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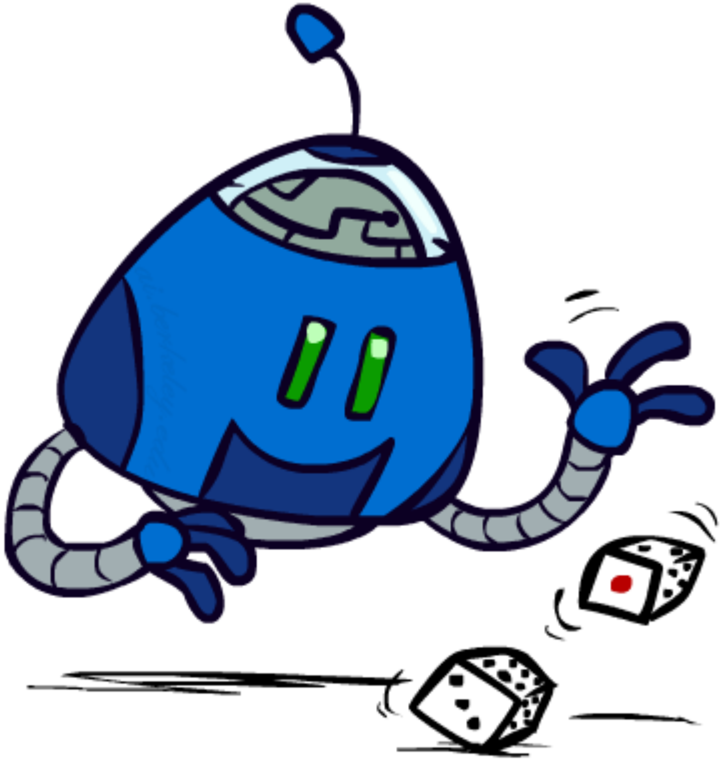
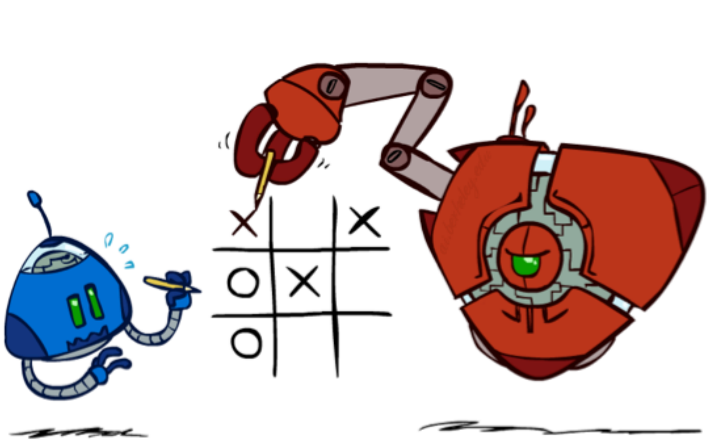
Uncertainty and Utilities

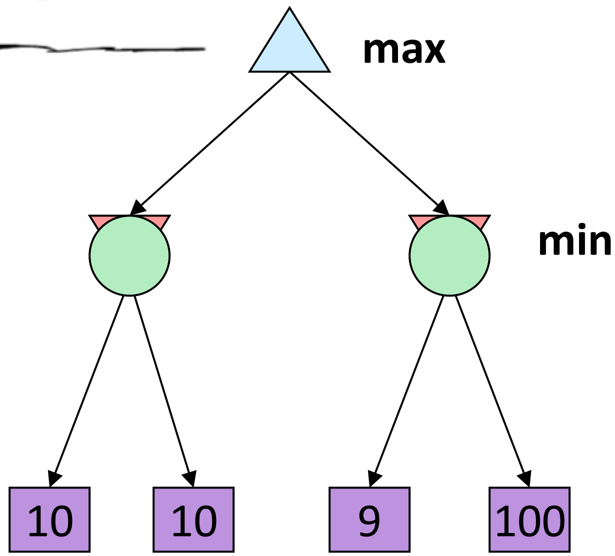
April 4th, 2018

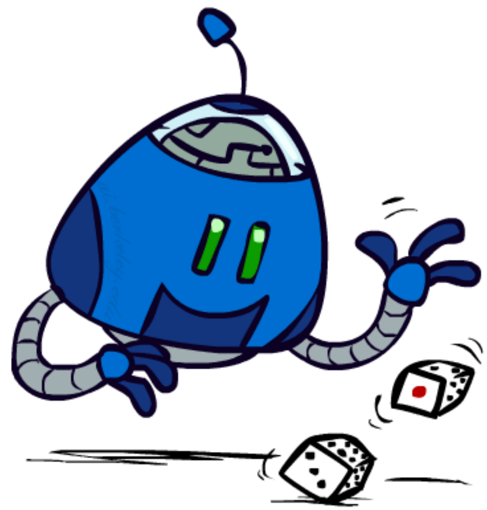
Uncertain Outcomes



Worst-Case vs. Average Case







Idea: Uncertain outcomes controlled by chance.

Reminder: Probabilities

- A random variable represents an event whose outcome is unknown
- A probability distribution is an assignment of weights to outcomes

- Example: Traffic on freeway
  - Random variable: T = whether there's traffic
  - Outcomes: T in {none, light, heavy}
  - Distribution: P(T=none) = 0.25, P(T=light) = 0.50, P(T=heavy) = 0.25
- Some laws of probability:
  - Probabilities are always non-negative
  - Probabilities over all possible outcomes sum to one
- As we get more evidence, probabilities may change:
  - P(T=heavy) = 0.25, P(T=heavy | Hour=8am) = 0.60
  - We'll talk about methods for reasoning and updating probabilities later

0.25

0.50

0.25

Reminder: Expectations

- The expected value of a function of a random variable is the average, weighted by the probability distribution over outcomes

- Example: How long to get to the airport?

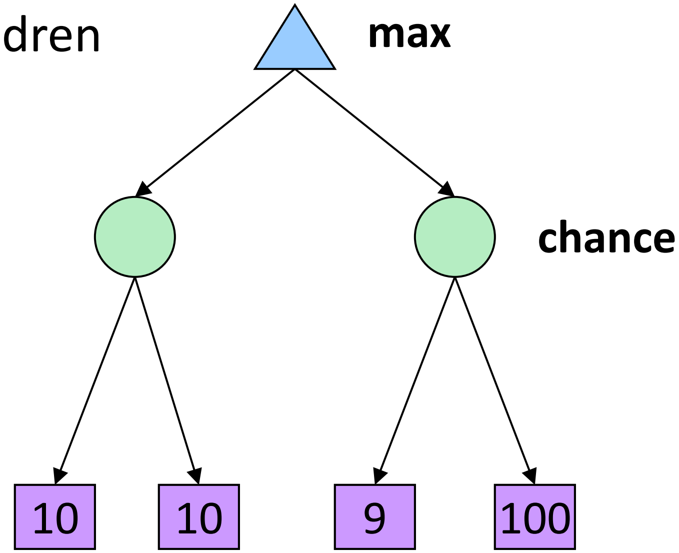
20 min x 0.25 + 30 min x 0.50 + 60 min x 0.25 ⇒ 35 min

Expectimax Search

- Why wouldn't we know what the result of an action will be?
  - Explicit randomness: rolling dice
  - Unpredictable opponents: the ghosts respond randomly
  - Actions can fail: when moving a robot, wheels might slip

- Values should now reflect average-case (expectimax) outcomes, not worst-case (minimax) outcomes

- Expectimax search: compute the average score under optimal play
  - Max nodes as in minimax search
  - Chance nodes are like min nodes but the outcome is uncertain
  - Calculate their expected utilities
  - I.e. take weighted average (expectation) of children



Expectimax Pseudocode

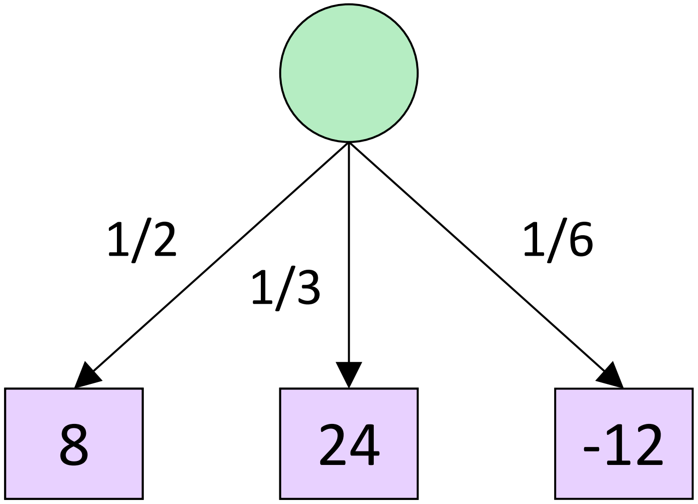
```
def value(state):
    if the state is a terminal state: return the state's utility
    if the next agent is MAX: return max-value(state)
    if the next agent is EXP: return exp-value(state)
```

```
def max-value(state):
    initialize v = -∞
    for each successor of state:
        v = max(v, value(successor))
    return v
```

```
def exp-value(state):
    initialize v = 0
    for each successor of state:
        p = probability(successor)
        v += p * value(successor)
    return v
```

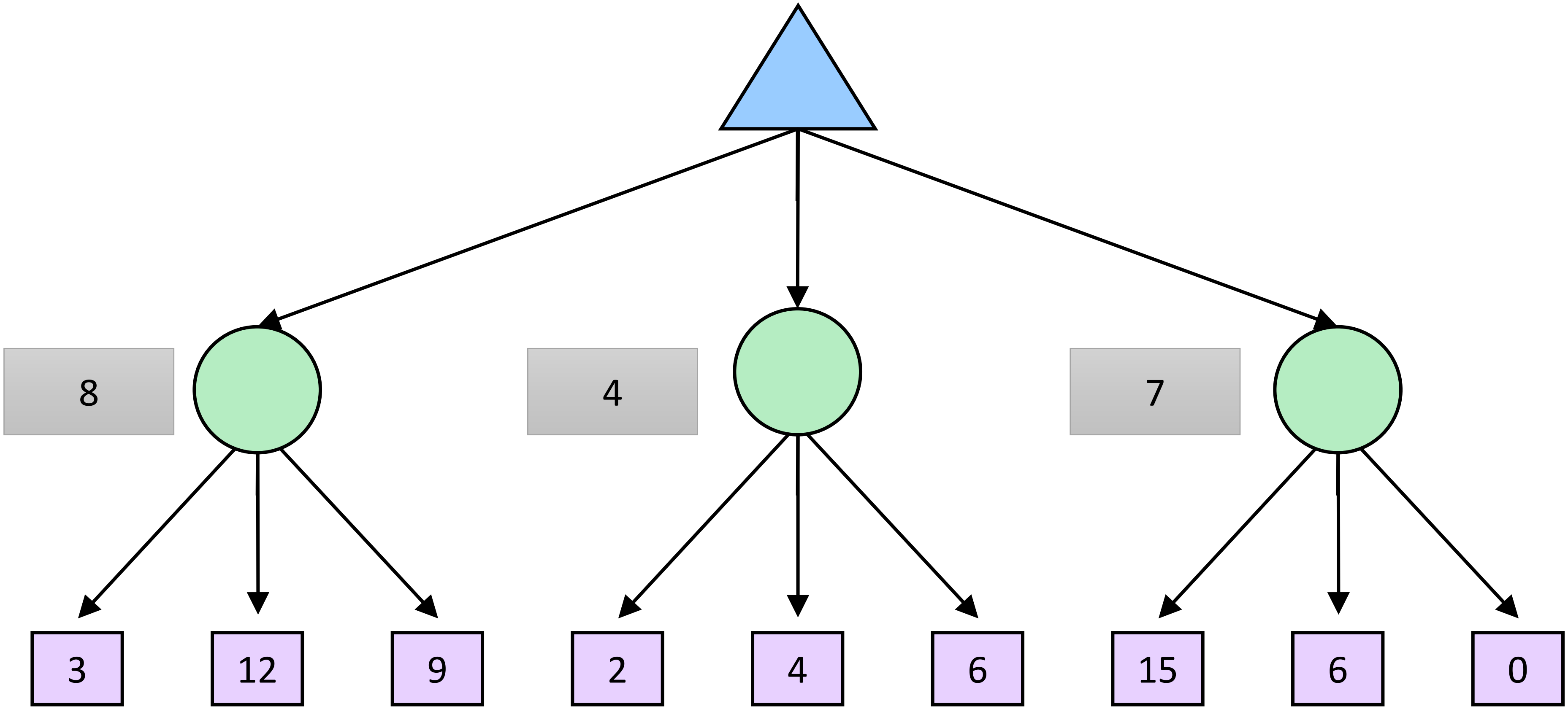
Expectimax Pseudocode

```
def exp-value(state):
    initialize v = 0
    for each successor of state:
        p = probability(successor)
        v += p * value(successor)
    return v
```

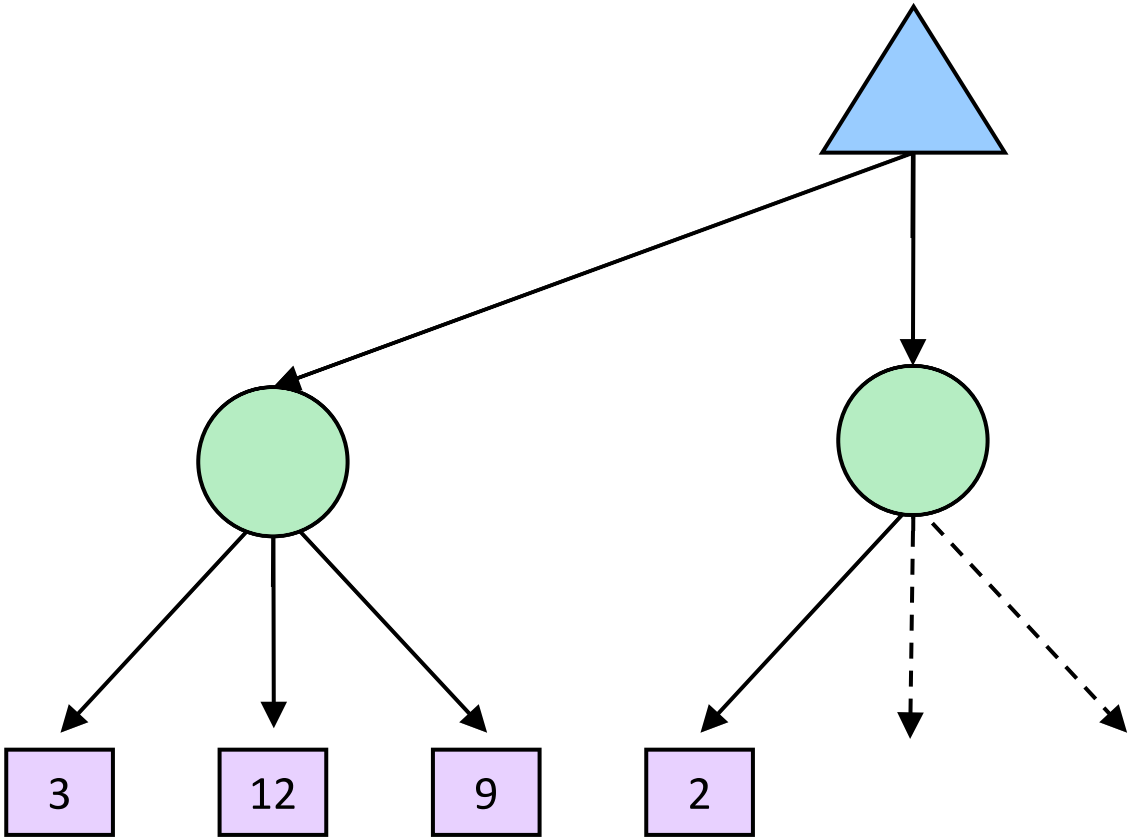


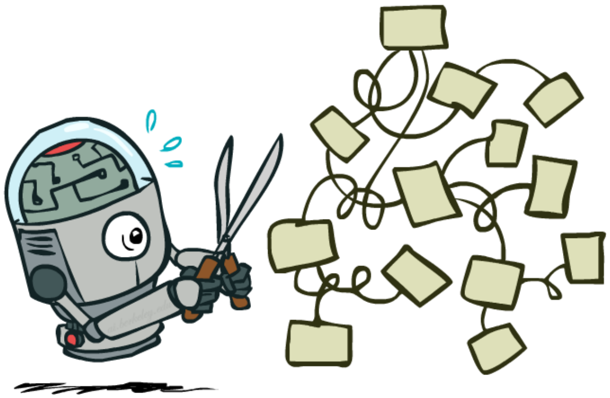
v = (1/2) (8) + (1/3) (24) + (1/6) (-12) = 10

Expectimax Example



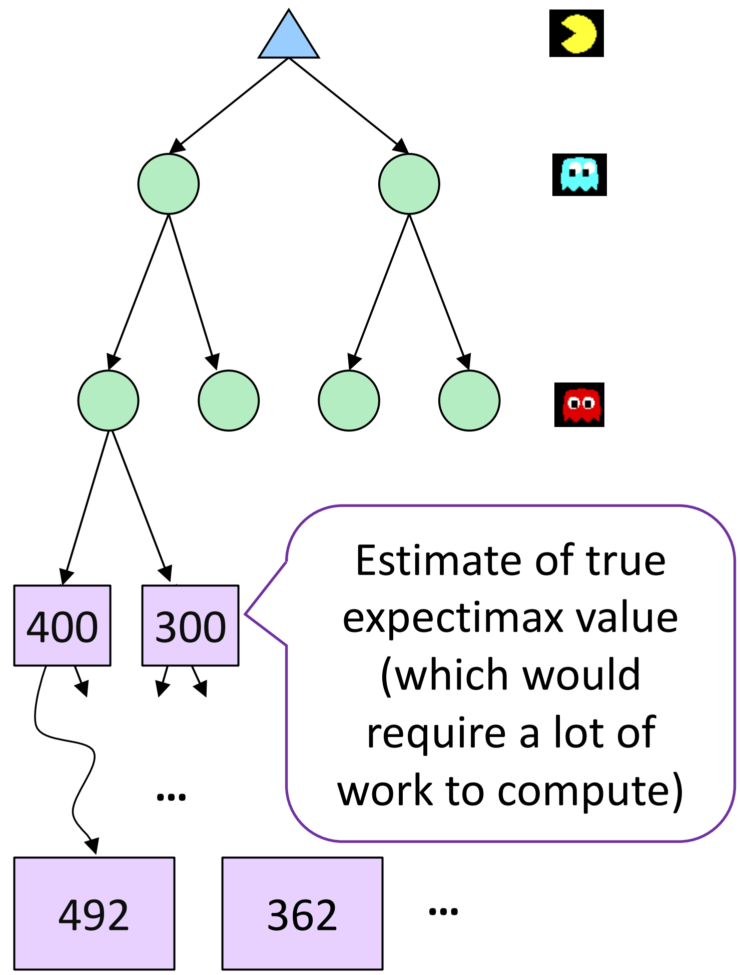
Expectimax Pruning?





All Children nodes are involved.

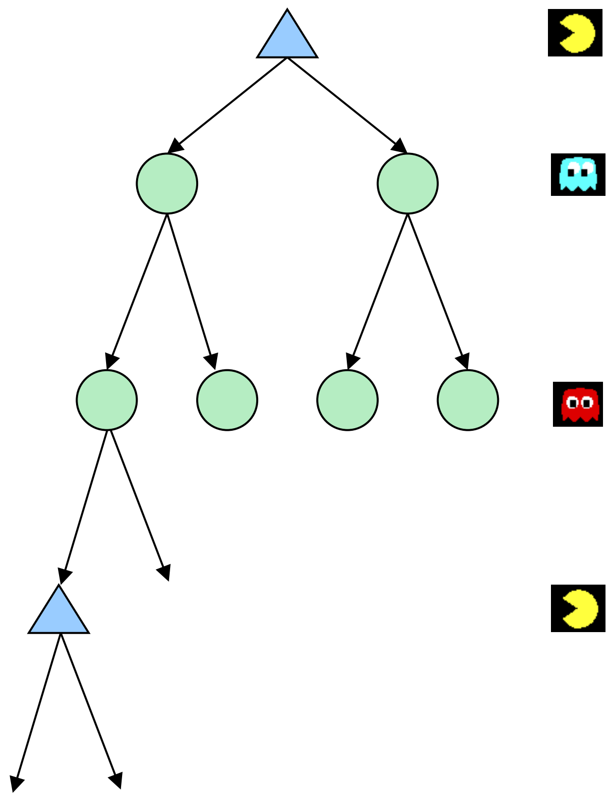
Depth-Limited Expectimax



What Probabilities to Use?

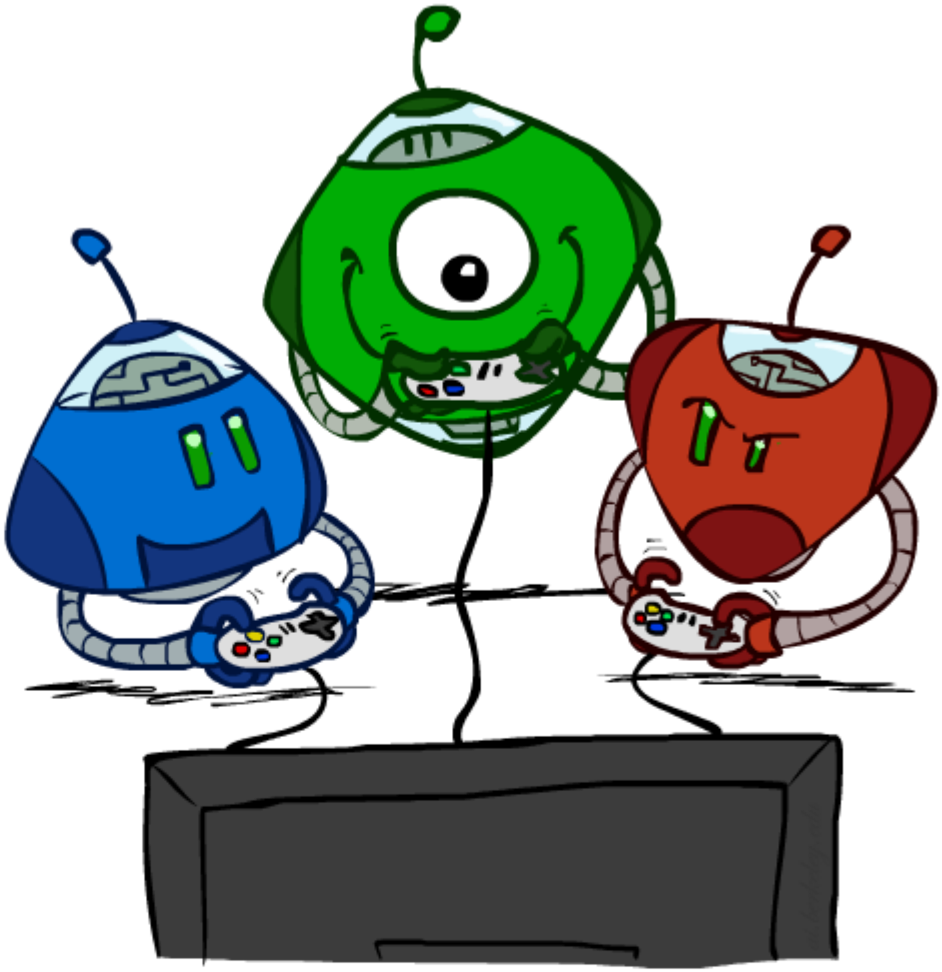
- In expectimax search, we have a probabilistic model of how the opponent (or environment) will behave in any state
  - Model could be a simple uniform distribution (roll a die)
  - Model could be sophisticated and require a great deal of computation
  - We have a chance node for any outcome out of our control: opponent or environment

- For now, assume each chance node magically comes along with probabilities that specify the distribution over its outcomes



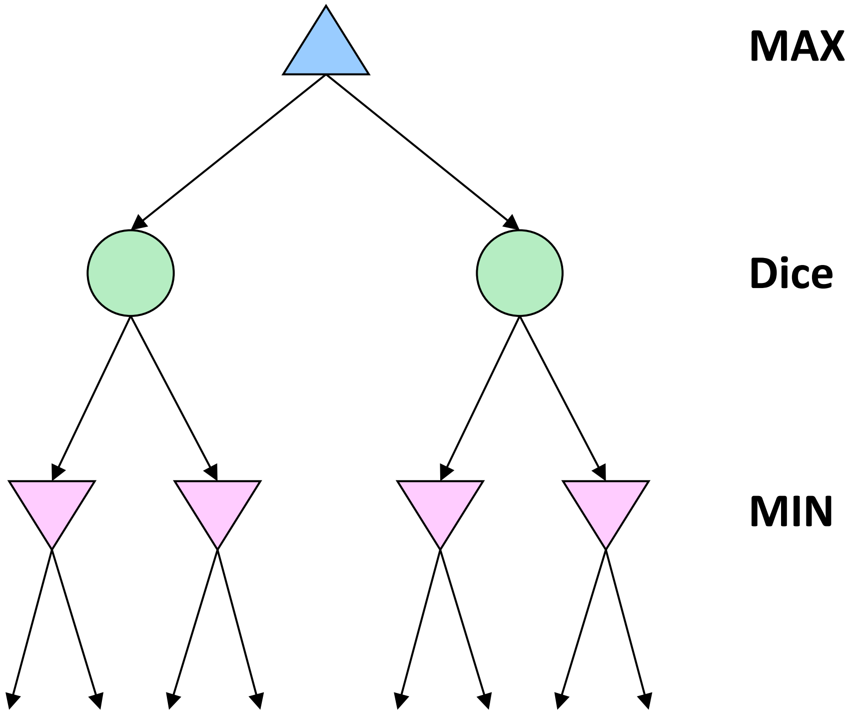
Having a probabilistic belief about another agent's action does not mean that the agent is flipping any coins!

Other Game Types



Mixed Layer Types

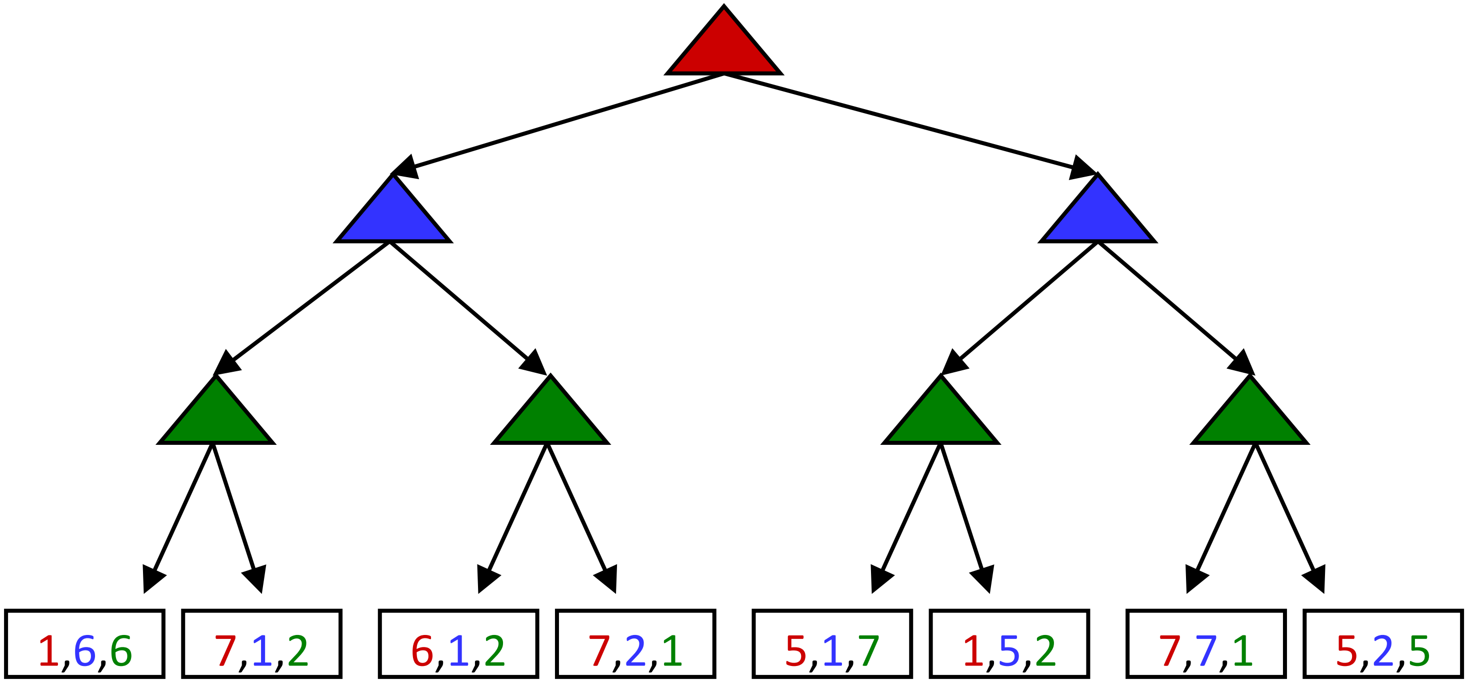
- E.g. Monopoly
- Expectiminimax
  - Environment is an extra “random agent” player that moves after each min/max agent
  - Each node computes the appropriate combination of its children



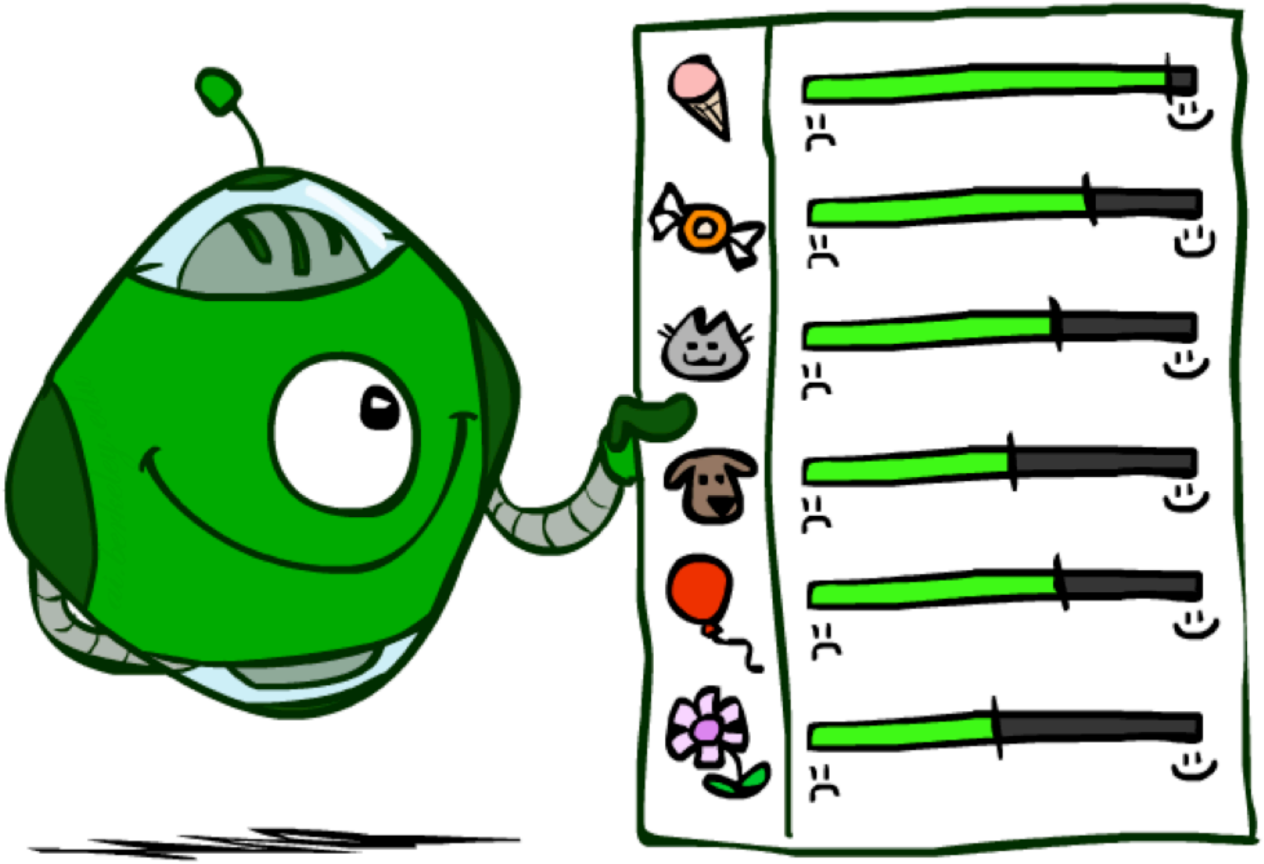
Multi-Agent Utilities

- What if the game is not zero-sum, or has multiple players?

- Generalization of minimax:
  - Terminals have utility tuples
  - Node values are also utility tuples
  - Each player maximizes its own component
  - Can give rise to cooperation and competition dynamically…



Utilities



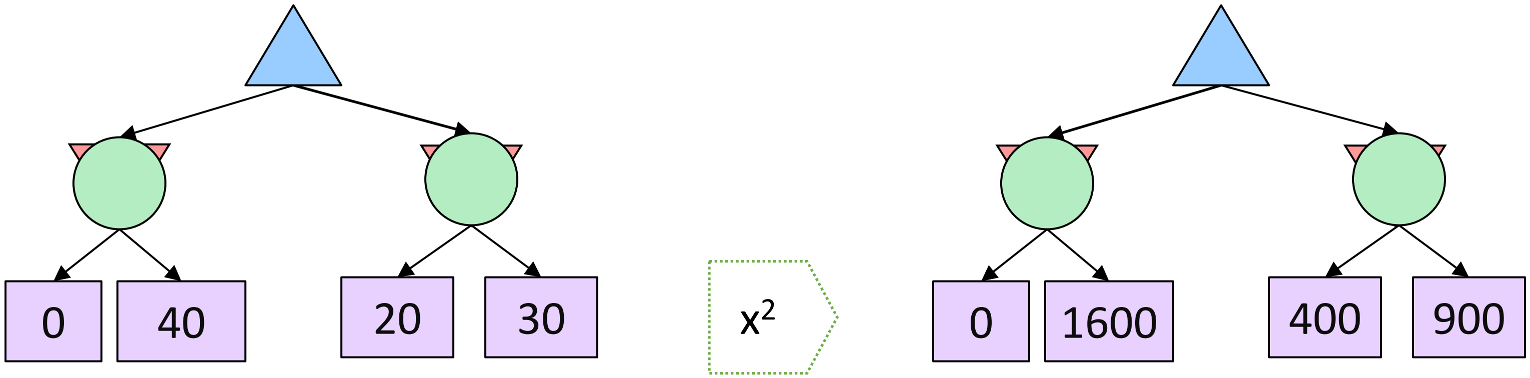
Maximum Expected Utility

- Principle of maximum expected utility:
  - A rational agent should chose the action that maximizes its expected utility, given its knowledge

$$action = argmax\ ExpectedUtility(a|e)$$

- Questions:
  - Where do utilities come from?
  - How do we know such utilities even exist?
  - How do we know that averaging even makes sense?
  - What if our behavior (preferences) can't be described by utilities?

What Utilities to Use?



- For worst-case minimax reasoning, terminal function scale doesn't matter
  - We just want better states to have higher evaluations (get the ordering right)

- For average-case expectimax reasoning, we need *magnitudes* to be meaningful

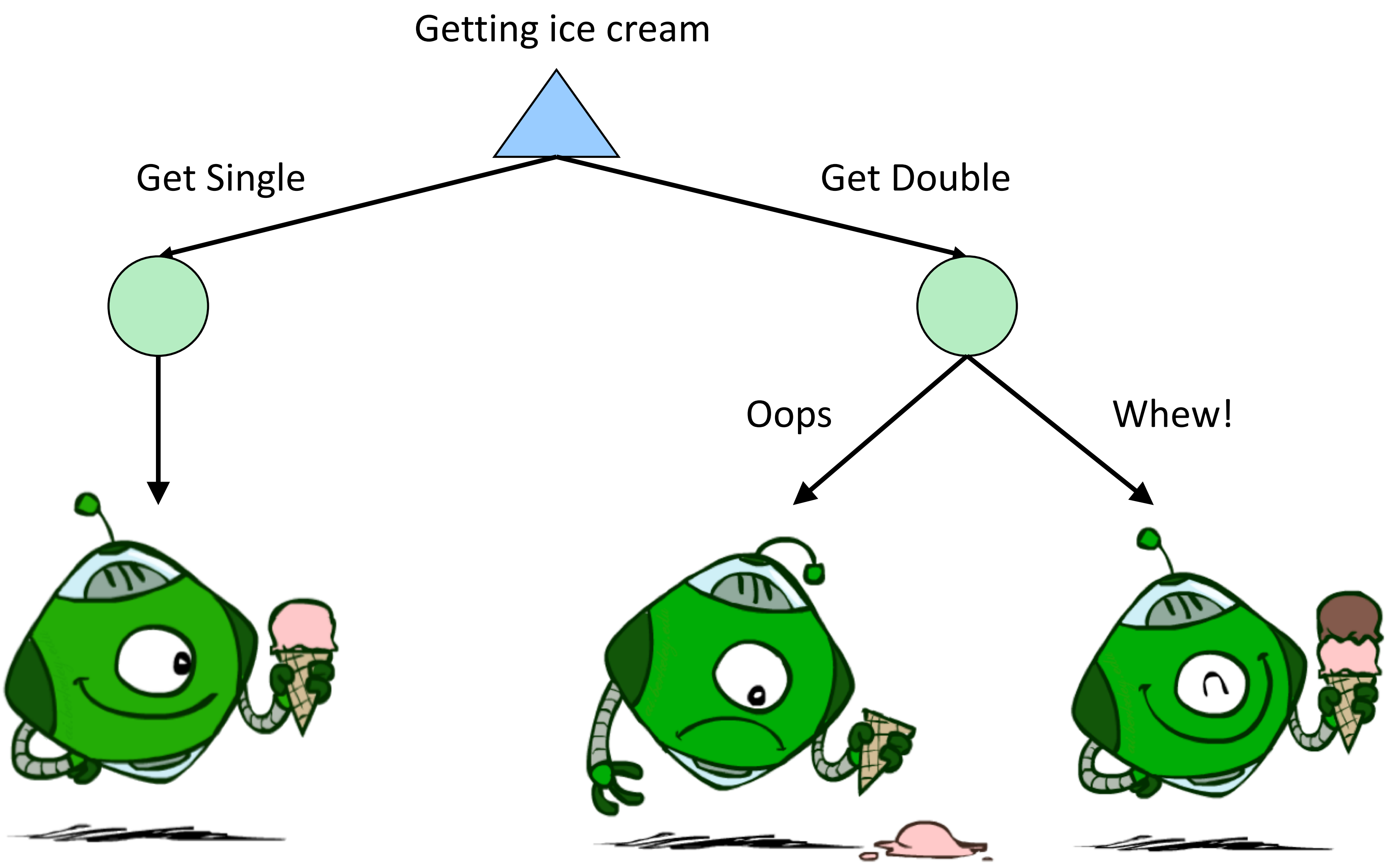
Utilities

- Utilities are functions **from outcomes (states of the world) to real numbers** that describe an agent's preferences

- Where do utilities come from?
  - In a game, may be simple (+1/-1)
  - Utilities summarize the agent's goals
  - Theorem: any "rational" preferences can be summarized as a utility function

- We hard-wire utilities and let behaviors emerge

Utilities: Uncertain Outcomes



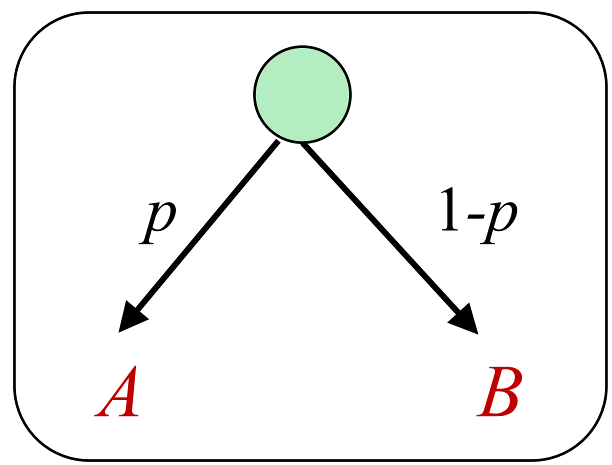
Preferences

- An agent must have preferences among:
  - Prizes: A, B, etc.
  - Lotteries: situations with uncertain prizes

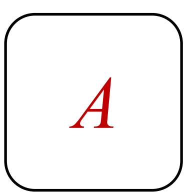
$$L = [p, A;\ (1-p), B]$$

- Notation:
  - Preference: $A \succ B$
  - Indifference: $A \sim B$

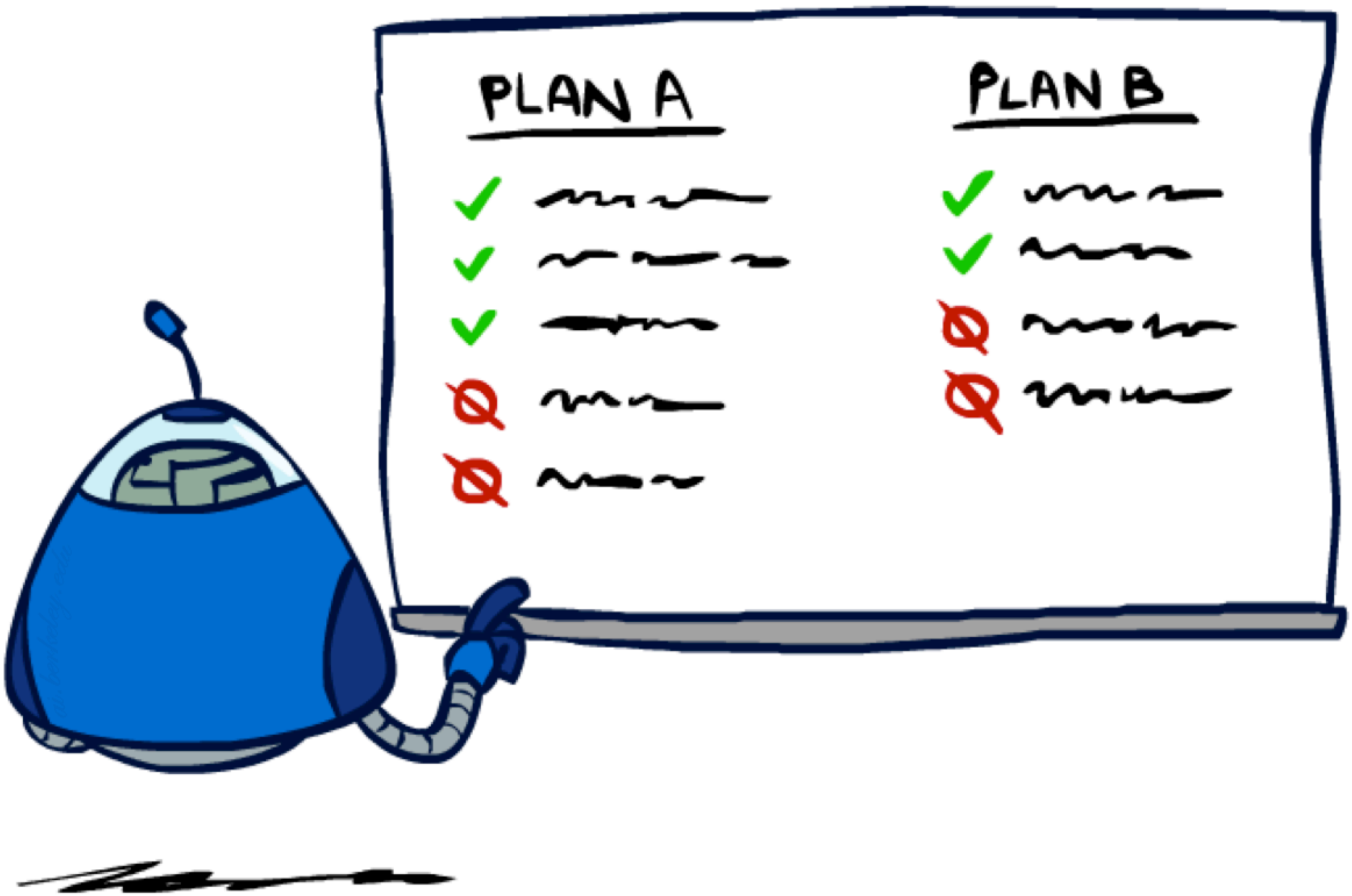
A Lottery



A Prize



Rationality

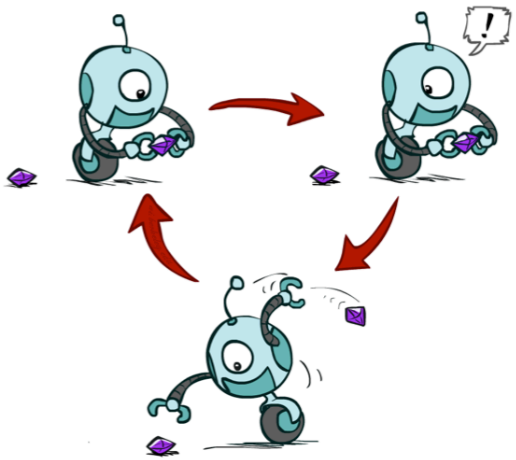


Rational Preferences

- We want some constraints on preferences before we call them rational, such as:

$$\text{Axiom of Transitivity: } (A \succ B) \wedge (B \succ C) \Rightarrow (A \succ C)$$

- For example: an agent with intransitive preferences can be induced to give away all of its money
  - If B > C, then an agent with C would pay (say) 1 cent to get B
  - If A > B, then an agent with B would pay (say) 1 cent to get A
  - If C > A, then an agent with A would pay (say) 1 cent to get C



The Axioms of Rationality

Orderability

$$(A \succ B) \vee (B \succ A) \vee (A \sim B)$$

Transitivity

$$(A \succ B) \wedge (B \succ C) \Rightarrow (A \succ C)$$

Continuity

$$A \succ B \succ C \Rightarrow \exists p \; [p, A; \; 1-p, C] \sim B$$

Substitutability

$$A \sim B \Rightarrow [p, A; \; 1-p, C] \sim [p, B; 1-p, C]$$

Monotonicity

$$A \succ B \Rightarrow$$
$$(p \geq q \Leftrightarrow [p, A; \; 1-p, B] \succeq [q, A; \; 1-q, B])$$

Theorem: Rational preferences imply behavior describable as maximization of expected utility → Rationality!

MEU Principle

- Theorem [Ramsey, 1931; von Neumann & Morgenstern, 1944]
  - Given any preferences satisfying these constraints, there exists a real-valued
  - function U such that:

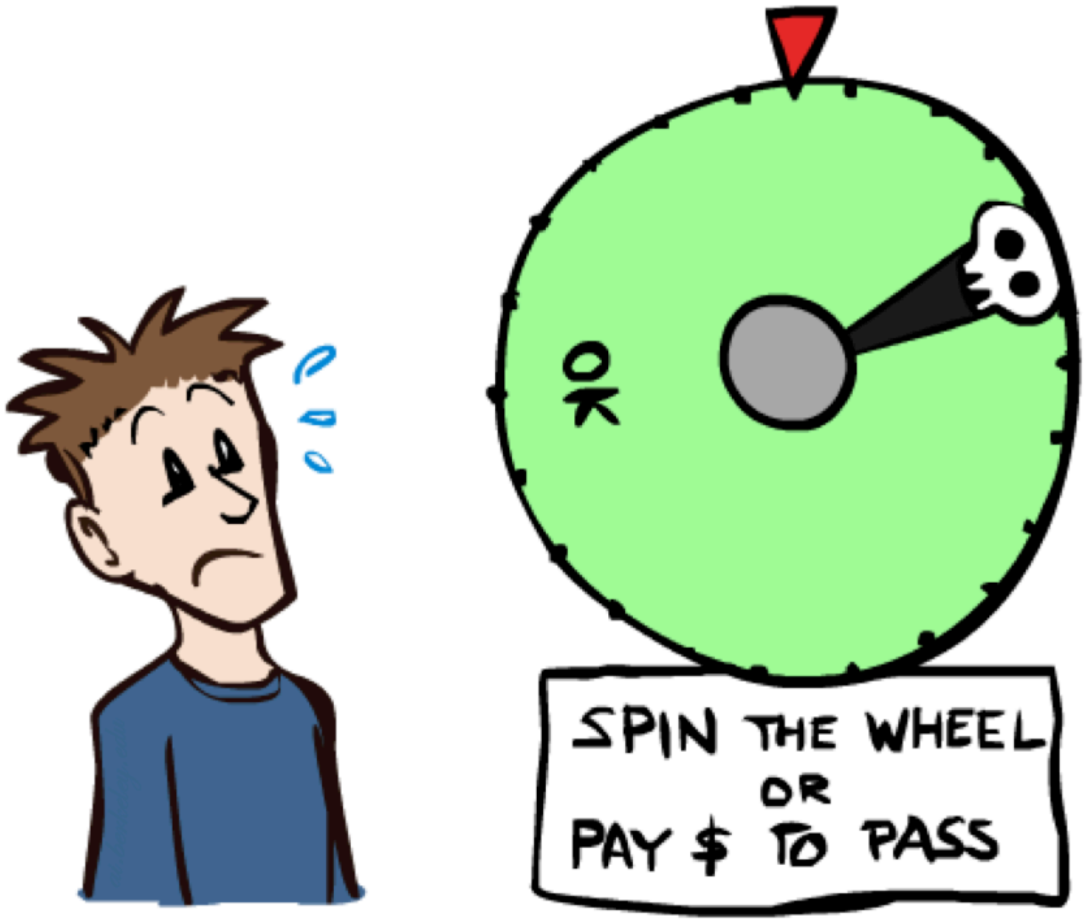
$$U(A) \geq U(B) \;\Leftrightarrow\; A \succeq B$$

$$U([p_1, S_1; \; \ldots \; ; \; p_n, S_n]) = \sum_i p_i U(S_i)$$

  - I.e. values assigned by U preserve preferences of both prizes and lotteries!

- Maximum expected utility (MEU) principle:
  - Choose the action that maximizes expected utility
  - Note: an agent can be entirely rational (consistent with MEU) without ever representing or manipulating utilities and probabilities
  - E.g., a lookup table for perfect tic-tac-toe, a reflex vacuum cleaner

Human Utilities



Utility Scales

- Normalized utilities: $u_+ = 1.0$, $u_- = 0.0$

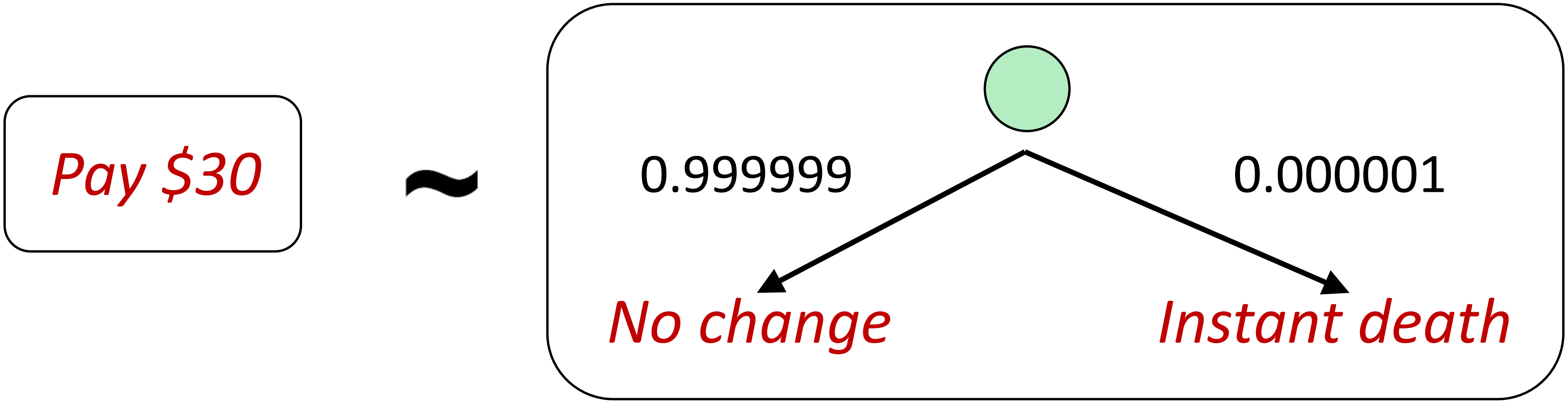
- Micromorts: one-millionth chance of death, useful for paying to reduce product risks, etc.

- QALYs (quality adjusted life year): quality-adjusted life years, useful for medical decisions involving substantial risk

- With deterministic prizes only (no lottery choices), only ordinal utility can be determined, i.e., total order on prizes

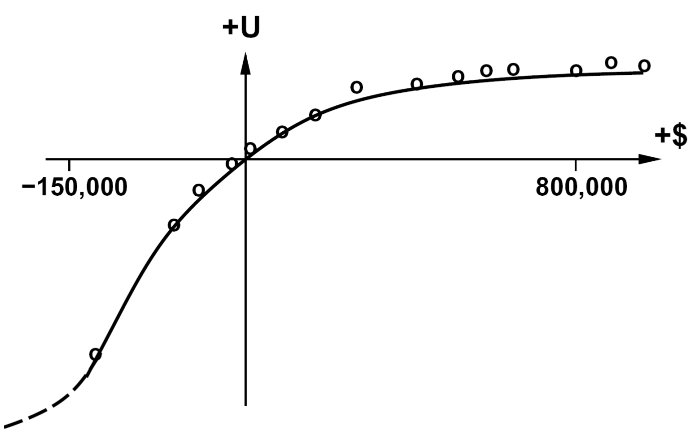
Human Utilities

- Utilities map states to real numbers. Which numbers?
- Standard approach to assessment of human utilities:
  - Compare a prize A to a standard lottery L_p between
    - "best possible prize" u_+
    - "worst possible catastrophe" u_-
  - Adjust lottery probability p until indifference: A ~ L_p
  - Resulting p is a utility in [0,1]



Money

- We can use having money (or being in debt) as the the utility.
- Given a lottery L = [p, $X; (1-p), $Y]
  - The expected monetary value EMV(L) is p*X + (1-p)*Y
  - U(L) = p*U($X) + (1-p)*U($Y)
  - Typically, U(L) < U(EMV(L))
  - In this sense, people are risk-averse
  - When deep in debt, people are risk-seeking



Example: Insurance

- Consider the lottery [0.5, $1000; 0.5, $0]
  - What is its expected monetary value? ($500)
  - What is its certainty equivalent?
    - Monetary value acceptable in lieu of lottery
    - $400 for most people
  - Difference of $100 is the insurance premium
    - There's an insurance industry because people will pay to reduce their risk
    - If everyone were risk-neutral, no insurance needed!
  - It's win-win: you'd rather have the $400 and the insurance company would rather have the lottery (their utility curve is flat and they have many lotteries)

Example: Human Rationality?

- Famous example of Allais (1953)
  - A: [0.8, \$4k;   0.2, \$0]
  - B: [1.0, \$3k;   0.0, \$0]

  - C: [0.2, \$4k;   0.8, \$0]
  - D: [0.25, \$3k;   0.75, \$0]

- Most people prefer B > A, C > D

- But if U(\$0) = 0, then
  - B > A $\Rightarrow$ U(\$3k) > 0.8 U(\$4k)
  - C > D $\Rightarrow$ 0.8 U(\$4k) > U(\$3k)