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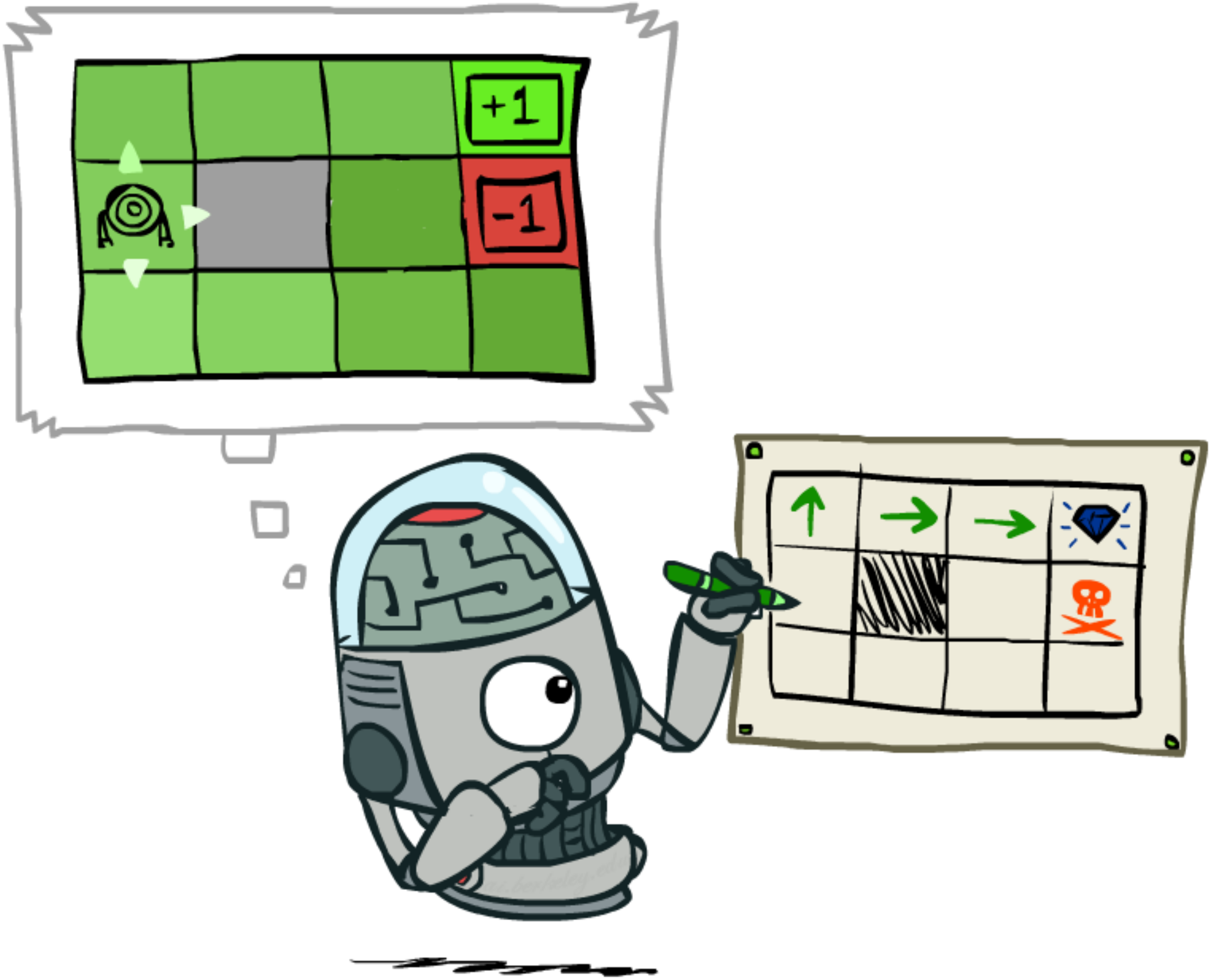
Markov Decision Processes II

April 25th, 2018

Policy Extraction



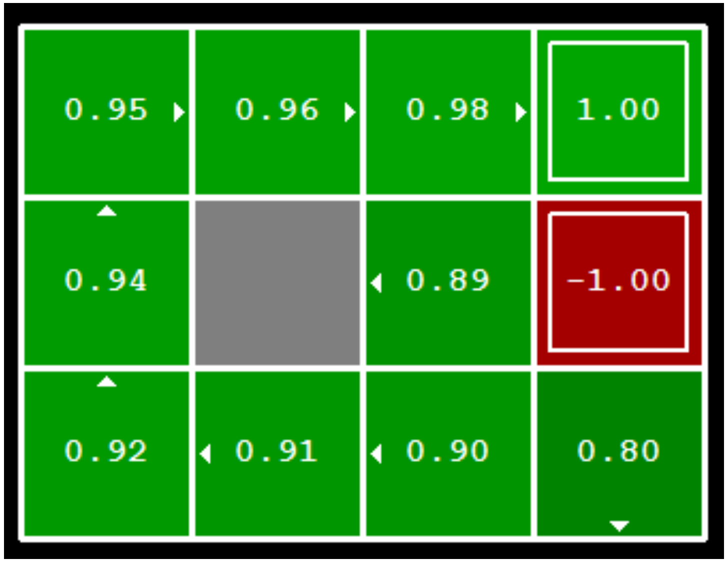
Computing Actions from Values

- Let's imagine we have the optimal values V*(s)
- How should we act?

- We need to do an expectimax (one step)

$$\pi^*(s) = \underset{a \in A(s)}{\operatorname{argmax}} \sum_{s'} P(s'|s,a) V^*(s')$$

This is called policy extraction, since it gets the policy implied by the values



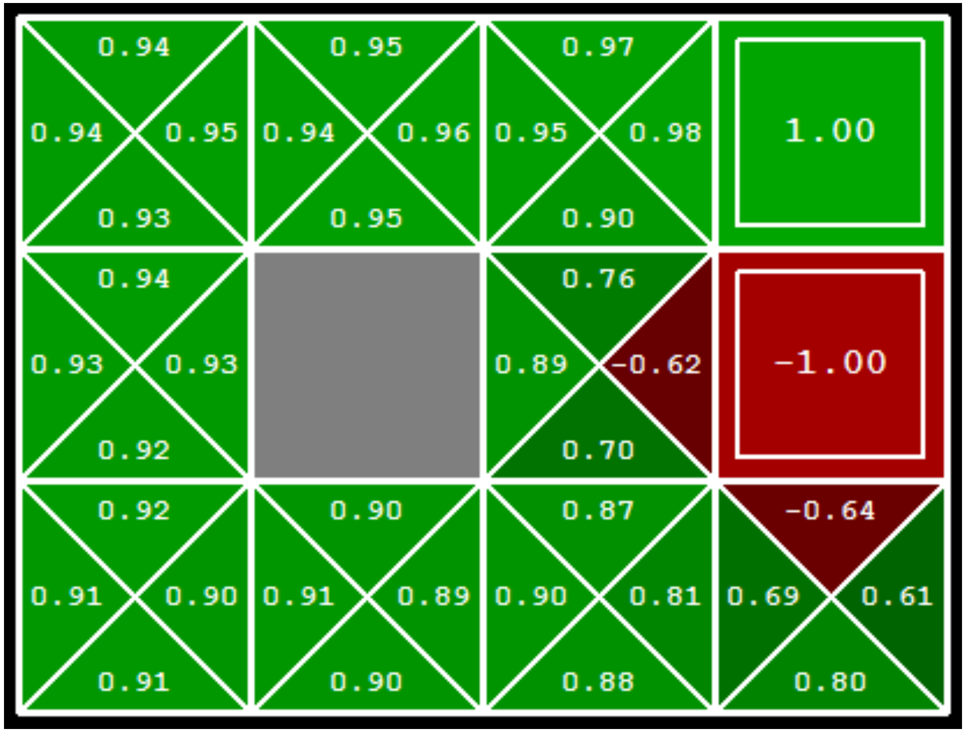
Computing Actions from Q-Values

- Let's imagine we have the optimal q-values:

- How should we act?

$$\pi^*(s) = \arg\max_a Q^*(s,a)$$

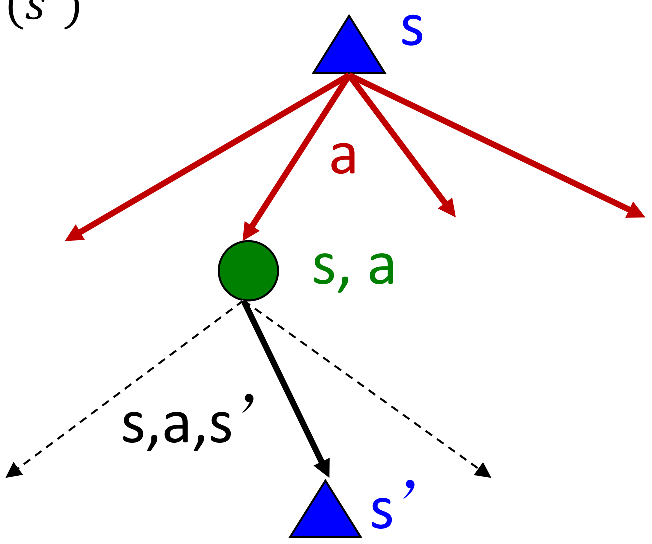
Important lesson: actions are easier to select from q-values than values!



Problems with Value Iteration

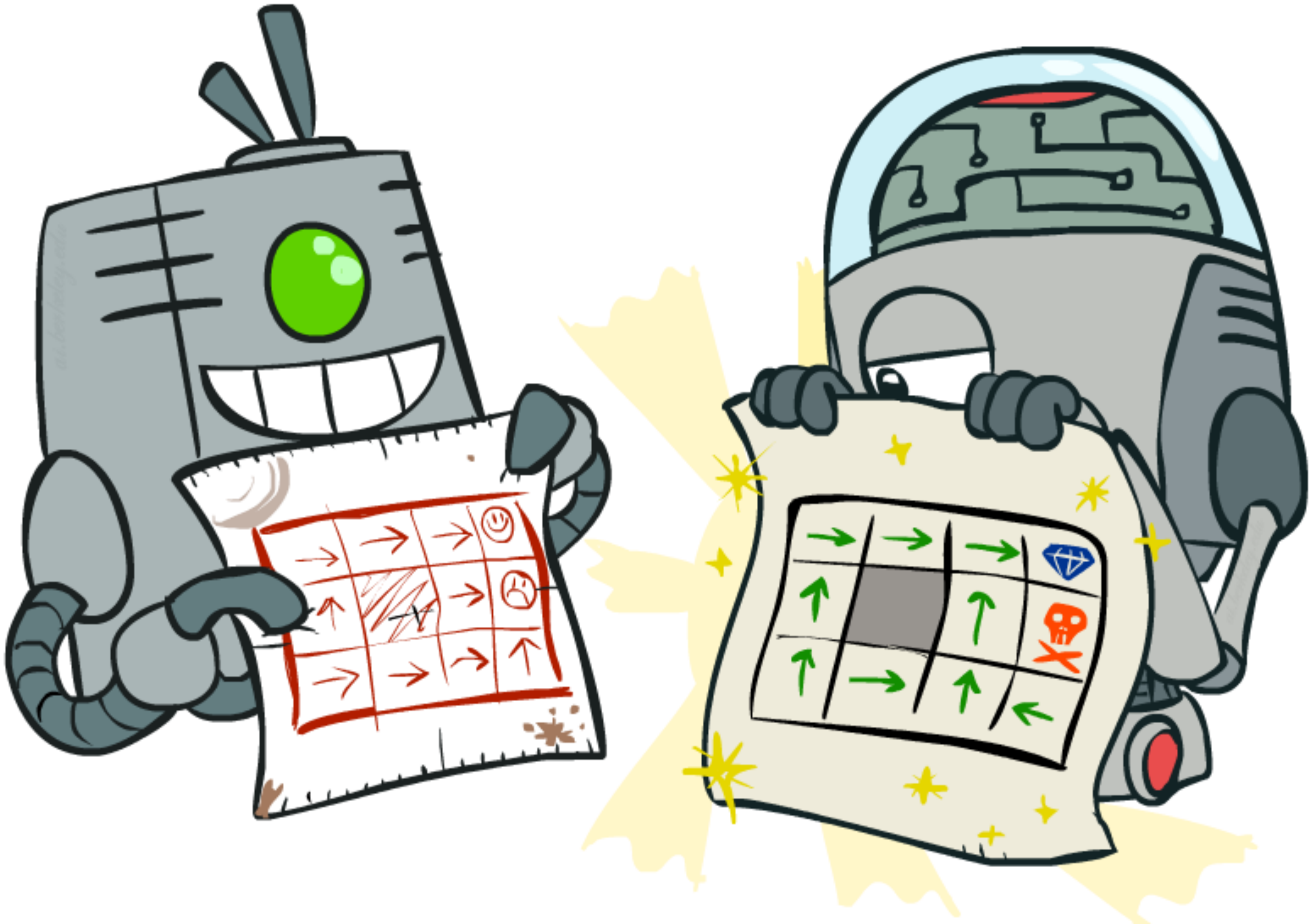
- Value iteration repeats the Bellman updates:

$$V_{k+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s'|s,a) V_k(s')$$

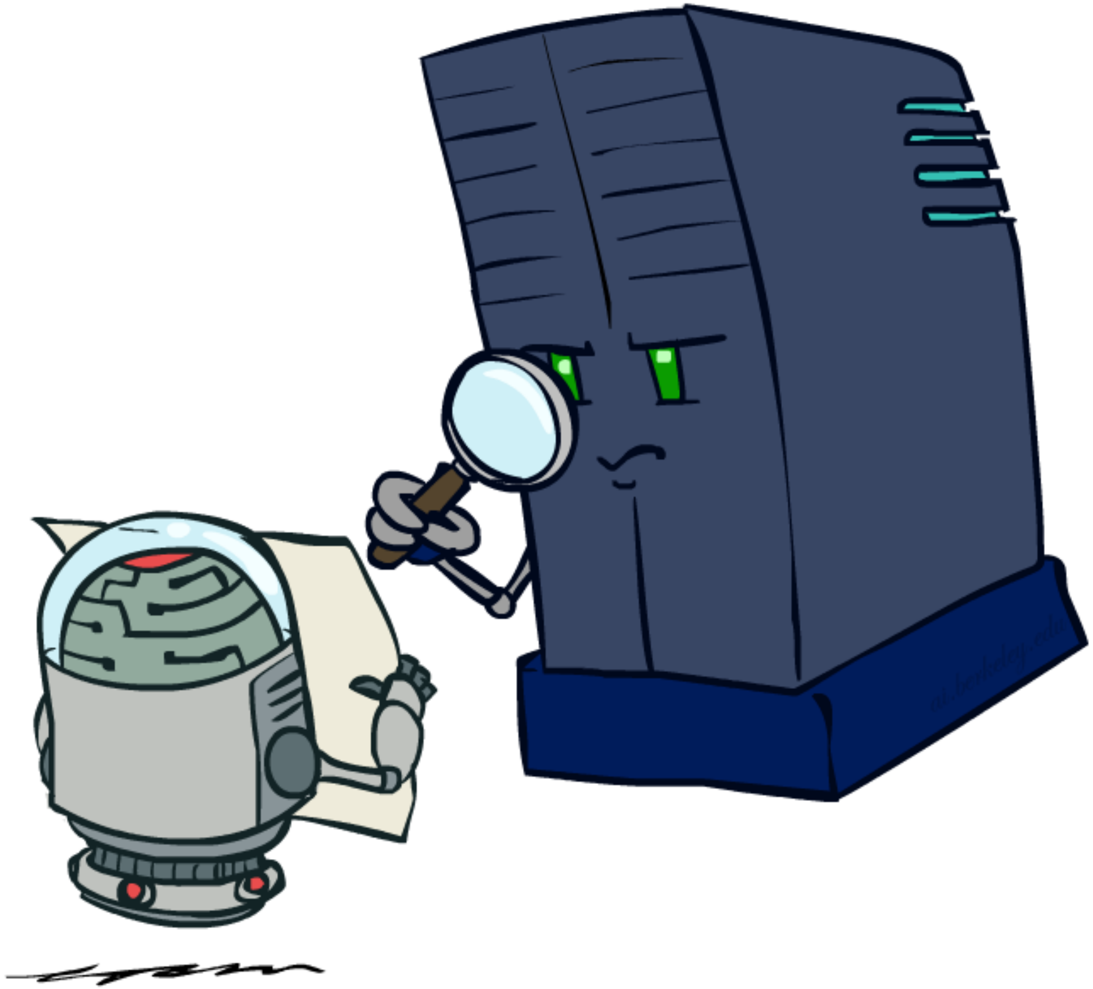


- Problem 1: It's slow – $O(S^2A)$ per iteration
- Problem 2: The "max" at each state rarely changes
- Problem 3: The policy often converges long before the values

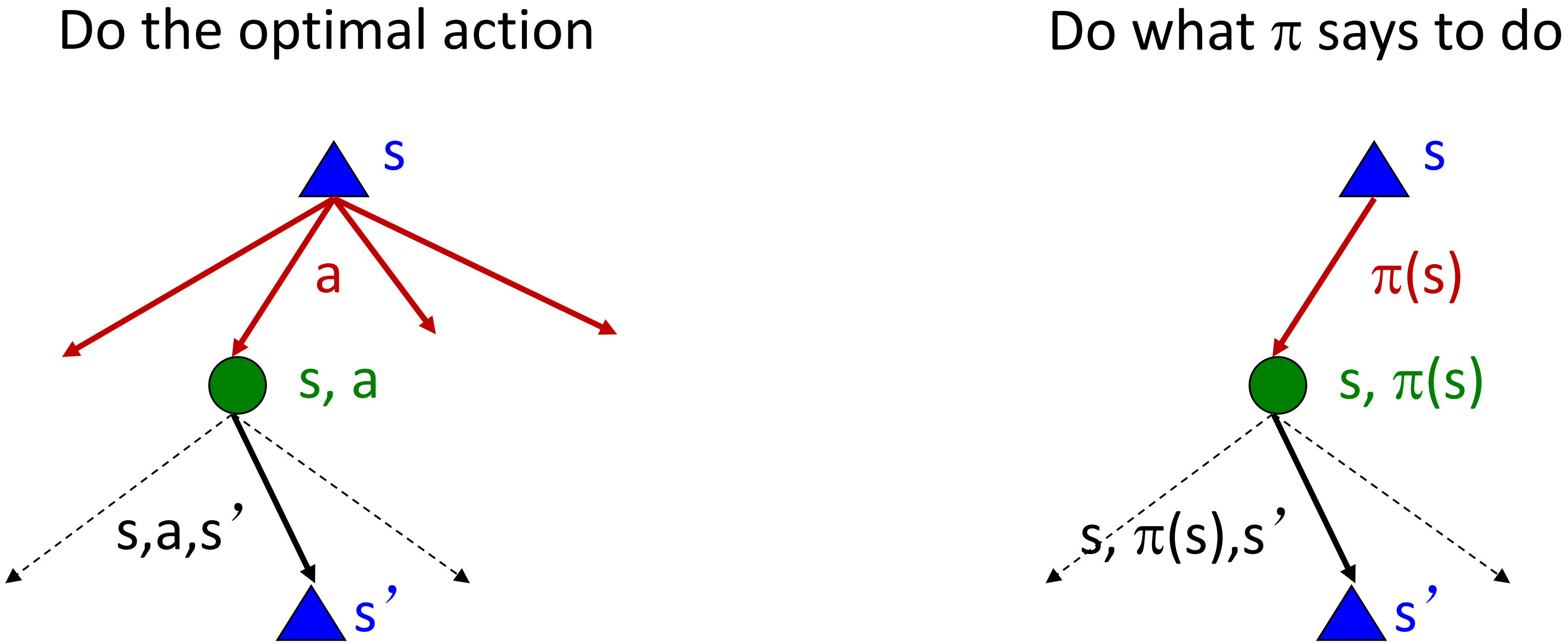
Policy Methods



Policy Evaluation



Fixed Policies



- Expectimax trees max over all actions to compute the optimal values
- If we fixed some policy $\pi(s)$, then the tree would be simpler – only one action per state
  - … though the tree's value would depend on which policy we fixed

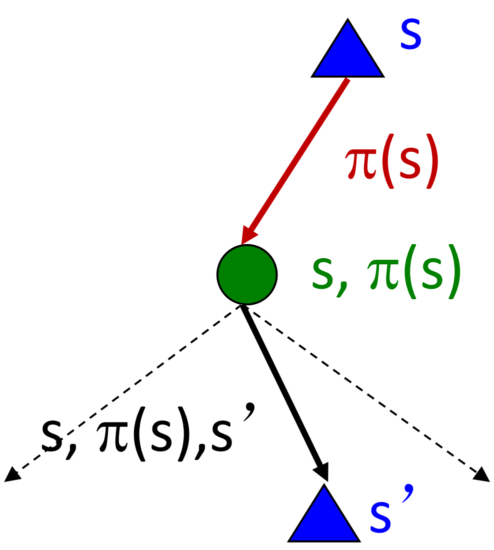
Utilities for a Fixed Policy

- Another basic operation: compute the utility of a state s under a fixed (generally non-optimal) policy

- Define the utility of a state s, under a fixed policy π:
  - $V^{\pi}(s)$ = expected total discounted rewards starting in s and following π

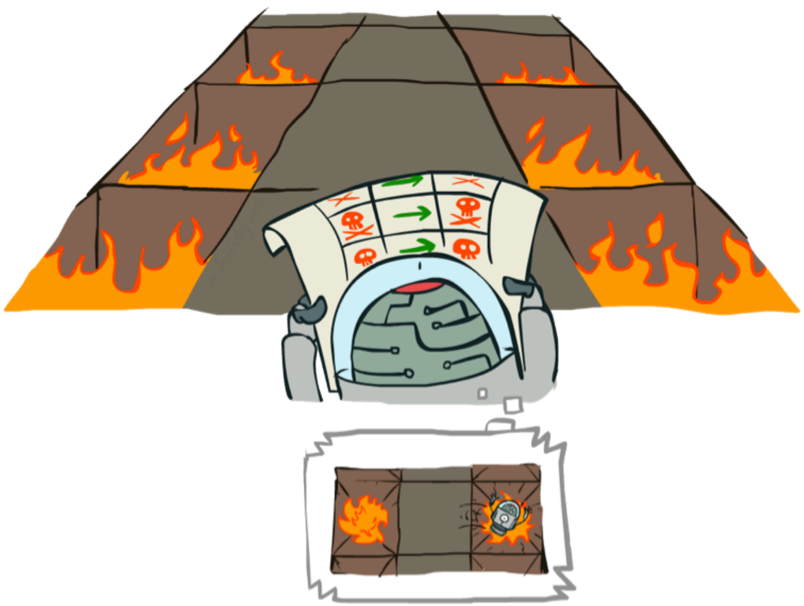
- Recursive relation (one-step look-ahead / Bellman equation):

$$V^{\pi}(s) \leftarrow R(s) + \gamma \sum_{s'} P(s'|s,a) V^{\pi}(s')$$

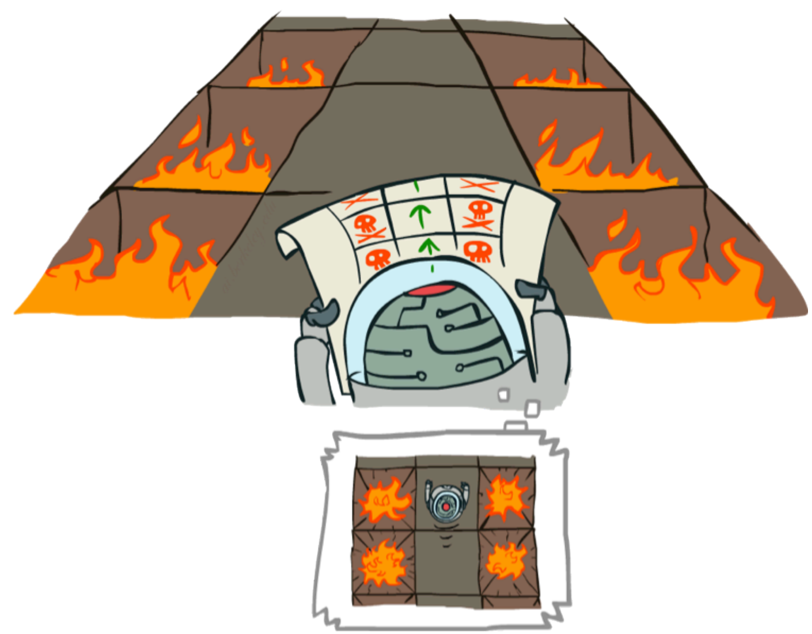


Example: Policy Evaluation

Always Go Right



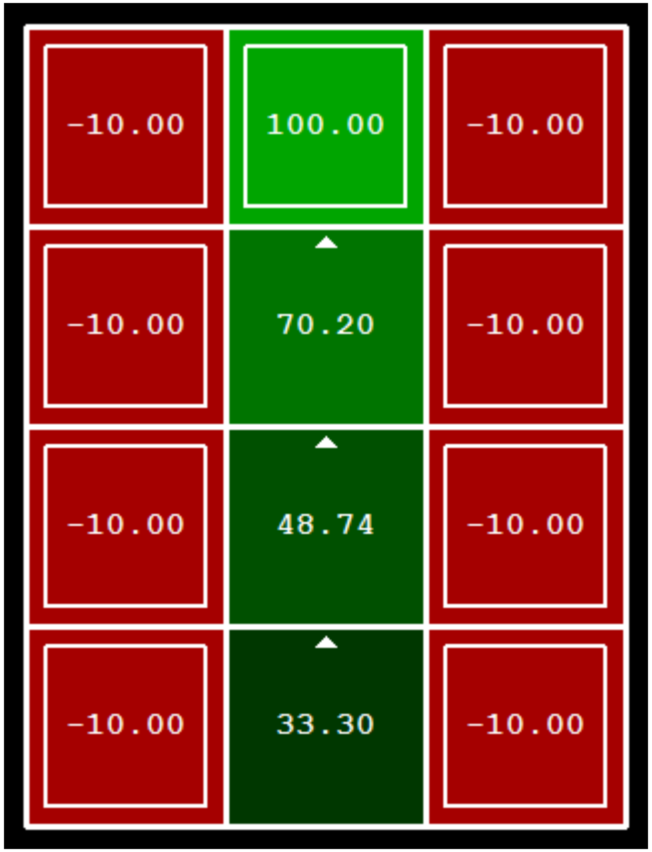
Always Go Forward



Example: Policy Evaluation

Always Go Right

Always Go Forward



Bellman Equation (policy) in Matrix Form

- The Bellman equation can be expressed concisely using matrices

$$v = \mathrm{R} + \gamma P v$$

- Where v is a column vector with one entry per state

$$\begin{bmatrix} v(1) \\ \vdots \\ v(n) \end{bmatrix} = \begin{bmatrix} \mathcal{R}_1 \\ \vdots \\ \mathcal{R}_n \end{bmatrix} + \gamma \begin{bmatrix} \mathcal{P}_{11} & \dots & \mathcal{P}_{1n} \\ \vdots & & \\ \mathcal{P}_{11} & \dots & \mathcal{P}_{nn} \end{bmatrix} \begin{bmatrix} v(1) \\ \vdots \\ v(n) \end{bmatrix}$$

Solving the Bellman Equation (policy)

- The Bellman equation (policy) is a linear equation
- It can be solved directly

$$\begin{aligned} v &= \mathcal{R} + \gamma \mathcal{P} v \\ (I - \gamma \mathcal{P})\, v &= \mathcal{R} \\ v &= (I - \gamma \mathcal{P})^{-1} \mathcal{R} \end{aligned}$$

- Computational complexity is $O(n^3)$ for n states
- Direct solution only possible for small MRPs

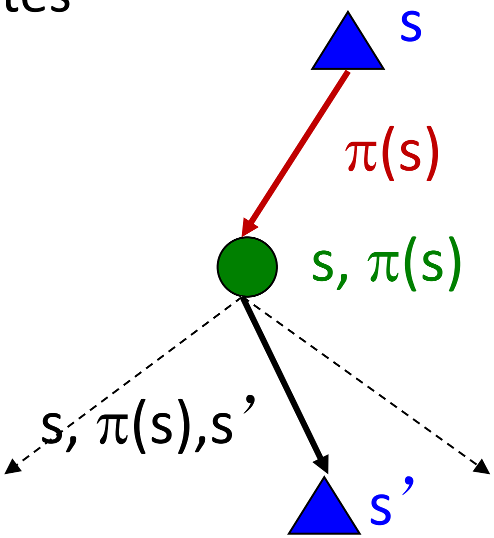
Solving the Bellman Equation (policy)

- Iteration: Turn recursive Bellman equations into updates (like value iteration)

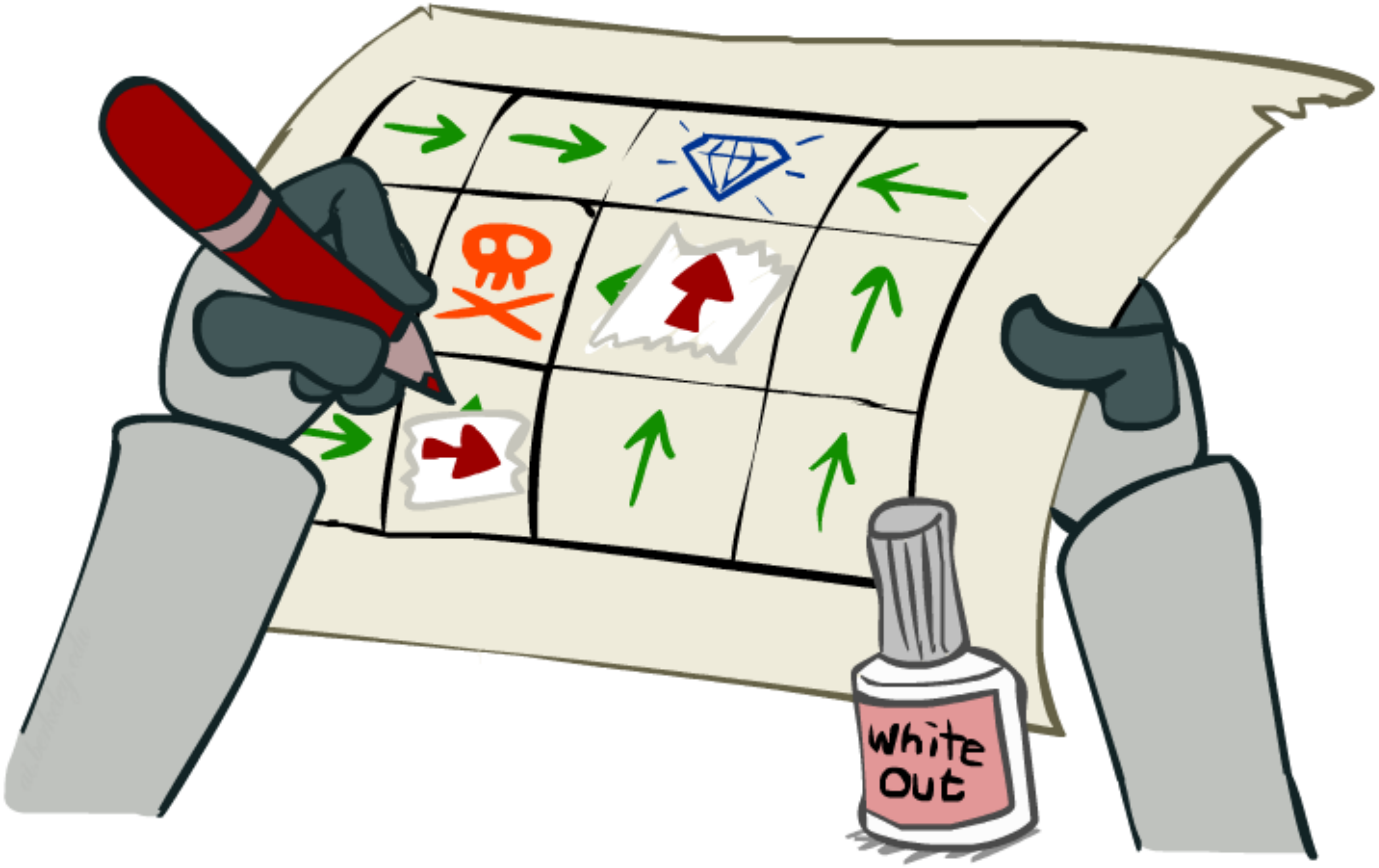
$$V_0^{\pi}(s) = 0$$

$$V_{k+1}^{\pi}(s) \leftarrow R(s) + \gamma \sum_{s'} P(s'|s,a) V_k^{\pi}(s'))$$

- Efficiency: $O(S^2)$ per iteration



Policy Iteration



Policy Iteration

- Alternative approach for optimal values:
  - Step 1: Policy evaluation: calculate utilities for some fixed policy (not optimal utilities!) until convergence
  - Step 2: Policy improvement: update policy using one-step look-ahead with resulting converged (but not optimal!) utilities as future values
  - Repeat steps until policy converges

- This is policy iteration
  - It's still optimal!
  - Can converge (much) faster under some conditions

Policy Iteration

- Evaluation: For fixed current policy π, find values with policy evaluation: $U_i = U^{\pi_i}$
  - Iterate until values converge:

$$U_{i+1}(s) \leftarrow R(s) + \gamma \sum_{s'} P(s' \mid s, \pi_i(s)) U_i(s')$$

- Improvement: For fixed values, get a better policy using policy extraction
  - One-step look-ahead:

$$\pi[s] \leftarrow \operatorname*{argmax}_{a \in A(s)} \sum_{s'} P(s' \mid s, a)\, U[s']$$

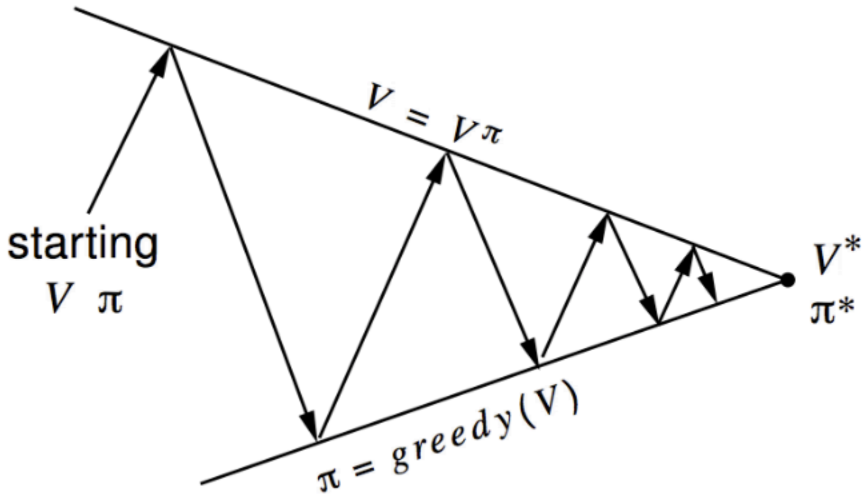
Policy Iteration

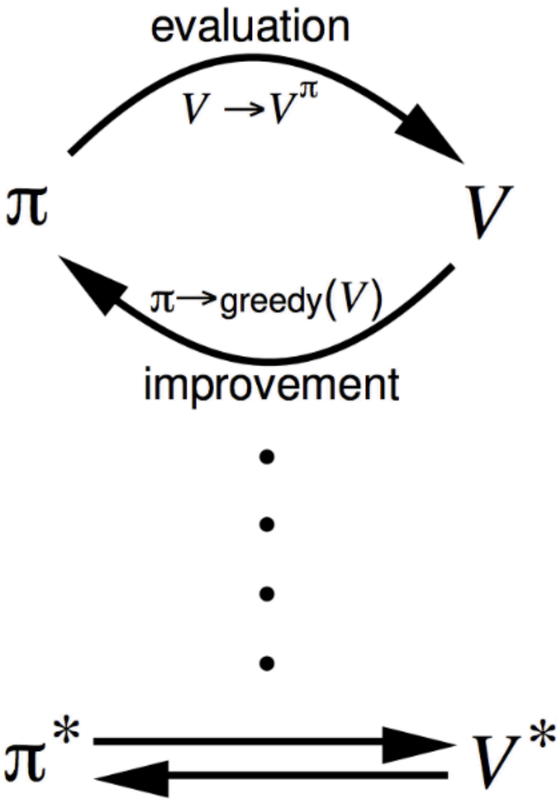
function POLICY-ITERATION(mdp) **returns** a policy
  inputs: mdp, an MDP with states S, actions $A(s)$, transition model $P(s' \mid s, a)$
  local variables: U, a vector of utilities for states in S, initially zero
    π, a policy vector indexed by state, initially random

  repeat
    $U \leftarrow$ POLICY-EVALUATION(π, U, mdp)
    $unchanged? \leftarrow$ true
    for each state s **in** S **do**
      if $\max_{a \in A(s)} \sum_{s'} P(s' \mid s, a)\ U[s'] > \sum_{s'} P(s' \mid s, \pi[s])\ U[s']$ **then do**
        $\pi[s] \leftarrow \operatorname{argmax}_{a \in A(s)} \sum_{s'} P(s' \mid s, a)\ U[s']$
        $unchanged? \leftarrow$ false
  until $unchanged?$
  return π

Figure 17.7 The policy iteration algorithm for calculating an optimal policy.

Policy Iteration





Policy evaluation Estimate v_π

Iterative policy evaluation

Policy improvement Generate $\pi' \geq \pi$

Greedy policy improvement

Modified Policy Iteration

- Does policy evaluation need to converge to v^{π}?
  - Or should we introduce a stopping condition
    - E.g. epsilon-convergence of value function
  - Or simply stop after k iterations of iterative policy evaluation?

- Why not update policy every iteration? i.e. stop after k = 1
  - This is equivalent to value iteration.

Comparison

- Both value iteration and policy iteration compute the same thing (all optimal values)

- In value iteration:
  - Every iteration updates both the values and (implicitly) the policy
  - We don’t track the policy, but taking the max over actions implicitly re-computes it

- In policy iteration:
  - We do several passes that update utilities with fixed policy (each pass is fast because we consider only one action, not all of them)
  - After the policy is evaluated, a new policy is chosen (slow like a value iteration pass)
  - The new policy will be better

- Both are dynamic programs for solving MDPs

Summary: MDP Algorithms

- So you want to….
  - Compute optimal values: use value iteration or policy iteration
  - Compute values for a particular policy: use policy evaluation
  - Turn your values into a policy: use policy extraction (one-step lookahead)

- These all look the same!
  - They are all variations of Bellman updates
  - They all use one-step look-ahead expectimax fragments
  - They differ only in whether we plug in a fixed policy or max over actions

Synchronous Dynamic Programming Algorithms

- Both value iteration and policy iteration used synchronous backups
  - i.e. all states are backup up in parallel
- Asynchronous DP backs up states individually, in any order
  - For each selected state, apply the appropriate backup
  - Can significantly reduce computation
  - Guaranteed to converge if all states continue to be selected

Synchronous Dynamic Programming Algorithms

- Three simple ideas for asynchronous dynamic programming:
  - In-place dynamic programming
  - Prioritiesd sweeping
  - Real-time dynamic programming

In-place Dynamic Programming

- Synchronous value iteration stores two copies of value function
  - For all s in S

$$v_{new}(s) \leftarrow \max_{a \in \mathcal{A}} \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_{old}(s') \right)$$

$$v_{old} \leftarrow v_{new}$$

- In-place value iteration only stores one copy of value function
  - For all s in S

$$v(s) \leftarrow \max_{a \in \mathcal{A}} \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v(s') \right)$$

Prioritised Sweeping

- Use magnitude of Bellman error to guide state selection, e.g.

$$\left| \max_{a \in \mathcal{A}} \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v(s') \right) - v(s) \right|$$

- Backup the state with the largest remaining Bellman error
- Update Bellman error of affected states after each backup
- Can be implemented efficiently by maintaining a priority queue

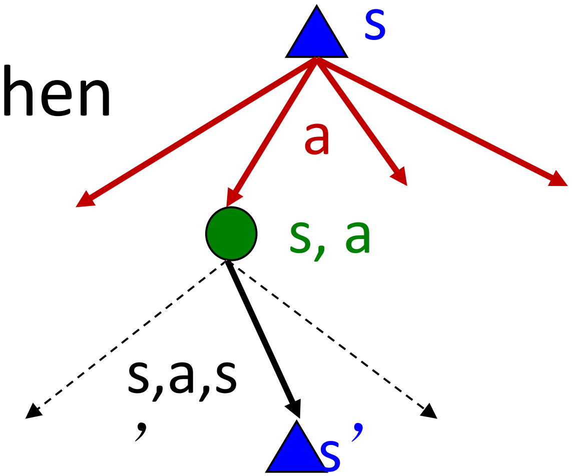
Real-Time Dynamic Programming

- Idea: only states that are relevant to agent
- Use agent's experience to guide the selection of states
- After each time-step
- Backup the state S_t
- $$v(S_t) \leftarrow \max_{a \in \mathcal{A}} \left(\mathcal{R}^a_{S_t} + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}^a_{S_t s'} v(s') \right)$$
- Focus the DP's backups onto parts of states that are most relevant to the agents

Full-Width Backups

- DP uses full-width backups
- For each backup (sync or async)
  - Every successor state and action is considered
  - Using knowledge of the MDP transitions and reward function

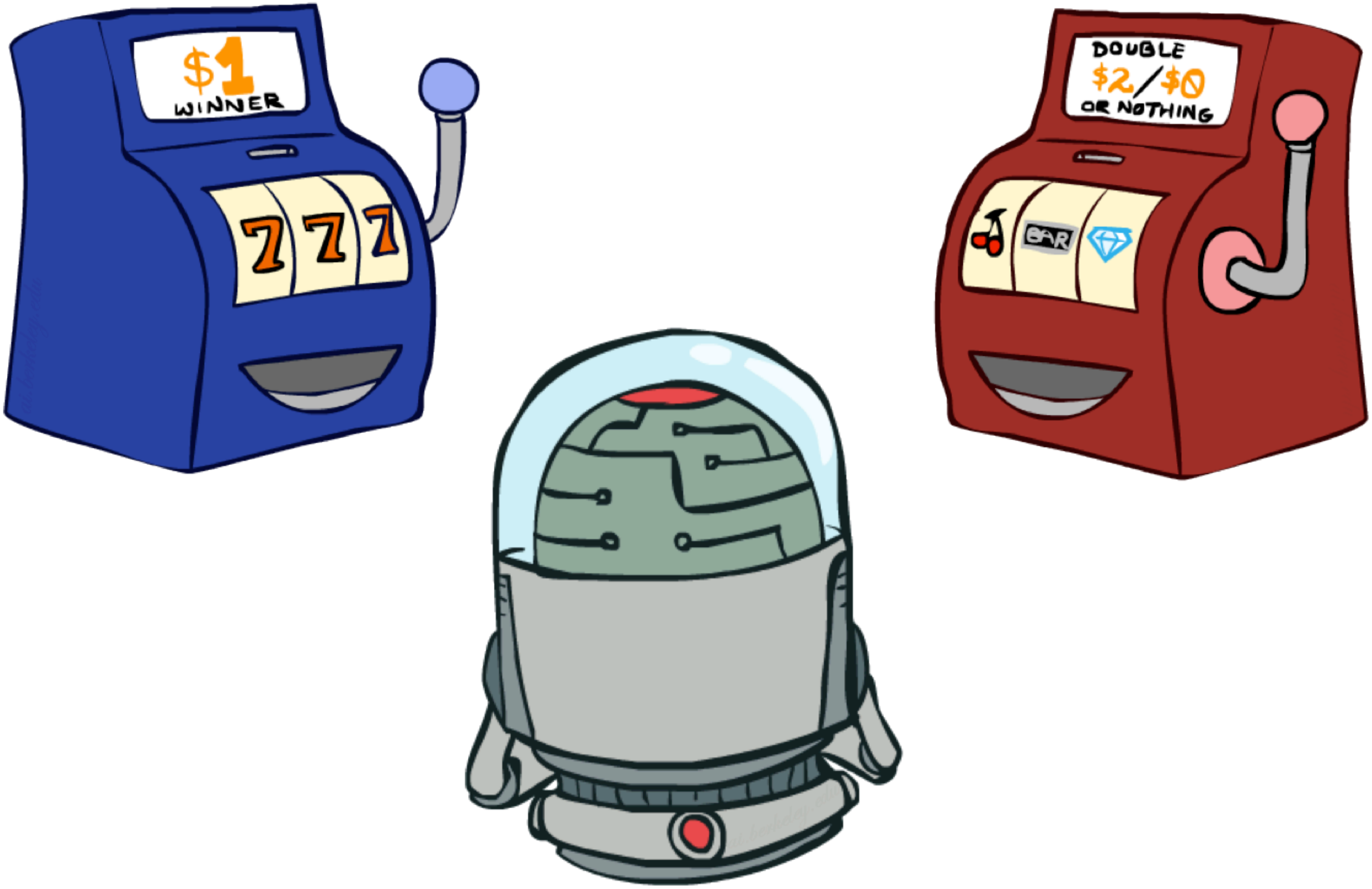
- DP is effective for medium-sized problems (millions of states)
- For large problems DP suffers Bellman's curse of dimensionality
  - Number of states n = |S| grows exponentially with number of state variables
- Each one backup can be too expensive then



Sample Backups

- Using sample rewards and sample transitions instead of reward function R and transition dynamics P
- Advantages:
  - Model free: no advance knowledge of MDP required
  - Breaks the curse of dimensionality through sampling
  - Cost of backup is constant, independent of $n = |S|$

Double Bandits



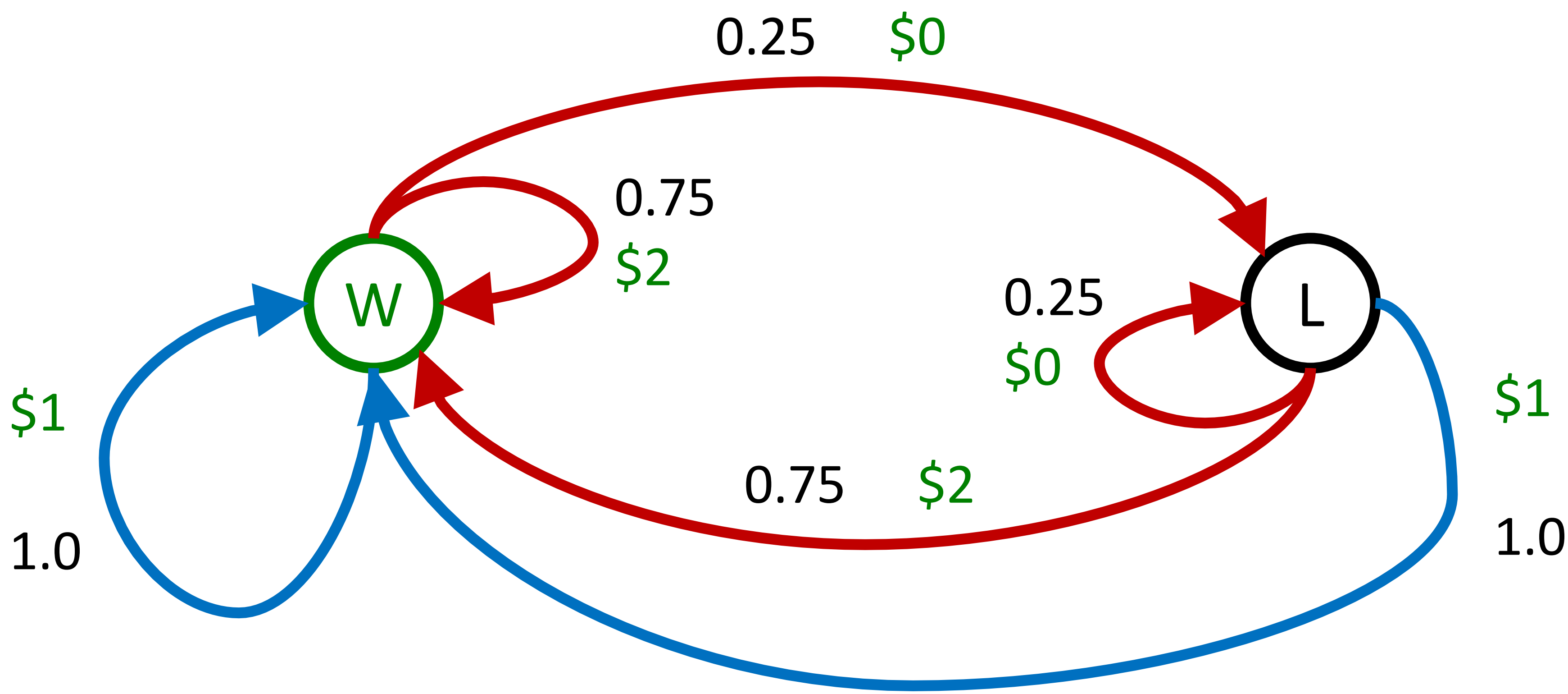
Double-Bandit MDP

- Actions: *Blue*, *Red*
- States: Win, Lose

No discount

100 time steps

Both states have the same value



Offline Planning

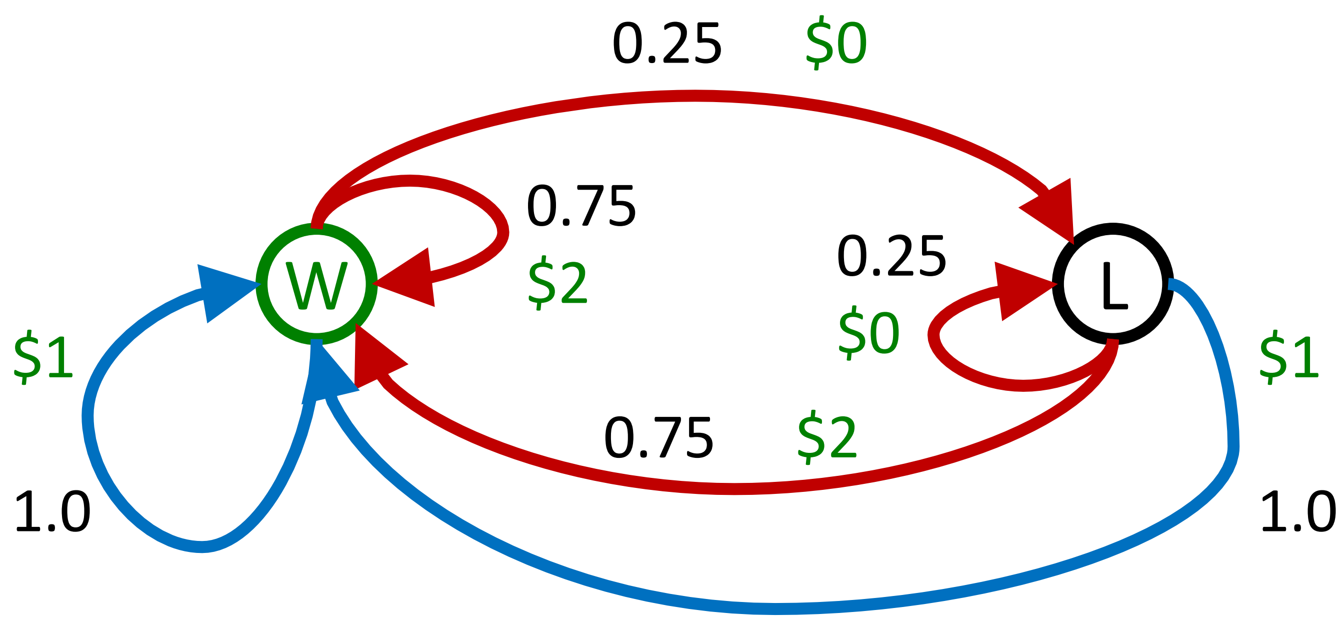
- Solving MDPs is offline planning
  - You determine all quantities through computation
  - You need to know the details of the MDP
  - You do not actually play the game!

No discount

100 time steps

Both states have the same value

	Value
Play Red	150
Play Blue	100



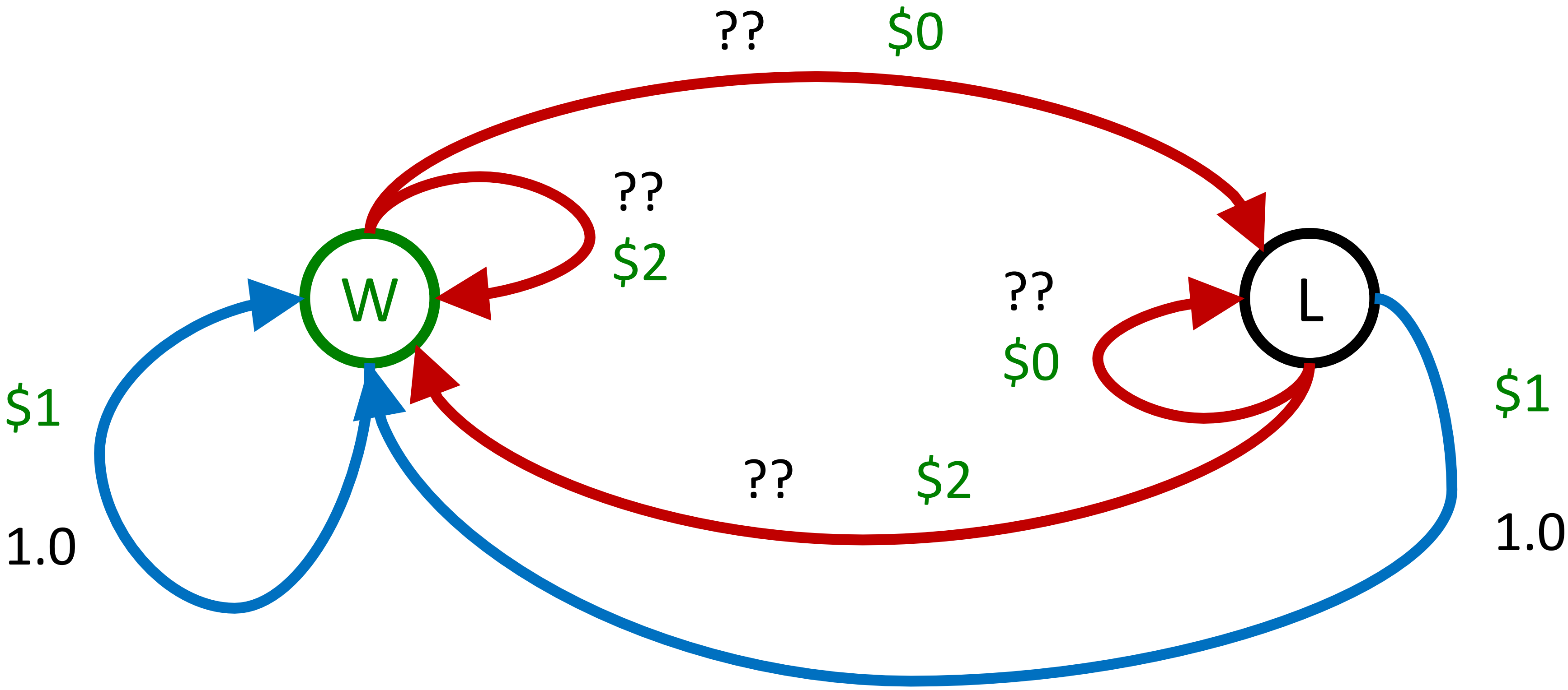
Let's Play!

$2 $2 $0 $2 $2

$2 $2 $0 $0 $0

Online Planning

- Rules changed! Red's win chance is different.



Let's Play!

$0	$0	$0	$2	$0
$2	$0	$0	$0	$0

What Just Happened?

- That wasn't planning, it was learning!
  - Specifically, reinforcement learning
  - There was an MDP, but you couldn't solve it with just computation
  - You needed to actually act to figure it out

- Important ideas in reinforcement learning that came up
  - Exploration: you have to try unknown actions to get information
  - Exploitation: eventually, you have to use what you know
  - Regret: even if you learn intelligently, you make mistakes
  - Sampling: because of chance, you have to try things repeatedly
  - Difficulty: learning can be much harder than solving a known MDP