Laplace's eqn on a rectangle

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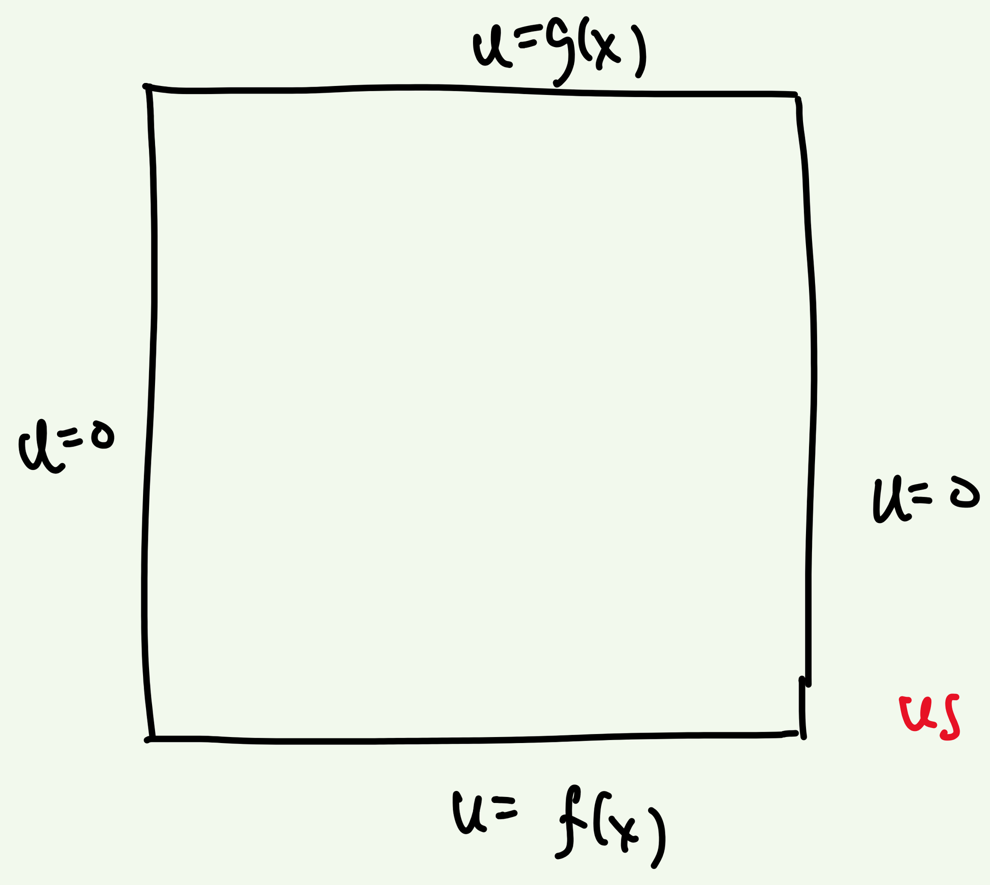
- 4.43 Laplace's eqn on a rectangle

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < L_x, \quad 0 < y < L_y$$

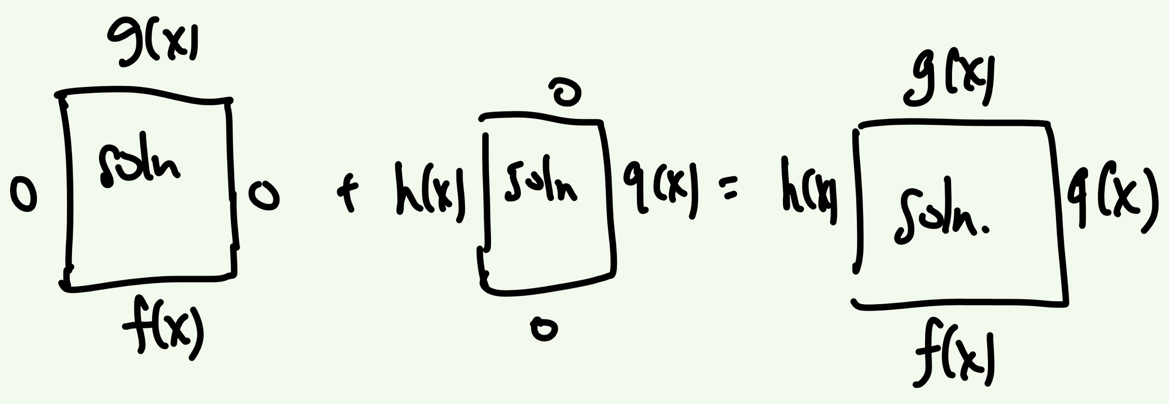
$u(x,0) = f(x)$, $u(x, L_y) = g(x)$ =>

$u(0,y) = 0$, $u(L_x, y) = 0$ => $M(0) = 0$, $M(L_x) = 0$



Find a Fourier series solution.

Note: solving this and the flipped B.C. problem allows us to solve Laplace's eqn. with non-zero B.C.



$u(x,y) = M(x)N(y)$

$N M_{xx} + M N_{yy} = 0$

$$\frac{M_{xx}}{M} = \frac{-N_{yy}}{N} = -\lambda$$

$$\frac{\partial^2 M}{\partial x^2} = -\lambda M, \quad \frac{\partial^2 N}{\partial y^2} = \lambda N$$

For $\lambda > 0$, this has soln.

$M(x) = A\cos(\sqrt{\lambda}\, x) + B\sin(\sqrt{\lambda}\, x)$

Applying B.C. $\quad \lambda_k = \left(\frac{k\pi}{L_x}\right)^2$

$M_k(x) = \sqrt{\frac{2}{L_x}} \sin\left(\frac{k\pi}{L_x} x\right)$

For N:

$$\frac{\partial^2 N}{\partial y^2} = \lambda N$$

this has solution

$N_k(y) = A_k \cosh(\sqrt{\lambda}\, y) + B_k \sinh(\sqrt{\lambda}\, y)$

$u(x,0) = \sum_k^\infty M_k(x) N_k(0) = f(x)$

$\sum_k^\infty A_k \cdot \sqrt{\frac{2}{L_x}} \sin\left(\frac{k\pi x}{L_x}\right) = f(x)$

$A_k = (f, M_k)$

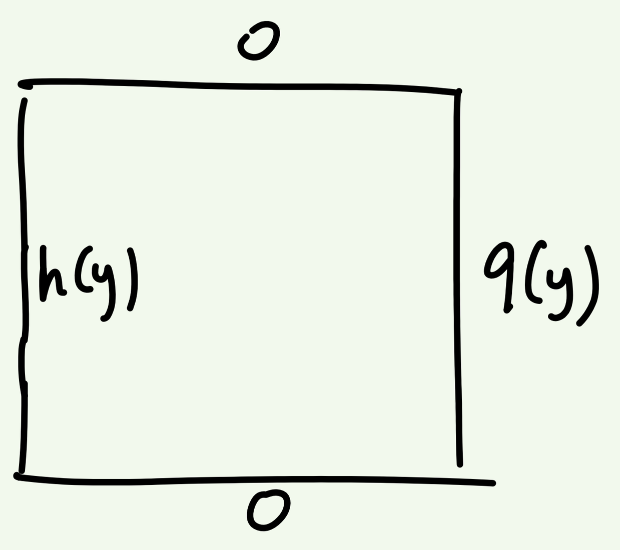
$u(x, L_y) = \sum_k^\infty M_k(x) N_k(L_y) = g(x)$

$\sum_k^\infty \left(A_k \cosh\left(\frac{k\pi L_y}{L_x}\right) + B_k \sinh\left(\frac{k\pi L_y}{L_x}\right)\right) \sqrt{\frac{2}{L_x}} \sin\left(\frac{k\pi x}{L_x}\right) = g(x)$

$A_k \cosh\left(\frac{k\pi L_y}{L_x}\right) + B_k \sinh\left(\frac{k\pi L_y}{L_x}\right) = (g, M_k)$

$$B_k = \frac{(g, M_k) - (f, M_k)\cosh\left(\frac{k\pi L_y}{L_x}\right)}{\sinh\left(\frac{k\pi L_y}{L_x}\right)}$$

what if the B.C. were on y?



$u(x,0) = 0 = u(x, L_y)$

$u(0,y) = h(y)$, $u(L_x, y) = q(y)$

$u(x,y) = M(y)N(x)$

(now we have the eigenfunction in y-direction)

$M_k(y) = \sqrt{\frac{2}{L_y}} \sin\left(\frac{k\pi y}{L_y}\right)$, $\lambda_k = \left(\frac{k\pi}{L_y}\right)^2$

$N_k(x) = A_k \cosh\left(\frac{k\pi x}{L_y}\right) + B_k \sinh\left(\frac{k\pi x}{L_y}\right)$