 & I_xI_y \\ I_xI_y & I_y^2 \end{pmatrix}$

2.1).

$\det(N) = I_x^2 I_y^2 - (I_xI_y)^2 = I_x^2 I_y^2 - I_x^2 I_y^2 = 0$

$\mathrm{trace}(N) = I_x^2 + I_y^2$.

We can prove that $\det(N) = \lambda_1 \cdot \lambda_2$ and $\mathrm{trace}(N) = \lambda_1 + \lambda_2$.

WTS $\det(N) = \lambda_1 \lambda_2$:

since N is symmetric, $N = V \Sigma V^{-1}$ by the ~~eigenvalue~~ eigen decomposition where $\Sigma = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$.

$\Rightarrow \det(N) = \det(V) \det(\Sigma) \det(V^{-1})$

$= \det(V) \det(\Sigma) \frac{1}{\det(V)} = \det(\Sigma) = \det \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \lambda_1 \cdot \lambda_2$.

since $\det(V) \det(V^{-1}) = \det(V \cdot V^{-1}) = \det(I) = 1$

WTS $\mathrm{trace}(N) = \lambda_1 + \lambda_2$

since N is symmetric, $N = V \Sigma V^{-1}$ by eigen decomposition, $\Sigma = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$.

$\Rightarrow \mathrm{trace}(N) = \mathrm{trace}(V \Sigma V^{-1}) = \mathrm{trace}(\Sigma V^{-1} V) = \mathrm{trace}(\Sigma I) = \mathrm{trace}(\Sigma)$

$= \mathrm{trace} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \lambda_1 + \lambda_2$

since $\mathrm{trace}(AB) = \mathrm{trace}(BA)$

so $\lambda_1 \lambda_2 = 0$ and $\lambda_1 + \lambda_2 = I_x^2 + I_y^2$

$\Rightarrow \lambda_1 = 0$ and $\lambda_2 = I_x^2 + I_y^2$ (without loss of generality).

2.2). WTS. M is positive semi-definite.

i.e. WTS $\vec{v}^T M \vec{v} \geq 0$ for any $\vec{v}$

Let $\vec{v}$ be any vector $\in \mathbb{R}^2$

$\vec{v}^T M \vec{v} = \vec{v}^T \left(\sum_x \sum_y \left[w(x,y) \begin{pmatrix} I_x^2 & I_xI_y \\ I_xI_y & I_y^2 \end{pmatrix} \right] \right) \vec{v}$

$= \vec{v}^T \left(\sum_x \sum_y \left[w(x,y) \begin{pmatrix} I_x^2 & I_xI_y \\ I_xI_y & I_y^2 \end{pmatrix} \vec{v} \right] \right)$ } by the distributive property of matrix multiplication.

$= \sum_x \sum_y \left[\vec{v}^T w(x,y) \underbrace{\begin{pmatrix} I_x^2 & I_xI_y \\ I_xI_y & I_y^2 \end{pmatrix}}_{N} \vec{v} \right]$

Since N is symmetric and N's eigenvalues $\lambda_1, \lambda_2 \geq 0$

$\Rightarrow$ N is positive semidefinite for any x, y.

$\Rightarrow$ $\vec{v}^T N \vec{v} \geq 0$ for all $\vec{v}, x, y$.

also, since $w(x,y) \geq 0$ for all x, y pairs,

$\Rightarrow$ $\vec{v}^T w(x,y) N \vec{v} \geq 0$ for all $x, y, \vec{v}$. ✪

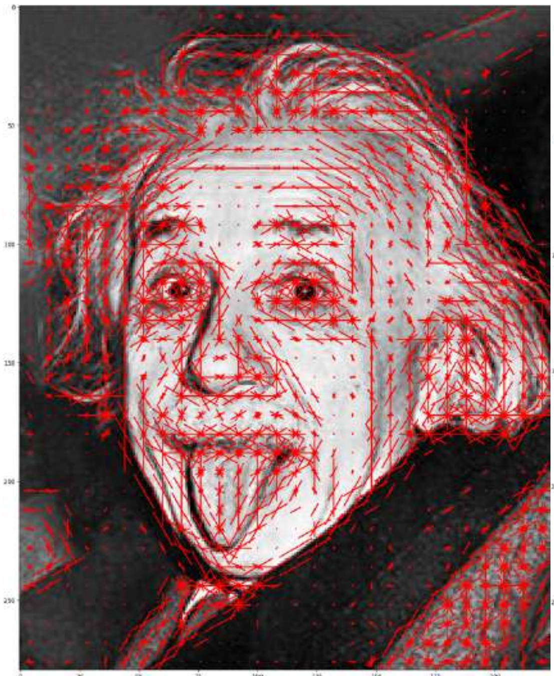
we had previously that

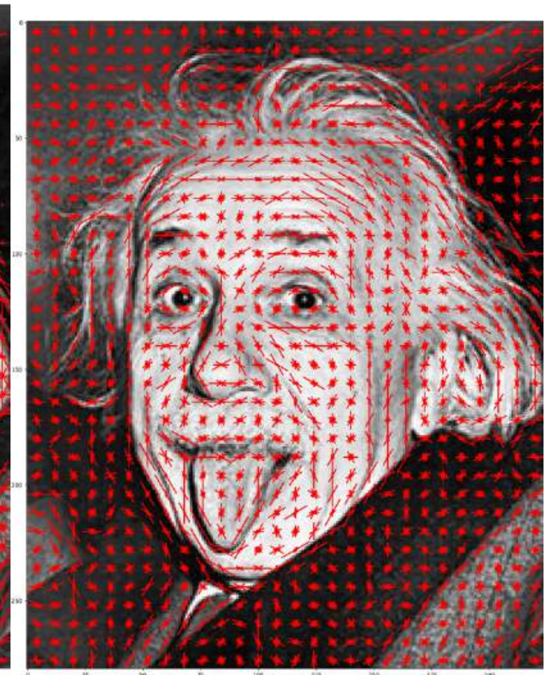
$$\vec{v}^T M \vec{v} = \sum_x \sum_y \left[\vec{v}^T w(x,y) \underbrace{\begin{pmatrix} I_x^2 & I_x I_y \\ I_x I_y & I_y^2 \end{pmatrix}}_{N} \vec{v} \right].$$

along with ✪, $\Rightarrow$ each term of the summation is positive or zero. $\Rightarrow$ $\vec{v}^T M \vec{v} \geq 0$ as wanted //.

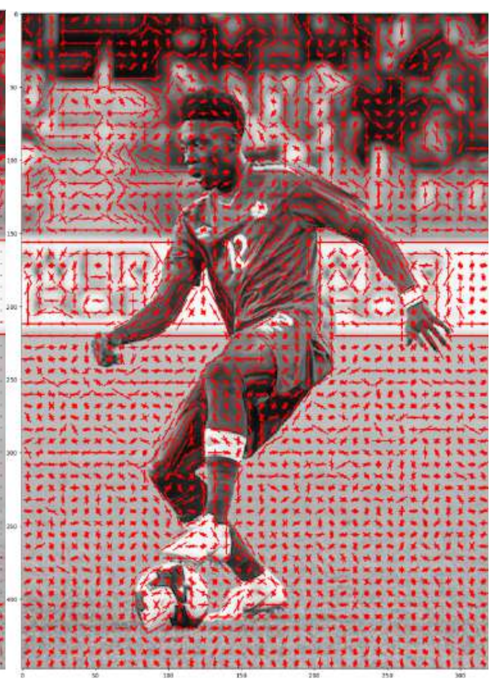
Q3)
See code for implementation details

Threshold = 500, tao = 8

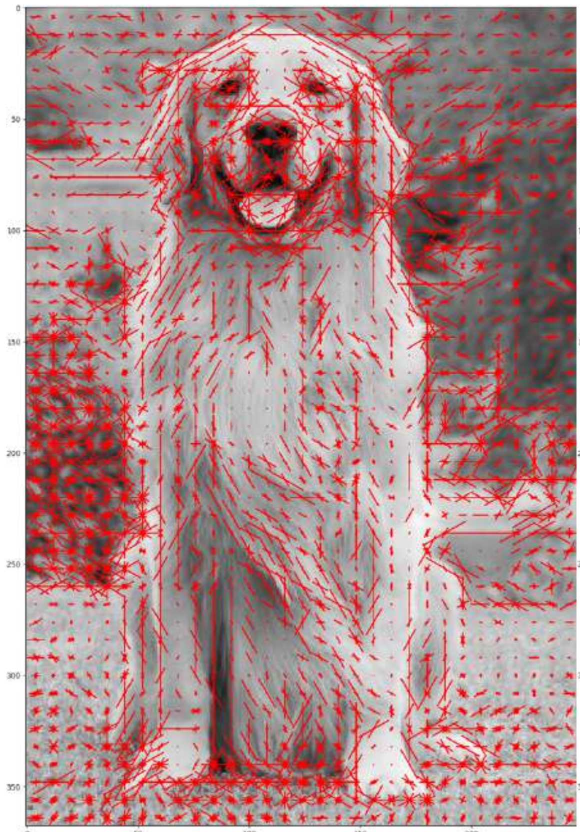




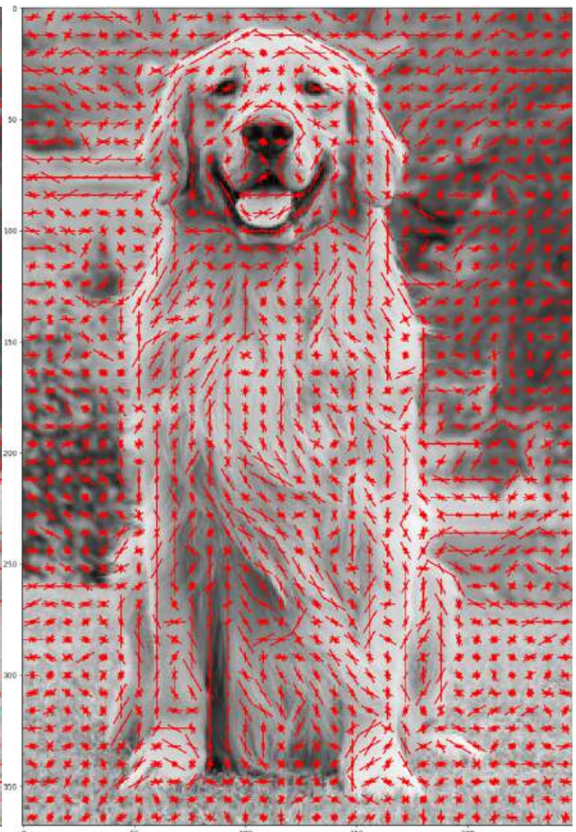
(magnitude histogram) (count histogram)



(magnitude histogram) (count histogram)



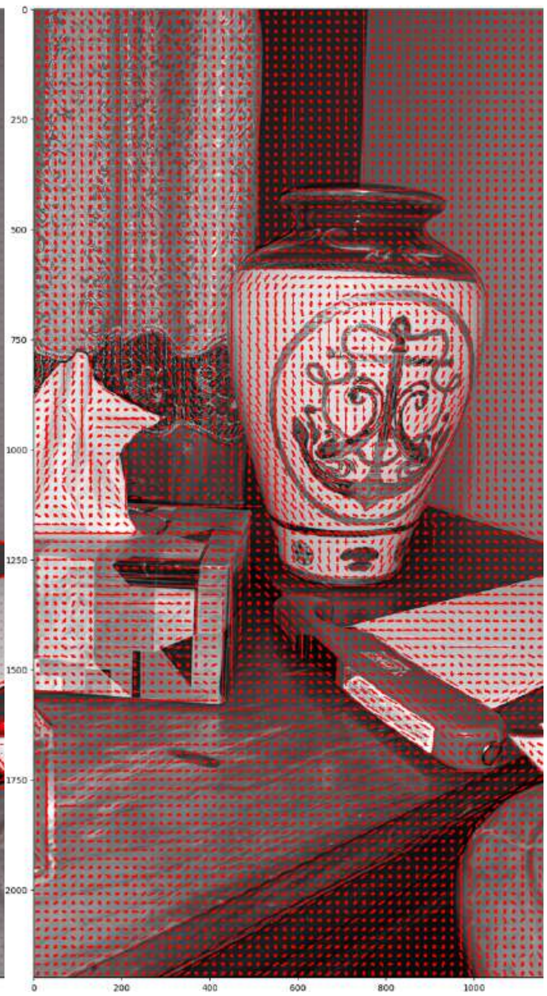
(magnitude)



(count)

Image with flash

Threshold = 1000, tao = 20



(magnitude) (count)

Without Flash

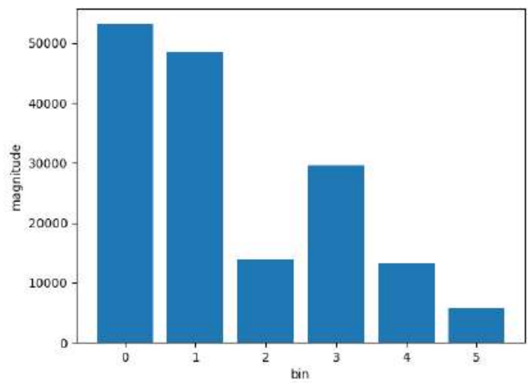
Magnitude

Count

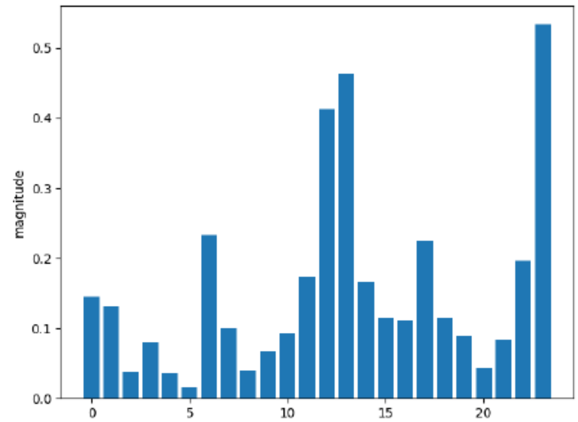
Normalization is applied to the Magnitude version.

Normalizing HOG is beneficial since it can normalize the contrast in local regions on the image. For example, in the flash light image example, the center of the picture (on the vase) is much brighter than its surroundings, so we can clearly see in the magnitude histogram's plot that the center has less gradients compared to that of the image with no flash. After normalizing the HOG, both images appear to be similar.

The following shows histograms for cell [0, 1] before and after normalizing.

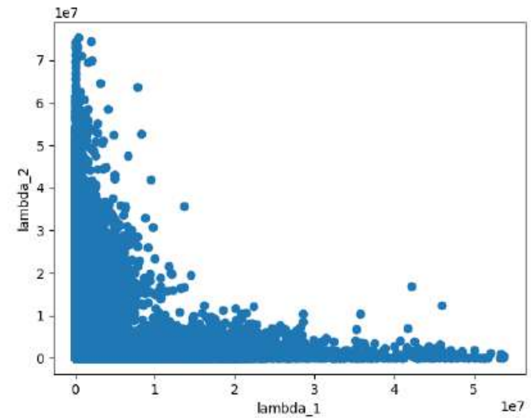


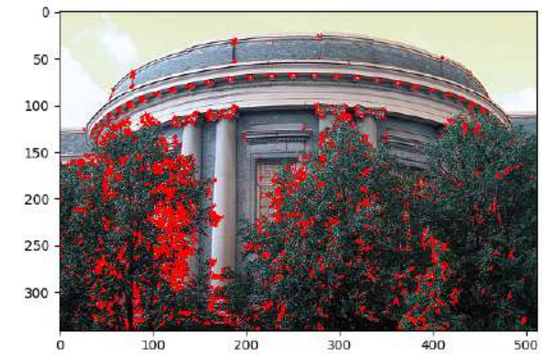
before normalizing



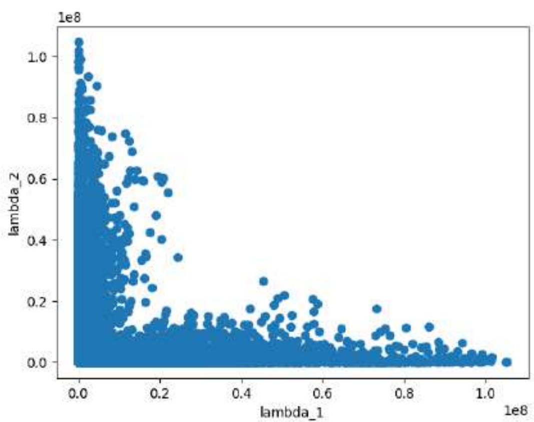
after normalizing

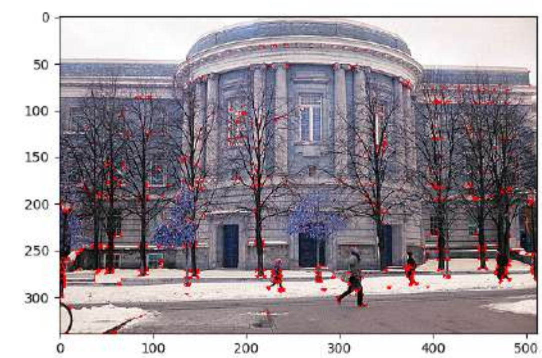
Q4)





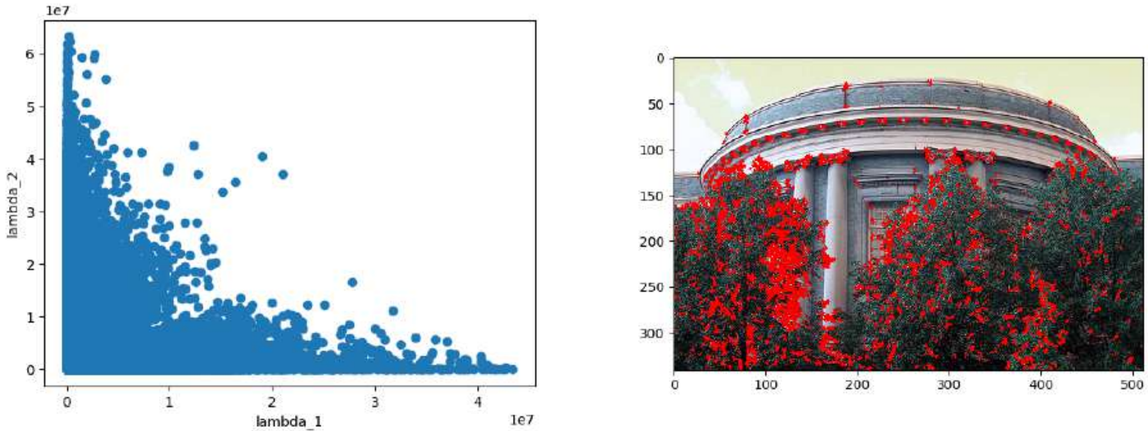
thres_e1 = 0.9e6, thres_e2 = 0.9e6, sigma = 0.7



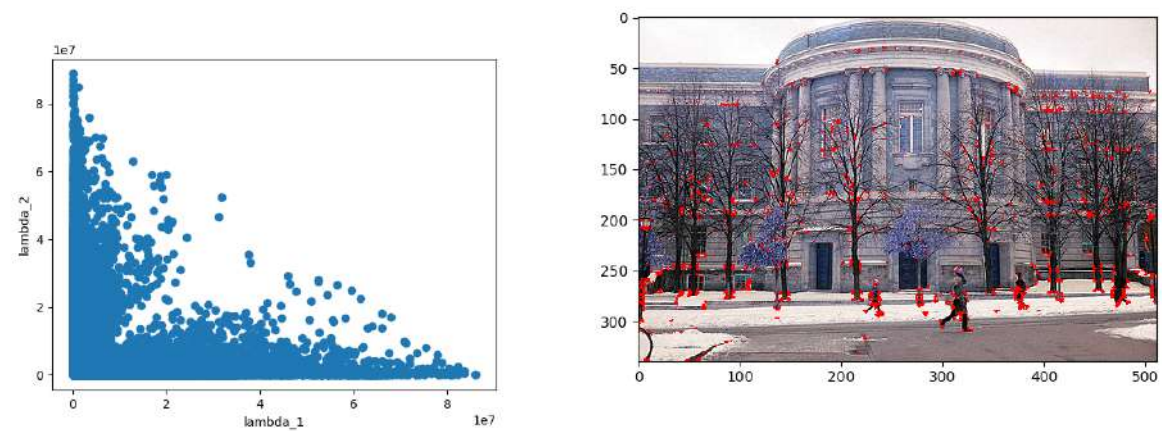


thres_e1 = 0.2e7, thres_e2 = 0.2e7, sigma = 0.7

Trying with different window function (sigma):



thres_e1 = 0.9e6, thres_e2 = 0.9e6, sigma=10



thres_e1 = 0.2e7, thres_e2 = 0.2e7, sigma = 10

We can see that as sigma increases, eigen value points are more centered (e1 and e2 are both larger) in the left graph. So with the same threshold, we see more corners identified.