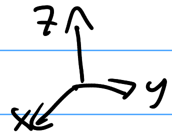
Lecture 4, Jan 15, 2026

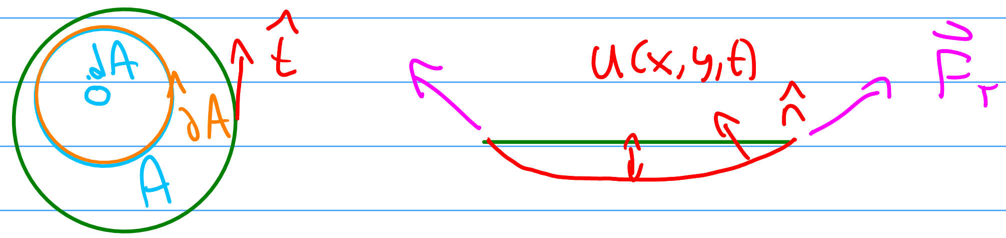
Today
- Vibrating membrane
- Solution to the wave Eqn



1.5 Vibrating Membrane

Consider a circular drum

dA is a small patch on the membrane



Top Side

$u(x,y,t)$: vertical displacement $[u] = m$ (only in the vertical)

$\rho(x,y)$: mass density $[\rho] = kg/m^2$

$\hat{t}$: unit tangent vector in the xy plane

$\hat{n}$: unit upward normal vector

$\vec{F}_T$: tensile (line) force which is tangent to the membrane and acts outwards.

The tensile force is in a plane tangent to the surface of the membrane. It is orthogonal to $\hat{t}$ and can be written as

$$\boxed{\vec{F}_T = T_0 \, \hat{t} \times \hat{n}}$$

The vertical component of the tensile force is

$$\boxed{\vec{F}_T \cdot \hat{z} = T_0 (\hat{t} \times \hat{n}) \cdot \hat{z}}$$

We use Newton's 2nd law to derive the PDE,

$ma = F.$

Consider a small patch of our surface.

The mass of this small area is $\rho_0 \, dA$

The vertical acceleration is $\dfrac{\partial^2 u}{\partial t^2}$

Hence, the LHS of Newton's law is

$$ma \approx \rho_0 \, dA \frac{\partial^2 u}{\partial t^2}$$

To find the net contribution over the whole surface, we integrate over an arbitrary area A,

$$\boxed{\iint_A \rho_0 \frac{\partial^2 u}{\partial t^2} \, dA}$$

this the $m \times a$ for our subsurface A

The tensile force in the vertical direction at a point is $T_0 (\hat{t} \times \hat{n}) \cdot \hat{z}$. This is the force at a point on the bdry. To find the total force we must sum (integrate) over the bdry of our area,

$$\boxed{\oint_{\partial A} T_0 (\hat{t} \times \hat{n}) \cdot \hat{z} \, ds \underset{\text{identity}}{\overset{\text{using}}{=}} \oint_{\partial A} T_0 (\hat{n} \times \hat{z}) \cdot \hat{t} \, ds}$$

We combine the two terms & get,

$$\boxed{\iint_A \rho_0 \frac{\partial^2 u}{\partial t^2} \, dA = \oint_{\partial A} T_0 (\hat{n} \times \hat{z}) \cdot \hat{t} \, ds}$$

Use Stokes thm to rewrite the RHS,

$$\boxed{\iint_A \rho_0 \frac{\partial^2 u}{\partial t^2}\, dA = \iint_A \vec{\nabla} \times [T_0 (\hat{n} \times \hat{z})] \cdot \hat{n}\, dA}$$

Since our surface is given by $z = u(x,y,t)$ we can find a normal vector to the surface,

$$\vec{N} = \vec{\nabla}(z - u) = (-\partial u/\partial x, -\partial u/\partial y, 1)$$

The unit normal is then,

$$\boxed{\hat{n} = \frac{\vec{N}}{\|N\|} = \frac{(-\partial u/\partial x, -\partial u/\partial y, -1)}{\sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}}}$$

We assume the slopes are small $\left(\frac{\partial u}{\partial x}\right)^2, \left(\frac{\partial u}{\partial y}\right)^2 \ll 1.$

$$\Rightarrow \boxed{\hat{n} \approx (-\partial u/\partial x, -\partial u/\partial y, +1)}$$

Need $\hat{n} \times \hat{z} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\partial u/\partial x & -\partial u/\partial y & 1 \\ 0 & 0 & 1 \end{vmatrix} = \left(-\frac{\partial u}{\partial y}, \frac{\partial u}{\partial x}, 0\right)$

Next, $\vec{\nabla} \times (\hat{n} \times \hat{z}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\partial u/\partial y & \partial u/\partial x & 0 \end{vmatrix} = \left(0, 0, \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$

$$= \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\hat{z}$$

Finally, $\vec{\nabla} \times (\hat{n} \times \hat{z}) \cdot \hat{n} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \nabla_H^2 u$ horizontal Laplacian

We plug our expressions into our integral eqn,

$$\iint_A \left[\rho_0 \frac{\partial^2 u}{\partial t^2} - T_0 \nabla_H^2 u \right] dA = 0$$

Since this must be true for all A, we deduce using our lemma that the integrand is zero,

$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla_H^2 u$ if $c^2 \equiv T_0/\rho_0$ $\Leftarrow$ $$\boxed{\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \nabla_H^2 u}$$

2D wave eqn (linear eqn)

2.2.1 Wave Eqn (solving this eqn)

The wave eqn in 1D can be written as

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \iff \left(\frac{\partial^2}{\partial t^2}\right) u - c^2 \left(\frac{\partial^2}{\partial x^2}\right) u = 0$$

$$\iff \left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right) u = 0$$

This is similar to the quadratic eqn

$$(t^2 - c^2 x^2) = (t - cx)(t + cx)$$

We can do the same thing with differential operators,

$$\boxed{\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\partial_t + c\frac{\partial}{\partial x}\right) u = 0}$$

Aside: If we define $\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u = V$ then the eqn is $\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)V = 0$

This rewrites a 2nd order scalar PDE in terms of 2 1st order PDEs.

Recall, $\frac{\partial V}{\partial t} - c\frac{\partial V}{\partial x} = 0$ (soln moves to the left)

$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$ (soln moves to the right)

To solve the wave eqn we choose variables:

$$\xi = x - ct \qquad \Rightarrow \qquad x = \tfrac{1}{2}(\xi + \eta)$$
$$\eta = x + ct \qquad \qquad t = \tfrac{1}{2c}(\eta - \xi)$$

To change variables, we use the chain rule:

$$\frac{\partial u}{\partial \xi} = \frac{\partial x}{\partial \xi}\frac{\partial u}{\partial x} + \frac{\partial t}{\partial \xi}\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial u}{\partial x} - \frac{1}{2c}\frac{\partial u}{\partial t} = -\frac{1}{2c}\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)u$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial x}{\partial \eta}\frac{\partial u}{\partial x} + \frac{\partial t}{\partial \eta}\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{2c}\frac{\partial u}{\partial t} = \frac{1}{2c}\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u$$

This shows, $\frac{\partial}{\partial \xi} = -\frac{1}{2c}\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) \Rightarrow \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) = -2c\frac{\partial}{\partial \xi}$

$\frac{\partial}{\partial \eta} = \frac{1}{2c}\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) \Rightarrow \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) = 2c\frac{\partial}{\partial \eta}$

We sub these eqns into our factored wave eqn

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u = \left(-2c\frac{\partial}{\partial \xi}\right)\left(2c\frac{\partial}{\partial \eta}\right)u = -4c^2\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$$

Divide by $-4c^2$, $\boxed{\dfrac{\partial^2 u}{\partial \xi \partial \eta} = 0}$ (PDE)

First, we integrate w.r.t. η,

$$\frac{\partial u}{\partial \xi} = \alpha'(\xi)$$

Next, integrate w.r.t. ξ,

$$u = \alpha(\xi) + \beta(\eta)$$

or

$$\boxed{u = \alpha(x - ct) + \beta(x + ct)}$$ d'Alembert's soln

moves to the right at speed c — moves to the left with speed c

This is valid of the wave eqn on the real line.

To find a unique soln, we need two initial conditions,

$$u(x,0) = f(x)$$
$$\frac{\partial u}{\partial t}(x,0) = g(x)$$

We can find α & β in terms of f & g.

$$u(x,0) = \alpha(x) + \beta(x) = f(x) \quad (1)$$

$$\frac{\partial u}{\partial t}(x,0) = -c\alpha'(x) + c\beta'(x) = g(x)$$

integrate,

$$-c\alpha(x) + c\beta(x) = \int_0^x g(s)\,ds \quad (2)$$

$$c(1) + (2) \Rightarrow \quad 2c\beta = cf + \int_0^x g(s)\,ds$$

$$\beta = \frac{1}{2}f + \frac{1}{2c}\int_0^x g(s)\,ds$$

$$c(1) - (2) \Rightarrow \quad 2c\alpha = cf - \int_0^x g(s)\,ds$$

$$\alpha = \frac{1}{2}f - \frac{1}{2c}\int_0^x g(s)\,ds$$

we then have our soln to the wave eqn that satisfies our initial conditions,

$$u(x,t) = \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_0^{x-ct} g(s)\,ds$$

$$+ \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(s)\,ds$$

$$\boxed{u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds}$$