

Lecture 8

Expectation Value of Operators

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Introduction to State Tomography

- We wish to consider the average value of an observable corresponding to the operator \hat{A} when performed on a particular state $|\psi\rangle$.
- We know from Postulate 3 of QM that the measurement will yield one of the possible eigenvalues of the observable with probability $P_n = |\langle a_n | \psi \rangle|^2$ (Postulate 4), where $\{|a\rangle\}$ represent the eigenbasis of \hat{A} . The new state after measurement will be $|\psi'\rangle = |a_n\rangle$ (Postulate 5).
- Each measurement projects or “collapses” the initial quantum state $|\psi\rangle$ to one of the eigenstates of \hat{A} .
- Therefore, to determine the original $|\psi\rangle$, we must repeat the measurement many times and determine the probabilities for each outcome. This means we need many identical copies of $|\psi\rangle$ in order to reconstruct it.
- From the distribution of measurement outcomes, we can reconstruct the original $|\psi\rangle$ up to some overall phase factor.

The Expectation Value

- We express the state $|\psi\rangle$ in the eigenbasis of operator \hat{A} .

$$|\psi\rangle = \sum_n c_n |a_n\rangle$$

- We denote the average or “expected” value of the measurement of \hat{A} for the state $|\psi\rangle$ as $\langle\hat{A}\rangle$.

$$\langle\hat{A}\rangle = \sum_n P_n a_n \quad (1)$$

- The average value of \hat{A} for the state $|\psi\rangle$ is equal to the sum over all eigenvalues $\{a\}$ of \hat{A} multiplied by the corresponding probability of measurement.

$$\langle\hat{A}\rangle = \sum_n a_n |\langle a_n | \psi \rangle|^2 \quad (2)$$

$$= \sum_n a_n |c_n|^2 = \sum_n a_n \langle \psi | \hat{P}_n | \psi \rangle \quad (3)$$

where $\hat{P}_n = |a_n\rangle\langle a_n|$ is the projection operator for state $|a_n\rangle$ (See Lecture 7).

The Expectation Value

- Using spectral decomposition (Lecture 7), we express operator \hat{A} using projection operators.

$$\hat{A} = \sum_n^{\text{all states}} a_n |a_n\rangle\langle a_n| \quad (4)$$

- Inserting eq.(4) into eq.(3), we find that the expectation value of operator \hat{A} for state $|\psi\rangle$ is

$$\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle \quad (5)$$

- The expectation value of an operator for a particular state is a physical quantity. Therefore, it cannot depend on our choice of basis for expressing $|\psi\rangle$.

Root-Mean-Square (RMS) Variation of Measurements

- Consider the expectation or average values $\langle S_z \rangle$ and $\langle S_x \rangle$ in the $|\pm\rangle$ basis.

$$\langle \pm | S_z | \pm \rangle = \pm \frac{\hbar}{2} \langle \pm | \pm \rangle = \pm \frac{\hbar}{2}$$

$$\begin{aligned}\langle + | S_x | + \rangle &= \langle + | \left(\frac{\hbar}{2} |+x\rangle\langle+x| - \frac{\hbar}{2} |-x\rangle\langle-x| \right) |+ \rangle \\ &= \frac{1}{\sqrt{2}} \left(\frac{\hbar}{2} \right) \langle + | (|+x\rangle - |-x\rangle) = 0\end{aligned}$$

$$\begin{aligned}\langle - | S_x | - \rangle &= \langle - | \left(\frac{\hbar}{2} |+x\rangle\langle+x| - \frac{\hbar}{2} |-x\rangle\langle-x| \right) | - \rangle \\ &= \frac{1}{\sqrt{2}} \left(\frac{\hbar}{2} \right) \langle - | (|+x\rangle + |-x\rangle) = 0\end{aligned}$$

- We see that the expectation value of S_x for the states $|\pm\rangle$ is zero. We know from S-G exp. 2 that if we prepare a state in an eigenstate of one component of the angular momentum operator, then make a second measurement along an orthogonal direction, we obtain a 50%/50% probability of measuring spin up or down. Therefore, the average value of the second measurements will yield zero.

Root-Mean-Square (RMS) Variation of Measurements

- The RMS value of a set of measurements of operator A is defined as

$$\Delta A = \sqrt{\langle (A - \langle A \rangle)^2 \rangle}$$

$$\Delta A = \sqrt{\langle A^2 \rangle - 2\langle A \rangle \langle A \rangle + \langle A \rangle^2}$$

$$= \sqrt{\langle A^2 \rangle - 2\langle A \rangle \langle A \rangle + \langle A \rangle^2}$$

$$\Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$$

where $\langle A^2 \rangle$ = expectation value of A^2 .

Root-Mean-Square (RMS) Variation of Measurements

- Consider the RMS deviation for measurements of S_z for the S_z eigenkets $|\pm\rangle$.

$$\langle \pm | S_z^2 | \pm \rangle = \frac{\hbar^2}{4}$$

$$\Rightarrow \Delta S_z = \sqrt{\langle \pm | S_z^2 | \pm \rangle - (\langle \pm | S_z | \pm \rangle)^2} = 0$$

- Recall that ΔS_z is the average distance between a particular measurement of S_z and the average value of S_z for a collection of measurements performed on the states $|\pm\rangle$. Because, each measurement yields the same result, either $+\frac{\hbar}{2}$ or $-\frac{\hbar}{2}$, the variation between measurements is zero.

$$\langle + | S_z | + \rangle \pm \Delta S_z \Rightarrow \frac{\hbar}{2} \pm 0$$

$$\langle - | S_z | - \rangle \pm \Delta S_z \Rightarrow -\frac{\hbar}{2} \pm 0$$

Root-Mean-Square (RMS) Variation of Measurements

- Consider the RMS deviation of the operators S_x and S_y for the states $|\pm\rangle$.
- We have found that $\langle \pm | S_x | \pm \rangle = 0$. Likewise, you can also show that $\langle \pm | S_y | \pm \rangle = 0$.
- To calculate ΔS_x and ΔS_y , we need to calculate $\langle S_x^2 \rangle$ and $\langle S_y^2 \rangle$.
- There are at least two equivalent ways of calculating expectation values. Previously, I used the spectral decomposition of S_x to calculate $\langle S_x \rangle$. We can use this approach again to calculate S_x^2 .

$$S_x = \frac{\hbar}{2} |+x\rangle\langle+x| - \frac{\hbar}{2} |-x\rangle\langle-x|$$

$$S_x^2 = \left(\frac{\hbar}{2} |+x\rangle\langle+x| - \frac{\hbar}{2} |-x\rangle\langle-x|\right)\left(\frac{\hbar}{2} |+x\rangle\langle+x| - \frac{\hbar}{2} |-x\rangle\langle-x|\right)$$

$$\begin{aligned} &= \frac{\hbar^2}{4} |+x\rangle\langle+x| |+x\rangle\langle+x| - \frac{\hbar^2}{4} |+x\rangle\langle+x| |-x\rangle\langle-x| - \frac{\hbar^2}{4} |-x\rangle\langle-x| |+x\rangle\langle+x| \\ &\quad + \frac{\hbar^2}{4} |-x\rangle\langle-x| |-x\rangle\langle-x| \end{aligned}$$

$$= \frac{\hbar^2}{4} (|+x\rangle\langle+x| + |-x\rangle\langle-x|) = \frac{\hbar^2}{4} \mathbb{1} \quad (\text{Completeness})$$

Root-Mean-Square (RMS) Variation of Measurements

- We can also calculate S_x^2 using the matrix representation of the operator

$$S_x^2 = \frac{\hbar^2}{4} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\hbar^2}{4} \mathbb{1}$$

- The expectation value $\langle \pm | S_x^2 | \pm \rangle = \frac{\hbar^2}{4}$.

$$\Delta S_x = \sqrt{\langle \pm | S_x^2 | \pm \rangle - (\langle \pm | S_x | \pm \rangle)^2} = \frac{\hbar}{2}$$

$$\langle + | S_x | + \rangle \pm \Delta S_x \Rightarrow 0 \pm \frac{\hbar}{2}$$

$$\langle - | S_x | - \rangle \pm \Delta S_x \Rightarrow 0 \pm \frac{\hbar}{2}$$

Root-Mean-Square (RMS) Variation of Measurements

- We likewise have for the operator S_y

$$S_y^2 = \frac{\hbar^2}{4} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\hbar^2}{4} \mathbb{1}$$

- The expectation value $\langle \pm |S_y^2| \pm \rangle = \frac{\hbar^2}{4}$.

$$\Delta S_y = \sqrt{\langle \pm |S_y^2| \pm \rangle - (\langle \pm |S_y| \pm \rangle)^2} = \frac{\hbar}{2}$$

$$\begin{aligned}\langle + |S_y| + \rangle \pm \Delta S_y &\Rightarrow 0 \pm \frac{\hbar}{2} \\ \langle - |S_y| - \rangle \pm \Delta S_y &\Rightarrow 0 \pm \frac{\hbar}{2}\end{aligned}$$

- These set of results demonstrate something very basic and profound about QM. Namely, we cannot simultaneously determine measurement outcomes for two non-commuting observable with arbitrary accuracy. If we determine the state of one operator, e.g., S_z with zero uncertainty, then a second measurement in an orthogonal direction, e.g., S_x or S_y is uncertain. In the next lecture, we will derive a rigorous relationship that quantifies this uncertainty relationship.