

Lecture 4

Quantum State Vectors

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Hilbert Space

- The ket $|\psi\rangle$, that mathematically represents the quantum state of the system, belongs to a Hilbert space.
- A Hilbert space is a generalization of Euclidean space, having finite or infinite number of dimensions, and can be real or complex.
- A Hilbert space is **complete**, by which we mean that we can define a distance using an inner product.
- For the spin-1/2 system, there are two states, represented by the kets $|+\rangle$ and $|-\rangle$, that define two independent directions.
- We can construct **any** state vector $|\psi\rangle$ as a linear superposition of states $|+\rangle$ and $|-\rangle$.

$$|\psi\rangle = a|+\rangle + b|-\rangle$$

where, a and b are complex constants.

- The dimension of the Hilbert space for a spin-1/2 particle is 2.

Inner Product

- We define the inner product as $\langle \phi | \psi \rangle$, which measures the projection of the quantum state $|\phi\rangle$ on the state $|\psi\rangle$.
- We refer to the object $\langle \cdot \cdot |$ as a “bra”.

$$\begin{aligned}(\text{“bra”}) + (\text{“ket”}) &= \langle \cdot \cdot | \cdot \cdot \rangle \\ &= \text{“braket”}\end{aligned}$$

- Mathematically, the bra is defined as

$$\langle \psi | = (|\psi\rangle)^\dagger$$

where the “dagger” † represents the conjugate transpose operator, sometimes also called the Hermitian transpose.

Likewise,

$$|\psi\rangle = (\langle \psi |)^\dagger$$

Inner Product

- We will see the significance of the conjugate transpose operator when using a matrix representation of the state vectors.
- For a linear superposition, we calculate the bra $\langle\psi|$ corresponding to the ket $|\psi\rangle$ in the following way

$$\begin{aligned}|\psi\rangle &= a|+\rangle + b|-\rangle \\ (|\psi\rangle)^\dagger &= (a|+\rangle + b|-\rangle)^\dagger \\ \langle\psi| &= a^*(|+\rangle)^\dagger + b^*(|-\rangle)^\dagger \\ \langle\psi| &= a^*\langle+| + b^*\langle-|\end{aligned}$$

- 1 The basis vectors are orthogonal.

$$\langle+|-\rangle = \langle-|+\rangle = 0$$

- 2 The basis vectors have a norm = 1.

$$\langle+|+\rangle = \langle-|-\rangle = 1$$

Inner Product

- **Completeness:** Any state vector in the Hilbert space of a spin-1/2 particle can be expressed as a linear superposition of the basis vectors.

$$|\psi\rangle = a|+\rangle + b|-\rangle$$

- From the properties of the inner product, and using the orthogonality and norm properties we find

$$\begin{aligned}\langle+|\psi\rangle &= \langle+|(a|+\rangle + b|-\rangle) \\ &= a \underbrace{\langle+|+\rangle}_{=1} + b \underbrace{\langle+|-\rangle}_{=0}\end{aligned}$$

- We define the **probability amplitudes**:

$$a = \langle+|\psi\rangle$$

$$b = \langle-|\psi\rangle$$

- We also find

$$\begin{aligned}\langle\psi|+\rangle &= (\langle+|\psi\rangle)^\dagger = (a^* \langle+| + b^* \langle-|) |+\rangle \\ &= a^* \langle+|+\rangle + b^* \langle-|+\rangle = a^*\end{aligned}$$

Postulate 4 (Spin-1/2 System)

The probability of obtaining the value $\pm \frac{\hbar}{2}$ in a measurement of the observable S_z of a system in the state $|\psi\rangle$ is

$$P_{\pm} = |\langle \pm | \psi \rangle|^2$$

- If you think back to Young's double slit experiment, we can make a correspondence with the probability amplitude and the complex field amplitude of an electromagnetic wave.

$$E_0 e^{i\phi} \Longleftrightarrow a$$

- The intensity of a an electromagnetic wave is proportional to the square of the magnitude of the electric field amplitude. If you recall, we said that the **probability** of a photon hitting the screen at a given position is proportional to the intensity of light at that position.

$$I = |E|^2 = E_0^2 \Longleftrightarrow |a|^2$$

Normalization

- There are only two measurement outcomes that are possible for a spin-1/2 particle. Therefore, we require that the sum of the probabilities of these two outcomes equal 1.

$$|\langle +|\psi\rangle|^2 = |a|^2 \Rightarrow \text{Probability that the state is } |+\rangle$$

$$|\langle -|\psi\rangle|^2 = |b|^2 \Rightarrow \text{Probability that the state is } |-\rangle$$

$$P_+ + P_- = 1$$

$$|\langle +|\psi\rangle|^2 + |\langle -|\psi\rangle|^2 = 1$$

$$\therefore |a|^2 + |b|^2 = 1$$

- **Normalization:** The preceding statement requires that $\langle \psi | \psi \rangle = 1$.

$$\begin{aligned}\langle \psi | \psi \rangle &= (a^* \langle + | + b^* \langle - |) (a | + \rangle + b | - \rangle) \\&= aa^* \underbrace{\langle + | + \rangle}_{=1} + a^* b \underbrace{\langle + | - \rangle}_{=0} + b^* a \underbrace{\langle - | + \rangle}_{=0} + bb^* \underbrace{\langle - | - \rangle}_{=1} \\&= |a|^2 + |b|^2 = 1\end{aligned}$$

Example: Normalization of a State Vector

Problem: Find the normalized state vector corresponding to the state

$$|\psi\rangle = 3|+\rangle + 2i|-\rangle$$

We define the normalized state vector $|\psi\rangle_N$ corresponding to the state $|\psi\rangle$ as

$$|\psi\rangle_N := C|\psi\rangle$$

where C is the normalization constant. To find C , apply the normalization condition to $|\psi\rangle_N$.

$${}_N\langle\psi|\psi\rangle_N = 1$$

$$|C|^2(3\langle+|-2i\langle-|)(3|+\rangle+2i|-\rangle) = 1$$

$$|C|^2(9+4) = 1$$

$$|C| = \frac{1}{\sqrt{13}}$$

$$C = \frac{1}{\sqrt{13}}e^{i\theta}$$

Important Note on Normalization

Multiplying a state vector by an overall phase factor $e^{i\theta}$ does not change the probability of a measurement.

$$\text{Let } |\psi_1\rangle = e^{i\theta} |\psi\rangle$$

Probability to observe state $|a\rangle$: $P = |\langle a|\psi_1\rangle|^2$

$$\begin{aligned} P &= \left| e^{i\theta} \langle a|\psi\rangle \right|^2 \\ &= \left(e^{i\theta} \langle a|\psi\rangle \right) \left(e^{-i\theta} \langle a|\psi\rangle^* \right) \\ &= |\langle a|\psi\rangle|^2 \\ \therefore P &= |\langle a|\psi_1\rangle|^2 = |\langle a|\psi\rangle|^2 \end{aligned}$$

Therefore, without loss of generality, we can set $\theta = 0$ and the overall phase factor of any quantum state $e^{i\theta} = 1$ without changing the outcome of measurement probabilities.

Example: Normalization of a State Vector

Continued ...

$$C = \frac{1}{\sqrt{13}} e^{i\theta}$$

Set $\theta = 0$

$$C = \frac{1}{\sqrt{13}}$$

$$|\psi\rangle_N = \frac{1}{\sqrt{13}}(3|+\rangle + 2i|-\rangle)$$

Matrix Notation

We represent a ket as a column matrix and a bra as a row matrix.

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\langle +| = (1 \quad 0), \quad \langle -| = (0 \quad 1)$$

Unit Norm:

$$\langle +|+\rangle = (1 \quad 0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1, \quad \langle -|-\rangle = (0 \quad 1) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1$$

Orthogonality:

$$\langle +|-\rangle = (1 \quad 0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0, \quad \langle -|+\rangle = (0 \quad 1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

Matrix Notation

A general state $|\psi\rangle = a|+\rangle + b|-\rangle$ can be represented in matrix form as

$$|\psi\rangle = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

The corresponding bra $\langle\psi|$ is

$$\langle\psi| = (|\psi\rangle)^\dagger = (a^* \quad b^*)$$

The inner product in matrix form is thus

$$\langle\psi|\psi\rangle = (a^* \quad b^*) \begin{pmatrix} a \\ b \end{pmatrix} = |a|^2 + |b|^2$$

The inner product $\langle\psi|\psi\rangle$ is a measure of the square of the magnitude of the quantum state vector $|\psi\rangle$.