Wishart Processes and Multidimensional Stochastic Volatility Model

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Overview

Introduction

Definition

An affine process is a stochastic process satisfying the folloing SED:

$$
\begin{aligned}
dX_t &= (\bar{\alpha} + B(X_t))dt + (\sqrt{X_t}dW_t a + a^T dW_t^T \sqrt{X_t}), \\
X_0 &= x.
\end{aligned}
$$

Definition

A Wishart process is an affine process, of which we have

$$\exists \alpha \geq 0, \text{ s.t. } \bar{\alpha} = \alpha a^T a, \text{ and}$$

$$\exists b \in \mathcal{M}_d(\mathbb{R}), \text{ s.t. } B(x) = bx + xb^T.$$

Introduction

We denote by

$$\mathrm{WIS}_d(x, \alpha, b, a) \quad (\text{respectively, } \mathrm{AFF}_d(x, \bar{\alpha}, B, a))$$

the law of Wishart (and affine) process $(X_t^x)_{t \geq 0}$, and

$\mathrm{WIS}_d(x, \alpha, b, a; t) \quad (\text{respectively, } \mathrm{AFF}_d(x, \bar{\alpha}, B, a; t))$

the marginal law of Wishart (and affine) process at time t.

Simulation of CIR Process

A Cox-Ingersoll-Ross (CIR) process Process is an 1-dim Wishart process.

SDE of CIR process

$$X_t^x = x + \int_0^t (a - kX_s^x)ds + \sigma \int_o^t \sqrt{X_s^x} dW_s, \tag{1}$$

$x \geq 0,\ k \geq 0,\ a > 0,\ \sigma > 0.$

Exact Simulation of CIR Process

We define

$$\nu := \frac{4a}{\sigma^2},$$

$$\eta_k(h) := \frac{4k\exp(-kh)}{\sigma^2(1-\exp(-kh))} \text{ for } k \neq 0, \;\; := \frac{4}{\sigma^2 h} \text{ for k=0.}$$

Proposition

$$X_t^x | X_s^x \overset{law}{=} \frac{\exp(-k(t-s))}{\eta_k(t-s)} K_s^t,$$

where $K_s^t \sim \chi^2_\nu\,(\lambda = X_s \eta_k(t-s))$.

Exact Simulation of CIR Process

Iteration of exact simulation

$$X_{t_{i+1}} = \frac{\exp(-k\delta t)}{\eta_k(\delta t)} * K_{t_i}^{\delta t},$$

where $K_{t_i}^{\delta t} \sim \chi_\nu^2\left(X_{t_i}\eta_k(\delta t)\right)$, and $X_{t_0} = x$.

Discretization Schemes

Definition

A **potential weak** r**th order scheme** for the operator L is a discretization scheme s.t.

$$\forall f \in \mathcal{C}^{\infty}, R_{r+1} f(x) := \mathbb{E}[f(\hat{X}_t^x)] - \left[f(x) + \sum_{k=1}^{r} \frac{1}{k!} t^k L^k f(x) \right]$$

is of order $r + 1$.

Discretization Schemes

Infinitesimal generator of CIR processes

$$
\begin{aligned}
Lf(x) &= (a - kx)\frac{d}{dx}f(x) + \frac{1}{2}\sigma^2 x \frac{d^2}{dx^2}f(x), \\
&= (V_0 + \frac{1}{2}(V_1)^2)f(x).
\end{aligned}
$$

Here we decompose the generator into two parts, V_0 and V_1, where

$$
V_0 f(x) = (a - kx - \frac{\sigma^2}{4}\frac{d}{dx})f(x), \tag{2}
$$

$$
V_1 f(x) = \sigma\sqrt{x}\frac{d}{dx}f(x). \tag{3}
$$

Discretization Schemes

ODEs associated to V_0 and V_1 could be solved explicitly:

Solution of ODEs

$$X_0(t,x) = xe^{-kt} + (a - \frac{\sigma^2}{4})\psi_k(t),$$
$$X_1(t,x) = \left((\sqrt{x} + \frac{\sigma}{2}t)^+\right)^2.$$

Où

ψ_k

$$\psi_k(t) := \frac{1 - e^{-kt}}{k} \text{ for } k \neq 0, := t \text{ for } k = 0.$$

Weak 2nd Order Scheme

We define

Definition of $\phi(x, t, w)$

$$\phi(x,t,w) := e^{-\frac{k}{2}} \times \left(\sqrt{(a - \frac{\sigma^2}{4})\psi_k(\frac{t}{2}) + e^{-\frac{kt}{2}} x} + \frac{\sigma}{2} w \right)^2 + (a - \frac{\sigma^2}{4})\psi_k(\frac{t}{2})$$

We have

$$\phi(x, t, \sqrt{t}N) = X_0\left(\frac{t}{2}, X_1\left(\sqrt{t}N, X_0(\frac{t}{2}, x)\right)\right).$$

Weak 2nd Order Scheme

Proposition

If $\sigma^2 \leq 4a$, $\phi(x, t, \sqrt{t}N)$ is well-defined and is a potential weak 2nd order scheme for L. Where $N \sim \mathcal{N}(0, 1)$.

For $\sigma^2 > 4a$, use bounded variable Y, s.t.

$$\mathbb{E}[Y^m] = \mathbb{E}[N^m], \quad N \sim \mathcal{N}(0, 1), m = 1, \ldots, 5,$$

separate into 2 cases:

- x is far from 0. i.e., $x \geq \mathbf{K}_2(t)$.
- x is near 0. i.e. $x < \mathbf{K}_2(t)$.

Where $\mathbf{K}_2$ is the boundary function related to Y.

Weak 2nd Order Scheme

Y

$\mathbb{P}[Y=\sqrt{3}]=\frac{1}{6}$, $\mathbb{P}[Y=-\sqrt{3}]=\frac{1}{6}$ and $\mathbb{P}[Y=0]=\frac{2}{3}$.

$\mathbf{K}_2$

$$\mathbf{K}_2(t) := \mathbf{1}_{\sigma^2>4a} e^{\frac{kt}{2}} \times \left((\frac{\sigma^2}{4}-a)\psi_K(\frac{t}{2}) + \left[\sqrt{e^{\frac{kt}{2}}[(\frac{\sigma^2}{4}-a)\psi_k(\frac{t}{2})]} + \frac{\sigma}{2}\sqrt{3t}\right]^2 \right)$$

Weak 2nd Order Scheme

Proposition

For the case where $\sigma^2 > 4a$*, if* $x \geq \mathrm{K}_2(t)$*,* $\phi(x, t, \sqrt{t}Y)$ *is well defined and is a potential weak 2nd order scheme for* L*.*

Let $\tilde{u}_q(t,x) := \mathbb{E}[(X_t^x)^q]$ for $q \in \mathbb{N}$, $\Delta(t,x) := 1 - \frac{\tilde{u}_1(t,x)^2}{\tilde{u}_2(t,x)}$, and $\pi(t,x) := \frac{1-\sqrt{\Delta(t,x)}}{2}$.

Proposition

Let $U \sim \mathcal{U}([0,1])$*. Then* $\hat{X}_t^x = g(t,x) := 1_{U \leq \pi(t,x)} \frac{\tilde{u}_1(t,x)}{2\pi(t,x)} + 1_{U > \pi(t,x)} \frac{\tilde{u}_1(t,x)}{2(1-\pi(t,x))}$ *is a potential second order scheme on* $x \in [0, \mathrm{K}_2(t)]$*.*

Weak 3rd Order Scheme

3rd generater

$$\tilde{L} = \frac{\sigma}{\sqrt{2}} \sqrt{\left| a - \frac{\sigma^2}{4} \right|} \partial_x,$$

$$\tilde{X}(t,x) := x + t \frac{\sigma}{\sqrt{2}} \sqrt{\left| a - \frac{\sigma^2}{4} \right|}$$

Weak 3rd Order Scheme

Similar to the weak 2nd order scheme, we choose Y matching the first 7 moments, and a boundary function $\mathbf{K}_3$.

Y

$\mathbb{P}[Y = \sqrt{3+\sqrt{6}}] = \mathbb{P}[Y = -\sqrt{3+\sqrt{6}}] = \frac{\sqrt{6}-2}{4\sqrt{6}}$, and
$\mathbb{P}[Y = \sqrt{3-\sqrt{6}}] = \mathbb{P}[Y = -\sqrt{3-\sqrt{6}}] = \frac{1}{2} - \frac{\sqrt{6}-2}{4\sqrt{6}}$.

Weak 3rd Order Scheme

$\mathbf{K}_3$

$$
\begin{aligned}
\mathbf{K}_3(t) &:= \psi_{-k}(t) \\
&\times \Bigg[\mathbf{1}_{4a/3<\sigma^2<4a} \left(\sqrt{\frac{\sigma^2}{4} - a + \frac{\sigma}{\sqrt{2}} \sqrt{a - \frac{\sigma^2}{4}}} + \frac{\sigma}{2}\sqrt{3+\sqrt{6}} \right)^2 \\
&\quad + \mathbf{1}_{\sigma^2 \le 4a/3} \frac{\sigma}{\sqrt{2}} \sqrt{a - \frac{\sigma^2}{4}} \\
&\quad + \mathbf{1}_{4a<\sigma^2} \left(\frac{\sigma^2}{4} - a + \left(\sqrt{\frac{\sigma}{\sqrt{2}} \sqrt{\frac{\sigma^2}{4} - a}} + \frac{\sigma}{2}\sqrt{3+\sqrt{6}} \right)^2 \right) \Bigg]
\end{aligned}
$$

Weak 3rd Order Scheme

For $x \geq \mathbf{K}_3(t)$, let $\epsilon \sim \mathcal{U}(\{-1,1\})$ and $\zeta \sim \mathcal{U}(\{1,2,3\})$, we define for $\hat{\sigma 2} \leq 4a$ (resp. $\sigma^2 > 4a$),

$$\hat{X}_t^{x,k=0} =$$

$$\begin{cases} \tilde{X}(\epsilon t, X_0(t, X_1(\sqrt{t}Y, x))) & \text{(resp. } \tilde{X}(\epsilon t, X_1(\sqrt{t}Y, X_0(t,x))))\text{if } \zeta = 1, \\ X_0(t, \tilde{X}(\epsilon t, X_1(\sqrt{t}Y, x))) & \text{(resp. } X_1(\sqrt{t}Y, \tilde{X}(\epsilon t, X_0(t,x))))\text{if } \zeta = 2, \\ X_0(t, X_1(\sqrt{t}Y, \tilde{X}(\epsilon t, x))) & \text{(resp. } X_1(\sqrt{t}Y, X_0(t, \tilde{X}(\epsilon t, x))))\text{if } \zeta = 3. \end{cases}$$

Proposition

For $x \geq \mathrm{K}_3(t)$, the scheme

$$\hat{X}_t^x = e^{-kt}\hat{X}_{\psi_{-k}(t)}^{x,k=0}$$

is well defined and is a potential 3rd order scheme.

Weak 3rd Order Scheme

For $x \in [0, \mathbf{K}_3(t)]$, set $m_i := \mathbb{E}[X^i]$. Let $s = \frac{m_3 - m_1 m_3}{m_2 - m_1^2}$, $p = \frac{m_1 m_3 - m_2^2}{m_2 - m_1^2}$, and $\Delta = s^2 - 4p$. We have $\Delta > 0$, and therefore we define $x_\pm = \frac{s \pm \sqrt{\Delta}}{2}$ and $\pi = \frac{m_1 - x_-}{x_+ - x_-}$.

Proposition

For $x \in [0, \mathrm{K}_3(t)]$, let $U \sim \mathcal{U}([0,1])$. Then

$$\hat{X}_t^x := 1_{U \leq \pi(t,x)} x_+(t,x) + 1_{U > \pi(t,x)} x_-(t,x)$$

is well defined and is a potential 3rd order scheme.

Simulation of Wishart Process

Definition

A Wishart process is an affine process satisfying:

$$\begin{aligned} dX_t &= (\bar{\alpha} + B(X_t))dt + (\sqrt{X_t}dW_t a + a^T dW_t^T \sqrt{X_t}), \\ X_0 &= x. \end{aligned}$$

where we have

$$\exists \alpha \geq 0, \text{ s.t. } \bar{\alpha} = \alpha a^T a, \text{ and}$$

$$\exists b \in \mathcal{M}_d(\mathbb{R}), \text{ s.t. } B(x) = bx + xb^T.$$

Important properties of Wishart: 1

Proposition

For each Wishart process, its infinitesimal generator depends on a through $a^T a$, i.e.,

$$AFF_d(x, \bar{\alpha}, B, a) \stackrel{law}{=} AFF(x, \bar{\alpha}, B, \sqrt{a^T a}).$$

Important properties of Wishart: 2

Proposition

Let $n = RK(a)$ be the rank of $a^T a$. Then there exists a diagonal matrix δ and a non-singular matrix $u \in \mathcal{G}_d(\mathbb{R})$ (the general linear group on $\mathbb{R}$ of dimension d), s.t. $\bar{\alpha} = u^T \delta u$, $a^T a = u^T I_d^n u$, and we have

$$AFF_d(x, \bar{\alpha}, B, a) \stackrel{law}{=} u^T AFF_d((u^{-1})^T x u^{-1}, \delta, B_u, I_d^n) u,$$

where $B_u(x) := (u^T)^{-1} B(u^T x u) u^{-1}$, and I_d^n is the matrix of which the first n diagonal entries are 1 and all the other entries are 0.

Here, $I_d^n = \sum_{i=1}^n e_d^i$ and $e_d^i = (1_{i=j=n})_{1i,l \leq d}$.

Important properties of Wishart: 3

Proposition

Let $t > 0$, and $\alpha \geq d-1$. Let $m_t := \exp(tb)$, $q_t := \int_0^t \exp(sb) a^T a \exp sb^T ds$ and $n = RK(q_t)$. Then there exists $\theta_t \in \mathcal{G}_d(\mathbb{R})$, s.t. $q_t = t\theta_t I_d^n \theta_t^T$, and we have

$$WIS_d(x, \alpha, b, a; t) \stackrel{law}{=} \theta_t WIS_d(\theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T, \alpha, 0, I_d^n; t)\theta_t^T$$

Exact simulation of Wishart process

Theorem

Let L be the generator of the Wishart process $WIS_d(x, \alpha, 0, I_d^n)$, and $L_{e_d^i}$ the generator of $WIS_d(x, \alpha, 0, e_d^i)$, then we have

$$L = \sum_{i=1}^{n} L_{e_d^i}, \tag{4}$$

and $\forall i, j$, $L_{e_d^i} L_{e_d^j} = L_{e_d^j} L_{e_d^i}$.

With this theorem, a Wishart process $\mathrm{WIS}_d(x, \alpha, 0, I_d^n; t)$ can be simulated step by step as :

$$\begin{aligned}
X_t^{1,x} &\sim \mathrm{WIS}_d(x,\ \alpha,\ 0,\ e_d^1;\ t) \\
X_t^{2,X_t^{1,x}} &\sim \mathrm{WIS}_d(X_t^{1,x},\ \alpha,\ 0,\ e_d^2;\ t) \\
X_t^{n,\dots^{X_t^{1,x}}} &\sim \mathrm{WIS}_d(X_t^{n-1,\dots^{X_t^{1,x}}},\ \alpha,\ 0,\ e_d^n;\ t)
\end{aligned}$$

Exact simulation of Wishart process

Proposition

Let $X_t^{n,\dots X_t^{1,x}}$ be defined above. Then

$$X_t^{n,\dots X_t^{1,x}} \sim WIS_d(x, \alpha, 0, I_d^n; t)$$

Lemma

$\mathrm{AFF}_d(x, \bar{\alpha}, B, a) \overset{\mathrm{law}}{=} q^T \mathrm{AFF}_d({q^{-1}}^T x q^{-1}, {q^{-1}}^T \bar{\alpha} q^{-1}, B_q, a q^{-1}) q$
where $B_q(x) = (q^T)^{-1} B(q^T x q) q^{-1}$.

Exact simulation for $WIS_d(x,\ \alpha,\ 0,\ e_d^1;\ t)$

By explicit calculation, $WIS_d(x,\ \alpha,\ 0,\ e_d^1;\ t)$ is of form:

$$
\begin{pmatrix} 1 & 0 & 0 \\ 0 & c_r & 0 \\ 0 & k_r & I_{d-r-1} \end{pmatrix}
$$

$$
\times \quad \begin{pmatrix} (U_t^u)_{1,1} + \sum_{k=1}^{r}((U_t^u)_{\{1,k+1\}}))^2 & ((U_t^u)_{\{1,l+1\}}))_{1\le l\le r}^T & 0 \\ ((U_t^u)_{\{1,l+1\}}))_{1\le l\le r} & I_r & 0 \\ 0 & k_r & I_{d-r-1} \end{pmatrix}
$$

$$
\times \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_r & 0 \\ 0 & k_r & I_{d-r-1} \end{pmatrix}
$$

where p, c_r, k_r is the extended Cholesky decomposition of $(x_{i,j})_{2\le i,j\le d}$ and r the corresponding dimension.

Simulation of U

For the simulation of $\mathrm{WIS}_d(x,\ \alpha,\ 0,\ e_d^1;\ t)$, we have:

$$
\begin{aligned}
d(U_t^u)_{1,1} &= (\alpha\ -\ r)dt\ +\ 2\sqrt{(U_t^u)_{1,1}}dZ_t^1 \\
d((U_t^u)_{\{1,l+1\}})_{1\leq l\leq r} &= (dZ_t^{l+1})_{1\leq l\leq r}
\end{aligned}
\tag{5}
$$

More precisely, $(U_t^u)_{1,1}$ is a CIR process with following parameters:

$$a = \alpha - r, k = 0, \sigma = 2$$

and $(Z_t^l)_{1\leq l\leq r+1}$ is a vector of independent standard Brownien motions.

Discretization scheme for Wishart process

- Use 2nd order discretization (resp. 3rd order discretization) CIR process to simulate $U_{1,1}$ and 2nd (resp. 3rd) order bounded Gaussian distribution to define $U_{\{1,l+1\}}$ as in 27.
- Define $\mathrm{WIS}(x, \alpha, 0, e_d^1, t)$ with 2nd (resp. 3rd) order U, and the Cholesky decomposition of $x_{i,j:2\leq i,j\leq d}$.
- Based on 3.2, simulate $\mathrm{WIS}(x, \alpha, 0, I_d^n; t)$ step by step.
- Based en 3.3, simulate 2nd (resp. 3rd) $\mathrm{WIS}(x, \alpha, b, a; t)$.
- Let x be the simulated value of $X_t^{1,x}$, repeat from the first step.

Verification of the correctness

Proposition

Let $X_t^x \sim WIS_d(x, \alpha, b, a; t)$*,* $m_t := \exp(tb)$ *and* $q_t := \int_0^t \exp(sb)a^T a \exp sb^T ds$ *and* $\mathcal{D}_{b,a;t} = \{v \in \mathcal{S}_d(\mathbb{R}), \mathbb{E}[exp(Tr(vX_t^x)] \leq \inf\}$*.* $\mathcal{D}_{b,a;t}$ *can be given explicitly by*

$$\mathcal{D}_{b,a;t} = \{v \in \mathcal{S}_d(\mathbb{R}), \forall s \in [0, t], I_d - 2q_s v \in \mathcal{G}_d(\mathbb{R})\}$$

Then the Laplace transform of X_t^x *is well defined for* $v = v_R + iv_I$ *with* $v_R \in \mathcal{D}_{b,a;t}, v_I \in \mathcal{S}_d(\mathbb{R})$ *and is given by*

$$\mathbb{E}[\exp(Tr(vX_t^x))] = \frac{\exp(Tr[v(I_d - 2q_t v)^{-1} m_t x m_t^T])}{\det(I_d - 2q_t v)^{\alpha/2}} \qquad (6)$$

CIR results

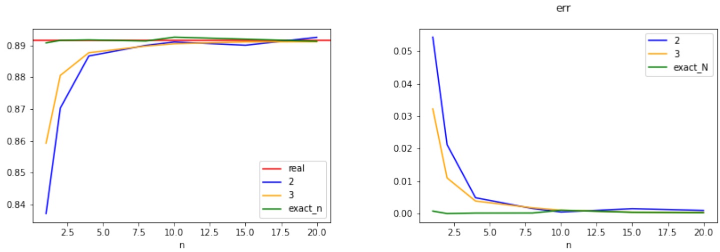

Figure: Comparison of precision of CIR process simulation using different method, with parameters $x_0 = 0.3, k = 0.1, a = 0.04, \sigma = 2$.

Wishart results

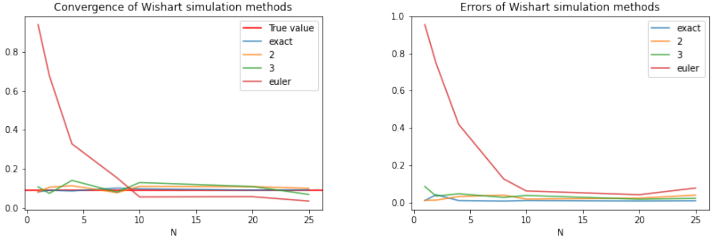

Figure: Comparison of precision of Wishart process simulation using different method, with parametres $x = 0.4Id, \alpha = 4.5, a = Id, b = 0$

First application: Gourieroux and Sufana model

Formulation of Gourieroux-Sufana model

$$dS_t = rS_t + (\sqrt{X_t} dB_t)^T S_t,$$
$$dX_t = (\alpha a^T a + bX_t + X_t b^T)dt + (\sqrt{X_t} dW_t a + a^T dW_t^T \sqrt{X_t}).$$

Simulation for Gourieroux-Sufana model

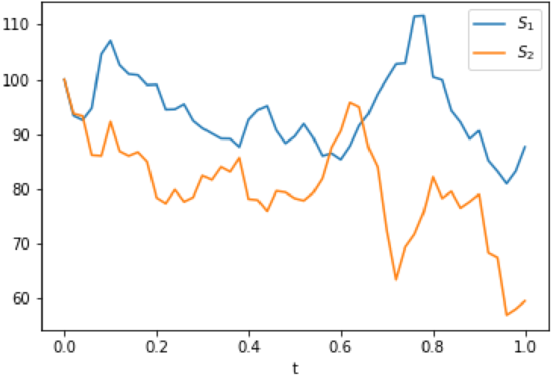

Figure: Example of GS model, $S_0 = (100, 100)$, $r = 0.02$, $X_0 = 0.04 I_2 + 0.02\mathbf{1}_{i \neq j}$, $\alpha = 4.5$, $a = 0.2 I_2$, $b = 0.5 I_2$, $T = 1$.

Conclusion