Singularities and the minimal exponent

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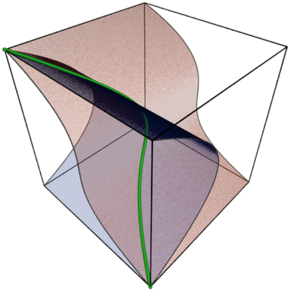
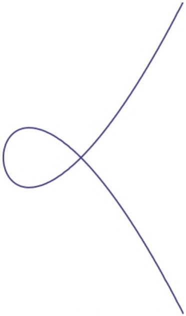
Introduction

Algebraic Variety = zero set of polynomial functions in $\mathbf{C}^n$.

Singularities = locus where the tangent space cannot be regularly defined.



Twisted cubic

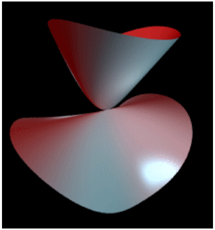


Nodal curve
$f = y^2 - x^2(x+1)$

Introduction

Study singularities is inevitable in algebraic geometry:

1. They are interesting!
2. Even if mainly interested in smooth varieties, one is naturally led to study varieties with mild singularities.



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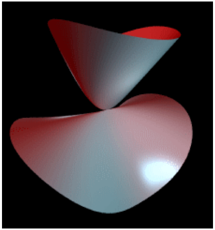

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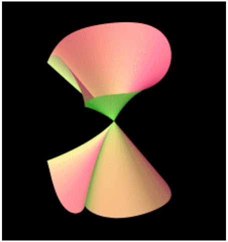

$$f = (x^2 - y^3)^2 - (z^2 - y^2)^3$$
$$\mathrm{mult}_0(f) = 4$$

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Examples

Multiplicity, Log canonical threshold, Milnor number...

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Goal of this talk

Introduce a useful invariant for singularities, called **minimal exponent**, which originates from Hodge theory for singular varieties.

Bernstein-Sato Polynomial

Let D be a nonempty hypersurface (zero set) defined by a polynomial function f on $\mathbf{C}^n$ (or, more generally, an open subset X of $\mathbf{C}^n$).

Theorem (Bernstein)

There is a polynomial $b(s) \in \mathbf{Q}[s]$ such that

$$b(s) \cdot f^s \in \mathscr{D}[s] \cdot f^{s+1},$$

where D denotes the ring of algebraic differential operators, i.e. $\mathscr{D} = \mathbf{C}[x_1, \cdots, x_n]\langle \partial_{x_1}, \cdots, \partial_{x_n} \rangle$.

Here, f^{s+1} is a formal symbol, on which the differential operators act in the usual way:

$$\partial_{x_i} \cdot f^{s+1} = (s+1)\frac{\partial f}{\partial x_i} f^s.$$

Definition

The monic $b(s)$ of the minimal degree satisfying $b(s) \cdot f^s \in \mathscr{D}[s] \cdot f^{s+1}$ is called the **Bernstein-Sato polynomial** or **b-function** of f, and denoted by $b_f(s)$.

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• Bernstein-Sato polynomials plays a role in the celebrated **Strong Monodromy Conjecture**, which is on the cross road of number theory, topology, analysis and algebraic geometry.

Examples

- If $f = x_i$ then $b_f(s) = s + 1$ as

$$(s+1) \cdot x_i^s = \partial_{x_i} \cdot x_i^{s+1}.$$

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$$(s+1)(s+\frac{n}{2})f^s = \frac{1}{4}(\partial_{x_1}^2 + \cdots + \partial_{x_n}^2)f^{s+1}.$$

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- If $f = x_1x_2x_3$, then $b_f(s) = (s+1)^3$.
- If $f = x_1^2 + x_2^3$, then $b_f(s) = (s+1)(s+\frac{5}{6})(s+\frac{7}{6})$.

Some facts

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2. $b_f(s)$ is divisible by $(s+1)$.
Evaluate at $s = -1$:

$$b_f(-1) f^{-1} \in D \cdot f^0 \subset \mathbf{C}[x_1, \ldots, x_n]$$

3. [Lichtin-Kollár] The **log canonical threshold** of f
$\mathrm{lct}(f) = -(\text{largest root of } b_f(s))$.

Recall that $\mathrm{lct}(f) = \sup\{s \colon 1/|f|^{2s} \text{ is locally integrable.}\}$.
(Can also be defined algebraically.)

The minimal exponent for hypersurfaces

Definition (Saito)

The minimal exponent for f is

$$\widetilde{\alpha}(f) = -\big(\textit{largest root of } \widetilde{b}_f(s) = b_f(s)/(s+1)\big) \in \mathbf{Q}_{>0} \cup \{\infty\}$$

under the convention that $\widetilde{\alpha}(f) = \infty$ if $b_f(s) = s+1$. Also written as $\widetilde{\alpha}(D)$, where D is the zero set of f.

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History

The minimal exponent was studied by Loeser, Steenrbink, and Varchenko for isolated singularities in the 1980s under the name **Arnold exponent** using asymptotic expansions of integrals over (vanishing cycles). Recently, it was studied by Mustață-Popa and Saito using Hodge theory.

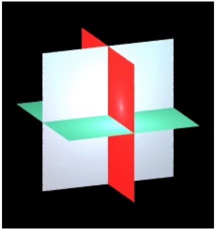
Slogan

Larger minimal exponent

means

better singularities!

Examples of minimal exponent



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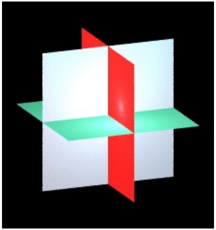

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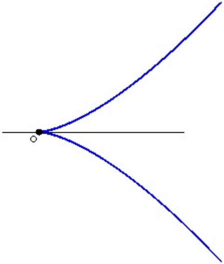

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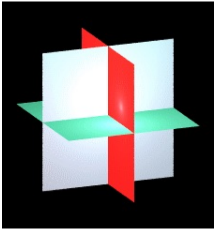

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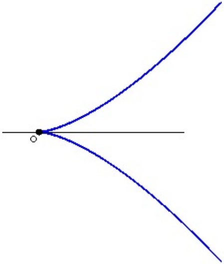

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- More generally, if $f = x_1^{a_1} + \cdots + x_n^{a_n}$ with $a_i \geq 2$ then

$$\widetilde{\alpha}(f) = \frac{1}{a_1} + \cdots + \frac{1}{a_n}.$$

Properties of minimal exponent

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The minimal exponent for complete intersections

We generalized the story of minimal exponent to complete intersections.

Let Z be a variety of codimension r in (an open set X of) $\mathbf{C}^n$ defined by polynomial functions $f_1, \ldots, f_r$.

Definition/Theorem. C.-Dirks-Mustață-Olano

The minimal exponent of Z is $\widetilde{\alpha}(Z) = \widetilde{\alpha}(g)$, where $g = f_1 y_1 + \cdots + f_r y_r$ is viewed as a function on $X \times (\mathbf{C}^r \setminus \{0\})$, and $y_1, \ldots, y_r$ are coordinates on $\mathbf{C}^r$.

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The definition depends on the embedding to the ambient space in a predictable way:

$$\widetilde{\alpha}(Z) - \dim X$$

is independent of the embedding. So a more precise notation is $\widetilde{\alpha}(X, Z)$.

Key features of $\widetilde{\alpha}(Z)$

Theorem. C.-Dirks-Mustață-Olano

Let Z be a complete intersection of codimension r in (an open set X of) $\mathbf{C}^n$.

- $\mathrm{lct}(X, Z) = \min\{\widetilde{\alpha}(Z), r\}$.

- $\widetilde{\alpha}(Z) = \infty$ if and only if Z is smooth.

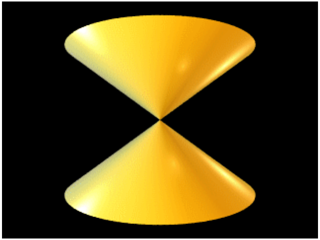
- Restriction Theorem:
  $\widetilde{\alpha}_p(Z \cap H) \leq \widetilde{\alpha}_p(Z)$ for a smooth hypersurface H containing p.
  Have " $=$ " if H is general.

- Lower semicontinuous in families.

- $\mathrm{codim}_Z Z_{\mathrm{sing}} \geq 2\lfloor \widetilde{\alpha}(Z) \rfloor - 2r + 1$ if Z is singular.

Example

Let Z be a complete intersection cut out by homogeneous hypersurfaces of degree d in $\mathbf{C}^n$. Suppose that Z has a unique singularity at the origin. Then $\widetilde{\alpha}(Z) = n/d$.



Affine cone

Connection to Hodge Theory

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Tools to do Hodge theory on singular varieties.

- Du Bois complex.
- Resolution of singularities.
- Saito's theory of Mixed Hodge modules.

Review of Hodge Theory for smooth varieties

Let Z be a complex manifold of dimension n.

- (Poincaré Lemma) The holomorphic de Rham complex

$$\Omega_Z^\bullet = [\mathscr{O}_Z \to \Omega_Z \to \cdots \to \Omega_Z^n]$$

is a resolution of the constant sheaf $\mathbf{C}_Z$.

- (Hodge decomposition) Moreover, if Z is projective, then the Hodge spectral sequence degenerates at the first page:

$$E_1^{p,q} = H^q(Z, \Omega_Z^p) \Rightarrow H^{p+q}(Z, \mathbf{C}).$$

The Du Bois complex

For a singular variety Z, there is a replacement of the de Rham complex, called the **Du Bois complex** $\underline{\Omega}_Z^\bullet$ with a natural filtration F, which behaves similarly to the de Rham complex but very hard to define.

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$$E_1^{p,q} = \mathbf{H}^q(Z, \underline{\Omega}_Z^p) \Rightarrow H^{p+q}(Z, \mathbf{C}).$$

Du Bois singularities

Definition (Steenbrink 83)

We say Z has (at worst) Du Bois singularities if the natural map $\mathscr{O}_Z \to \underline{\Omega}_Z^0$ is an isomorphism.

The isomorphism $\mathscr{O}_Z \to \underline{\Omega}_Z^0$ means that $\mathscr{O}_Z = \mathcal{H}^0(\underline{\Omega}_Z^0)$ and $\mathcal{H}^q(\underline{\Omega}_Z^0) = 0$ for $q \geq 1$.

One property

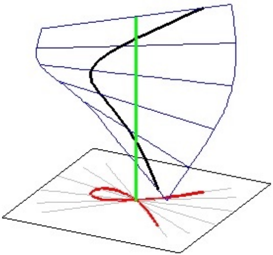
If Z is projective and has Du Bois singularities then $H^k(Z, \mathbf{C}) \to H^k(Z, \mathscr{O}_Z)$ is surjective for every k.

It implies a version of Kodaira vanishing on such varieties.

Hironaka's resolution of singularities

Existence of Resolution of singularities

For any singular variety Z, there is a proper birational (bimeromorphic) morphism $\pi\colon \widetilde{Z} \to Z$ from a smooth $\widetilde{Z}$.



Blow up the node

Rational singularities

Definition (Artin 66, Kempf 73)

We say Z has (at worst) rational singularities if there exist a (hence, for any) resolution of singularities $\pi\colon \widetilde{Z} \to Z$ such that the natural map $\mathscr{O}_Z \to \mathbf{R}\pi_*\mathscr{O}_{\widetilde{Z}}$ is an isomorphism.

The isomorphism $\mathscr{O}_Z \to \mathbf{R}\pi_*\mathscr{O}_{\widetilde{Z}}$ means that $\pi_*\mathscr{O}_{\widetilde{Z}} = \mathscr{O}_Z$ and $R^q\pi_*\mathscr{O}_{\widetilde{Z}} = 0$ for $q \geq 1$.

One property

If Z has rational singularities, then $H^k(\widetilde{Z}, \pi^*\mathcal{F}) = H^k(Z, \mathcal{F})$ is $\mathcal{F}$ is a vector bundle.

Hierarchy of singularities

Theorem. Kovács 99, Saito 2000

$$\text{Rational singularities} \implies \text{Du Bois singularities}$$

In general, its hard to determine whether varieties have rational or Du Bois singularities.

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For complete intersections, the minimal exponent can help!

Theorem (Saito 93, Budur-Mustață-Saito 06)

Let Z be a complete intersection in (an open subset X in) $\mathbf{C}^n$ of codimension r.

$\widetilde{\alpha}(Z) \geq r$ *if and only if Z has Du Bois singularities.*

$\widetilde{\alpha}(Z) > r$ *if and only if Z has rational singularities.*

Examples for hypersurfaces

The nodal curve $x^2 + y^2 = 0$ has Du Bois but non-rational singularities.
The cusp curve $x^2 + y^3 = 0$ doesn't have Du Bois singularities.
Use the fact: if $f = x_1^{a_1} \cdots + x_n^{a_n}$ with all $a_i \geq 2$, then $\widetilde{\alpha}(f) = \sum \frac{1}{a_i}$.

Other Examples

- Varieties with normal crossings have Du Bois singularities
- Quotient singularities (smooth manifold quotient by a finite group action) have rational singularities
- [Artin] If a surface has rational singularities then the preimage of a singular point in the minimal resolution of singularities is a tree of smooth rational curves.

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*We say Z has k-**Du Bois Singularities** if the natural maps $\Omega_Z^p \to \underline{\Omega}_Z^p$ are isomorphisms for $p \leq k$.*

Definition (Friedman-Laza)

*We say Z has k-**rational singularities** if the natural maps $\Omega_Z^p \to \mathbf{R}\pi_*\Omega_{\widetilde{Z}}^p(\log E)$ are isomorphisms for $p \leq k$ for a strong resolution of singularities $\pi\colon \widetilde{Z} \to Z$ where $E = (\pi^{-1}Z_{\mathrm{sing}})_{\mathrm{red}}$ is reduced preimage of the singular locus.*

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- $k = 0$ recovers the previous notions.
- Friedman-Laza came up the notion when they study the deformation of singular Calabi-Yau varieties.

Characterization by the minimal exponent

Theorem (Mustaţă-Popa 22a, C.-Dirks-Mustaţă-Olano 22, C.-Dirks-Mustaţă 23)

If Z is a complete intersection in (some open subset X of) $\mathbf{C}^n$ of codimension r, then

1. $\widetilde{\alpha}(Z) \geq k + r$ *if and only if Z has k-Du Bois singularities.*
2. $\widetilde{\alpha}(Z) > k + r$ *if and only if Z has k-rational singularities.*

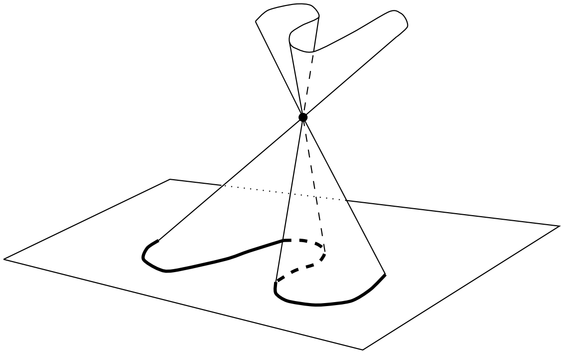
- For hypersurfaces, the theorem is a result of Mustaţă-Olano-Popa-Witaszek 21, Jung-Kim-Saito-Yoon 21, Friedman-Laza-Saito 22, Mustaţă-Popa 22b.

Corollary

$k - \text{Rational} \Rightarrow k - \text{DB} \Rightarrow (k-1) - \text{Rational}$

Example: Affine cone. C.-Dirks-Olano 24

Let $Z = H_1 \cap H_2$ be a complete intersection in the affine space $\mathbf{A}^8$ with an isolated singularity at the origin, where H_1, H_2 are homogeneous hypersurfaces of degree 2 and 3 respectively. Then $\widetilde{\alpha}(Z) = 3$, i.e. Z has 1-Du Bois singularities but non-1-rational singularities; in particular



An Inversion of Adjunction property

Adjunction: the process of inferring statements on a hypersurface from the knowledge of the ambient variety.

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Theorem (C. 24)

If D is a hypersurface in a complete intersection Z in $\mathbf{C}^n$ intersecting with a hyperplane H such that $\widetilde{\alpha}(Z \setminus D) > \widetilde{\alpha}(D)$, then $\widetilde{\alpha}(Z) > \alpha(D)$.

It recovers the Inversion of Adjunction for Du Bois and rational singularities for complete intersections by Elkik and Schwede.
Similar statements have been proved for log canonicity (for example, [Kawakita 07], [Hacon 14]).

An application: Deformation of varieties

Theorem (Friedman-Laza 22)

Let $f\colon Y \to S$ be a flat proper family of varieties and let $s \in S$. Suppose that the fiber Y_s has k-Du Bois local complete intersection singularities. Then, after shrinking S to a neighborhood of s, the sheaves $R^q f_* \Omega^p_{Y/S}$ are vector bundles and compatible with base change for $0 \le p \le k$ and all $q \ge 0$.

Corollary (Friedman-Laza 22)

Let Y be a canonical Calabi-Yau variety which is additionally a scheme with 1-Du Bois local complete intersection singularities. Then the functor $\mathbf{Def}(Y)$ is unobstructed.

One open problem: birational description

Let f be a polynomial function on some open subset X in $\mathbf{C}^n$

Strong monodromy conjecture

The poles of the topological (motivic) Zeta function Z_f are zeros of the Bernstein-Sato polynomial $b_f(s)$.

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Strong monodromy conjecture

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Let $\pi\colon \widetilde{X} \to X$ be a birational morphism from a smooth variety such that $f \circ \pi$ is a monomial under a suitable coordinates and that π is an isomorphism outside the zero set D of f. Assume D is reduced. We can write

$$\mathrm{div}(f \circ \pi) = \widetilde{D} + \sum a_i E_i \quad \text{and} \quad \mathrm{div}(\mathrm{Jac}(\pi)) = \sum k_i E_i$$

with $\widetilde{D}$ the strict transform and E_i are prime components.

Implication of strong monodromy conjecture

The strong monodromy conjecture predicts some roots of $b_f(s)$ are of the form $-\frac{k_i+1}{a_i}$.

A conjecture of Mustață-Popa

We have $\widetilde{\alpha}(f) = \frac{k_i+1}{a_i}$ for some i.

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Evidence

• $\operatorname{lct}(f) = \min_i(\frac{k_i+1}{a_i}, 1)$.
• [Lichtin] every zeros of $b_f(s)$ is of the form $-\frac{k_i+\ell}{a_i}$ for some positive integer ℓ.
• [Mustață-Popa] $\widetilde{\alpha}(f) \geq \min_{E_i} \frac{k_i+1}{a_i}$ if D is reduced.
• In an ongoing project with Mustață, we described the minimal exponent of the hypersurfaces using the direct images of the twisted log forms on embedded resolutions of singularities.

Thank You!