AMATH 361
CONTINUUM MECHANICS
Course Notes

K. Lamb

Department of Applied Mathematics
University of Waterloo

Contents

Chapter 1

WHAT IS CONTINUUM MECHANICS?

Continuum mechanics is the study of how matter deforms and flows at scales large compared to intermolecular distances. Examples of problems that come into the purview of continuum mechanics can be split into two broad classes of problems:

- **Fluid flow:** Flow of air around airplanes and cars, flow of water around ships, large scale flow in the atmosphere (weather, climate), surface water waves, large scale ocean circulation, sound waves in air and water, flow in buildings (e.g., ventilation systems), flow around buildings, flow of molten plastic into injection molds, blood flow, ...

- **Deformation of solids:** Solids deform under various loads (external forces or under their own weight. Examples include flow of glaciers, flow in the earth's mantle, rise and fall of the Tibetan plateau and the Himalayas, earthquakes, vibrations in beams and other structures, deformation of the brain in cerebropalsy, ...

Sometimes the categorization of matter is not so clear cut, e.g., flow of glass, plastercine, tar, polymers or glaciers, and sometimes it includes elements of both (flow of lava from a volcano may have a fluid and solid component). Even flow of solid rock in the Earth's mantle has been studied on extremely long time scales by treating it as an extremely viscous fluid.

In general, a fluid is matter which changes shape to that of its container. Fluids include gases and liquids. The latter, of course, have a fixed volume and may only partially fill their container and will take the shape of the part of the container they fill only when subjected to an external body force such as gravity. Fluids are unable to withstand shearing or stretching forces without deforming and they continue to deform as long as these forces act on them. When the external force is removed a fluid continues to move until viscous forces bring it to rest. Solids, on the other hand, undergo a finite deformation when subjected to a shearing or stretching force. They do not continually deform, but instead stop deforming when inner forces balance the externally applied force. When the force is removed the solid will go back to its original shape if perfectly elastic. Fluids have no memory and do not return to their original shape after the cessation of applied forces.

The major focus of this course will be on developing models to describe the motion of a fluid or the deformation of a solid. These models will be unlike ones you have considered in previous courses in that most of this course will be spent on simply deriving the models. There will be time for only a few limited applications. These models are the simplest realistic models for the subject at hand and form the basis for more complicated models and for more complicated problems (e.g., magnetohydrodynamics which has the added complication of involving the motion of an electrically charged plasma). The model derivations will make heavy use of vector calculus, much of which was in fact derived in early attempts to develop models for the study of fluid flow. We will need some new concepts, concepts which some of the greatest mathematicians and scientists of the past struggled with and we will need ideas from physics including Cartesian tensors and some elementary thermodynamics. Although we have the use of hindsight and a century and a half of experience you will undoubtedly have to struggle with some aspects of this course – do not let yourself get behind. Some ideas take time to sink in. The course will pull together much of what you have learned and open the doors to an understanding of an overwhelming wealth of physical phenomena in the world around us.

A complete mathematical description of the motion of a fluid or solid is impossible and would in any case be too complicated to be of any use. Simplifications, involving a number of assumptions or hypotheses, must be made which results in a useful model — one which is simple enough to be solvable (in some sense - perhaps numerically) and which includes the essential physics.

Consider the motion of a fluid. Fluid flows that we might wish to investigate are quite varied. Examples include

1. Atmospheric circulation:
   (a) for climate modelling (global scales, long time periods),
   (b) for long range weather predictions (e.g., El-Niño events)
   (c) small scale, short-term local weather forecasts.
2. Ocean circulation:

   (a) for climate modelling (global scales, long time periods),
   (b) tidal forecasts
   (c) large scale currents, eddies ($\geq$ 100 km)
   (d) small-scale events ($\leq$ 10 km)

3. Surface waves: shoaling waves and impact on shorelines (erosion), impact on structures (boats, oil rigs, piers etc.)
4. Flow around an object: airplanes, cars, boats
5. Flow through pipes
6. Heating/air conditioning systems
7. Flow in rivers
8. Flow of magma
9. Flow of molten rock in the Earth's inner core
10. Flow of rock in the Earth's mantle on very long time scales (1000's of years)
11. Blood flow

The mathematical models used in the study of these problems are generally different. The effects of the Earth's rotation is important for (1), (2), (9) and (10) but is generally unimportant for the others. Compressibility of water is important for large scale ocean circulation but is usually not considered (never to my knowledge) in models of flow on the continental shelf. The density of the fluid may be treated as constant in (3), (6) and (7) as well as for flow around cars and slow moving airplanes, but not for flow past high-speed aircraft.

All the models used to study the above fluid flows are based on simplifications and/or extentions of the Navier-Stokes equations first derived in 1822 by Claude-Louis Navier (1785–1836). A major objective of this course is to derive the Navier-Stokes equations. There will be a heavy emphasis on understanding the simplifying assumptions and approximations used in deriving it. This understanding is essential for knowing when the model is appropriate and also for determining how the model might be modified for a particular problem not governed by these equations. Our second goal is to derive models for the study of the deformation of solids (elasticity theory). The derivation of these models follows the same path for a considerable part of the course as they share some assumptions and are both based on the conservation laws of mass, momentum, energy and angular momentum.

All the problems mentioned above have one thing in common: they all consider phenomena on lengths scales extremely large compared to intermolecular distances (a situation where this would not be true is the re-entry of spacecraft into the atmosphere where, for a time, intermolecular distances are comparable to the length scale of the spacecraft). In these problems we are not interested in how individual molecules move. Indeed, a thimble full of water contains on the order of 10^{22} molecules of water. It is absurd to imagine we can attempt to predict the motion of each individual molecule even if we had appropriate initial conditions such as the energy level, position and velocity of each molecule. In the study of continuum mechanics we ignore the molecular structure of matter and treat matter as a continuum. This results in a major simplification – indeed, a necessary one as it turns the problem from an impossible one to one on which we can make some progress.

1.1 The Continuum Hypothesis

Throughout mankind's early history, fluids and solids were regarded as a continuum. Eventually it was realized that they were made up of a large number of molecules separated by very small distances (approximately 10^{-8} m for air at atmospheric pressure and 10^{-9} m for water although these values were not determined until relatively recently). The Greeks (Epicurus 340-270 BC) first put forward the idea that matter was make up of small indivisible particles but this idea was largely forgotten until resurrected by the French philosopher Pierre Gassendi (1592-1655) and others. The Navier-Stokes equations for fluid motion were first derived by Navier (1822) and Poisson (1829) using a simplified molecular model for gases. Nowadays scientist usually use the continuum model based on the approach of Saint-Venant (1843) and Stokes (1845).

To talk about how matter moves and deforms we need to talk about such quantities as its density $\rho(\vec{x},t)$, velocity $\vec{u}(\vec{x},t)$, pressure $p(\vec{x},t)$ etc. What do we mean by these? One can talk about the average density of matter in a volume V — it is simply the total mass in the volume divided by the mass M of matter in V. Given the true molecular nature of matter what can we possibly mean by the density of a piece of matter, say air, at a point $\vec{x}$?

Suppose we take a cube of material with sides of length l centred at $\vec{x}$. Let $M_l(\vec{x})$ be the mass of the material in the cube. The average density of the matter in the cube is

$$\rho_l(\vec{x}) = \frac{M_l(\vec{x})}{l^3}. \tag{1.1}$$

Suppose we plot $\rho_l(\vec{x})$ as a function of l. We would get something like this:

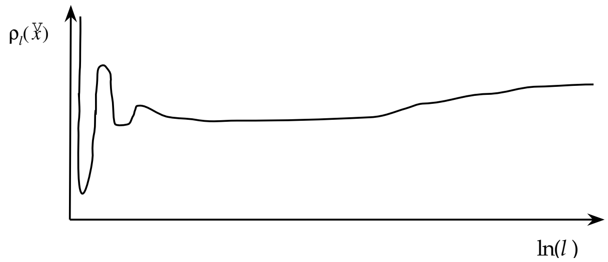


Figure 1.1: Average density as a function of l.

The average density ρ_l varies slowly as l decreases from large values (imagine a large cube of water in a lake that includes light warm surface water and colder, denser water at depth. If the cube is centred in the cold dense water, as l decreases ρ_l will increase as the warm water is excluded from the cube). At an intermediate range of values, including very small values, ρ_l becomes constant. At extremely small values of l, comparable to intermolecular distances, the average density begins to fluctuate widely as there are now not many molecules inside the volume. A small decrease in l can exclude one molecule resulting in a large change in the average density.

Definition: *The density field $\rho(\vec{x}, t)$ is defined to be the average density of a very small volume of matter centred at $\vec{x}$. The volume l^3 is such that l is in the intermediate range — i.e., the volume is extremely small but still contains a large number of molecules.*

Other scalar and vector fields are similarly defined, although some care may be needed to define the velocity field for mixtures. We will not consider this case here. The definition of the pressure field $p(\vec{x}, t)$ also requires an average in time. The pressure is the average force exerted by molecules as they bounce off a surface. One must average in time to include a large number of collisions.

Chapter 2

KINEMATICS

Kinematics is the study of motion without consideration of its causes (that is the subject of dynamics). Because the forces causing the motion are not considered, everything in this section applies to all continuums.

Definition: *A* **particle** *is a very small volume of matter. One so small that it can be treated mathematically as a point yet large enough to contain a large number of molecules. The density $\rho(\vec{x}, t)$ is the average density of a particle. The velocity of the particle at $\vec{x}$ at time t is $\vec{u}(\vec{x}, t)$.*

This definition was first given in the context of fluid flow by Leonhard Euler (1707–1783) and it was an important step forward. A particle is a physical point, not simply a mathematical point. It has mass and is subjected to forces. It makes sense to discuss its motion.

Definition: *A* **pathline** *is the trajectory traced out by a particle as it moves.*

If the position of a particle is given by $\vec{x}(t)$ then its velocity is given by

$$\frac{d\vec{x}}{dt} = \vec{u}(\vec{x}(t), t). \tag{2.1}$$

Given $\vec{u}(\vec{x}, t)$, (2.1) gives a coupled set of three (two in R^2) ODEs to solve. The ODEs are usually nonlinear and impossible to solve analytically.

Example 1: Given

$$\vec{u}(\vec{x}, t) = \vec{x} \tag{2.2}$$

find the particle paths.

Solution: Here the velocity field is independent of time. In component form we have

$$\begin{aligned}\frac{dx_1}{dt} &= x_1,\\ \frac{dx_2}{dt} &= x_2,\\ \frac{dx_3}{dt} &= x_3.\end{aligned} \tag{2.3}$$

In this example the three ODEs are uncoupled and linear. The solution is easily found:

$$\vec{x} = (a_1 e^t, a_2 e^t, a_3 e^t), \tag{2.4}$$

where $\vec{a} = (a_1, a_2, a_3)$ is the initial position of the particle. The pathlines are straight lines emanating from the origin. In two dimensions we have

$$\frac{x_1}{x_2} = \frac{a_1}{a_2} = \text{constant}. \tag{2.5}$$

Particles move along the pathlines away from the origin. Note that at the origin $\vec{u} = 0$ so the particle initially at the origin stays there. In the theory of DEs such a point is called a fixed point or equilibrium point. In this case it is an unstable fixed point because if the particle is moved off the origin it moves off to infinity.

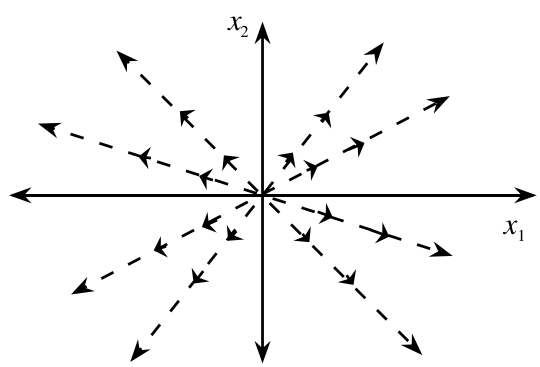


Figure 2.1: Pathlines for Example 1 are the straight lines x_1/x_2 =constant.

Example 2: Given the two-dimensional velocity field

$$\vec{u}(\vec{x}, t) = \Big(\frac{x_1}{t+\alpha}, x_1 + \frac{x_2}{t+\alpha}\Big), \qquad \text{for } t \geq 0, \alpha > 0 \tag{2.6}$$

find the particle paths.

Solution: In component form we have

$$\frac{dx_1}{dt} = \frac{x_1}{t+\alpha}, \tag{2.7}$$

$$\frac{dx_2}{dt} = x_1 + \frac{x_2}{t+\alpha}. \tag{2.8}$$

The first of these is uncoupled from the second. It is a separable ODE which can be written as

$$\frac{dx_1}{x_1} = \frac{dt}{t+\alpha},$$

which can be integrated to give

$$\ln|x_1| = \ln|t+\alpha| + C,$$

or

$$x_1(t) = A(t+\alpha), \tag{2.9}$$

where A is an arbitrary constant. Using this result the equation for $x_2(t)$ can be written as

$$\frac{dx_2}{dt} - \frac{x_2}{t+\alpha} = A(t+\alpha). \tag{2.10}$$

Recall that the general form of the solution of a linear ODE is a linear combination of solutions of the homogeneous ODE (obtained by removing the forcing term) plus any particular solution. The homogeneous ODE for x_2 is identical to the ODE for x_1. Thus, x_2 has the form

$$x_2(t) = B(t+\alpha) + x_p(t), \tag{2.11}$$

where B is an arbitrary constant (the multiple of the homogeneous solution) and $x_p(t)$ is any particular solution of the nonhomogeneous ODE. An inspection of (2.10) suggests that we should try to find a solution of the form

$$x_p(t) = C(t+\alpha)^2. \tag{2.12}$$

Substituting this into (2.10) gives

$$2C(t+\alpha) - C(t+\alpha) = A(t+\alpha),$$

so $C = A$. Thus, the general solution for $x_2(t)$ is

$$x_2(t) = B(t+\alpha) + A(t+\alpha)^2. \tag{2.13}$$

Therefore the pathlines for the flow given by (2.25) are

$$\Big(x_1(t), x_2(t)\Big) = \Big(A(t+\alpha), B(t+\alpha) + A(t+\alpha)^2\Big), \tag{2.14}$$

where A and B are constants which can be determined from the initial conditions. At $t = 0$ the particle position is

$$\vec{x}(0) = \Big(A\alpha, B\alpha + A\alpha^2\Big). \tag{2.15}$$

Setting

$$\vec{x}(0) = (a_1, a_2), \tag{2.16}$$

gives $A = a_1/\alpha$ and $B = a_2/\alpha - a_1$ so we can write the solution in terms of the initial particle position (a_1, a_2) as

$$\vec{x}(t) = \Big(\frac{a_1}{\alpha}(t+\alpha), \big(\frac{a_2}{\alpha} - a_1\big)(t+\alpha) + \frac{a_1}{\alpha}(t+\alpha)^2\Big). \tag{2.17}$$

A pathline is a curve in space. In the above examples, solutions (2.4) and (2.17) give the pathlines in the form $\vec{x}(t)$. That is, the curves are given in a parameterized form using time as the parameter. We can also think of the solutions as being functions of $\vec{a}$, in which case the pathlines have the form $\vec{x}(\vec{a}, t)$. The time t is still the parameter, that is, along the curve t varies while $\vec{a}$ is fixed. In general different values of $\vec{a}$ will give different pathlines. Differentiating with respect to the parameter t gives a vector which is tangent to the curve. Thus

$$\frac{d\vec{x}}{dt}(t) = \vec{u}(\vec{x}, t), \tag{2.18}$$

is tangent to the pathline. Here the dependence on $\vec{a}$ has been suppressed. More generally we could write

$$\frac{\partial\vec{x}}{\partial t}(\vec{a}, t) = \vec{u}(\vec{x}, t), \tag{2.19}$$

where $\vec{x}$ and $\vec{a}$ are related via $\vec{x} = \vec{x}(\vec{a}, t)$.

2.1 Eulerian and Lagrangian Descriptions of Flow

There are two standard ways to describe a flow: The Lagrangian and Eulerian descriptions.

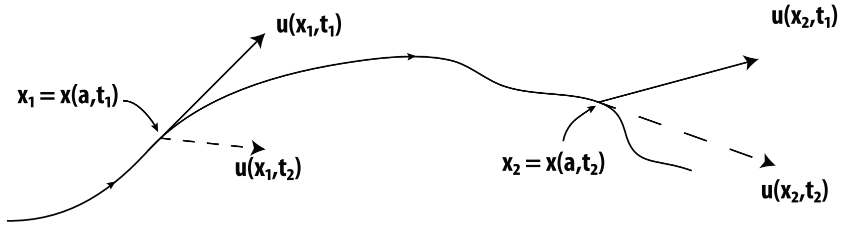


Figure 2.2: Pathline fand velocity vectors at different times. $\vec{u}(\vec{x}_1, t)$ is tangent to the pathline at time t_1 when the particle is at $\vec{x}_1$. $\vec{u}(\vec{x}_2, t)$ is tangent to the pathline at time t_2 when the particle is at $\vec{x}_2$.

In the **Lagrangian description** the flow is described in terms of particles. Each particle has a label, or set of coordinates, $\vec{a}$ say, and the flow is given as a function of these coordinates. One normally takes $\vec{a}$ to be the particle position at some fixed reference time, usually $t = 0$. We will use a subscript L to denote the use of Lagrangian coordinates. Thus

$$\vec{u}_L(\vec{a}, t) = \text{the velocity of particle } \vec{a} \text{ at time } t,$$

and

$$\rho_L(\vec{a}, t) = \text{the density of particle } \vec{a} \text{ at time } t,$$

etc. The density $\rho_L(\vec{a}, t)$, for fixed $\vec{a}$, gives the density of a fixed particle (specified by the value of $\vec{a}$) as it moves.

In the previous example of pathlines

$$\vec{x}(\vec{a}, t) = \Big(\frac{a_1}{\alpha}(t+\alpha), \big(\frac{a_2}{\alpha} - a_1\big)(t+\alpha) + \frac{a_1}{\alpha}(t+\alpha)^2\Big),$$

is the position of particle $\vec{a} = (a_1, a_2)$ as a function of time. The velocity of particle $\vec{a}$ is

$$\vec{u}_L(\vec{a}, t) = \frac{\partial \vec{x}}{\partial t}(\vec{a}, t) = \Big(\frac{a_1}{\alpha}, \frac{a_2}{\alpha} - a_1 + 2\frac{a_1}{\alpha}(t+\alpha)\Big). \tag{2.20}$$

Definition: *A* **Lagrangian measurement** *is taken while moving with a particle, i.e., while moving with the fluid.*

For example, one can imagine taking a Lagrangian measurement of temperature in the atmosphere from a tiny air balloon which moves with the wind (real balloons, of course, don't move exactly with the fluid).

In the **Eulerian description** the flow is described in terms of fixed spatial coordinates. We have already been using this coordinate description when we originally

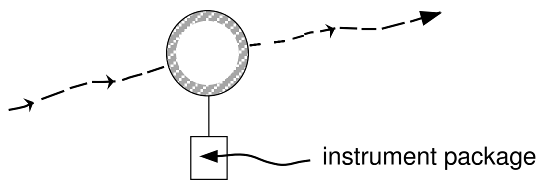


Figure 2.3:

defined $\vec{u}(\vec{x}, t)$ and $\rho(\vec{x}, t)$, etc. The velocity $\vec{u}(\vec{x}, t)$ is the velocity of the fluid particle which is at position $\vec{x}$ and time t. At different times (with $\vec{x}$ fixed) it gives the velocity of different particles.

Definition: *An* **Eulerian measurement** *is taken at a fixed position in space.*

For example, an Eulerian temperature measurement could be obtained at the top of a weather tower.

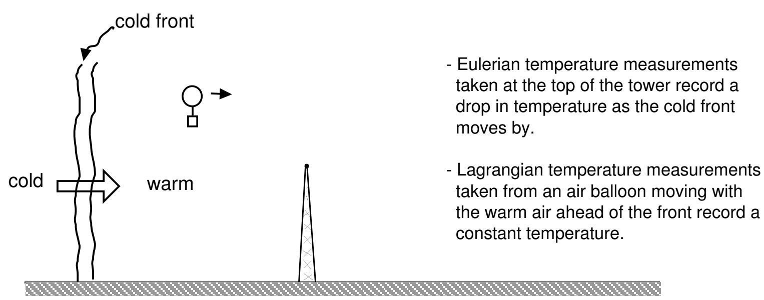


Figure 2.4:

The Eulerian description is the most common.

Converting coordinate systems: To convert from Eulerian to Lagrangian variables one must find the particle paths as functions of the Lagrangian variables (normally initial particle position). Thus one must first solve

$$\frac{d\vec{x}}{dt} = \vec{u}(\vec{x}, t)$$

to obtain $\vec{x}(\vec{a}, t)$. Then replace $\vec{x}$ by $\vec{x}(\vec{a}, t)$ to obtain a function of $\vec{a}$ and t. For example $\rho_L(\vec{a}, t) = \rho\big(\vec{x}(\vec{a}, t), t\big)$. The Lagrangian velocity fields can be obtained by making this substition in $\vec{u}(\vec{x}, t)$ however it is usually simpler to differentiate $\vec{x}(\vec{a}, t)$ w.r.t t as in (2.20).

To convert from Lagrangian to Eulerian variables invert $\vec{x} = \vec{x}(\vec{a}, t)$ and find $\vec{a}(\vec{x}, t)$. Then replace $\vec{a}$ by $\vec{a}(\vec{x}, t)$. For example, $\rho(\vec{x}, t) = \rho_L\big(\vec{a}(\vec{x}, t), t\big)$ and $\vec{u}(\vec{x}, t) = \vec{u}_L\big(\vec{a}(\vec{x}, t), t\big)$.

Example 3: Given the Eulerian velocity field

$$\vec{u}(\vec{x}, t) = \vec{x} \tag{2.21}$$

find the Lagrangian velocity field.

Solution: This is the same velocity field used in example 1. To find $\vec{u}_L(\vec{a}, t)$ we use the fact that $\vec{u}_L(\vec{a}, t) = \vec{u}\big(\vec{x}(\vec{a}, t), t\big) = \frac{\partial \vec{x}}{\partial t}(\vec{a}, t)$. From example 1 we have

$$\vec{x} = (a_1 e^t, a_2 e^t, a_3 e^t).$$

Thus,

$$\vec{u}_L(\vec{a}, t) = \frac{\partial \vec{x}}{\partial t} = (a_1 e^t, a_2 e^t, a_3 e^t).$$

Example 4: Find the Lagrangian velocity field corresponding to the Eulerian velocity field in example 2, namely,

$$\vec{u}(\vec{x}, t) = \Big(\frac{x_1}{t+\alpha}, x_1 + \frac{x_2}{t+\alpha}\Big), \qquad \text{for } t \geq 0,\ \alpha > 0.$$

Solution: From example 2 (equation (2.17) we know that the particle paths are

$$\vec{x}(\vec{a}, t) = \Big(\frac{a_1}{\alpha}(t+\alpha), \big(\frac{a_2}{\alpha} - a_1\big)(t+\alpha) + \frac{a_1}{\alpha}(t+\alpha)^2\Big).$$

To find $\vec{u}_L(\vec{a}, t)$ one can substitute $\vec{x}(\vec{a}, t)$ for $\vec{x}$ in the expression for $\vec{u}(\vec{x}, t)$ (exercise), or use $u_L = \frac{\partial \vec{x}}{\partial t}(\vec{a}, t)$.Doing so gives

$$\vec{u}_L(\vec{a}, t) = \Big(\frac{a_1}{\alpha}, \big(\frac{a_2}{\alpha} - a_1\big) + 2\frac{a_1}{\alpha}(t+\alpha)\Big).$$

Example 5: Given the pathlines

$$\vec{x}(t) = \Big(a_1 e^{-t^2}, a_2 e^{t^2}\Big), \tag{2.22}$$

(i) Sketch the pathlines,

(ii) Find the Lagrangian velocity field

(iii) Find the Eulerian velocity field

Solution: Eliminating t we have $x_1 x_2 = a_1 a_2$ which is constant. Thus, the pathlines are hyperbolas in the $x_1 x_2$-plane. As $t \to \infty$ $x_1(t) \to 0$ and $x_2(t) \to \pm\infty$, depending on the sign of a_2. Thus, the flow is in the direction indicated in Figure 2.5.

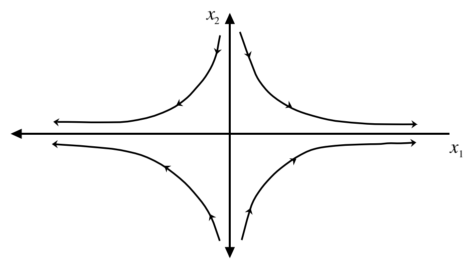


Figure 2.5: Some pathlines for example 5.

To find the Lagrangian velocity field we simply take the derivative of $\vec{x}(t)$ with respect to t giving

$$\vec{u}_L(\vec{a}, t) = \frac{\partial \vec{x}}{\partial t}(t) = \left(-2a_1 t e^{-t^2}, 2a_2 t e^{t^2}\right). \tag{2.23}$$

To find the Eulerian velocity field $\vec{u}(\vec{x}, t)$ we need to express the fluid velocity as a function of $\vec{x}$ instead of $\vec{a}$. Thus, we simply have to express a_1 and a_2 in terms of $\vec{x}$ and t. From the expression for the path lines we have $a_1 = x_1 e^{t^2}$ and $a_2 = x_2 e^{-t^2}$ hence

$$\vec{u}(\vec{x}, t) = \vec{u}_L(\vec{a}(\vec{x}, t), t) = (-2x_1 t, 2x_2 t).$$

Definition: *A* **streamline** *is a curve $\vec{x}(s)$ which is everywhere tangent to the velocity field $\vec{u}(\vec{x}, t)$ at a fixed time t.*

In this definition the parameter s could be any parameter. It does not necessarily have to be length along the curve.

Since $\frac{d\vec{x}}{ds}$ is tangent to the streamline $\vec{x}(s)$ it follows that $\frac{d\vec{x}}{ds}$ is parallel to $\vec{u}\big(\vec{x}(s), t\big)$. Wlog we can set

$$\frac{d\vec{x}}{ds}(s) = \vec{u}(\vec{x}(s), t), \tag{2.24}$$

and solve for $\vec{x}(s)$. Note the important difference between equation (2.1) for a pathline and equation (2.24) for a streamline. In the latter, t, the second argument of $\vec{u}$, is fixed and does not vary along the curve. For a pathline t is the parameter and it does vary along the curve. A streamline is a curve which is everywhere tangent to the velocity field at a fixed instant in time. The pathline is the path followed by a particle. The pathline is tangent to the velocity field at $\vec{x}$ only at the time t when the particle is at $\vec{x}$, but not necessarily at any other time.

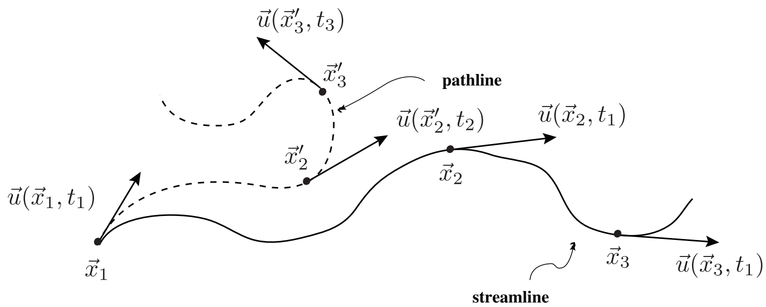


Figure 2.6: Streamline at time t_1 passing through point $\vec{x}_1$ and pathline passing through $\vec{x}_1$ at time t_1.

Example 6: Given the two-dimensional velocity field

$$\vec{u}(\vec{x}, t) = \Big(\frac{x_1}{t+\alpha}, x_1 + \frac{x_2}{t+\alpha}\Big), \qquad \text{for } t,\, \alpha \geq 0 \tag{2.25}$$

find the streamlines.

Solution: This is the same velocity field considered in example 2 above where we computed the path lines. We now want to find the streamlines. We now wish to solve

$$\frac{dx_1}{ds} = \frac{x_1}{t+\alpha}, \tag{2.26}$$

$$\frac{dx_2}{ds} = x_1 + \frac{x_2}{t+\alpha}. \tag{2.27}$$

Compare this with the set of ODEs (2.7)–(2.8) in example 2. Here t is a fixed parameter, we want to find $\vec{x}(s)$, not $\vec{x}(t)$. The first of the equations for the streamlines is easily solved giving

$$x_1(s) = Ae^{s/(t+\alpha)}.$$

Using this the problem for $x_2(s)$ is

$$\frac{dx_2}{ds} - \frac{x_2}{t+\alpha} = Ae^{s/(t+\alpha)},$$

which has the solution

$$x_2(s) = Be^{s/(t+\alpha)} + Ase^{s/(t+\alpha)}.$$

Pathlines and streamlines are generally not the same. Pathlines and streamlines will be the same if $\vec{u}(\vec{x},t)$ is independent of t or if $\vec{u}(\vec{x},t)$ changes in time such that the direction is constant and only the magnitude changes.

Example 7: Consider the velocity field $\vec{u} = (-2\omega t x_2, 2\omega t x_1)$. The particle paths are

$$\vec{x}(t) = \Big(r\cos(\omega t^2 + \theta), r\sin(\omega t^2 + \theta)\Big)$$

where (r, θ) are the initial polar coordinates of the particle (exercise). Particles move in circular paths with angular velocity ωt^2. The streamlines are also circles centred on the origin.

Exercise: Show that the streamlines and pathlines for the flow in example 5 coincide.

Definition: *A flow is* **steady** *provided all flow quantities are independent of time in the Eulerian reference frame.*

Not only is the velocity field $\vec{u}(\vec{x})$ independent of time but so are all other fields such as density, temperature, pressure, etc.

2.2 The Material Derivative

Consider a fluid particle with position $\vec{x}(t)$ and let $f(\vec{x},t)$ be any property of the flow (density, pressure, a velocity component, etc.). The value of f at the location of (or 'seen by') a moving particle $\vec{a}$, denoted by $f_L(\vec{a},t)$, is a function of time and is given by

$$f_L(\vec{a},t) = f\big(\vec{x}(\vec{a},t),t\big) = f\big(x_1(\vec{a},t), x_2(\vec{a},t), x_3(\vec{a},t), t\big). \tag{2.28}$$

The rate of change of f_L with time is

$$\begin{aligned} \frac{\partial f_L}{\partial t} &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial f}{\partial x_3}\frac{dx_3}{dt}, \\ &= \frac{\partial f}{\partial t} + \Big(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \frac{dx_3}{dt}\Big) \cdot \Big(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\Big), \\ &= \frac{\partial f}{\partial t} + \frac{d\vec{x}}{dt} \cdot \vec{\nabla} f. \end{aligned} \tag{2.29}$$

But $d\vec{x}/dt = \vec{u}(\vec{x}, t)$ so

$$\frac{\partial f_L}{\partial t} = \frac{\partial f}{\partial t} + \vec{u} \cdot \vec{\nabla} f. \tag{2.30}$$

Definition: *The* **material derivative** *is the operator*

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u} \cdot \vec{\nabla}. \tag{2.31}$$

The material derivative of a function f gives the rate of change of f with time in a reference frame moving with the velocity $\vec{u}$ of the material at the point in question.

Interpretation: Recall from vector calculus that

1. $\vec{\nabla} f$ is a vector pointing in the direction in which f is increasing most rapidly with magnitude equal to the rate of change of f with distance in that direction.

2. The directional derivative $\hat{T} \cdot \vec{\nabla} f$, where $\hat{T}$ is a unit vector, is the rate of change of f with distance in the direction of $\hat{T}$.

3. Suppose an observer moves with velocity $\vec{V}$. Then

$$\vec{V} \cdot \vec{\nabla} f = \left(\frac{\vec{V}}{|\vec{V}|} \cdot \vec{\nabla} f\right)|\vec{V}|,$$

   that is, it equals the rate of change of f with distance in the direction of motion (as $\vec{V}/|\vec{V}|$ is a unit vector) times the rate of change of distance with time. This is the rate of change of f with time measured by the observer moving with velocity $\vec{V}$, assuming f is independent of time.

Thus

$$\frac{Df}{Dt} = f_t + \vec{u} \cdot \vec{\nabla} f$$

is the total rate of change with respect to time seen by an observer moving with the flow velocity $\vec{u}$. Contributions to Df/Dt come from the rate of change of f with respect to time at a fixed position (f_t) plus the rate of change of f with respect to time due to motion through the spatial variations in f ($\vec{u} \cdot \vec{\nabla} f$).

Example 8: Acceleration of a particle. Given $\vec{u}(\vec{x}, t) = (x, yt^2, z + t)$ find the acceleration of the particle which is at $\vec{x} = (x, y, z) = (1, 2, 3)$ at $t = 1$.

Solution: Let $\vec{x}(t)$ be the position of a particle. Then by definition, the particle's velocity is

$$\frac{d\vec{x}}{dt} = \vec{u}\Big(x(t), y(t), z(t), t\Big).$$

Consider the x component. Letting $\vec{u} = (u, v, w)$ we have

$$\frac{dx}{dt} = u\Big(x(t), y(t), z(t), t\Big).$$

The x-component of the acceleration is

$$\frac{d^2x}{dt^2} = \frac{d}{dt}u\Big(x(t), y(t), z(t), t\Big) = \frac{Du}{Dt} = u_t + \vec{u} \cdot \vec{\nabla} u.$$

Similarly

$$\frac{d^2y}{dt^2} = \frac{Dv}{Dt} = v_t + \vec{u} \cdot \vec{\nabla} v,$$

and

$$\frac{d^2z}{dt^2} = \frac{Dw}{Dt} = w_t + \vec{u} \cdot \vec{\nabla} w.$$

Thus, the acceleration of the particle is

$$\frac{D\vec{u}}{Dt} = \Big(\frac{Du}{Dt}, \frac{Dv}{Dt}, \frac{Dw}{Dt}\Big) = \frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u}. \tag{2.32}$$

Here, $\vec{u} \cdot \vec{\nabla} \vec{u}$ means $(\vec{u} \cdot \vec{\nabla})\vec{u} = (\vec{u} \cdot \vec{\nabla} u, \vec{u} \cdot \vec{\nabla} v, \vec{u} \cdot \vec{\nabla} w)$. $D\vec{u}/Dt$ is a function of $\vec{x}$ and t. It is the particle acceleration field $\vec{a}(\vec{x}, t)$ (not to be confused with particle labels $\vec{a}$ used earlier) of the flow.

For $\vec{u} = (x, yt^2, z + t)$ the first component of the particle acceleration field is

$$\begin{aligned} \frac{Du}{Dt} &= u_t + uu_x + vu_y + wu_z, \\ &= \frac{\partial u}{\partial t} + x\frac{\partial x}{\partial x} + yt^2\frac{\partial x}{\partial y} + (z + t)\frac{\partial x}{\partial z}, \\ &= 0 + x \cdot 1 + yt^2 \cdot 0 + (z + t) \cdot 0, \\ &= x. \end{aligned}$$

The second component of the acceleration field is

$$\begin{aligned}
\frac{Dv}{Dt} &= v_t + uv_x + vv_y + wv_z, \\
&= \frac{\partial(yt^2)}{\partial t} + x\frac{\partial(yt^2)}{\partial x} + yt^2\frac{\partial(yt^2)}{\partial y} + (z+t)\frac{\partial(yt^2)}{\partial z}, \\
&= 2yt + x\cdot 0 + yt^2\cdot t^2 + (z+t)\cdot 0, \\
&= 2yt + yt^4.
\end{aligned}$$

The last component is

$$\begin{aligned}
\frac{Dw}{Dt} &= w_t + uw_x + vw_y + ww_z, \\
&= \frac{\partial}{\partial t}(z+t) + x\frac{\partial}{\partial x}(z+t) + yt^2\frac{\partial}{\partial y}(z+t) + (z+t)\frac{\partial}{\partial z}(z+t), \\
&= 1 + x\cdot 0 + yt^2\cdot 0 + (z+t)\cdot 1, \\
&= 1 + z + t.
\end{aligned}$$

Thus, the particle acceleration field is

$$\vec{a}(\vec{x},t) = \frac{D\vec{u}}{Dt} = \Big(x, 2yt + yt^4, 1 + z + t\Big).$$

Evaluating at $\vec{x} = (1,2,3)$ and $t = 1$ gives

$$\frac{D\vec{u}}{Dt}(1,2,3,1) = \Big(1,6,5\Big),$$

as the acceleration of the particle at $\vec{x} = (1,2,3)$ at time $t = 1$.

As we shall see shortly, the material derivative plays an important role in the equations of motion which we will begin deriving next. The particle acceleration field $D\vec{u}/Dt$ is the acceleration used in the application of Newton's second law.

2.3 Material Volumes

Definition: *A* **material volume** *is a fixed piece of matter (a subset of R^3) which moves with the flow. It is comprised of the same particles for all time. In Lagrangian coordinates a material volume is a fixed region, i.e., is independant of time.*

For example, imagine that a patch of water in a river is dyed red. The red patch of water is a material volume. Over short periods of time it moves and deforms with the flow [1]. At different times the red patches occupies different volumes in Eulerian (e.g., Cartesian) coordinates. In Lagrangian coordinates the red patch always occupies the same volume, i.e., the Lagrangian coordinates of the patch don't change with time.

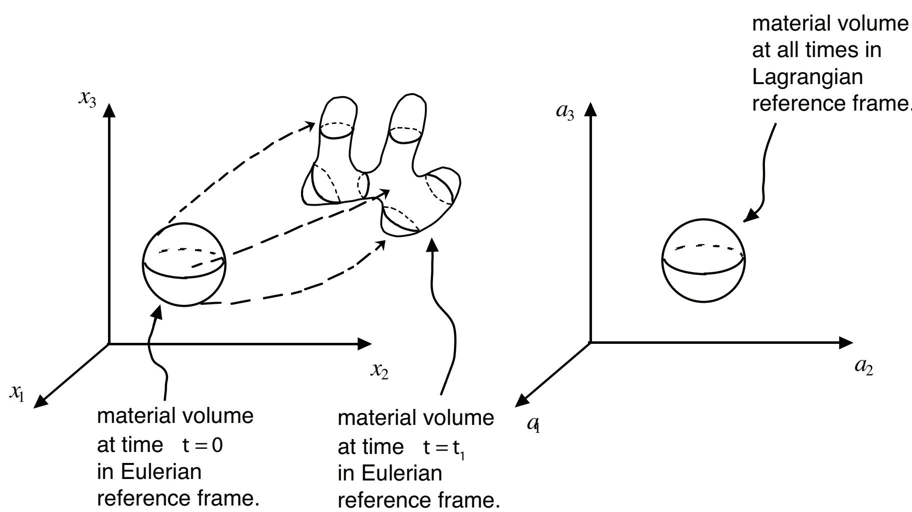


Figure 2.7:

In order to apply conservation laws we will need to find the time evolution of volume integrals of the form

$$I(t) = \iiint_{W(t)} f(\vec{x}, t)\, dV, \tag{2.33}$$

where $W(t)$ is a material volume. The difficulty here is that the region of integration is a function of time. To take the time derivative we must make a change of variables so that the integral in the new coordinate system is over a fixed region. Then the time derivative can be brought under the integral sign. We will use Lagrangian coordinates for this purpose.

[1] In reality the dye will diffuse making the boundary between the red patch and the water surrounding it blurry so this picture is only useful for times sufficiently short that diffusion is not significant. How short is short enough will depend on the length scales of interest.

Definition: *Let $\Phi(\cdot, t) : R^3 \to R^3$ be the mapping that maps the initial position $\vec{a}$ of a particle to its position at time t. That is, $\Phi(\vec{a}, t) = \vec{x}(\vec{a}, t)$.*

Assumptions:

1. $\Phi(\cdot, t)$ is 1-1 and hence invertible.

2. $\Phi(\cdot, t)$ and its inverse are continuously differentiable functions of $\vec{x}$ and t.

The first of these implies that two different particles of matter can't be at the same place at a later time or that one particle can't split into two. By definition of a material volume every particle in $W(0)$ gets mapped to a particle in $W(t)$ and every particle in $W(t)$ comes from a particle in $W(0)$.

Let $\Omega(t)$ be the region in space occupied by the matter in question and let $W(t) \in \Omega(t)$ be a material volume. Then $\Phi(\cdot, t)$ maps $W(0)$ to $W(t)$. That is, $\Phi\big(W(0), t\big) = W(t)$. More generally, $W(0)$ could be any subset of $\Omega(0)$. If it is a curve then $W(t) = \Phi\big(W(0), t\big)$ is called a **material curve**. If it is a surface, then $W(t) = \Phi\big(W(0), t\big)$ is called a **material surface**.

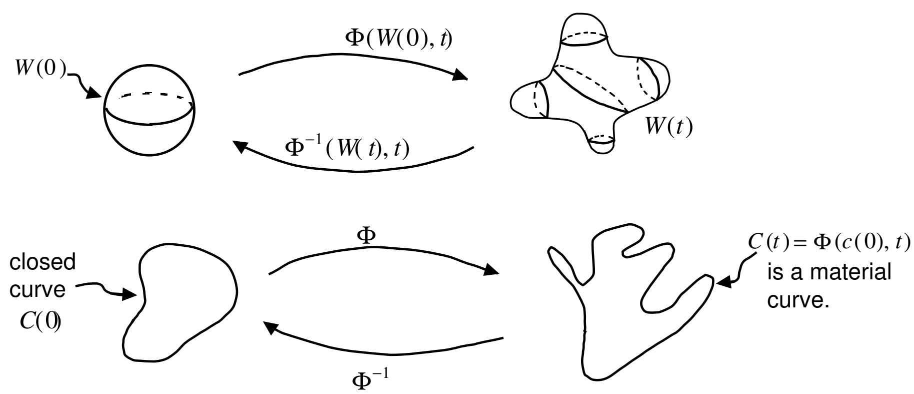


Figure 2.8:

Pictorially one can imagine colouring a region $W(0)$ at time $t = 0$. As the matter moves and deforms so does the coloured region. In reality, in the case of a fluid (to a much lesser extent for solids) molecules enter and leave $W(t)$ (diffusion). A dyed region of fluid will lose its identity after a sufficiently long time. We assume that the material volume/surface/curve retains its identity over the time interval of interest.

2.4 The Transport Theorem

Now, consider the volume integral

$$I(t) = \iiint\limits_{W(t)} f(\vec{x}, t)\, dV, \tag{2.34}$$

where the volume of integration, $W(t)$, is a material volume which is a function of time. We wish to find the time derivative of $I(t)$. To do this we map $W(t)$, the volume over which we're integrating, to $W(0)$ which is independent of time. This is done using the mapping $\Phi(\cdot, t)$ which maps $W(0)$ to $W(t)$. Recall that, from second year calculus,

$$\iiint\limits_{W(t)} f(\vec{x}, t)\, dV = \iiint\limits_{W(0)} f\big(\Phi(\vec{a}, t), t\big) J(\vec{a}, t)\, dV, \tag{2.35}$$

where

$$J(\vec{a}, t) = \frac{\partial(\Phi_1, \Phi_2, \Phi_3)}{\partial(a_1, a_2, a_3)}, \tag{2.36}$$

is the Jacobian of the transformation $\vec{x} = \Phi(\vec{a}, t) = \big(\Phi_1(\vec{a}, t), \Phi_2(\vec{a}, t), \Phi_3(\vec{a}, t)\big)$. You may wonder why the absolute value of the Jacobian is not used. This is because of the following lemma which says that $J > 0$.

Lemma: *If Φ is invertible and C^2 then*

(i) $J > 0$,

(ii) $\frac{\partial J}{\partial t} = (\vec{\nabla} \cdot \vec{u})J$ *where $\vec{u}$ is the flow velocity* $\frac{\partial \vec{x}}{\partial t}$.

Proof:

(i) At $t = 0$ we have $\Phi(\vec{a}, 0) = \vec{a}$, i.e., $\Phi(\cdot, 0)$ is the identity map. Thus $J(\vec{a}, 0) = 1$. Since $\Phi(\cdot, 0)$ is invertible and C^2, the Jacobian J is a continuous function of t and is never zero. Thus J is always positive.

(ii) By definition the Jacobian is

$$J = \frac{\partial(\Phi_1, \Phi_2, \Phi_3)}{\partial(a_1, a_2, a_3)} = \begin{vmatrix} \Phi_{1,1} & \Phi_{2,1} & \Phi_{3,1} \\ \Phi_{1,2} & \Phi_{2,2} & \Phi_{3,2} \\ \Phi_{1,3} & \Phi_{2,3} & \Phi_{3,3} \end{vmatrix},$$

where $\Phi_{i,j}(\vec{a}, t) = \frac{\partial \Phi_i}{\partial a_j}$. We can write the above as

$$J = \begin{vmatrix} \vec{\nabla}_a^T \Phi_1 & \vec{\nabla}_a^T \Phi_2 & \vec{\nabla}_a^T \Phi_3 \end{vmatrix},$$

where $\vec{\nabla}_a = (\frac{\partial}{\partial a_1}, \frac{\partial}{\partial a_2}, \frac{\partial}{\partial a_3})$ is the gradient operator with respect to the Lagrangian variables and the superscript T is the transpose operator. Using the fact that the determinant operator is linear in rows and columns

$$\begin{aligned}\frac{\partial J}{\partial t} &= \left| \begin{array}{ccc} \frac{\partial}{\partial t}\vec{\nabla}_a^T \Phi_1 & \vec{\nabla}_a^T \Phi_2 & \vec{\nabla}_a^T \Phi_3 \end{array} \right| \\ &\quad + \left| \begin{array}{ccc} \vec{\nabla}_a^T \Phi_1 & \frac{\partial}{\partial t}\vec{\nabla}_a^T \Phi_2 & \vec{\nabla}_a^T \Phi_3 \end{array} \right| \\ &\quad\quad + \left| \begin{array}{ccc} \vec{\nabla}_a^T \Phi_1 & \vec{\nabla}_a^T \Phi_2 & \frac{\partial}{\partial t}\vec{\nabla}_a^T \Phi_3 \end{array} \right|, \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}.\end{aligned} \tag{2.37}$$

Consider the first term. Since

$$\frac{\partial}{\partial t}\Phi_{i,j} = \frac{\partial^2}{\partial t \partial a_j}\Phi_i = \frac{\partial}{\partial a_j}\frac{\partial \Phi_i}{\partial t} = \frac{\partial}{\partial a_j} u_i(\vec{\Phi}(\vec{a},t),t) = \frac{\partial u_i}{\partial x_k}\frac{\partial \Phi_k}{\partial a_j} \tag{2.38}$$

(summing over the repeated index k) we have

$$\frac{\partial}{\partial t}\vec{\nabla}_a^T \Phi_1 = \begin{pmatrix} \frac{\partial}{\partial t}\Phi_{1,1} \\ \frac{\partial}{\partial t}\Phi_{1,2} \\ \frac{\partial}{\partial t}\Phi_{1,3} \end{pmatrix} = \begin{pmatrix} \frac{\partial u_1}{\partial x_k}\frac{\partial \Phi_k}{\partial a_1} \\ \frac{\partial u_1}{\partial x_k}\frac{\partial \Phi_k}{\partial a_2} \\ \frac{\partial u_1}{\partial x_k}\frac{\partial \Phi_k}{\partial a_3} \end{pmatrix} = \frac{\partial u_1}{\partial x_k}\vec{\nabla}_a^T \Phi_k. \tag{2.39}$$

Thus, using the notation $u_{i,j} = \frac{\partial u_i}{\partial x_j}$,

$$\mathrm{I} = \left| \begin{array}{ccc} u_{1,1}\vec{\nabla}_a^T \Phi_1 + u_{1,2}\vec{\nabla}_a^T \Phi_2 + u_{1,3}\vec{\nabla}_a^T \Phi_3 & \vec{\nabla}_a^T \Phi_2 & \vec{\nabla}_a^T \Phi_3 \end{array} \right|. \tag{2.40}$$

Since the determinant operator is linear in columns, and determinants with repeated columns are zero,

$$\mathrm{I} = u_{1,1}\left| \begin{array}{ccc} \vec{\nabla}_a^T \Phi_1 & \vec{\nabla}_a^T \Phi_2 & \vec{\nabla}_a^T \Phi_3 \end{array} \right| = u_{1,1}J.$$

Similarly,

$$\mathrm{II} = u_{2,2}J,$$

and

$$\mathrm{III} = u_{3,3}J.$$

Hence

$$\frac{\partial J}{\partial t} = \mathrm{I} + \mathrm{II} + \mathrm{III} = u_{1,1}J + u_{2,2}J + u_{3,3}J = (\vec{\nabla}\cdot\vec{u})J. \tag{2.41}$$

The Transport Theorem: *If Φ is invertible, $\vec{u}(\vec{x},t)$ is C^1 and $f(\vec{x},t)$ is C^1 then*

$$\frac{dI}{dt} = \frac{d}{dt}\iiint\limits_{W(t)} f(\vec{x},t)\,dV = \iiint\limits_{W(t)} \left\{\frac{Df}{Dt} + f\vec{\nabla}\cdot\vec{u}\right\} dV. \tag{2.42}$$

Proof: Have

$$\begin{aligned} \frac{dI}{dt} = \frac{d}{dt}\iiint\limits_{W(t)} f(\vec{x},t)\,dV &= \frac{d}{dt}\iiint\limits_{W(0)} f\Big(\Phi(\vec{a},t),t\Big)J(\vec{a},t)\,dV \\ &= \iiint\limits_{W(0)} \left\{\left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_i}\frac{\partial \Phi_i}{\partial t}\right]J + f\frac{\partial J}{\partial t}\right\} dV. \end{aligned} \tag{2.43}$$

Now $\Phi_i(\vec{a},t) = x_i(\vec{a},t)$ so

$$\frac{\partial \Phi_i}{\partial t}(\vec{a},t) = u_i\Big(\Phi(\vec{a},t),t\Big). \tag{2.44}$$

Using this and the results of the previous lemma,

$$\begin{aligned} \frac{dI}{dt} &= \iiint\limits_{W(0)} \left\{\left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_i}u_i\right]J + f\,\vec{\nabla}\cdot\vec{u}J\right\} dV \\ &= \iiint\limits_{W(0)} \left\{\frac{Df}{Dt}J + f\vec{\nabla}\cdot\vec{u}J\right\} dV \\ &= \iiint\limits_{W(0)} \left\{\frac{Df}{Dt} + f\vec{\nabla}\cdot\vec{u}\right\} J\,dV \\ &= \iiint\limits_{W(t)} \left\{\frac{Df}{Dt} + f\vec{\nabla}\cdot\vec{u}\right\} dV \end{aligned}$$

where in the last step we have undone the coordinate transformation. This proves the result. Using $\vec{u}\cdot\vec{\nabla}f + f\vec{\nabla}\cdot\vec{u} = \vec{\nabla}\cdot(f\vec{u})$ we can express the result in an alternative form:

$$\frac{dI}{dt} = \iiint\limits_{W(t)} \left\{\frac{Df}{Dt} + f\vec{\nabla}\cdot\vec{u}\right\} dV = \iiint\limits_{W(t)} \left\{f_t + \vec{\nabla}\cdot(f\vec{u})\right\} dV. \tag{2.45}$$

Using the divergence theorem the latter gives

$$\frac{dI}{dt} = \iiint\limits_{W(t)} f_t \, dV + \iint\limits_{\partial W(t)} f\vec{u} \cdot \hat{n} \, dS. \tag{2.46}$$

The interpretation of (2.46) is that the rate of change of I is equal to the integral of f_t over the fixed (constant in time) region $W(t)$ plus the flux of f out of $W(t)$.

2.5 Problems

1. The velocity field of a certain flow is given by

$$\vec{u}(x, y, z, t) = (x^2 t, yt^2, xzt). \tag{2.47}$$

   (a) Find the pathlines. Express as a function of the initial particle position (a, b, c).

   (b) Find the streamlines.

   (c) Find the acceleration of the fluid particle at $(x, y, z) = (1, 1, 1)$ at $t = 2$.

   (d) Find the Lagrangian velocity field $u_L(a, b, c, t)$.

2. Consider the 2-D velocity field

$$\vec{u} = (x_1, -x_2 + x_1 e^{-t}).$$

   (a) Find the acceleration field $\vec{a}(\vec{x}, t)$. What is the acceleration of the fluid particle located at $(x_1, x_2) = (1, -1)$ at time $t = 0$?

   (b) Find the pathlines as functions of the initial particle position (a_1, a_2).

   (c) Find the streamlines. Make a sketch showing the velocity vector field and the streamlines at two different times. Explain why the streamlines and pathlines are different.

   (d) Find the Lagrangian velocity and acceleration fields $\vec{u}_L(\vec{a}, t)$ and $\vec{a}_L(\vec{a}, t)$ (the latter is the acceleration of particle $\vec{a}$ at time t).

   (e) Is the flow steady? Is it incompressible or not?

   (f) Is it possible for the mass density for this flow to be $\rho(x, y, t) = \rho_o/(1 + x^2)$ where ρ_o is a constant?

3. In Lagrangian coordinates the velocity field of a flow is

$$\vec{u}_L(\vec{a},t) = (-a_2 e^{-t}, -a_3, 2t). \tag{2.48}$$

   (a) Find the pathlines, using $\vec{a} = (a_1, a_2, a_3)$ as the particle position at $t = 0$.

   (b) Find $\vec{a}_L(\vec{a},t)$, the acceleration in Lagrangian coordinates.

   (c) Find $\vec{a}_E(\vec{x},t)$, the acceleration field in Eulerian coordinates.

4. Let $f, g : R^3 \times T \to R$ and $h : R \to R$ be continuous and differentiable maps. Let D/Dt be the material derivative operator. Verify that

   (a)
$$\frac{D}{Dt}\Big(f+g\Big) = \frac{Df}{Dt} + \frac{Dg}{Dt}. \tag{2.49}$$

   (b)
$$\frac{D}{Dt}\Big(fg\Big) = \frac{Df}{Dt}g + f\frac{Dg}{Dt}. \tag{2.50}$$

   (c)
$$\frac{D}{Dt}\Big(h \circ g\Big) = \Big(h' \circ g\Big)\frac{Dg}{Dt}. \tag{2.51}$$

5. The water temperature in a lake is

$$T = 10 + 4.0\bigg(1 + \tanh\Big(\frac{z+5}{5.0}\Big)\bigg), \tag{2.52}$$

   where z, in meters, is the vertical coordinate (positive up) and the temperature is in degrees Celsius. The lake surface is at $z = 0$. A scuba diver swims downward at an angle 30° below the horizontal at a constant speed of 0.5 m s^{-1}. Find the extreme value (positive or negative) of the time rate of change of temperature experienced by the diver (this requires units!).

6. Very small amplitude 2-D water waves may be described, to a very good approximation, by the velocity field $\vec{u}(x,z,t) = (u,w) = \vec{\nabla}\phi$ where

$$\phi(x,z,t) = a\frac{\omega}{k}\frac{\cosh\big(k(z+h)\big)}{\sinh(kh)}\cos(kx - \omega t).$$

   Find a sensible approximation for the pathlines using the fact that the wave amplitude a is very small (hint: assume particles move only a small distance from some average position (x_o, z_o)). Here (x,z) are the horizontal and vertical Cartesian coordinates and (u,w) are the corresponding horizontal and vertical velocity components. ϕ is called a velocity potential. The wave amplitude a is very small. The bottom boundary is at $z = -h$. What is the value of w at the bottom?.

7. (a) Find the area

$$A = \iint_R dx\,dy = \iint_R J\,dr\,d\theta \tag{2.53}$$

of the ellipse $x^2/a^2 + y^2/b^2 = 1$ by computing the Jacobian J of the transformation

$$(x, y) = (ar\cos\theta, br\sin\theta) \tag{2.54}$$

and evaluating the second integral with suitable limits on r and θ.

(b) Let

$$I(t) = \iint_R f(x, y, t)\,dx\,dy, \tag{2.55}$$

where R, the elliptical region of integration, is as in part (a) except that now it is time dependant, i.e., both $a(t)$ and $b(t)$ are functions of time. Show that

$$\frac{dI}{dt} = \iint_R \left[f_t + \vec{v}\cdot\vec{\nabla} f\right] dx\,dy + \iint_R f\frac{\partial J}{\partial t}\,dr\,d\theta \tag{2.56}$$

where

$$\vec{v} = \left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}\right). \tag{2.57}$$

(c) Write $\vec{v}$ in terms of x and y and show that

$$\frac{\partial J}{\partial t} = (\vec{\nabla}\cdot\vec{v})J, \tag{2.58}$$

and hence that

$$\begin{aligned} \frac{dI}{dt} &= \iint_R f_t\,dx\,dy + \iint_R \vec{\nabla}\cdot(f\vec{v})\,dx\,dy \\ &= \iint_R f_t\,dx\,dy + \int_{\partial R} f\vec{v}\cdot\hat{n}\,ds. \end{aligned} \tag{2.59}$$

8. A certain flow has velocity field

$$\vec{u} = (x + xy, \frac{1}{2}y - \frac{1}{2}y^2, -z + x). \tag{2.60}$$

The components of $\vec{u}$ are in m s^{-1}, units of x, y, and z are meters. A material volume has an initial volume of 3 m^3. What is its volume 2 s later?

9. Let $\vec{u} = (-y, x)$ be a two-dimensional velocity field.

(a) Is the flow incompressible? Justify.

(b) Calculate the vorticity $\zeta = \vec{\nabla} \times \vec{u}$ (which is a scalar in 2D).

(c) Suppose the density field $\rho(x, y, t)$ is a function of the radial distance $r = \sqrt{x^2 + y^2}$ only, that is $\rho = F(r)$ for some function F. For what functions F does ρ satisfy the continuity equation?

(d) Sketch the streamlines and the velocity field. Using these explain your previous result.

(e) Find the counterclockwise circulation

$$C = \oint \vec{u} \cdot \hat{t}\, ds$$

around the closed curve $x^2 + y^2 = R^2$. Do this by calculating the line integral directly.

(f) Find the counterclockwise circulation around the ellipse $x^2/a^2+y^2/b^2 = 1$ (a and b constant) by using Stokes' Theorem.

(g) Find the volume flux $\oint \vec{u} \cdot \hat{n}\, ds$ through the same ellipse using what ever method is simplest.

10. (a) Consider

$$I(t) = \int_a^b h(x, t)\, dx,$$

where $a(t)$ and $b(t)$ are constants. By using the definition of the derivative of $I(t)$

$$I'(t) = \lim_{h\to 0} \frac{I(t + h) - I(t)}{h},$$

show that

$$I'(t) = \int_a^b h_t(x, t))\, dx.$$

(b) Now consider the integral

$$I(t) = \int_{a(t)}^{b(t)} h(x,t)\,dx.$$

Such an integral can arise in a 1-D continuum mechanics problem. For example, consider a bar of metal being stretched and compressed in the along-bar direction. $a(t)$ and $b(t)$ could be the position of two particles. Then if $h(x,t)$ is the mass density, $I(t)$ is the mass of material between the two particles which should remain constant in time.

(c) Introduce a change of variable which make the limits of integration time independent via

$$x(t,\alpha) = a(t) + \alpha\Big(b(t) - a(t)\Big).$$

This gives

$$I(t) = \int_0^1 h(x(t,\alpha),t)J(t)\,d\alpha,$$

where

$$J = \frac{\partial x}{\partial \alpha} = b(t) - a(t),$$

is the Jacobian of the transformation. From part (a) we have

$$I'(t) = \int_0^1 \left\{ \left[h_t + h_x \frac{\partial x}{\partial t} \right] J(t) + h \frac{dJ}{dt} \right\} d\alpha.$$

Defining

$$v = \frac{\partial x}{\partial t},$$

show that

$$\frac{dJ}{dt} = \frac{\partial v}{\partial x} J. \qquad (*)$$

Hence, show that

$$I'(t) = \int_{a(t)}^{b(t)} \Big(h_t + v h_x + v_x h \Big)\,dx.$$

In this expression v is a function of x and t. Show that

$$v = a'(t) + \frac{b'(t) - a'(t)}{b(t) - a(t)}\big(x - a(t)\big).$$

Integrating the last two terms show that

$$I'(t) = \int_{a(t)}^{b(t)} h_t\,dx + h\big(t, b(t)\big)b'(t) - h\big(t, a(t)\big)a'(t).$$

(d) Note: there is another (easier?) way to do this (can you think of one?) but this solution method is similar to the two dimensional case considered in the next question and to the three dimensional case which will be done in class.

11. **A second transport theorem.**

(a) Let

$$I(t) = \oint_{C(t)} \vec{f}(\vec{x}, t) \cdot d\vec{x},$$

where $C(t)$ is a material curve. Then, using $\vec{x} = \Phi(\vec{a}, t)$ we can rewrite I as

$$I(t) = \oint_{C(t)} f_i(\vec{x}, t)\, dx_i = \oint_{C(0)} f_i\Big(\Phi(\vec{a}, t), t\Big) \frac{\partial \Phi_i}{\partial a_k}\, da_k.$$

Taking the time derivative show that

$$\frac{dI}{dt} = \oint_{C(0)} \left[\frac{Df_i}{Dt} \frac{\partial \Phi_i}{\partial a_k} + f_i \frac{\partial u_i}{\partial x_j} \frac{\partial \Phi_j}{\partial a_k} \right] da_k,$$

and hence that

$$\frac{dI}{dt} = \oint_{C(t)} \frac{D\vec{f}}{Dt} \cdot d\vec{x} + \oint_{C(t)} f_i \frac{\partial u_i}{\partial x_j}\, dx_j. \qquad (*)$$

Using $du_i = \frac{\partial u_i}{\partial x_j} dx_j$ this can be written as

$$\frac{dI}{dt} = \oint_{C(t)} \frac{D\vec{f}}{Dt} \cdot d\vec{x} + \oint_{C(t)} \vec{f} \cdot d\vec{u}.$$

(b) **Special Case:** If $\vec{f} = \vec{u}$, the flow velocity, $I(t)$ is the circulation of the flow around the closed curve $C(t)$. We usually use the symbol $\Gamma(t)$ to denote the circulation, i.e.

$$\Gamma(t) = \oint_{C(t)} \vec{u}(\vec{x}, t) \cdot d\vec{x}.$$

Using (*) show that the rate of change of the circulation around a material curve is

$$\frac{d\Gamma}{dt} = \oint_{C(t)} \frac{D\vec{u}}{Dt} \cdot d\vec{x}.$$

12. The water temperature in a lake warms during the day due to solar radiation from the sun and typically cools at night due to the combined effects of long wave radiation emission and latent and sensible heat transfer (i.e., by evaporation and transfer of heat between the air and water). The latter three processes also act during the day. Suppose that during daylight hours the temperature varies as

$$T(z,t) = 20 + \frac{5}{2}e^{z/\lambda}\big(1-\cos(2\sigma t)\big); \qquad \text{for } 0 \le t \le \frac{\tau}{2}.$$

Here $\lambda = 5$ m is a positive constant and the water surface is at $z = 0$. Time runs from $t = 0$ at sun rise and $t = \tau/2$ at sunset where τ is the length of a day. The frequency σ is $2\pi/\tau$, i.e., one oscillation per day. The exponential decay with depth occurs because sunlight is absorbed by the water and hence decreases exponentially with depth. The absorbed radiation is what warms the water up.

Consider a zooplankton swimming upward at an angle of 45° to the horizontal. Find an expression for the rate of change of temperature experienced by the zoo-plankton at height z at time t.

Chapter 3

DERIVATION OF THE GOVERNING EQUATIONS I

We now begin the derivation of the governing equations of motion. The derivation is based on conservation laws: conservation of mass, linear momentum, angular momentum and energy. Some assumptions will also be made, the validity of which must be tested by comparing model predictions with observations. We begin with conservation of mass.

3.1 Conservation of mass: the continuity equation

$$M(t) = \iiint\limits_{W(t)} \rho \, dV, \tag{3.1}$$

is the mass of the material volume $W(t)$. Conservation of mass implies it is constant so, using the transport theorem

$$\frac{dM}{dt} = \iiint\limits_{W(t)} \Big[\frac{D\rho}{Dt} + \rho \vec{\nabla} \cdot \vec{u} \Big] \, dV = 0. \tag{3.2}$$

This is true for all material volumes $W(t)$. By the assumed continuity of the integrand the Dubois-Reymond Lemma says that the integrand must be everywhere zero. Hence

$$\frac{D\rho}{Dt} + \rho \vec{\nabla} \cdot \vec{u} = 0. \tag{3.3}$$

This is called the continuity equation. The continuity equation can also be written as

$$\rho_t + \vec{\nabla} \cdot (\rho \vec{u}) = 0. \tag{3.4}$$

Physical interpretation of $\vec{\nabla} \cdot \vec{u}$

$$V(t) = \iiint\limits_{W(t)} dV, \tag{3.5}$$

is the volume of $W(t)$. Taking the time derivative gives

$$\frac{dV}{dt} = \iiint\limits_{W(t)} \Big[\frac{D1}{Dt} + \vec{\nabla} \cdot \vec{u}\Big]\, dV = \iiint\limits_{W(t)} \vec{\nabla} \cdot \vec{u}\, dV. \tag{3.6}$$

Thus, $\vec{\nabla} \cdot \vec{u}$ is the rate of change of volume with respect to time per unit volume. The continuity equation can be written as

$$\frac{1}{\rho}\frac{D\rho}{Dt} = -\vec{\nabla} \cdot \vec{u}. \tag{3.7}$$

When $\vec{\nabla} \cdot \vec{u} > 0$ the material volume is expanding and the density is decreasing.

Theorem: *Let $\rho(\vec{x}, t)$ and $f(\vec{x}, t)$ be C^1 scalar fields and $\vec{u}(\vec{x}, t)$ be a C^1 velocity field of a flow $\Phi(\vec{a}, t)$. If mass is conserved then*

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint\limits_{W(t)} \rho f \, dV = \iiint\limits_{W(t)} \rho \frac{Df}{Dt}\, dV. \tag{3.8}$$

Proof: Exercise.

3.2 Conservation of linear momentum

For a solid, rigid object of mass m, considered as a point, with velocity $\vec{v}$, Newton's second law states that

$$\frac{d}{dt}\Big(m\vec{v}\Big) = \vec{F}, \tag{3.9}$$

where $\vec{F}$ is the net force acting on the object. Newton published his laws of motion in 1687 however it took almost a century until Leonhard Euler realized how they could be applied to a deforming continuum. In 1776 came the publication of Euler's laws of mechanics:

1. The total force acting upon a body equals the rate of change of the total (linear) momentum.

2. The total torque acting upon a body equals the rate of change of the total moment of momentum, where both the torque and the moment are taken with respect to the same fixed point.

Euler stated that these laws apply to all bodies or systems of bodies, and to every part of every body, whether or not they can be treated as a continuum or whether or not they are deforming[1]. As discussed briefly later, the second of Euler's laws is not valid for all continuums.

Applying the first of Euler's laws of mechanics to a material volume gives

$$\frac{d}{dt}\iiint\limits_{W(t)} \rho\vec{u}\,dV = \text{net force acting on } W(t), \tag{3.10}$$

since the volume integral is the total linear momentum of the matter occuping the material volume $W(t)$.

We must now investigate the forces acting on $W(t)$. There are three types of forces.

1. **Body forces** act throughout the volume. Examples of body forces include gravitational and electromagnetic forces. The only body force we will consider is the gravitational force

$$\vec{F}_g = \iiint\limits_{W(t)} \rho\vec{g}\,dV = -\iiint\limits_{W(t)} \rho\vec{\nabla}\Pi\,dV, \tag{3.11}$$

where $\vec{g} = -\vec{\nabla}\Pi$ is the gravitational force per unit mass and Π is the gravitational potential.

2. **Surface forces.** Matter outside $W(t)$ exerts a force on the matter in $W(t)$. We assume this force can be thought to act on the surface of $W(t)$ (denoted by $\partial W(t)$). It is due to short range forces between molecules and to movement of molecules across $\partial W(t)$.

3. **Line forces, i.e., surface tension.** Surface tension acts on the interface between a liquid and a gas or between two immiscible liquids. These forces do not appear in the equations of motion. They only arise in boundary conditions so we will not consider them now.

[1] Lin & Segel, 1974. Mathematics applied to deterministic problems in the natural sciences.

Definition: *The* **stress vector** $\vec{t}(\vec{x}, t, \hat{n})$ *is the force per unit area acting on a surface element at* $(\vec{x}, t)$ *with unit outward normal* $\hat{n}$. *The force is exerted by the material into which* $\hat{n}$ *points and acts on the material from which* $\hat{n}$ *points.*

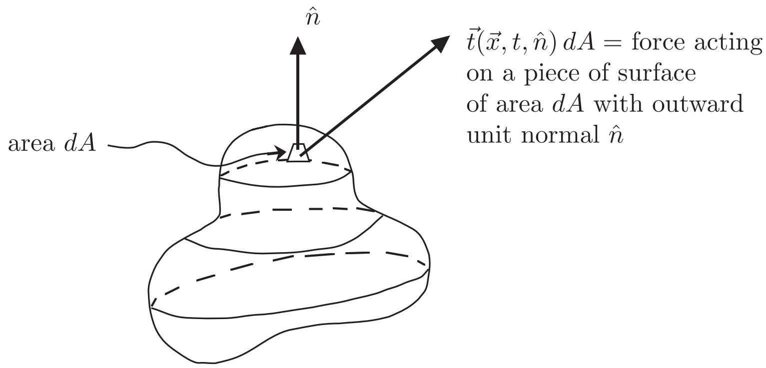


Figure 3.1: A small surface element of area dA on the surface of a volume W of matter has unit outward normal $\hat{n}$. The force exerted by the matter outside W acting on the surface element is $\vec{t}(\vec{x}, t, \hat{n})\, dA$.

The balance of linear momentum gives

$$\frac{d}{dt} \iiint\limits_{W(t)} \rho \vec{u}\, dV = - \iiint\limits_{W(t)} \rho \vec{\nabla} \Pi\, dV + \iint\limits_{\partial W(t)} \vec{t}(\vec{x}, t, \hat{n})\, dS. \tag{3.12}$$

Before taking a closer look at what $\vec{t}(\vec{x}, t, \hat{n})$ is we will consider some simple examples of surface forces.

3.3 Simple Examples of Surface Forces in Fluids

Example 9: Consider a simple liquid of constant density ρ at rest under gravity (e.g., water).

The fluid below height z exerts a force per unit area equal to $p\hat{k}$ on the fluid above. In this case the outward unit normal is $\hat{n} = -\hat{k}$ and we have

$$\vec{t}(\vec{x}, t, -\hat{k}) = p\hat{k}. \tag{3.13}$$

This is what balances the forces (gravitational plus surface air pressure) acting on the fluid above z. The fluid above height z exerts equal and opposite force, equal to $-p\hat{k}$ per unit area. Thus

$$\vec{t}(\vec{x}, t, \hat{k}) = -p\hat{k}. \tag{3.14}$$

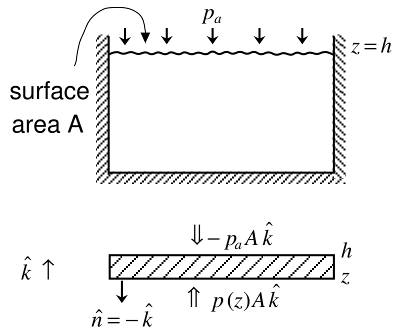


Figure 3.2:

More generally, the stress vector associated with hydrostatic pressure in a simple liquid and in a gas at rest is (see Figure 3.3)

$$\vec{t}(\vec{x}, t, \hat{n}) = -p(z)\hat{n}. \tag{3.15}$$

The surface force clearly depend on the orientation of the surface, so $\vec{t}$ must depend on $\hat{n}$.

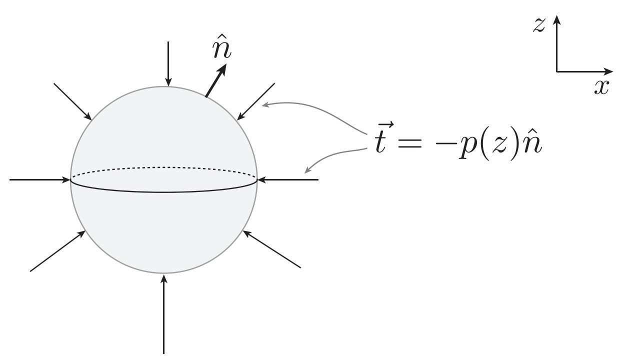


Figure 3.3: Surface force exerted by fluid outside a spherical volume of fluid. Fluid is at rest so the only surface force is due to the pressure which acts normally to the surface. Hence the surface stress $\vec{t}$ acts in the direction of $-\hat{n}$. With gravity acting down (negative z direction) the pressure decreases with z so the force acting on top of the volume is lower in magnitude than that acting on the bottom of the volume.

Comment: The fact that there is no component of $\vec{t}$ tangent to the surface is dependent on our restriction to a 'simple fluid', which is technically called a Newtonian fluid. A formal definition of a Newtonian fluid will be given later. Many materials at rest can support tangential forces, including all solids and substances such as chilled whipped cream, butter or margarine, ketchup and possibly blood.

Example 10: Viscosity of a simple liquid of constant density ρ at rest under gravity (e.g., water).

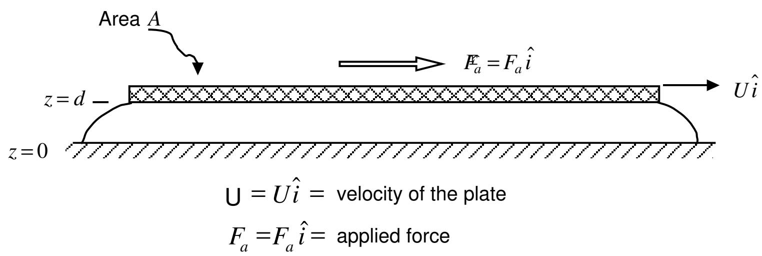


Figure 3.4:

Consider a flat plate separated by a layer of fluid from a flat horizontal table. The plate and table are a uniform distance d apart. Under the influence of gravity the flat plate will settle slowly and squeeze the fluid. We ignore this, and hence assume that d is constant in time (or alternatively a vertical force can be applied to the plate to hold its vertical position). A tangential applied force $\vec{F}_a = F_a\hat{i}$ acts on the plate. The plate accelerates to a constant velocity $U\hat{i}$. Let the contact area between the liquid and the plate and table be equal to A, assumed constant in time. For simple fluids, such as air, water, simple oils, experiments show that at steady state:

1. $U \propto F_a$,
2. $U \propto 1/A$
3. $U \propto d$
4. The fluid moves with the boundaries and the velocity of the fluid is linear in z away from the edges of the plate. Thus, the fluid velocity field is $u(z)\hat{i}$ where
$$u = \frac{U}{d}z. \tag{3.16}$$

From these results $UA/d \propto F_a$ which implies that

$$\mu = \frac{F_a}{\frac{UA}{d}} = \text{constant}. \tag{3.17}$$

The quantity μ is called the shear viscosity (or viscosity) of the fluid.

$$\text{Units:} \quad [\mu] = \frac{\text{kg}}{\text{m} \cdot \text{s}} = \frac{\text{N} \cdot \text{s}}{\text{m}^2}.$$

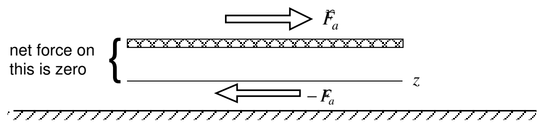


Figure 3.5:

From the above experimental results

$$\vec{F}_a = \mu \frac{U}{d} A \hat{\imath} = \mu \frac{A}{d} \vec{U}. \tag{3.18}$$

Let us now consider the forces acting on various parts of the system. Since the plate has constant horizontal velocity, the net horizontal force on the plate must be zero. Thus, the net horizontal force exerted by the fluid on the plate must be

$$\vec{F}_{fp} = -\vec{F}_a. \tag{3.19}$$

By Newton's third law the plate exerts a horizontal force $\vec{F}_a$ on the fluid. Now consider the layer of fluid lying above height z (Figure 3.5).

Since the flow is steady the net force acting on the fluid above height z is zero. The plate exerts a horizontal force $\vec{F}_a$ on it, so the fluid below height z exerts a horizontal force $-\vec{F}_a$ on the fluid above height z. Newton's third law then says that the fluid above height z exerts a horizontal force $\vec{F}_a$ on the fluid below height z. Now, the fluid below height z must also support the weight of the fluid and the plate lying above it, that is, the net vertical force on the matter above height z must be zero. Thus, the fluid below height z must exert a vertical force $pA\hat{k}$ where p is the pressure in the fluid. From these considerations, and noting that we must divide F_a by the area A to get the tangential force per unit area, the stress vector acting on a surface with unit normal $\pm\hat{k}$ is (Figure 3.6)

$$\vec{t}(z, \pm\hat{k}) = \mp p\hat{k} \pm \mu \frac{U}{d} \hat{\imath}. \tag{3.20}$$

Note that because the velocity is linear in z we can write this result as

$$\hat{t}(z, \pm\hat{k}) = \mp p\hat{k} \pm \mu \frac{du}{dz} \hat{\imath}. \tag{3.21}$$

This expresses the tangential component, the shear stress, as we shall see, in a more fundamental form. This example illustrates the fact that the stress vector need not act normally to the surface as in the previous example. It will, in general, have both tangential and normal components.

3.4 Problems

1. Consider a 2D flow in the xy-plane given by $\vec{u} = (u, v) = (x, y)$. Is the flow incompressible? Are the following density fields possible for a flow that conserves mass? Explain why.

   (a) $\rho(x, y, t) = \rho_0 \frac{e^{-t}}{1+x^2+y^2}$.

   (b) $\rho(x, y, t) = \rho_0 e^{-2t}$.

2. Show that the continuity equation (in 2-D for simplicity) is invariant under a rotation of the coordinate axes through an angle θ in the positive (i.e., counterclockwise) direction.

3. Let $c(x, y, z, t)$ be the concentration of a conserved chemical in the ocean (e.g., salts, a tracer or pollutant) with units of mass of the chemical per unit mass of fluid (i.e., the water plus any chemicals in it, not just mass of water molecules). What is the chemical concentration H in units of mass per unit volume (i.e., what is H in terms of c and something else)? *Derive* equations for c and H following the procedure used to derive the continuity equation. Simplify them as much as possible.

4. A thin, horizontal flat plate lies between a horizontal flat table and a second horizontal flat plate. The space between the lower plate and the table is filled by a layer of oil of thickness d_l and viscosity μ_l. The space between the lower and upper plates is filled with a second layer of oil of a different type with thickness d_u and viscosity μ_u. Both plates have the same surface area A in contact with the oil. A horizontal force $F_a\hat{\imath}$ is applied to the upper plate until a steady state is reached. Here, $\hat{\imath}$ is a unit horizontal vector, parallel to the table and the plates. At steady state the upper and lower plates move with velocity $V\hat{\imath}$ and $U\hat{\imath}$ (the table is at rest). Find an expression for the velocity of the plates in terms of F_a. Find an expression for U/V and verify that it

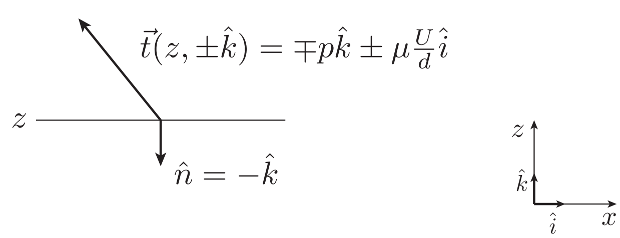


Figure 3.6:

makes sense in when $\mu_l \gg \mu_u$ or $\mu_l \ll \mu_u$. As usual, ignore edge effects and settling of the plates.

Chapter 4

CAUCHY'S FUNDAMENTAL THEOREM FOR STRESS

Recall we had the equation

$$\frac{d}{dt}\iiint\limits_{W(t)} \rho\vec{u}\,dV = -\iiint\limits_{W(t)} \rho\vec{\nabla}\Pi\,dV + \iint\limits_{\partial W(t)} \vec{t}(\vec{x},t,\hat{n})\,dS. \tag{4.1}$$

Cauchy's Fundamental Theorem for Stress: *Equation (4.1) implies that $\vec{t}(\vec{x},t,\hat{n})$ is linear in the components of $\hat{n}$, i.e.,*

$$t_j = \tau_{ij} n_i, \tag{4.2}$$

for some scalars $\tau_{ij}(\vec{x},t)$. Here, the summation convention, whereby repeated indices are summed from 1 to 3, has been used.

Proof: Let $W(t_o)$ be a tetrahedron with corners at $\vec{x}$, $\vec{x}+(a,0,0)$, $\vec{x}+(0,b,0)$ and $\vec{x}+(0,0,c)$ where a, b, and c are positive constants. Let face F_i, with area A_i be the face with unit outward normal $-\hat{i}_i$ for $i = 1,2,3$. Let face F, with area A, be the fourth face of the tetrahedron and denote its outward unit normal by $\hat{n} = (n_1, n_2, n_3)$.

Next, let the volume of $W(t_o)$ be l^3 where l has dimensions of length. Then $A = \gamma l^2$ for some constant γ. Now, the projected area of a piece of a plane of area A is $A_p = |\hat{n}\cdot\hat{n}_p|A$ where $\hat{n}$ is a unit normal to the plane and $\hat{n}_p$ is a unit normal of the projected plane. Thus, we have

$$A_j = |\hat{n}\cdot\hat{i}_j|A = n_j A = n_j\gamma l^2. \tag{4.3}$$

From (3.8)

$$\frac{d}{dt}\iiint\limits_{W(t)} \rho\vec{u}\,dV = \iiint\limits_{W(t)} \rho\frac{D\vec{u}}{Dt}\,dV, \tag{4.4}$$

so from (4.1)

$$\iint\limits_{\partial W(t)} \vec{t}(\vec{x},t,\hat{n})\,dS = \iiint\limits_{W(t)} \left[\rho\frac{D\vec{u}}{Dt} + \rho\vec{\nabla}\pi\right] dV. \tag{4.5}$$

By continuity the magnitude of the right hand side of (4.5) is bounded by Ml^3 for some M (the integrand is bounded), so

$$\lim_{l\to 0}\frac{1}{l^2}\iint\limits_{\partial W(t)} \vec{t}(\vec{x},t,\hat{n})\,dS = \lim_{l\to 0}\frac{1}{l^2}\iiint\limits_{W(t)} \left[\rho\frac{D\vec{u}}{Dt} + \rho\vec{\nabla}\pi\right] dV \le \lim_{l\to 0}\frac{Ml^3}{l^2} = 0. \tag{4.6}$$

Now we take the limit $l \to 0$ in such a manner that a, b and $c \to 0$ and the tetrahedron shrinks to the point $\vec{x}$. Expanding the surface integral into a sum over the four faces of the tetrahedron gives

$$\lim_{l\to 0}\left\{\sum_{j=1}^{3}\frac{1}{l^2}\iint\limits_{F_j}\vec{t}(\vec{x},t,-\hat{i}_j)\,dS + \frac{1}{l^2}\iint\limits_{F}\vec{t}(\vec{x},t,\hat{n})\,dS\right\} = 0. \tag{4.7}$$

Consider the k^{th} component of this equation. Using the mean value theorem and the fact that the area of F_j is $n_j\gamma l^2$ (see (4.3)),

$$\begin{aligned}\frac{1}{l^2}\iint\limits_{F_j} t_k(\vec{x},t,-\hat{i}_j)\,dS &= \frac{1}{l^2}t_k(\vec{x}_j,t,-\hat{i}_j)\,n_j\gamma l^2 \\ &= t_k(\vec{x}_j,t,-\hat{i}_j)n_j\gamma, \end{aligned} \tag{4.8}$$

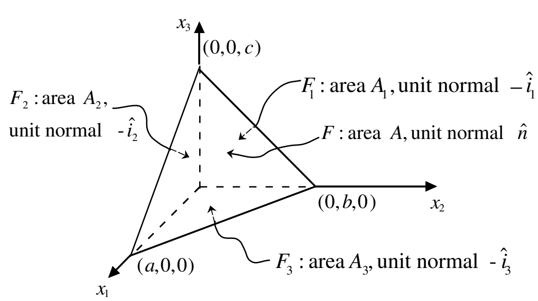


Figure 4.1:

for some point $\vec{x}_j$ on face F_j. Similarly

$$\frac{1}{l^2}\iint\limits_F t_k(\vec{x}, t, \hat{n})\, dS = t_k(\vec{x}_o, t, \hat{n})\gamma, \tag{4.9}$$

for some point $\vec{x}_o$ on face F. Hence, the k^{th} component of (4.7) can be written as

$$\lim_{l\to 0}\left\{\sum_{j=1}^{3} t_k(\vec{x}_j, t, -\hat{i}_j)n_j\gamma + t_k(\vec{x}_o, t, \hat{n})\gamma\right\} = 0. \tag{4.10}$$

Taking the limit, and noting that as $l \to 0$ each $\vec{x}_j$ and $\vec{x}_o$ must go to $\vec{x}$ as the tetrahedron shrinks to the point $\vec{x}$, we have

$$\sum_{j=1}^{3} t_k(\vec{x}, t, -\hat{i}_j)n_j\gamma + t_k(\vec{x}, t, \hat{n})\gamma = 0, \tag{4.11}$$

from which we have

$$t_k(\vec{x}, t, \hat{n}) = -\sum_{j=1}^{3} t_k(\vec{x}, t, -\hat{i}_j)n_j = \sum_{j=1}^{3} t_k(\vec{x}, t, \hat{i}_j)n_j, \tag{4.12}$$

which shows that $t_k(\vec{x}, t, \hat{n})$ is linear in the components of $\hat{n}$.

Definition: *Let*

$$\tau_{jk}(\vec{x}, t) = t_k(\vec{x}, t, \hat{i}_j). \tag{4.13}$$

The τ_{jk} form the components of the **stress tensor** τ *(more on Cartesian tensors later)*

$$\tau = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix}. \tag{4.14}$$

τ_{jk} is the k^{th} component of the force per unit area acting on a surface with outward normal $\hat{i}_j$. By 'outward' normal, we mean that the material into which $\hat{i}_j$ points is exerting the force on the material from which $\hat{i}_j$ points.

Using this definition, (4.12) can be written as

$$t_k = \tau_{jk}n_j. \tag{4.15}$$

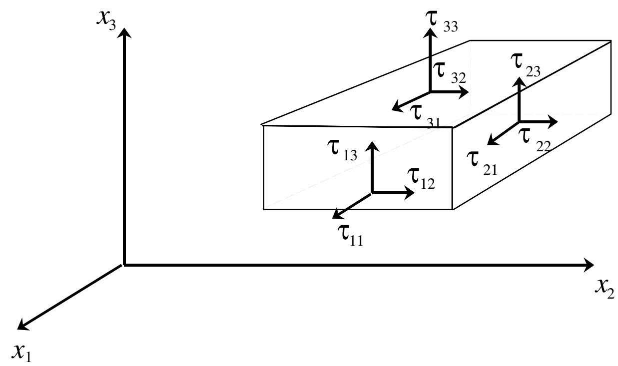


Figure 4.2: Stresses acting on surface of parallelepiped with faces parallel to the Cartesian coordinate planes are components of the stress tensor.

4.1 Conservation of linear momentum: the momentum equations

Using (4.13) and the transport theorem, the balance of linear momentum (4.1) can be written as

$$\iiint\limits_{W(t)} \rho \frac{D\vec{u}}{Dt}\, dV = -\iiint\limits_{W(t)} \rho \vec{\nabla}\Pi\, dV + \iint\limits_{\partial W(t)} \hat{n}\cdot\tau\, dS. \tag{4.16}$$

The k^{th} component of the last term is

$$\iint\limits_{\partial W(t)} \tau_{ik} n_i\, dS = \iiint\limits_{W(t)} \frac{\partial}{\partial x_i}\tau_{ik}\, dV, \tag{4.17}$$

so we can write (4.16) as

$$\iiint\limits_{W(t)} \rho \frac{D\vec{u}}{Dt}\, dV = -\iiint\limits_{W(t)} \rho \vec{\nabla}\Pi\, dV + \iiint\limits_{W(t)} \vec{\nabla}\cdot\tau\, dV, \tag{4.18}$$

where

$$\vec{\nabla}\cdot\tau = \Big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\Big) \begin{pmatrix} \tau_{1,1} & \tau_{1,2} & \tau_{1,3} \\ \tau_{2,1} & \tau_{2,2} & \tau_{2,3} \\ \tau_{3,1} & \tau_{3,2} & \tau_{3,3} \end{pmatrix} = (\frac{\partial \tau_{i1}}{\partial x_i}, \frac{\partial \tau_{i2}}{\partial x_i}, \frac{\partial \tau_{i3}}{\partial x_i}). \tag{4.19}$$

Since this result is true for all material volumes we have

$$\rho \frac{D\vec{u}}{Dt} = -\rho\vec{\nabla}\Pi + \vec{\nabla}\cdot\tau. \tag{4.20}$$

This is the second basic equation of continuum mechanics and is called the **momentum equation**. The i^{th} component of the momentum equation is

$$\rho \frac{Du_i}{Dt} = -\rho \frac{\partial \Pi}{\partial x_i} + \frac{\partial \tau_{ki}}{\partial x_k}. \tag{4.21}$$

This momentum equation applies to all continuums for which

- mass is conserved; and
- gravity is the only body force.

The form of τ does depend on the type of matter in question. In particular, it is different for solids and fluids.

So far we have derived four equations: the continuity equation and the three components of the momentum equation. There are 13 unknowns: the density, the three velocity components and the nine components of τ. Our work is far from complete!

4.2 Symmetry of the stress tensor

In this course we will make the following assumption:

Assumption: *For a continuum the rate of change of angular momentum equals the net torque.*

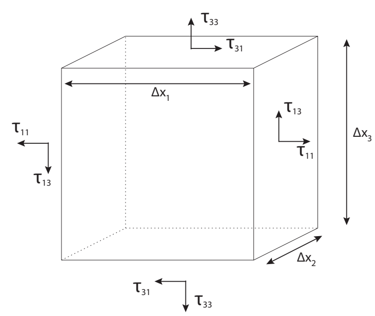
That is, we assume that Euler's second law of mechanics applies to our continuum. It turns out that this assumption is not true for all continuums. For example, it is not satisfied for polymer fluids, which have long molecules, or for liquid crystals. For these types of matter some of the applied torque can result in changes in the angular momentum of the molecules, without changing the angular momentum of a particle, something that is not taken into account in the continuum model.

Assuming Euler's second law holds, we have

$$\frac{d}{dt} \iiint\limits_{W(t)} \rho(\vec{x} \times \vec{u})\, dV = - \iiint\limits_{W(t)} \rho(\vec{x} \times \vec{\nabla}\Pi)\, dV + \iint\limits_{\partial W(t)} \vec{x} \times \vec{t}\, dS. \tag{4.22}$$

Theorem: *If (4.22) holds then the stress tensor is symmetric, i.e.,* $\tau_{ij} = \tau_{ji}$. *This implies than there are six, rather than nine, independent components of the stress tensor.*

Proof: A detailed proof is relegated to Appendix C. Here we will simply provide a sketch of a proof. Consider a small cubical volume centred at $\vec{x} = 0$ with sides of length Δx_i, each face of the cube being parallel to one of the coordinate planes. As before, we let l^3 be the volume of the cube and we will be interested in taking the limit as $l \to 0$ by letting the cube shrinks to the origin.



Consider the torque about the centre. The first term in equation (4.22) goes to zero like l^4 since the integrand goes to zero like l (due to the $\vec{x}$ term) and the volume of $W(t)$ goes to zero like l^3. Similarly the first term on the right hand side of (4.22) goes to zero like l^4. Thus the last term of (4.22) does as well. That is, as $l \to 0$,

$$\iint_{\partial W(t)} \vec{x} \times \vec{t}\, dS \to 0 \quad \text{like } l^4. \tag{4.23}$$

Consider the counter-clockwise torque about $\hat{i}_2$. It is given by the second component of (4.23) which is equal to

$$\iint_{\partial W(t)} (x_3 t_1 - x_1 t_3)\, dS. \tag{4.24}$$

On the face with outward unit normal $\hat{i}_1$, $t_1 = t_1(\vec{x}, t, \hat{i}_1) = \tau_{11}$ and, by the mean value theorem, the integral of $x_3 t_1$ is $x_{o3}\tau_{11}(\vec{x}_o)\Delta x_2\, \Delta x_3$ for some point $\vec{x}_o$ on the face. As $l \to 0$, this goes to zero *faster* than l^3 since Δx_2 and Δx_3 go to zero like l and $x_{o3} \to 0$ faster than l. The latter is because as $l \to 0$ the point x_{o3} must go to the centre of the face, so $x_{o3} \to 0$ both because the range of x_3 values on the face goes to zero like l and because x_{o3} moves to the centre of the face. Thus the contribution to the integral of $x_3 t_1$ goes to zero faster than l^3.

Consider now the second piece. Again, on the face with outward unit normal $\hat{i}_1$, $t_3 = t_3(\vec{x}, t, \hat{i}_1) = \tau_{13}$ and, by the mean value theorem, the integral of $x_1 t_3$ is $x_{o1}\tau_{13}(\vec{x}_o)\Delta x_2\,\Delta x_3$ for some point $\vec{x}_o$ on the face. In this case, as $l \to 0$ $x_{o1} \to \Delta x_1/2$. Thus, for small l, $x_1 t_3 \approx \tau_{13}(\Delta x_1/2)\Delta x_2\,\Delta x_3$ which goes to zero like l^3. Thus, the dominant contribution to the torque about $\hat{i}_2$ on the face with unit normal $\hat{i}_1$ is

$$\frac{1}{2}\tau_{13}\Delta x_1 \Delta x_2\,\Delta x_3. \tag{4.25}$$

Similar considerations show that the dominant contribution to the torque on the side with normal $-\hat{i}_1$ is the same. On the faces with unit normals $\pm\hat{i}_3$ the dominant contribution is

$$-\frac{1}{2}\tau_{31}\Delta x_1 \Delta x_2\,\Delta x_3. \tag{4.26}$$

Thus the dominant contribution to the second component of the torque is

$$(\tau_{13} - \tau_{31})\Delta x_1 \Delta x_2\,\Delta x_3 = (\tau_{13} - \tau_{31})l^3, \tag{4.27}$$

which must go to zero faster than l^3. This means that as $l \to 0$ $\tau_{13} - \tau_{31}$ must go to zero and hence that $\tau_{13}(\vec{x}) = \tau_{31}(\vec{x})$ at $\vec{x} = 0$, the centre of the cube. By a translation of the coordinate system the same is true at every point $\vec{x}$. Similarly $\tau_{12} = \tau_{21}$ and $\tau_{23} = \tau_{32}$.

Chapter 5

DERIVATION OF THE GOVERNING EQUATIONS II

So far we have considered conservation of mass, linear momentum and angular momentum. We now consider conservation of energy.

Taking the dot product of the momentum equation with the velocity gives

$$\rho u_i \frac{Du_i}{Dt} = -\rho u_i \frac{\partial \Pi}{\partial x_i} + u_i \frac{\partial}{\partial x_k} \tau_{ki}. \tag{5.1}$$

Using the continuity equation

$$\begin{aligned} \rho u_i \frac{Du_i}{Dt} &= \rho \frac{D}{Dt}\Big(\frac{1}{2} u_i u_i\Big) \\ &= \frac{D}{Dt}\Big(\frac{\rho}{2} u_i u_i\Big) - \frac{1}{2} u_i u_i \frac{D\rho}{Dt} \\ &= \frac{D}{Dt}\Big(\frac{\rho}{2} u_i u_i\Big) + \Big(\frac{\rho}{2} \vec{u} \cdot \vec{u}\Big) \vec{\nabla} \cdot \vec{u}, \end{aligned}$$

so

$$\frac{DK}{Dt} + K \vec{\nabla} \cdot \vec{u} = -\rho \vec{u} \cdot \vec{\nabla} \Pi + \vec{u} \cdot \Big(\vec{\nabla} \cdot \tau\Big) \tag{5.2}$$

where

$$K = \frac{\rho}{2} \vec{u} \cdot \vec{u}, \tag{5.3}$$

is the kinetic energy density (the kinetic energy per unit volume).

Interpretation:

1. DK/Dt is the rate of change of the kinetic energy density of a particle.

2. $-K\vec{\nabla}\cdot\vec{u}$: as a material volume expands ($\vec{\nabla}\cdot\vec{u} > 0$) the kinetic energy density decreases. As a material volume shrinks the kinetic energy density increases.

3. $-\rho\vec{u}\cdot\vec{\nabla}\Pi$: $-\rho\vec{\nabla}\Pi$ is the body force per unit volume, so $-\rho\vec{u}\cdot\vec{\nabla}\Pi$ is the rate at which work is done (per unit volume) by the body force on the material volume.

4. $\vec{u}\cdot\vec{\nabla}\cdot\tau$: is the rate work is done by the surface forces.

Using $DK/Dt = K_t + \vec{u}\cdot\vec{\nabla}K$, the kinetic energy equation can be written in the form

$$\frac{\partial K}{\partial t} + \vec{\nabla}\cdot(\vec{u}K) = -\rho\vec{u}\cdot\vec{\nabla}\Pi + \vec{u}\cdot(\vec{\nabla}\cdot\tau). \tag{5.4}$$

At a fixed location K can change if $\vec{\nabla}\cdot(\vec{u}K) \neq 0$, i.e., by transport of K from one location to another. That is, over a fixed volume V_o

$$\begin{aligned}\frac{d}{dt}\iiint_{V_o} K\,dV = \iiint_{V_o}\frac{\partial K}{\partial t}\,dV &= -\iiint_{V_o}\vec{\nabla}\cdot\big(\vec{u}K\big)\,dV + \text{other terms},\\ &= -\iint_{\partial V_o} K\vec{u}\cdot\hat{n}\,dS + \text{other terms}.\end{aligned}$$

The first term on the last line is the flux, or transport, of K out of V_o.

Exercise: Defining the potential energy density P as

$$P = \rho\Pi, \tag{5.5}$$

and the total mechanical energy density E as

$$E = K + P, \tag{5.6}$$

show that the kinetic energy equation can be written as

$$\frac{DE}{Dt} + E\vec{\nabla}\cdot\vec{u} = \vec{u}\cdot\vec{\nabla}\cdot\tau. \tag{5.7}$$

5.1 Internal Energy

The total mechanical energy is not conserved. Another form of energy must be included, namely the *internal energy*. Internal energy is the same thing as heat. It is the energy associated with molecular motion. There are three forms of molecular motion:

- Translational motion. The mean velocity of molecules in a small volume may be zero ($\bar{\vec{u}}(\vec{x}, t) = 0$), however the molecules are still moving and the mean of their kinetic energy is not zero: $\overline{\vec{u} \cdot \vec{u}} \neq \bar{\vec{u}} \cdot \bar{\vec{u}}$;
- rotational motion;
- vibrational motion.

Definition: *Let $e(\vec{x}, t)$ be the internal energy per unit mass. Then ρe is the internal energy per unit volume.*

For an ideal gas, $e = C_v T$ where C_v is the specific heat at constant volume and T is the temperature in °K. The energy per unit mass required to increase the temperature of an ideal gas, kept at constant volume, by ΔT °K is $C_V \Delta T$.

5.2 Balance of Energy

We now postulate that the total energy density, mechanical plus internal, is conserved. Thus

$$\frac{d}{dt} \iiint\limits_{W(t)} \Big(E + \rho e\Big)\, dV \quad = \quad \text{rate at which work is done by surface and thermodynamic forces.}$$

Work done by the body force is excluded as the body force converts potential energy to kinetic energy and doesn't change the total mechanical energy E. Contributions to the right hand side include

- **Surface Forces**:

$$\begin{aligned}
\iint\limits_{\partial W(t)} \vec{u} \cdot \vec{t}\, dS &= \int\limits_{\partial W(t)} u_i \tau_{ji} n_j \, dS \\
&= \iiint\limits_{W(t)} \frac{\partial}{\partial x_j} \Big(u_i \tau_{ji}\Big)\, dV \\
&= \iiint\limits_{W(t)} \vec{\nabla} \cdot \Big(\vec{u} \cdot \tau^T\Big)\, dV \\
&= \iiint\limits_{W(t)} \vec{\nabla} \cdot \Big(\vec{u} \cdot \tau\Big)\, dV
\end{aligned}$$

The latter step follows from the symmetry of τ.

- **Thermodynamic Forces:** Internal energy, or heat, is transferred from hot to cold. Let $\vec{q}$ be the local internal flux density. Then $\vec{q} \cdot \hat{n}\, dA$ is the heat flux through a surface element of area dA in the direction of the unit normal $\hat{n}$. The rate at which work is done by thermodynamic forces is the rate at which heat is transferred through the surface and is

$$\iint\limits_{\partial W(t)} -\vec{q} \cdot \hat{n}\, dS = -\iiint\limits_{W(t)} \vec{\nabla} \cdot \vec{q}\, dV,$$

the minus sign being used because we want heat flux into $W(t)$ (i.e., in direction $-\hat{n}$).

Combining the above, the balance of energy gives

$$\frac{d}{dt} \iiint\limits_{W(t)} \Big[E + \rho e\Big]\, dV = \iiint\limits_{W(t)} \vec{\nabla} \cdot \Big(-\vec{q} + \vec{u} \cdot \tau\Big)\, dV.$$

which, after using the transport theorem, can be written as

$$\iiint\limits_{W(t)} \Big[\frac{D}{Dt}\Big(E + \rho e\Big) + \vec{\nabla} \cdot \vec{u}\Big(E + \rho e\Big)\Big]\, dV = \iiint\limits_{W(t)} \vec{\nabla} \cdot \Big(-\vec{q} + \vec{u} \cdot \tau\Big)\, dV.$$

Thus

$$\frac{D}{Dt}\Big(E + \rho e\Big) + \Big(E + \rho e\Big)\vec{\nabla} \cdot \vec{u} = -\vec{\nabla} \cdot \Big(\vec{q} - \vec{u} \cdot \tau\Big), \tag{5.8}$$

or

$$\frac{\partial}{\partial t}\Big(E + \rho e\Big) + \vec{\nabla} \cdot \Big((E + \rho e)\vec{u} + \vec{q} - \vec{u} \cdot \tau\Big) = 0. \tag{5.9}$$

These equations say that the total energy, $E + \rho e$ being the energy density per unit volume, is conserved.

In general a PDE of the form

$$\frac{\partial R}{\partial t} + \vec{\nabla} \cdot \Big(\vec{F}(R)\Big) = 0, \tag{5.10}$$

is said to be in conservation form. It implies that R is conserved, or, more accurately, that the integral of R, i.e., the total amount of R, is conserved. For example, suppose R is constant outside a region $\Omega \subset V_o$ then, using (5.10) and the divergence theorem,

$$\frac{d}{dt} \iiint\limits_{V_o} R\, dV = -\iiint\limits_{V_o} \vec{\nabla} \cdot \vec{F}\, dV = -\iint_{\partial V_o} \vec{F}(R) \cdot \hat{n}\, dS. \tag{5.11}$$

But R is constant, say equal to R_o, outside Ω. In particular R is constant on the surface ∂V_o, in which case the latter integral is zero (exercise). Thus, the total amount of R is conserved. $\vec{F}(R)$ is the transport, or flux, of R. Equation (5.10) says that R can change only if the amount of R leaving a region is different from that entering the same region. There are no sources and sinks.

5.3 Summary of Equations so far

So far, using conservation of mass, linear momentum, angular momentum and energy we have derived the following equations:

- **Continuity Equation:**

$$\frac{D\rho}{Dt} + \rho\vec{\nabla}\cdot\vec{u} = 0 \tag{5.12}$$

- **Momentum Equation:**

$$\rho\frac{D\vec{u}}{Dt} = -\rho\vec{\nabla}\Pi + \vec{\nabla}\tau. \tag{5.13}$$

- **Energy Equation:**

$$\frac{D}{Dt}\Big(E + \rho e\Big) + \Big(E + \rho e\Big)\vec{\nabla}\cdot\vec{u} = -\vec{\nabla}\cdot\Big(\vec{q} - \vec{u}\cdot\tau\Big). \tag{5.14}$$

We have five equations for 14 unknowns:

- the density ρ;
- the three velocity components u_i;
- the six components of τ, as τ is symmetric;
- the internal energy e;
- the three components of $\vec{q}$.

Our next goal is to try and relate the components of τ to the flow field. To do so we must learn something about two other tensors, the strain rate tensor and the rotation tensor. First, however, it is necessary to learn what a tensor is!

Chapter 6

CARTESIAN TENSORS

You are all familiar with first-order tensors. Velocity, force, and acceleration are examples of first order tensors. A first-order tensor is a vector. In three dimensions a vector $\vec{v}$ can be described with three numbers, e.g., $\vec{v} = (v_1, v_2, v_3)$. The numbers v_i are the components of $\vec{v}$ relative to a specific coordinate system. The v_i can not be anything we choose because a vector $\vec{v}$ has a physical meaning which is independent of the coordinate system used to describe it. For example, a moving particle moves in a manner which does not depend on a coordinate description used to describe its movement. It moves with a certain speed in a particular direction, both of which are independent of the coordinate system used to describe it. The description of the particles velocity, however, does depend on the coordinate system used. The v_i are the velocities in the direction of the coordinate basis vectors. When the coordinate system is changed the values of the three components change according to a specific rule.

Second-order tensors have nine components in 3-D, say A_{ij} where i and j range from 1 to 3. They can be written in the form of a 3×3 matrix. Like first-order tensors, a second-order tensor has a physical meaning which is independent of its coordinate system. As a result, the components of a second-order tensor change according to a specific rule if the coordinate system is changed. We restrict ourselves to Cartesian coordinate systems, which is what the use of the name Cartesian tensors implies.

6.1 The transformation rule for first-order tensors.

Consider two orthogonal coordinate systems with coordinate vectors $\hat{\imath}_1$, $\hat{\imath}_2$, $\hat{\imath}_3$ and $\hat{\imath}'_1$, $\hat{\imath}'_2$, $\hat{\imath}'_3$. Let α_{ij} be the angle between $\hat{\imath}_i$ and $\hat{\imath}'_j$ measured counterclockwise from

$\hat{i}_i$. The α_{ij} are not all independent as the basis vectors are orthogonal. For example, in 2-D $\alpha_{11} = \alpha_{22}$, and, $\alpha_{12} - \alpha_{11} = \pm\pi/2$.

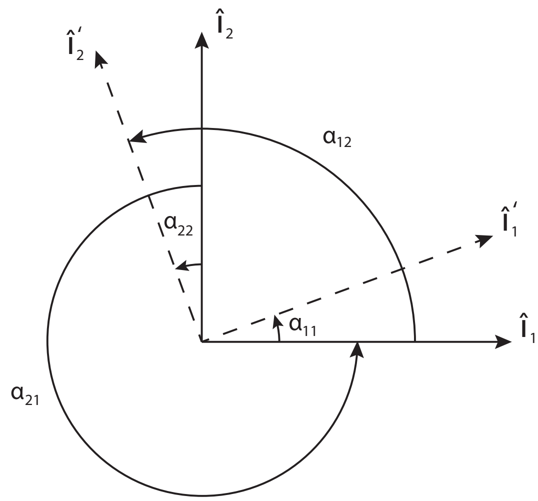


Figure 6.1: Illustration of angles α_{ij} between basis vectors for two Cartesian coordinate systems in 2D.

Projecting $\hat{i}'_i$ onto the $\hat{i}_k$ we have

$$\hat{i}'_i = (\hat{i}'_i \cdot \hat{i}_k)\hat{i}_k, \qquad i = 1, 2, 3.$$

Here, and in the following, repeated indices are summed over 1 to 3 unless otherwise stated. Using $\hat{i}'_i \cdot \hat{i}_k = |\hat{i}'_i||\hat{i}_k| \cos(\alpha_{ki}) = \cos(\alpha_{ki})$ gives

$$\hat{i}'_i = \cos(\alpha_{ki})\hat{i}_k, \qquad i = 1, 2, 3. \tag{6.1}$$

Similarly, projecting $\hat{i}_j$ onto the $\hat{i}'_i$ gives

$$\hat{i}_j = \cos(\alpha_{ji})\hat{i}'_i, \qquad j = 1, 2, 3. \tag{6.2}$$

Substituting (6.1) into (6.2) gives

$$\hat{i}_j = \cos(\alpha_{ji})\cos(\alpha_{ki})\hat{i}_k. \tag{6.3}$$

It follows that $\cos(\alpha_{ji})\cos(\alpha_{ki}) = \delta_{jk}$. Similarly $\hat{i}'_j = \cos(\alpha_{ij})\cos(\alpha_{il})\hat{i}'_l$ and hence $\cos(\alpha_{ij})\cos(\alpha_{il}) = \delta_{jl}$. In summary, defining

$$C_{ij} = \cos(\alpha_{ij}), \tag{6.4}$$

and C to be the rotation matrix

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} \tag{6.5}$$

we have

$$\cos(\alpha_{ji})\cos(\alpha_{ki}) = C_{ji}C_{ki} = \left(C \cdot C^T\right)_{jk} = \delta_{jk}.$$

Hence

$$C \cdot C^T = \mathrm{I}, \tag{6.6}$$

where I is the identity matrix.

Exercise: In 2-D let $\alpha_{11} = \theta$. Show that

$$C = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is the rotation matrix. What is its inverse?

Now, consider a vector $\vec{x} = x_i \hat{\imath}_i = x'_k \hat{\imath}'_k$. The x_i are the components of $\vec{x}$ in the unprimed coordinate system while the x'_i are the components in the primed system. We will now find the relationship between the x_i and the x'_i. Using (6.1) we have

$$x_i \hat{\imath}_i = x'_k \left(C_{ik} \hat{\imath}_i\right) = C_{ik} x'_k \hat{\imath}_i,$$

hence

$$x_i = C_{ik} x'_k. \tag{6.7}$$

Similarly

$$x'_i = C_{ki} x_k. \tag{6.8}$$

In matrix form, these equations are

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = C \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} \qquad \text{or, equivalently,} \qquad \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = C^T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

This is the transformation rule which says how the components of a first-order tensor change under a rotation of the coordinate system.

6.2 The transformation rule for second-order tensors.

The stress tensor is an example of a second-order tensor. When the coordinate system is rotated its components transform according to a specific rule. This transformation rule applies to all second-order Cartesian tensors.

Recall that $\hat{t}(\vec{x}, t, \hat{n}) = \hat{n} \cdot \tau$ is the force per unit area acting on a surface with outward unit normal $\hat{n}$. The stress tensor τ is a linear function which acts on a vector (e.g., $\hat{n}$) and returns a vector (e.g., $\hat{t}$).

In the unprimed system

$$t_i = \tau_{ji} n_j. \tag{6.9}$$

In the primed system

$$t'_i = \tau'_{ji} n'_j. \tag{6.10}$$

We wish to determine how the τ'_{ji} and the τ_{ij} are related. We can do this using knowledge of how the t_i and t'_i and the n_i and n'_i are related. First, using the rule for transforming vectors we have

$$t'_i = C_{ki} t_k = \tau'_{ji} n'_j = \tau'_{ji} \Big(C_{lj} n_l \Big),$$

so

$$C_{ki} t_k = C_{lj} \tau'_{ji} n_l.$$

Next, multiplying by C_{mi} and using (6.6) gives

$$\delta_{mk} t_k = C_{mi} C_{lj} \tau'_{ji} n_l,$$

or

$$t_m = \Big(C_{mi} C_{lj} \tau'_{ji} \Big) n_l.$$

But $t_m = \tau_{lm} n_l$ by definition, so

$$\tau_{lm} = C_{lj} C_{mi} \tau'_{ji}. \tag{6.11}$$

This gives the rule for transforming the components of a second-order tensor under a rotation of the coordinate system. It can be written as

$$\tau_{lm} = C_{lj} \tau'_{ji} C^T_{im},$$

or

$$\tau = C \tau' C^T. \tag{6.12}$$

Exercise: Show that $\tau' = C^T \tau C$.

Notice the pattern: Unprimed $\rightarrow$ primed

$$x'_i = C_{ki} x_k$$

and

$$\tau'_{ji} = C_{lj} C_{mi} \tau_{lm}.$$

In both cases the summation is over the first index of C. The sum over l replaces the index l of τ by the index j of τ'.

Primed $\rightarrow$unprimed

$$x_j = C_{jk} x'_k$$

and

$$\tau_{lm} = C_{lj} C_{mi} \tau'_{ji}.$$

In both cases the summation is over the second index of C.

To show that some 2-index quatity A_{ij} is a second-order tensor one must simply show that under a rotation of the coordinate system, with rotation matrix C, the A_{ij} become

$$A'_{ji} = C_{lj} C_{mi} A_{lm}.$$

Higher order tensors exist. The transformation rule for higher-order tensors is similar. For example, a 4$^{\text{th}}$ order tensor has four indices and transforms according to

$$A'_{ijkl} = C_{pi} C_{qj} C_{rk} C_{sl} A_{pqrs} \tag{6.13}$$

6.3 Special useful tensors.

1. **Kronecker Delta:**

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \tag{6.14}$$

2. **Permutation or alternating tensor**

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ is an even permutation of 123} \\ -1 & \text{if } ijk \text{ is an odd permutation of 123} \\ 0 & \text{otherwise} \end{cases} \tag{6.15}$$

This third order tensor arises in cross products. Consider

$$\vec{a} \times \vec{b} = (a_i \hat{i}_i) \times (b_j \hat{i}_j) = a_i b_j (\hat{i}_i \times \hat{i}_j).$$

Dotting both sides with $\hat{i}_k$ to obtain the k^{th} component gives

$$(\vec{a} \times \vec{b})_k = \left[\hat{i}_k \cdot (\hat{i}_i \times \hat{i}_j)\right] a_i b_j = \epsilon_{kij} a_i b_j, \tag{6.16}$$

where

$$\epsilon_{ijk} = \hat{i}_k \cdot (\hat{i}_i \times \hat{i}_j). \tag{6.17}$$

It is straight forward to show that ϵ has the values given by (6.15, proof of which is left as an exercise. In terms of the ϵ_{ijk} the curl of a vector has components

$$\left(\vec{\nabla} \times \vec{u}\right)_k = \epsilon_{kij} \frac{\partial u_k}{\partial x_i} \tag{6.18}$$

A useful identity is

$$\epsilon_{ijk} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}. \tag{6.19}$$

6.4 Contractions and tensor products

If two indices of a tensor are equated and summed over the repeated index then the result is a new, lower order tensor. For example, if Q is a third order tensor then D, defined by

$$D_i = Q_{ikk} = Q_{i11} + Q_{i22} + Q_{i33},$$

is a first order tensor.

The product of two tensors yields a higher-order tensor. For example, if A and B are second and third order tensors then D, with components

$$D_{ijklm} = A_{ij} B_{klm}$$

is a fifth order tensor.

Exercise: Demonstrate that the previous statement is true.

6.5 Symmetric and Antisymmetric Tensors

Definition: *A second-order tensor B is* **symmetric** *if $B_{ij} = B_{ji}$ for all i and j. It is* **antisymmetric** *if $B_{ij} = -B_{ji}$ for all i and j.*

Any second-order tensor can be written as the sum of a symmetric and an antisymmetric tensor via

$$B = \frac{1}{2}\Big(B + B^T\Big) + \frac{1}{2}\Big(B - B^T\Big) = S + A, \tag{6.20}$$

were the symmetric part S has components

$$S_{ij} = \frac{1}{2}\Big(B_{ij} + B_{ji}\Big),$$

and the antisymmetric part has components

$$A_{ij} = \frac{1}{2}\Big(B_{ij} - B_{ji}\Big).$$

The antisymmetric second order tensor A_{ij} has zeros down the diagonal and hence has only three independent components. Every antisymmetric tensor can be associated with a vector. To see this, let $\vec{v} = v_k \hat{i}_k$ be a vector and set $R_{ij} = -\epsilon_{ijk} v_k$.

Exercise: Show that R_{ij} is a tensor by proving that its components transform under a rotation of the coordinate system according to the transformation rule (6.11. Show that $v_k = -\frac{1}{2}\epsilon_{ijk} R_{ij}$.

Fact: Any symmetric real second-order tensor can be diagonalized. This means that under a suitable rotation of the coordinate system the matrix made up of the components of the tensor is diagonal.

Proof: Let

$$B = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix}, \tag{6.21}$$

be a symmetric, real second-order tensor.

From linear algebra there exist a complete set of orthogonal eigenvectors $\hat{i}'_i$ with eigenvalues λ_i. The three eigenvectors, normalized to have unit length, can be chosen so that $\hat{i}'_1$, $\hat{i}'_2$ and $\hat{i}'_3$ form a right-handed system. Let C be the rotation matrix for the rotation from the $\hat{i}_j$ to the $\hat{i}'_j$. Then from (6.1, $\hat{i}'_j = C_{ij}\hat{i}_i$, i.e.,

$$i'_j = \begin{pmatrix} C_{1j} \\ C_{2j} \\ C_{3j} \end{pmatrix}, \tag{6.22}$$

is the column vector representation of $\hat{i}'_j$ in terms of the initial unprimed basis. Thus

$$B \begin{pmatrix} C_{1j} \\ C_{2j} \\ C_{3j} \end{pmatrix} = \lambda_i \begin{pmatrix} C_{1j} \\ C_{2j} \\ C_{3j} \end{pmatrix}. \tag{6.23}$$

so

$$B_{lj}C_{ji} = \lambda_i C_{li} \qquad \text{(no sum on } i\text{)},$$

or

$$(BC)_{li} = \lambda_i C_{li}. \qquad \text{(no sum on } i\text{)}. \tag{6.24}$$

Now from the rule for transforming components of a second-order tensor

$$B' = C^T BC,$$

hence

$$B'_{li} = (C^T)_{jl}(BC)_{li} = C_{lj}\lambda_i C_{li} = C_{lj}C_{li}\lambda_i = \delta_{ji}\lambda_i$$

where we have used $CC^T = I$. It follows that

$$B' = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}. \tag{6.25}$$

That is, the tensor is diagonal in the primed coordinate system.

Definition: *The eigenvectors of a symmetric second-order tensor are called the principal axes.*

Example 11: Find the eigenvalues and principal axes of the second-order tensor

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{11}{4} & -\frac{\sqrt{3}}{4} \\ 0 & -\frac{\sqrt{3}}{4} & \frac{9}{4} \end{pmatrix}. \tag{6.26}$$

Solution: First find the eigenvalues. The determinant of $\lambda I - B$ is

$$\det \begin{pmatrix} \lambda - 1 & 0 & 0 \\ 0 & \lambda - \frac{11}{4} & \frac{\sqrt{3}}{4} \\ 0 & \frac{\sqrt{3}}{4} & \lambda - \frac{9}{4} \end{pmatrix} = (\lambda - 1)(\lambda - 2)(\lambda - 3), \tag{6.27}$$

so the eigenvalues of B are $\lambda = 1$, 2 and 3. Now we need to find the eigenvectors.

$\lambda = 1$: By inspection

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{11}{4} & -\frac{\sqrt{3}}{4} \\ 0 & -\frac{\sqrt{3}}{4} & \frac{9}{4} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

so

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

$\lambda = 2$: Need to solve

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{11}{4} & -\frac{\sqrt{3}}{4} \\ 0 & -\frac{\sqrt{3}}{4} & \frac{9}{4} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ \frac{11}{4} - \frac{\sqrt{3}}{4}c \\ -\frac{\sqrt{3}}{4}b + \frac{9}{4}c \end{pmatrix} = 2 \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Solving gives $a = 0$ and $c = \sqrt{3}b$. Normalizing gives

$$\vec{v}_2 = \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}.$$

$\lambda = 3$: Need to solve

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{11}{4} & -\frac{\sqrt{3}}{4} \\ 0 & -\frac{\sqrt{3}}{4} & \frac{9}{4} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ \frac{11}{4} - \frac{\sqrt{3}}{4}c \\ -\frac{\sqrt{3}}{4}b + \frac{9}{4}c \end{pmatrix} = 3 \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Solving gives $a = 0$ and $b = -\sqrt{3}c$. Normalizing gives

$$\vec{v}_3 = \begin{pmatrix} 0 \\ -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}.$$

It is easy to verify that $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ form a right-handed system so take $\hat{i}'_j = \vec{v}_j$. In the primed coordinate system the tensor B is diagonal and the eigenvectors corresponding to $\hat{i}'_1$, $\hat{i}'_2$ and $\hat{i}'_3$ are the diagonal elements. Hence,

$$B' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}. \tag{6.28}$$

Definition: *For the stress tensor τ, which is a symmetric second-order tensor, the eigenvalues are called the* **principal stresses***.*

In a reference frame in which τ is diagonal

$$\tau = \begin{pmatrix} \tau'_{11} & 0 & 0 \\ 0 & \tau'_{22} & 0 \\ 0 & 0 & \tau'_{33} \end{pmatrix}. \tag{6.29}$$

Now the surface force per unit area, $\vec{t'}$, acting on a surface with outward unit normal $\hat{n}'$ has components $t'_k = \tau'_{ik} n'_i$. Using the fact that τ' is diagonal, this reduces to $t'_k = \tau'_{kk} n'_k$ (no sum over k). This means that in this coordinate system, surfaces parallel to one of the coordinate planes have only normal surface forces. For example, if $\hat{n}' = \hat{i}'_3$ then $t'_1 = t'_2 = 0$ and $t'_3 = \tau'_{33}$. Thus, an infinitesimal volume of material is subjected to normal forces (i.e., compression or tension) along the direction of the coordinate axes in the reference frame in which τ is diagonal. Now the forces acting on a small volume are independent of the coordinate system used to describe them.

The surface forces acting on an infinitesimal volume are made up of three compressive or tensile forces acting in three orthogonal directions. The principal axes $\vec{v}_j$ of the stress tensor give the directions. The eigenvalues λ_j give the force per unit area in the direction of $\vec{v}_j$. If eigenvalue λ_j is positive/negative, the material is subjected to a tensile/compressive surface force in the direction of $\vec{v}_j$. This is the physical interpretation of the stress tensor. It is in this sense that the physical meaning of the stress tensor is independent of the coordinate system.

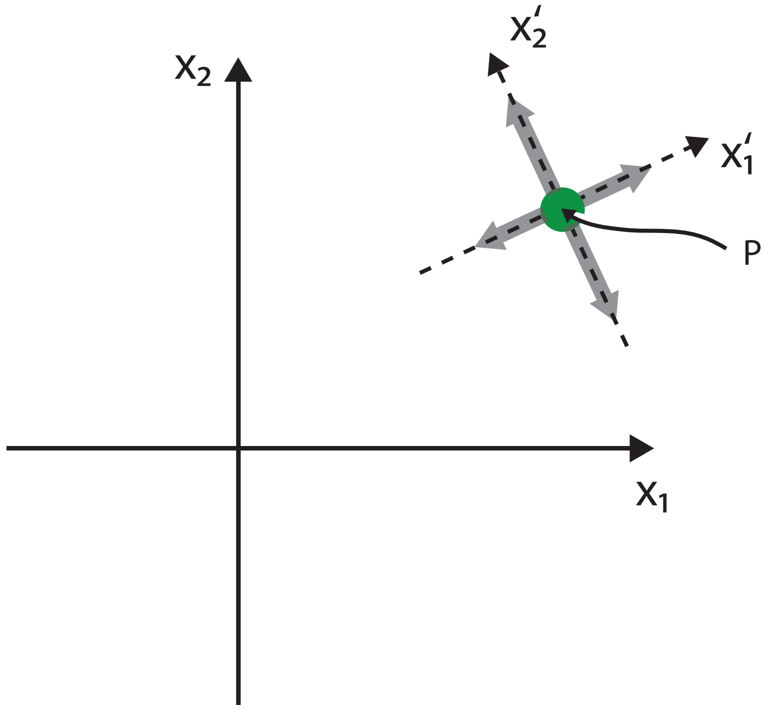


Figure 6.2: x'_1 and x'_2 are coordinate axes in a reference frame in which τ is symmetric at the point P. Hence surface forces acting on a small volume at P are directed along the x'_1 and x'_2 axes. They are compressive along x'_1 if $\lambda_1 < 0$ and tensile along x'_2 if $\lambda_2 > 0$ as in the case illustrated here.

Chapter 7

STRAIN RATE TENSOR AND ROTATION TENSOR

We will now look at deformation rates of small material volumes. A description of how material deform is of interest from a purely kinematic point of view. It is of particular interest in the study of fluids because for a fluid, the stress tensor is related directly to the rates of deformation. In solids, on the other hand, the stress tensor is related to how much the material has been deformed.

Definition: *The* **Linear Strain Rate:** *The linear strain rate in the direction $\hat{i}_k$ is the rate of change of length per unit length of a material volume in the direction $\hat{i}_k$.*

The linear strain rate is illustrated in Figure 7.1. Here a piece of material lies between x_1 and $x_1 + \delta x_1$ has length $l = \delta x_1$ in the x_1 direction. Matter at position x_1 is moving with velocity $u_1(x_1)$ while the other end of the piece of material moves with velocity $u_1(x_1 + \delta x_1)$. A time δt later piece of material has moved and stretched. The endpoints are now at $x_1 + u_1(x_1)\delta t$ and $x_1 + \delta x_1 + u_1(x_1 + \delta x_1)\delta t$. The change in length is

$$\begin{aligned} \Delta l &= [(x_1 + \delta x_1 + u_1(x_1 + \delta x_1)\delta t) - (x_1 + u_1(x_1)\delta t)] - \delta x_1, \\ &= [u_1(x_1 + \delta x_1) - u_1(x_1)]\delta t \approx \frac{\partial u_1}{\partial x_1}\delta x_1 \delta t. \end{aligned} \tag{7.1}$$

The change in length per unit length is $\Delta l / l$ so, since $\delta x_1 = l$, the linear strain is

$$\lim_{\delta t \to 0} \frac{\Delta l}{l \delta t} = \frac{\partial u_1}{\partial x_1}. \tag{7.2}$$

The linear strain rate in the direction $\hat{i}_1$ is $\frac{\partial u_1}{\partial x_1}$. In the direction $\hat{i}_k$ it is $\frac{\partial u_k}{\partial x_k}$ *with no summation over k.*

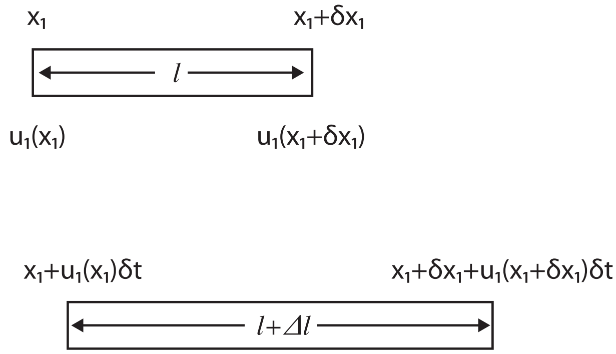


Figure 7.1: Linear Strain Rate: a small piece of material of length $\delta x_1 = l$ in the x_1 direction is stretched in the x_1 direction. Initially the ends are at x_1 and $x_1 + \delta x_1 = x_1 + l$. The ends move with velocity $u_1(x_1)$ and $u_1(x_1 + \delta x_1)$ respectively.

Shear Strain Rate: Consider two mutually perpendicular lines $\overline{OA}$ and $\overline{OB}$ meeting at $O = \vec{x}$. Assume the lines are parallel to the x_1 and x_2 axes at time t and let their lengths be δx_1 and δx_2. At time $t + \delta t$ they are at $\overline{O'A'}$ and $\overline{O'B'}$. We wish to describe the rate at which the angle between the two lines is decreasing (the shear strain rate) and the average rotation rate of the two lines. The former is used to determine the rate of deformation, i.e., the rate at which a material volume is changing shape. The latter does not contribute to the deformation, it gives information on how the material is spinning without changing shape.

Figure 7.2 illustrates the deformation of some material via the change in angle between two initially mutually perpendicular lines $\overline{OA}$ and $\overline{OB}$ which move to $\overline{O'A'}$ and $\overline{O'B'}$ in time δt.

If $\vec{u}(\vec{x})$ is the velocity field at time t, we can estimate the positions of O', A' and B' via

$$\begin{aligned}
O' &= \vec{x} + \vec{u}(\vec{x})\delta t, \\
A' &= \vec{x} + \delta x_1 \hat{i}_1 + \vec{u}(\vec{x} + \delta x_1 \hat{i}_1)\delta t \\
&\approx \vec{x} + \delta x_1 \hat{i}_1 + \Big[\vec{u}(\vec{x}) + \frac{\partial \vec{u}}{\partial x_1}(\vec{x})\delta x_1\Big]\delta t, \\
B' &= \vec{x} + \delta x_2 \hat{i}_2 + \vec{u}(\vec{x} + \delta x_2 \hat{i}_2)\delta t \\
&\approx \vec{x} + \delta x_2 \hat{i}_2 + \Big[\vec{u}(\vec{x}) + \frac{\partial \vec{u}}{\partial x_2}(\vec{x})\delta x_2\Big]\delta t.
\end{aligned}$$

In general O, A and B may move in three-dimensions. Consider the projection of O', A', and B' onto the x_1x_2-plane. The projected points are O'_p, A'_p and B'_p respectively. Let $d\alpha$ be the angle $\overline{O'_pA'_p}$ makes with $\hat{i}_1$, measured positive counter-clockwise (and ccw about $\hat{i}_3$ which points out of the page), and let $d\beta$ be the angle

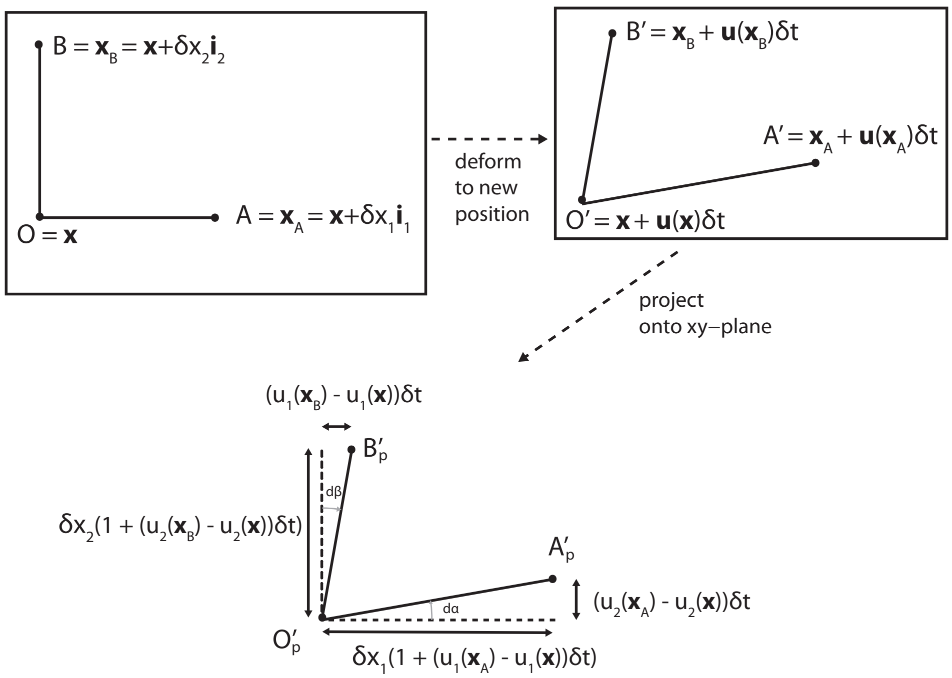


Figure 7.2: Shear Strain Rate.

$\overline{O'_pB'_p}$ makes with $\hat{i}_2$, measured positive clockwise. Both $d\alpha$ and $d\beta$ are positive in Figure 7.2. Now for small δx_1, δx_2, δt these angles are small so

$$d\alpha \approx \tan d\alpha \approx \frac{\frac{\partial u_2}{\partial x_1}\delta x_1 \delta t}{(1+\frac{\partial u_1}{\partial x_1}\delta t)\delta x_1} \approx \frac{\partial u_2}{\partial x_1}\delta t.$$

The rate of change of $d\alpha$ is

$$\frac{d\alpha}{dt} \approx \frac{\partial u_2}{\partial x_1}. \tag{7.3}$$

Similarly, the rate of change of $d\beta$ is

$$\frac{d\beta}{dt} \approx \frac{\partial u_1}{\partial x_2}. \tag{7.4}$$

These become exactly equal as δx_1, δx_2 and δt go to zero.

Definition: *The* **shear strain rate** *is the rate of decrease of the angle formed by two mutually perpendicular lines. For lines parallel to the x_1 and x_2 axes the shear* *strain rate is*

$$\frac{d\alpha}{dt} + \frac{d\beta}{dt} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}.$$

Definition: *The* **strain rate tensor** *is the tensor with components*

$$e_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big). \tag{7.5}$$

Note that the diagonal elements are $e_{11} = \partial u_1/\partial x_1$, $e_{22} = \partial u_2/\partial x_2$, and $e_{33} = \partial u_3/\partial x_3$ which are the linear strain rates. The off-diagonal element are one half the shear strain rates. The trace of e is $e_{ii} = \partial u_i/\partial x_i = \vec{\nabla} \cdot \vec{u}$.

The average rotation rate of the two lines counter clockwise about the x_3 axis is

$$\frac{1}{2}\Big(\frac{d\alpha}{dt} - \frac{d\beta}{dt}\Big) = \frac{1}{2}\Big(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\Big).$$

Similarly, that average counterclockwise rotation rates about the x_1 and x_2 axes are

$$\frac{1}{2}\Big(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\Big) \quad \text{and} \quad \frac{1}{2}\Big(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\Big).$$

Definition: *The* **vorticity vector** ω *is defined by*

$$\vec{\omega} = \vec{\nabla} \times \vec{u} = \Big(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\Big). \tag{7.6}$$

It is twice the angular velocity of a small fluid volume.

Example: Solid body rotation

Figure 7.3 illustrates solid body rotation. A small material volume spins about its centre of volume while rotating around the axis of rotation. The two-dimensional velocity field is

$$\vec{u} = -\omega y\hat{i} + \omega x\hat{j} = (-\omega y, \omega x, 0) \tag{7.7}$$

and the vorticity is

$$\vec{\omega} = \Big(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\Big)\hat{k} = \Big(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\Big)\hat{k} = 2\omega\hat{k}. \tag{7.8}$$

The shear strain rate is

$$\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} = \frac{\partial}{\partial y}(-\omega y) + \frac{\partial}{\partial x}(\omega x) = 0,$$

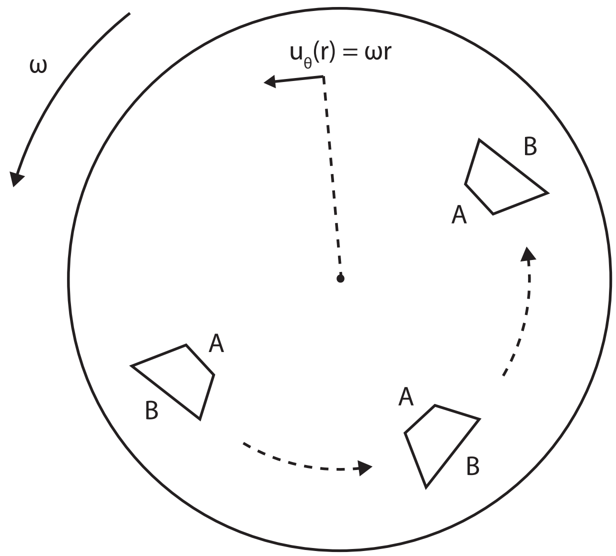


Figure 7.3: Solid body rotation with angular velocity ω. A quadrilateral material volume is shown at three different times. It spins about its centre of volume with angular velocity ω, while its centre of volume rotates about the axis of rotation with the same angular velocity ω. The material volume does not change shape, only its orientation and the location of its centre of volume changes.

while the linear strain rates are

$$\frac{\partial u_1}{\partial x_1} = \frac{\partial}{\partial x}(-\omega y) = 0,$$

and

$$\frac{\partial u_2}{\partial x_2} = \frac{\partial}{\partial y}(\omega x) = 0.$$

All the strain rates are zero because there is no deformation of material volumes. Every material volume preserves its shape while spinning with counter-clockwise angular velocity ω equal to half the vorticity.

7.1 Relative motion near a point.

Consider the motion of a point P at $\vec{x} + d\vec{x}$ relative to the motion of a point O at $\vec{x}$. The velocity of P *relative* to O is

$$d\vec{u} = \vec{u}(\vec{x} + d\vec{x}) - \vec{u}(\vec{x}),$$

or, in component form,

$$du_i = u_i(\vec{x} + d\vec{x}) - u_i(\vec{x}) \approx \frac{\partial u_i}{\partial x_j} dx_j.$$

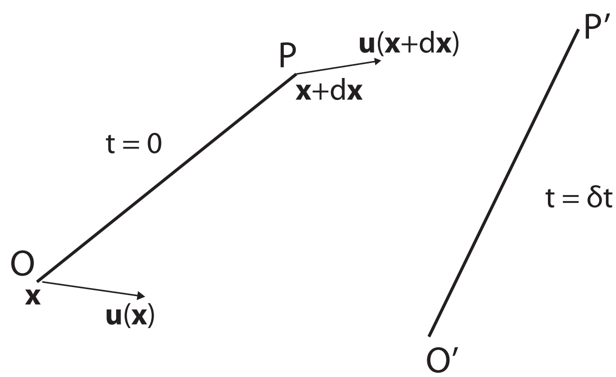


Figure 7.4: Relative motion of two points O and P at $t = 0$ which move to O' and P' at time δt.

Definition: *The* **velocity gradient tensor** G *is the tensor with components*

$$G_{ij} = \frac{\partial u_i}{\partial x_j}, \tag{7.9}$$

so

$$G = \begin{pmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{pmatrix} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{pmatrix}, \tag{7.10}$$

Decomposing G into symmetric and antisymmetric parts gives

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big) + \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\Big) = e_{ij} + \frac{1}{2} r_{ij}. \tag{7.11}$$

The symmetric part, e_{ij}, is the strain rate tensor. The antisymmetric tensor r_{ij} is called the rotation tensor. The vector associated with r_{ij} is the vorticity vector, the two being related by

$$\omega_k = -\frac{1}{2}\epsilon_{ijk} r_{ij} \quad \text{and} \quad r_{ij} = -\epsilon_{ijk}\omega_k. \tag{7.12}$$

In terms of $\vec{\omega}$, r can be written as

$$r = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}. \tag{7.13}$$

The relative velocity of P and O can be written as

$$du_i = \frac{\partial u_i}{\partial x_j} dx_j = e_{ij} dx_j + \frac{1}{2} r_{ij} dx_j.$$

Now

$$r_{ij} dx_j = -\epsilon_{ijk}\omega_k dx_j = \left(\vec{\omega} \times d\vec{x}\right)_i,$$

so

$$du_i = e_{ij} dx_j + \frac{1}{2}\left(\vec{\omega} \times d\vec{x}\right)_i.$$

Since $\vec{\omega} \times d\vec{x}$ is perpendicular to $d\vec{x}$, the second term represents the rotation of the point P about the point O with angular velocity $\vec{\omega}/2$. The strain rate tensor is symmetric. Rotating to a reference frame in which e'_{ij} is diagonal (at $\vec{x}$), we have

$$e' = \begin{pmatrix} e'_{11} & 0 & 0 \\ 0 & e'_{22} & 0 \\ 0 & 0 & e'_{33} \end{pmatrix},$$

and hence

$$du'_i = e'_{ij} dx'_j + \frac{1}{2}\left(\vec{\omega}' \times d\vec{x}'\right)_i.$$

From the first term, assuming for now that $\vec{\omega} = 0$, we have

$$\begin{aligned} du'_1 &= e'_{11} dx'_1, \\ du'_2 &= e'_{22} dx'_2, \\ du'_3 &= e'_{33} dx'_3. \end{aligned}$$

Suppose $\overline{OP}$ is parallel to the x_1 axis. Then $dx'_2 = dx'_3 = 0$ and the velocity of P relative to O is $d\vec{u}' = (e'_{11} dx'_1, 0, 0)$. This implies that P is moving directly away from O along the x_1' axis at a rate e_{11}' times its distance from O. Similar results are obtained if $\overline{OP}$ is parallel to either of the other two principle axes.

Deformation of a spherical volume: Consider a small spherical volume of radius r centred at $O = \vec{x}$. The surface of the sphere is made up of points $\vec{x}' + d\vec{x}'$ (in the reference frame in which e is diagonal) where $d\vec{x}'$ satisfies

$$\frac{dx_i' dx_i'}{r^2} = 1. \tag{7.14}$$

After a small time dt a point $d\vec{x}'$ (relative to O) goes to

$$d\tilde{\vec{x}} = d\vec{x}' + (e_{11}' dx_1', e_{22}' dx_2', e_{33}' dx_3')dt.$$

Using

$$d\tilde{x}_i = (1 + e_{ii}' dt)dx_i' \qquad \text{no sum on } i,$$

we have

$$dx_i' = \frac{d\tilde{x}_i}{1 + e_{ii}' dt} \qquad \text{no sum on } i.$$

Substituting into (7.14) we have

$$\frac{d\tilde{x}_1^2}{(1 + e_{11}' dt)^2 \, r^2} + \frac{d\tilde{x}_2^2}{(1 + e_{22}' dt)^2 \, r^2} + \frac{d\tilde{x}_3^2}{(1 + e_{33}' dt)^2 \, r^2} = 1.$$

Hence, the surface of the sphere becomes an ellipsoid with semi-major axes

$$\begin{aligned} a &= (1 + e_{11}' dt)r, \\ b &= (1 + e_{22}' dt)r, \\ c &= (1 + e_{33}' dt)r, \end{aligned}$$

along the principal axes of the strain rate tensor. If $e_{11}' > 0$ then $a > r$, i.e., sphere is being stretched in the x_1' direction. If $e_{11}' < 0$ then $a < r$ and the sphere is being compressed in the x_1' direction. Behaviour is similar in other directions. If the vorticity is nonzero the ellipsoid would be rotated about $\vec{\omega}$ as well. Examples of the deformation of a circular region in two dimensions are shown in Figure 7.5.

Exercise: Show that the time rate of change of the volume of the sphere, per unit volume, is $e_{ii}' = \vec{\nabla} \cdot \vec{u}$.

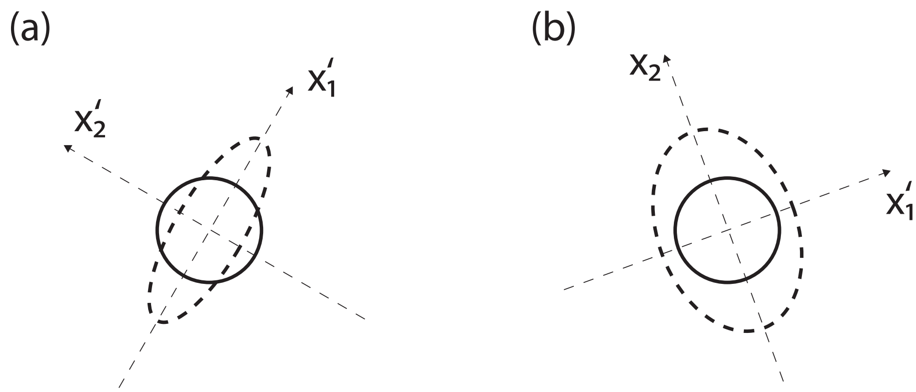


Figure 7.5: Deformation of a small circular volume. (a) $e'_{22} < 0 < e'_{11}$. (b) $0 < e'_{11} < e'_{22}$. Solid circles show initial volume. Dashed ellipses show the deformed volume a short time later.

Chapter 8

CONSTITUTIVE EQUATIONS FOR A NEWTONIAN FLUID

Thus far our modelling has been quite general. To close the set of equations we must focus on a particular material. This requires linking the stress tensor with the deformation of the material. The connection between the two is fundamentally different for fluids and solids. For simple fluids the stress tensor depends on the rate of deformation. For elastic solids it depends on the amount of deformation. For some materials the stress tensor depends on both the amount and the rate of deformation. We first consider simple fluids.

Recall our simple problem of a flat plate separated from a flat horizontal table by a thin layer of oil. Experiments show that at steady state

$$\vec{t}(z, \pm\hat{k}) = \mp p\hat{k} \pm \mu \frac{du}{dz}, \tag{8.1}$$

hence, for this simple flow,

$$\tau_{31} = \mu \frac{du}{dz} \quad \text{and} \quad \tau_{33} = -p. \tag{8.2}$$

Since we are working in the x_1x_3-plane we can ignore the stress tensor components coming for surfaces with unit normal in the $\hat{i}_2$ direction. Hence, the stress tensor is

$$\tau = \begin{pmatrix} \tau_{11} & \tau_{13} \\ \tau_{31} & \tau_{33} \end{pmatrix} = \begin{pmatrix} -p & \mu\frac{du}{dz} \\ \mu\frac{du}{dz} & -p \end{pmatrix} = -p\mathcal{I} + \mu \begin{pmatrix} 0 & \frac{du}{dz} \\ \frac{du}{dz} & 0 \end{pmatrix}, \tag{8.3}$$

where $\mathcal{I}$ is the identity matrix. Now the strain rate tensor for this flow is

$$\mathbf{e} = \begin{pmatrix} \frac{\partial u_1}{x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) & \frac{\partial u_3}{\partial x_1} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2}\frac{du}{dz} \\ \frac{1}{2}\frac{du}{dz} & 0 \end{pmatrix}. \tag{8.4}$$

Thus, for this simple flow we have

$$\tau = -p\mathcal{I} + 2\mu\mathbf{e}. \tag{8.5}$$

For more complicated flows we make the assumption that

$$\tau_{ij} = -p\delta_{ij} + \sigma_{ij}, \tag{8.6}$$

where p is the thermodynamic pressure (assuming this makes sense for nonequilibrium states). σ_{ij} is called the **deviatoric stress tensor**. For our simple case study σ is linearly related to the only nonzero component of $\mathbf{e}$, namely du/dz.

Definition: *A* **Newtonian fluid** *is a fluid for which the σ_{ij} are linearly related to the components $\mathbf{e}_{kl}$ of the strain rate tensor.*

This is *an assumption* — one based on simple experiments using simple fluids. The predictions of this model must be confirmed with experimental evidence. The Newtonian assumption turns out to be excellent for fluids with simple molecular structures (simple fluids) such as air, water and many oils, except possibly for special phenomena such as shock waves. There are many non-Newtonian fluids, including airplane fuel and many liquids you can find in the grocery store, such as ketchup and salad dressing. There are many ways for a fluid to be non-Newtonian. The deviatoric tensor and the stain rate tensor can be nonlinearly related. Some fluids exhibit hysteresis, i.e., they have 'memory' meaning their past state plays some role in their future evolution. A perfectly elastic solid is a good example of this. Internal forces, (the stress tensor), are acting to try to return it to its original shape.

If the σ_{ij} are linearly related to the $\mathbf{e}_{kl}$, then

$$\sigma_{ij} = K_{ijkl}\mathbf{e}_{kl}, \tag{8.7}$$

for some scalars $K_{ijkl}(\vec{x}, t)$ which are components of a fourth order tensor.

Assumption: The fluid is isotropic. That is, its properties are the same in each direction. This implies that the stress-strain relationship is independent of the orientation of the orthogonal coordinate system. This means that the components K_{ijkl} have the same value in every Cartesian coordinate system. That is

$$K'_{ijkl} = K_{ijkl}. \tag{8.8}$$

Since σ is symmetric, (8.7) implies that K_{ijkl} is symmetric in i and j.

Theorem: Any fourth-order isotropic tensor K_{ijkl} has the form

$$K_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}. \tag{8.9}$$

Since K_{ijkl} is symmetric in i and j, it follows that $\gamma = \mu$ and hence

$$K_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\Big(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\Big), \tag{8.10}$$

for some scalars $\lambda(\vec{x}, t)$, $\mu(\vec{x}, t)$ (see Aris, 1962 'Vectors, Tensor and the Basic Equations of Fluid Mechanics).

As a consequence

$$\begin{aligned}
\sigma_{ij} &= \lambda\delta_{ij}\delta_{kl}\mathbf{e}_{kl} + \mu\Big(\delta_{ik}\delta_{jl}\mathbf{e}_{kl} + \delta_{il}\delta_{jk}\mathbf{e}_{kl}\Big) \\
&= \lambda\delta_{ij}\mathbf{e}_{kk} + \mu\Big(\delta_{ik}\mathbf{e}_{kj} + \delta_{il}\mathbf{e}_{jl}\Big) \\
&= \lambda\big(\vec{\nabla}\cdot\vec{u}\big)\delta_{ij} + \mu\Big(\mathbf{e}_{ij} + \mathbf{e}_{ji}\Big)
\end{aligned}$$

Thus,

$$\tau_{ij} = -p\delta_{ij} + \lambda\big(\vec{\nabla}\cdot\vec{u}\big)\delta_{ij} + 2\mu\mathbf{e}_{ij}. \tag{8.11}$$

For our simple case study $\vec{\nabla}\cdot\vec{u} = 0$, in which case $\tau_{ij} = -p\delta_{ij} + 2\mu\mathbf{e}_{ij}$ as we derived earlier.

The average of the diagonal terms of τ is

$$\frac{1}{3}\tau_{ii} = -p + \lambda\vec{\nabla}\cdot\vec{u} + \frac{2}{3}\mu\mathbf{e}_{ii} = -p + \Big(\lambda + \frac{2}{3}\mu\Big)\vec{\nabla}\cdot\vec{u}. \tag{8.12}$$

We now have a relationship between the stress tensor and the thermodynamic pressure p and the strain rate tensor $\mathbf{e}$. The relationship involves two new scalars:

- The **shear viscosity** μ which is easy to measure.
- The **bulk viscosity** $K = \lambda + \frac{2}{3}\mu$ which is difficult to measure. This is because λ is multiplied by $\vec{\nabla}\cdot\vec{u}$ which for many fluids is very small.

A simple monatomic perfect gas model predicts that $K = 0$. Setting $K = 0$ is commonly done (many fluids texts make this assumption in fact). The assumption that $K = 0$ is called the Stokes assumption. It is not a correct assumption in general. For air, comprised largely of diatomic gases, $K \approx 0.6\mu$ and for water $K \approx 3\mu$. Because λ is multiplied by $\vec{\nabla}\cdot\vec{u}$, the bulk viscosity is only important when compression/expansion of the fluid is important, e.g., for sound waves.

$$\text{If } K = 0 \text{ then} \quad \tau_{ij} = -\Big(p + \frac{2}{3}\mu\vec{\nabla}\cdot\vec{u}\Big)\delta_{ij} + 2\mu\mathbf{e}_{ij}.$$

8.1 The Momentum Equations for a Newtonian Fluid

We now substitute our expression for τ into the momentum equations. Recall that the i$^{\text{th}}$ component of the momentum equations is

$$\rho\frac{Du_i}{Dt} = -\rho\frac{\partial\Pi}{\partial x_i} + \frac{\partial\tau_{ki}}{\partial x_k}.$$

For a Newtonian fluid

$$\begin{aligned}\frac{\partial\tau_{ki}}{\partial x_k} &= \frac{\partial}{\partial x_k}\Big(-p\delta_{ki} + \lambda\vec{\nabla}\cdot\vec{u}\delta_{ki} + 2\mu e_{ki}\Big)\\ &= -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_i}\Big(\lambda\vec{\nabla}\cdot\vec{u}\Big) + 2\frac{\partial}{\partial x_k}\Big(\mu\mathbf{e}_{ki}\Big).\end{aligned}$$

Hence

$$\rho\frac{Du_i}{Dt} = -\rho\frac{\partial\Pi}{\partial x_i} - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_i}\Big(\lambda\vec{\nabla}\cdot\vec{u}\Big) + 2\frac{\partial}{\partial x_k}\Big(\mu\mathbf{e}_{ki}\Big). \tag{8.13}$$

This is a general form of the momentum equation for a Newtonian fluid.

The six unknown components of τ have been replaced by three unknowns, the pressure and the two viscosities μ and λ. The viscosities are properties of the fluid and hence are regarded as known, being determined by experiment. They are functions of the temperature and pressure but not of the fluid velocity. Thus, there is only one new variable, the pressure p, which depends on the flow. In many circumstances the temperature and pressure variations are sufficiently small that λ and μ may be treated as constants. In that case

$$\begin{aligned}\frac{\partial}{\partial x_k}\Big(\mu\mathbf{e}_{ki}\Big) &= \mu\frac{1}{2}\frac{\partial}{\partial x_k}\Big(\frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k}\Big)\\ &= \frac{1}{2}\mu\Big(\frac{\partial^2 u_k}{\partial x_i\partial x_k} + \frac{\partial^2 u_i}{\partial x_k^2}\Big)\\ &= \frac{1}{2}\mu\frac{\partial}{\partial x_i}\vec{\nabla}\cdot\vec{u} + \frac{1}{2}\mu\nabla^2 u_i,\end{aligned}$$

so that

$$\rho\frac{Du_i}{Dt} = -\rho\frac{\partial\Pi}{\partial x_i} - \frac{\partial p}{\partial x_i} + (\lambda+\mu)\frac{\partial}{\partial x_i}\vec{\nabla}\cdot\vec{u} + \mu\nabla^2 u_i, \tag{8.14}$$

or, in vector form

$$\rho\frac{D\vec{u}}{Dt} = -\rho\vec{\nabla}\Pi - \vec{\nabla}p + (\lambda+\mu)\vec{\nabla}(\vec{\nabla}\cdot\vec{u}) + \mu\nabla^2\vec{u}. \tag{8.15}$$

8.2 Energy Equation for a Newtonian Fluid

We have seen two energy equations. One for the mechanical energy,

$$\frac{DE}{Dt} + E\vec{\nabla}\cdot\vec{u} = \vec{u}\cdot(\vec{\nabla}\cdot\tau).$$

and one for the total energy,

$$\frac{D}{Dt}\Big(E+\rho e\Big) + \Big(E+\rho e\Big)\vec{\nabla}\cdot\vec{u} = -\vec{\nabla}\cdot\Big(\vec{q}-\vec{u}\cdot\tau\Big).$$

Taking the difference of the two equations we obtain

$$\frac{D}{Dt}\Big(\rho e\Big) + \Big(\rho e\Big)\vec{\nabla}\cdot\vec{u} = -\vec{\nabla}\cdot\vec{q} + \vec{\nabla}\cdot(\vec{u}\cdot\tau) - \vec{u}\cdot(\vec{\nabla}\cdot\tau),$$

which, after using the continuity equation, can be written as

$$\rho\frac{De}{Dt} = -\vec{\nabla}\cdot\vec{q} + \vec{\nabla}\cdot(\vec{u}\cdot\tau) - \vec{u}\cdot(\vec{\nabla}\cdot\tau).$$

Now,

$$\vec{\nabla}\cdot(\vec{u}\cdot\tau) - \vec{u}\cdot\vec{\nabla}\cdot\tau = \frac{\partial}{\partial x_j}\Big(u_i\tau_{ij}\Big) - u_i\frac{\partial\tau_{ij}}{\partial x_j} = \frac{\partial u_i}{\partial x_j}\tau_{ij} = \Big(\mathbf{e}_{ij} + \frac{1}{2}\mathbf{r}_{ij}\Big)\tau_{ij}.$$

Since τ is symmetric and $\mathbf{r}$ is antisymmetric, it follows that $\tau_{ij}\mathbf{r}_{ij} = 0$, (exercise). Thus

$$\rho\frac{De}{Dt} = -\vec{\nabla}\cdot\vec{q} + \tau_{ij}\mathbf{e}_{ij}.$$

Using the expression for τ for a Newtonian fluid,

$$\begin{aligned}\tau_{ij}\mathbf{e}_{ij} &= -p\delta_{ij}\mathbf{e}_{ij} + \lambda\vec{\nabla}\cdot\vec{u}\delta_{ij}\mathbf{e}_{ij} + 2\mu\mathbf{e}_{ij}\mathbf{e}_{ij}\\ &= -p\vec{\nabla}\cdot\vec{u} + \lambda(\vec{\nabla}\cdot\vec{u})^2 + 2\mu\Big(\mathbf{e}_{ij}\Big)^2,\end{aligned}$$

where $(\mathbf{e}_{ij})^2 = \mathbf{e}_{ij}\mathbf{e}_{ij}$.

Experimental evidence suggests that $\vec{q} = -k\vec{\nabla}T$ where T is the temperature and $k(\vec{x},t)$ is the thermal conductivity. Thus, internal energy goes from hot regions to cold regions directly down the temperature gradient at a rate proportional to the temperature gradient. Using this we have the internal energy equation

$$\rho\frac{De}{Dt} = \vec{\nabla}\cdot(k\vec{\nabla}T) - p\vec{\nabla}\cdot\vec{u} + \phi, \tag{8.16}$$

where

$$\phi = \lambda(\vec{\nabla} \cdot \vec{u})^2 + 2\mu(\mathbf{e}_{ij})^2, \tag{8.17}$$

is called the viscous dissipation.

The mechanical energy equation for a Newtonian fluid is

$$\frac{DE}{Dt} + E\vec{\nabla} \cdot \vec{u} = \vec{\nabla} \cdot \left(\vec{u} \cdot \tau\right) + p\vec{\nabla} \cdot \vec{u} - \phi, \tag{8.18}$$

the derivation of which is left as an exercise.

Interpretation:

- $\vec{\nabla} \cdot (k\vec{\nabla}T)$ is the convergence of internal energy flux.
- $p\vec{\nabla} \cdot \vec{u}$ is the work done by volume expansions. As a small volume expands ($\vec{\nabla} \cdot \vec{u} > 0$) it cools down. Internal energy decreases and mechanical energy increases. For example, if fluid is initially at rest and a small volume is heated by some external source, then the pressure goes up and the volume starts to expand. To expand, the fluid must start moving and hence the kinetic energy (and hence E) must increase. Internal energy is converted to mechanical energy.
- ϕ is the viscous dissipation. It represents an irreversible transfer of kinetic energy to internal energy. The viscous dissipation is always positive. Using $\lambda = K - 2\mu/3$, it can be written as $\phi = K(\vec{\nabla} \cdot \vec{u})^2 + 2\mu(\mathbf{e}_{ij} - \frac{1}{3}\vec{\nabla} \cdot \vec{u}\delta_{ij})^2$. The second law of thermodynamics implies that this must be positive, hence K and μ must both be positive.

8.3 SUMMARY: Governing Equations for a Newtonian Fluid

So far, using conservation of mass, linear momentum, angular momentum and energy and the assumption that the fluid is isotropic and that the stress tensor and deviatoric tensor are linearly related, we have derived the following equations:

- **Continuity Equation:**

$$\frac{D\rho}{Dt} + \rho\vec{\nabla} \cdot \vec{u} = 0.$$

- **Momentum Equation:**

$$\rho \frac{Du_i}{Dt} = -\rho \frac{\partial \Pi}{\partial x_i} - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_i}\Big(\lambda \vec{\nabla} \cdot \vec{u}\Big) + 2\frac{\partial}{\partial x_k}\Big(\mu \mathbf{e}_{ki}\Big),$$

  or, if μ and λ can be treated as constant,

$$\rho \frac{D\vec{u}}{Dt} = -\rho \vec{\nabla}\Pi - \vec{\nabla}p + (\lambda + \mu)\vec{\nabla}(\vec{\nabla} \cdot \vec{u}) + \mu \nabla^2 \vec{u},$$

- **Internal Energy Equation:**

$$\rho \frac{De}{Dt} = \vec{\nabla} \cdot (k\vec{\nabla}T) - p\vec{\nabla} \cdot \vec{u} + \phi,$$

  where ϕ is given by (8.17).

The mechanical energy equation can be derived from the above set of five equations. We now have seven unknowns: the density ρ, the pressure p, the internal energy e, the temperature T and the velocity $\vec{u}$. In addition, there are three scalar fields k, λ, and μ which need to be determined experimentally. To make further progress we will need to consider some elementary thermodynamics. This will give two relationships among the thermodynamic variables ρ, p, e, and T.

Chapter 9

SIMPLE NEWTONIAN FLOWS FOR A FLUID OF CONSTANT DENSITY

The we have derived the following equations:

- **Continuity Equation:**
$$\frac{D\rho}{Dt} + \rho\vec{\nabla}\cdot\vec{u} = 0.$$

- **Momentum Equation:**
$$\rho\frac{Du_i}{Dt} = -\rho\frac{\partial\Pi}{\partial x_i} - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_i}\Big(\lambda\vec{\nabla}\cdot\vec{u}\Big) + 2\frac{\partial}{\partial x_k}\Big(\mu\mathbf{e}_{ki}\Big),$$
or, if μ and λ can be treated as constant,
$$\rho\frac{D\vec{u}}{Dt} = -\rho\vec{\nabla}\Pi - \vec{\nabla}p + (\lambda+\mu)\vec{\nabla}(\vec{\nabla}\cdot\vec{u}) + \mu\nabla^2\vec{u},$$

- **Internal Energy Equation:**
$$\rho\frac{De}{Dt} = \vec{\nabla}\cdot(k\vec{\nabla}T) - p\vec{\nabla}\cdot\vec{u} + \phi,$$

where
$$\phi = \lambda(\vec{\nabla}\cdot\vec{u})^2 + 2\mu(\mathbf{e}_{ij})^2,$$
is the viscous dissipation.

We will consider some simple flows of the simplest possible Newtonian Fluid: one with constant density. The assumption that the density of the fluid is constant is an excellent approximation for many liquids in our daily experience. By implication the temperature of the fluid is constant as well and so are μ and λ. The continuity equation becomes

$$\vec{\nabla} \cdot \vec{u} = 0, \tag{9.1}$$

while the momentum equations simplify to

$$\rho \frac{D\vec{u}}{Dt} = -\rho \vec{\nabla}\Pi - \vec{\nabla}p + \mu \nabla^2 \vec{u}. \tag{9.2}$$

The expression for the stress tensor simplifies to

$$\tau_{ij} = -p\delta_{ij} + 2\mu e_{ij}. \tag{9.3}$$

We have a closed set of four equations for four unknowns; the pressure p and the three components of $\vec{u}$. We have vastly simplified the problem, yet, the general solution of these equations is unknown. Solutions of these equations can have extremely complicated behaviour. The chief cause of the rich interesting behaviour exhibited by even constant density flows is also reponsible for the difficulty in solving them: the nonlinear term $\vec{u} \cdot \vec{\nabla}\vec{u}$ in the momentum equation. Because of this term, two known solutions can't be combined to give another solution. None of the methods used to solve linear problems (Fourier Series, separation of variables, Laplace Transforms, ...) can be applied except in special cases.

There are several special solutions of these equations, some of which we discuss in the following. In the following I will use the convention that z is the vertical coordinate, in the opposite direction to gravity. Setting $\vec{u} = (u, v, w)$, $\vec{x} = (x, y, z)$ and $\Pi = gz$ the equations of motion are

$$\begin{aligned}
\rho\Big(u_t + uu_x + vu_y + wu_z\Big) &= -p_x + \mu\nabla^2 u, \\
\rho\Big(v_t + uv_x + vv_y + wv_z\Big) &= -p_y + \mu\nabla^2 v, \\
\rho\Big(w_t + uw_x + vw_y + ww_z\Big) &= -p_z - \rho g + \mu\nabla^2 w, \\
u_x + v_y + w_z &= 0.
\end{aligned}$$

One way to find solutions of these equations is to consider very simple flows — flows for which the nonlinear term is zero.

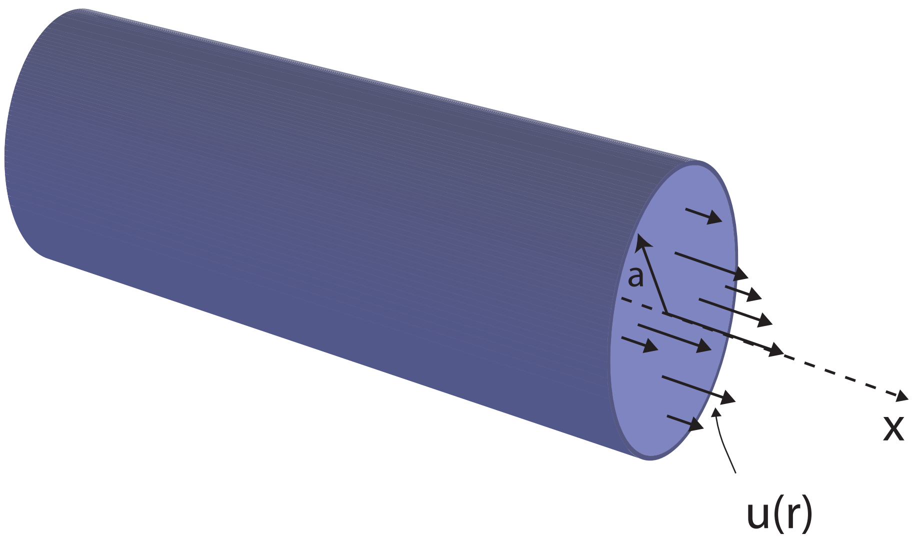


Figure 9.1: Steady laminar flow $u(r)$ through a circular pipe of radius a aligned with the x-axis.

9.1 Circular Poiseuille Flow: Steady laminar pipe flow

The first flow we will consider is the problem of steady, laminar, pressure driven flow through a circular pipe of radius a. The pipe is centred on the x-axis. Flow is along the pipe, so $v = w = 0$, and by symmetry the along pipe flow u is a function of r only, where $r = \sqrt{y^2 + z^2}$ is the distance from the axis of the pipe. The nonlinear term in the momentum equations

$$\vec{u} \cdot \vec{\nabla} \vec{u} = u \frac{\partial \vec{u}}{\partial x} + v \frac{\partial \vec{u}}{\partial y} + w \frac{\partial \vec{u}}{\partial z} = 0,$$

since $v = w = 0$ and the velocity field is independent of x. Hence, the equations of motion are linear. The vertical component of the momentum equation is

$$p_z = -\rho g, \tag{9.4}$$

i.e., the pressure is hydrostatic. Integrating gives

$$p = -\rho g z + F(x). \tag{9.5}$$

The horizontal component of the momentum equation is

$$p_x = F'(x) = \mu \nabla^2 u = \mu \frac{1}{r} \frac{d}{dr} \Big(r \frac{du}{dr} \Big), \tag{9.6}$$

with boundary conditions $u = 0$ at $r = a$. The only way a function of x can equal a function of r is if both functions are constants. Setting

$$p_x = -G, \tag{9.7}$$

where $-G$ is the constant horizontal pressure gradient along the pipe (G is positive if the pressure is decreasing in the positive x direction, i.e., at the left end of the pipe segment in Figure 9.1). Hence, we have

$$\frac{d}{dr}\left(r\frac{du}{dr}\right) = -\frac{G}{\mu}r, \tag{9.8}$$

Integrating once gives

$$r\frac{du}{dr} = -\frac{Gr^2}{2\mu} + A. \tag{9.9}$$

Since du/dr must be finite at $r = 0$, $A = 0$. Integrating again and applying the boundary condition that $u(a) = 0$ gives

$$u = -\frac{G}{4\mu}\left(r^2 - a^2\right). \tag{9.10}$$

Exercise: *Find the maximum velocity and the volume flux through the pipe (ans: volume flux is $\pi G a^4/(8\mu)$).*

Flow through a pipe becomes turbulent if $U_{max}a/\nu >\approx 1500 - 25,000$, where

$$\nu = \mu/\rho, \tag{9.11}$$

is called the kinematic viscosity. The dimensionless quantity $Re = U_{max}a/\nu$ is called the Reynolds number. This in an extremely important parameter for characterizing flows of all types. There are many different Reynolds numbers but they all have the form of *velocity scale$\times$length scale$/$kinematic viscosity*. The value of Re at which transition to turbulence occurs is very sensitive to the experimental conditions - hence the large range of values cited.

Stress on the walls: In cylindrical coordinates σ_{rx} is the x component of the deviatoric stress exerted on fluid with outward unit normal in the radial, i.e., $\hat{e}_r$, direction, hence the tangential stress acting on the pipe by the fluid is

$$-\sigma_{rx} = -2\mu e_{rx}. \tag{9.12}$$

Here

$$e_{rx} = \frac{1}{2}\Big(\frac{\partial u_r}{\partial x} + \frac{\partial u_x}{\partial r}\Big). \tag{9.13}$$

You may be surprised that the expression for e_{rx} in cylindrical coordinates is so similar to those for e_{ij} in Cartesian coordinates. This is not true for all components of the Strain Rate Tensor. For example,

$$e_{r\theta} = \frac{1}{2}\left(r\frac{\partial}{\partial r}\Big(\frac{u_\theta}{r}\Big) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right).$$

For circular Poiseuille flow

$$\sigma_{rx} = \mu\frac{du}{dr} = -\frac{G}{2}r. \tag{9.14}$$

We can use this to derive the equation of motion. Consider an annulus of length l centred on the x-axis with inner radius r and outer radius $r + dr$. The net force in the $\hat{i}$ direction is due to a combination of the pressure acting on the ends of the annular cylinder and the shear stress acting on its inner and outer surfaces. The contribution from the pressure is simply the pressure drop Gl times the surface area of the ends of the annulus. There is no net contribution from the hydrostatic part of the pressure field as it is the same at the two ends of the annulus and the two contributions cancel. Thus, the net force in the $\hat{i}$ direction is

$$\begin{aligned} &Gl\,2\pi r\,dr\hat{i} + \Big(\mu\frac{du}{dr}(r+dr)l\,2\pi(r+dr) - \mu\frac{du}{dr}(r)l\,2\pi r\Big)\hat{i} \\ &= \Big(Gl\,2\pi r\,dr + \mu\Big[\frac{du}{dr}(r+dr)\Big(r+dr\Big) - \frac{du}{dr}(r)r\Big]2\pi l\hat{i}, \\ &= \Big[Gr + \mu\frac{d}{dr}\big(r\frac{du}{dr}\big)\Big]2\pi l\,dr\hat{i} \end{aligned}$$

Since the acceleration is zero, the net force must be zero, which gives equation (9.8). This should have been anticipated: the momentum equations are a statement of conservation of linear momentum.

Exercise: Show that the total tangential force exerted by the fluid on a piece of pipe of length l is $Gl\pi a^2\hat{i}$ by calculating the shear stress on the pipe (can you think of another way to do it?). What externally applied force must be applied to the pipe to hold it in position.

Exercise: Find the analogous laminar, pressure driven flow between two horizontal flat plates.

9.2 Flow above an Impulsively Started Flat Plate

We next will consider a time dependent problem that can be solved analytically: flow in a semi-infinite fluid above an impulsively started flat plate. The fluid lies above an infinitely long plate at $z = 0$. Fluid fills the region $z > 0$ and is initially at rest. At $t = 0$ the plate impulsively starts to move with velocity U_o. Because of viscosity the fluid at the plate moves with the plate and hence has velocity U_o for $t > 0$

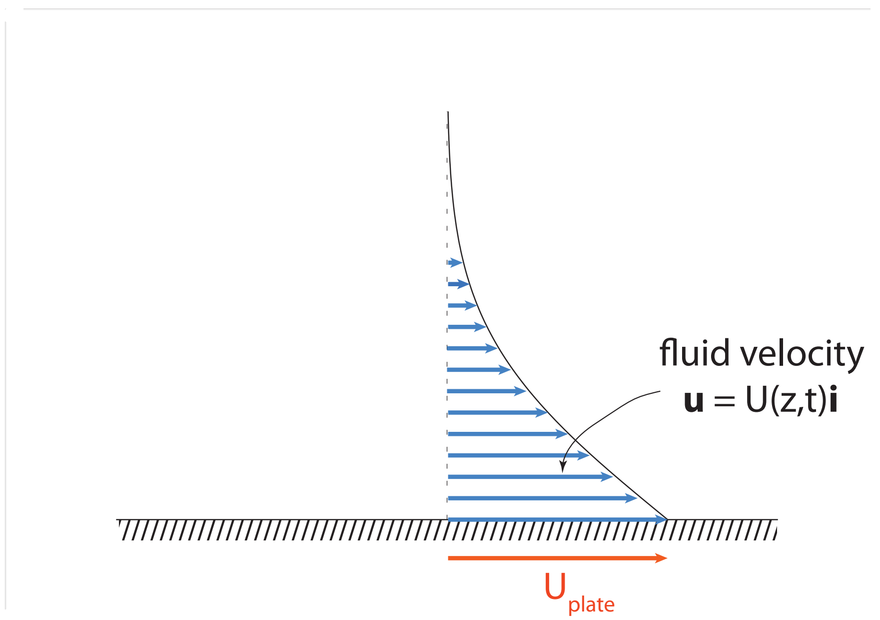


Figure 9.2: Flow above an impulsively started flat plat moving tangentially with velocity $U_{plate} = U_0 H(t)$.

Since $v = w = 0$, and the flow is independent of x, the nonlinear terms are once again equal to zero. We assume that p is independent of x as well. The equation of motion we need to solve is the now time-dependent horizontal momentum equation

$$u_t = \nu u_{zz}, \tag{9.15}$$

with boundary conditions

$$u(0, t) = U_o H(t), \tag{9.16}$$

where $H(t)$ is the Heaviside function, and

$$u(\infty, t) = 0, \tag{9.17}$$

which is the far field condition. The initial condition is

$$u(z,0) = 0. \tag{9.18}$$

In order to solve this problem, we first define a nondimensional velocity variable $\tilde{u}$ via

$$\tilde{u} = \frac{u}{U_o}. \tag{9.19}$$

This removes U_o from the problem as the problem for $\tilde{u}$ is

$$\begin{aligned} \tilde{u}_t &= \nu \tilde{u}_{zz}, \\ \tilde{u}(z,0) &= 0, \\ \tilde{u}(\infty,t) &= 0, \\ \tilde{u}(0,t) &= H(t), \end{aligned}$$

The solution $\tilde{u}(z,t)$ depends only on z, t and ν. Since $\tilde{u}$ is dimensionless, changing the dimensions (units) of z, t or ν cannot change the value of $\tilde{u}$. Thus, $\tilde{u}$ can depend only on z, t and ν in combinations which are dimensionless. These quantities have units of

$$[t] = \text{s}, \qquad [z] = \text{m}, \qquad [\nu] = \text{m}^2\ \text{s}^{-1}. \tag{9.20}$$

The only dimensionless combination is some power of $\nu t/z^2$. A convenient choice is

$$\eta = \frac{z}{2\sqrt{\nu t}}. \tag{9.21}$$

We now look for a solution of the form

$$\tilde{u} = f(\eta). \tag{9.22}$$

Such a solution exists because there is no length scale in the problem. Substituting into the PDE for $\tilde{u}$ we have

$$\tilde{u}_t = f'(\eta)\eta_t = -\frac{z}{4\sqrt{\nu}t^{3/2}} f'(\eta) = -\frac{\eta}{2t} f'(\eta),$$

and

$$\tilde{u}_z = f'(\eta)\frac{1}{2\sqrt{\nu t}} \quad \Rightarrow \quad \tilde{u}_{zz} = f''(\eta)\frac{1}{4\nu t}.$$

Thus,

$$\tilde{u}_t - \nu \tilde{u}_{zz} = -\frac{\eta}{2t} f'(\eta) - \frac{f''(\eta)}{4t} = 0,$$

which implies that

$$f''(\eta) + 2\eta f'(\eta) = 0. \tag{9.23}$$

Note only η appears in the equation — as it must for there to be a similarity solution. In the far field we have the boundary conditions

$$f(\infty) = 0, \tag{9.24}$$

which includes both $\tilde{u}(\infty, t) = 0$ and $\tilde{u}(z, 0) = 0$ since $\eta = \infty$ when $z = \infty$ for $t > 0$ and $\eta = \infty$ when $t = 0$ for $z > 0$. The second boundary condition is

$$f(0) = 1. \tag{9.25}$$

Rearranging (9.23), we have

$$\frac{df'}{f'} = -2\eta\, d\eta,$$

so

$$f' = Ae^{-\eta^2}.$$

Integrating this gives

$$f(\eta) = A \int_0^\eta e^{-s^2}\, ds + B.$$

From the boundary condition $f(0) = 1$ we have $B = 1$. From the second boundary condition we have

$$0 = f(\infty) = A \int_0^\infty e^{-s^2}\, ds + 1 = A\frac{\sqrt{\pi}}{2} + 1,$$

so

$$f(\eta) = 1 - \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-s^2}\, ds = 1 - \mathrm{erf}(\eta). \tag{9.26}$$

The solution is shown in FIgure 9.3. From $u/U_o = f(\eta)$ we see that the velocity profile has the same shape for all time. It simply gets stretched in the z direction as time increases (as t increases, z must increase to keep η and hence u/U_o constant). Because the solution is self-similar (i.e., has the same shape) in time, this type of solution is called a similarity solution.

Let us calculate the vorticity in the flow. The vorticity is $\omega = (w_y - v_z, u_z - w_x, v_x - u_y)$. Only the y-component is nonzero. Referring to this as the vorticity, the vorticity is

$$u_z = U_o \tilde{u}_z = -\frac{U_o}{\sqrt{\pi\nu t}} e^{-z^2/4\nu t}.$$

Note that

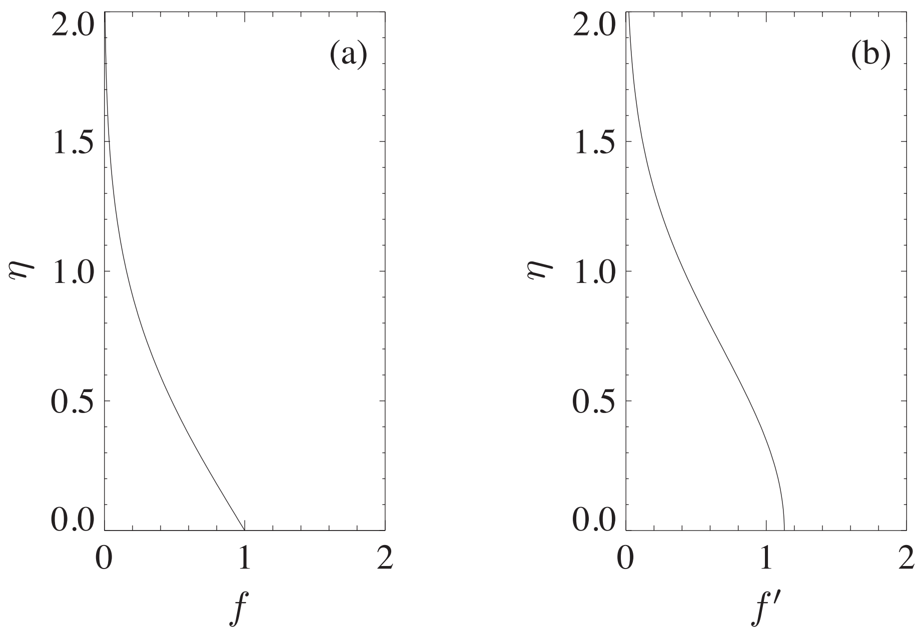


Figure 9.3: Similarity solution for impulsively started flat plate. (a) Nondimensional similarity profile $f(\eta)$. (b) Scaled vorticity $f'(\eta)$.

- the vorticity decays super-exponentially with z;
- the vorticity at any height z initially increases in magnitude until $t = z^2/2\nu$ and then decreases going to zero as $t \to \infty$;
- the integrated vorticity (per unit length in x-direction) $\int_0^\infty u_z \, dz = u|_0^\infty = -U_o$, the difference in the far field fluid velocity and the fluid velocity at the lower boundary, which is independent of time;
- All the vorticity is at the boundary at $t = 0$.

This illustrates an important fact: *all the vorticity is created instantaneously when the plate starts moving*. Thereafter it diffuses out.

For small times the vorticity (and flow) is confined to a thin boundary layer along the plate. A **boundary layer thickness** can be defined as the distance from the wall at which $u/U_o = 0.05$, i.e., the distance at which the fluid velocity is 5% of the velocity of the boundary. This occurs when $f(\eta) = 0.05$, or at

$$\eta \approx 2 \Rightarrow z = 4\sqrt{\nu t}.$$

Other definitions of the boundary layer thickness could be used however they would all give the result that the boundary layer thickness increases in time like $t^{1/2}$ and is proportional to $\sqrt{\nu}$.

Chapter 10

VORTICITY

A quantity of fundamental importance in fluid dynamics is the vorticity

$$\vec{\omega} = \vec{\nabla} \times \vec{u}. \tag{10.1}$$

The vorticity is twice the angular velocity of a fluid particle. Thus the vorticity is a measure of how fluid particles are spinning.

Straight parallel flows can have nonzero vorticity. For laminar flow along a channel of the form $\vec{u} = (u(y), 0, 0)$ the vorticity is $\vec{\omega} = (0, 0, -u'(y))$ so if $u'(y) \neq 0$ the flow has vorticity (see Figure 10.1).

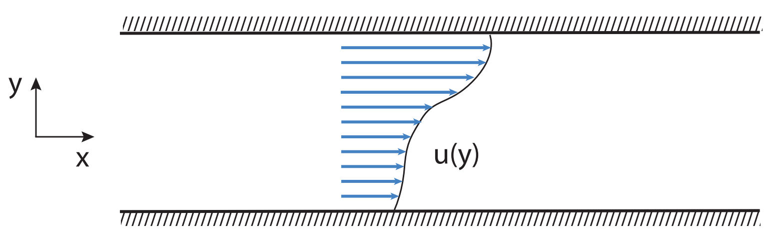


Figure 10.1: Shear flow $u(y)$ with non-zero vorticity.

As discussed in the section on the Strain Rate Tensor, the vorticity is twice the average angular velocity of two mutually perpendicular lines. This suggests a simple vorticity meter for 2-D flows: a paddle wheel (Figure 10.2).

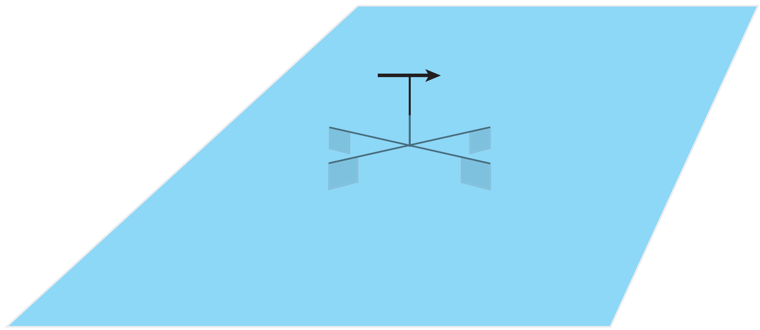


Figure 10.2: A simple paddle wheel used to visualize vorticity.

10.1 Circular Flows

Circular flows can have zero or nonzero vorticity. For a circular flow of the form $\vec{u} = u_\theta(r)\hat{e}_\theta$ the vorticity is $\vec{\omega} = \zeta\hat{k}$ where

$$\zeta = \frac{1}{r}\frac{d}{dr}\Big(r u_\theta(r)\Big). \tag{10.2}$$

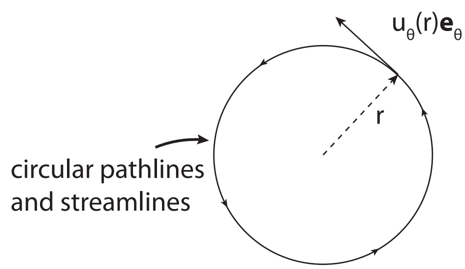


Figure 10.3: Circular flow.

Case I: Solid body rotation. For solid body rotation with angular velocity ω_o, the angular component of the velocity $u_\theta(r) = \omega_o r$ is proportional to the distance from the axis of rotation. The vorticity is

$$\zeta = \frac{1}{r}\frac{d}{dr}\Big(\omega_o r^2\Big) = 2\omega_o,$$

which is twice the angular velocity. In solid body rotation fluid particles all revolve around one another with angular velocity ω_o. Fluid volumes do not deform, the Strain Rate Tensor is identically zero.

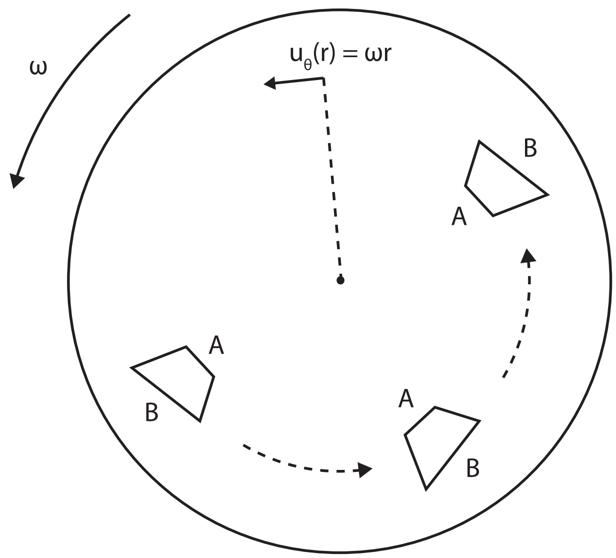


Figure 10.4: Figure 7.3 repeated: solid body rotation with angular velocity ω. A quadrilateral material volume is shown at three different times. It spins about its centre of volume with angular velocity ω, while its centre of volume rotates about the axis of rotation with the same angular velocity ω. The material volume does not change shape, only its orientation and the location of its centre of volume changes.

Case II: Irrotational Vortex. For an irrotational vortex the vorticity is zero (this is what irrotational means!). Thus

$$\zeta = \frac{1}{r}\frac{d}{dr}\Big(r u_\theta(r)\Big) = 0.$$

This implies that $r u_\theta(r)$ is a constant. Taking this constant to be $\Gamma/2\pi$ we have

$$u_\theta(r) = \frac{\Gamma}{2\pi r}. \tag{10.3}$$

Note that $u_\theta \to 0$ as $r \to \infty$ and that $u_\theta \to \infty$ as $r \to 0$. The flow has a singularity at the origin. Consider Stokes Theorem, which says that

$$\oint_{\partial R} \vec{u} \cdot d\vec{s} = \iint_R \vec{\nabla} \times \vec{u} \cdot d\vec{A},$$

provided $\vec{u}$ is continuously differentiable on the closed, simply connected region R. This states that the circulation around a simple closed curve is equal to the flux of vorticity through the curve. For the irrotational vortex, if R does not contain the origin $\vec{\nabla} \times \vec{u} = 0$ on R so

$$\oint_{\partial R} \vec{u} \cdot d\vec{s} = 0.$$

If R is a disk of radius a centred at the origin then

$$\int_{\partial R} \vec{u} \cdot d\vec{s} = u_\theta(a) \cdot 2\pi a = \Gamma.$$

In fact this is true for all closed curves that contain the origin. The irrotational vortex is know as a **point vortex**. Its vorticity is zero everywhere except at the origin where it is infinite (given by a 2-D Dirac Delta function).

In reality vortices with singularities don't exist. Viscosity prevents them from occuring. Many vortices (i.e., those in a draining bathtub, atmospheric cyclones and tornadoes) behave like irrotational vortices far from their centre and have approximately solid body rotation near their centre.

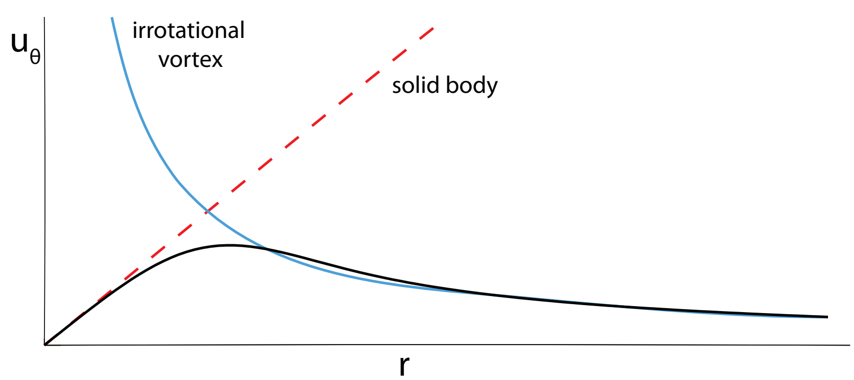


Figure 10.5: Velocity profiles for idealized vortices: solid body rotation (dashed red curve) and an irrotational vortex (light blue). The velocity profile for a more realistic vortex is shown in black.

An idealized model of this is the Rankine Vortex, which has solid body rotation for $r < R$ and is irrotational for $r > R$. Many flows consist of patches of vorticity embedded in a non-vortical fluid. For a 2-D flow, the velocity field far away from the vortex patches can be approximated by replacing the vortex patches with point vortices with circulation equal to the vorticity flux through the patch.

For an incompressible fluid with constant viscosity the viscous term

$$\mu \nabla^2 \vec{u},$$

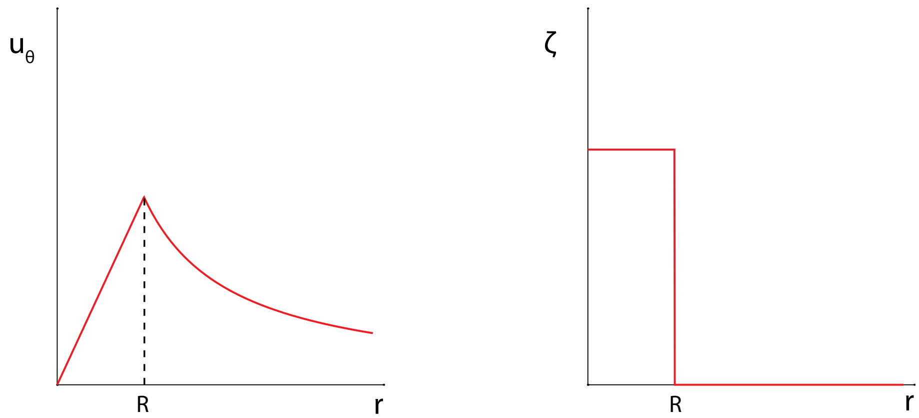


Figure 10.6: Velocity and vorticity profiles for a rankine vortex

is equal to

$$-\mu\left(\vec{\nabla} \times \vec{\omega}\right). \tag{10.4}$$

Here we have used the vector identity,

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}, \tag{10.5}$$

with $\vec{A} = \vec{u}$, making use of the fact that $\vec{\nabla} \cdot \vec{u} = 0$.

Thus, in the case of solid body rotation, with $\vec{\omega}$ constant everywhere, the viscous term is zero. In fact all the shear stresses are zero because there is no deformation of fluid volumes. For an irrotational vortex the shear stresses are nonzero (fluid elements deform!), but, away from the origin, the net viscous force is zero. This is analogous to linear horizontal shear flow. At the origin viscous forces are infinite, so a singularity is immediately smoothed out (and hence never actually exists). Point vortices can, however, provide a useful mathematical model of a flow.

For both solid body rotation and for an irrotational vortex (except at $r = 0$), the momentum equations reduce to

$$\rho \frac{D\vec{u}}{Dt} = -\vec{\nabla} p - \rho \vec{g}. \tag{10.6}$$

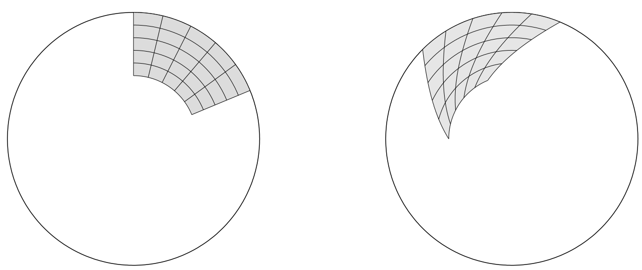


Figure 10.7: In an irrotational vortex a piece of material (shaded grey) is sheared and deformed as it moves about the axis of rotation.

10.2 Surfaces of Constant Pressure (ρ =constant)

Imagine a fluid with a free surface exposed to the atmosphere at constant pressure P_a. Barring surface tension effects, the surface of the fluid will be at constant pressure P_a. If the fluid is moving in a circular motion about a vertical axis the surface will be deformed. What shape does the free surface take? Since the free surface is at constant pressure we can answer this question by determining the shape of surfaces of constant pressure in the fluid.

For circular motion $\vec{u} = u_\theta(r)\hat{e}_\theta$, and fluid particles move along circular paths with constant angular velocity. Hence fluid particles are accelerating inward. The radial component of the momentum equation in cylindrical coordinates is a statement of this: $\rho\times$*inward acceleration* $=$ *inward pressure gradient*. Mathematically

$$\rho\frac{u_\theta^2}{r} = \frac{\partial p}{\partial r}. \tag{10.7}$$

The vertical component of the momentum equation is

$$p_z = -\rho g, \tag{10.8}$$

i.e., the pressure is hydrostatic.

Case I: Solid Body Rotation. For solid body rotation $u_\theta = \omega_o r$. Integrating (10.7) gives

$$p = \frac{\rho\omega_o^2}{2}r^2 + f(z),$$

Since p is hydrostatic in the vertical, $f'(z) = -\rho g$, so

$$p = \frac{\rho\omega_o^2}{2}r^2 - \rho g z + C. \tag{10.9}$$

Solving for z in terms of p gives

$$z = \frac{\omega_o^2}{2g}r^2 - \frac{p}{\rho g} + C \tag{10.10}$$

(new C). Surfaces of constant pressure, including the free surface, are paraboloids.

As an alternative derivation consider a fluid parcel lying between r and $r + dr$ in the radial direction, between and z and $z + dz$ in the vertical, and between θ and $\theta + d\theta$ in the angular direction (see figure below). The parcel has mass $\rho r d\theta\, dr\, dz$. Its angular velocity is $\omega_o r$, which implies an inward acceleration of $u_\theta^2/r = \omega_o^2 r$. There must be an inward force acting on the fluid parcel equal to the acceleration times the mass, i.e., equal to

$$\omega_o^2 r \times \rho r\, d\theta\, dr\, dz. \tag{10.11}$$

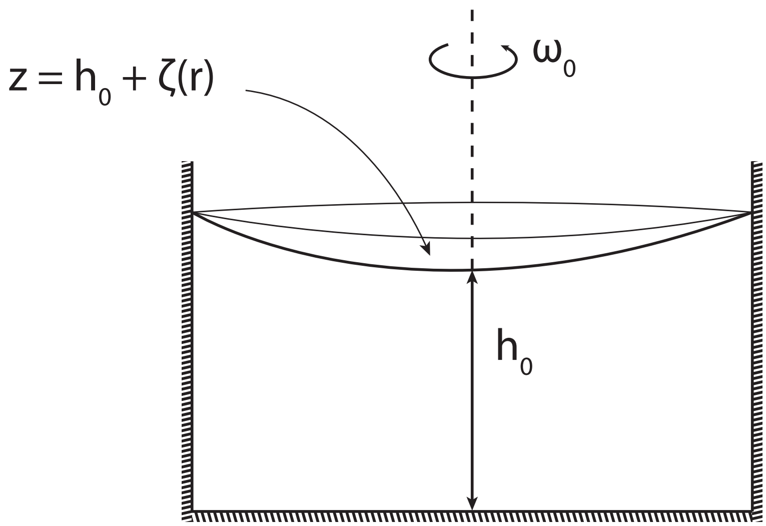


Figure 10.8: Surface of constant pressure $z = h_0 + \zeta(r) = h_0 + \frac{\omega_0^2}{2g} r^2$ in volume of fluid in solid body rotation.

This force is provided by the pressure difference across the volume (see Figure 10.9). The pressure acting on the surface at r is $P_a + (h(r) - z)\rho g$. The pressure acting on the surface at $r+dr$ is $P_a + (h(r+dr) - z))\rho g$. The difference multiplied by the area $r\, d\theta\, dz$ gives a net inward force equal to

$$\Big(h(r+dr) - h(r)\Big)\rho g r\, d\theta\, dz. \tag{10.12}$$

Equation (10.11) and (10.12), assuming dr is small, gives

$$h'(r)\,\mathrm{d}r\,\rho g r\,\mathrm{d}\theta\,\mathrm{d}z = \omega_o^2 r \times \rho r\,\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}z,$$

which gives

$$h'(r) = \frac{\omega_o^2 r}{g},$$

so

$$h(r) = h_o + \frac{\omega_o^2 r^2}{2g}, \tag{10.13}$$

which recovers the previous solution.

Case II: Irrotational Vortex.

Exercise: Show that surfaces of constant pressure are hyperboloids given by

$$z = -\frac{\Gamma^2}{8g\pi^2 r^2} - \frac{p}{\rho g} + C. \tag{10.14}$$

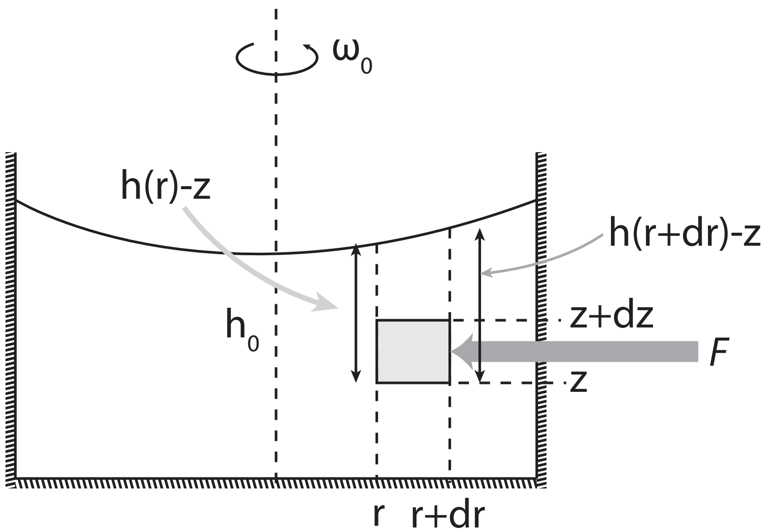


Figure 10.9: Higher pressure on the outer surface of a material volume due to the greater height of fluid results in a net inward force F which accelerates the material volume around its circular path.

The vortex created in a draining bathtub is almost an irrotational vortex!

10.3 Vortex Dynamics

We now turn to a more general treatment of vorticity. First some definitions. **Definition:** *A vortex line is a curve which is everywhere tangent to the vorticity field.*

A vortex line is to the vorticity as a streamline is to the velocity field.

Definition: *A vortex tube is the set of vortex lines passing through any simple closed curve.*

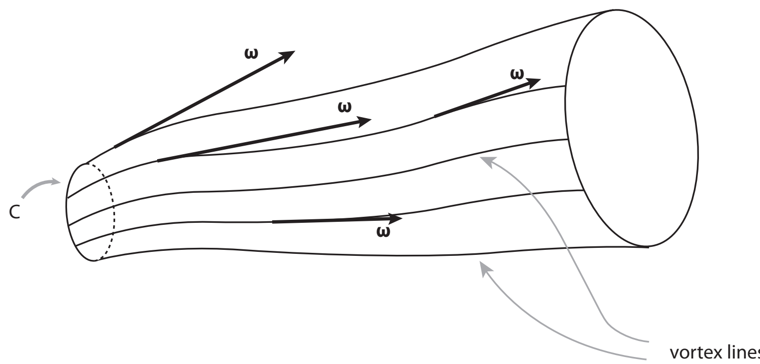


Figure 10.10: A vortex tube passing through a closed curve C.

Theorem: *The circulation around any closed curve C which passes once around the tube is independent of the curve.*

Proof: Consider a closed surface S formed by a section of a vortex tube and its two caps (Figure 10.11. Call these three surfaces S_3, S_1 and S_2 (see diagram). Surfaces S_1 and S_2 are bounded by curves C_1 and C_2 which pass once around the vortex tube.

Now

$$\iint_S \vec{\nabla} \times \vec{u} \cdot \hat{n}\, dA = 0, \qquad (10.15)$$

since S is a closed surface. Separating S into its three parts gives

$$\iint_{S_1} \vec{\nabla} \times \vec{u} \cdot \hat{n}_1\, dA + \iint_{S_2} \vec{\nabla} \times \vec{u} \cdot \hat{n}_2\, dA + \iint_{S_3} \vec{\nabla} \times \vec{u} \cdot \hat{n}_3\, dA = 0. \qquad (10.16)$$

By definition $\omega = \vec{\nabla} \times \vec{u}$ is tangent to S_3, so $\vec{\nabla} \times \vec{u} \cdot \hat{n}_3 = 0$ on S_3. Thus

$$\iint_{S_1} \vec{\nabla} \times \vec{u} \cdot \hat{n}_1\, dA = -\iint_{S_2} \vec{\nabla} \times \vec{u} \cdot \hat{n}_2\, dA. \qquad (10.17)$$

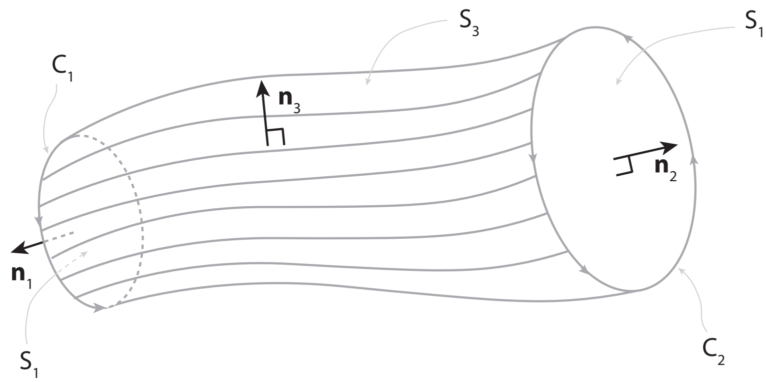


Figure 10.11: A part of a vortex tube with surface S_3 between end caps S_1 and S_2. The unit normals to the surfaces are $\hat{n}_1$, $\hat{n}_2$ and $\hat{n}_3$ respectively. The curves C_1 and C_2 bound surfaces S_1 and S_2.

Noting that $-\hat{n}_1$ and $\hat{n}_2$ point the same way along the tube we can write this as

$$\iint_{S_1} \vec{\omega} \cdot (-\hat{n}_1)\, dA = \iint_{S_2} \vec{\omega} \cdot \hat{n}_2\, dA, \tag{10.18}$$

which says that the vorticity flux through the two cross sections of the tube is identical. Using Stokes Theorem

$$\oint_{C_1} \vec{u} \cdot\, d\vec{s} = \oint_{C_2} \vec{u} \cdot\, d\vec{s}, \tag{10.19}$$

where the circulations around the two curves are both in the same direction around the vortex tube (i.e., counter-clockwise around $-\hat{n}_1$ and $\hat{n}_2$).

The result that the vorticity flux through any cross-section of a vortex tube is the same means that a vortex tube can never end! All vortex tubes must close on itself (e.g., a smoke ring) or end at a boundary (e.g., a tornado). This is a kinematic results. It is independent of any forces acting on the fluid and hence is true for all C^1 flows of all continuums.

10.4 Kelvin's Circulation Theorem

Let $C(t)$ be a closed material curve and let

$$\Gamma(t) = \oint_{C(t)} \vec{u} \cdot d\vec{s}, \tag{10.20}$$

be the circulation around C. How does the circulation vary with time? To find out we need to find an expression for $d\Gamma/dt$. From an assignment we have seen that

$$\frac{d\Gamma}{dt} = \oint_{C(t)} \frac{D\vec{u}}{Dt} \cdot d\vec{s}. \tag{10.21}$$

This result is a kinematic results — valid for all continuums. We now use the momentum equation in its general form,

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\vec{\nabla}p - \vec{\nabla}\Pi + \frac{1}{\rho}\vec{F}, \tag{10.22}$$

where $\vec{F} = \vec{\nabla} \cdot \sigma$ is the divergence of the deviatoric stress tensor, which we can think of as a generalized viscous term. Using this we have

$$\frac{d\Gamma}{dt} = -\oint_{C(t)} \frac{\vec{\nabla}p}{\rho} \cdot d\vec{s} - \oint_{C(t)} \vec{\nabla}\Pi \cdot d\vec{s} + \oint_{C(t)} \frac{1}{\rho}\vec{F} \cdot d\vec{s}. \tag{10.23}$$

The second term on the right is zero (true for all conservative body forces). Using Stokes Theorem and the fact that

$$\vec{\nabla} \times \Big(-\frac{1}{\rho}\vec{\nabla}p\Big) = \vec{\nabla}\Big(-\frac{1}{\rho}\Big) \times \vec{\nabla}p - \frac{1}{\rho}\vec{\nabla} \times \vec{\nabla}p = \frac{\vec{\nabla}\rho \times \vec{\nabla}p}{\rho^2}, \tag{10.24}$$

we have

$$\frac{d\Gamma}{dt} = \iint_{S(t)} \frac{\vec{\nabla}\rho \times \vec{\nabla}p}{\rho^2} \cdot \hat{n}\, dS + \iint_{S(t)} \vec{\nabla} \times \Big(\frac{1}{\rho}\vec{F}\Big) \cdot \hat{n}\, dS. \tag{10.25}$$

Definition: *A fluid is* **barotropic** *if the density and pressure are functionally related, i.e.,* $f(p, \rho) = 0$ *for some function* f*. Two important examples are a constant density fluid (*$f(p, \rho) = \rho - \rho_o$ *where* ρ_o *is the constant density) and an ideal gas at constant entropy, in which case* $p = A\rho^\gamma$ *where* A *and* γ *are constants.*

Kelvin's Circulation Theorem: *For a barotropic, inviscid flow with conservative body forces*

$$\frac{d\Gamma}{dt} = 0. \tag{10.26}$$

The proof is straightforward. If $f(p, \rho) = 0$ for some function f then $\vec{\nabla}f = f_p\vec{\nabla}p + f_\rho\vec{\nabla}\rho = 0$. Hence $\vec{\nabla}p$ and $\vec{\nabla}\rho$ are parallel or one of them is zero. In either

	air		water	
T (°C)	μ (kg m^{-1}s^{-1})	ν (m^{2}s^{-1})	μ (kg m^{-1}s^{-1})	ν (m^{2}s^{-1})
10	1.76×10^{-5}	1.41×10^{-5}	1.307×10^{-3}	1.307×10^{-6}
20	1.81×10^{-5}	1.50×10^{-5}	1.003×10^{-3}	1.005×10^{-6}

Table 10.1: Viscosities of air and water

case $\vec{\nabla}\rho \times \vec{\nabla}p = 0$. If the fluid is inviscid then $\vec{F} = 0$. Equations (10.25) gives the result.

When the conditions of Kelvin's Circulation Theorem are met, the strength of a vortex tube is constant in time. Equation (10.25) includes two mechanisms by which the circulation around a vortex tube can change.

First, consider the viscous term. Taking $\vec{F} = \mu\nabla^2\vec{u}$ we have

$$\vec{\nabla} \times \left(\frac{1}{\rho}\vec{F}\right) = \vec{\nabla} \times \left(\nu\nabla^2\vec{u}\right),$$

where $\nu = \mu/\rho$ is the kinematic viscosity. If we can treat ν as a constant (implying small temperature variations) then we have

$$\vec{\nabla} \times \left(\frac{1}{\rho}\vec{F}\right) = \nu\vec{\nabla} \times \nabla^2\vec{u} = \nu\nabla^2\vec{\omega}.$$

Both μ and ν are both often treated as constants even if the density is not constant. For air/water ν varies more/less than μ does for 'normal' temperatures. The interpretation of the viscous term in (10.25) is that the circulation changes due to the diffusion of vorticity associated with viscosity. Diffusion reduces local maxima of $|\vec{\omega}|$ and increases local minima of $|\vec{\omega}|$, hence changing the circulation. The diffusion of vorticity is illustrated in Figure 10.12.

Flows that are not barotropic are sometimes called baroclinic (this word, unfortunately, can have somewhat different meanings in different contexts). For a baroclinic flow, $\vec{\nabla}\rho \times \vec{\nabla}p \neq 0$ and circulation (hence vorticity) can be generated or destroyed.

10.4.1 Implications of Kelvin's Theorem

Theorem: *For an inviscid, barotropic flow with conservative body forces, if $\vec{\omega} = 0$ in a material volume $W(t)$ at some initial time $t = 0$ then $\vec{\omega}$ remains zero in the material volume.*

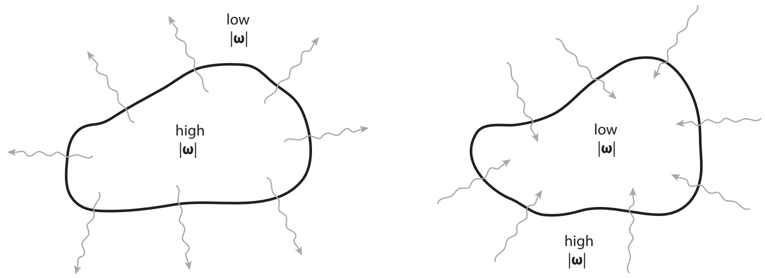


Figure 10.12: Diffusion of vorticity from regions of high vorticity to regions of low vorticity. In the schematic to the left vorticity diffuses out of a fluid volume with high $|\vec{\omega}|$ weakening the circulation around the boundary. In the the schematic at the right vorticity diffuses into a region of low $|\vec{\omega}|$: the fluid outside this volume spins up the fluid inside it via viscous forces.

Proof: If not, $\vec{\omega} \neq 0$ at some point $\vec{x}$ in $W(t)$. Then by continuity of $\vec{\omega}$ there is a surface S inside $W(t)$ such that

$$\iint_S \vec{\omega} \cdot \hat{n}\, dS \neq 0.$$

Then there is non-zero circulation around the curve C which forms the boundary of S. This curve came from a curve C_o inside $W(0)$. The circulation around this curve at $t = 0$ is zero, since $\vec{\omega} = 0$ inside $W(0)$. This contradicts Kelvin's Circulation Theorem which says that the circulation can't change in time.

Helmholtz Vortex Theorems: *Under the conditions of Kelvins Circulation Theorem*

(a) *Vortex lines move with the fluid.*

(b) *Vortex tubes move with the fluid.*

What these mean is that a material line which is initially a vortex line is always a vortex line and a material surface which is initially a vortex tube is always a vortex tube.

Proof of (b): Let $A(t)$ be a material surface. At $t = 0$ it forms a vortex tube. We need to show that $A(t)$ is a vortex tube as well. To do this we show that $\vec{\omega}$ is tangent to $A(t)$. Let $S(t)$ a material surface which is a small piece of $A(t)$. Let $C(t)$ be its boundary.

Now

$$\oint_{C(0)} \vec{u} \cdot d\vec{r} = \iint_{S(0)} \vec{\omega} \cdot \hat{n}\, dS = 0, \tag{10.27}$$

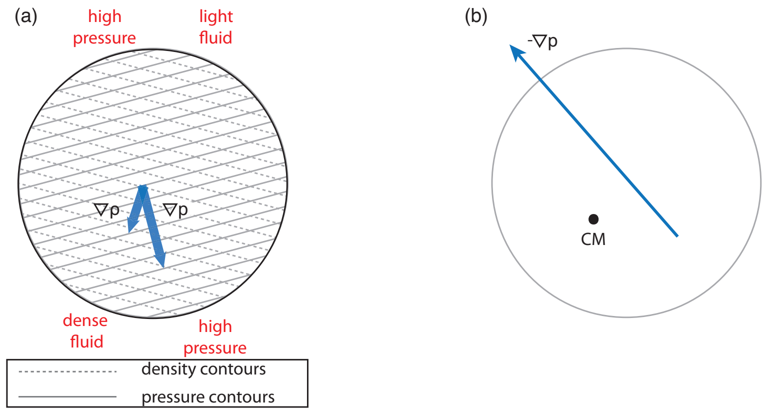


Figure 10.13: Baroclinic generation of circulation and vorticity. (a) When density contours and density contours are at an angle to one another so that $\vec{\nabla}\rho \times \vec{\nabla}p \neq 0$ vorticity is changed. In this example $\vec{\nabla}\rho \times \vec{\nabla}p$ points out of the page and the counter-clockwise circulation increases. (b) For the density and pressure fields illustrated in (a) the pressure force acts through the geometric centre of a small spherical volume which the centre of mass (CM) is off centre. This results in a counter-clockwise torque about the centre of mass increasing the counter-clockwise circulation.

since $\vec{\omega}$ is tangent to $A(0)$. Kelvin's Circulation Theorem then gives

$$\oint_{C(t)} \vec{u}\cdot d\vec{r} = \iint_{S(t)} \vec{\omega}\cdot\hat{n}\, dS = 0, \tag{10.28}$$

This is true for all $S(t)$ lying on $A(t)$, hence $\omega\cdot\hat{n} = 0$ on $A(t)$, i.e., $\vec{\omega}$ is tangent to $A(t)$.

To prove (a), let the vortex tube collapse to a line (an infinitesimally thin tube).

10.4.2 The Vorticity Equation

We will now derive the equation governing the time evolution of the vorticity field. Consider the general case of a compressible fluid with viscous force $\vec{F}$. We have

$$\vec{u}_t + \vec{u}\cdot\vec{\nabla}\vec{u} = -\frac{1}{\rho}\vec{\nabla}p - \vec{\nabla}\Pi + \frac{1}{\rho}\vec{F}. \tag{10.29}$$

Using

$$\vec{u}\cdot\vec{\nabla}\vec{u} = \vec{\omega}\times\vec{u} + \frac{1}{2}\vec{\nabla}(\vec{u}\cdot\vec{u}),$$

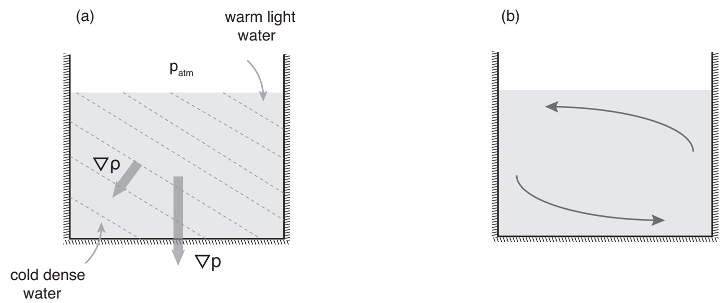


Figure 10.14: Baroclinic generation of circulation and vorticity. (a) A container of water is initially at rest and stratified with warm light water in the upper right and relatively cold dense water in the lower left. (b) The warm light water spreads over the surface and the colder, denser water spreads over the bottom creating counter-clockwise circulation.

which was proved on an assignment, the curl of (10.29) gives

$$\vec{\omega}_t + \vec{\nabla} \times (\vec{\omega} \times \vec{u}) = -\vec{\nabla} \times \left(\frac{\vec{\nabla} p}{\rho}\right) + \vec{\nabla} \times \left(\frac{\vec{F}}{\rho}\right).$$

Using the vector identity

$$\vec{\nabla} \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \vec{\nabla})\vec{A} - (\vec{A} \cdot \vec{\nabla})\vec{B} + (\vec{\nabla} \cdot \vec{B})\vec{A} - (\vec{\nabla} \cdot \vec{A})\vec{B}, \qquad (10.30)$$

we have, using the fact that $\vec{\nabla} \cdot \vec{\omega} = 0$,

$$\begin{aligned} \vec{\omega}_t + \vec{\nabla} \times (\vec{\omega} \times \vec{u}) &= \omega_t + \vec{u} \cdot \vec{\nabla}\vec{\omega} - \vec{\omega} \cdot \vec{\nabla}\vec{u} + (\vec{\nabla} \cdot \vec{u})\vec{\omega} \\ &= \frac{D\vec{\omega}}{Dt} - \vec{\omega} \cdot \vec{\nabla}\vec{u} + (\vec{\nabla} \cdot \vec{u})\vec{\omega}. \end{aligned}$$

Thus,

$$\frac{D\vec{\omega}}{Dt} = \vec{\omega} \cdot \vec{\nabla}\vec{u} - (\vec{\nabla} \cdot \vec{u})\vec{\omega} + \frac{\vec{\nabla}\rho \times \vec{\nabla} p}{\rho^2} + \vec{\nabla} \times \left(\frac{\vec{F}}{\rho}\right). \qquad (10.31)$$

This equation is called the vorticity equation. The four terms on the right represent four different mechanisms for changing the vorticity. The last two we have already seen: baroclinic generation and diffusion. The first two terms combined are referred to as the tilting/stretching terms although the latter is a bit of a misnomer.

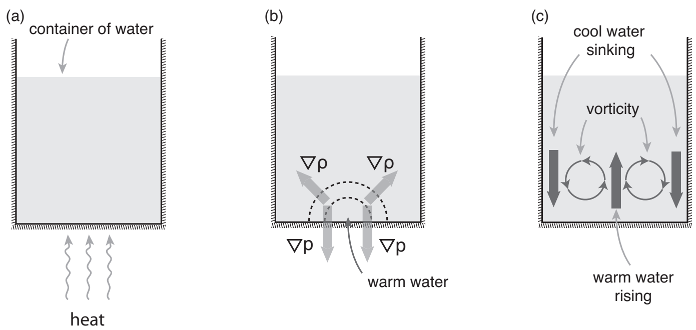


Figure 10.15: Baroclinic generation of circulation and vorticity. (c) A container of water which is initially at rest with uniform temperature. A patch of water is heated at the bottom. (d) The warmer water at the bottom is less dense than the surrounding fluid resulting in non-parallel density and pressure gradients. (e) Baroclinic generation of vorticity as the warm water rises in the centre and cold water sinks at the sides. In 3D a vortex ring would be generated if a circular patch of water is warmed.

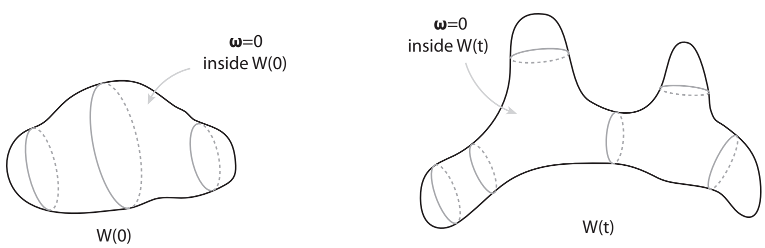


Figure 10.16: Vorticity inside a material volume $W(t)$ remains zero if conditions of Kelvin's Theorem are met.

Interpretation of $\vec{\omega} \cdot \vec{\nabla}\vec{u} - (\vec{\nabla} \cdot \vec{u})\vec{\omega}$.

Without loss of generality, choose a reference frame so that $\vec{\omega} = (0, 0, \zeta)$ at a point P. Then at P

$$
\begin{aligned}
(\vec{\omega} \cdot \vec{\nabla})\vec{u} - (\vec{\nabla} \cdot \vec{u})\vec{\omega} &= \zeta \frac{\partial \vec{u}}{\partial z} - (\vec{\nabla} \cdot \vec{u})(0, 0, \zeta), \\
&= \big(\zeta u_z, \zeta v_z, \zeta w_z\big) - \big(0, 0, \zeta(u_x + v_y + w_z)\big) \\
&= \big(\zeta u_z, \zeta v_z, -\zeta(u_x + v_y)\big)
\end{aligned}
$$

Note that this result is valid for both compressible and incompressible flow. If the flow is incompressible then we could replace $-(u_x + v_y)$ by w_z.

How do we interpret the various components?

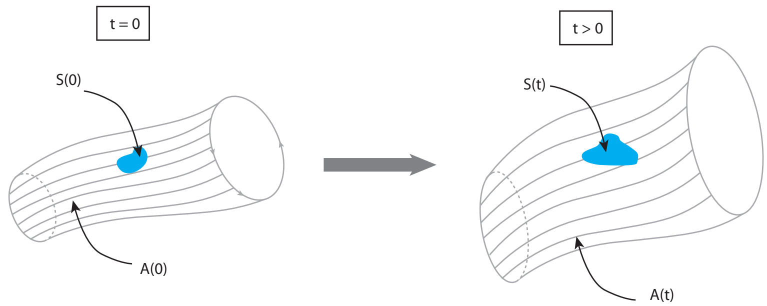


Figure 10.17: A piece of a vortex tube A at two times. S is a small material surface lying on A.

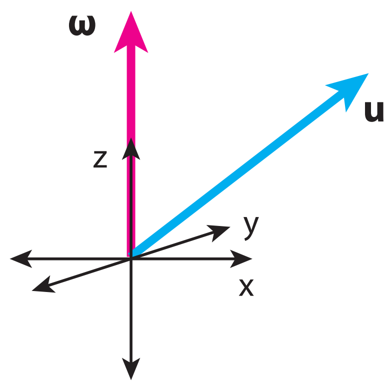


Figure 10.18: The vorticity $\vec{\omega}$ and velocity field $\vec{u}$ in a reference frame with $\vec{\omega} = (0, 0, \zeta)$.

ζu_z: This is generation of x-component of vorticity due to tilting of vortex lines into the x-direction.
ζv_z: This is generation of y-component of vorticity due to tilting of vortiex lines into the y-direction.

$-\zeta(u_x + v_y)$: The factor $u_x + v_y$ is the rate of expansion of matter in the xy-plane. Conservation of angular momentum about the z-axis implies that if the fluid contracts, the angular velocity, and hence the vorticity, must increase. It does so at the rate $-\zeta(u_x + v_y)$. This can be predicted from previous results: the vorticity flux through a vortex tube is constant in time for a barotropic inviscid fluid. If the cross-sectional area of the vortex tube decreases then the vorticity must increase. This is a consequence of conservation of angular momentum.

This mechanism is often called the stretching term since for an incompressible flow compression in the xy-plane implies stretching of a vortex line in the z direction. For a compressible flow, expansion/contraction in the xy-direction is not

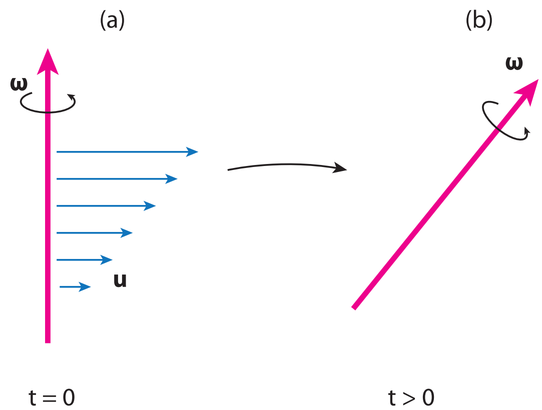


Figure 10.19: Vertical shear tilts a vertical vortext line which moves with the fluid barring baroclinic generation and diffusion. This transfers vertical vorticity into horizontal vorticity components.

necessarily accompanied by stretching in the z-direction.

Special Case: For a 2D, incompressible, barotropic, inviscid flow

$$\frac{D\vec{\omega}}{Dt} = 0, \tag{10.32}$$

i.e., the vorticity of a fluid particle is constant. For a collection of point vortices, the strength of each vortex is constant in time and the vorticity is zero everywhere except at the points. Vortex lines move with the flow, hence point vortices do as well. Irrotational incompressible flow (next topic) satisfies Laplace's equation, hence flow contributions from individual vortices can be superimposed.

10.4.3 Vorticity for Constant Density Flows

There are several remarkable features of constant density flows, one being that the velocity field is determined by the vorticity field and hence the three components of the vorticity equation form a closed system of equations. Recall the vector identity (10.5) which, for an incompressible flow, gives

$$\vec{\nabla} \times \vec{\omega} = -\nabla^2 \vec{u}. \tag{10.33}$$

Taking the i^{th} component gives

$$\nabla^2 u_i = -(\vec{\nabla} \times \vec{\omega})_i. \tag{10.34}$$

Thus, if the vorticity field is known, we need only solve Poisson's equation to find the velocity field. Note that we have only used incompressibility, constant density is not necessary. If the density *is* constant, then $\vec{\nabla}\rho = 0$ and the vorticity equation becomes

$$\vec{\omega}_t = -\vec{u} \cdot \vec{\nabla}\vec{\omega} + \vec{\omega} \cdot \vec{\nabla}\vec{u} + \nu \nabla^2 \vec{\omega}. \tag{10.35}$$

If $\vec{\omega}$ is known and (10.34) is solved for $\vec{u}$, the right hand side is known. Thus $\vec{\omega}_t$ is known and we can integrate forward in time. Thus, *the time evolution of the velocity and vorticity fields can be found without consideration of the pressure field. For an constant density flow we need only solve three equations, not four (three components of the momentum equation plus incompressibility equation) as originally thought!* This is a truly remarkable result, particularly when one considers the physical meaning of the momentum equation. The momentum equation is a statement of Newton's 2nd law: mass times acceleration equals the net force. The pressure gradient is part of the net force – surely we need to know what the force acting of a particle is in order to determine how it will accelerate? The above discussion shows that this is not the case!

Solution of Poisson's Equation in an Unbounded Domain

In order to solve the vorticity equation we need to solve Poisson's Equation (10.34) for $\vec{u}$. This can be done. In a closed vessel this is a difficult problem. We consider the case of an unbounded domain.

Consider Poisson's Equation

$$\nabla^2 f(\vec{x}) = S(\vec{x}) \tag{10.36}$$

in an unbounded domain. We assume that S goes to zero as $|\vec{x}| \to \infty$ sufficiently rapidly. We will see how rapidly it must go to zero shortly. To find the solution of (10.36) we first consider some elementary solutions of Laplace's Equation, the homogeneous form of Poisson's Equation. Recall that homogeneous solutions of a non-homogeneous ODE play an important role in finding the solution of the non-homogeneous ODE. The same is true for Poisson's Equation. The homogeneous form of Poisson's Equation is Laplaces equation for which

$$G(\vec{x}) = \frac{1}{4\pi|\vec{x}|} \tag{10.37}$$

is an elementary solution in R^3, except at $|\vec{x}| = 0$. To check this note that

$$\frac{\partial^2 G}{\partial x_i^2} = \frac{\partial}{\partial x_i}\Big(-\frac{x_i}{4\pi|\vec{x}|^3}\Big) = -\frac{1}{4\pi|\vec{x}|^3} + 3\frac{x_i^2}{4\pi|\vec{x}|^5}.$$

Summing over i gives

$$\nabla^2 G = -\frac{3}{4\pi|\vec{x}|^3} + 3\frac{x_i x_i}{4\pi|\vec{x}|^5} = -\frac{3}{4\pi|\vec{x}|^3} + \frac{3}{4\pi|\vec{x}|^3} = 0. \tag{10.38}$$

The function G is a solution of Laplace's Equations in three dimensions only. The 2D version will be discussed later.[1]

Since $\nabla^2 G = 0$, except at the origin, we have

$$0 = \iiint\limits_{R^3 - B_\epsilon(\vec{x})} \nabla_y^2 G(\vec{x}-\vec{y}) f(\vec{y}) d\vec{y}, \tag{10.39}$$

where $B_\epsilon(\vec{x})$ is the ball of radius ϵ centred at $|\vec{x}|$ (see Figure 10.20). The integration is with respect to the y's. That is $\vec{x}$ is held fixed. In the integrand of the above equations $G(\vec{x}-\vec{y})$ is undefined at $\vec{y} = \vec{x}$ so we have removed a small spherical volume centred at $\vec{x}$. Now

$$f\nabla^2 G = \vec{\nabla}\Big(f\vec{\nabla}G - G\vec{\nabla}f\Big) + G\nabla^2 f, \tag{10.40}$$

so, using the divergence theorem and (10.36), we have

$$\begin{aligned} &\iint\limits_{\partial B_\epsilon} \Big(f(\vec{y})\vec{\nabla}_y G(\vec{x}-\vec{y}) - G(\vec{x}-\vec{y})\vec{\nabla}_y f(\vec{y})\Big)\cdot\hat{n}\,dS \\ &\qquad - \lim_{R_o\to\infty} \iint\limits_{|\vec{x}-\vec{y}|=R_o} \Big(f(\vec{y})\vec{\nabla}_y G(\vec{x}-\vec{y}) - G(\vec{x}-\vec{y})\vec{\nabla}_y f(\vec{y})\Big)\cdot\hat{n}\,dS \\ &\qquad = \iiint\limits_{R^3 - B_\epsilon} G(\vec{x}-\vec{y}) S(\vec{y})\,d\vec{y}. \end{aligned} \tag{10.41}$$

[1]In two dimensions the sum over i of $1/4\pi|\vec{x}|^3$ would give $2/4\pi|\vec{x}|^3$.

Here $\hat{n}$ points away from $\vec{x}$ on both surfaces.

The solution of Poisson's Equation is not unique, since there are many solutions of the homogeneous Laplace Equation, e.g., $f_h = a + b_i x_i + c_1 x_1^2 + c_2 x_2^2 + c_3 x_3^2$ for any constants a, b_i and c_i provided $c_1 + c_2 + c_3 = 0$. To obtain a unique solution of (10.36) it is necessary to specify the far field behaviour. We will *assume* that $f \to 0$ as $|\vec{x}| \to \infty$.

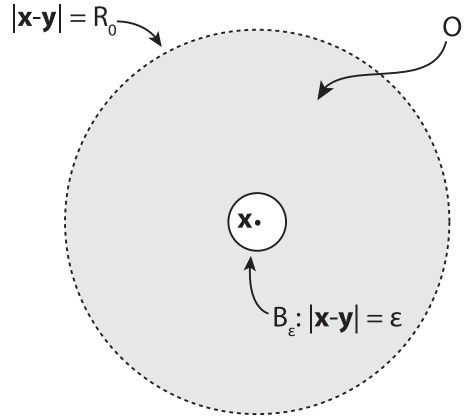


Figure 10.20:

Now $G \to 0$ like $1/r$ as $r = |\vec{x} - \vec{y}| \to \infty$ and $\vec{\nabla} G \to 0$ like $1/r^2$. Since the surface area of $|\vec{x} - \vec{y}| = R_o$ is $4\pi R_o^2$, the assumption that $f \to 0$ as $r \to \infty$ means that the surface integral over $r = R_o$ vanishes in the limit $R_o \to 0$. This leaves us with

$$\iint\limits_{\partial B_\epsilon} \Big(f(\vec{y}) \vec{\nabla}_y G(\vec{x} - \vec{y}) - G(\vec{x} - \vec{y}) \vec{\nabla}_y f(\vec{y}) \Big) \cdot \hat{n}\, dS = \iiint\limits_{R^3 - B_\epsilon} G(\vec{x} - \vec{y}) S(\vec{y})\, d\vec{y}, \tag{10.42}$$

Now

$$\vec{\nabla}_y G(\vec{x} - \vec{y}) = -\frac{1}{4\pi} \frac{\vec{y} - \vec{x}}{|\vec{x} - \vec{y}|^3}, \tag{10.43}$$

and

$$\hat{n} = \frac{\vec{y} - \vec{x}}{|\vec{y} - \vec{x}|} = \frac{\vec{y} - \vec{x}}{\epsilon}, \tag{10.44}$$

so

$$\vec{\nabla}_y G(\vec{x} - \vec{y}) \cdot \hat{n} = -\frac{(\vec{y} - \vec{x}) \cdot (\vec{y} - \vec{x})}{4\pi\epsilon^4} = -\frac{1}{4\pi\epsilon^2}, \tag{10.45}$$

on ∂B_ϵ, which is the surface $|\vec{y} - \vec{x}| = \epsilon$. Note the denominator is the surface area of B_ϵ. Using these results we have

$$\iint\limits_{\partial B_\epsilon} f(\vec{y})\vec{\nabla}_y G(\vec{x}-\vec{y})\cdot\hat{n}\,dS = -\iint\limits_{\partial B_\epsilon} \frac{f(\vec{y})}{4\pi\epsilon^2}\,dS, \tag{10.46}$$

and,

$$\iint\limits_{\partial B_\epsilon} G(\vec{x}-\vec{y})\vec{\nabla}_y f(\vec{y})\cdot\hat{n}\,dS = \iint\limits_{\partial B_\epsilon} \frac{1}{4\pi\epsilon}\vec{\nabla}_y f(\vec{y})\cdot\hat{n}\,dS. \tag{10.47}$$

In the latter we can pull out the factor $1/4\pi\epsilon$ and use the divergence theorem again, using B_ϵ as the volume, to get

$$\iint_{\partial B_\epsilon} G(\vec{x}-\vec{y})\vec{\nabla}_y f(\vec{y})\cdot\hat{n}\,dS = \frac{1}{4\pi\epsilon}\iiint\limits_{B_\epsilon} \nabla^2 f(\vec{y})\,dV. \tag{10.48}$$

This assumes that the second derivatives of f exist inside B_ϵ. We now use the mean value theorem to say that

$$-\frac{1}{4\pi\epsilon^2}\iint\limits_{B_\epsilon} \nabla^2 f(\vec{y})\,dV = -f(\vec{y}'), \tag{10.49}$$

for some $\vec{y}'$ on the surface $|\vec{x}-\vec{y}| = 0$, and that

$$\frac{1}{4\pi\epsilon}\iint\limits_{B_\epsilon} \nabla^2 f(\vec{y})\,dV = \frac{\epsilon^2}{3}\nabla^2 f(\vec{y}''), \tag{10.50}$$

for some $\vec{y}''$ inside B_ϵ. This step assumes that f is C^2 (or more precisely than $\nabla^2 f = S$ is continuous).

We now take the limit $\epsilon \to 0$ of equation (10.42). The limit of the first part of the integral on the left hand side gives, using (10.46) and (10.49),

$$\lim_{\epsilon\to 0}\iint\limits_{\partial B_\epsilon} f(\vec{y})\vec{\nabla}_y G(\vec{x}-\vec{y})\cdot\hat{n}\,dS = \lim_{\epsilon\to 0} f(\vec{y}') = f(\vec{x}), \tag{10.51}$$

since $\vec{y}' \to \vec{x}$ as $\epsilon \to 0$. Next, the second piece of the integral, using (10.47) and (10.50), is

$$-\lim_{\epsilon\to 0}\iint\limits_{\partial B_\epsilon} G(\vec{x}-\vec{y})\vec{\nabla}_y f(\vec{y})\cdot\hat{n}\,dS = -\lim_{\epsilon\to 0}\frac{\epsilon^2}{3}\nabla^2 f(\vec{y}'') = 0, \tag{10.52}$$

since $f(\vec{y}'')$ also goes to $f(\vec{x})$ as $\epsilon \to 0$. Thus,

$$f(\vec{x}) = -\iiint\limits_{R^3} G(\vec{x}-\vec{y})S(\vec{y})\,d\vec{y} = -\iiint\limits_{R^3} \frac{S(\vec{y})}{4\pi|\vec{x}-\vec{y}|}\,d\vec{y}. \tag{10.53}$$

This, then, gives the solution $f(\vec{x})$ in terms of an integral over R^3 involving the source term $S(\vec{y})$. The integrand has a singularity at $\vec{y} = \vec{x}$. From the above derivation we see that the integral must be defined by removing a ball of radius ϵ centred at $\vec{x}$ and taking the limit as $\epsilon \to 0$. To derive this result we have assumed that S is continuous and that $f \to 0$ as $|\vec{x}| \to \infty$. The latter implies that S goes to zero faster than $1/|\vec{x}|^2$.

The function G is called a Green's Function. In R^2 the Green's Function is

$$G(\vec{x}) = \frac{\ln|\vec{x}|}{2\pi}, \tag{10.54}$$

and

$$f(\vec{x}) = \iint\limits_{R^2} \frac{\ln|\vec{x}-\vec{y}|}{2\pi} S(\vec{y})\,d\vec{y}. \tag{10.55}$$

Verification is left as a useful exercise.

Back to the vorticity

We now return to equation (10.34) which we want to solve for u_i:

$$\nabla^2 u_i = -(\vec{\nabla}\times\vec{\omega})_i. \tag{10.56}$$

Using (10.53)

$$u_i(\vec{x}) = \iiint\limits_{R^3} \frac{\epsilon_{ijk}}{4\pi|\vec{x}-\vec{y}|}\frac{\partial\omega_k}{\partial y_j}(\vec{y})\,d\vec{y}. \tag{10.57}$$

Integrating by parts, assuming that $\vec{\omega}$ and its derivatives go to zero at infinity,

$$u_i(\vec{x}) = -\iiint\limits_{R^3} \frac{\epsilon_{ijk}\omega_k}{4\pi}\frac{\partial}{\partial y_j}\frac{1}{|\vec{x}-\vec{y}|}\,d\vec{y} = \iiint\limits_{R^3} \frac{\epsilon_{ijk}\omega_k}{4\pi}\frac{y_j - x_j}{|\vec{x}-\vec{y}|^3}\,d\vec{y}. \tag{10.58}$$

Now $\epsilon_{ijk}\omega_k(y_j - x_j)$ is the i^{th} component of $(\vec{x}-\vec{y})\times\vec{\omega}$ so

$$\vec{u}(\vec{x}) = -\iiint\limits_{R^3} \frac{(\vec{x}-\vec{y})\times\vec{\omega}}{4\pi|\vec{x}-\vec{y}|^3}\,d\vec{y}. \tag{10.59}$$

This is called the Biot-Savart Law. This integral also arises in electromagnetism.

Chapter 11

APPLICATIONS

11.1 The Bernoulli Function and Bernoulli's Equation

Consider the momentum equation

$$\vec{u}_t + \vec{\nabla}\Big(\frac{1}{2}\vec{u}\cdot\vec{u}\Big) + \vec{\omega}\times\vec{u} = -\frac{1}{\rho}\vec{\nabla}p - \vec{\nabla}\Pi + \nu\nabla^2\vec{u}. \tag{11.1}$$

Assume that $\rho = \rho(p)$. This includes the two important cases of a constant density fluid and an ideal gas of constant entropy. Let

$$F(p) = \int^p \frac{ds}{\rho(s)}. \tag{11.2}$$

Then

$$\vec{\nabla}F(p) = \frac{dF}{dp}\vec{\nabla}p = \frac{1}{\rho}\vec{\nabla}p. \tag{11.3}$$

Thus (11.1) can be written as

$$\vec{u}_t + \vec{\nabla}\Big(\frac{1}{2}\vec{u}\cdot\vec{u} + F(p) + \Pi\Big) = \vec{u}\times\vec{\omega} + \nu\nabla^2\vec{u},$$

or as

$$\vec{u}_t + \vec{\nabla}B = \vec{u}\times\vec{\omega} + \nu\nabla^2\vec{u}, \tag{11.4}$$

where

$$B = \frac{1}{2}\vec{u}\cdot\vec{u} + F(p) + \Pi, \tag{11.5}$$

is called the Bernoulli function.

SPECIAL CASES:

Case A: *Steady inviscid flow.* If the flow is steady and the viscous term $\nu\nabla^2\vec{u}$ is zero then

$$\vec{\nabla}B = \vec{u} \times \vec{\omega}. \tag{11.6}$$

Now $\vec{u}\times\vec{\omega}$ is perpendicular to both the streamlines and to the vortex lines, since $\vec{u}$ and $\vec{\omega}$ are tangent to these curves. Thus B is constant along both streamlines and vortex lines in a *steady, inviscid* flow. B in general will have different values along different streamlines and vortex lines.

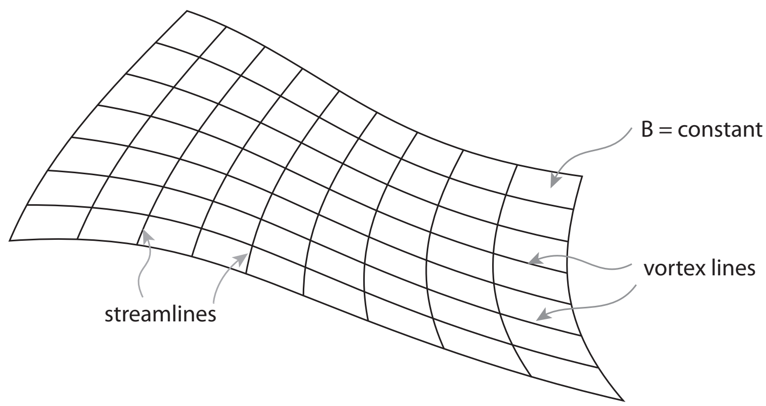


Figure 11.1: Streamlines and vortex lines on a surface of constant B.

Case B: *Steady, irrotational, inviscid flow*. If the flow is also irrotational then

$$\vec{\nabla}B = 0,$$

and hence B is constant everywhere, i.e.,

$$\frac{1}{2}\vec{u}\cdot\vec{u} + F(p) + \Pi = \text{constant}. \tag{11.7}$$

Note that if the density is constant the vector identity $\vec{\nabla}\times\vec{\nabla}\times\vec{u} = \vec{\nabla}(\vec{\nabla}\cdot\vec{u}) - \nabla^2\vec{u}$ implies that $\mu\nabla^2\vec{u} = -\mu\vec{\nabla}\times\vec{\omega} = 0$ if the flow is irrotational. Hence for a constant density fluid we do not need to that the fluid is inviscid if the flow is irrotational. Irrotational and incompressible flows are very special.

Case C: *Unsteady, irrotational, inviscid flow*. If $\vec{\nabla}\times\vec{u} = \vec{\omega} = 0$ then $\vec{u} = \vec{\nabla}\phi$ for some function ϕ (called the velocity potential), so we have

$$\vec{\nabla}\phi_t + \vec{\nabla}B = 0.$$

Hence

$$\phi_t + B = G(t), \tag{11.8}$$

for some function G of t. Since only spatial gradients of ϕ are physically meaningful $G(t)$ can be taken to be zero.

Equations (11.6), (11.7), and (11.8) are all different versions of *Bernoulli's Equation*.

Example 1: Consider a fluid at rest. Then $\vec{u} = \vec{\omega} = 0$ and $B = F(\rho) + \Pi$ is a constant. For a constant density fluid $F(\rho) = p/\rho$ and using $\Pi = gz$ we have, after multiplying B by the constant ρ,

$$p = -\rho g z + C,$$

for some constant C. This is simply hydrostatic pressure.

Example 2: Consider steady laminar flow down a horizontal pipe. A pitot tube is inserted through a hole in the pipe (see diagram). Consider two points 1 and 2 in the fluid, where point 2 is at the stagnation point at the front of the pitot tube and point 1 is upstream along the streamline that terminates at the stagnation point 2. Fluid fills the pipe to a height h_2 above the top of the pipe. Fluid also fills a vertical column above point 1 to a height h_1.

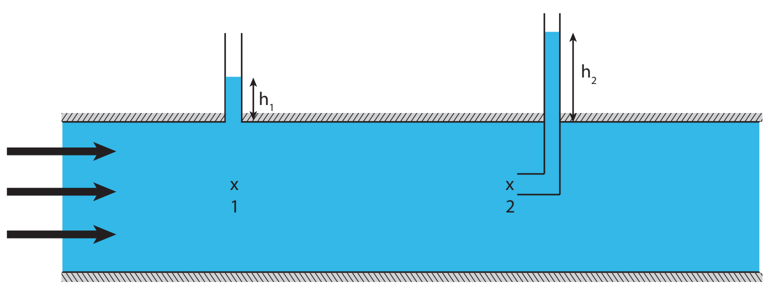


Figure 11.2: A pitot tube.

Now, *if* viscous effects are negligible, which implies that the viscous boundary layer along the walls of the pipe is thin — i.e., points 1 and 2 are not in the boundary layer, or the viscous term is negligible compared with the pressure gradient between points 1 and 2, then B is constant along the streamline passing through points 1 and 2. Hence

$$\frac{1}{2}q_1^2 + F(p_1) + gz_1 = \frac{1}{2}q_2^2 + F(p_2) + gz_2,$$

where $q = |\vec{u}|$ is the speed of the fluid and the subscripts refer to values at points 1 and 2. Normally $z_1 = z_2$. This would certainly be the case if point 1 is in the centre of a circular pipe and the portion of the pitot tube running along the pipe is long compared with the pitot tube diameter.

Special Case: If the fluid has constant density we have

$$\frac{1}{2}q_1^2 + \frac{p_1}{\rho} = \frac{1}{2}q_2^2 + \frac{p_2}{\rho}.$$

But $q_2 = 0$ so

$$\frac{1}{2}q_1^2 = \frac{p_2 - p_1}{\rho}.$$

Now in the vertical direction the fluid pressure is hydrostatic above point 1 (assuming flow is in x direction only). Thus $p_1 = p_a + \rho g(d + h_1$ where d is the distance of point 1 from the top of the pipe. In the pitot tube the fluid is not moving and hence $\vec{\nabla} p + \rho g \hat{k} = \vec{\nabla}(p + \rho g z) = 0$ inside the tube. This means that $p_2 = p_a + \rho g(d + h_2)$. Hence

$$q_1 = \sqrt{2g(h_2 - h_1)}. \tag{11.9}$$

11.2 Potential Flow

A problem of central and historic importance in fluid mechanics is that of flow past a solid body B. This problem was initially modeled assuming the flow is

- incompressible and of constant density,
- inviscid,
- irrotational.

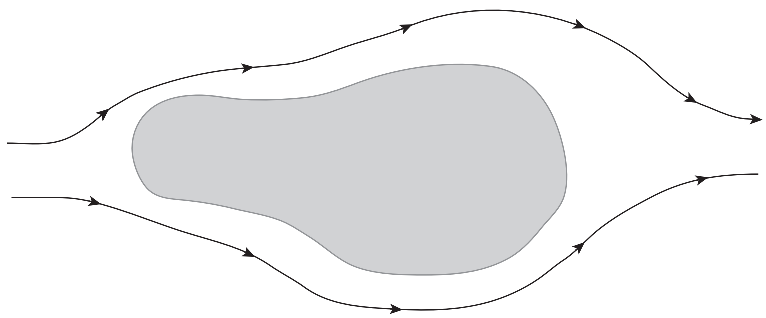


Figure 11.3: Flow past a bluff body.

Since the vorticity $\vec{\omega} = \vec{\nabla} \times \vec{u}$ is assumed to be zero it follows that a velocity potential ϕ exists such that $\vec{u} = \vec{\nabla}\phi$. Incompressibility then gives $\vec{\nabla} \cdot \vec{u} = \vec{\nabla} \cdot \vec{\nabla}\phi = 0$, so the velocity potential ϕ is a solution of Laplace's equation

$$\nabla^2 \phi = 0. \tag{11.10}$$

Usually we will be interested in flow past an object which has no angular velocity (e.g., an airplane or, in 2-D, an airplane wing). In this case it is useful to use a reference frame fixed with the object. Assuming uniform flow $\vec{U}$ at ∞ (so the velocity of the object through the fluid at rest is $-\vec{U}$), the problem for ϕ becomes: solve Laplace's equation outside the object B (we assume the fluid fills all of space outside the object) subject to the no normal flow boundary condition

$$\vec{\nabla} \cdot \hat{n} = 0 \quad \text{on } \partial B, \tag{11.11}$$

and the far field condition

$$\vec{\nabla}\phi \to \vec{U} \quad \text{as } \vec{x} \to \infty. \tag{11.12}$$

Note that the no-slip boundary condition has been dropped. For an inviscid fluid the only boundary condition that must be satisfied at a solid boundary is that of no normal flow. The loss of a boundary condition is due to the fact that the viscous terms in the momentum equation involve second-order derivatives. Without these terms the highest derivatives appearing in the equation are first order. A reduction in the order of the equation implies a reduction in the number of boundary conditions. Another point to make is that gravity plays no role.

This problem has great mathematical appeal: (i) It is linear, (ii) lots is known about Laplace's equation, (iii) time appears as a parameter only ($\vec{U}$ could be a function of time). Because of this it was the subject of much study. It can be proved that any solution of (11.10)–(11.12) provides no force on the object: e.g., air can not exert a force on an airplane to balance the gravitational force. This result was used by some mathematicians at the turn of the century to prove that it is impossible to build an airplane that can fly (this despite the obvious that some birds glide and soar without flapping their wings). How was the case argued for the validity of the potential flow equations? It is not to difficult to show that for low Mach number flow ($M^2 << 1$ where $M = U/c$ is the ratio of the velocity of the object through a fluid at rest and c is the sound speed in the fluid) the flow can very accurately be treated as incompressible and of constant density. The problem then, is the latter two assumptions which really belong together. It was argued that for large Reynolds' number flow the viscous term in the momentum equations can be neglected. Without viscosity there is no vorticity generation in a boundary layer and hence if the flow is initially irrotational it remains so. It turns out that the 'small' viscous terms are crucial to explain flight. The viscous terms are in fact far from small in the boundary layer. One may ask how terms that are only important in a very thin boundary layer can have such important repercussions. It is related to the reduction in order of the PDE governing the flow. Dropping terms of order 1/Re for large Reynolds number flow is a *singular perturbation* problem. We will have to leave this story for later.

Why we should study potential flow theory if it is wrong? There are several reasons. First, when an object starts moving through a fluid the fluid flow is initially a potential flow. Boundary layers will form with time, however if they stay thin, such as for flow past a streamlined object such as an airplane wing (Figure 11.4), the flow outside the boundary layer is well approximated by potential theory. Real flows have boundary layers along solid boundaries which are prone to separation. Potential flow theory can be used to predict when and where the flow will separate. For this reason, current state-of-the-art numerical models used to compute flow around an airplane use the potential flow solution as their starting point. Some flows such as water waves are well modeled by potential flow theory since the oscillating flow along the bottom boundary layer continually produces vorticity of opposite signs resulting in a nearly irrotational flow above the bottom. The study of potential flow theory also gives us an understanding of such things as added mass. Thus, potential flow theory gives great insight and provides a useful starting point for more complicated, more realistic flows.

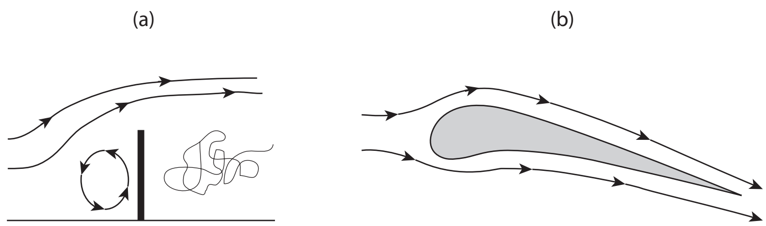


Figure 11.4: (a) For flow past a fence potential theory will be useless. (b) For flow past a streamlined aerofoil potential flow can be an excellent approximation.

Solutions of (11.10)–(11.12) are unique if the region occupied by the fluid is simply connected. By unique I mean up to an additive constant. Because only gradients of ϕ are physically meaningful (ϕ itself has no physical meaning) we are not concerned with additive constants.

Note that if an object moves with translational velocity $\vec{U}(t)$ then time only appears as a parameter in the far field condition. If $\vec{U}(t)$ changes then $\vec{u} = \vec{\nabla}\phi$ changes everywhere instantaneously. This is an artifact of the assumption of incompressibility which makes the sound speed infinite. In a compressible flow, and all flows are compressible, even those of water, information about changes in the object's velocity propagates at the speed of sound.

As a final note, in potential flow theory gravity is usually neglected. This is because if the fluid density is constant any conservative body force can be absorbed into the pressure term via a redefinition of the pressure by

$$\frac{1}{\rho}\vec{\nabla}p + \vec{\nabla}\Pi = \frac{1}{\rho}\vec{\nabla}\Big(p + \rho\Pi\Big) = \frac{1}{\rho}\vec{\nabla}\tilde{p}. \tag{11.13}$$

For the rest of this chapter, we will refer to $\tilde{p}$ as the pressure.

11.2.1 Boundary Conditions for More General Motion

Before considering two special cases, consider an object B moving through a fluid which is at rest at infinity. In general the object will have translational and rotational velocity. Let $\vec{U}_o$ and $\vec{\Omega}$ be the translational and angular velocity of B. Also, let V_o be the volume of B and

$$\vec{x}_o = \frac{1}{V_o}\iiint\limits_B \vec{x}\, dV \tag{11.14}$$

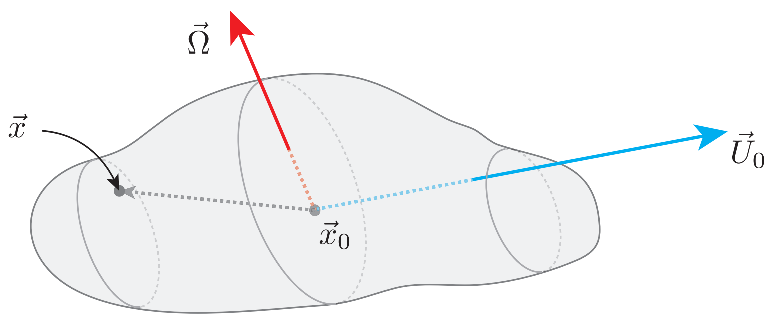


Figure 11.5:

be the centre of volume, or geometric centre, of B. A piece of the surface of B at $\vec{x}$ will have velocity $\vec{U}_s = \vec{U}_o + \vec{\Omega} \times (\vec{x} - \vec{x}_o)$.

No flow through the boundary means that the fluid velocity and the surface velocity of the object must have the same normal velocity (fluid may slip along the boundary as we're assuming an inviscid fluid) so $\vec{u} \cdot \hat{n} = \vec{U}_s \cdot \hat{n}$ on ∂B. Thus, the problem for ϕ is now solve Laplace's equation subject to

$$\vec{\nabla}\phi \cdot \hat{n} = \vec{U}_s \cdot \hat{n} \quad \text{on } \partial B, \tag{11.15}$$

and

$$\vec{\nabla}\phi \to 0 \quad \text{as } |\vec{x}| \to \infty. \tag{11.16}$$

This expresses the problem in a reference frame fixed with the fluid at infinity.

11.2.2 Potential flow past a sphere

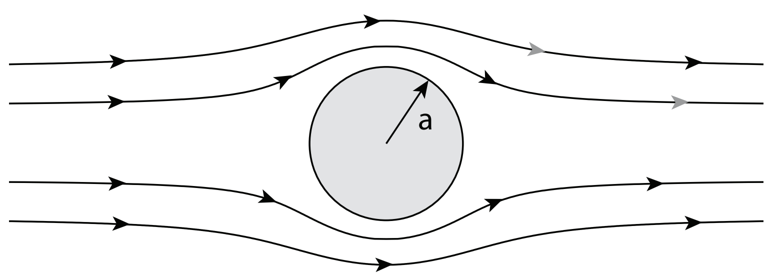


Figure 11.6: Sample streamlines for potential flow past a cylinder.

Consider uniform flow past a sphere of radius a. Streamlines in the xy-plane would look similar to those for flow past a cylinder shown in Figure 11.6. Let the fluid velocity at infinity be given by $\vec{U} = U\hat{\imath}$, that is in the positve x direction, assuming $U > 0$. The problem for ϕ is

$$\nabla^2\phi = 0, \tag{11.17}$$

subject to the boundary conditions

$$\vec{\nabla}\phi \cdot \hat{n} = 0, \quad \text{on } |\vec{x}| = a, \tag{11.18}$$

and

$$\vec{\nabla}\phi \to U\hat{\imath}, \quad \text{as } |\vec{x}| \to \infty. \tag{11.19}$$

It is simplest to solve this problem using spherical coordinates. Let θ be the angle from the $-\hat{\imath}$ and β be the angle from $\hat{\imath}$–$\hat{\jmath}$ plane (Figure 11.7). By symmetry $\phi = \phi(r, \theta)$. Thus

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) = 0. \tag{11.20}$$

Boundary Conditions:

1. $u_r = \frac{\partial\phi}{\partial r}$ and $u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta}$ are periodic in θ with period 2π.
2. $u_\theta \to U\sin\theta$ and $u_r \to -U\cos\theta$ as $r \to \infty$.
3. $u_r = \frac{\partial\phi}{\partial r} = 0$ on the boundary $r = a$.

Because the region outside the sphere is simply connected the flow is unique. Thus we can guess at a solution and if it works we are done. Using the fact that $u_r \to -U\cos\theta$ as $r \to \infty$ as inspiration lets guess that

$$\phi = R(r)\cos\theta, \tag{11.21}$$

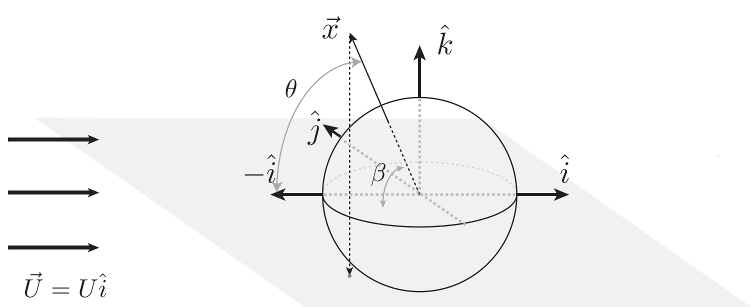


Figure 11.7: Coordinate system for flow past a sphere.

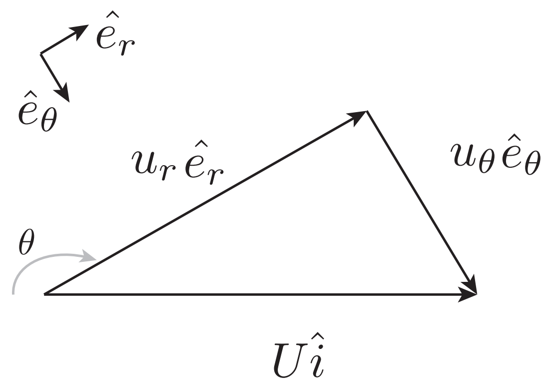


Figure 11.8: Relationship between $U\hat{i}$, u_θ and u_r.

for some function $R(r)$. Such an expression has the right θ dependence in the farfield, satisfies the periodicity requirement and if $R'(a) = 0$ it can satisfy the boundary condition at $r = a$. Substituting this expression into Laplace's equation gives

$$\nabla^2\phi = \left[\frac{1}{r^2}\Big(r^2R'\Big)' - 2\frac{R}{r^2}\right]\cos\theta. \tag{11.22}$$

Thus, $R(r)$ must satisfy the ODE

$$R'' + \frac{2}{r}R' - \frac{2}{r^2}R = 0. \tag{11.23}$$

Letting $R = r^\alpha$ results in two linearly independent solution with $\alpha = 1$ and $\alpha = -2$. Thus

$$\phi = \Big(Ar + \frac{B}{r^2}\Big)\cos\theta. \tag{11.24}$$

We now need to see if A and B can be chosen so that all the boundary conditions are satisfied. The first, periodicity of u_r and u_θ, is obviously satisfied. For the far field behaviour we have

$$u_r = \frac{\partial\phi}{\partial r} = \Big(A - \frac{2B}{r^3}\Big)\cos\theta \to -U\cos\theta \quad \text{as } r \to \infty, \tag{11.25}$$

hence we must have $A = -U$. The other far field condition is

$$u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\Big(A + \frac{B}{r^3}\Big)\sin\theta \to U\sin\theta \quad \text{as } r \to \infty, \tag{11.26}$$

is also satisfied with $A = -U$. The final condition will determine the value of B. We have

$$u_r = \Big(A - 2\frac{B}{r^3}\Big)\cos\theta = 0 \quad \text{at } r = a, \tag{11.27}$$

so $B = -Ua^3/2$. Thus

$$\phi = -U\Big(r + \frac{a^3}{2r^2}\Big)\cos\theta, \tag{11.28}$$

is the velocity potential for flow past a sphere of radius a centred at the origin. In Cartesian coordinates, using $x = -r\cos\theta$,

$$\phi = U\Big(1 + \frac{a^3}{2r^3}\Big)x, \tag{11.29}$$

Writing $Ux = \vec{U}\cdot\vec{x}$, since the far field velocity is in the x direction, we can write the velocity potential as

$$\phi = \Big(1 + \frac{a^3}{2r^3}\Big)\vec{U}\cdot\vec{x}, \tag{11.30}$$

This expression, which is a scalar, is invariant under a coordinate transformation and hence is the velocity potential for arbitrary far field flow $\vec{U}$.

The velocity potential in a reference frame fixed with the fluid at infinity, i..e, in which the sphere has velocity $\vec{U}_0 = -\vec{U}$, is

$$\phi = \Big(1 + \frac{a^3}{2r^3}\Big)\vec{U}\cdot\vec{x} - \vec{U}\cdot\vec{x} = \frac{a^3}{2r^3}\vec{U}\cdot\vec{x}. \tag{11.31}$$

or, in terms of the velocity of the sphere $\vec{U}_0$,

$$\phi = \vec{U}_0\cdot\vec{\Phi} \tag{11.32}$$

where

$$\vec{\Phi}(\vec{x}) = -\frac{1}{2}\frac{a^3}{r^3}\vec{x} = \frac{1}{2}a^3\frac{\vec{x}}{|\vec{x}|^3}. \tag{11.33}$$

This is at the time when the sphere is centred at the origin. For a sphere centred at $\vec{x}_0(t)$ moving with velocity $\vec{U}_0(t) = \frac{\mathrm{d}\vec{x}_0}{\mathrm{d}t}$ the velocity potential is

$$\phi(\vec{x},t) = \vec{U}_0(t)\cdot\vec{\Phi}\big(\vec{x} - \vec{x}_0(t)\big). \tag{11.34}$$

This is true for arbitrary $\vec{U}_0(t)$.

Velocity and pressure fields

In the reference frame fixed with the sphere the velocity field is obtained by taking the gradient of ϕ. Thus

$$u_r = \frac{\partial\phi}{\partial r} = -U\Big(1 - \frac{a^3}{r^3}\Big)\cos\theta, \tag{11.35}$$

and

$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial r} = U\Big(1 + \frac{a^3}{2r^3}\Big)\sin\theta, \tag{11.36}$$

are the velocity fields. Using Bernoulli's Theorem for steady flow we have

$$p + \frac{\rho}{2}(u_r^2 + u_\theta^2) = p_\infty + \frac{\rho}{2}U^2,$$

which gives

$$p - p_\infty = \frac{\rho U^2}{2}\Big[-\frac{a^3}{r^3}\Big(1 - 3\cos^2\theta\Big) - \frac{a^6}{r^6}\Big(\frac{1}{4} + \frac{3}{4}\cos^2\theta\Big)\Big]. \tag{11.37}$$

On the surface of the sphere

$$p - p_\infty = \frac{\rho U^2}{2}\Big(1 - \frac{u_\theta^2}{U^2}\Big) = \frac{\rho U^2}{2}\Big(1 - \frac{9}{4}\sin^2\theta\Big) \tag{11.38}$$

and

$$u_\theta = \frac{3}{2}U\sin\theta. \tag{11.39}$$

On the surface of the sphere there are two stagnation points (points where $\vec{u} = 0$): one at the front ($\theta = 0$) and one at the back ($\theta = \pi$). The excess pressure $p - p_\infty$ is largest at these points. The velocity is maximized and the excess pressure is minimized at $\theta = \pi/2$. By symmetry there is no net force on the sphere.

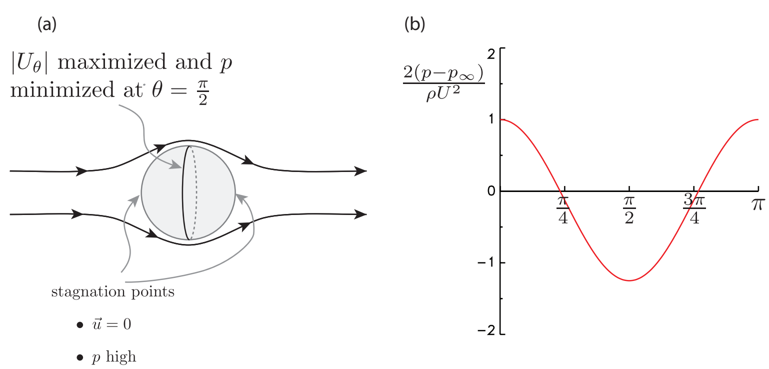


Figure 11.9: (a) Potential flow past a sphere. (b) Pressure distribution as a function of θ.

11.2.3 Potential flow past a cylinder

We now consider potential flow past a cylinder of radius a centred at the origin. The far field flow is in the x direction with speed U.

Cartesian coordinates:

In Cartesian coordinates the problem for the velocity potential ϕ is

$$\begin{aligned}
\nabla^2\phi &= 0, && (11.40)\\
\vec{\nabla}\phi\cdot\hat{n} &= 0 \quad \text{on } |\vec{x}| = a, && (11.41)\\
\vec{\nabla}\phi &\to (U,0) \quad \text{as } |\vec{x}|\to\infty, && (11.42)
\end{aligned}$$

while the problem for the streamfunction Ψ is

$$\begin{aligned}
\nabla^2\psi &= 0, && (11.43)\\
\psi &= 0 \quad \text{on } |\vec{x}| = a, && (11.44)\\
\vec{\nabla}\Psi = (\psi_x,\psi_y) = (-v,u) &\to (0,U) \quad \text{as } |\vec{x}|\to\infty. && (11.45)
\end{aligned}$$

Polar coordinates:

In polar coordinates the problem for the velocity potential ϕ is

$$\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial\phi}{\partial r}\Big) &= 0, && (11.46)\\
u_r = \phi_r &= 0 \quad \text{on } r = a, && (11.47)\\
(u_r,u_\theta) = (\phi_r,\frac{1}{r}\phi_\theta) &\to (U\cos\theta, -U\sin\theta) \quad \text{as } r\to\infty, && (11.48)
\end{aligned}$$

while the problem for the streamfunction Ψ is

$$\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial\psi}{\partial r}\Big) &= 0, && (11.49)\\
\psi &= 0 \quad \text{on } r = a, && (11.50)\\
(u_r,u_\theta) = (\frac{1}{r}\psi_\theta, -\psi_r) &\to (U\cos\theta, -U\sin\theta) \quad \text{as } r\to\infty. && (11.51)
\end{aligned}$$

Figure 11.10 shows the two coordinate systems.

The velocity potential

We could proceed as we did for the problem of potential flow past a sphere by guessing but in the 2D case the solution is not unique because the domain $r\ge a$ is not simply connected. Looking for separable solutions of Laplaces equation of the form $\phi = R(r)\Theta(\theta)$ leads to

$$\frac{\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right)}{R} + \frac{1}{r^2}\frac{d^2\Theta}{d\theta^2} = 0 \qquad (11.52)$$

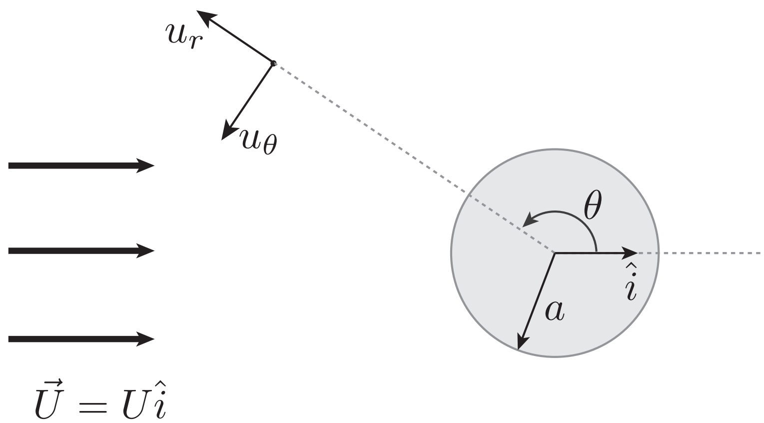


Figure 11.10: Coordinate system for potential flow past a cylinder of radius a. Far field flow is in the $\hat{\imath}$ direction (positive x).

or

$$\frac{r(rR')'}{R} + \frac{\Theta''}{\theta} = 0. \tag{11.53}$$

This has the form of a function of r plus a function of θ is equal to zero hence both terms must be constant, so

$$\frac{\Theta''}{\theta} = -\frac{r(rR')'}{R} = \lambda \tag{11.54}$$

where λ is an undetermined constant.

Case (a): If $\lambda > 0$ then $\Theta = Ae^{\sqrt{\lambda}} + Be^{-\sqrt{\lambda}}$. The radial velocity u_r must be periodic in θ (the potential ϕ need not be). Thus $u_r = R'\Theta$ must be periodic. This implies that Θ is a periodic function of θ hence θ so $A = B = 0$. So $\lambda > 0$ is not possible.
Case (b): If $\lambda < 0$ then $\Theta = A\cos\sqrt{-\lambda} + B\sin\sqrt{-\lambda})$. This will give a periodic u_r provided $\sqrt{-\lambda}$ is an integer, i.e., $\lambda = -n^2$ for $n = 1, 2, 3, \ldots$ and $\Theta = A\cos(n\theta) + B\sin(n\theta)$. The problem for R is $r(rR')' = n^2R$ or $r^2R'' + rR' - n^2R = 0$. Letting $R = r^\alpha$ leads to $\alpha = \pm n$. Thus $R = Cr^n + Dr^{-n}$. Thus

$$\Big(Cr^n + Dr^{-n}\Big)\Big(A\cos(n\theta) + B\sin(n\theta)\Big) \tag{11.55}$$

is a solution of Laplaces equation for each integer n. Since Laplace's equation is linear we can take a linear combination of such solutions yielding

$$\sum_{n=1}^{\infty}\Big(C_nr^n + D_nr^{-n}\Big)\Big(A_n\cos(n\theta) + B_n\sin(n\theta)\Big). \tag{11.56}$$

Case (c): If $\lambda = 0$ then $\Theta'' = 0$ so $\Theta = A_0 + B_0\theta$. The problem for R is $(rR')' = 0$ which has the solution $R = C_0 \ln r + D_0$. Thus for $\lambda = 0$ we have

$$\Big(C_0 \ln r + D_0\Big)\Big(A_0 + B_0\theta\Big) \tag{11.57}$$

as a solution of Laplace's equation. Note that the requirement that $u_r = \frac{1}{r}C_0\Big(A_0 + B_0\theta\Big)$ is periodic in θ requires that $C_0 = 0$ or $B_0 = 0$. We will see shortly that $C_0 = 0$.

The general solution of Laplace's equation is

$$\phi = \Big(D_0 + C_0 \ln r\Big)\Big(A_0 + B_0\theta\Big) + \sum_{n=1}^{\infty}\Big(C_n r^n + D_n r^{-n}\Big)\Big(A_n \cos(n\theta) + B_n \sin(n\theta)\Big). \tag{11.58}$$

We now apply the boundary conditions. At $r = a$ we have $\phi_r = 0$. Using our general expression for ϕ this gives

$$\frac{C_0}{a}\Big(A_0 + B_0\theta\Big) + \sum_{n=1}^{\infty}\Big(nC_n a^{n-1} - \frac{nD_n}{a^{n+1}}\Big)\Big(A_n \cos(n\theta) + B_n \sin(n\theta)\Big) = 0. \tag{11.59}$$

By uniqueness of trigonometric series and periodicity in θ we have $C_0 = 0$ and $D_n = C_n a^{2n}$. Thus, after redefining A_0 and B_0 to absorb D_0, the solution is

$$\phi = A_0 + B_0\theta + \sum_{n=1}^{\infty} C_n\big(r^n + \frac{a^{2n}}{r^n}\big)\Big(A_n \cos(n\theta) + B_n \sin(n\theta)\Big). \tag{11.60}$$

Next we use $\phi_r \to U\cos\theta$ as $r \to \infty$ to obtain $C_2 = C_3 = C_4 = \cdots = 0$ and $B_1 = 0$. We can also drop the constant A_0 as it is not physically relevant to get the final solution

$$\phi = B_0\theta + C_1\big(r + \frac{a^2}{r}\big)\cos\theta. \tag{11.61}$$

From this

$$u_r = \phi_r = C_1\big(1 - \frac{a^2}{r^2}\big)\cos\theta \to C_1\cos\theta \tag{11.62}$$

as $r \to \infty$ hence the boundary condition $u_r \to U\cos\theta$ as $r \to \infty$ gives $C_1 = U$. Thus our final solution is

$$\phi = B_0\theta + U\big(r + \frac{a^2}{r}\big)\cos\theta. \tag{11.63}$$

The constant B_0 is undetermined. The solution is not unique.

Circulation around the cylinder

The circulation about a circular path of radius $b > a$ centred on the cylinder is

$$\begin{aligned}\Gamma = \oint r = b\vec{u}\cdot d\vec{r} &= \int\limits_{r=b} u_\theta \, ds \\ &= \int_0^{2\pi} \Big[\frac{B_0}{b} - U(1+\frac{a^2}{b^2})\sin\theta\Big] a\, d\theta \\ &= 2\pi B_0.\end{aligned} \tag{11.64}$$

The constant B_0 determines the circulation around the cylinder. In terms of Γ the velocity potential is

$$\phi = \frac{\Gamma}{2\pi}\theta + U\big(r + \frac{a^2}{r}\big)\cos\theta. \tag{11.65}$$

The streamfunction

We could proceed as before however it is simpler to use our expression for ϕ. From this we know that

$$\psi_r = -u_\theta = -\frac{1}{r}\phi_\theta = -\frac{\Gamma}{2\pi r} + U\big(1 - \frac{a^2}{r^2}\big)\sin\theta. \tag{11.66}$$

Integrating gives

$$\psi = -\frac{\Gamma}{2\pi}\ln r + U\big(r - \frac{a^2}{r}\big)\sin\theta + F(\theta). \tag{11.67}$$

Using $\psi_\theta = ru_r = r\phi_r$ then gives $F' = 0$ so F is a constant. Setting $\psi = 0$ on $r = a$ determines the value of the constant F. This gives the solution

$$\psi = -\frac{\Gamma}{2\pi}\big(\ln r - \ln a\big) + U\big(r - \frac{a^2}{r}\big)\sin\theta.$$

The value of ψ on the boundary $r = a$ does not matter physically (only gradients of ψ and ϕ are physically relevant) so the $\ln a$ term is usually neglected and we take

$$\psi = -\frac{\Gamma}{2\pi}\ln r + U\big(r - \frac{a^2}{r}\big)\sin\theta. \tag{11.68}$$

Complex potential

In two-dimensions irrotational incompressible flow has a complex potential $W(z) = \phi + i\psi$. Using our solutions above it can be shown that

$$W(z) = -i\frac{\Gamma}{2\pi}\ln z + U\big(z + \frac{a^2}{z}\big). \tag{11.69}$$

This is left as an exercise.

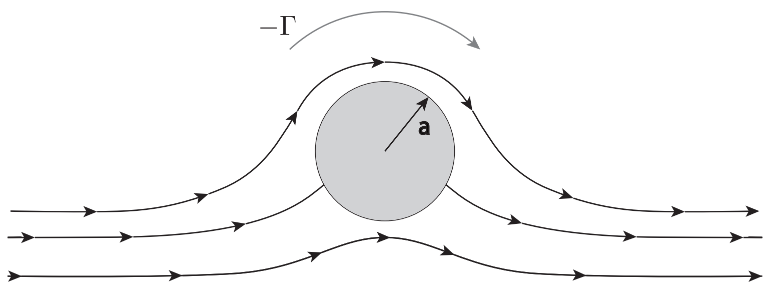


Figure 11.11: Sample streamlines for potential flow past a cylinder with negative (clockwise) circulation.

Interpretation

For $\Gamma = 0$ the flow is symmetric about the x-axis as shown in Figure 11.12). The stagnation points where $u_r = u_\theta$ are on the x-axis at $(x, y) = (\pm a, 0)$. These are the locations of highest pressure. The maximum velocities occur at $(x, y) = (0, \pm a)$ where the pressure is lowest.

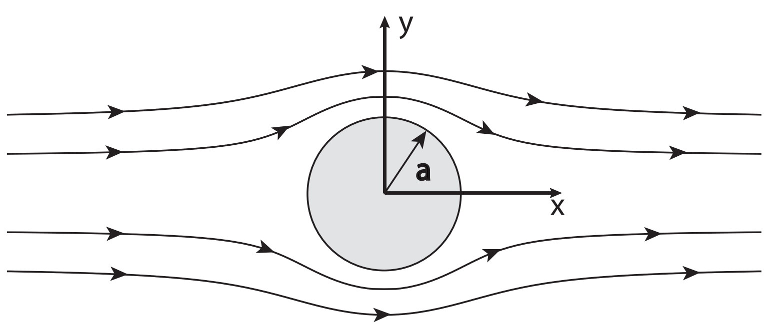


Figure 11.12: Sample streamlines for potential flow past a cylinder for the case with no circulation ($\Gamma = 0$).

$-\frac{\Gamma}{2\pi} \ln r$ is the complex potential for a point vortex centred at the origin. We can add a point vortex of arbitrary strength because $r = 0$ is a streamline. A schematic showing streamlines for $\Gamma < 0$ (clockwise circulation) is shown in Figure 11.11. The streamlines terminating at the stagnation points are shown.

For $\Gamma \neq 0$

- the flow is not symmetric about $y = 0$.
- streamlines are symmetric about $x = 0$.
- stagnation points are below the x axis is $\Gamma < 0$ and above if $\Gamma > 0$.

Chapter 12

THERMODYNAMICS AND EQUATIONS OF STATE

The state of a given mass of fluid is in equilibrium if its properties, e.g., temperature, density, pressure, viscosity, etc., are spatially and temporally uniform. These properties of the fluid are called state variables. In many cases (water, dry air) the state of the fluid can be uniquely specified by the values of two state variables. For example, the state of a fixed mass of an ideal gas, or of pure water, is specified by its temperature and pressure or by its pressure and volume. Other properties of the fluid can be expressed as a function of any two state variables. For example, for an ideal gas and for pure water the density is a function of the temperature and pressure, that is

$$\rho = \rho(p, T). \tag{12.1}$$

Many important liquids require more than two state variables. The density of sea water depends on the pressure, temperature and on the salinity. Technically the density depends on the concentrations of several different salts, however in the ocean the relative concentrations of the different salts are almost uniform so the dependence can be lumped into one term, the salinity s. Another important fluid which requires more than two state variables is air in the atmosphere, which depends on the humidity (and on the concentration of various pollutants) in addition to the pressure and temperature.

In the following discussion we will restrict our attention to fluids whose state is determined by two state variables. As a familiar simple example we will largely focus on ideal gases. Two convenient state variables are the pressure p and the specific volume $v = V/m = 1/\rho$ where V and m are the volume and mass of the gas. For an ideal gas an equation of state is

$$pV = nRT \quad \text{or} \quad T = \frac{1}{R_g} pv, \tag{12.2}$$

where

$$R_g = \frac{nR}{m}. \tag{12.3}$$

12.1 First law of thermodynamics: Conservation of Energy

The first law of thermodynamics is a statement of conservation of energy. It says that

$$\Delta e = \Delta Q + W, \tag{12.4}$$

where

- Δe is the change in internal energy per unit mass;
- ΔQ is the heat added per unit mass;
- W is the work done on the fluid per unit mass.

Example: Consider a fluid in a cylinder with a movable piston of area A.

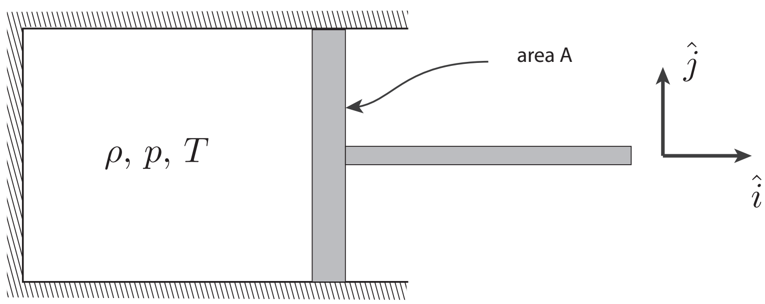


Figure 12.1:

An external force $-pA\,\hat{\imath}$ must be applied to the piston to keep it in position. Suppose the fluid is brought to another state, (p_f, v_f), through a series of small steps such that the gas is always at equilibrium. The path taken by the state of the fluid can be indicated by a curve on a pv-diagram (Figures 12.2).

Work done: For a small piston displacement $dx\,\hat{\imath}$ the work done by the external force is $-pA\hat{\imath}\cdot dx\,\hat{\imath} = -pA\,dx = -p\,dV$ where dV is the change in volume. Thus, the work done per unit mass is

$$dW = -\frac{p\,dV}{m} = -p\,dv.$$

The total work done, per unit mass, is then

$$W = -\int_{v_o}^{v_f} p(v)\,dv.$$

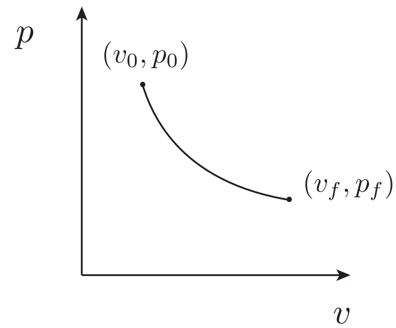


Figure 12.2: Possible state path followed by a fluid as it goes from (v_0, p_0) to (v_f, p_f).

Internal energy is a state variable, that is, it depends only on the state of the gas and not on how the state was reached. Thus, the change in internal energy Δe between the initial and final states depends only on the initial and final states, it does not depend on the path between them. The work done does depend on the path, and hence so does ΔQ, the heat added to the system. W and ΔQ are not state variables.

Definition: *Curves on the pv-diagram along which the temperature is constant are called* **isotherms.**

To find the isotherms, place a cylinder of fluid in thermal contact with an infinite body at a fixed temperature T_o. At equilibrium the temperature of the fluid is also T_o. Next, increase v while allowing heat exchange with the infinite body so that the temperature of the fluid stays constant. The path traced out by the state of the fluid on the pv-diagram is $p(v, T_o)$. Such curves are isotherms. Generally, e.g., for an ideal gas, p decreases as v increases.

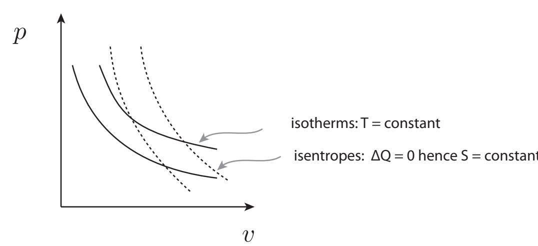


Figure 12.3: Schematic of isentropes and isotherms for an ideal gas.

Definition: *Another set of curves can be constructed by changing the state of the fluid without allowing any heat exchange. These curves are called* **isentropes**. *They are curves of constant* **entropy**.

12.2 Entropy and the Second law of thermodynamics

There is a property of matter S, called entropy, such that

$$T\,dS = dQ, \tag{12.5}$$

for *reversible* processes. Here T is in degrees Kelvin. The second law states that for any change of state S can only increase for an isolated system.

Example: Consider two pieces of matter in thermal contact but thermally isolated from anything else. One is cold with temperature T_c and entropy S_c. The other is hot with temperature $T_h > T_c$ and entropy S_h. A small amount of heat dQ is transferred from the hot piece to the cold piece. The second law says that

$$T_c\,dS_c = dQ = -T_h\,dS_h.$$

The total entropy change of the system is

$$dS = dS_c + dS_h = dQ\Big(\frac{1}{T_c} - \frac{1}{T_h}\Big) > 0. \tag{12.6}$$

The second law ensures that heat can only go from the hot piece of matter to the cold piece.

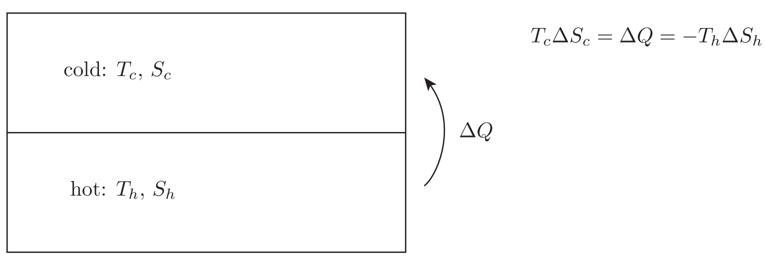


Figure 12.4: Heat ΔQ can only be transferred from hot regions to cold regions so that the entropy of the system increases.

12.3 Ideal Gases

The ideal gas law states that

$$pV = nRT, \tag{12.7}$$

where

- p = pressure
- V = volume
- n = number of moles of gas (1 mol $\approx 6.022 \times 10^{23}$ molecules)
- R = 8314.36 J kmol^{-1} K^{-1} is a universal constant
- T = temperature in K (T_C, the temperature in °C $= T - 273.15$).

The mass of gas is $m = nM$ where M is the molecular mass. The ideal gas law can be written as

$$pv = \left(\frac{nR}{m}\right)T = \frac{R}{M}T = R_g T, \tag{12.8}$$

where R_g is a constant which depends on the gas. The ideal gas law can also be written as

$$p = R_g \rho T. \tag{12.9}$$

Definition: The specific heats at constant volume (pressure) C_v (C_p) are defined by

$$C_v = \left(\frac{\delta Q}{\delta T}\right)_V \qquad \text{and} \qquad C_p = \left(\frac{\delta Q}{\delta T}\right)_p. \tag{12.10}$$

The subscripts indicate which state variable is being constant. That is, the heat required to raise the temperature of one unit mass of fluid by δT is

$$\delta Q = C_v\, \delta T \qquad \text{if the volume of the matter is kept constant,}$$

and

$$\delta Q = C_p\, \delta T \qquad \text{if the pressure of the matter is kept constant.}$$

Note that the δ's denote 'tiny change in'. Since heat added is not a function of any state variable it is incorrect to think of $\delta Q/\delta T$ and $\delta Q/\delta v$ as derivatives.

From the first law of thermodynamics we have

$$de = dQ - p\, dv.$$

Thus

1. If v is kept constant

   $$de = dQ = C_v\, dT$$

   hence

   $$C_v = \frac{de}{dT} = \left(\frac{\partial e}{\partial T}\right)_v. \tag{12.11}$$

Since the internal energy e is a function of the state variable we are now using derivatives. Again, since we could think of e as a function of T and v or as a function of T and p, say, we use a subscript to indicate the second independent variable.

2. If p is kept constant then

$$de = dQ - pdv = C_p\,dT - R_g\,dT$$

where we have used the fact that p is constant to give $p\,dv = d(pv) = d(R_gT)$. Thus

$$\left(\frac{\partial e}{\partial T}\right)_p = C_p - R_g. \tag{12.12}$$

For an ideal gas the internal energy is a function of T only, hence

$$\left(\frac{\partial e}{\partial T}\right)_p = \left(\frac{\partial e}{\partial T}\right)_v = \frac{de}{dT}.$$

Hence

$$C_p - C_v = R_g. \tag{12.13}$$

In general C_v and C_p are functions of temperature, however they are almost constant for T between about 250 and 400 K so we will assume they are constant. Thus,

$$e = C_vT. \tag{12.14}$$

We now find an expression for the entropy of an ideal gas. Since

$$de = dQ - p\,dv = T\,dS - p\,dv,$$

if T is constant it follows from (12.14) that $de = 0$. Thus

$$\left(\frac{dS}{dv}\right)_T = \frac{p}{T}. \tag{12.15}$$

Now S is a state variable, and hence can be considered to be a function of T and v. Thus, letting dS and dv go to zero we have,

$$\left(\frac{\partial S}{\partial v}\right)_T = \frac{p}{T} = \frac{R_g}{v}. \tag{12.16}$$

Integrating gives

$$S = F(T) + R_g\ln(v). \tag{12.17}$$

If v is constant then $de = dQ = T\,dS$. Hence, $C_v\,dT = T\,dS$ and

$$\left(\frac{\partial S}{\partial T}\right)_v = \frac{C_v}{T}, \tag{12.18}$$

from which we have

$$S = C_v \ln(T) + G(v). \tag{12.19}$$

Combining these results gives $S = C + C_v \ln(T) + R_g \ln(v)$ or

$$e^S = AT^{C_v} v^{R_g}, \tag{12.20}$$

where A is a constant. Only changes in S are of interest so we do not care what the value of A is. Using the ideal gas law and (12.13) it can be shown that

$$e^S = B\Big(pv^\gamma\Big)^{C_v}, \tag{12.21}$$

where

$$\gamma = \frac{C_p}{C_v} > 1, \tag{12.22}$$

is the ratio of specific heats. For diatomic gases such as O_2 and N_2 (and hence for air), $\gamma \approx 1.4$.

From (12.21) we can see that the isentropes for an ideal gas are given by $pv^\gamma =$ constant, or $p = B/v^\gamma$. In terms of p and ρ the isentropes are given by $p/\rho^\gamma =$ constant. Since $\gamma > 1$, the isentropes on the pv-diagram fall more rapidly than the isotherms $p = R_g T/v$ (see Figure 12.3.

Definition: *An* **adiabatic***, or* **isentropic***, reversible process is one which involve no heat exchange ($\delta Q = 0$). That is, it is a reversible process at constant entropy. For an ideal gas an isentropic process has*

$$\frac{p}{\rho^\gamma} = \text{constant}. \tag{12.23}$$

12.4 Reformulation of the Internal Energy Equation for a Newtonian Fluid

Recall that the internal energy equation is

$$\rho\frac{De}{Dt} = \vec{\nabla}\cdot(k\vec{\nabla}T) - p\vec{\nabla}\cdot\vec{u} + \phi,$$

where

$$\phi = 2\mu e_{ij}^2 + \lambda\Big(\vec{\nabla}\cdot\vec{u}\Big)^2,$$

is the viscous dissipation. Using the second law of thermodynamics (12.5) we have

$$\begin{aligned}\frac{De}{Dt} &= T\frac{DS}{Dt} - p\frac{Dv}{Dt}\\ &= T\frac{DS}{Dt} + \frac{p}{\rho^2}\frac{D\rho}{Dt}\\ &= T\frac{DS}{Dt} - \frac{p}{\rho}\vec{\nabla}\cdot\vec{u},\end{aligned}$$

where we have used $v = 1/\rho$ and the continuity equation. Hence, the internal energy equation can be written as

$$\rho T\frac{DS}{Dt} = \vec{\nabla}\cdot(k\vec{\nabla}T) + \phi. \tag{12.24}$$

This introduces a new variable into our set of equations. To eliminate S, we consider it to be a function of T and p. Then we can write

$$\frac{DS}{Dt} = \Big(\frac{\partial S}{\partial T}\Big)_p\frac{DT}{Dt} + \Big(\frac{\partial S}{\partial p}\Big)_T\frac{Dp}{Dt}. \tag{12.25}$$

Now at constant pressure $dQ = T\,dS = C_p\,dT$, so

$$\Big(\frac{\partial S}{\partial T}\Big)_p = \frac{C_p}{T}. \tag{12.26}$$

Also

$$de = T\,dS - p\,dv = d(TS) - S\,dT - d(pv) + v\,dp,$$

so that

$$d(e + pv - TS) = -S\,dT + v\,dp. \tag{12.27}$$

The quantity

$$G = e + pv - TS, \tag{12.28}$$

is called the Gibb's Free energy. From (12.27) we have

$$\Big(\frac{\partial G}{\partial T}\Big)_p = -S \quad \text{and} \quad \Big(\frac{\partial G}{\partial p}\Big)_T = v. \tag{12.29}$$

Equating mixed partials, $G_{pT} = G_{Tp}$ we have

$$\Big(-\frac{\partial S}{\partial p}\Big)_T = \Big(\frac{\partial v}{\partial T}\Big)_p. \tag{12.30}$$

Definition:

$$\beta = \frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_p = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p, \tag{12.31}$$

is the coefficient of thermal expansion. It is the rate of change of volume per unit volume wrt temperature at constant pressure.

In terms of the coefficient of thermal expansion

$$\left(\frac{\partial S}{\partial p}\right)_T = -v\beta. \tag{12.32}$$

Using (12.26) and (12.32), (12.25) can be written as

$$\frac{DS}{Dt} = \frac{C_p}{T}\frac{DT}{Dt} - v\beta\frac{Dp}{Dt}, \tag{12.33}$$

and (12.24) becomes

$$\rho C_p\frac{DT}{Dt} - \beta T\frac{Dp}{Dt} = \vec{\nabla}\cdot(k\vec{\nabla}T) + \phi. \tag{12.34}$$

12.5 Equations of Motion for a Newtonian Fluid

We have now obtained our final set of equations. Using conservation of mass, linear momentum, angular momentum and energy; the assumptions that the fluid is isotropic and that the stress tensor and deviatoric tensor are linearly related; and some ideas from thermodynamics including the assumption that the state of the fluid is determined by two state variables, we have derived the equations of motion for a Newtonian fluid. In their most general form they are:

- **the Continuity Equation:**

$$\frac{D\rho}{Dt} + \rho\vec{\nabla} \cdot \vec{u} = 0.$$

- **the Momentum Equations:**

$$\rho\frac{Du_i}{Dt} = -\rho\frac{\partial Pi}{\partial x_i} - \frac{\partial p}{\partial x_i} - \frac{\partial}{\partial x_i}\left(\lambda\vec{\nabla} \cdot \vec{u}\right) + 2\frac{\partial}{\partial x_j}\left(\mu e_{ji}\right),$$

- **The Energy Equation:**

$$\rho C_p\frac{DT}{Dt} - \beta T\frac{Dp}{Dt} = \vec{\nabla} \cdot (k\vec{\nabla}T) + \phi.$$

  where

$$\phi = 2\mu e_{ij}^2 + \lambda\left(\vec{\nabla} \cdot \vec{u}\right)^2.$$

These comprise five equations for six unknowns ρ, p, T and $\vec{u}$ which are closed using an equation of state

$$f(\rho, p, T) = 0.$$

In addition, there are five scalars k, λ, μ, C_p and β, which are themselves functions of the thermodynamic variables (assumed known).

Chapter 13

SOUND WAVES

Waves in fluids arise when restoring forces act to bring fluid particles back to their equilibrium positions. Different types of waves exist, due to the variety of restoring mechanisms that are present in many fluid systems. Examples include:

- **Sound Waves:** due to pressure fluctuations associated with compressibility of fluids or due to elasticity of solids.
- **Surface Water Waves:** due to gravity acting on vertically displaced fluid at the air-water interface.
- **Internal Gravity Waves:** due to gravity acting on vertically displaced fluid in a stratified fluid.
- **Capillary Waves:** Waves on an interface between two immiscible fluids driven by surface tension.
- **Rossby Waves:** Large scale waves in the ocean and atmosphere which arise due to the Earth's rotation (a consequence of vorticity, or angular momentum, stored in the fluid).

Each of these waves are modified by other restoring forces which may be present. For example, long surface gravity waves are modified by the Earth's rotation, so much so that they have a special name (Poincaré waves). The combination of the Earth's rotation and gravity combine to produce a new type of wave called a Kelvin Wave. Sound waves are modified by gravity and the Earth's rotation, however these modifications are only significant for very long, low frequency sound waves. At audible frequencies these effects are negligible. All the above waves are damped by viscosity and thermal diffusion.

For simplicity we will consider sound waves in their purest form by ignoring gravity, rotation, diffusion and viscosity. This is the logical starting point for an investigation of sound waves, allowing the simplest possible analysis while preserving the essential forcing mechanism. Thus, the governing equations we shall consider are

$$\rho_t + \vec{\nabla} \cdot (\rho \vec{u}) = 0, \tag{13.1}$$

$$\rho \frac{D\vec{u}}{Dt} + \vec{\nabla} p = 0, \tag{13.2}$$

and

$$\frac{DS}{Dt} = 0, \tag{13.3}$$

where $\vec{u}$ is the velocity field and ρ, p and S are the density, pressure and entropy respectively. This set of nonlinear equations is in general very difficult to solve. For waves in sea water an additional equation for the salinity would be required.

Consider an undisturbed state for which $\rho = \rho_o$ and $p = p_o = \bar{p}(\rho_o)$ are constant and the fluid is at rest. To this basic state we will add a small disturbance. Since entropy is then initially uniform, (13.3) implies it remains constant and uniform provided any disturbance is generated without changing the entropy. We assume this to be the case, giving an equation of state

$$p = \bar{p}(\rho). \tag{13.4}$$

We now add a small perturbation, or disturbance, by letting

$$p = p_o + \epsilon p', \tag{13.5}$$

$$\rho = \rho_o + \epsilon \rho', \tag{13.6}$$

and

$$\vec{u} = \epsilon \vec{u}', \tag{13.7}$$

where ϵ is a small dimensionless parameter measuring the smallness of the perturbation.

Substituting (13.5)–(13.7) into the continuity equation gives

$$\epsilon \rho'_t + \vec{\nabla} \cdot \Big((\rho_o + \epsilon \rho') \epsilon \vec{u}' \Big) = 0$$

or

$$\epsilon \rho'_t + \epsilon \vec{\nabla} \cdot (\rho_o \vec{u}') = -\epsilon^2 \vec{\nabla} \cdot (\rho' \vec{u}'),$$

or, upon division by ϵ,

$$\rho'_t + \rho_o \vec{\nabla} \cdot \vec{u}' = -\epsilon \vec{\nabla} \cdot (\rho' \vec{u}'). \tag{13.8}$$

Substituting (13.5)–(13.7) into the momentum equation results in

$$\rho_o \vec{u}'_t + \vec{\nabla} p' = -\epsilon \rho' \vec{u}'_t - \epsilon \rho_o \vec{u}' \cdot \vec{\nabla} \vec{u}' - \epsilon^2 \rho' \vec{u}' \cdot \vec{\nabla} \vec{u}'. \tag{13.9}$$

The equation of state gives

$$\begin{aligned} p_o + \epsilon p' &= \bar{p}(\rho_o + \epsilon \rho') \\ &= \bar{p}(\rho_o) + \epsilon \frac{d\bar{p}}{d\rho}(\rho_o)\rho' + \frac{\epsilon^2}{2}\frac{d^2\bar{p}}{d\rho^2}(\rho_o)\rho'^2 + \cdots \end{aligned}$$

so that

$$p' - c^2 \rho' = \frac{1}{2}\epsilon \frac{d^2\bar{p}}{d\rho^2}(\rho_o) + \cdots , \tag{13.10}$$

where

$$c^2 = \frac{d\bar{p}}{d\rho}(\rho_o). \tag{13.11}$$

Equations (13.8), (13.9), and (13.10) give five equations for the three velocity components of $\vec{u}'$, ρ' and p'. The solutions depend on the value of ϵ, that is, they are dependent on the amplitude of the perturbation. This is because the equations are nonlinear and this is, in part, why nonlinear equations are so difficult to solve. Since ϵ is assumed to be small it makes sense to expand $\rho'(\vec{x}, t, \epsilon)$ (and the other variables) in a perturbation series (which you may have seen in AMATH 351) in powers of ϵ. Thus, one would set

$$\rho'(\vec{x}, t) = \rho^{(0)}(\vec{x}, t) + \epsilon \rho^{(1)}(\vec{x}, t) + \epsilon^2 \rho^{(2)}(\vec{x}, t) + \cdots ,$$

with similar expansions for the other variables. Substituting the expansions into equations (13.8), (13.9), and (13.10) then gives a hierarchy of problems. The first, given by the coefficients of ϵ^0, is a linear homogeneous problem for $\vec{u}^{(0)}$, $\rho^{(0)}$ and $p^{(0)}$. The second, given by the coefficients of ϵ^1 is an inhomogeneous problem for $\vec{u}^{(1)}$, $\rho^{(1)}$ and $p^{(1)}$ with forcing terms which are functions of $\vec{u}^{(0)}$, $\rho^{(0)}$ and $p^{(0)}$. This results in what is known as a weakly-nonlinear solution. One can continue by systematically solving other problems obtained from the coefficients of ϵ^2, ϵ^3, etc., to obtain more and more terms in the series. Some use of this method will be made in the fourth year fluid mechanics course when you study surface gravity waves.

13.1 The Linear Problem

The equations for $\vec{u}^{(0)}$, $\rho^{(0)}$ and $p^{(0)}$ can easily be obtained by dropping all terms in (13.8), (13.9), and (13.10) that contain an ϵ. This is based on the assumption that

ϵ is very small, so that the terms on the right hand side of these equations are small compared to one of the terms on the left. This results in three linear equations:

$$\rho'_t + \rho_o \vec{\nabla} \cdot \vec{u}' = 0, \tag{13.12}$$

$$\rho_o \vec{u}'_t + \vec{\nabla} p' = 0, \tag{13.13}$$

and

$$p' - c^2 \rho' = 0. \tag{13.14}$$

The solution of these is, of course, only an approximation to $\vec{u}'$, ρ' and p'. They are in fact the functions $\vec{u}^{(0)}$, $\rho^{(0)}$ and $p^{(0)}$ discussed above. The approximation improves as ϵ gets smaller.

Equation for ρ': Using (13.14) to eliminate p' in (13.13) reduces the problem to two equations for ρ' and $\vec{u}'$. A single equation for ρ' can be obtained by taking $\partial(13.12)/\partial t$ and then eliminating $\vec{u}'$ by using (13.13) to replace $\rho_o \vec{u}'_t$ with $-c^2 \vec{\nabla} \rho'$. This gives

$$\rho'_{tt} - c^2 \nabla^2 \rho' = 0. \tag{13.15}$$

This is the well known linear wave equation in three dimensions which is studied in detail in the fourth year PDEs class AMATH 453. I'll call it the 3D wave equation, although there are many other types of wave equations, some of which are discussed in the fourth year fluid mechanics course.

Equation for $\vec{u}'$: Taking the partial derivative of (13.13) wrt t (after elimination of p' using (13.14)) and using (13.12) to eliminate ρ'_t results in

$$\vec{u}'_{tt} - c^2 \vec{\nabla}(\vec{\nabla} \cdot \vec{u}') = 0. \tag{13.16}$$

Thus the components of $\vec{u}'$ *do not*, in general satisfy the 3D wave equation.

Equation for the vorticity: Taking the curl of (13.13) gives

$$\vec{\omega}'_t = 0, \tag{13.17}$$

where $\vec{\omega}' = \vec{\nabla} \times \vec{u}'$. Thus *the vorticity field is independent of time*. As a special case consider the case where $\vec{\omega}'$ is initially zero. Then (13.17) implies in remains zero. Since $\vec{\nabla} \times \vec{u}' = 0$ it follows that $\vec{u}' = \vec{\nabla}\phi$ for some potential ϕ. Hence, $\vec{\nabla} \cdot \vec{u}' = \nabla^2 \phi$ and (13.16) gives

$$\vec{\nabla}\Big(\phi_{tt} - c^2 \nabla^2 \phi\Big) = 0.$$

Thus

$$\phi_{tt} - c^2 \nabla^2 \phi = B(t),$$

for some function $B(t)$. Now only the spatial gradient of ϕ has any physical significance — it gives the velocity field and does not depend on B — so we can take $B = 0$. Thus

$$\phi_{tt} - c^2\nabla^2\phi = 0. \tag{13.18}$$

Taking the gradient gives

$$\vec{u}'_{tt} - c^2\nabla^2\vec{u}' = 0. \tag{13.19}$$

Thus, *if the vorticity is zero then the velocity field satisfies the 3D wave equation*. There is obviously something very different about vortical and non-vortical velocity fields (i.e., velocity fields with and without vorticity) — a hint of the importance of the concept of vorticity (turbulence is inherently vortical — no vorticity implies no turbulence).

In general one can split $\vec{u}'$ into non-vortical and vortical components via

$$\vec{u}' = \vec{\nabla}\phi + \vec{u}_v$$

were $\vec{u}_v$ is the vortical part of the velocity field. The non-vortical part $\vec{\nabla}\phi$ propagates away leaving the vortical part behind — inclusion of nonlinearities are essential for the vortical part to do anything interesting.

13.1.1 The 1D case

For simplicity we now consider the one-dimensional case in detail. Thus, we set $\vec{u}' = \big(u'(x,t), 0, 0\big)$, and $\rho' = \rho'(x,t)$. For this case the vorticity is identically zero. The equations for ρ' and u' are

$$\rho'_t + \rho_o u'_x = 0, \tag{13.20}$$

$$\rho_o u'_t + c^2\rho'_x = 0, \tag{13.21}$$

from which one can easily derive

$$\rho'_{tt} - c^2\rho'_{xx} = 0, \tag{13.22}$$

and

$$u'_{tt} - c^2 u'_{xx} = 0. \tag{13.23}$$

Hence, ρ' and u' both satisfy the 1D wave equation. The general solution of (13.22) is

$$\rho' = F(x+ct) + G(x-ct). \tag{13.24}$$

That is, ρ' is the sum of two waves, one, $F(x+ct)$, propagating to the left with speed c and one, $G(x-ct)$, propagating to the right with speed c. Thus, c, given by (13.11), is the propagation speed of the waves and is called the sound speed.

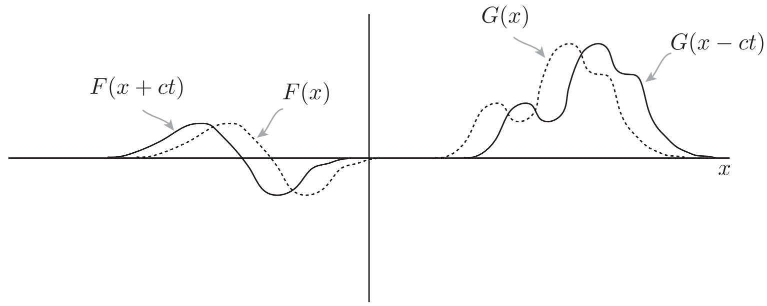


Figure 13.1: Two solutions of the linear wave equation $\eta_{tt} - c^2\eta_{xx} = 0$. Dashed lines show wave forms at $t = 0$, solid at time $t > 0$. Wave forms propagate with speed $c > 0$ with $\eta = F(x+ct)$ propagating leftward and $\eta = G(x-ct)$ propagating rightward.

From (13.20) we have

$$u'_x = -\frac{c}{\rho_o}F'(x+ct) + \frac{c}{\rho_o}G'(x-ct)$$

(the primes on F and G denote differentiation) which, upon integration, gives

$$u' = -\frac{c}{\rho_o}F(x+ct) + \frac{c}{\rho_o}G(x-ct) + B(t). \tag{13.25}$$

Substitution into (13.21) shows that $B'(t) = 0$ hence B is spatially and temporally constant. This represents a uniform, time independent background flow which we can take to be zero by changing reference frames to one moving with speed B. Thus, with $B = 0$, we have

$$u' = -\frac{c}{\rho_o}F(x+ct) + \frac{c}{\rho_o}G(x-ct). \tag{13.26}$$

If the fluid in question is an ideal gas, then the ideal gas law, and use of $p' =$

$c^2\rho'$ from (13.14), gives

$$
\begin{aligned}
T = T_o + \epsilon T' = \frac{p}{R_g \rho} &= \frac{p_o + \epsilon p'}{R_g\big(\rho_o + \epsilon\rho'\big)} \\
&= \frac{1}{R_g\rho_o}\big(p_o + \epsilon p'\big)\big(1 - \epsilon\frac{\rho'}{\rho_o} + \epsilon^2\frac{\rho'^2}{\rho_o^2} + \cdots\big) \\
&= \frac{p_o}{R_g\rho_o} + \epsilon\frac{p' - p_o\rho'/\rho_o}{\rho_o R_g} + \cdots \\
&= \frac{p_o}{R_g\rho_o} + \epsilon\frac{\gamma p_o\rho'/\rho_o - p_o\rho'/\rho_o}{\rho_o R_g} + \cdots \\
&= \frac{p_o}{R_g\rho_o} + \epsilon\frac{(\gamma-1)p_o\rho'}{\rho_o^2 R_g} + \cdots,
\end{aligned}
$$

where we have used $c^2 = \gamma p_o/\rho_o$. Using $\gamma - 1 = C_p/C_v - 1 = R_g/C_v$ we have

$$
T' = \frac{p_o}{C_v\rho_o^2}\rho'. \tag{13.27}
$$

Hence, from (13.14) and (13.27) we see that the temperature, pressure and density perturbations are all in phase.

13.1.2 Plane Waves

Special *plane wave* solutions of the 1D wave equation are sinusoidal waves of the form

$$
\rho' = a\cos(kx - \sigma t). \tag{13.28}
$$

Because the equation is linear with real, constant coefficients, it is customary to write

$$
\rho' = ae^{i(kx-\sigma t)}, \tag{13.29}
$$

where it is understood that only the real part is physically meaningful (we could also use the imaginary part). In these expressions σ is the wave frequency (rad s^{-1}) and k is called the wave number. They may have either sign. The wave period and wavelength are given by $\tau = 2\pi/|\sigma|$ and $\lambda = 2\pi/|k|$ respectively. We can write (13.29) as

$$
\rho' = ae^{i\left(k(x-\frac{\sigma}{k}t)\right)} \tag{13.30}
$$

Since this must have the form of $F(x+ct)$ or $G(x-ct)$ we see that we must have

$$\sigma = \pm ck. \tag{13.31}$$

This is called the *dispersion relation*. It relates the wave frequency σ to the wave number k. The velocity field, using (13.26) and (13.31), is

$$u' = a\frac{\sigma}{k\rho_o}e^{i(kx-\sigma t)} = \frac{\sigma}{k\rho_o}\rho', \tag{13.32}$$

so that u' is in phase with ρ' for a rightward propagating wave ($\sigma/k > 0$) and out of phase with ρ' for a leftward propagating wave ($\sigma/k < 0$). In otherwords, u' is in the direction of propagation in regions of positive ρ' and in the opposite direction in regions of negative ρ'. Some thought shows that this is necessary in order to have the fluid converging/diverging so that the density may increase/decrease as the wave propagates.

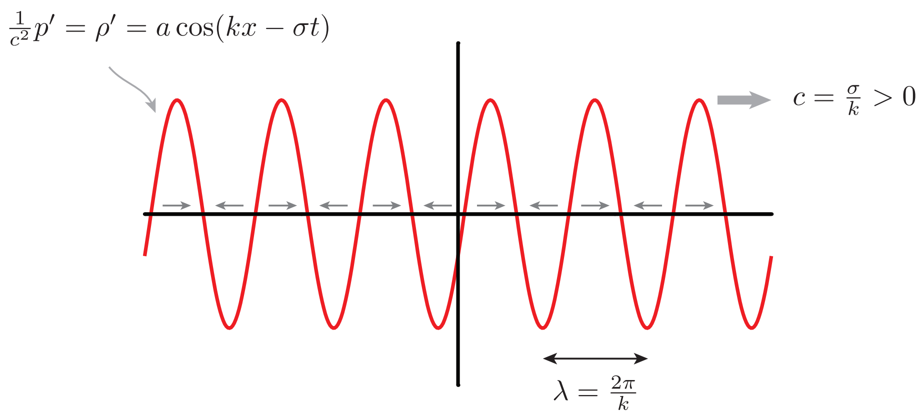


Figure 13.2: A simple sinusoidal plane wave propagating to the right with speed c. The red sinusoid shows the density perturbation ρ which is equal to p/c^2. The leftward and rightward arrows depict the corresponding fluid velocity $u' = \frac{k}{\rho_0'}\rho'$.

In three dimensions, the 3D wave equation (13.15) has plane wave solutions of the form

$$\rho' = ae^{i(kx+ly+mz-\sigma t)},$$

or, more succinctly,

$$\rho' = ae^{i(\vec{k}\cdot\vec{x}-\sigma t)}, \tag{13.33}$$

where

$$\vec{k} = (k, l, m) \tag{13.34}$$

is called the wave vector. Substitution of (13.33) into (13.15) gives

$$\sigma^2 = c^2(k^2 + l^2 + m^2) = c^2\vec{k}\cdot\vec{k}.$$

Thus

$$\sigma = \pm c|\vec{k}|. \tag{13.35}$$

The two dimensional case, with $m = 0$, is sketched in Figure 13.3. The wave vector $\vec{k}$ is perpendicular to lines of constant ρ'. These lines propagate in the direction of $\pm\vec{k}$ with speed c. A rotation of the coordinate system so that the x' axis points in the direction of $\vec{k}$ reduces the equation to the one dimensional case.

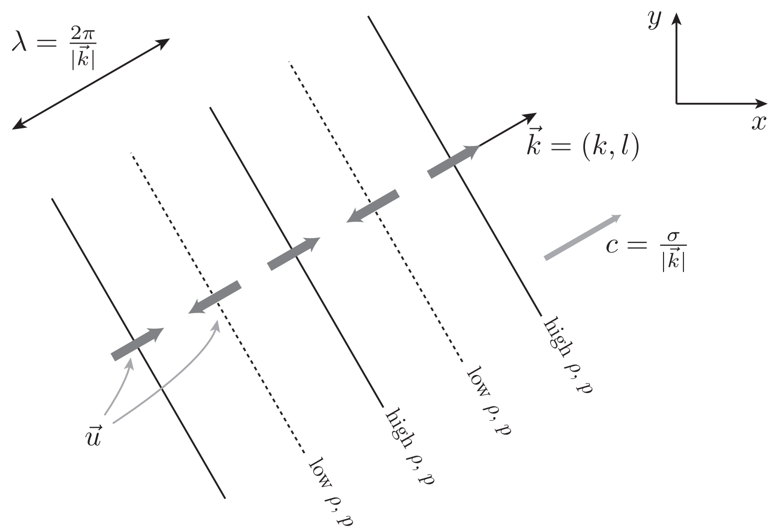


Figure 13.3:

Estimating the effects of dissipation and diffusion

We now briefly consider the importance of the neglected dissipation and diffusion terms. For one-dimensional waves, the linearized equations, including viscosity and thermal diffusion, are

$$\rho'_t + \rho_o u'_x = 0, \tag{13.36}$$

$$\rho_o u'_t + c^2\rho'_x = (\frac{4}{3}\mu + \mu_B)u'_{xx}, \tag{13.37}$$

and

$$T'_t = \kappa T'_{xx} - \frac{p_o}{\rho_o C_v}u'_x, \tag{13.38}$$

where μ is the shear viscosity and $\mu_B = 2/3\mu + \lambda$ is the bulk viscosity. For a monatomic gas μ_B is approximately zero, however in general it is nonzero. In water, μ_B is about three times the value of μ. Note the thermal diffusivity $\kappa = k/(\rho_o C_v)$ where this k, from the internal energy equation, should not be confused with the wave number k above. For our linear plane wave solution with no viscosity and diffusion we have, for rightward propagating waves,

$$\begin{aligned} \rho' &= a\cos(kx - \sigma t), \\ u' &= \frac{c}{\rho_o} a\cos(kx - \sigma t), \end{aligned}$$

and, for an ideal gas,

$$T' = \frac{p_o}{C_v \rho_o} a\cos(kx - \sigma t).$$

We use these to determine the size of the viscous and diffusive terms in equations (13.36)–(13.38) to see when they are small compared to the other terms. Thus, in (13.37), we have

$$(\frac{4}{3}\mu + \mu_B)u'_{xx} = -(2\mu - \lambda)\frac{ck^2}{\rho_o} a\cos(kx - \sigma t)$$

which has a magnitude of

$$(\frac{4}{3}\mu + \mu_B)\frac{ck^2}{\rho_o} a \tag{13.39}$$

The first term on the left of (13.37) has magnitude

$$\sigma c a. \tag{13.40}$$

This is large compared with (13.39) provided

$$\sigma \gg (\frac{4}{3}\mu + \mu_B)\frac{\sigma^2}{c^2 \rho_o}. \tag{13.41}$$

Using the fact that μ_B is usually comparable in magnitude to μ, this can be written as

$$\sigma \ll \frac{c^2}{\nu}. \tag{13.42}$$

where $\nu = \mu/\rho_0$. For dry air at atmospheric pressure and at 20°C, $c \approx 340$ m/s and $\nu \approx 1.5 \times 10^{-5}$ m^2 s^{-1} so (13.42) says that the viscous term in (13.37) is negligible if the wave frequency is alot smaller than about 10^{10} s^{-1}, or about 10 GHz. This is extremely large (audible frequencies span the range from about 20–20,000 Hz) justifying the neglect of this term. A similar consideration of the thermal diffusion term leads to the condition that

$$\sigma \ll \frac{c^2}{\kappa} \tag{13.43}$$

which, since κ is similar to ν, leads to the same restriction on σ.

The above argument shows that the amount of damping, or the attenuation rate, increases with wave frequency. The presence of polyatomic molecules with their greater degrees of freedom (implying more forms of internal energy) can affect attenuation rates. For example, water vapour can significantly affect attenuation rates, so much that in a humid concert hall high frequency sounds can be noticeably diminished.

Because light does not penetrate very far in water, acoustical techniques are heavily used in oceanography to detect anything from small objects such as sediment particles and phytoplankton, up to fish (most of the reflection being off the 'gas bubble' in their swim bladder) and submarines. Very high frequency sound is required to detect small particles (as high as 200 MHz) because small wavelengths are required to get significant reflection off small particles. High frequencies are attenuated more rapidly than low frequencies so small particles can only be detected at short distances. Low frequency sound can travel great distances. In fact the time taken for sound waves (of around 200 Hz) to travel over distances as large as 10,000 km are used to detect average temperature changes of the ocean. In water, thermal conductivity effects are negligible. Dissolved salts in sea water result in attenuation rates which are a two orders of magnitude larger than in fresh water. This is due to the presence of boric acid and magnesium sulfate (Clay and Medwin, 1977).

13.2 Nonlinear, One-Dimensional Sound Waves

Here we present some simple ideas about nonlinear sound waves. The technique described is useful for many different types of waves. When I say simple I mean simple in the context of nonlinear waves — you will find it rather challenging reading but hopefully interesting. This section is here primarily to give you a taste for something beyond linear waves. You are not responsible for this material for the exam.

Consider the nonlinear, one dimensional inviscid equations for a constant entropy gas, which are

$$\rho_t + u\rho_x + \rho u_x = 0, \tag{13.44}$$

and

$$\rho u_t + \rho u u_x + c^2 \rho_x = 0, \tag{13.45}$$

where we have used the equation of state $p = \bar{p}(\rho)$ which gives

$$p_x = \frac{d\bar{p}}{d\rho}\rho_x = c^2 \rho_x. \tag{13.46}$$

Note that c is a function of ρ and hence of x and t. Equations (13.44) and (13.45) comprise a set of coupled, first-order nonlinear hyperbolic partial differential equations. The theory for these equations is well developed (see, for example, the book 'Linear and Nonlinear Waves', by G. B. Whitham). Taking c/ρ times the first equation plus $\pm 1/\rho$ times the second equation gives

$$\frac{c}{\rho}\Big(\rho_t + u\rho_x\Big) + c u_x \pm \Big(u_t + u u_x + \frac{c^2}{\rho}\rho_x\Big) = 0 \tag{13.47}$$

which, with a slight rearrangement, gives

$$\frac{c}{\rho}\Big(\rho_t + \big(u \pm c\big)\rho_x\Big) \pm \Big(u_t + \big(u \pm c\big)u_x\Big) = 0. \tag{13.48}$$

These equations have the form of c/ρ times an operator acting on ρ, plus or minus the same operator acting on u. There is a technique for putting couple sets of hyberbolic equations in this form which is discussed in Whitham.

The trick to solving (13.48) is to move through the gas with a special velocity and describe what is happening in that reference frame. In particular, consider a path $x(t)$ through the gas defined by

$$\frac{dx}{dt} = u(x(t), t) + c(x(t), t), \tag{13.49}$$

that is, move through the gas with a velocity equal to the local current plus the local speed of sound. Of course, $x(t)$ can't be found just yet because we don't know u and c. Along this special path

$$u \;=\; \bar{u}(t) = u(x(t), t), \tag{13.50}$$

$$\rho \;=\; \bar{\rho}(t) = \rho(x(t), t), \tag{13.51}$$

and

$$c = \bar{c}(t) = c(x(t), t). \tag{13.52}$$

Now

$$\frac{d\bar{u}}{dt} = u_t + u_x\frac{dx}{dt} = u_t + (u + c)u_x \tag{13.53}$$

and, similarly,

$$\frac{d\bar{\rho}}{dt} = \rho_t + \rho_x \frac{dx}{dt} = \rho_t + (u+c)\rho_x \tag{13.54}$$

where u and ρ are evaluated at $(x(t), t)$. Thus, using the $+$ sign in (13.48), on $x = x(t)$, we have

$$\frac{c}{\rho}\frac{d\bar{\rho}}{dt} + \frac{d\bar{u}}{dt} = 0. \tag{13.55}$$

Using $c^2 = A\gamma\rho^{\gamma-1}$,

$$\begin{aligned}
\frac{c}{\rho}\frac{d\rho}{dt} &= \sqrt{A\gamma}\rho^{(\frac{\gamma-1}{2}-1)}\frac{d\rho}{dt} \\
&= \sqrt{A\gamma}\frac{2}{\gamma-1}\frac{d}{dt}\rho^{\frac{\gamma-1}{2}} \\
&= \frac{2}{\gamma-1}\frac{d}{dt}\Big(\sqrt{A\gamma}\rho^{\frac{\gamma-1}{2}}\Big) \\
&= \frac{2}{\gamma-1}\frac{dc}{dt}
\end{aligned}$$

so (13.55) becomes

$$\frac{d}{dt}\Big(\frac{2}{\gamma-1}\bar{c} + \bar{u}\Big) = 0. \tag{13.56}$$

In summary, (dropping the bars)

$$\frac{d}{dt}\Big(\frac{2}{\gamma-1}c + u\Big) = 0 \qquad \text{on} \qquad \frac{dx}{dt} = u + c. \tag{13.57}$$

This means that $2c/(\gamma-1) + u$ is constant along the path $dx/dt = u + c$.

We next consider how the flow field varies in time in a reference frame moving with the local fluid velocity minus the local sound speed, i.e, along

$$\frac{dx}{dt} = u(x(t), t) - c(x(t), t). \tag{13.58}$$

Following a similar procedure as above gives the result that

$$\frac{d}{dt}\Big(\frac{2}{\gamma-1}c - u\Big) = 0 \qquad \text{on} \quad \frac{dx}{dt} = u - c. \tag{13.59}$$

The conserved quantities $2c/(\gamma-1) \pm u$ are called Riemann invariants. The paths $dx/dt = u \pm c$ are called characteristics. If you know the two characteristics passing through a point (x, t) then using the conserved quantities you know c and

u and hence everything else (ρ, p, etc.). The two characteristics are referred to as left and right going characteristics - they move with speed c to the left or right relative to the local fluid velocity u. Information travels with the local sound speed $\pm c$ relative to the gas.

Simple Waves: A simple wave is a special solution for which one of the Riemann invariants is initially spatially uniform, in which case it remains spatially and temporally constant. These give nonlinear waves propagating to the right or to the left.

Suppose that at $t = 0$ the Riemann invariant $c/(\gamma - 1) - u$ is spatially uniform and assume the disturbance is contained in a finite region R. Outside R, $\rho = \rho_o$, $p = p_o$, $c^2 = c_o^2 = \gamma p_o/\rho_o$ and $u = 0$. Thus,

$$\frac{2}{\gamma - 1}c - u = \frac{2}{\gamma - 1}c_o = \frac{2\gamma}{\gamma - 1}\frac{p_o}{\rho_o}. \tag{13.60}$$

This holds everywhere. On the rightward propagating characteristic

$$\frac{2}{\gamma - 1}c + u = \text{constant} \qquad \text{on} \qquad \frac{dx}{dt} = u + c. \tag{13.61}$$

Using (13.60) we can say that

$$\frac{2}{\gamma - 1}c_o + 2u = \text{constant} \qquad \text{on} \qquad \frac{dx}{dt} = \frac{\gamma + 1}{2}u + c_o,$$

or, since c_o is a constant,

$$u = \text{constant} \qquad \text{on} \qquad \frac{dx}{dt} = \frac{\gamma + 1}{2}u + c_o.$$

But if u is constant so is dx/dt and so is c, from (13.61), and hence so are ρ and p. Hence the characteristic $x(t)$ is a straight line and all flow fields are constant along it. All quantities propagate with speeds

$$\begin{aligned} V = c_o + \frac{\gamma + 1}{2}u &= c_0 + \frac{\gamma + 1}{2}\Big(\frac{2}{\gamma - 1}(c - c_0)\Big) \\ &= -\frac{2}{\gamma - 1}c_o + \frac{\gamma + 1}{\gamma - 1}\sqrt{A\gamma}\rho^{\frac{\gamma - 1}{2}}. \end{aligned} \tag{13.62}$$

The propagation speed increases with u or, equivalently, with ρ.

Consider the initial state

$$\rho(x,0) = \bar{\rho}(x).$$

Values of ρ propagate with speed $V(\rho)$ given by (13.62). A value of $\rho = a$ at time $t = 0$ will move to the right a distant $V(a)t$ in time t, hence,

$$\rho(x,t) = \bar{\rho}\big(x - V(\rho)t\big).$$

Another way to write this is to use a coordinate, say ξ, to label each characteristic. A convenient choice is the x-coordinate of the characteristic at $t = 0$. Since ρ is constant along each characteristic ρ is a function of ξ, in particular

$$\rho(x,t) = \bar{\rho}(\xi). \tag{13.63}$$

The characteristic is

$$x = \xi + V(\xi)t. \tag{13.64}$$

The latter equation implicitly determines $\xi(x,t)$. From (13.63)

$$\rho_x = \bar{\rho}'\xi_x$$

and, differentiating (13.64) w.r.t x,

$$1 = \big(V't + 1\big)\xi_x.$$

Thus

$$\rho_x = \frac{\bar{\rho}'}{1 + V't}. \tag{13.65}$$

Breaking occurs when ρ_x becomes infinite. This first occurs when $1 + V't = 0$ for the first time. The breaking time t_B is given by

$$t_B = \min \frac{-1}{V'}.$$

On a diagram showing the characterstics, the breaking time is the minimum time at which two characteristics meet. After this point the solution becomes double valued implying the physics has broken down. The solution requires a jump discontinuity, called a shock, and jump conditions across the shock must be derived. A well known example of such a shock in air is a sonic boom.

References

Kinsler, L. E., A. R. Frey, A. B. Coppens, and J. V. Sanders. *Fundamentals of Acoustics*, John Wiley and Sons. Third ed. 1982.

Clay, D. S., and H. Medwin. *Acoustical Oceanography: Principles and Applications*, John Wiley and Sons, 1977. (There is an updated 1997(?) edition).

Whitham, G. B., *Linear and Nonlinear Waves*, Wiley, 1974.

Chapter 14

INTRODUCTION TO ELASTICITY THEORY

Recall the momentum equations in their general form

$$\rho\frac{D\vec{u}}{Dt} = -\rho\vec{\nabla}\Pi + \vec{\nabla}\cdot\tau. \tag{14.1}$$

For a Newtonian fluid the stress tensor τ was decomposed into two parts: the contribution from the pressure, which acts normally to a surface, and the deviatoric stress tensor which is linearly related to the strain rate tensor. That is, τ_{ij} depends linearly on the *rate* of deformation of fluid volumes. For an elastic solid τ_{ij} depends on the *amount* of deformation. In a solid the molecules are arranged in a particular structure. Any deformation of the solid results in a deformation of this structure. In an elastic solid internal restoring forces arise which act to bring these molecular structures back to their equilibrium form.

14.1 Strain Tensors

Before proceeding further it is necessary to introduce some tensors used to describe deformations.

Let a_i and $a_i + da_i$ be the locations of points P and P' in an undeformed solid. Let $Q = \vec{x}(\vec{a})$ and $Q' = \vec{x}(\vec{a} + d\vec{a}) = \vec{x} + d\vec{x}$ be the locations of these points after the deformation (Figure 14.2). The distance between Q and Q' is

$$ds = |QQ'| = \sqrt{dx_i dx_i}, \tag{14.2}$$

while the undisturbed distance between the two points is

$$ds_o = |PP'| = \sqrt{da_i da_i}. \tag{14.3}$$

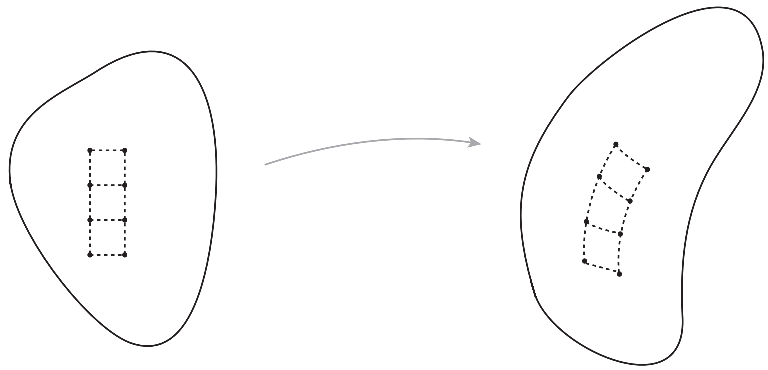


Figure 14.1: Deformed molecular structures exert intermolecular forces which act to restore the original shape.

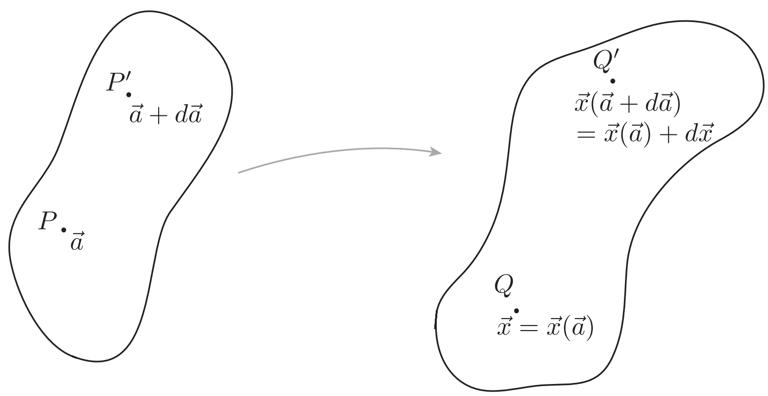


Figure 14.2:

Then

$$ds^2 - ds_o^2 = dx_i dx_i - da_i da_i.$$

We now discuss two ways to describe $ds^2 - ds_o^2$: as a function of the Lagrangian variables a_i or in terms of the Eulerian variables x_i.

Lagrangian Description: Have

$$x_i = x_i(\vec{a})$$

and

$$x_i + dx_i = x_i(\vec{a} + d\vec{a}) \approx x_i(\vec{a}) + \frac{\partial x_i}{\partial a_j} da_j.$$

Thus, for small da_j,

$$dx_i = \frac{\partial x_i}{\partial a_j} da_j. \tag{14.4}$$

and

$$ds^2 - ds_o^2 = \Big(\frac{\partial x_i}{\partial a_j} da_j\Big)\Big(\frac{\partial x_i}{\partial a_k} da_k\Big) - da_i da_i.$$

Now $da_i da_i = da_k da_k = \delta_{jk} da_j da_k$ so we can write this as

$$ds^2 - ds_o^2 = \Big(\frac{\partial x_i}{\partial a_j}\frac{\partial x_i}{\partial a_k} - \delta_{jk}\Big) da_j da_k = 2E_{jk} da_j da_k. \tag{14.5}$$

The symmetric tensor

$$E_{jk} = \frac{1}{2}\Big(\frac{\partial x_i}{\partial a_j}\frac{\partial x_i}{\partial a_k} - \delta_{jk}\Big) \tag{14.6}$$

is called the Green-St. Venant strain tensor.

Eulerian Description: Proceeding in a similar fashion

$$ds^2 - ds_o^2 = (\delta_{jk} - \frac{\partial a_i}{\partial x_j}\frac{\partial a_i}{\partial x_k}\Big) dx_j dx_k = 2e_{jk} dx_j dx_k, \tag{14.7}$$

where

$$e_{jk} = \frac{1}{2}(\delta_{jk} - \frac{\partial a_i}{\partial x_j}\frac{\partial a_i}{\partial x_k}\Big) \tag{14.8}$$

is the symmetric Almansi-Hamel strain tensor.

We now introduce the displacement vector $\vec{q} = \vec{x} - \vec{a}$. In terms of $\vec{q}$

$$\begin{aligned}
\frac{\partial x_i}{\partial a_j}\frac{\partial x_i}{\partial a_k} &= \frac{\partial (q_i + a_i)}{\partial a_j}\frac{\partial (q_i + a_i)}{\partial a_k} \\
&= \Big(\frac{\partial q_i}{\partial a_j} + \delta_{ij}\Big)\Big(\frac{\partial q_i}{\partial a_k} + \delta_{ik}\Big) \\
&= \frac{\partial q_i}{\partial a_j}\frac{\partial q_i}{\partial a_k} + \delta_{ij}\frac{\partial q_i}{\partial a_k} + \delta_{ik}\frac{\partial q_i}{\partial a_j} + \delta_{ij}\delta_{ik} \\
&= \frac{\partial q_i}{\partial a_j}\frac{\partial q_i}{\partial a_k} + \frac{\partial q_j}{\partial a_k} + \frac{\partial q_k}{\partial a_j} + \delta_{jk}
\end{aligned}$$

so

$$E_{jk} = \frac{1}{2}\Big(\frac{\partial q_j}{\partial a_k} + \frac{\partial q_k}{\partial a_j} + \frac{\partial q_i}{\partial a_j}\frac{\partial q_i}{\partial a_k}\Big). \tag{14.9}$$

Similarly,

$$e_{jk} = \frac{1}{2}\Big(\frac{\partial q_j}{\partial x_k} + \frac{\partial q_k}{\partial x_j} - \frac{\partial q_i}{\partial x_j}\frac{\partial q_i}{\partial x_k}\Big). \tag{14.10}$$

14.2 Linear Elasticity Theory

The Green-St. Venant and Almansi-Hamel tensors are, in general, different tensors. Because of the quadratic terms they are nonlinear and hence difficult to work with. For small displacements $\vec{q}$ they become linear and are approximately the same. To show this, first use the chain rule to get

$$\frac{\partial q_j}{\partial a_k} = \frac{\partial q_j}{\partial x_l}\frac{\partial x_l}{\partial a_k} = \frac{\partial q_j}{\partial x_l}\frac{\partial}{\partial a_k}(q_l + a_l) = \frac{\partial q_j}{\partial x_k} + \frac{\partial q_j}{\partial x_l}\frac{\partial q_l}{\partial a_k}.$$

Using this and a similar expression for $\frac{\partial q_k}{\partial a_j}$ we can rewrite the Green-St. Venant tensor as

$$E_{jk} = \frac{1}{2}\Big(\frac{\partial q_j}{\partial x_k} + \frac{\partial q_k}{\partial x_j} + \frac{\partial q_j}{\partial x_l}\frac{\partial q_l}{\partial a_k} + \frac{\partial q_k}{\partial x_l}\frac{\partial q_l}{\partial a_j} + \frac{\partial q_i}{\partial a_j}\frac{\partial q_i}{\partial a_k}\Big). \tag{14.11}$$

For small displacements, q_i and all its derivatives are small so

$$E_{jk} \approx \frac{1}{2}\Big(\frac{\partial q_j}{\partial x_k} + \frac{\partial q_k}{\partial x_j}\Big),$$

and

$$e_{jk} \approx \frac{1}{2}\Big(\frac{\partial q_j}{\partial x_k} + \frac{\partial q_k}{\partial x_j}\Big).$$

Thus, for small displacements

$$e_{jk} \approx E_{jk} \approx \frac{1}{2}\Big(\frac{\partial q_j}{\partial x_k} + \frac{\partial q_k}{\partial x_j}\Big), \tag{14.12}$$

which is called the Cauchy strain tensor. The time derivative of this tensor is the strain rate tensor.

Interpretation

Example A: Consider a stretching in the x_1 direction.

$$ds^2 - ds_o^2 = 2E_{jk}da_j da_k = 2E_{11}da_1^2.$$

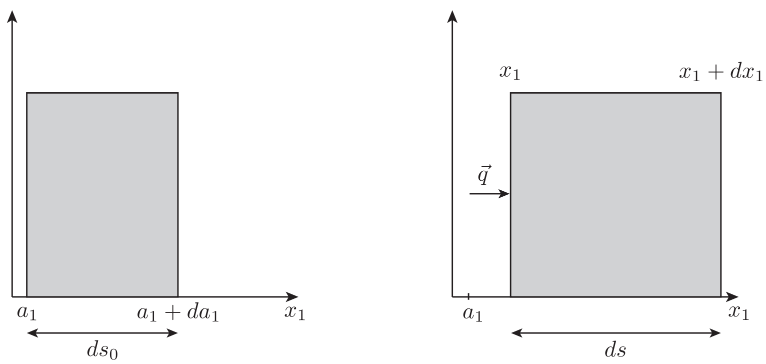


Figure 14.3: Stretching in x_1 direction.

Define the extension E_1 in the x_1 direction as

$$E_1 = \frac{\text{change in length}}{\text{original length}} = \frac{ds - ds_o}{ds_o} = \frac{2E_{11}da_1^2}{(ds + ds_o)ds_o}.$$

Now $ds_o = da_1$ so

$$E_1 = \frac{2E_{11}da_1^2}{(ds + da_1)da_1},$$

and hence

$$E_{11} = \frac{1}{2}E_1\Big(\frac{ds + da_1}{da_1}\Big) = \frac{1}{2}E_1(E_1 + 2) = \frac{1}{2}\Big[(1 + E_1)^2 - 1\Big].$$

Thus E_{11} is related to the extension E_1. For small extensions $E_1^2 \ll E_1$ and $E_{11} \approx E_1$.

Example B: Consider a shear deformation in the x_1 direction.

The displacements are

$$\vec{x}(\vec{a}) = \vec{a} + \alpha(a_2, 0, 0),$$

which can be rearranged to give

$$\vec{a}(\vec{x}) = \vec{x} - \alpha(x_2, 0, 0).$$

From (14.9), using

$$\vec{q} = \vec{x}(\vec{a}) - \vec{a} = \alpha(a_2, 0, 0)$$

we find that

$$E = \left(\begin{array}{ccc} 0 & \frac{\alpha}{2} & 0 \\ \frac{\alpha}{2} & \frac{\alpha^2}{2} & 0 \\ 0 & 0 & 0 \end{array} \right). \tag{14.13}$$

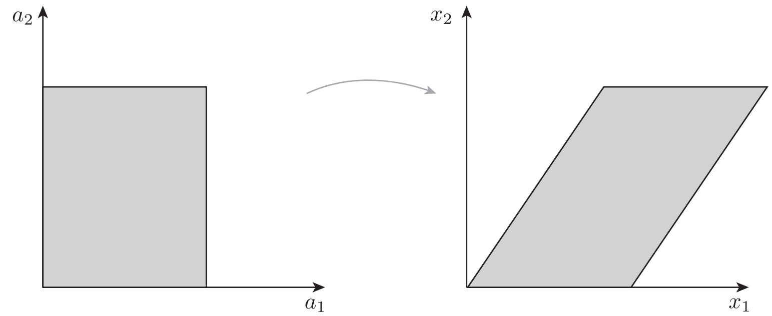


Figure 14.4: Shear deformation

From (14.10), using

$$\vec{q} = \vec{x} - \vec{a}(\vec{x}) = \alpha(x_2, 0, 0),$$

we find that

$$e = \begin{pmatrix} 0 & \frac{\alpha}{2} & 0 \\ \frac{\alpha}{2} & -\frac{\alpha^2}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{14.14}$$

The only difference between the two tensors is in the sign of the quadratic term. The terms linear in α are identical.

We can now write down two expressions for $ds^2 - ds_o^2$. First, in terms of the E_{ij}, we have

$$ds^2 - ds_o^2 = 2E_{jk}da_j da_k = 2E_{12}da_1 da_2 + 2E_{21}da_2 da_1 + 2E_{22}da_2 da_2, \tag{14.15}$$

Using (14.13),

$$ds^2 - ds_o^2 = 2\alpha da_1 da_2 + \alpha^2 da_2^2. \tag{14.16}$$

Alternatively, in terms of the e_{jk} we have

$$ds^2 - ds_o^2 = 2e_{jk}dx_j dx_k = 2e_{12}dx_1 dx_2 + 2e_{21}dx_2 dx_1 + 2e_{22}dx_2 dx_2, \tag{14.17}$$

which, using (14.14), gives

$$ds^2 - ds_o^2 = 2\alpha dx_1 dx_2 - \alpha^2 dx_2^2. \tag{14.18}$$

In elasticity it is usually easier to measure the strain than the actual displacements, i.e., to measure the change in distance between two points that the actual displacements of the points. For small displacements, which is all we will consider, the α^2 terms are negligible. Knowing da_1, da_2 and hence s_o, subjecting the material to a shear strain and then measuring ds, enables us to determine E_{12} using (14.16).

Exercise: For large displacements, for which the α^2 term is not negligible, how could you determine E_{22} and E_{12} using an appropriate set of initial points $\vec{a}$ and $\vec{a} + d\vec{a}$?

14.2.1 Equations of Motion for Linear Elasticity Theory

Continuity Equation: $\rho = \rho_o =$constant is a good approximation for solids.

Momentum Equations: For solids the velocities are normally extremely small so

$$\frac{D\vec{u}}{Dt} = \vec{u}_t + \vec{u} \cdot \vec{\nabla}\vec{u} \approx \vec{u}_t = \frac{\partial^2 \vec{q}}{\partial t^2},$$

and the momentum equation becomes

$$\rho_o \frac{\partial^2 \vec{q}}{\partial t^2} = -\rho_o \vec{\nabla}\Pi + \vec{\nabla} \cdot \tau. \tag{14.19}$$

Constitutive Theory

For a Newtonian fluid we related the stress tensor τ to the strain rate tensor, i.e., to the rate at which material volumes are being deformed. For an elastic solid the stress tensor is related to the amount of deformation.

Consider a small bar of material of length l. When it is subjected to a tensile force of magnitude T it will stretch. Let its new length be $l + \Delta l$. If T is plotted as a function of Δl (i.e., how much tension is needed to give a specified elongation Δl), there are many possible forms of the graph. One example is sketched below.

For most materials there is a part of the curve OP, where O is the origin, along which $T \propto \Delta l$, i.e., the tension and elongation are linearly related. If the tension is removed Δl goes to zero, i.e., the material returns to its original state. This is what we mean by saying a material is *elastic*.

Beyond the point P many things can happen. Usually the relationship between T and Δl becomes nonlinear, but remains elastic until some point Y. Beyond Y the deformation becomes *nonelastic*. Unloading (removal of tensile force) leaves a permanent elongation - the material does not return to its original shape. The material is said to be *plastic*.

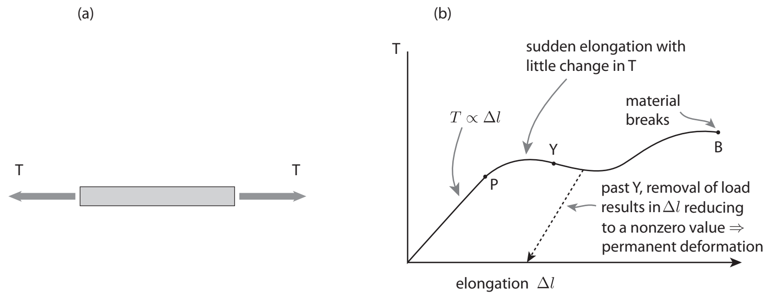


Figure 14.5: (a) Piece of material subject to a tensile force. (b) Schematic illustrating tensile force T as a function of the elongation Δl. At first T varies linearly with Δl (Hooke's law). For larger deformations the material may undergo a sudden stretching and the tension necessary for an elongation Δl can decrease. For elongations larger than a critical value Y the material is permanently deformed: when the load is removed Δl remains positive. Past some point the material breaks.

For our purposes the key point is that for small elongations the tensile force is linearly related to the elongation. For more complicated but small deformations (e.g., twisting) we postulate that the stresses are linearly related to the strain, i.e., the τ_{ij} are linearly related to the the Cauchy Strain tensor e_{ij}. Assuming the medium is isotropic and using the symmetry of τ we have

$$\tau_{ij} = 2\mu e_{ij} + \lambda e_{kk}\delta_{ij}. \tag{14.20}$$

This has the same form that we had for a Newtonian fluid. The difference is that here e is the strain tensor, not the strain rate tensor. The constants μ and λ are called Lamé Constants.

Alternative constants, related to μ and λ are

$$\text{Young's Modulus:} \qquad E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu},$$

and

$$\text{Poisson's Ratio:} \qquad \nu = \frac{\lambda}{2(\lambda + \mu)},$$

which is dimensionless. In terms of Young's modulus and the Poisson ratio we can write (14.20) as

$$\tau_{ij} = \frac{E}{1+\nu}\Big(e_{ij} + \frac{\nu}{1-2\nu} e_{kk}\delta_{ij}\Big). \tag{14.21}$$

This can be solved to express the e_{ij} in terms of the τ_{ij} as

$$e_{ij} = \frac{1+\nu}{E}\tau_{ij} - \frac{\nu}{E}\tau_{kk}\delta_{ij}. \tag{14.22}$$

material	E (Pa)	ν
Steel	2×10^{11}	0.29
Al Alloys	0.7×10^{11}	0.31
Copper	1.1×10^{11}	0.33
Glass	0.5×10^{11}	0.25
Concrete	0.25×10^{11}	0.20

Table 14.1: Properties of some common materials

We now have a closed set of equations to solve:

$$\rho_o \frac{\partial^2 \vec{q}}{\partial t^2} = -\rho_o \vec{\nabla}\Pi + \vec{\nabla} \cdot \tau, \tag{14.23}$$

$$\tau_{ij} = 2\mu e_{ij} + \lambda e_{kk}\delta_{ij}, \tag{14.24}$$

and

$$e_{jk} = \frac{1}{2}\Big(\frac{\partial q_j}{\partial x_k} + \frac{\partial q_k}{\partial x_j}\Big). \tag{14.25}$$

There are fifteen equations for fifteen unknowns: the 6 components of the symmetric stress tensor τ, the 6 components of the symmetric strain tensor e and the three components of the displacement vector $\vec{q}$.

Typical Static Problem

We now describe a solution procedure for a typical problem in which the objective is to find the steady state deformation for a given applied force. Consider, for example, a cylinder of material hanging from a horizontal, flat surface (Figure 14.6). It is deformed under the influence of gravity.

At steady state the momentum equation reduces to

$$-\frac{\partial \Pi}{\partial x_i} + \frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j}. \tag{14.26}$$

Here the body force is gravity. We assume that the surface stresses are known. Thus we need to find the τ_{ij} which satisfy this equation and the known surface stresses $\vec{t} = \tau \cdot \hat{n}$ on all the surfaces. In particular, for this example $\vec{t} = 0$ along the sides and bottom and $\vec{t} = \rho g l \hat{k}$ on the top surface. A problem here is that we have three equations for six unknowns, hence we can only solve for three of the τ_{ij} or for three linear combinations of the τ_{ij}.

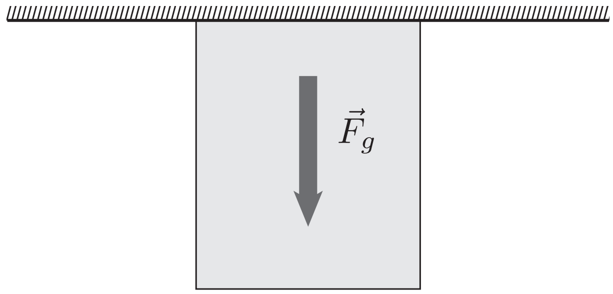


Figure 14.6:

Once the τ_{ij} are determined we solve (14.20) for the e_{ij}. This is a set of six equations for six unknowns.

Finally, we solve (14.40), that is

$$e_{jk} = \frac{1}{2}\Big(\frac{\partial q_j}{\partial x_k} + \frac{\partial q_k}{\partial x_j}\Big),$$

for the q_i. This involves six equations for three unknowns. But, the 'known' e_{ij} involve three undetermined functions (from the first problem). There are really six unknowns. The e_{ij} must satisfy three compatibility conditions which determine the three unknown functions. We now look at this in more detail.

The compatibility problem

Suppose that the e_{ij} are known and we wish to find the q_i.

The six equations to be solved are

$$\begin{aligned}
q_{1,1} = \frac{\partial q_1}{\partial x_1} &= e_{11},\\
q_{2,2} = \frac{\partial q_2}{\partial x_2} &= e_{22},\\
q_{3,3} = \frac{\partial q_3}{\partial x_3} &= e_{33},\\
q_{1,2} + q_{2,1} = \frac{\partial q_1}{\partial x_2} + \frac{\partial q_2}{\partial x_1} &= 2e_{12},\\
q_{2,3} + q_{3,2} = \frac{\partial q_2}{\partial x_3} + \frac{\partial q_3}{\partial x_2} &= 2e_{23},\\
q_{3,1} + q_{1,3} = \frac{\partial q_3}{\partial x_1} + \frac{\partial q_1}{\partial x_3} &= 2e_{31},
\end{aligned}$$

where we use the ',i' notation to indicate partial differentiation with respect to x_i.

Theorem: The system of PDEs

$$\begin{aligned} q_{1,1} &= e_{11}, \\ q_{2,2} &= e_{22}, \\ q_{1,2} + q_{2,1} &= 2e_{12}, \end{aligned}$$

has a single valued solution if

$$e_{11,22} + e_{22,11} = 2e_{12,12}. \tag{14.27}$$

Proof: Suppose we try to solve the system. If there is a solution then

$$q_{1,1} = e_{11} \Rightarrow q_1 = \int_a^{x_1} e_{11}\, dx_1 + F(x_2), \tag{14.28}$$

and

$$q_{2,2} = e_{22} \Rightarrow q_2 = \int_b^{x_2} e_{22}\, dx_2 + G(x_1). \tag{14.29}$$

Since equations do not involve derivatives wrt x_3 we do not explicitely write the x_3 dependence of F and G.

Substituting these expressions in the the third equation gives

$$q_{1,2} + q_{2,1} = \int^{x_1} e_{11,2}\, dx_1 + \int^{x_2} e_{22,1}\, dx_2 + F'(x_2) + G'(x_1) = 2e_{12}.$$

Taking $\partial^2/\partial x_1 \partial x_2$ gives $2e_{12,12} - e_{11,22} - e_{22,11} = 0$. Thus, (14.27) is a necessary condition in order for a solution to exist. On the otherhand, if (14.27) is satisfied it follows that, after integrating wrt x_1 and then x_2,

$$2e_{12} - \int_a^{x_1} e_{11,2}\, dx_1 - \int_b^{x_2} e_{22,1}\, dx_2 = A(x_1) + B(x_2).$$

From (14.28) and (14.29) it follows that $F' = B$ and $G' = A$, or $F = \int^{x_2} B\, dx_2$ and $G = \int^{x_1} B\, dx_1$. This determines q_1 and q_2 in terms of two arbitrary functions F and G, which must be determined from boundary conditions.

In this manner one can solve for q_1, q_2, and q_3. There are three compatibility conditions (rotating indices in (14.27)).

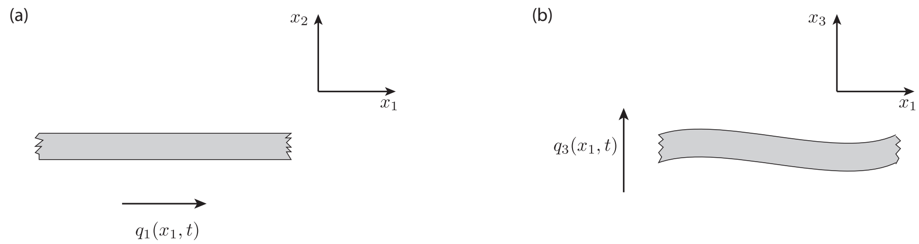


Figure 14.7: (a) Longitudinal vibrations in a bar. Material displacements and direction of wave propagation are aligned with the bar in the x_1 direction. (b) Transverse waves in a bar. Material displacements are in the x_3 direction, perpendicular to the direction of wave propagation x_1.

Example: Longitudinal Vibrations of a Bar

Consider longitudinal vibrations in a bar (see Figure 14.7(a)). The displacement in the bar has the form $\vec{q} = (q_1(x_1, t), 0, 0)$. We have

$$e = \begin{pmatrix} \frac{\partial q_1}{\partial x_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{14.30}$$

and

$$\tau = 2\mu e + \lambda e_{kk} I = \begin{pmatrix} (2\mu + \lambda)\frac{\partial q_1}{\partial x_1} & 0 & 0 \\ 0 & \lambda\frac{\partial q_1}{\partial x_1} & 0 \\ 0 & 0 & \lambda\frac{\partial q_1}{\partial x_1} \end{pmatrix}. \tag{14.31}$$

Question: What does the non-zero values of τ_{22} and τ_{33} imply about the forces necessary to give the assumed deformation?

The first component of the momentum equation (ignoring the body force) gives

$$\frac{\partial^2 q_1}{\partial t^2} = \frac{1}{\rho_o}\frac{\partial \tau_{1j}}{\partial x_j} = \Big(\frac{2\mu + \lambda}{\rho_o}\Big)\frac{\partial^2 q_1}{\partial x_1^2},$$

or

$$\frac{\partial^2 q_1}{\partial t^2} - C_D^2 \frac{\partial^2 q_1}{\partial x_1^2} = 0, \tag{14.32}$$

where

$$C_D^2 = \frac{2\mu + \lambda}{\rho_o}. \tag{14.33}$$

These waves are called *longitudinal waves*, which means the particle motion is back and forth in the same direction as wave propagation. These waves are sound waves and C_D is the sound speed. For steel its value is around 6×10^3 m/s.

Exercise: Check the other components of the momentum equations. What do they give?

Example: Transverse Waves of a Bar

Now imagine the displacement in a bar has the form $\vec{q} = (0, 0, q_3(x_1, t))$ (Figure 14.7(b)). Then

$$e = \begin{pmatrix} 0 & 0 & \frac{1}{2}\frac{\partial q_3}{\partial x_1} \\ 0 & 0 & 0 \\ \frac{1}{2}\frac{\partial q_3}{\partial x_1} & 0 & 0 \end{pmatrix}, \tag{14.34}$$

and

$$\tau = \begin{pmatrix} 0 & 0 & \mu\frac{\partial q_3}{\partial x_1} \\ 0 & 0 & 0 \\ \mu\frac{\partial q_3}{\partial x_1} & 0 & 0 \end{pmatrix}. \tag{14.35}$$

The third component of the momentum equation gives (ignoring the body force)

$$\frac{\partial^2 q_3}{\partial t^2} = \frac{1}{\rho_o}\frac{\partial \tau_{3j}}{\partial x_j} = \frac{\mu}{\rho_o}\frac{\partial^2 q_3}{\partial x_1^2},$$

or

$$\frac{\partial^2 q_3}{\partial t^2} - C_T^2\frac{\partial^2 q_3}{\partial x_1^2} = 0, \tag{14.36}$$

where

$$C_T^2 = \frac{\mu}{\rho_o}. \tag{14.37}$$

These waves are called *tranvserse waves*. The particle motion is transverse (perpendicular) to the direction of wave propagation. In general, $C_D \neq C_T$, so sound waves and transverse waves have different propagation speeds.

Example: General elastic waves in an unbounded media.

We can rewrite the momentum equation (ignoring the body force)

$$\rho_o\frac{\partial^2 \vec{q}}{\partial t^2} = \vec{\nabla} \cdot \tau, \tag{14.38}$$

completely in terms of the q_i by using the constitutive relation

$$\tau_{ij} = 2\mu e_{ij} + \lambda e_{kk}\delta_{ij}, \tag{14.39}$$

and

$$e_{jk} = \frac{1}{2}\Big(\frac{\partial q_j}{\partial x_k} + \frac{\partial q_k}{\partial x_j}\Big). \tag{14.40}$$

Doing so we find that (exercise)

$$\rho_o \frac{\partial^2 \vec{q}}{\partial t^2} = \mu \nabla^2 \vec{q} + (\mu + \lambda)\vec{\nabla}\big(\vec{\nabla}\cdot\vec{q}\big). \tag{14.41}$$

From this we can derive wave equations for two types of waves. First, taking the divergence gives

$$\rho_o \frac{\partial^2}{\partial t^2}\frac{\partial q_i}{\partial x_i} = \mu \nabla^2 \frac{\partial q_i}{\partial x_i} + (\mu + \lambda)\nabla^2 \frac{\partial q_k}{\partial x_k}.$$

Letting

$$\delta = \frac{\partial q_i}{\partial x_i} = \vec{\nabla}\cdot\vec{q}, \tag{14.42}$$

which is called the *dilation*, we have

$$\frac{\partial^2 \delta}{\partial t^2} - C_D^2 \nabla^2 \delta = 0. \tag{14.43}$$

This is the 3-D wave equation for the dilation δ. Solutions of this equation are called dilatational waves, or P-waves in siesmology (P for pressure).

We now consider a different type of wave. Taking the partial derivative with respect to x_j of (14.41) we have

$$\rho_o \frac{\partial^2}{\partial t^2}\frac{\partial q_i}{\partial x_j} = \mu \nabla^2 \frac{\partial q_i}{\partial x_j} + (\mu + \lambda)\frac{\partial^2}{\partial x_i \partial x_j}\Big(\vec{\nabla}\cdot\vec{q}\Big).$$

Switching i and j gives

$$\rho_o \frac{\partial^2}{\partial t^2}\frac{\partial q_j}{\partial x_i} = \mu \nabla^2 \frac{\partial q_j}{\partial x_i} + (\mu + \lambda)\frac{\partial^2}{\partial x_j \partial x_i}\Big(\vec{\nabla}\cdot\vec{q}\Big).$$

The difference of these two equations gives

$$\frac{\partial^2 \omega_{ij}}{\partial t^2} - C_T^2 \nabla^2 \omega_{ij} = 0. \tag{14.44}$$

where

$$\omega_{ij} = \frac{\partial q_i}{\partial x_j} - \frac{\partial q_j}{\partial x_i}. \tag{14.45}$$

Note that the time derivative of ω_{ij} is the vorticity, so ω_{ij} is a measure of how much a particle has rotated. The waves described by (14.44) are called equivoluminal waves, transverse waves, shear waves and S-waves in seismology.

The preceding examples of longitudinal vibrations and transverse waves in a bar are special examples of dilatational and shear waves.

Recall that the vector field $\vec{q}$ can be decomposed into a divergence free part and an irrotational part (i.e., a gradient). That is, we can set

$$\vec{q} = \vec{v} + \vec{\omega},$$

where

$$\vec{\nabla} \cdot \vec{\omega} = 0,$$

and

$$\vec{\nabla} \times \vec{v} = 0.$$

You should convince yourself that the components of $\vec{v}$, the irrotational-free part of $\vec{q}$ satisfy (14.43) while the components of $\vec{\omega}$, the dilation free part of $\vec{q}$, satisfy (14.44). The dilation-free and irrotational-free parts of the deformation propagate at different speeds.

Chapter 15

NON-NEWTONIAN FLUIDS: THE MAXWELL MODEL

There are many ways in which a fluid can be non-Newtonian or a solid can behave in a non-elastic way. Here we will briefly discuss a model for a medium that can behave like both a solid and a viscous fluid in appropriate limits. Many other types of behaviour, requiring different models, are possible.

Figure 15.1: (a) Longitudinal vibrations in a bar. Material displacements and direction of wave propagation are aligned with the bar in the x_1 direction. (b) Transverse waves in a bar. Material displacements are in the x_3 direction, perpendicular to the direction of wave propagation x_1.

Consider again a solid undergoing a shear deformation $\vec{q} = (\gamma x_2, 0, 0)$. For a linear elastic solid, the stress tensor (Figure 15.1) is

$$\tau = \begin{pmatrix} 0 & \mu\gamma & 0 \\ \mu\gamma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{15.1}$$

Because of intermolecular forces which act to try to return molecular structures back to their initial shape, constant forces must be applied to the material to maintain the deformation. In reality, if the deformation is held fixed for long periods of time, the molecules will slowly adjust by moving and by transferring bonds from one molecule to another. Thus, the applied force necessary to maintain the deformation decreases with time. The time scale can be extremely long, hundreds or thousands of years. Once can also think of liquids behaving in a similar manner, in which case the time scale is extremely short (e.g., estimated to be 10^{-12} s for water).

If the shear deformation depicted above occurs instantaneously at time t then the shear stress necessary to maintain the deformation at time $t + T > t$ can be modelled as

$$\tau_{12}(t + T) = \mu\gamma f(T), \tag{15.2}$$

where $F(T)$ is a decreasing function of time with $f(0) = 1$.

Suppose the deformation was not instantaneous but occured over a finite time interval $[t_o, t]$. In particular, let $d\gamma(t')$ be the change in the shear deformation that occured between time t' and time $t' + dt'$. The change in the applied stress necessary to give this change in deformation at t' will be $\mu d\gamma$, however by time t the contribution to the total applied stress necessary to hold the deformation will have decreased to $\mu d\gamma f(t - t') = \mu \frac{d\gamma}{dt}(t') f(t - t')\, dt'$. We assume that all the changes in deformation and the applied stress are independent of one another and cumulate linearly. Thus, the total deformation at time t will be

$$\gamma = \int_{t_o}^{t} \frac{d\gamma}{dt}(t')\, dt', \tag{15.3}$$

and the total shear stress necessary to maintain the deformation will be

$$\tau_{12}(t) = \int_{t_o}^{t} \mu \frac{d\gamma}{dt}(t') f(t - t')\, dt'. \tag{15.4}$$

If f decays very slowly, meaning that f is essentially constant over the interval, then on $[t_o, t]$ we have $f(t - t') \approx f(0) = 1$ and

$$\tau_{12}(t) \approx \int_{t_o}^{t} \mu \frac{d\gamma}{dt}(t')\, dt' = \mu\gamma. \tag{15.5}$$

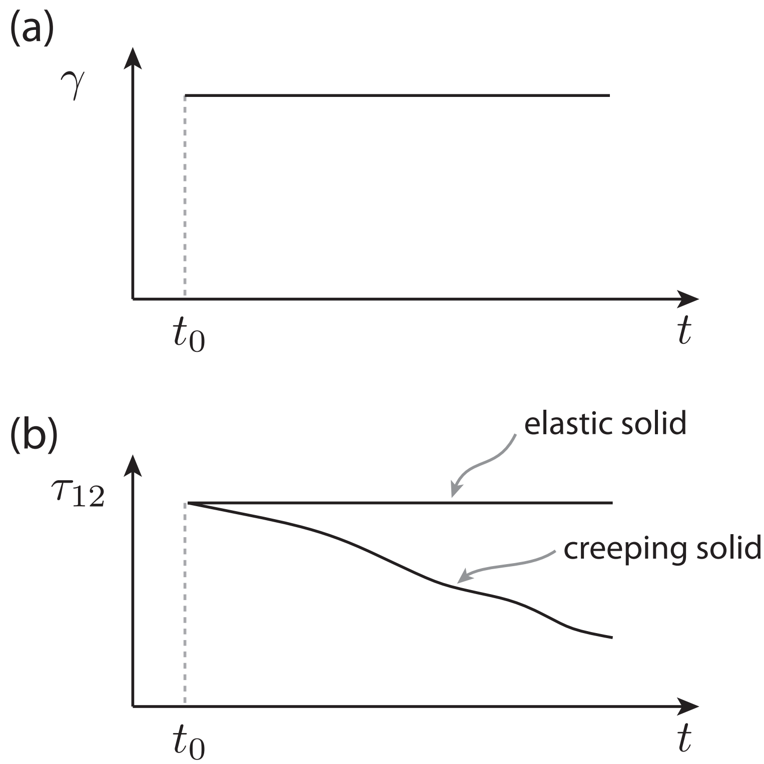


Figure 15.2: (a) A solid is deformed by a fixed amount for $t > t_0$. (b) For an elastic solid the stress τ_{12} is proportional to the displacement and remains constant for $t > t_0$. For a creeping solid the stress required to maintain the deformation slowly decreases in time.

This is the behaviour for sudden deformations — deformations which occur rapidly relative to the time scale over which f decays. It is the elastic limit.

If, on the other hand, f varies very rapidly compared with γ (the deformation is slow), then, setting $T = t - t'$, we can write $\tau_{12}(t)$ as

$$\begin{aligned}
\tau_{12}(t) &= \mu \int_0^{t-t_o} \frac{d\gamma}{dt}(t-T) f(T)\, dT, \\
&\approx \mu \int_0^{\Delta t} \frac{d\gamma}{dt}(t-T) f(T)\, dT, \\
&\approx \mu \frac{d\gamma}{dt}(t) \int_0^{\Delta t} f(T)\, dT, \\
&\approx \mu \frac{d\gamma}{dt}(t) \int_0^{\infty} f(T)\, dt.
\end{aligned}$$

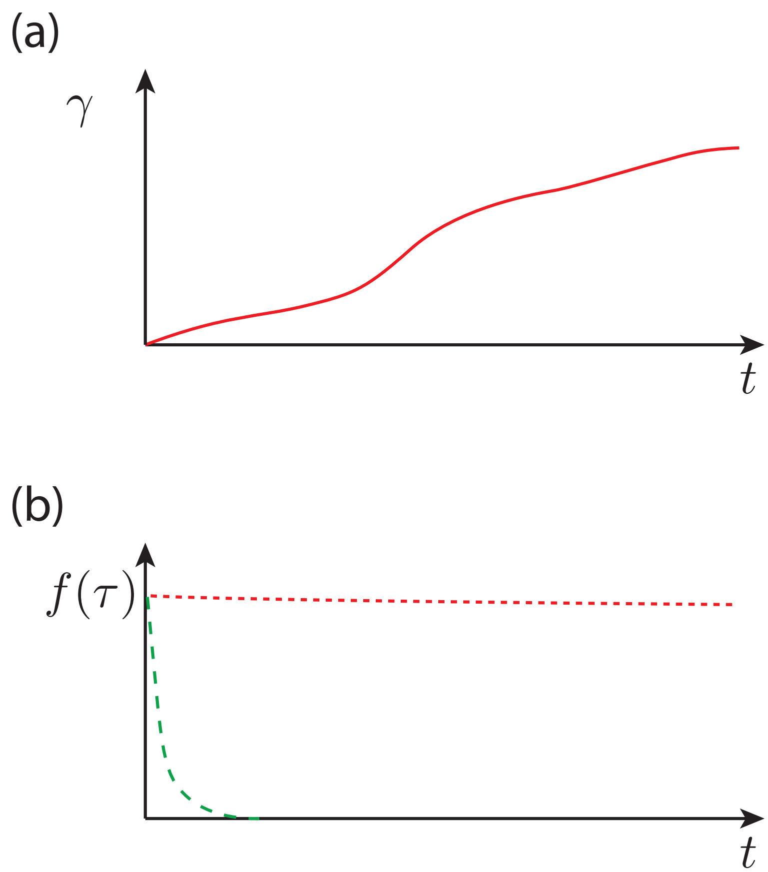


Figure 15.3: (a) Deformation as a function of time. (b) Force $f(\tau)$ required to give the deformation. f can vary slowly (red short-dashed curve) compared with γ or rapidly (green long-dashed curve).

where Δt is small enough that $\frac{d\gamma}{dt}$ is approximately constant over a time interval of length Δt and large enough that $f(T) \approx 0$ for $T > \Delta t$. Defining

$$T_r = \int_o^\infty f(t)\,dt, \tag{15.6}$$

to be the relaxation time (note f is dimensionless so the integral has dimensions of time), we have

$$\tau_{12}(t) = \mu T_r \frac{d\gamma}{dt}. \tag{15.7}$$

In this limit, the shear stress is proportional to the rate of deformation and is the same expression we had for a Newtonian fluid where μT_r is the shear viscosity.

This model has the feature that on short time scales matter behaves like an elastic solid while on long time scales it behaves like a viscous fluid. Such behaviour is called **viscoelastic**. This is one way to model such things as rock, ice and glass. Think of the motion of rock in the Earth's mantle (e.g., plate tectonics and the growth of mountains) or the flow of glaciers which occurs on long time scales.

The Maxwell Model

One particular model for the function f is exponential decay

$$f(T) = e^{-T/T_r}. \tag{15.8}$$

Using this in (15.4) gives

$$\tau_{12}(t) = \mu \int_{t_o}^{t} \frac{d\gamma}{dt}(t') e^{-(t-t')/T_r}\, dt'.$$

Differentiating leads to

$$\frac{d\tau_{12}}{dt} + \frac{1}{T_r}\tau_{12} = \mu \frac{d\gamma}{dt}. \tag{15.9}$$

In the limit $T_r \to 0$, assuming that $T_r \frac{d\tau_{12}}{dt}$ becomes small compared with τ_{12}, we have $\tau_{12} = \mu T_r \frac{d\gamma}{dt}$, the viscous fluid limit, while in the limit $T_r \to \infty$ we have $\frac{d\tau_{12}}{dt} = \mu \frac{d\gamma}{dt}$, or $\tau_{12} = \mu\gamma$, the elastic limit.

Appendix A

Appendix A: The Chain Rule

Consider the change of variable

$$x = x(\alpha, t) = a(t) + \alpha\Big(b(t) - a(t)\Big), \tag{A.0.1}$$

used in the first problem of assignment 1. This is used to convert functions of x and t to functions of α and t. To be very clear we should write

$$x = f(\alpha, t) = a(t) + \alpha\Big(b(t) - a(t)\Big), \tag{A.0.2}$$

with inverse

$$\alpha = g(x, t), \tag{A.0.3}$$

but we usually use a somewhat sloppy notation and say that

$$x = x(\alpha, t). \tag{A.0.4}$$

Thus, sometimes x is a function. Sometimes it is an independent variable. One must interpret the meaning from the context in which it is used. This 'sloppy' notation is useful because x, the independent variable, and $x(\alpha, t)$, the function, *have the same physical meaning*.

The Jacobian of the transformation is

$$J(\alpha, t) = \frac{\partial f}{\partial \alpha}(\alpha, t). \tag{A.0.5}$$

In the assignment problem, the function $v = \frac{\partial x}{\partial t}$ is defined. To be clearer we define

$$\tilde{v}(\alpha, t) = \frac{\partial f}{\partial t}(\alpha, t), \tag{A.0.6}$$

and let $v(x,t)$ be defined by

$$v(x,t) = \tilde{v}\big(g(x,t),t\big). \tag{A.0.7}$$

While $v(x,t)$ and $\tilde{v}(\alpha,t)$ are different functions ($v(1,2)$ and $\tilde{v}(1,2)$ would in general have different values), they have the same physical meaning. Thus, authors almost always will not use the tilde. They will used the functions $v(x,t)$ and $v(\alpha,t)$ where, again, which function is being referred to is to be understood from the context in which it is used.

What is $\partial v/\partial x$ in terms of $\tilde{v}$ or f?

$$\begin{aligned} v_x = \frac{\partial v}{\partial x}(x,t) &= \frac{\partial}{\partial x}\tilde{v}\big(g(x,t),t\big) \\ &= \frac{\partial g}{\partial x}\frac{\partial \tilde{v}}{\partial \alpha} \\ &= \frac{\partial g}{\partial x}\frac{\partial^2 f}{\partial \alpha \partial t} \\ &= \frac{\partial g}{\partial x}\frac{\partial^2 f}{\partial t \partial \alpha}. \end{aligned} \tag{A.0.8}$$

Here, it is understood that the functions of α and t are evaluated at $\alpha = g(x,t)$. Using definition (A.0.5) of the Jacobian, we have the result

$$\frac{\partial v}{\partial x}(x,t) = \frac{\partial g}{\partial x}\frac{\partial J}{\partial t}. \tag{A.0.9}$$

Since $f(g(x,t),t) = x$ (the functions f and g are inverses of each other)

$$1 = \frac{\partial x}{\partial x} = f_\alpha g_x = J g_x, \tag{A.0.10}$$

and so

$$g_x = 1/J. \tag{A.0.11}$$

Thus

$$\frac{\partial v}{\partial x}(x,t) = \frac{1}{J}\frac{\partial J}{\partial t}, \tag{A.0.12}$$

Using the 'sloppy' notation (which you should get used to) one could get the same result as follows. First,

$$J_t = \frac{\partial^2 x}{\partial t \partial \alpha} = \frac{\partial^2 x}{\partial \alpha \partial t} = \frac{\partial}{\partial \alpha}\frac{\partial x}{\partial t} = \frac{\partial}{\partial \alpha} v. \tag{A.0.13}$$

Here, *it must be understood that $v = x_t$ is a function of α and t, since $x(\alpha,t)$ is!* We want an expression involving $v(x,t)$. Using the chain rule

$$J_t = \frac{\partial x}{\partial \alpha}\frac{\partial v}{\partial x} = J v_x. \tag{A.0.14}$$

Appendix B

Appendix B: Vector Calculus Highlights

This course makes heavy use of vector calculus. Vector calculus is used mainly as a theoretical tool to derive the partial differential equations governing the behaviour of a continuum. In these notes those parts of vector calculus that we will use are briefly reviewed. You may also need to review relevent ideas from your vector calculus course notes.

B.1 Vector Calculus on the Plane

We will start with vector calculus in two dimensions. Consider a curve C on the xy-plane (not necessarily closed). The integral of a function $f(x, y)$ along the curve is denoted by

$$\int_C \Phi(x, y)\, ds. \tag{B.1.1}$$

Here s is arc length along the curve. To evaluate the integral one must first parameterize the curve. If $\vec{x}(\alpha)$, for $\alpha \in [a, b]$, is such a parameterization, then one has

$$ds = \sqrt{dx^2 + dy^2} = \left(\left(\frac{dx}{d\alpha}\right)^2 + \left(\frac{dy}{d\alpha}\right)^2 \right)^{1/2} d\alpha, \tag{B.1.2}$$

where (dx, dy) is the change in (x, y) as one moves along a small segment of the curve of length ds. This can be written as

$$ds = \left| \frac{d\vec{x}}{d\alpha} \right| d\alpha. \tag{B.1.3}$$

Hence,

$$\int_C f(x,y)\,ds = \int_a^b \Phi\big(x(\alpha), y(\alpha)\big)\left(\left(\frac{dx}{d\alpha}\right)^2 + \left(\frac{dy}{d\alpha}\right)^2\right)^{1/2} d\alpha. \tag{B.1.4}$$

The line integral has been converted into an integral on an interval of the real line, to be evaluated by standard techniques.

The function Φ often has two special forms: the tangential and normal components of a vector field relative to the curve. In these cases, the line integrals have the forms

$$\int_C \vec{f}\cdot\hat{t}\,ds, \tag{B.1.5}$$

and

$$\int_C \vec{f}\cdot\hat{n}\,ds, \tag{B.1.6}$$

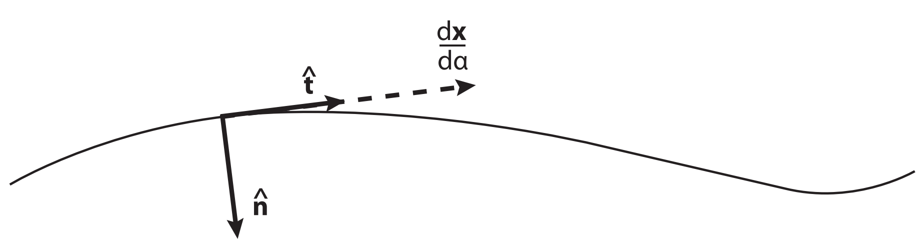
respectively where $\hat{t}$ and $\hat{n}$ are unit vectors which are respectively tangent and normal to the curve. The unit tangent vector can point in either direction along the curve. The unit normal vector can point to either side of the curve. One must always indicate which direction these unit vectors are in.

If there is a need to evaluate the line integral the curve must be parameterized. If $\vec{x}(\alpha)$ parameterizes the curve, the vector

$$\frac{d\vec{x}}{d\alpha} = \left(\frac{dx}{d\alpha}, \frac{dy}{d\alpha}\right), \tag{B.1.7}$$

is tangent to the curve. A unit vector is then

$$\hat{t} = \frac{\frac{d\vec{x}}{d\alpha}}{\left|\frac{d\vec{x}}{d\alpha}\right|}. \tag{B.1.8}$$



This may lead to a complicated expression, however what is really needed is $\hat{t}\,ds$ and using (B.1.3) this is given by

$$\hat{t}\,ds = \frac{d\vec{x}}{d\alpha}\,d\alpha = \left(\frac{dx}{d\alpha}, \frac{dy}{d\alpha}\right)d\alpha. \tag{B.1.9}$$

Note that the middle expression can be written as simply $d\vec{x}$ so the integral is often written as

$$\int_C \vec{f}\cdot\hat{t}\,ds = \int_C \vec{f}\cdot d\vec{x}. \tag{B.1.10}$$

These two notations are equivalent. Either way, to evaluate the integral the integral must be converted to a regular integral over a segment of the real line using a parameterization of the curve. The result is

$$\int_C \vec{f}\cdot\hat{t}\,ds = \int_a^b \vec{f}(\vec{x}(\alpha)\cdot\frac{d\vec{x}}{d\alpha}\,d\alpha. \tag{B.1.11}$$

Note that no messy square roots appear! Note also that $\hat{t}$ has been assumed to point in the same direction as $d\vec{x}/d\alpha$. That is, the parameter α must increase in the direction of $\hat{t}$. If $\hat{t}$ points in the opposite direction then you must multiply by minus one.

The unit normal vector is found by rotating $\hat{t}$ by $\pi/2$ which gives

$$\hat{n}\,ds = \pm\Big(\frac{dy}{d\alpha}, -\frac{dx}{d\alpha}\Big)\,d\alpha. \tag{B.1.12}$$

The sign depends on the direction of $\hat{n}$ and of $d\vec{x}/d\alpha$. Choose the sign so $\hat{n}$ points in the desired direction.

When the curve C is a simple closed curve the line integrals of $\vec{f}\cdot\hat{t}$ or $\vec{f}\cdot\hat{n}$ are equal to area integrals over the region R bounded by the curve provided the components of $\vec{f}$ satisfy some smoothness conditions. In the following we set $\vec{f} = (f, g)$ where f and g are C^1 functions.

Stokes' Theorem on the plane says that

$$\oint_C \vec{f}\cdot\hat{t}\,ds = \iint\limits_R \Big\{\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\Big\}\,dA, \tag{B.1.13}$$

where the line integral is **counter-clockwise** around C. The Divergence Theorem on the plane states that

$$\oint_C \vec{f}\cdot\hat{n}\,ds = \iint\limits_R \Big\{\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\Big\}\,dA, \tag{B.1.14}$$

where $\hat{n}$ points out of the region R. These theorems are equivalent. One can easily be obtained from the other. Suppose $\hat{t}\,ds = (dx, dy)$ points counterclockwise around C. Then if $\hat{n}$ is the outward unit normal, $\hat{n}\,ds$ is obtained by rotating $\hat{t}\,ds$

clockwise by $\pi/2$. This gives $\hat{n}\,ds = (dy, -dx)$. Thus

$$\begin{aligned}(f,g)\cdot\hat{n}\,ds &= (f,g)\cdot(dy,-dx),\\ &= -g\,dx + f\,dy\\ &= (-g,f)\cdot(dx,dy)\\ &= (-g,f)\cdot\hat{t}\,ds.\end{aligned}$$

Exercise: *Using this result, given (B.1.13) derive (B.1.14) and visa-versa.*

B.2 Vector Calculus in three dimensions

Line integrals in 3-D of the form (B.1.1) and (B.1.5) where C no longer lies in a plane are evaluated in exactly the manner discussed above. Line integrals of the form (B.1.6 are no longer very common. They do occur in situations where the curve C lies on a surface in which case one can have a unit normal vector $\hat{n}$ which is normal to the curve and tangent to the surface or which is normal to both the curve and the surface. The first of these situation arises when surface tension forces are considered on a curved interface between two liquids.

In three dimensions Stokes' Theorem and the Divergence Theorem are different. Stokes' Theorem says that

$$\oint_C \vec{f}\cdot\hat{t}\,ds = \iint\limits_S \left(\vec{\nabla}\times\vec{f}\right)\cdot\hat{n}\,dS, \tag{B.2.1}$$

where S is any smooth surface bounded by C and $\hat{n}$ is a unit normal vector to S such that the line integral is counterclockwise around $\hat{n}$. In the special case where $\vec{f} = \big(f(x,y), g(x,y), 0\big)$ and the curve lies in the (x,y) plane this reduces to

$$\begin{aligned}\oint_C \vec{f}\cdot\hat{t}\,ds &= \iint\limits_S \left(\vec{\nabla}\times\vec{f}\right)\cdot\hat{k}\,dS,\\ &= \iint\limits_S \Big(0,0,g_x - f_y\Big)\cdot\hat{k}\,dS,\\ &= \iint\limits_S \Big(g_x - f_y\Big)\,dA,\end{aligned}$$

which is (B.1.13).

In three dimensions the Divergence Theorem relates an integral over a closed surface to a volume integral. It states that

$$\iint\limits_{\partial R} \vec{f} \cdot \hat{n}\, dS = \iiint\limits_{R} \vec{\nabla} \cdot \vec{f}\, dV, \tag{B.2.2}$$

where $\hat{n}$ is a unit vector pointing out of the volume R with surface ∂R.

For both Stokes' Theorem and the Divergence Theorem, the components of the vector field are assumed to be C^1.

Appendix C

Appendix C: Balance of Angular Momentum and Symmetry of the Stress Tensor

Here we prove that the stress tensor is symmetric if the balance of angular moment holds, i.e., it the rate of change of the angular momentum of a material volume is equal to the total torque acting on it. The total torque is comprised of two parts. The torque associated with the body force and that associated with the surface force. Mathematically, we have

$$\frac{d}{dt}\iiint\limits_{W(t)} \rho\vec{x}\times\vec{u}\,dV = \iiint\limits_{W(t)} -\rho\Big(\vec{x}\times\vec{\nabla}\Pi\Big)\,dV + \iint\limits_{\partial W(t)} \vec{x}\times\vec{t}\,dS, \qquad \text{(C.0.1)}$$

Using the transport theorem, the left side is

$$\begin{aligned}
\iiint\limits_{W(t)} \rho\frac{D}{Dt}\Big(\vec{x}\times\vec{u}\Big)\,dV &= \iiint\limits_{W(t)} \rho\left(\frac{D\vec{x}}{Dt}\times\vec{u}+\vec{x}\times\frac{D\vec{u}}{Dt}\right)dV \\
&= \iiint\limits_{W(t)} \rho(\vec{u}\times\vec{u}+\vec{x}\times\frac{D\vec{u}}{Dt}\Big)\,cV \\
&= \iiint\limits_{W(t)} \vec{x}\times\Big(-\rho\vec{\nabla}\Pi+\vec{\nabla}\cdot\tau\Big)\,dV.
\end{aligned}$$

Exercise: *Prove that $D\vec{x}/Dt=\vec{u}$, a result used above.*

Using this in (C.0.1) we have

$$\iiint\limits_{W(t)} \vec{x} \times \left(\vec{\nabla} \cdot \tau\right) dV = \iint\limits_{\partial W(t)} \vec{x} \times \vec{t}\, dS. \tag{C.0.2}$$

Now the i$^{\text{th}}$ component of $\vec{x} \times \vec{\nabla} \cdot \tau$ is

$$\begin{aligned}
\left(\vec{x} \times \vec{\nabla} \cdot \tau\right)_i = \epsilon_{ijk} x_j \left(\vec{\nabla} \cdot \tau\right)_k \quad &= \quad \epsilon_{ijk} x_j \frac{\partial}{\partial x_l} \tau_{lk}, \\
&= \quad \epsilon_{ijk} \Big[\frac{\partial}{\partial x_l}\big(x_j \tau_{lk}\big) - \tau_{lk} \frac{\partial}{\partial x_l} x_j\Big], \\
&= \quad \frac{\partial}{\partial x_l}\Big(\epsilon_{ijk} x_j \tau_{lk}\Big) - \epsilon_{ijk} \delta_{jl} \tau_{lk}, \\
&= \quad \frac{\partial}{\partial x_l}\Big(\epsilon_{ijk} x_j \tau_{lk}\Big) - \epsilon_{ijk} \tau_{jk},
\end{aligned}$$

so the i$^{\text{th}}$ component of the l.h.s. of (C.0.2) is

$$\iiint\limits_{W(t)} \left(\vec{x} \times \vec{\nabla} \cdot \tau\right)_i dV = \iiint\limits_{W(t)} \frac{\partial}{\partial x_l}\Big(\epsilon_{ijk} x_j \tau_{lk}\Big)\, dV - \iiint\limits_{W(t)} \epsilon_{ijk} \tau_{jk}, dV.$$

Using the divergence theorem we have

$$\iiint\limits_{W(t)} \left(\vec{x} \times \vec{\nabla} \cdot \tau\right)_i dV = \iint\limits_{\partial W(t)} \epsilon_{ijk} x_j \tau_{lk} n_l \, dS - \iiint\limits_{W(t)} \epsilon_{ijk} \tau_{jk}, dV. \tag{C.0.3}$$

Turning to the r.h.s. of (C.0.2), with the use of Cauchy's Fundamental Theorem for Stress, the i$^{\text{th}}$ component of $\vec{x} \times \vec{t}$ is

$$\left(\vec{x} \times \vec{t}\right)_i = \epsilon_{ijk} x_j t_k = \epsilon_{ijk} x_j \tau_{lk} n_l.$$

The integral of this over $\partial W(t)$ cancels with the first tem on the right hand side of (C.0.3) so the i$^{\text{th}}$ component of (C.0.2) reduces to

$$\iiint\limits_{W(t)} \epsilon_{ijk} \tau_{jk}\, dV = 0. \tag{C.0.4}$$

Since this is true for all material volumes $W(t)$ and we assume the integrand is continuous,

$$\tau_{jk} = \tau_{kj}, \tag{C.0.5}$$

i.e., the stress tensor is symmetric.

Appendix D

Appendix D: Equations in other coordinate systems

D.1 Cylindrical Polar Coordinates

Coordinates: (r, θ, x) where r is the radial distance from the x axis and θ is the azimuthal angle positive ccw about the positive x axis. The corresponding unit vectors are $\hat{i}_r$, $\hat{i}_\theta$ and $\hat{i}_x$. The direction of $\hat{i}_x$ is the same every where but the directions of $\hat{i}_r$ and $\hat{i}_\theta$ depend on location.

Velocity vector:

$$\vec{u} = (u_r, u_\theta, u) = u_r\,\hat{i}_r + u_\theta\,\hat{i}_\theta + u\,\hat{i}_x \tag{D.1.1}$$

Gradient of a scalar:

$$\vec{\nabla}\phi = (\frac{\partial \phi}{\partial r}, \frac{1}{r}\phi_\theta, \frac{\partial \phi}{\partial x}) \tag{D.1.2}$$

Laplacian of a scalar:

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial \phi}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial x^2} \tag{D.1.3}$$

Divergence of a vector:

$$\vec{\nabla}\cdot\vec{u} = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u}{\partial x} \tag{D.1.4}$$

Curl of a vector:

$$\vec{\nabla} \times \vec{u} = \Big(\frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{\partial u_\theta}{\partial x}\Big)\hat{i}_r + \Big(\frac{\partial u_r}{\partial x} - \frac{\partial u}{\partial r}\Big)\hat{i}_\theta + \Big(\frac{1}{r}\frac{\partial (r u_\theta)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\Big)\hat{i} \quad \text{(D.1.5)}$$

Laplacian of a vector:

$$\nabla^2 \vec{u} = \Big(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\Big)\hat{i}_r + \Big(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\Big)\hat{i}_\theta + \nabla^2 u\,\hat{i} \quad \text{(D.1.6)}$$

Strain Rate Tensor e:

$$e_{rr} = \frac{\partial u_r}{\partial r}, \quad \text{(D.1.7)}$$

$$e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \quad \text{(D.1.8)}$$

$$e_{xx} = \frac{\partial u}{\partial x}, \quad \text{(D.1.9)}$$

$$e_{r\theta} = \frac{1}{2}\Big(r\frac{\partial}{\partial r}\Big(\frac{u_\theta}{r}\Big) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\Big), \quad \text{(D.1.10)}$$

$$e_{rx} = \frac{1}{2}\Big(\frac{\partial u_r}{\partial x} + \frac{\partial u}{\partial r}\Big), \quad \text{(D.1.11)}$$

$$e_{\theta x} = \frac{1}{2}\Big(\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial u_\theta}{\partial x}\Big) \quad \text{(D.1.12)}$$

Deviatoric Stress Tensor for constant density fluid: $\sigma = 2\mu e$.

Continuity equation: Writing it in the form $\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0$ we have

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\big(\rho r u_r\big) + \frac{1}{r}\frac{\partial}{\partial \theta}\big(\rho u_\theta\big) + \frac{\partial}{\partial x}\big(\rho u\big) = 0. \quad \text{(D.1.13)}$$

Navier-Stokes equations for constant ρ and $\nu = \mu/\rho$ and no body force:

$$\frac{\partial u_r}{\partial t} + (\vec{u}\cdot\vec{\nabla})u_r - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\Big(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\Big) \quad \text{(D.1.14)}$$

$$\frac{\partial u_\theta}{\partial t} + (\vec{u}\cdot\vec{\nabla})u_\theta + \frac{u_\theta u_r}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\Big(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\Big) \quad \text{(D.1.15)}$$

$$\frac{\partial u}{\partial t} + (\vec{u}\cdot\vec{\nabla})u = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u \quad \text{(D.1.16)}$$

where

$$\vec{u}\cdot\vec{\nabla} = u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u\frac{\partial}{\partial x} \quad \text{(D.1.17)}$$