PHYS 234: Quantum Physics 1 (Winter 2026)
Assignment 1 Solutions

1. Matrix Operations.

$$A = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(a)

$$AB = \begin{pmatrix} i & 1 \\ 2 & -i \end{pmatrix}, \qquad BA = \begin{pmatrix} -i & 2 \\ 1 & i \end{pmatrix},$$

so $AB \neq BA$.

(b) $A^\dagger = (A^*)^T = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix} = A.$

(c) Yes. Since $A^\dagger = A$, A is Hermitian.

(d) $\det(A) = 1 \cdot 2 - (i)(-i) = 2 - 1 = 1$ and $\mathrm{Tr}(A) = 1 + 2 = 3$.

2. Inverse and Identity.

$$M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

(a) $\det(M) = 2 \cdot 1 - 1 \cdot 1 = 1$, so

$$M^{-1} = \frac{1}{\det(M)} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

(b) Multiply:

$$MM^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

(c) For $\begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix}$, $\det = a^2 - 1$. Not invertible when $a = \pm 1$.

3. Eigenvalue Problem.

$$H = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

(a) $\det(H - \lambda I) = \det \begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 1 = \lambda(\lambda - 2)$, so $\lambda = 0, 2$.

(b) For $\lambda = 2$, an eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, so $|v_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. For $\lambda = 0$, an eigenvector is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, so $|v_0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

(c) Yes: $\langle v_2 | v_0 \rangle = 0$.

(d) Yes. It has two distinct eigenvalues (hence two independent eigenvectors), so it is diagonalizable.

4. Complex Arithmetic. $z = 1 + i$.

(a) $|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$, and $\arg(z) = \pi/4$ (principal value).

(b) $z = \sqrt{2}\, e^{i\pi/4}$.

(c) $z^2 = (1+i)^2 = 2i$. Also $iz = i(1+i) = i - 1$, so $e^{iz} = e^{-1} e^{i}$.

(d) $zz^* = (1+i)(1-i) = 1 - i^2 = 2 = |z|^2$.

5. Euler's Formula.

(a) From $e^{i\theta} = \cos\theta + i \sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$, add them: $e^{i\theta} + e^{-i\theta} = 2\cos\theta$, hence the result.

(b) Subtract: $e^{i\theta} - e^{-i\theta} = 2i\sin\theta$, so

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

(c) $\cos(\pi/3) = \frac{e^{i\pi/3} + e^{-i\pi/3}}{2} = \frac{2\cos(\pi/3)}{2} = \frac{1}{2}$.

6. Inner Products.

$$|\psi\rangle = \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad |\phi\rangle = \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

(a) $\langle \psi | = (|\psi\rangle)^\dagger = \begin{pmatrix} 1 & -i \end{pmatrix}$.

(b) $\langle \psi | \phi \rangle = \begin{pmatrix} 1 & -i \end{pmatrix} \begin{pmatrix} i \\ 1 \end{pmatrix} = i - i = 0$.

(c) Yes, since $\langle \psi | \phi \rangle = 0$.

(d) $\langle \psi | \psi \rangle = 1 + 1 = 2$, so

$$|\psi_{\text{norm}}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

7. Hermitian Conjugation.

$$A = \begin{pmatrix} 2 & 1+i \\ 1-i & 3 \end{pmatrix}.$$

(a) $A^\dagger = \begin{pmatrix} 2 & 1+i \\ 1-i & 3 \end{pmatrix} = A$.

(b) Since $A^\dagger = A$, it is Hermitian.

(c) $\mathrm{Tr}(A) = 2 + 3 = 5$ and $\det(A) = 2 \cdot 3 - (1+i)(1-i) = 6 - 2 = 4$.

(d) $\mathrm{Tr}(A) = 5 \in \mathbb{R}$ (in fact, the trace of any Hermitian matrix is real).

8. Non-Commuting Matrices.

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a)

$$AB = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \qquad BA = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(b)

$$[A, B] = AB - BA = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}.$$

(c) $B^\dagger = B$ so B is Hermitian. Also $A^\dagger = -A$, so A is anti-Hermitian (not Hermitian).

(d) $[A, B]^\dagger = [A, B]$ (it is real symmetric), so the commutator is Hermitian.

9. Explicit Inversion.

$$M = \begin{pmatrix} 1+i & 1 \\ 1 & 1-i \end{pmatrix}.$$

(a) $\det(M) = (1+i)(1-i) - 1 = 2 - 1 = 1$.

(b)

$$M^{-1} = \begin{pmatrix} 1-i & -1 \\ -1 & 1+i \end{pmatrix}.$$

(c) Since $\det(M) = 1$, multiplication gives $MM^{-1} = I$.

(d) No: $(M^{-1})^\dagger = \begin{pmatrix} 1+i & -1 \\ -1 & 1-i \end{pmatrix} \neq M^{-1}$.

10. Parameter Dependence.

$$A(\lambda) = \begin{pmatrix} \lambda & i \\ -i & \lambda \end{pmatrix}.$$

(a) $\det(A) = \lambda^2 - (i)(-i) = \lambda^2 - 1$.

(b) Not invertible when $\lambda^2 - 1 = 0$, i.e. $\lambda = \pm 1$.

(c) For $\lambda \neq \pm 1$,

$$A^{-1} = \frac{1}{\lambda^2 - 1} \begin{pmatrix} \lambda & -i \\ i & \lambda \end{pmatrix}.$$

11. Complex Matrix Eigenvalues.

$$H = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

(a) Solve $\det(H - \lambda I) = \det \begin{pmatrix} -\lambda & i \\ -i & -\lambda \end{pmatrix} = \lambda^2 - 1$, so $\lambda = \pm 1$.

(b) For $\lambda = 1$, an eigenvector is $\begin{pmatrix} i \\ 1 \end{pmatrix}$, so $|v_+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}$. For $\lambda = -1$, an eigenvector is $\begin{pmatrix} -i \\ 1 \end{pmatrix}$, so $|v_-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix}$.

(c) Yes, ± 1 are real.

(d) Yes: $H^\dagger = H$.

12. Trace and Determinant Check.

$$A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}.$$

(a) Characteristic polynomial: $\det(A - \lambda I) = \det \begin{pmatrix} 3-\lambda & 2 \\ 1 & 2-\lambda \end{pmatrix} = (3-\lambda)(2-\lambda) - 2 = \lambda^2 - 5\lambda + 4 = (\lambda - 1)(\lambda - 4)$. So $\lambda = 1, 4$.

(b) $\lambda_1 + \lambda_2 = 1 + 4 = 5 = \mathrm{Tr}(A)$.

(c) $\lambda_1 \lambda_2 = 4 = \det(A)$.

(d) Not necessarily. Here eigenvectors for $\lambda = 1$ and 4 can be chosen as $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, whose dot product is $-1 \neq 0$. (Orthogonality is guaranteed for Hermitian matrices, not for this one.)

13. Polar Representation. $z = -1 + i\sqrt{3}$.

(a) $|z| = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$.

(b) The point is in quadrant II with reference angle $\pi/3$, so $\arg(z) = 2\pi/3$ (principal).

(c) $z = 2e^{i(2\pi/3)}$.

(d) $z^3 = 2^3 e^{i(2\pi)} = 8$.

14. Complex Conjugation. $z = a + ib$.

(a) $z + z^* = (a + ib) + (a - ib) = 2a$.

(b) $z - z^* = (a + ib) - (a - ib) = 2ib$.

(c) $\dfrac{z}{z^*} = \dfrac{a + ib}{a - ib}$ has modulus $\frac{|z|}{|z|} = 1$, so it lies on the unit circle.

(d) $\dfrac{1+i}{1-i} = \dfrac{(1+i)^2}{(1-i)(1+i)} = \dfrac{2i}{2} = i = e^{i\pi/2}$ (principal).

15. Trigonometric Identities from Exponentials.

(a) Using $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$, $\sin^2\theta = \frac{1 - \cos(2\theta)}{2}$.

(b) $\cosh(ix) = \frac{e^{ix}+e^{-ix}}{2} = \cos x$.

(c) $\sin(2\theta) = \dfrac{e^{2i\theta} - e^{-2i\theta}}{2i}$.

(d) $\sin(\pi/4) = \dfrac{e^{i\pi/4} - e^{-i\pi/4}}{2i} = \frac{\sqrt{2}}{2}$.

16. Oscillatory Functions.

(a) $e^{i(\omega t+\phi)} = \cos(\omega t + \phi) + i\sin(\omega t + \phi)$.

(b) Real part: $\cos(\omega t + \phi)$; imaginary part: $\sin(\omega t + \phi)$.

(c) $\left|e^{i\theta}\right| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$.

(d) They package sines/cosines into a single exponential, making algebra (especially with phases and differential equations) much cleaner.

17. Normalization.

$$|\psi\rangle = \begin{pmatrix} 2 \\ -i \end{pmatrix}.$$

(a) $\langle\psi|\psi\rangle = 2^*2 + (-i)^*(-i) = 4 + 1 = 5$.

(b) $|\psi_{\text{norm}}\rangle = \dfrac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -i \end{pmatrix}$.

(c) $\langle\psi_{\text{norm}}|\psi_{\text{norm}}\rangle = 1$ by construction.

(d) The total probability of finding the particle somewhere (in the given basis) is 1.

18. Orthogonality.

$$|\phi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |\phi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(a) $\langle\phi_1|\phi_2\rangle = \frac{1}{2}(1 \cdot 1 + 1 \cdot (-1)) = 0$.

(b) Yes.

(c) Yes: each has norm 1.

(d) Yes. They are orthonormal and linearly independent, hence form a basis for $\mathbb{C}^2$.