Wishart Processes and Multidimensional Stochastic Volatility Model

Yiming QIN, Benxin ZHONG

École Polytechnique

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Overview

Introduction

The volatility of assets

CIR process:

$$dX_t = (a - kX_t)dt + \sigma\sqrt{X_t}dW_t.$$

Wishart process:

$$dX_t = (\bar{\alpha} + B(X_t))dt + (\sqrt{X_t}dW_t a + a^T dW_t^T \sqrt{X_t}).$$

where for Wishart process we have

$$\exists \alpha \geq 0, \text{ s.t. } \bar{\alpha} = \alpha a^T a, \text{ and}$$

$$\exists b \in \mathcal{M}_d(\mathbb{R}), \text{ s.t. } B(x) = bx + xb^T.$$

Introduction

The asset price and volatility

GS model:

$$\begin{cases} dS_t = \mathbf{diag}(S_t)(rdt + \sqrt{X_t} dB_t), \\ dX_t = (\bar{\alpha} + bX_t + X_t b^T)dt + \sqrt{X_t} dW_t a + a^T dW_t^T \sqrt{X_t}. \end{cases}$$

Fonseca model:

$$\begin{cases} dS_t = \mathbf{diag}(S_t)(rdt + \sqrt{X_t} dZ_t), \\ dX_t = (\bar{\alpha} + bX_t + X_t b^T)dt + \sqrt{X_t} dW_t a + a^T dW_t^T \sqrt{X_t} \end{cases}$$

Gourieroux-Sufana model

SDE of Gourieroux-Sufana model

$$\begin{cases} dS_t = \mathbf{diag}(S_t)(rdt + \sqrt{X_t}dB_t), \\ dX_t = (\bar{\alpha} + bX_t + X_t b^T)dt + \sqrt{X_t}dW_t a + a^T dW_t^T \sqrt{X_t}. \end{cases}$$

Principle

Extension of Heston model to multi-dimensions.

Simulation : split the generator

Infinitesimal generator

$$\mathcal{L} = \mathcal{L}_S + \mathcal{L}_X$$

where $\mathcal{L}_X$ describes a Wishart process whose 2nd-order discretisation scheme is implemented before, and $\mathcal{L}_S$ has an explicit solution: $S_t^l = S_0^l \exp[(r - x_{l,l}/2)t + (\sqrt{x}B_t)_l]$.

Simulation : split the generator

Lemma

Let L_1, L_2 be two generators defined on same domain $\mathbb{D}$. Let $\hat{X}_t^{1,x}$ and $\hat{X}_t^{2,x}$ denote, respectively, two potential weak νth-order schemes on $\mathbb{D}$ for L_1 and L_2.

1. *If $L_1 L_2 = L_2 L_1$, ${\hat{X}_t}^{2,{\hat{X}_t}^{1,x}}$ is a potential weak νth-order discretisation scheme for $L_1 + L_2$.*
2. *Let B be an independent Bernoulli variable of parameter $1/2$. If $\nu \geq 2$, then*
    - $B{\hat{X}_t}^{2,{\hat{X}_t}^{1,x}} + (1-B){\hat{X}_t}^{1,{\hat{X}_t}^{2,x}}$
    - $\hat{X}_{t/2}^{2,{\hat{X}_t}^{1,\hat{X}_{t/2}^{2,x}}}$

    are potential weak second-order schemes for $L_1 + L_2$.

Convergence

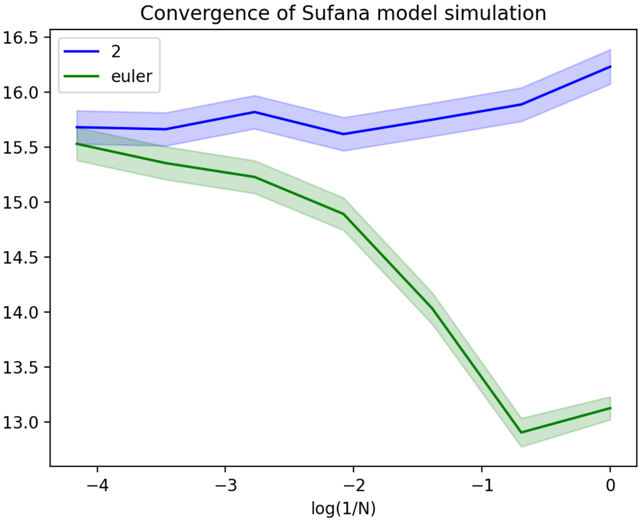

Figure: Convergence of GS model

Fonseca model

SDE

$$\begin{cases} dS_t = \mathbf{diag}(S_t)((r+\lambda_t)dt + \sqrt{X_t}dZ_t), \\ dX_t = (\bar{\alpha} + bX_t + X_t b^T)dt + \sqrt{X_t}dW_t a + a^T dW_t^T \sqrt{X_t} \end{cases}$$

Where $dZ_t = \bar{\rho}dB_t + dW_t\rho$.

Principle

Brownian motions of the assets' returns and those driving their instantaneous covariance matrix are linearly correlated.

Fonseca model

SDE

$$
\begin{cases}
dS_t = \mathbf{diag}(S_t)(rdt + \sqrt{X_t}dZ_t), \\
dX_t = (\bar{\alpha} + bX_t + X_t b^T)dt + \sqrt{X_t}dW_t a + a^T dW_t^T \sqrt{X_t}
\end{cases}
$$

Where $dZ_t = \bar{\rho}dB_t + dW_t\rho$.

Fonseca model

Let $Y_t := \log S_t$, i.e. the returns of assets.

SDE

$$\begin{cases} dY_t = (r - \frac{1}{2}\mathbf{diag}(X_t))dt + \sqrt{X_t}dZ_t \\ dX_t = (\bar{\alpha} + bX_t + X_t b^T)dt + \sqrt{X_t}dW_t a + a^T dW_t^T \sqrt{X_t} \end{cases}$$

Where $dZ_t = \bar{\rho} dB_t + dW_t \rho$.

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Let $Y_t := \log S_t$, i.e. the returns of assets.

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dX_t = (\bar{\alpha} + bX_t + X_t b^T)dt + \sqrt{X_t}dW_t a + a^T dW_t^T \sqrt{X_t}
\end{cases}
$$

Where $dZ_t = \bar{\rho}dB_t + dW_t\rho$.

Problem

Wt does not participate directly in the simulation of X_t.

Extension of Linear Gaussian model

We introduce another model with linear inner correlation:

SDE

$$\begin{cases} dY_t = \kappa(\theta - Y_s)dt + c\sqrt{X_t}dZ_t, \\ dX_t = (\bar{\alpha} + bX_t + X_t b^T)dt + \epsilon[\sqrt{X_t}dW_t \mathrm{I}_{\mathrm{d}}^{\mathrm{n}} + \mathrm{I}_{\mathrm{d}}^{\mathrm{n}} dW_t^T \sqrt{X_t}] \end{cases}$$

Where $dZ_t = \bar{\rho}dB_t + dW_t\rho$.

The 2nd order simulation scheme of ELGM is given.

Extension of Linear Gaussian model

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Where $dZ_t = \bar{\rho}dB_t + dW_t\rho$.

The 2nd order simulation scheme of ELGM is given.

Remark

The ELGM is a Fonseca model with $a = \mathrm{I}_n^d$.

Extension of Linear Gaussian model

Decomposition of ELGM

$$\mathcal{L} = \mathcal{L}' + \mathcal{L}'' + \sum_{q=1}^{n} \mathcal{L}_q^c$$

- $\mathcal{L}'$: Generator of X, assuming Y as constant.
- $\mathcal{L}''$: Generator of Y, assuming X as constant.
- $\mathcal{L}^c$: Generator with inter correlation between X and Y. Calculated one-dim by one-dim.

Extension of Linear Gaussian model

Decomposition of $\mathcal{L}_q^c$

$$\begin{cases} dY_t = \rho_q \sqrt{X_t} dW_t g_d^q, \\ dX_t = (d-1)\epsilon^2 e_d^q dt + \epsilon(\sqrt{X_t} dW_t e_d^q + e_d^q dW_t^T \sqrt{X_t}) \end{cases}$$

Where $(e_d^q)_{i,j} := 1_{i=j=q}$, and $(g_d^q)_i := 1_{i=q}$.

Therefore only the q th row and column of X_t is modified, and we get that

$$(\sqrt{X_t} dW_t)_{i,q} g_d^q = (X_t)_{q,i} - x_{q,i}, \quad \text{if } i \neq q,$$

$$(\sqrt{X_t} dW_t)_{q,q} g_d^q = \frac{1}{2}((X_t)_{q,q} - x_{q,q} - \epsilon^2(d-1)t),$$

we get then

$$\begin{cases} (Y_t)_i = y_i + \frac{\rho_q}{\epsilon}((X_t)_{q,i} - x_{q,i}), \quad \text{if } i \neq q, \\ (Y_t)_q = y_i + \frac{\rho_q}{2\epsilon}((X_t)_{q,i} - x_{q,i} - \epsilon^2(d-1)t). \end{cases}$$

Decomposition

Infinitesimal generator

$$\mathcal{L}_{Y,X} = \mathcal{L}_Y + \mathcal{L}_X + \mathcal{L}_{<Y,X>}, \tag{1}$$

where

$$\begin{aligned}
\mathcal{L}_Y &= \frac{1}{2}\nabla_Y x \nabla_Y^T + \nabla_Y (r - \frac{1}{2}\mathbf{diag}(x)), \\
\mathcal{L}_X &= \mathbf{Tr}[(\bar{\alpha} + bx + xb^T)\mathcal{D}_X + 2x\mathcal{D}_X a^T a\mathcal{D}_X], \\
\mathcal{L}_{<Y,X>} &= 2\mathbf{Tr}[\mathcal{D}_X a^t \rho \nabla_Y x],
\end{aligned}$$

where $(\mathcal{D}_X)_{ij} := \frac{\partial}{\partial x_{ij}}$.

Back to Fonseca model

Introduce another process $\tilde{Y}_t$:

$$d\tilde{Y}_t = \sqrt{X_t} dW_t \rho. \tag{2}$$

Proposition

$$\mathcal{L}_{\tilde{Y}} = \frac{1}{2}\|\rho\|^2 \nabla_{\tilde{Y}} x \nabla_{\tilde{Y}}^T,$$

and that

$$\mathcal{L}_{<\tilde{Y},X>} = \mathcal{L}_{<Y,X>}.$$

Decomposition

Corollary

$$\mathcal{L}_{Y,X} = \mathcal{L}_{\tilde{Y},X} + \mathcal{L}_1, \tag{3}$$

where

$$\mathcal{L}_1 = \nabla_Y (r - \frac{1}{2}\mathbf{diag}(x)) + \frac{\bar{\rho}^2}{2} \nabla_Y x \nabla_Y^T$$

is the generator of a stochastic process of which the volatility is constant.

The process of $\mathcal{L}_1$ is

$$Y_t = (r - \frac{1}{2}\mathbf{diag}(x))t + \bar{\rho} x B_t.$$

Transformation

Now focus on the process of $\mathcal{L}_{\tilde{Y},X}$. Abuse a little the notion, let $\tilde{Y}_t$ denoted by Y_t.

New SDE

$$
\begin{cases} dY_t = \sqrt{X_t}dW_t\rho \\ dX_t = (\bar{\alpha} + bX_t + X_t b^T)dt + \sqrt{X_t}dW_t a + a^T dW_t^T \sqrt{X_t} \end{cases}
$$

Lemma

Since $X_t \sim \text{WIS}(X_0, \bar{\alpha}, b, a; t)$, there exists $u \in \mathcal{G}_d(\mathbb{R})$, s.t. $a^T a = u^T I_d^n u$, and $\bar{\alpha} = u^T \bar{\delta} u$.
Let $V_t \sim \text{WIS}(V_0, \bar{\delta}, b_u, I_d^n; t)$, where $V_0 = (u^T)^{-1} X_0 u^{-1}$, and $b_u = (u^T)^{-1} b u^T$, we have $X_t \overset{\text{law}}{=} u^T V_t u$.

Transformation

Consider the two matrix-variables x and v, where $x = u^T vu$, we have $\frac{\partial x_{ij}}{\partial v_{kl}} = (u^T)_{ik} u_{lj} = u_{ki}(u^T)_{jl}$, then

$$\frac{\partial}{\partial v_{kl}} = \sum_{i,j} \frac{\partial}{\partial x_{ij}} \frac{\partial x_{ij}}{\partial v_{kl}} = \sum_{i,j} u_{ki} \frac{\partial}{\partial x_{ij}} (u^T)_{jk},$$

hence

$$\mathcal{D}_V = u \mathcal{D}_X u^T.$$

Transformation

Consider the process (V_t, Y_t).

Proposition

$$\mathcal{L}_Y = \frac{\|\rho\|_2^2}{2}\nabla_Y x \nabla_Y^T = \frac{\|\rho\|_2^2}{2}(\nabla_Y u^T) v (u \nabla_Y^T), \tag{4}$$

$$\begin{aligned}
\mathcal{L}_{<Y,X>} &= 2\mathbf{Tr}[\mathcal{D}_X a^T \rho \nabla_Y x] \\
&= 2\mathbf{Tr}[\mathcal{D}_X u^T I_d^n \rho \nabla_Y u^T v u] \\
&= 2\mathbf{Tr}[u \mathcal{D}_X u^T I_d^n \rho \nabla_Y u^T v] \\
&= 2\mathbf{Tr}[\mathcal{D}_V I_d^n \rho (\nabla_Y u^T) v].
\end{aligned} \tag{5}$$

Transformation

Define $R_t := (u^T)^{-1} Y_t$.

Proposition

$$\nabla_R = \nabla_Y u^T.$$

Corollary

$$\mathcal{L}_Y = \mathcal{L}_R, \quad \mathcal{L}_{<Y,X>} = \mathcal{L}_{<R,V>}. \tag{6}$$

Transformation

SDE of (R_t, V_t)

$$\begin{cases} dR_t = \sqrt{V_t} dW_t \rho, \\ dV_t = (\bar{\delta} + b_u V_t + V_t b_u^T) dt + \sqrt{V_t} dW_t I_d^n + I_d^n dW_t^N \sqrt{V_t}. \end{cases}$$

The process (R_t, V_t) follows the ELGM.

1. Simulate (R_t, V_t).
2. Let $Y_t = u^T R_t$, and $X_t = u^T V_t u$.

Convergence

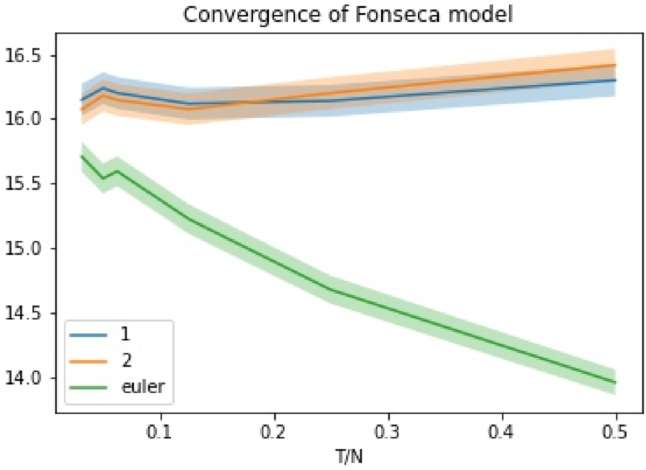


Figure: Convergence of Fonseca model.

Speed-up

Faster method and reguler method

- if $\bar{\alpha} - I_n^d \in S_d^+(\mathbb{R})$

$$X_t = U_t^T U_t$$
$$U_t = c + \epsilon W_t I_d^n$$

- else

$$X_t = \mathrm{WIS}(x, d-1, 0, e_d^q, \epsilon^2 t)$$

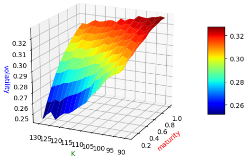
Results: for a simulation of Wishart process of $50,000$ simulations with 20 steps, the time reduced from 456.87 to 1570.93 seconds.

Smile Skew : calculate implied volatility

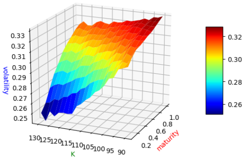
Calculation

$$
\begin{aligned}
S &= \mathbb{E}[(S_T - K)^+] \\
m &= \frac{\log(S)}{Ke^{-rT}} \\
\sigma_0 &= \sqrt{\frac{2m}{T}} \\
\sigma_n &= \sigma_{n-1} + \frac{C - C^{BS}(\sigma_{n-1})}{\frac{\partial C^{BS}(\sigma_{n-1})}{\partial \sigma_{n-1}}}
\end{aligned}
$$

Smile Skew



(a) Asset 1



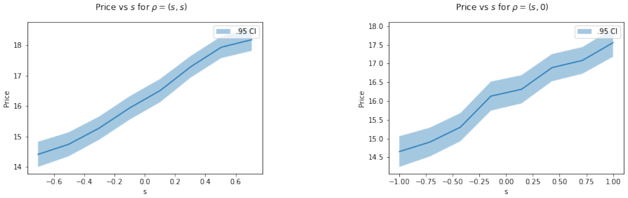
(b) Asset 2

Figure: Implied volatility

Remark

- A small smile effect could be detected when $K \approx 130$,
- An obvious skew effect could be detected.

Influence of ρ



(a) $\rho = (s, s)$ (b) $\rho = (s, 0)$

Figure: Positive correlation between a best-of option price and s

Remark

s negative :

- Price $\uparrow$ if Volatility $\downarrow$, Put option Price $\downarrow$ with high price,
- Price $\downarrow$ if Volatility $\uparrow$, Put option Price $\downarrow$ with high volatility.

Conlusion

Conclusion

- Simulation of CIR process
- Simulation of Wishart process
- Implementation of Gourieroux-Sufana model
- Implementation of Fonseca model through ELGM model
- Analysis on Fonseca model