Lecture 5
Representation and Change of Basis

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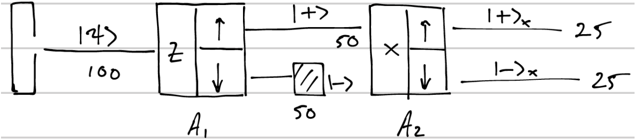


- We don't know yet the form of $\left|\pm\right\rangle_x$, however completeness requires that we can express $\left|\pm\right\rangle_x$ as a linear superposition of S_z-basis eigenvectors $\left|\pm\right\rangle$.

$$\left|+\right\rangle_x = a\left|+\right\rangle + b\left|-\right\rangle$$
$$\left|-\right\rangle_x = c\left|+\right\rangle + d\left|-\right\rangle$$

- To find the set of coefficients $\{a, b, c, d\}$, we need to analyze the outcome of the measurements.
- In QM, we can only measure probabilities not the probability amplitudes.

$$\left|\left\langle + \middle| + \right\rangle_x\right|^2 = \frac{1}{2} \quad \left|\left\langle + \middle| - \right\rangle_x\right|^2 = \frac{1}{2}$$

$$\left|\left\langle - \middle| + \right\rangle_x\right|^2 = \frac{1}{2} \quad \left|\left\langle - \middle| - \right\rangle_x\right|^2 = \frac{1}{2}$$

Basis States of the Observable S_x

$$|\langle +|\,(a\,|+\rangle + b\,|-\rangle)|^2 = |a|^2 = \tfrac{1}{2}$$

$$|\langle -|\,(a\,|+\rangle + b\,|-\rangle)|^2 = |b|^2 = \tfrac{1}{2}$$

$$|\langle +|\,(c\,|+\rangle + d\,|-\rangle)|^2 = |c|^2 = \tfrac{1}{2}$$

$$|\langle -|\,(c\,|+\rangle + d\,|-\rangle)|^2 = |d|^2 = \tfrac{1}{2}$$

$$|a| = |b| = |c| = |d| = \tfrac{1}{\sqrt{2}}$$

- The states $|\pm\rangle_x$ must be normalized, thus

$$\left|{}_x\langle +|+\rangle_x\right|^2 \Rightarrow |a|^2 + |b|^2 = 1$$

$$\left|{}_x\langle -|-\rangle_x\right|^2 \Rightarrow |c|^2 + |d|^2 = 1$$

- The states $|\pm\rangle_x$ must also be orthogonal, thus

$$_x\langle+|-\rangle_x = {}_x\langle-|+\rangle_x = 0$$

$$\Rightarrow (a^* \langle+| + b^* \langle-|)(c\,|+\rangle + d\,|-\rangle) = 0$$

$$a^* c + b^* d = 0$$

$$ac^* + bd^* = 0$$

- The most general way of expressing the probability amplitudes is

$$a = |a|e^{i\alpha}, \quad b = |b|e^{i\beta}$$
$$c = |c|e^{i\gamma}, \quad d = |d|e^{i\delta}$$

Basis States of the Observable S_x

- Using the orthogonality relationships, we find

$$\frac{1}{2}\left(e^{i(\gamma-\alpha)}+e^{i(\delta-\beta)}\right)=0$$

- Only the relative phase between the terms matters.
- The choice of phases is not unique, therefore we must adopt a convention for picking phases.
- The convention used is to make the make the coefficient of the $|+\rangle$ real and positive.
- Let $\alpha=\gamma=0 \Rightarrow \frac{1}{2}\left(1+e^{i(\delta-\beta)}\right)=0$
- Let $\beta=0$ and $\delta=\pi$ (again by convention)

$$\Rightarrow |+\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle+|-\rangle)$$

$$|-\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle-|-\rangle)$$

Note

See the end of the lecture for a derivation of the basis states of S_y.

Example: Probability of Measurement Outcomes

Example: Given the state $|\psi\rangle = 5\,|+\rangle + |-\rangle$, calculate the probability that a measurement of

1. S_z yields $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$
2. S_x yields $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$

- First, ensure that $|\psi\rangle$ is normalized so that the sum of the possible measurement outcomes sums to unity.

$$|\psi\rangle_N = C\,|\psi\rangle$$

$${}_N\langle\psi|\psi\rangle_N = 1$$

$$\Rightarrow |C|^2(5\,\langle+| + \langle-|)(5\,|+\rangle + |-\rangle) = 1$$

$$\Rightarrow 26|C|^2 = 1$$

$$C = \frac{1}{\sqrt{26}} \qquad \text{(Set overall phase factor} = 1)$$

$$|\psi\rangle_N = \frac{1}{\sqrt{26}}(5\,|+\rangle + |-\rangle)$$

Example: Probability of Measurement Outcomes

- Second, by Postulate 4, calculate the magnitude-squared of the projection of $|\psi\rangle_N$ on to appropriate eigenvector of S_z and S_x.

  - Probability that a measurement of S_z yields $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$

$$
\begin{aligned}
P\left(\tfrac{\hbar}{2}\right) &= \left|\langle + | \psi\rangle_N\right|^2 \\
&= \left|\tfrac{5}{\sqrt{26}}\langle + | + \rangle + \tfrac{1}{\sqrt{26}}\langle + | - \rangle\right|^2 \\
&= \tfrac{25}{26}
\end{aligned}
$$

$$
\begin{aligned}
P\left(-\tfrac{\hbar}{2}\right) &= \left|\langle - | \psi\rangle_N\right|^2 \\
&= \left|\tfrac{5}{\sqrt{26}}\langle - | + \rangle + \tfrac{1}{\sqrt{26}}\langle - | - \rangle\right|^2 \\
&= \tfrac{1}{26}
\end{aligned}
$$

Example: Probability of Measurement Outcomes

- Probability that a measurement of S_x yields $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$

$$
\begin{aligned}
P\left(\frac{\hbar}{2}\right) &= \left|{}_x\langle + | \psi \rangle_N\right|^2 \\
&= \left| \frac{1}{\sqrt{26}} \cdot \frac{1}{\sqrt{2}} (\langle + | + \langle - |)(5\,|+\rangle + |-\rangle) \right|^2 \\
&= \frac{1}{52} |5\,\langle + | + \rangle + \langle - | - \rangle|^2 \\
&= \frac{9}{13}
\end{aligned}
$$

$$
\begin{aligned}
P\left(-\frac{\hbar}{2}\right) &= \left|{}_x\langle - | \psi \rangle_N\right|^2 \\
&= \left| \frac{1}{\sqrt{26}} \cdot \frac{1}{\sqrt{2}} (\langle + | - \langle - |)(5\,|+\rangle + |-\rangle) \right|^2 \\
&= \frac{1}{52} |5\,\langle + | + \rangle - \langle - | - \rangle|^2 \\
&= \frac{4}{13}
\end{aligned}
$$

Change of Basis

Because the eigenvectors of S_z and S_x form a complete basis, we can represent an arbitrary state $|\psi\rangle$ using either basis.

- $|\psi\rangle$ represented in the S_z basis:

$$|\psi\rangle = a\,|+\rangle + b\,|-\rangle$$

- $|\psi\rangle$ represented in the S_x basis:

$$|\psi\rangle = a'\,|+\rangle_x + b'\,|-\rangle_x$$

- Solve for coefficient $\{a', b'\}$ in terms of $\{a, b\}$.

$$a' = {}_x\langle +|\psi\rangle = {}_x\langle +|\left(a\,|+\rangle + b\,|-\rangle\right)$$

$$b' = {}_x\langle -|\psi\rangle = {}_x\langle -|\left(a\,|+\rangle + b\,|-\rangle\right)$$

- We can organize the above quantity into a matrix equation

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \underbrace{\begin{pmatrix} {}_x\langle +|+\rangle & {}_x\langle +|-\rangle \\ {}_x\langle -|+\rangle & {}_x\langle -|-\rangle \end{pmatrix}}_{U} \begin{pmatrix} a \\ b \end{pmatrix}$$

- The matrix U is referred to as a transformation matrix.

$$U_{11} = {}_x\langle +|+\rangle = \frac{1}{\sqrt{2}}(\langle +| + \langle -|)\,|+\rangle = \frac{1}{\sqrt{2}}$$

$$U_{12} = {}_x\langle +|-\rangle = \frac{1}{\sqrt{2}}(\langle +| + \langle -|)\,|-\rangle = \frac{1}{\sqrt{2}}$$

$$U_{21} = {}_x\langle -|+\rangle = \frac{1}{\sqrt{2}}(\langle +| - \langle -|)\,|+\rangle = \frac{1}{\sqrt{2}}$$

$$U_{22} = {}_x\langle -|-\rangle = \frac{1}{\sqrt{2}}(\langle +| - \langle -|)\,|-\rangle = -\frac{1}{\sqrt{2}}$$

Change of Basis

- $U_{z\to x}$ transforms a vector from the $z-$basis to the $x-$ basis.

$$U_{z\to x} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Example: Given the normalized state vector $|\psi\rangle = \frac{1}{\sqrt{26}}(5\,|+\rangle + |-\rangle)$ in the S_z basis, calculate $|\psi\rangle$ in the S_x basis.

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \frac{1}{\sqrt{26}} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \frac{1}{\sqrt{52}}\begin{pmatrix} 6 \\ 4 \end{pmatrix} = \frac{1}{\sqrt{13}}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$|\psi\rangle = \frac{1}{\sqrt{13}}\left(3\,|+\rangle_x + 2\,|-\rangle_x\right)$$

The Transformation Matrix

- Let $|\psi\rangle$ be a quantum state that belongs to a Hilbert space H.
- Let $\{|a\rangle\}$ and $\{|b\rangle\}$ be two complete orthonormal bases that span H.
- We can represent the state $|\psi\rangle$ in both basis.

$$|\psi\rangle = \sum_n \langle a_n|\psi\rangle\,|a_n\rangle = \sum_n A_n\,|a_n\rangle$$

$$|\psi\rangle = \sum_n \langle b_n|\psi\rangle\,|b_n\rangle = \sum_m B_m\,|b_m\rangle$$

where $A_n = \langle a_n|\psi\rangle$ and $B_n = \langle b_n|\psi\rangle$ are the coefficients of the basis vectors in the a and b bases, respectively.

The Transformation Matrix

- We seek to find the transformation matrix $U_{a\to b}$ that transforms $|\psi\rangle$ from the a to the b basis.
- We note that the inner product $\langle\psi|\psi\rangle$ is independent of the choice of basis.
- Let's express $\langle\psi|\psi\rangle$ by expanding the bra $\langle\psi|$ in the b basis and the ket $|\psi\rangle$ in the a basis.

$$\langle\psi|\psi\rangle = \sum_m \sum_n (\langle b_m|\psi\rangle)^* \langle a_n|\psi\rangle \langle b_m|a_n\rangle = \sum_m \sum_n A_n B_m^* \langle b_m|a_n\rangle$$

- We can organize the above expression in matrix form

$$\langle\psi|\psi\rangle = \begin{pmatrix} B_1^* & B_2^* & \cdots \end{pmatrix} \underbrace{\begin{pmatrix} \langle b_1|a_1\rangle & \langle b_1|a_2\rangle & \cdots \\ \langle b_2|a_1\rangle & \langle b_2|a_2\rangle & \cdots \\ & \vdots & \end{pmatrix}}_{U_{a\to b}} \begin{pmatrix} A_1 \\ A_2 \\ \vdots \end{pmatrix}$$

$$\boxed{\underbrace{|\psi\rangle}_{b-\text{basis}} = U_{a\to b} \underbrace{|\psi\rangle}_{a-\text{basis}}}$$

- The transformation matrix $U_{a\to b}$ allows us to compute the representation of the state $|\psi\rangle$ in the $b-$basis, given that we know $|\psi\rangle$ in the $a-$basis.

The Transformation Matrix

- Likewise, we can also compute $|\psi\rangle$ in the $a-$basis starting from the $b-$basis by calculating the inverse of $U_{a\to b}$.

$$U^{-1}_{a\to b}\underbrace{|\psi\rangle}_{b-\text{basis}} = \underbrace{U^{-1}_{a\to b}U_{a\to b}}_{\mathbb{1}}\underbrace{|\psi\rangle}_{a-\text{basis}}$$

$$\underbrace{|\psi\rangle}_{a-\text{basis}} = U^{-1}_{a\to b}\underbrace{|\psi\rangle}_{b-\text{basis}} := U_{b\to a}\underbrace{|\psi\rangle}_{b-\text{basis}}$$

$$\therefore \boxed{U_{b\to a} = U^{-1}_{a\to b}}$$

- Conservation of probability requires that $\langle\psi|\psi\rangle$ is independent of the choice of basis. Therefore

$$\underbrace{\langle\psi|\psi\rangle}_{a-\text{basis}} = \underbrace{\langle\psi|\psi\rangle}_{b-\text{basis}}$$

$$\begin{aligned}\Rightarrow \underbrace{\langle\psi|\psi\rangle}_{a-\text{basis}} &= (U_{b\to a}|\psi\rangle)^\dagger(U_{b\to a}|\psi\rangle)\\ &= \langle\psi| U^\dagger_{b\to a}U_{b\to a}|\psi\rangle\end{aligned}$$

The Transformation Matrix

- For the above equality to hold, we require that $U^{\dagger}_{b\to a}U_{b\to a} = \mathbb{1}$.
- In general, transformation matrices are unitary, by which we mean that they preserve the norm of the state vector and satisfy the condition:

$$UU^{\dagger} = \mathbb{1}$$

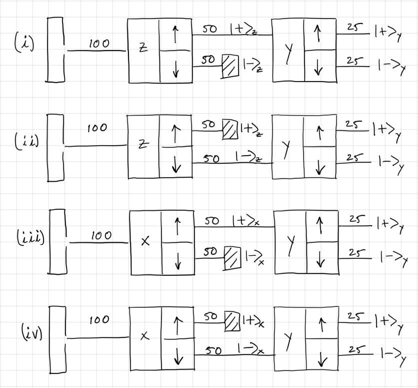
$$U^{\dagger}U = \mathbb{1}$$

- Since $UU^{\dagger} = \mathbb{1}$, $U^{\dagger}$ is must also satisfy

$$U^{\dagger} = U^{-1}$$

Eigenstates of the Operator S_y

- Consider the outcome of the following S-G measurements.



- We can express the basis states of the observable S_y in terms of the basis states of S_z.

$$|+\rangle_y = a\,|+\rangle + b\,|-\rangle$$

$$|-\rangle_y = c\,|+\rangle + d\,|-\rangle$$

Eigenstates of the Operator S_y

- Utilize the measurement outcomes from experiments (i) and (ii).

$$\left|\langle +|+\rangle_y\right|^2 = \tfrac{1}{2} \Rightarrow |a|^2 = \tfrac{1}{2}$$

$$\left|\langle -|+\rangle_y\right|^2 = \tfrac{1}{2} \Rightarrow |b|^2 = \tfrac{1}{2}$$

$$\left|\langle +|-\rangle_y\right|^2 = \tfrac{1}{2} \Rightarrow |c|^2 = \tfrac{1}{2}$$

$$\left|\langle -|-\rangle_y\right|^2 = \tfrac{1}{2} \Rightarrow |d|^2 = \tfrac{1}{2}$$

$$|a| = |b| = |c| = |d| = \frac{1}{\sqrt{2}}$$

- Choose $|a| = a$ and $|c| = c$, $b = |b|e^{i\beta}$ and $d = |d|e^{i\delta}$.

Eigenstates of the Operator S_y

- Utilize the measurement outcomes from experiments (iii) and (iv).

$$\left|{}_x\langle +|+\rangle_y\right|^2 = \left|\tfrac{1}{\sqrt{2}}(\langle +| + \langle -|)(a\,|+\rangle + b\,|-\rangle)\right|^2 = \tfrac{1}{2}$$

$$\Rightarrow |a+b|^2 = 1$$

$$\underbrace{|a|^2 + |b|^2}_{=1} + a^*b + ab^* = 1$$

$$a^*b + ab^* = 0 \Rightarrow \tfrac{1}{2}(e^{i\beta} + e^{-i\beta}) = \cos\beta = 0$$

$$\left|{}_x\langle +|-\rangle_y\right|^2 = \left|\tfrac{1}{\sqrt{2}}(\langle +| + \langle -|)(c\,|+\rangle + d\,|-\rangle)\right|^2 = \tfrac{1}{2}$$

$$\Rightarrow |c+d|^2 = 1$$

$$\underbrace{|c|^2 + |d|^2}_{=1} + c^*d + cd^* = 1$$

$$c^*d + cd^* = 0 \Rightarrow \tfrac{1}{2}(e^{i\delta} + e^{-i\delta}) = \cos\delta = 0$$

- Orthogonality condition

$$
{}_y\langle -|+\rangle_y = 0
$$

$$
(c^* \langle +| + d^* \langle -|)(a\,|+\rangle + b\,|-\rangle) = 0
$$

$$
c^* a + d^* b = 0
$$

$$
\Rightarrow 1 + e^{i(\beta-\delta)} = 0
$$

$$
\beta - \delta = \pi
$$

$$
\beta = \tfrac{\pi}{2}, \quad \delta = -\tfrac{\pi}{2}
$$

Eigenvectors of S_y

$$
|+\rangle_y = \tfrac{1}{\sqrt{2}}(|+\rangle + i\,|-\rangle)
$$

$$
|-\rangle_y = \tfrac{1}{\sqrt{2}}(|+\rangle - i\,|-\rangle)
$$