Quanto forward

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Background

A quanto is a type of derivative in which the underlying is denominated in one currency, but the instrument itself is settled in another currency at some rate.For example:

- a digital contract which pays one dollar at time T if the then British Petroleum stock price is larger than some pre-agreed stike;
- a forward contract,namely receiving the BP stock price at time T as if it were in dollars in exhange for paying a pre-agreed dollar amount;
- an option to receive the BP stock price less a strike price,in dollars.

Assumption

1. sterling stock price $S_t = S_0 e^{\sigma_1 W_1(t) + \mu t}$
2. the value of one pound in dollars $C_t = C_0 e^{\rho\sigma_2 W_1(t) + \bar{\rho}\sigma_2 W_2(t) + vt}$
3. dollar cash bond $B_t = e^{rt}$
4. sterling cash bond $D_t = e^{ut}$

where $\rho \in [-1, 1]$ and $\bar{\rho} = \sqrt{1-\rho^2}$,and σ_1,σ_2,μ,ρ,v,r,u are all constant

Change of measure

In this model,here are three tradables:

1. the dollar worth of the sterling bond:C_tD_t
2. the dollar worth of the stock:C_tS_t
3. the dollar cash bond:B_t

Writing down the first two of these tradables after discounting by the third,the numeraire,we have $Y_t = B_t^{-1}C_tD_t$ and $Z_t = B_t^{-1}C_tS_T$ respectively.

Change of measure

According to n-factor GCM,we need to find $\gamma_t = (\gamma_1(t), \gamma_2(t))$ to make

$$dY_t = Y_t \left(\rho\sigma_2 d\tilde{W}_1(t) + \bar{\rho}\sigma_2 d\tilde{W}_2(t)\right) \tag{1}$$

$$dZ_t = Z_t \left((\sigma_1 + \rho\sigma_2)d\tilde{W}_1(t) + \bar{\rho}\sigma_2 d\tilde{W}_2(t)\right) \tag{2}$$

Where

$$d\tilde{W}_1(t) = dW_1(t) + \gamma_1(t)dt \tag{3}$$
$$d\tilde{W}_2(t) = dW_2(t) + \gamma_2(t)dt \tag{4}$$

Thus under Q we can write the original process S_t as:

$$S_t = S_0 e^{\sigma_1 \tilde{W}_1(t) + (u - \rho\sigma_1\sigma_2 - \frac{1}{2}\sigma_1^2)t} \tag{5}$$

Pricing

The forward at time zero in dollars is

$$\begin{aligned} V_0 &= e^{-rT} E_Q(S_T - k) \\ &= e^{-rT} E_Q(S_0 e^{\sigma_1 \sqrt{T} Z + (u - \rho\sigma_1\sigma_2 - \frac{1}{2}\sigma_1^2)T} - k) \\ &= e^{-rT} (S_0 e^{(u+\rho\sigma_1\sigma_2)T} - k) \end{aligned} \tag{6}$$

where $Z \sim N(0,1)$ under Q.

For this to be on market,that is to have a value of zero,we must set k to be $S_0 e^{(u+\rho\sigma_1\sigma_2)T}$.

Implication

We know that for the local currency forward $F = S_0 e^{uT}$,so $k = Fe^{-\rho\sigma_1\sigma_2 T}$.It is clear that this quanto forward price is greater than the simple forward price if and only if the correlation between the stock and the exchange rate is negative.

Example

Suppose that the current value of the Nikkei sotck index is 15,000 yen,the 1-year dollar risk-free rate is 5%,the 1-year yen risk-free rate is 2%.And suppose that the volatility of the index is 20%,the volatility of the 1-year forward yen per dollar exchange rate is 12%,and the correlation between the two is 0.3.For simplicity:

$$s_0 = 15000$$
$$u = 0.02$$
$$T = 1$$
$$\sigma_1 = 0.2$$
$$\sigma_2 = 0.12$$
$$\rho = 0.3$$

$$F = S_0 e^{\mu T} = 15303.02$$
$$k = F e^{-\rho \sigma_1 \sigma_2 T} = 15193.23$$