Lecture 2
The Stern-Gerlach Experiment I

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The Stern-Gerlach Experiment

- We saw last lecture that a magnetic field gradient generates a force on a magnetic moment,

$$\boldsymbol{F} = -\boldsymbol{\nabla} U = \boldsymbol{\nabla}(\boldsymbol{\mu} \cdot \boldsymbol{B}) \tag{1}$$

where U is the potential energy of a magnetic moment $\boldsymbol{\mu}$ in an external magnetic field $\boldsymbol{B}$.

- $\boldsymbol{\mu}$ has two contributions coming from the orbital and intrinsic (spin) angular momenta:

$$\boldsymbol{\mu}_{\text{total}} = \boldsymbol{\mu}_{\text{orbital}} + \boldsymbol{\mu}_{\text{spin}} \tag{2}$$

$$\boldsymbol{\mu}_{\text{orbital}} = \left(\frac{q}{2m}\right) \boldsymbol{L} \tag{3}$$

$$\boldsymbol{\mu}_{\text{spin}} = g \left(\frac{q}{2m}\right) \boldsymbol{S} \tag{4}$$

$g \approx 2$ for an electron spin.

The Stern-Gerlach Experiment

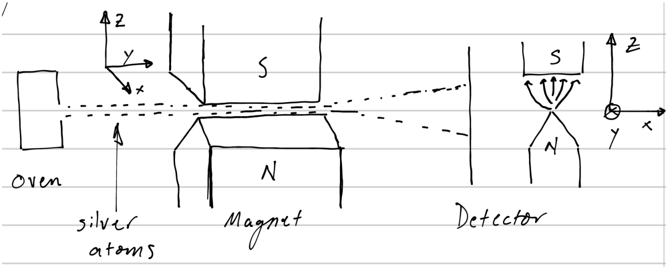


Figure: S-G apparatus.

- In their measurements, Stern and Gerlach used a beam of silver atoms produced by evaporating silver metal from an oven. The beam of neutral atoms exited from a small aperture and were sent through the poles of a permanent magnet that was designed to produce a strong component of the magnetic field gradient in the $z-$direction. A schematic of the experiment is shown above.

- Between the poles of the magnet, the spins experience a force given by

$$F_z = (\boldsymbol{\mu} \cdot \hat{z}) \left(\frac{\partial B_z}{\partial z} \right) = |\boldsymbol{\mu}| \cos\theta \left(\frac{\partial B_z}{\partial z} \right) \tag{5}$$

- Classically, we would expect a uniform distribution of θ for the silver atoms exiting the oven. That is, there should be no preferred orientation of silver atoms with respect to any direction in space.
- If θ is sampled uniformly over the range $\theta \in \{0, 2\pi\}$, then we expect the distribution like the one shown in the figure below, with a peak at $\theta = 90°$ (See Eq.5.)

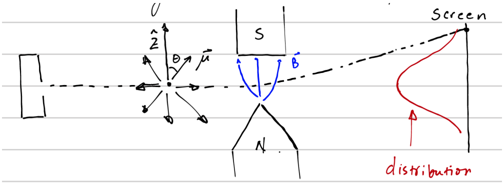
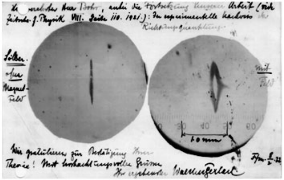


Figure: Classical distribution

- Instead, of a continuous peak having a $\cos\theta$ distribution, S-G observed two distinct peaks. (See figure below)



- The results were **not** consistent with the Bohr model, which predicted a continuous distribution of silver atoms.
- The spatial quantization of angular momentum was later understood by Ulenbeck and Goudsmit in 1925 as a consequence of the intrinsic "spin" angular momentum of the electron.
- Furthermore, it was recognized much later that the silver atom as an unpaired electron in the outer $5s$ orbital. Since s orbitals do not have an orbital angular momentum, the total angular momentum of the electron is from the spin.

$$\boldsymbol{\mu}_{\text{total}} = \boldsymbol{\mu}_{\text{spin}} = g\left(\frac{q}{2m}\right)\boldsymbol{S} \tag{6}$$

- This meant that the direction of $\boldsymbol{\mu}_{\text{spin}}$ had only 2 values, which could only be true if the intrinsic angular momentum of the electron was quantized, having only two values.
- A measurement of the spin angular momentum for the electron along any axis, yields two distinct values:

$$S_z = \pm\frac{\hbar}{2} \tag{7}$$

The z-subscript refers to the measurement axis. The choice is arbitrary.
- The unit of angular momentum is called Planck's constant:

$$\hbar = \frac{h}{2\pi} = 1.0546 \times 10^{-34}\ \text{J.s} \tag{8}$$

$$\text{For and electron: } \left|\boldsymbol{\mu}_{\text{spin}}\right| \approx \frac{\hbar\, e}{2m} = 9.27 \times 10^{-24}\ \frac{\text{J}}{\text{T}} \tag{9}$$

- We refer to particles with $|S_z| = \frac{\hbar}{2}$ as spin-$\frac{1}{2}$ particles.

The Quantum State Vector

- Since our choice of coordinates is arbitrary, this means that the outcome of the measurement would be $\pm\frac{\hbar}{2}$ regardless of the direction of the magnetic field.
- We use the Dirac notation, created by Paul Dirac, to represent the state of the spin.
- The "ket" $|+\rangle$ refers to spins deflected upward through the gradient magnet, and $|-\rangle$ refers to spins deflected downward; "up" and "down" are arbitrary.
- You will sometimes see these notations, all of which refer to spin-up:

$$|+\rangle = |+\hbar/2\rangle = |\uparrow\rangle = |+\hat{z}\rangle$$

- In general, the quantum state of the spin can be expressed as a linear combination of states $|+\rangle$ and $|-\rangle$

$$|\psi\rangle = a\,|+\rangle + b\,|-\rangle \tag{10}$$

where a and b are complex coefficients.

The Quantum State Vector

Postulate 1

The state of a quantum system, including all the information you can know about it, is represented mathematically by a normalized ket $|\psi\rangle$.

- Following the notation used by McIntyre, we represent the S-G experiment using the following simplified diagram.

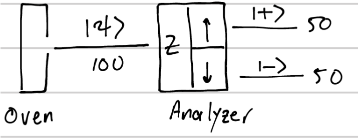


Figure: S-G apparatus diagram

- The diagram refers to a measurement made with the magnet oriented in the $z-$direction, and 2 outcomes: $S_z = \pm\frac{\hbar}{2}$, corresponding to the states $|+\rangle$ and $|-\rangle$. There is a $50/50$ probability of measuring either state.
- The order of events measured at the output of the Analyzer is completely random.

S-G Experiment 1

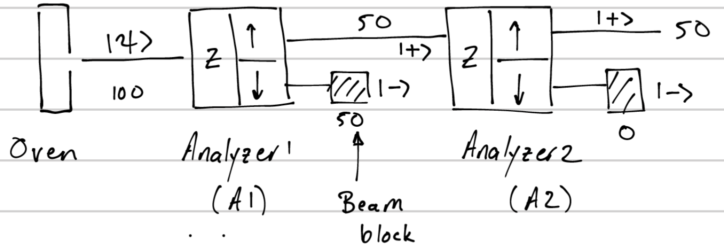


Figure: S-G Experiment 1

- Analyzer 1 prepares the beam in the $|+\rangle$ state with respect to the $z-$axis.
- If a second analyzer is placed, having the same orientation as the first, at the output of the first analyzer, then the state of the particles is not affected by passing through the second analyzer.

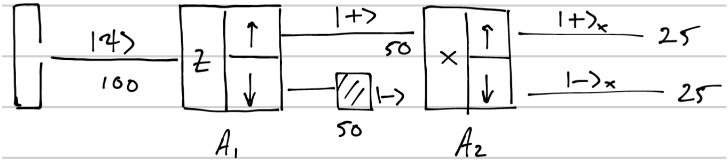


Figure: S-G Experiment 2

- Here, we prepare the state in the $|+\rangle$ state and rotate the measurement analyzer by 90° so that it's orientation is along the $x-$axis.
- We again find a $50/50$ chance of measuring either $|+\rangle_x$ or $|-\rangle_x$. We use the $x-$subscript to indicate the measurement axis.
- Note: (1) We would get the same $50/50$ distribution if we used the lower port ($|-\rangle$) of Analyzer 1.

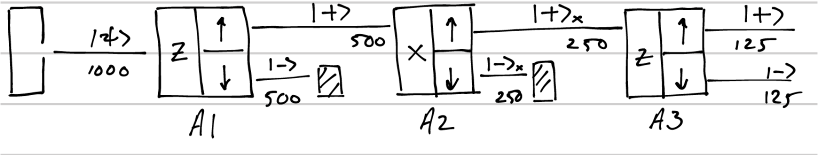


Figure: S-G Experiment 3

- Classically, you might expect that the output of $A3$ should be 100% polarized in the $|+\rangle$ state. However, we see that the spins that exit $A2$ retain no information of having been polarized in the $|+\rangle$ state by $A1$.
- This is because the measurement of S_x disturbs our knowledge of S_z. In quantum mechanics, we cannot simultaneously know S_x and S_z. Measurement of different orthogonal components of $\boldsymbol{S}$ are incompatible, i.e., a measurement of one component of $\boldsymbol{S}$ disturbs the outcome of the measurement of the other component.

Experiment 3 Comment

- We will see, later in the course, other sets of measurements that are incompatible.
- All measurements are compatible with themselves. For example, we see in S-G Experiment 1, that if we repeat the $z-$measurement, the output of A2 is 100% polarized in the $|+\rangle$ state.
- Suppose we repeat Experiment 3, this time switching the order of the last two analyzers.

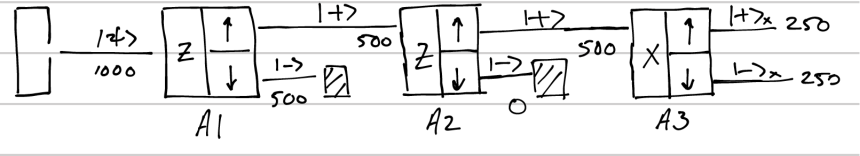


Figure: S-G Experiment 3 with the A2 and A3 switched.

Experiment 3 Comment

- The output of A3 is $2\times$ as large as before. This fact demonstrates a very important difference between classical and quantum measurements.
- Classical measurements do not disturb the state of the system. Therefore, we get the same outcome regardless of the order in which the measurements are performed.
- For quantum measurements, in contrast, the order can determine the outcome, provided some of the measurements are incompatible.