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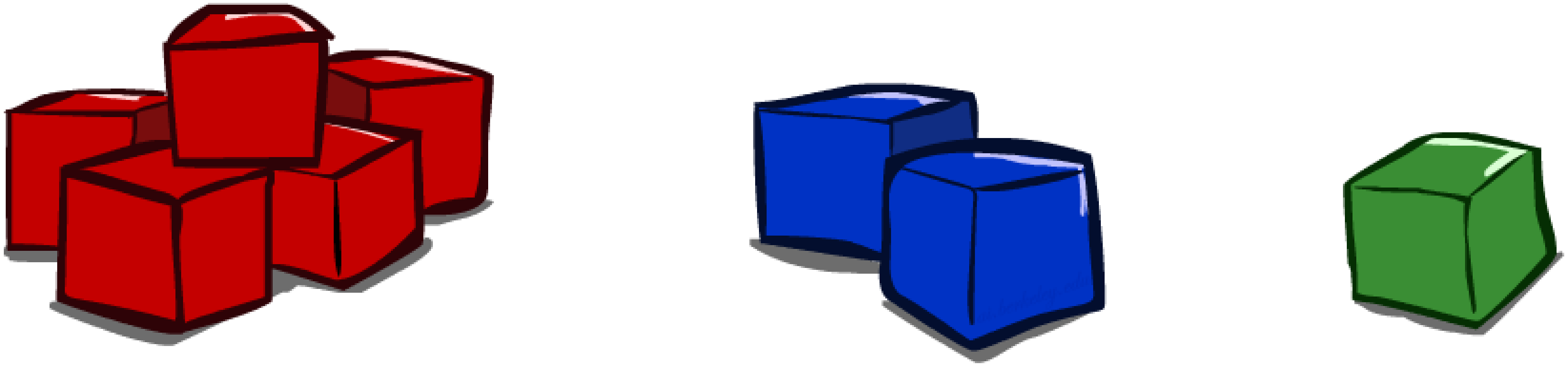
Bayes' Nets: Sampling

May 30th, 2018

Approximate Inference: Sampling



Sampling

- Sampling is a lot like repeated simulation
  - Predicting the weather, basketball games, …
- Basic idea
  - Draw N samples from a sampling distribution S
  - Compute an approximate posterior probability
  - Show this converges to the true probability P
- Why sample?
  - Inference: getting a sample is faster than computing the right answer (e.g. with variable elimination)
  - Learning: get samples from a distribution you don’t know

Sampling

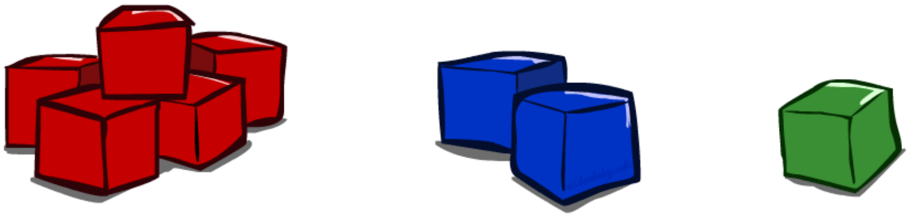
- Sampling from given distribution
  - Step 1: Get sample *u* from uniform distribution over [0, 1)
    - E.g. random() in python
  - Step 2: Convert this sample *u* into an outcome for the given distribution by having each outcome associated with a sub-interval of [0,1) with sub-interval size equal to probability of the outcome

- Example

C	P(C)
red	0.6
green	0.1
blue	0.3

$$0 \leq u < 0.6, \rightarrow C = red$$
$$0.6 \leq u < 0.7, \rightarrow C = green$$
$$0.7 \leq u < 1, \rightarrow C = blue$$

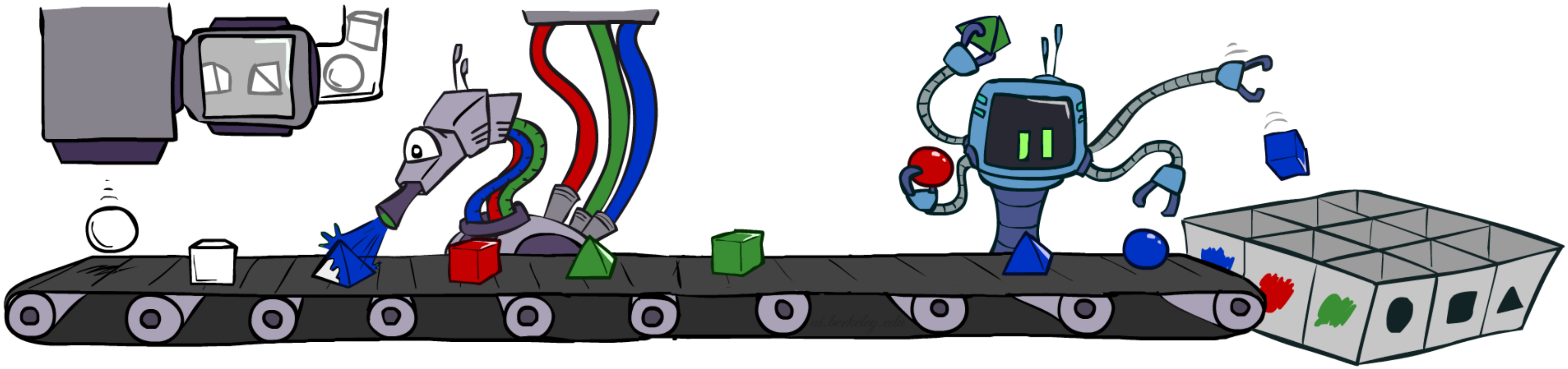
  - If random() returns *u* = 0.83, then our sample is *C* = blue
  - E.g, after sampling 8 times:



Sampling in Bayes' Nets

- Prior Sampling
- Rejection Sampling
- Likelihood Weighting
- Gibbs Sampling

Prior Sampling



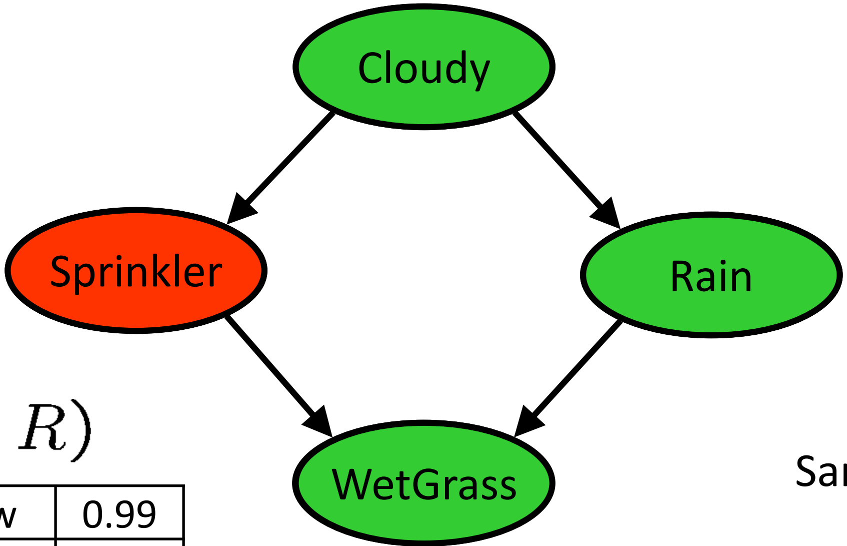
Prior Sampling

$P(C)$

+c	0.5
-c	0.5

$P(S|C)$

+c	+s	0.1
	-s	0.9
-c	+s	0.5
	-s	0.5



$P(R|C)$

+c	+r	0.8
	-r	0.2
-c	+r	0.2
	-r	0.8

$P(W|S,R)$

+s	+r	+w	0.99
		-w	0.01
	-r	+w	0.90
		-w	0.10
-s	+r	+w	0.90
		-w	0.10
	-r	+w	0.01
		-w	0.99

Samples:

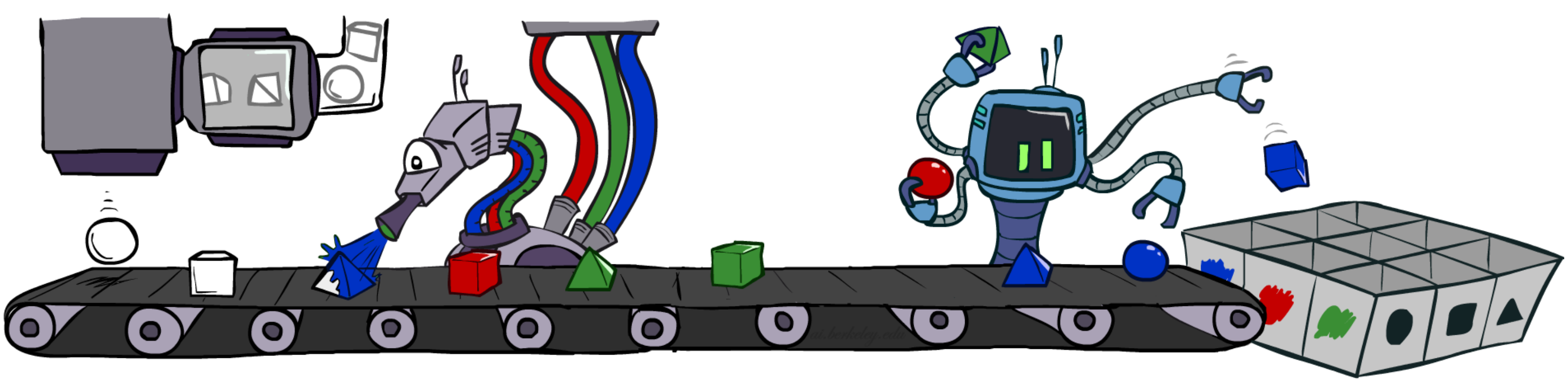
+c, -s, +r, +w

-c, +s, -r, +w

…

Prior Sampling

- For i=1, 2, …, n
  - Sample x_i from $P(X_i \mid Parents(X_i))$
- Return $(x_1, x_2, \ldots, x_n)$



Prior Sampling

- This process generates samples with probability:

$$S_{PS}(x_1 \ldots x_n) = \prod_{i=1}^{n} P(x_i|\text{Parents}(X_i)) = P(x_1 \ldots x_n)$$

...i.e. the BN's joint probability

- Let the number of samples of an event be $N_{PS}(x_1 \ldots x_n)$

- Then

$$\begin{aligned} \lim_{N\to\infty} \hat{P}(x_1, \ldots, x_n) &= \lim_{N\to\infty} N_{PS}(x_1, \ldots, x_n)/N \\ &= S_{PS}(x_1, \ldots, x_n) \\ &= P(x_1 \ldots x_n) \end{aligned}$$

- I.e., the sampling procedure is consistent

Example

- We'll get a bunch of samples from the BN:

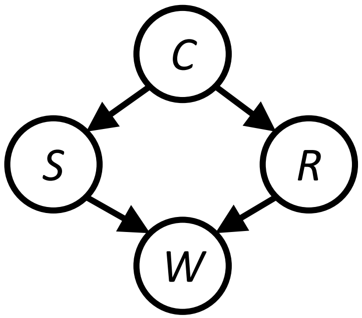
  +c, -s, +r, +w

  +c, +s, +r, +w

  -c, +s, +r, -w

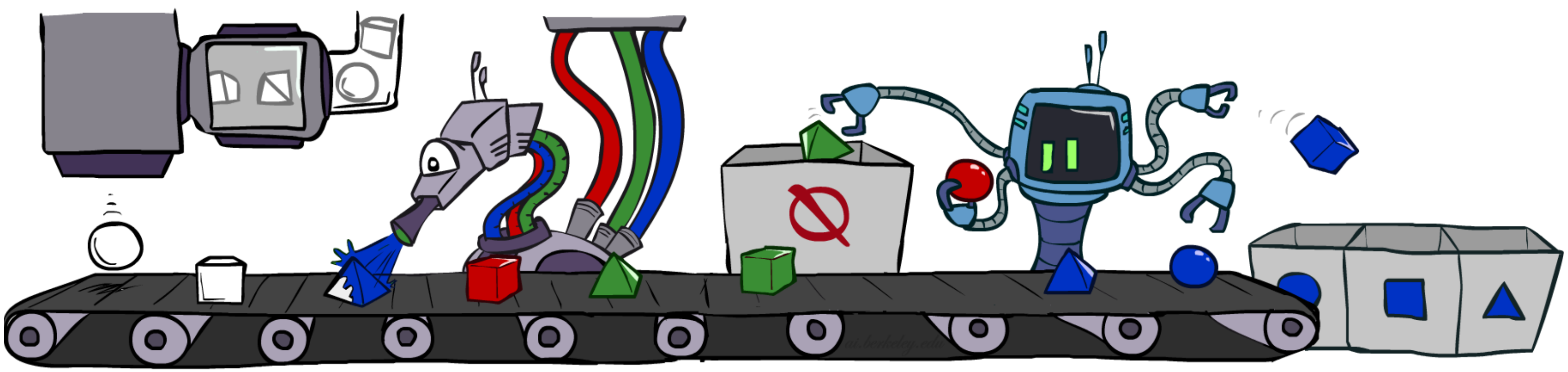
  +c, -s, +r, +w

  -c, -s, -r, +w



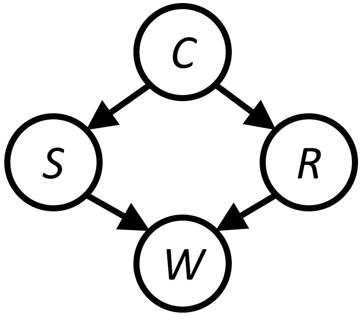
- If we want to know P(W)
  - We have counts <+w:4, -w:1>
  - Normalize to get P(W) = <+w:0.8, -w:0.2>
  - This will get closer to the true distribution with more samples
  - What about P(C| +w)? P(C| +r, +w)? P(C| -r, -w)?
  - Fast: can use fewer samples if less time (what's the drawback?)

Rejection Sampling



Rejection Sampling

- Let's say we want P(C)
  - No point keeping all samples around
  - Just use counts of C as we go

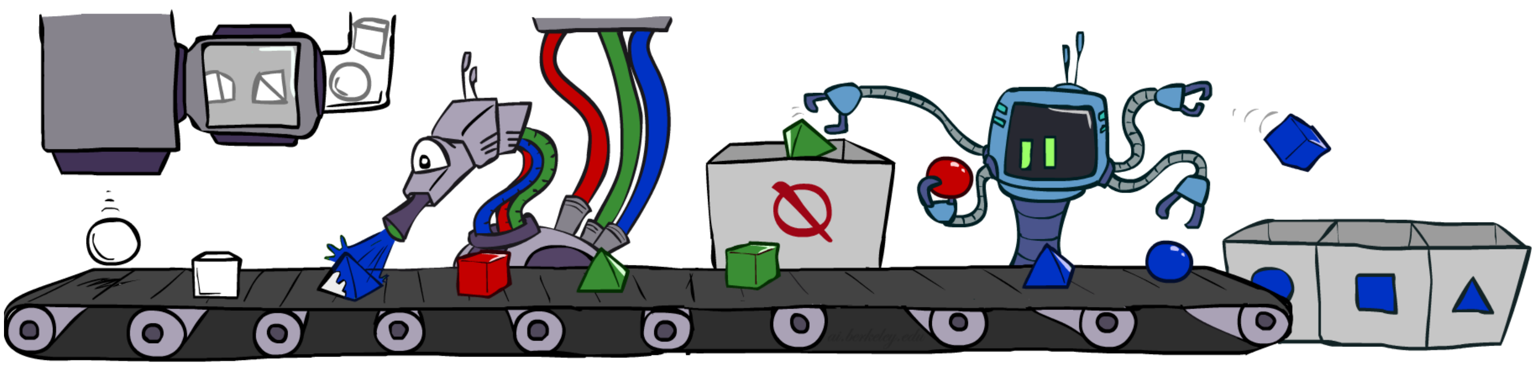


- Let's say we want P(C| +s)
  - Compute C outcomes, but ignore (reject) samples which don't have S=+s
  - This is called rejection sampling
  - It is also consistent for conditional probabilities (i.e., correct in the limit)

+c, -s, +r, +w
+c, +s, +r, +w
-c, +s, +r, -w
+c, -s, +r, +w
-c, -s, -r, +w

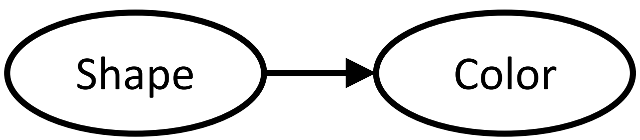
Rejection Sampling

- IN: evidence instantiation
- For i=1, 2, …, n
  - Sample x_i from $P(X_i \mid Parents(X_i))$
  - If x_i not consistent with evidence
    - Reject: Return, and no sample is generated in this cycle
- Return $(x_1, x_2, \ldots, x_n)$

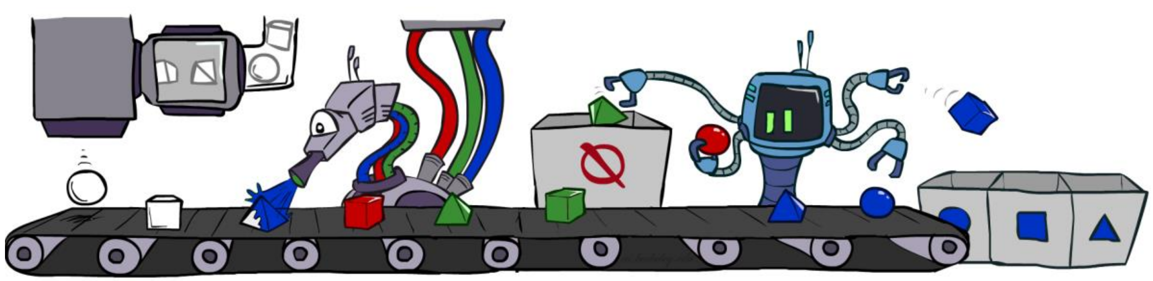


Problem with rejection sampling:

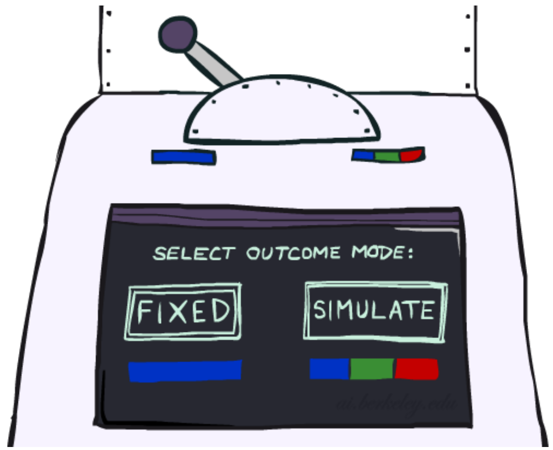
- If evidence is unlikely, rejects lots of samples
- Evidence not exploited as you sample
- Consider P(Shape|blue)

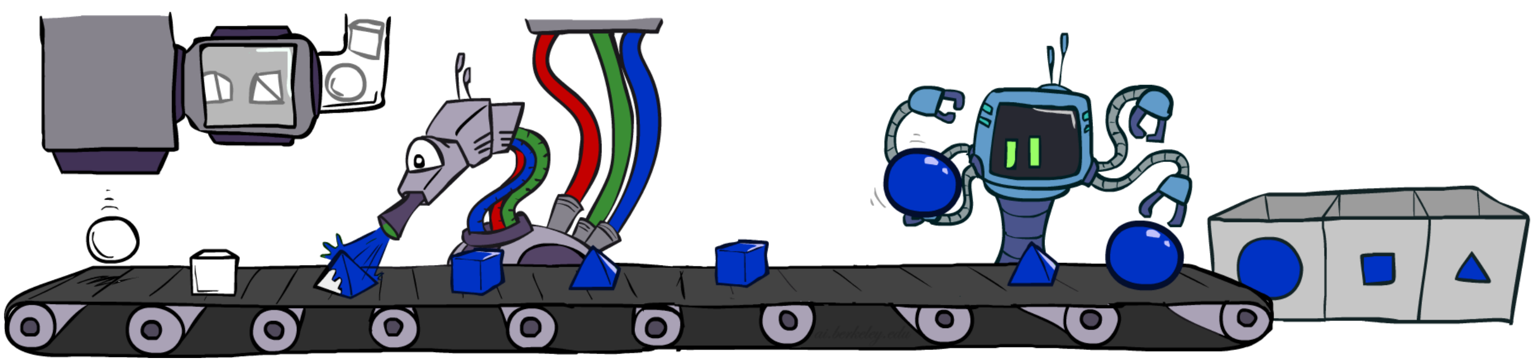


~~pyramid, green~~
~~pyramid, red~~
sphere, blue
~~cube, red~~
~~sphere, green~~



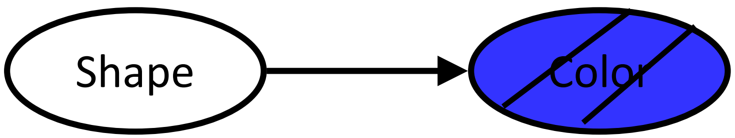
Likelihood Weighting



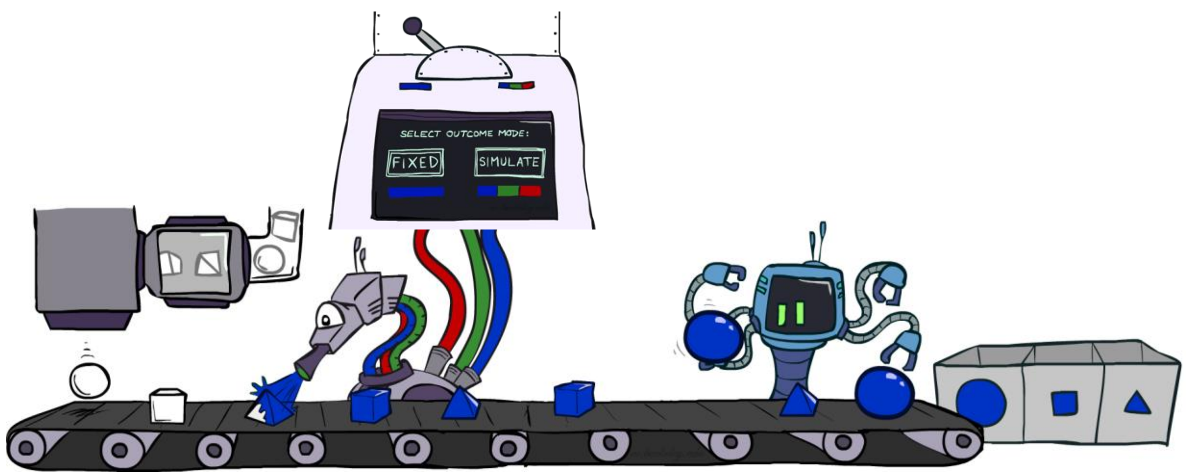


Likelihood Weighting

- Idea: fix evidence variables and sample the rest
  - Problem: sample distribution not consistent!
  - Solution: weight by probability of evidence given parents



pyramid, blue
pyramid, blue
sphere, blue
cube, blue
sphere, blue



Likelihood Weighting

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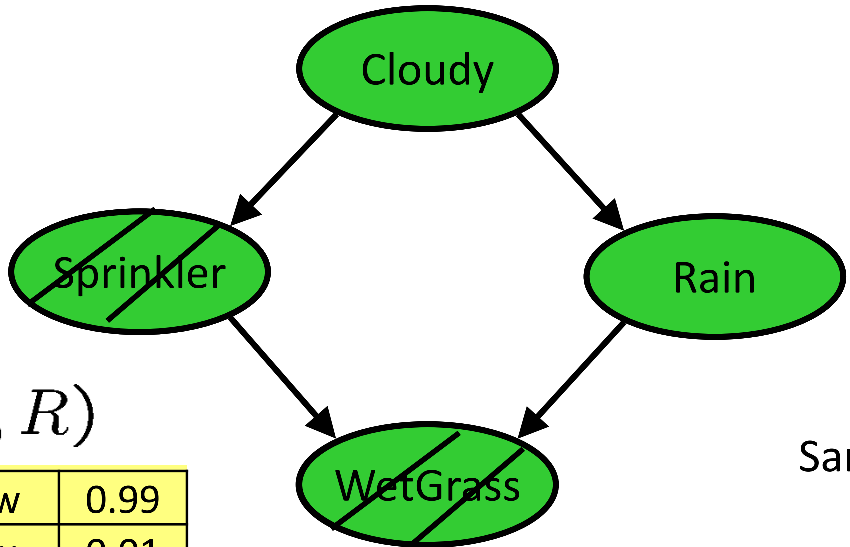

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-s	+r	+w	0.90
		-w	0.10
	-r	+w	0.01
		-w	0.99

Samples:

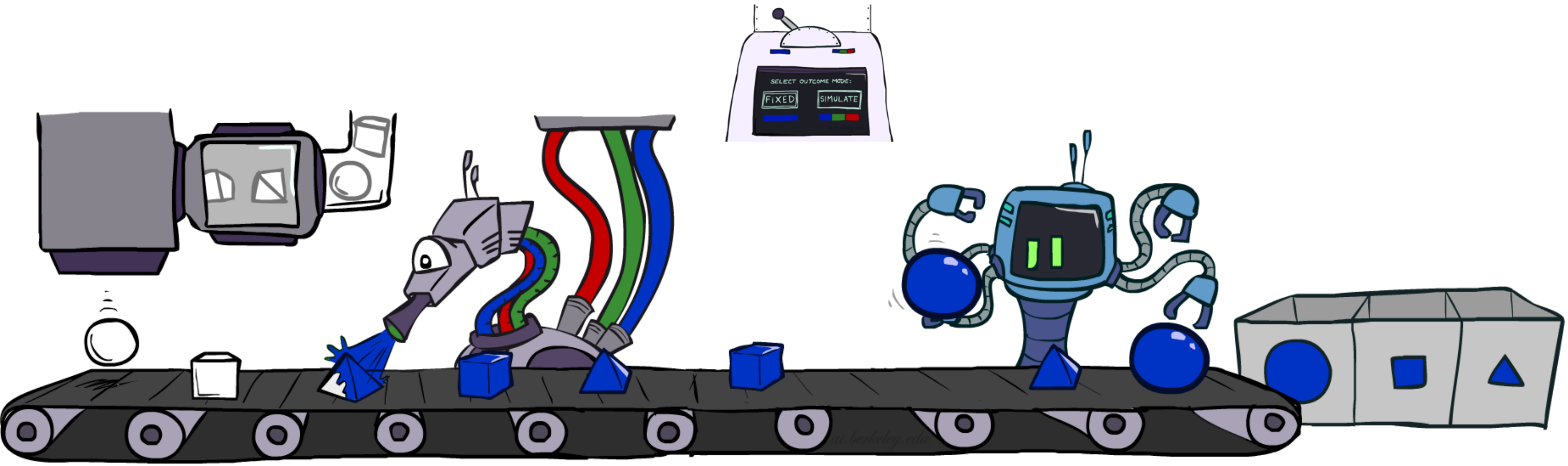
+c, +s, +r, +w

...

$$w = 1.0 \times 0.1 \times 0.99$$

Likelihood Weighting

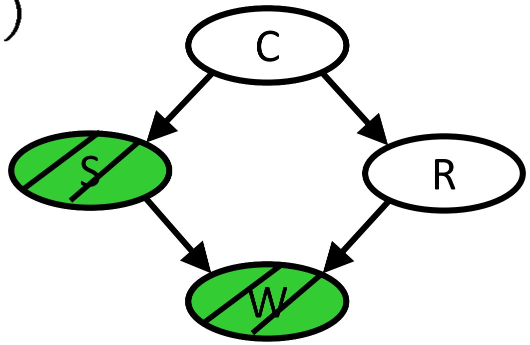
- IN: evidence instantiation
- w = 1.0
- for i=1, 2, …, n
  - if X_i is an evidence variable
    - X_i = observation x_i for X_i
    - Set w = w * $P(x_i \mid Parents(X_i))$
  - else
    - Sample x_i from $P(X_i \mid Parents(X_i))$
- return $(x_1, x_2, \ldots, x_n)$, w



Likelihood Weighting

- Sampling distribution if z sampled and e fixed evidence

$$S_{WS}(\mathbf{z}, \mathrm{e}) = \prod_{i=1}^{l} P(z_i|\text{Parents}(Z_i))$$



- Now, samples have weights

$$w(\mathbf{z}, \mathbf{e}) = \prod_{i=1}^{m} P(e_i|\text{Parents}(E_i))$$

- Together, weighted sampling distribution is consistent

$$S_{\mathrm{WS}}(z, e) \cdot w(z, e) = \prod_{i=1}^{l} P(z_i|\mathrm{Parents}(z_i)) \prod_{i=1}^{m} P(e_i|\mathrm{Parents}(e_i))$$

$$= P(\mathbf{z}, \mathrm{e})$$

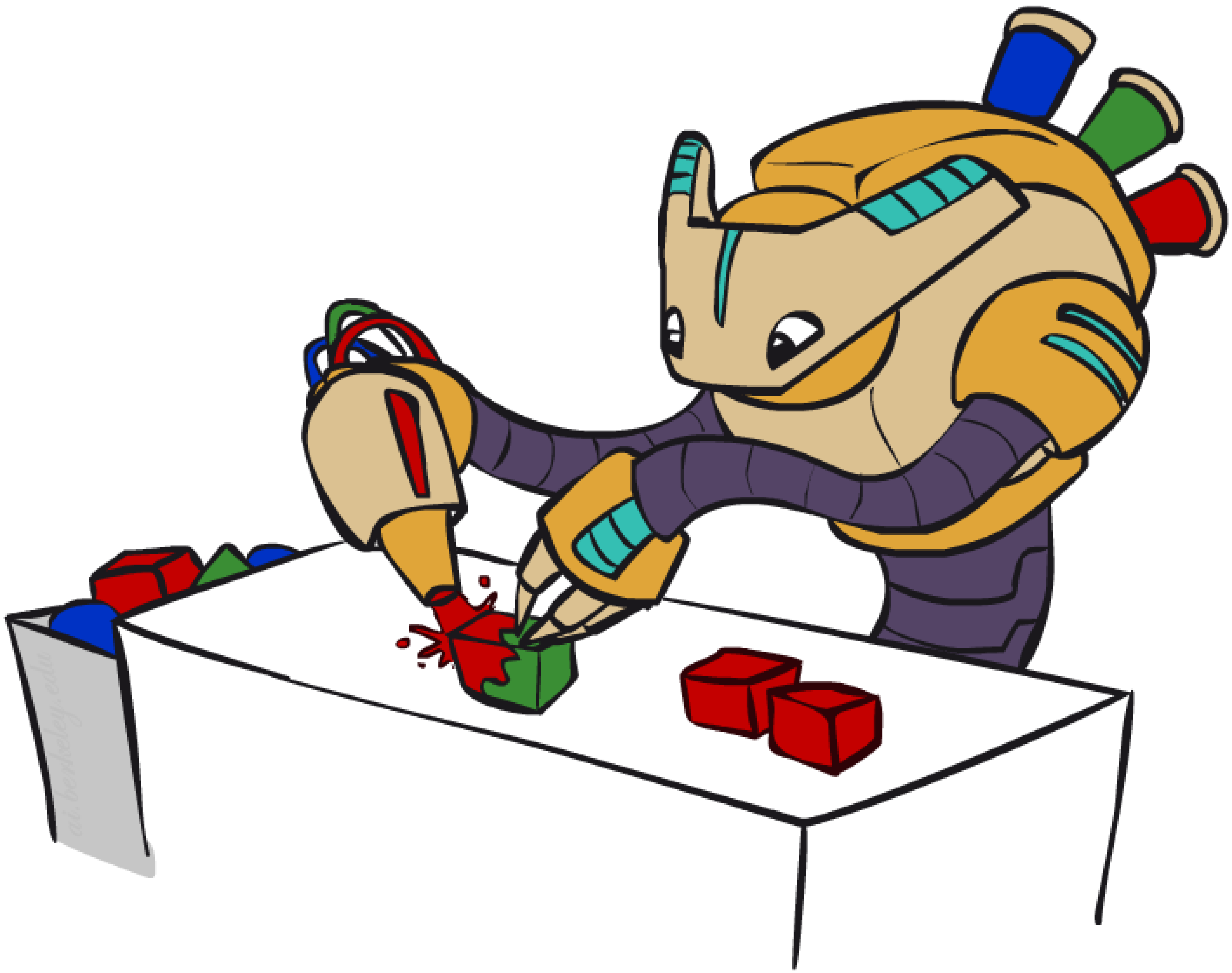
Likelihood Weighting

- Likelihood weighting is good
  - We have taken evidence into account as we generate the sample
  - More of our samples will reflect the state of the world suggested by the evidence

- Likelihood weighting doesn't solve all our problems
  - Evidence influences the choice of downstream variables, but not upstream ones (C isn't more likely to get a value matching the evidence)
- We would like to consider evidence when we sample every variable

  → Gibbs sampling

Gibbs Sampling



Gibbs Sampling

- *Procedure:*
  - keep track of a full instantiation $x_1, x_2, \ldots, x_n$.
  - Start with an arbitrary instantiation consistent with the evidence.
  - Sample one variable at a time, conditioned on all the rest, but keep evidence fixed.
  - Keep repeating this for a long time.

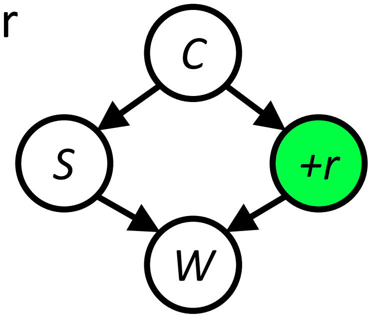
- *Property:* in the limit of repeating this infinitely many times the resulting sample is coming from the correct distribution

- *Rationale*: both upstream and downstream variables condition on evidence.

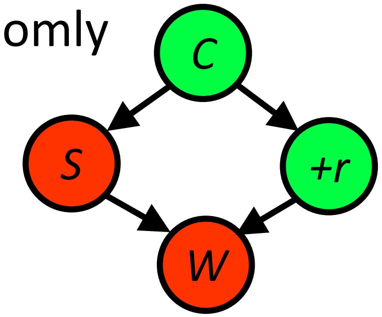
- In contrast: likelihood weighting only conditions on upstream evidence, and hence weights obtained in likelihood weighting can sometimes be very small. Sum of weights over all samples is indicative of how many "effective" samples were obtained, so want high weight.

Gibbs Sampling Example: P(S | +r)

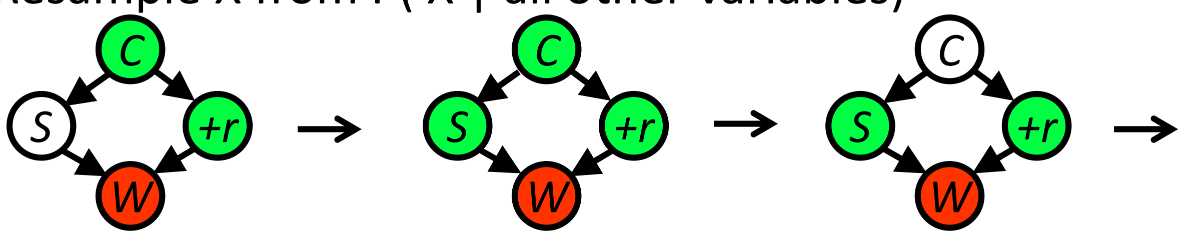
- Step 1: Fix evidence
  - R = +r



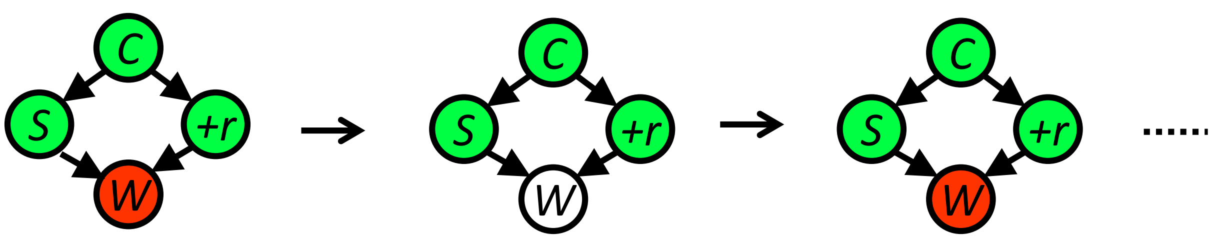
- Step 2: Initialize other variables
  - Randomly



- Steps 3: Repeat
  - Choose a non-evidence variable X
  - Resample X from P(X | all other variables)



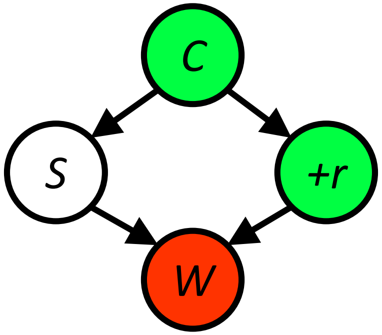
Sample from $P(S|+c, -w, +r)$ Sample from $P(C|+s, -w, +r)$



Sample from $P(W|+s, +c, +r)$

Efficient Resampling of One Variable

- Sample from P(S | +c, +r, -w)



$$
\begin{aligned}
P(S|+c,+r,-w) &= \frac{P(S,+c,+r,-w)}{P(+c,+r,-w)} \\
&= \frac{P(S,+c,+r,-w)}{\sum_s P(s,+c,+r,-w)} \\
&= \frac{P(+c)P(S|+c)P(+r|+c)P(-w|S,+r)}{\sum_s P(+c)P(s|+c)P(+r|+c)P(-w|s,+r)} \\
&= \frac{P(+c)P(S|+c)P(+r|+c)P(-w|S,+r)}{P(+c)P(+r|+c)\sum_s P(s|+c)P(-w|s,+r)} \\
&= \frac{P(S|+c)P(-w|S,+r)}{\sum_s P(s|+c)P(-w|s,+r)}
\end{aligned}
$$

- More generally: only CPTs that have resampled variable need to be considered, and joined together

Gibbs Sampling

- How is this better than sampling from the full joint?
  - In a Bayes' Net, sampling a variable given all the other variables (e.g. P(R|S,C,W)) is usually much easier than sampling from the full joint distribution
    - Only requires a join on the variable to be sampled (in this case, a join on R)
    - The resulting factor only depends on the variable's parents, its children, and its children's parents (this is often referred to as its Markov blanket)

Sampling in Bayes' Nets

- Prior Sampling
- Rejection Sampling
- Likelihood Weighting
- Gibbs Sampling

Further Reading on Gibbs Sampling

- Gibbs sampling produces sample from the query distribution P(Q | e) in limit of re-sampling infinitely often

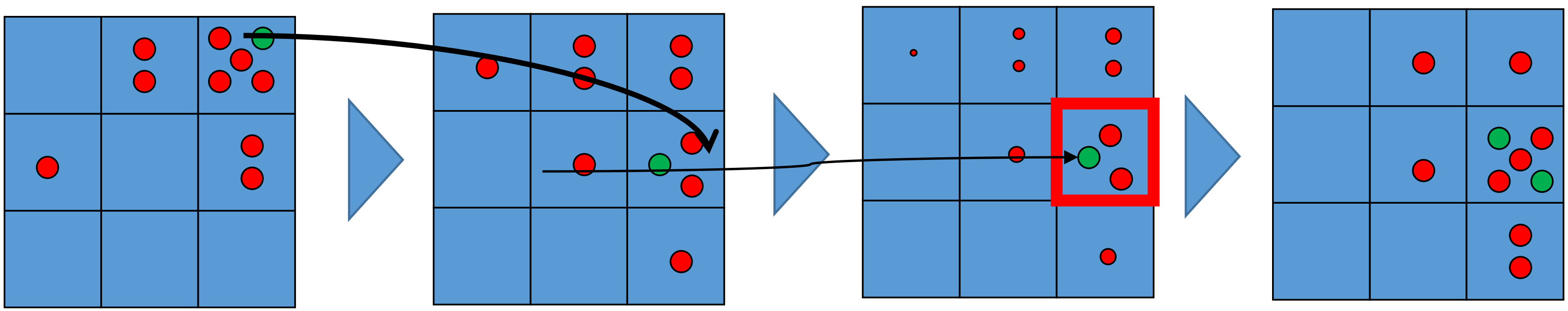
- Gibbs sampling is a special case of more general methods called Markov chain Monte Carlo (MCMC) methods

  - Metropolis-Hastings is one of the more famous MCMC methods (in fact, Gibbs sampling is a special case of Metropolis-Hastings)

How About Particle Filtering?

= likelihood weighting

Elapse | Weight | Resample

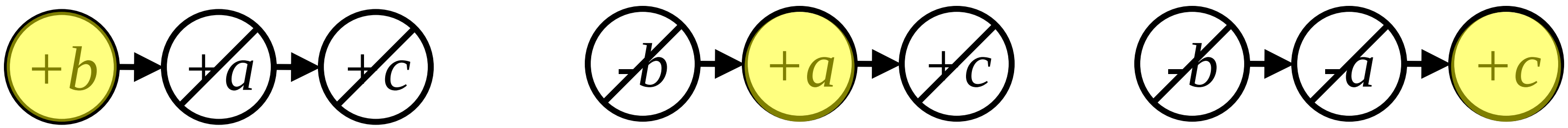


Particles:	Particles:	Particles:	(New) Particles:
(3,3)	(3,2)	(3,2) w=.9	(3,2)
(2,3)	(2,3)	(2,3) w=.2	(2,2)
(3,3)	(3,2)	(3,2) w=.9	(3,2)
(3,2)	(3,1)	(3,1) w=.4	(2,3)
(3,3)	(3,3)	(3,3) w=.4	(3,3)
(3,2)	(3,2)	(3,2) w=.9	(3,2)
(1,2)	(1,3)	(1,3) w=.1	(1,3)
(3,3)	(2,3)	(2,3) w=.2	(2,3)
(3,3)	(3,2)	(3,2) w=.9	(3,2)
(2,3)	(2,2)	(2,2) w=.4	(3,2)

Markov Chain Monte Carlo

- *Idea*: instead of sampling from scratch, create samples that are each like the last one.

- *Procedure*: resample one variable at a time, conditioned on all the rest, but keep evidence fixed. E.g., for P(b|c):



- *Properties*: Now samples are not independent (in fact they' re nearly identical), but sample averages are still consistent estimators!

- *What' s the point*: both upstream and downstream variables condition on evidence.