Question 2

(a)

$$\begin{aligned}
\frac{D(f+g)}{Dt} &= \frac{\partial(f+g)}{\partial t} + \boldsymbol{u}\cdot\nabla(f+g) \\
&= \frac{\partial f}{\partial t} + \frac{\partial g}{\partial t} + \boldsymbol{u}\cdot(\nabla f + \nabla g) \quad \text{(by vector identity)} \\
&= \frac{\partial f}{\partial t} + \frac{\partial g}{\partial t} + \boldsymbol{u}\cdot\nabla f + \boldsymbol{u}\cdot\nabla g \\
&= \frac{\partial f}{\partial t} + \boldsymbol{u}\cdot\nabla f + \frac{\partial g}{\partial t} + \boldsymbol{u}\cdot\nabla g \\
&= \frac{Df}{Dt} + \frac{Dg}{Dt}.
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{D(fg)}{Dt} &= \frac{\partial(fg)}{\partial t} + \boldsymbol{u}\cdot\nabla(fg) \\
&= f\frac{\partial g}{\partial t} + \frac{\partial f}{\partial t}g + \boldsymbol{u}\cdot(f(\nabla g) + (\nabla f)g) \quad \text{(by vector identity)} \\
&= \frac{\partial f}{\partial t}g + \boldsymbol{u}\cdot((\nabla f)g) + f\frac{\partial g}{\partial t} + \boldsymbol{u}\cdot(f(\nabla g)) \\
&= \frac{Df}{Dt}g + f\frac{Dg}{Dt}.
\end{aligned}$$

(c)

$$\begin{aligned}
\nabla(h\circ g) &= \left(\frac{\partial(h\circ g)}{\partial x_1}, \frac{\partial(h\circ g)}{\partial x_2}, \frac{\partial(h\circ g)}{\partial x_3}\right) \\
&= \left((h'\circ g)\frac{\partial g}{\partial x_1}, (h'\circ g)\frac{\partial g}{\partial x_2}, (h'\circ g)\frac{\partial g}{\partial x_3}\right) \\
&= (h'\circ g)\nabla g.
\end{aligned}$$

Then:

$$\begin{aligned}
\frac{D(h\circ g)}{Dt} &= \frac{\partial(h\circ g)}{\partial t} + \boldsymbol{u}\cdot\nabla(h\circ g) \\
&= (h'\circ g)\frac{\partial g}{\partial t} + [\boldsymbol{u}\cdot(h'\circ g)\nabla g] \\
&= (h'\circ g)\left(\frac{\partial g}{\partial t} + \boldsymbol{u}\cdot\nabla g\right) \\
&= (h'\circ g)\frac{Dg}{Dt}.
\end{aligned}$$

Question 3

To find the rate of change of temperature with respect to time as measured by the airplane, we solve for the material derivative of T w.r.t. t.

$$\begin{aligned}\frac{Df}{Dt} &= \frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \nabla T \\ &= \frac{\partial T}{\partial t} + u_x \frac{\partial T}{\partial x} + u_z \frac{\partial T}{\partial z}.\end{aligned}$$

For each component:

$$\frac{\partial T}{\partial t} = -\frac{24}{2000} \operatorname{sech}^2 \left(\frac{x + \alpha z - 3t}{2000} \right),$$

$$\begin{aligned}u_x \frac{\partial T}{\partial x} &= 10 \cos(30°) \cdot 8 \operatorname{sech}^2 \left(\frac{x + \alpha z - 3t}{2000} \right) \left(\frac{1}{2000} \right) \\ &= 5\sqrt{3} \cdot \frac{8}{2000} \operatorname{sech}^2 \left(\frac{x + \alpha z - 3t}{2000} \right) \\ &= \frac{40\sqrt{3}}{2000} \operatorname{sech}^2 \left(\frac{x + \alpha z - 3t}{2000} \right),\end{aligned}$$

$$\begin{aligned}u_z \frac{\partial T}{\partial z} &= 10 \sin(30°) \cdot 8 \operatorname{sech}^2 \left(\frac{x + \alpha z - 3t}{2000} \right) \frac{\partial}{\partial z} \left(\frac{x + \alpha z - 3t}{2000} \right) \\ &= 5 \cdot 8 \operatorname{sech}^2 \left(\frac{x + \alpha z - 3t}{2000} \right) \left(\frac{100}{2000} \right) \\ &= \frac{4000}{2000} \operatorname{sech}^2 \left(\frac{x + \alpha z - 3t}{2000} \right).\end{aligned}$$

Plugging in the numerical values:

$$\begin{aligned}\frac{Df}{Dt} &= \left(-\frac{24}{2000} + \frac{40\sqrt{3}}{2000} + \frac{4000}{2000} \right) \operatorname{sech}^2 \left(\frac{x + \alpha z - 3t}{2000} \right) \\ &= \frac{3976 + 40\sqrt{3}}{2000} \operatorname{sech}^2 \left(\frac{x + \alpha z - 3t}{2000} \right) \\ &\approx \frac{3976 + 69.28}{2000} \operatorname{sech}^2 \left(\frac{x + \alpha z - 3t}{2000} \right) \\ &\approx 2.023 \operatorname{sech}^2 \left(\frac{x + 100z - 3t}{2000} \right) \,°\mathrm{C}\,\mathrm{s}^{-1}.\end{aligned}$$

The extreme value occurs when the hyperbolic secant term is 1, the maximum rate of change of approximately $2.023\,°\mathrm{C}\,\mathrm{s}^{-1}$.

Question 4

(a) Computing the Jacobian J of the transformation:

$$
\begin{aligned}
J = \frac{\partial(x,y)}{\partial(r,\theta)} &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} \\
&= \begin{vmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{vmatrix} \\
&= (a\cos\theta)(br\cos\theta) - (-ar\sin\theta)(b\sin\theta) \\
&= abr(\cos^2\theta + \sin^2\theta) \\
&= abr.
\end{aligned}
$$

Substituting the transformation into the equation of the ellipse $x^2/a^2 + y^2/b^2 = 1$ yields the limits for r:

$$
\begin{aligned}
\frac{(ar\cos\theta)^2}{a^2} + \frac{(br\sin\theta)^2}{b^2} &= 1 \\
r^2(\cos^2\theta + \sin^2\theta) &= 1 \\
r^2 &= 1 \implies r = 1.
\end{aligned}
$$

The limits are $0 \le r \le 1$ and $0 \le \theta \le 2\pi$. The area A is therefore:

$$
\begin{aligned}
A &= \iint_R dx\,dy = \iint_R J\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 abr\,dr\,d\theta \\
&= ab\int_0^{2\pi} d\theta \int_0^1 r\,dr \\
&= ab(2\pi)\left[\frac{r^2}{2}\right]_0^1 \\
&= \pi ab.
\end{aligned}
$$

(b) We rewrite the integral $I(t)$ using the transformation to coordinates (r,θ). Since the limits for r (0 to 1) and θ (0 to 2π) are constant in time:

$$
I(t) = \int_0^{2\pi}\int_0^1 f(x(r,\theta,t), y(r,\theta,t), t)\,J(r,\theta,t)\,dr d\theta.
$$

Therefore:

$$\begin{aligned}\frac{dI}{dt} &= \frac{d}{dt}\int_0^{2\pi}\int_0^1 fJ\,drd\theta \\ &= \int_0^{2\pi}\int_0^1 \frac{\partial}{\partial t}(fJ)\,drd\theta.\end{aligned}$$

By the product rule:

$$\frac{\partial}{\partial t}(fJ) = \frac{df}{dt}J + f\frac{\partial J}{\partial t}.$$

By the chain rule:

$$\begin{aligned}\frac{df}{dt} &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \\ &= f_t + \nabla f \cdot \boldsymbol{v}, \quad \text{where } \boldsymbol{v} = \left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}\right).\end{aligned}$$

Substituting this back into the integral:

$$\begin{aligned}\frac{dI}{dt} &= \int_0^{2\pi}\int_0^1 \left[(f_t + \boldsymbol{v}\cdot\nabla f)J + f\frac{\partial J}{\partial t}\right]\,drd\theta \\ &= \int_0^{2\pi}\int_0^1 (f_t + \boldsymbol{v}\cdot\nabla f)J\,drd\theta + \int_0^{2\pi}\int_0^1 f\frac{\partial J}{\partial t}\,drd\theta.\end{aligned}$$

Since $J\,drd\theta = dxdy$:

$$\frac{dI}{dt} = \iint_R [f_t + \boldsymbol{v}\cdot\nabla f]\,dxdy + \iint_R f\frac{\partial J}{\partial t}\,drd\theta.$$

(c) To find $\boldsymbol{v}$:

$$\begin{aligned}v_x &= \frac{\partial x}{\partial t} = a'(t)r\cos\theta = \frac{a'(t)}{a(t)}x \\ v_y &= \frac{\partial y}{\partial t} = b'(t)r\sin\theta = \frac{b'(t)}{b(t)}y.\end{aligned}$$

Then:

$$\begin{aligned}\nabla\cdot\boldsymbol{v} &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\left(\frac{a'(t)}{a(t)}x, \frac{b'(t)}{b(t)}y\right) \\ &= \frac{a'(t)}{a(t)} + \frac{b'(t)}{b(t)}.\end{aligned}$$

From part a, we know that

$$J = a(t)b(t)r,$$

hence:

$$\frac{\partial J}{\partial t} = r(a'(t)b(t) + a(t)b'(t))$$

and

$$(\nabla \cdot \boldsymbol{v})J = \left(\frac{a'(t)}{a(t)} + \frac{b'(t)}{b(t)}\right) a(t)b(t)r = r(a'(t)b(t) + a(t)b'(t)) = \frac{\partial J}{\partial t}.$$

Substituting the results into the final expression in part b:

$$\begin{aligned}
\frac{dI}{dt} &= \iint_R [f_t + \boldsymbol{v} \cdot \nabla f]\, dxdy + \iint_R f(\nabla \cdot \boldsymbol{v})J\, drd\theta \\
&= \iint_R [f_t + \boldsymbol{v} \cdot \nabla f]\, dxdy + \iint_R f(\nabla \cdot \boldsymbol{v})\, dxdy \\
&= \iint_R f_t + \boldsymbol{v} \cdot \nabla f + f(\nabla \cdot \boldsymbol{v})\, dxdy.
\end{aligned}$$

Then splitting the integral and by vector identity:

$$\begin{aligned}
\frac{dI}{dt} &= \iint_R f_t\, dxdy + \iint_R \boldsymbol{v} \cdot \nabla f + f(\nabla \cdot \boldsymbol{v})\, dxdy \\
&= \iint_R f_t\, dxdy + \iint_R \nabla \cdot (f\boldsymbol{v})\, dxdy. \qquad (1)
\end{aligned}$$

We rewrite 1 with Divergence Theorem:

$$\iint_R f_t\, dxdy + \iint_R \nabla \cdot (f\boldsymbol{v})\, dxdy = \iint_R f_t\, dxdy + \int_{\partial R} f\boldsymbol{v} \cdot \hat{\boldsymbol{n}}\, ds.$$

Question 5

(a) Since the vector field has no y dependency:

$$\begin{aligned}
\nabla \cdot \boldsymbol{u} &= \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \\
&= \frac{\partial}{\partial x}[am\cos(kx + mz - \omega t)] + \frac{\partial}{\partial z}[-ak\cos(kx + mz - \omega t)] \\
&= -amk\sin(kx + mz - \omega t) - (-akm\sin(kx + mz - \omega t)) \\
&= -amk\sin(kx + mz - \omega t) + amk\sin(kx + mz - \omega t) \\
&= 0.
\end{aligned}$$

Since $\nabla \cdot \boldsymbol{u} = 0$, the flow is incompressible.

(b) Since the vector field has no y dependency:

$$\begin{aligned}
(\nabla \times \boldsymbol{u})_y &= \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \\
&= \frac{\partial}{\partial z}[am\cos(kx + mz - \omega t)] - \frac{\partial}{\partial x}[-ak\cos(kx + mz - \omega t)] \\
&= -am^2\sin(kx + mz - \omega t) - [-(-ak^2\sin(kx + mz - \omega t))] \\
&= -am^2\sin(kx + mz - \omega t) - ak^2\sin(kx + mz - \omega t) \\
&= -a(m^2 + k^2)\sin(kx + mz - \omega t).
\end{aligned}$$

Since the curl is not identically zero, the flow is not irrotational.

(c) From part a, we know that $\nabla \cdot \boldsymbol{u} = 0$, therefore we only need to check whether the material derivative satisfies:

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \nabla \rho = 0.$$

The derivatives are:

$$\begin{aligned}
\frac{\partial \rho}{\partial t} &= a\frac{N^2}{g}\frac{k}{\omega}\cos(kx + mz - \omega t)(-\omega) = -a\frac{N^2}{g}k\cos(kx + mz - \omega t) \\
\frac{\partial \rho}{\partial x} &= a\frac{N^2}{g}\frac{k}{\omega}\cos(kx + mz - \omega t)(k) = a\frac{N^2}{g}\frac{k^2}{\omega}\cos(kx + mz - \omega t) \\
\frac{\partial \rho}{\partial z} &= -\frac{N^2}{g} + a\frac{N^2}{g}\frac{k}{\omega}\cos(kx + mz - \omega t)(m).
\end{aligned}$$

We define $\phi = kx + mz - \omega t$. Substituting these into the material derivative:

$$
\begin{aligned}
\frac{D\rho}{Dt} &= \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + w\frac{\partial \rho}{\partial z} \\
&= -a\frac{N^2}{g}k\cos(\phi) + [am\cos(\phi)]\left(a\frac{N^2}{g}\frac{k^2}{\omega}\cos(\phi)\right) \\
&\quad + [-ak\cos(\phi)]\left(-\frac{N^2}{g} + a\frac{N^2}{g}\frac{mk}{\omega}\cos(\phi)\right) \\
&= -a\frac{N^2}{g}k\cos(\phi) + \frac{a^2mk^2N^2}{g\omega}\cos^2(\phi) + a\frac{N^2}{g}k\cos(\phi) - \frac{a^2mk^2N^2}{g\omega}\cos^2(\phi) \\
&= 0.
\end{aligned}
$$

Since $D\rho/Dt = 0$, mass is conserved, so this density field is possible.

Question 6

The continuity equation from the lecture is:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0.$$

Since ρ and t are scalars, the term $\partial\rho/\partial t$ is invariant. Then for $\nabla \cdot (\rho \boldsymbol{u})$ to be invariant, we only need $\nabla \cdot \boldsymbol{u}$ to be invariant. From the given transformation matrix, in terms of the new coordinates (x', y'):

$$\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta. \end{aligned}$$

The differential operators are:

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial x}\frac{\partial}{\partial y'} = \cos\theta\frac{\partial}{\partial x'} - \sin\theta\frac{\partial}{\partial y'} \\ \frac{\partial}{\partial y} &= \frac{\partial x'}{\partial y}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial y}\frac{\partial}{\partial y'} = \sin\theta\frac{\partial}{\partial x'} + \cos\theta\frac{\partial}{\partial y'}. \end{aligned}$$

The velocity components (u, v) in terms of the primed components (u', v') are:

$$\begin{aligned} u &= u'\cos\theta - v'\sin\theta \\ v &= u'\sin\theta + v'\cos\theta. \end{aligned}$$

Substituting the operators and velocity components into the divergence expression:

$$\begin{aligned} \nabla \cdot \boldsymbol{u} &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \\ &= \left(\cos\theta\frac{\partial}{\partial x'} - \sin\theta\frac{\partial}{\partial y'}\right)(u'\cos\theta - v'\sin\theta) \\ &\quad + \left(\sin\theta\frac{\partial}{\partial x'} + \cos\theta\frac{\partial}{\partial y'}\right)(u'\sin\theta + v'\cos\theta) \\ &= (\cos^2\theta)\frac{\partial u'}{\partial x'} - (\cos\theta\sin\theta)\frac{\partial v'}{\partial x'} - (\sin\theta\cos\theta)\frac{\partial u'}{\partial y'} + (\sin^2\theta)\frac{\partial v'}{\partial y'} \\ &\quad + (\sin^2\theta)\frac{\partial u'}{\partial x'} + (\sin\theta\cos\theta)\frac{\partial v'}{\partial x'} + (\cos\theta\sin\theta)\frac{\partial u'}{\partial y'} + (\cos^2\theta)\frac{\partial v'}{\partial y'} \\ &= (\cos^2\theta + \sin^2\theta)\frac{\partial u'}{\partial x'} + (\sin^2\theta + \cos^2\theta)\frac{\partial v'}{\partial y'} \\ &\quad + (-\cos\theta\sin\theta + \sin\theta\cos\theta)\frac{\partial v'}{\partial x'} + (-\sin\theta\cos\theta + \cos\theta\sin\theta)\frac{\partial u'}{\partial y'}. \end{aligned}$$

The cross terms cancel and the squared terms sum to 1:

$$\nabla \cdot \boldsymbol{u} = \frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = \nabla' \cdot \boldsymbol{u}'.$$

Since the divergence form is identical in the primed system, the continuity equation is invariant.

Question 7

From the lecture:

$$\frac{dV}{dt} = \iiint_{W(t)} \nabla \cdot \boldsymbol{u}\, dV.$$

We compute the divergence of the given velocity field $\boldsymbol{u} = (u, v, w)$:

$$\begin{aligned}
\nabla \cdot \boldsymbol{u} &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \\
&= \frac{\partial}{\partial x}(x + xy) + \frac{\partial}{\partial y}\left(\frac{1}{2}y - \frac{1}{2}y^2\right) + \frac{\partial}{\partial z}(-z + x) \\
&= (1 + y) + \left(\frac{1}{2} - y\right) + (-1) \\
&= 0.5\,\mathrm{s}^{-1}.
\end{aligned}$$

Since the divergence is a constant:

$$\begin{aligned}
\frac{dV}{dt} &= \iiint_{W(t)} 0.5\, dV \\
&= 0.5 \iiint_{W(t)} 1\, dV.
\end{aligned}$$

Therefore:

$$\frac{dV}{dt} = 0.5V.$$

Integrating:

$$\begin{aligned}
\int_{V_0}^{V(t)} \frac{dV}{V} &= \int_0^t 0.5\, dt \\
\ln\left(\frac{V(t)}{V_0}\right) &= 0.5t \\
V(t) &= V_0 e^{0.5t}.
\end{aligned}$$

Substituting the initial values $V_0 = 3\,\mathrm{m}^3$ and $t = 2\,\mathrm{s}$:

$$V(2) = 3e^{0.5(2)} = 3e^1 \approx 8.154\,\mathrm{m}^3.$$

The volume after 2 seconds is approximately $8.15\,\mathrm{m}^3$.

Question 8

By definition:

$$H = \frac{\text{mass of chemical}}{\text{volume}} = \frac{\text{mass of chemical}}{\text{total mass}} \times \frac{\text{total mass}}{\text{volume}} = c\rho.$$

We follow the procedure used to derive the continuity equation starting from mass conservation. Since the chemical is conserved:

$$\frac{d}{dt} \iiint_{W(t)} H \, dV = 0.$$

Applying the Reynolds Transport Theorem:

$$\iiint_{W(t)} \left(\frac{\partial H}{\partial t} + \nabla \cdot (H\boldsymbol{u}) \right) dV = 0.$$

By the localization theorem:

$$\frac{\partial H}{\partial t} + \nabla \cdot (H\boldsymbol{u}) = 0.$$

Substituting $H = \rho c$:

$$\frac{\partial (\rho c)}{\partial t} + \nabla \cdot (\rho c \boldsymbol{u}) = 0.$$

Expanding using the product rule and vector identity:

$$c\frac{\partial \rho}{\partial t} + \rho\frac{\partial c}{\partial t} + c\nabla \cdot (\rho\boldsymbol{u}) + \rho\boldsymbol{u} \cdot \nabla c = 0$$
$$c \underbrace{\left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\boldsymbol{u}) \right]}_{\text{Continuity Eq. for Fluid}} + \rho \left[\frac{\partial c}{\partial t} + \boldsymbol{u} \cdot \nabla c \right] = 0.$$

The first term is zero because the fluid satisfies mass conservation, hence:

$$\rho \left(\frac{\partial c}{\partial t} + \boldsymbol{u} \cdot \nabla c \right) = 0.$$

Since $\rho \neq 0$, $\left(\frac{\partial c}{\partial t} + \boldsymbol{u} \cdot \nabla c\right)$ must be zero. This is the material derivative Dc/Dt, so the simplified equation is:

$$\frac{Dc}{Dt} = 0.$$

Question 9

(a)

$$\nabla \cdot \boldsymbol{u} = \frac{\partial(-y)}{\partial x} + \frac{\partial(x)}{\partial y} = 0 + 0 = 0.$$

Since the divergence is zero everywhere, the flow is incompressible.

(b)

$$\zeta = (\nabla \times \boldsymbol{u})_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial(x)}{\partial x} - \frac{\partial(-y)}{\partial y} = 1 - (-1) = 2.$$

The flow has a constant vorticity of 2.

(c) Since ρ is a function of r:

$$\frac{\partial \rho}{\partial t} = 0.$$

Using the product rule:

$$\boldsymbol{u} \cdot \nabla \rho + \rho(\nabla \cdot \boldsymbol{u}) = 0.$$

From part (a), $\nabla \cdot \boldsymbol{u} = 0$, so the continuity equation reduces to $\boldsymbol{u} \cdot \nabla \rho = 0$. Given $\rho = F(r)$ where $r = \sqrt{x^2 + y^2}$, the gradient is radial:

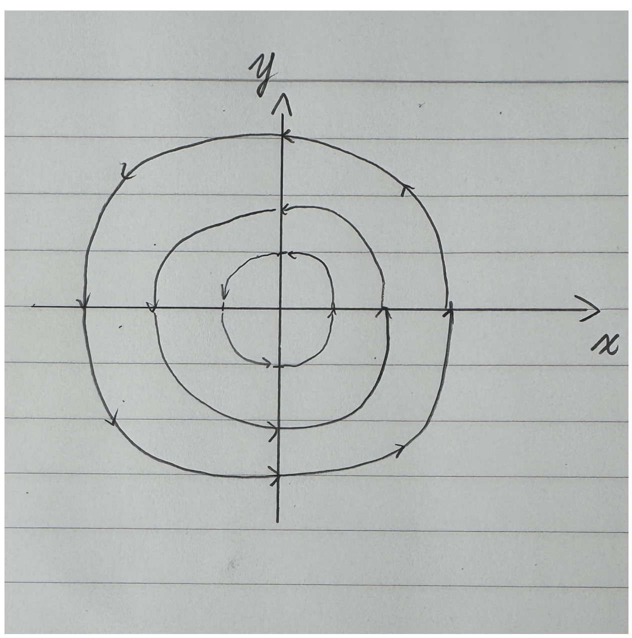
$$\nabla \rho = F'(r) \nabla r = F'(r) \left(\frac{x}{r}, \frac{y}{r} \right).$$

Therefore:

$$\boldsymbol{u} \cdot \nabla \rho = (-y, x) \cdot \frac{F'(r)}{r}(x, y) = \frac{F'(r)}{r}(-yx + xy) = 0.$$

This is zero for any differentiable function F. Therefore, any differentiable function $F(r)$ satisfies the continuity equation.

(d)



The velocity vectors are always tangent to the circles $r = \text{const}$. Since the density $\rho = F(r)$ is constant along these circles, the fluid particles move along contours of constant density. Thus, the density of a fluid particle does not change as it moves.

(e) We parametrize the curve with $x = R\cos\theta, y = R\sin\theta$ for $\theta \in [0, 2\pi]$. Then $\boldsymbol{u} = (-R\sin\theta, R\cos\theta)$ and $d\boldsymbol{s} = (-R\sin\theta, R\cos\theta)\,d\theta$. The circulation is:

$$\begin{aligned} C = \oint \boldsymbol{u} \cdot \hat{\boldsymbol{t}}\,ds &= \int_0^{2\pi} (-R\sin\theta, R\cos\theta) \cdot (-R\sin\theta, R\cos\theta)\,d\theta \\ &= \int_0^{2\pi} (R^2\sin^2\theta + R^2\cos^2\theta)\,d\theta \\ &= \int_0^{2\pi} R^2\,d\theta \\ &= 2\pi R^2. \end{aligned}$$

(f) Using Stokes' Theorem:

$$C = \oint \boldsymbol{u} \cdot d\boldsymbol{l} = \iint_A (\nabla \times \boldsymbol{u}) \cdot \hat{\boldsymbol{k}}\,dA.$$

From part b, $\nabla \times \boldsymbol{u} = 2$.

$$C = \iint_A 2\,dA = 2 \times (\text{Area of Ellipse}).$$

The area of an ellipse is πab, so:

$$C = 2\pi ab.$$

(g) Using the Divergence Theorem:

$$\text{Flux} = \iint_A (\nabla \cdot \boldsymbol{u})\,dA.$$

From part a, $\nabla \cdot \boldsymbol{u} = 0$. Therefore:

$$\text{Flux} = 0.$$

Question 10

(a) Given that:

$$I(t) = \oint_{C(0)} f_i(\boldsymbol{\Phi}(\boldsymbol{a}, t), t)\frac{\partial \Phi_i}{\partial a_k}\, da_k.$$

Since the integration domain $C(0)$ and the variable $\boldsymbol{a}$ are independent of time:

$$\begin{aligned}\frac{dI}{dt} &= \oint_{C(0)} \frac{\partial}{\partial t}\left(f_i \frac{\partial \Phi_i}{\partial a_k}\right)\, da_k \\ &= \oint_{C(0)} \left(\frac{df_i}{dt}\frac{\partial \Phi_i}{\partial a_k} + f_i \frac{\partial}{\partial t}\left(\frac{\partial \Phi_i}{\partial a_k}\right)\right)\, da_k.\end{aligned}$$

The time derivative of f_i holding $\boldsymbol{a}$ fixed is the material derivative by definition. For the second term:

$$\frac{\partial}{\partial t}\left(\frac{\partial \Phi_i}{\partial a_k}\right) = \frac{\partial}{\partial a_k}\left(\frac{\partial \Phi_i}{\partial t}\right) = \frac{\partial u_i}{\partial a_k}.$$

By the chain rule:

$$\frac{\partial u_i}{\partial a_k} = \frac{\partial u_i}{\partial x_j}\frac{\partial x_j}{\partial a_k} = \frac{\partial u_i}{\partial x_j}\frac{\partial \Phi_j}{\partial a_k}.$$

Substituting back:

$$\frac{dI}{dt} = \oint_{C(0)} \left[\frac{Df_i}{Dt}\frac{\partial \Phi_i}{\partial a_k} + f_i \frac{\partial u_i}{\partial x_j}\frac{\partial \Phi_j}{\partial a_k}\right]\, da_k.$$

From the lecture notes with Einstin summation notation:

$$dx_i = \frac{\partial \Phi_i}{\partial a_k}\, da_k \quad \text{and} \quad dx_j = \frac{\partial \Phi_j}{\partial a_k}\, da_k.$$

Applying this to the integrand:

$$\begin{aligned}\frac{dI}{dt} &= \oint_{C(t)} \frac{Df_i}{Dt}\, dx_i + \oint_{C(t)} f_i \frac{\partial u_i}{\partial x_j}\, dx_j \\ &= \oint_{C(t)} \frac{D\boldsymbol{f}}{Dt}\cdot d\boldsymbol{x} + \oint_{C(t)} f_i \left(\frac{\partial u_i}{\partial x_j} dx_j\right). \qquad (*)\end{aligned}$$

Using the substituion given by the question, the final rate of change is:

$$\frac{dI}{dt} = \oint_{C(t)} \frac{D\boldsymbol{f}}{Dt}\cdot d\boldsymbol{x} + \oint_{C(t)} \boldsymbol{f}\cdot d\boldsymbol{u}.$$

(b) Substituting $\boldsymbol{f} = \boldsymbol{u}$ into equation $(*)$:

$$\frac{d\Gamma}{dt} = \oint_{C(t)} \frac{D\boldsymbol{u}}{Dt}\cdot d\boldsymbol{x} + \oint_{C(t)} u_i \frac{\partial u_i}{\partial x_j}\, dx_j.$$

We know that

$$u_i \frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_j} \left(\frac{1}{2} u_i u_i \right) = \frac{\partial}{\partial x_j} \left(\frac{1}{2} |\boldsymbol{u}|^2 \right).$$

Therefore:

$$\oint_{C(t)} \boldsymbol{u} \cdot d\boldsymbol{u} = \oint_{C(t)} d \left(\frac{1}{2} |\boldsymbol{u}|^2 \right).$$

Since we are integrating around a closed loop, this integral vanishes:

$$\oint_{C(t)} d \left(\frac{1}{2} |\boldsymbol{u}|^2 \right) = 0.$$

Therefore, the rate of change of circulation simplifies to:

$$\frac{d\Gamma}{dt} = \oint_{C(t)} \frac{D\boldsymbol{u}}{Dt} \cdot d\boldsymbol{x}.$$