Lecture 7
Projection Operators

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The Projection Operator

- We seek an operator formalism for calculating the outcome of a measurement of a quantum system.
- Prior to a measurement, the state of the system can in general be described by the ket $|\psi\rangle$ that is a linear superposition of eignvectors of an observable $\hat{A}$.

$$|\psi\rangle = \sum_n c_n |a_n\rangle \qquad \text{where } c_n = \langle a_n|\psi\rangle$$

$$|\psi\rangle = \sum_n \langle a_n|\psi\rangle |a_n\rangle$$

$$= \sum_n |a_n\rangle (\langle a_n|\psi\rangle) = \sum_n \underbrace{(|a_n\rangle\langle a_n|)}_{\hat{P}_n} |\psi\rangle$$

- We define the projection operator

$$\hat{P}_n := |a_n\rangle\langle a_n| \tag{1}$$

- Consider the action of $\hat{P}_n$ on a state $|\psi\rangle$.

$$\hat{P}_n |\psi\rangle = |a_n\rangle\langle a_n| \psi\rangle = c_n |a_n\rangle$$

- How do we represent the projection operator?
- We have seen that the inner product, defined as $\langle\psi|\psi\rangle$, is a scalar.
- What is $|\psi\rangle\langle\psi|$?
- If v is a $n \times 1$ complex vector, then the outer product w is an $n \times n$ tensor.

$$\hat{w} := |v\rangle\langle v| = v \otimes v = vv^\dagger$$

$$\hat{w} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \begin{pmatrix} v_1^* & v_2^* & \cdots & v_n^* \end{pmatrix}$$

$$w_{ij} = v_i v_j^*$$

Example

- Find the projection operators for the S_z operator.

$$\hat{P}_+ = |+\rangle\langle+| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\hat{P}_- = |-\rangle\langle-| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\hat{P}_+ + \hat{P}_- = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$$

- We note an interesting fact: the sum of all projection operators corresponding to S_z is identity. This is a general feature for all observables, which we will prove shortly.

The Projection Operator

- Consider the expectation value of the operator $\hat{P}_n$ for state $|\psi\rangle$. I will discuss expectation values later in this lecture.

$$\begin{aligned}\langle\psi|\hat{P}_n|\psi\rangle &= \langle\psi|\underbrace{|a_n\rangle\langle a_n|}_{\hat{P}_n}|\psi\rangle \\ &= \langle\psi|a_n\rangle\,\langle a_n|\psi\rangle = (\langle a_n|\psi\rangle)^*\,\langle a_n|\psi\rangle \\ &= |\langle a_n|\psi\rangle|^2 \\ &= |c_n|^2\end{aligned} \tag{2}$$

- The expectation value $\langle\psi|\hat{P}_n|\psi\rangle$ returns the probability that $|\psi\rangle$ is in the state $|a_n\rangle$.

- Consider the quantity $\langle\psi|\psi\rangle$

$$\langle\psi|\psi\rangle = \sum_n\sum_m c_n^* c_m \underbrace{\langle a_n|a_m\rangle}_{\delta_{nm}} = \sum_n |c_n|^2$$

$$= \sum_n \langle\psi|\hat{P}_n|\psi\rangle = \langle\psi|\underbrace{\sum_n \hat{P}_n}_{\mathbb{1}}|\psi\rangle$$

Completeness

$$\sum_{n}^{\text{all states}} \hat{P}_n = \sum_{n}^{\text{all states}} |a_n\rangle\langle a_n| = \mathbb{1} \tag{3}$$

- This is a very useful expression that I will use many times in this course.
- We can use completeness to represent operators corresponding to observables in terms of their projection operators.

$$\hat{A}\mathbb{1} = \sum_n \hat{A}\,|a_n\rangle\langle a_n| = \sum_n a_n\,|a_n\rangle\langle a_n|$$

$$\boxed{\hat{A} = \sum_{n}^{\text{all states}} a_n\,|a_n\rangle\langle a_n|} \tag{4}$$

- Eq.(4) is referred to as spectral decomposition of operator $\hat{A}$.

Spectral Decomposition

- To see how the spectral decomposition of an operator $\hat{A}$ works, consider the action of $\hat{A}$ on an arbitrary state $|\psi\rangle$.

$$
\begin{aligned}
|\psi\rangle &= \sum_n \langle a_n|\psi\rangle\, |a_n\rangle \\
&= \sum_n c_n\, |a_n\rangle
\end{aligned}
$$

$$
\begin{aligned}
\hat{A}\,|\psi\rangle &= \sum_n \sum_m a_m c_n\, |a_m\rangle\langle a_m|\, a_n\rangle \\
&= \sum_n \sum_m a_m c_n\, \underbrace{\langle a_m|a_n\rangle}_{\delta_{nm}}\, |a_m\rangle \\
&= \sum_n c_n a_n\, |a_n\rangle
\end{aligned}
$$

- $\hat{A}\,|\psi\rangle$ corresponds to a different state vector $|\psi'\rangle$

$$|\psi'\rangle = \sum_n c'_n\,|a_n\rangle$$

with $c'_n = c_n\,a_n$.

- We note that if $|\psi\rangle$ represents an arbitrary quantum state, then in general

$$\hat{A}\,|\psi\rangle = |\psi'\rangle \neq \text{const.}\,|\psi\rangle$$

- Therefore, an arbitrary superposition of eigenstates of an operator is not an eigenstate of the operator.
- There is a physical relationship between the states $|\psi\rangle$ and $|\psi'\rangle$. We will explore this relationship in the next lecture.

Projection operators and measurement

- Let the initial quantum state prior to measurement be given by

$$|\psi_i\rangle = \sum_n c_n |a_n\rangle$$

- The measurement produces one of the possible eigenstates $|a_n\rangle$.
- We say that a measurement of $\hat{A}$ that yields a measurement result a_n, "collapses" or "projects" $|\psi_i\rangle$ to $|\psi_f\rangle = |a_n\rangle$.
- All subsequent measurements using observable $\hat{A}$ yield the state $|a_n\rangle$ with 100% probability, i.e.

$$\hat{A}|\psi_f\rangle = \hat{A}|a_n\rangle = a_n |a_n\rangle$$

- We express the measurement process using the projection operator corresponding to state $|a_n\rangle$.

$$|\psi_f\rangle = \frac{\hat{P}_n |\psi_i\rangle}{\sqrt{|\langle a_n|\psi_i\rangle|^2}} = \frac{|a_n\rangle\langle a_n|\,\psi_i\rangle}{|\langle a_n|\psi_i\rangle|} = \frac{\langle a_n|\psi_i\rangle\,|a_n\rangle}{|\langle a_n|\psi_i\rangle|} \tag{5}$$

- The denominator in eq.(5) ensures that the coefficient of $|a_n\rangle$ is normalized to be 1, up to an overall phase factor.
- If we express the inner product $\langle a_n|\psi_i\rangle = |\langle a_n|\psi_i\rangle| e^{i\alpha}$, then eq.(5) may be written as

$$\left|\psi_f\right\rangle = e^{i\alpha}\left|a_n\right\rangle$$

- Up to an overall phase factor, eq.(5) projects the state $|\psi_i\rangle$ to the state $|a_n\rangle$. Since a quantum state can at most be defined up to an overall phase factor, the states $|a_n\rangle$ and $e^{i\alpha}|a_n\rangle$ are equivalent as far as any measurements are concerned.

Postulate 5 of QM

Postulate 5

After a measurement of observable $\hat{A}$ that yields the result a_n, the quantum system is in a new state that is the normalized projection of the original system ket onto the ket(s) corresponding to the result of the measurement:

$$\left|\psi_f\right\rangle = \frac{\hat{P}_n \left|\psi_i\right\rangle}{\sqrt{\langle\psi_i|\hat{P}_n|\psi_i\rangle}}$$

- In writing Postulate 5, we have made use of eq.(2) to express the denominator in terms of the projection operator.

Expectation Value of an Operator

- We introduce the expectation value of an operator $\hat{A}$, which returns the average or "expected" value of the operator corresponding to the state $|\psi\rangle$.
- Consider an arbitrary state $|\psi\rangle$ expressed in the $\hat{A}$ basis.

$$|\psi\rangle = \sum_{n}^{\text{all states}} c_n \, |a_n\rangle \qquad \text{where } c_n = \langle a_n|\psi\rangle$$

where $\hat{A} = \sum_n a_n \, |a_n\rangle\langle a_n|$.

- The expectation value of $\hat{A}$ for the state $|\psi\rangle$ is given by

$$\begin{aligned}\langle\psi|\hat{A}|\psi\rangle &= \sum_n a_n \langle\psi\,|a_n\rangle\langle a_n|\,\psi\rangle \\ &= \sum_n a_n |\langle a_n|\psi\rangle|^2 = \sum_n a_n |c_n|^2\end{aligned}$$

- We see that the expectation value returns the eigenvalues of operator $\hat{A}$ weighted by the probabilities. Thus, it corresponds to the average value of the operator for a given state $|\psi\rangle$.

Analysis of S-G Experiment 3 Using Projection Operators

- We can view output ports of a S-G analyzer as projection operators that act on the input state.
- The act of going through the S-G analyzer does not constitute a measurement. The measurement process would involve putting a detector on one or both ports of the S-G analyzer to obtain information about the state of the atom. If we place a detector on one of the ports and register the presense of an atom, then we know the state of that atom with $\%100$ certainty.

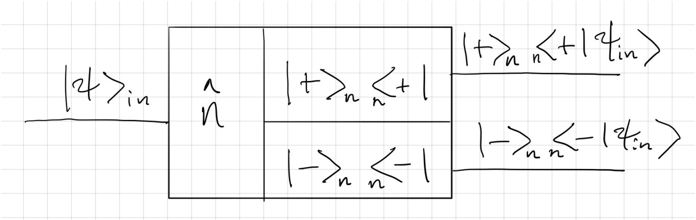
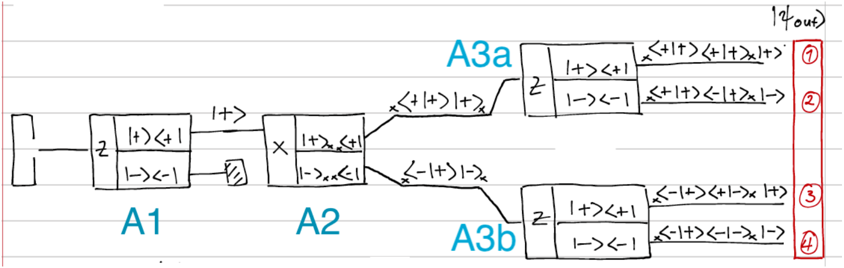


Figure: S-G Analyzer oriented in an arbitrary direction $\hat{\mathbf{n}}$.

Analysis of S-G Experiment 3 Using Projection Operators

- Consider S-G Experiment 3



- Output of $|+\rangle$ port of Analyzer 1:

$$\begin{aligned}|\psi_1\rangle &= |+\rangle\,\langle+|\psi\rangle_{\text{oven}}\\ &= |+\rangle\end{aligned}$$

- The prefactor $\langle+|\psi\rangle_{\text{oven}}$ is proportional to the number of spins in the $|+\rangle$ state that come out of the oven. I will set the prefactor to 1 to simplify the math.

Analysis of S-G Experiment 3 Using Projection Operators

- Output of $\left|+\right\rangle_x$ port of Analyzer 2:

$$\begin{aligned}\left|\psi_{2+}\right\rangle &= \left|+\right\rangle_x {}_x\!\left\langle+|\psi_1\right\rangle \\ &= {}_x\!\left\langle+|+\right\rangle \left|+\right\rangle_x\end{aligned}$$

- Output of $\left|-\right\rangle_x$ port of Analyzer 2:

$$\begin{aligned}\left|\psi_{2-}\right\rangle &= \left|-\right\rangle_x {}_x\!\left\langle-|\psi_1\right\rangle \\ &= {}_x\!\left\langle-|+\right\rangle \left|-\right\rangle_x\end{aligned}$$

- Output of $\left|+\right\rangle$ port of Analyzer 3a:

$$\begin{aligned}\left|\psi_{\text{out}}\right\rangle_1 &= \left|+\right\rangle\!\left\langle+\right| \left|\psi_{2+}\right\rangle \\ &= \left|+\right\rangle\!\left\langle+\right| \left({}_x\!\left\langle+|+\right\rangle \left|+\right\rangle_x\right) \\ &= {}_x\!\left\langle+|+\right\rangle \left\langle+|+\right\rangle_x \left|+\right\rangle \\ &= \left|\left\langle+|+\right\rangle_x\right|^2 \left|+\right\rangle = \frac{1}{2}\left|+\right\rangle\end{aligned}$$

Analysis of S-G Experiment 3 Using Projection Operators

- Output of $|-\rangle$ port of Analyzer 3a:

$$
\begin{aligned}
|\psi_{\text{out}}\rangle_2 &= |-\rangle\langle-|\,|\psi_{2+}\rangle \\
&= |-\rangle\langle-|\,\left({}_x\langle+|+\rangle\,|+\rangle_x\right) \\
&= {}_x\langle+|+\rangle\,\langle-|+\rangle_x\,|-\rangle = \tfrac{1}{2}\,|-\rangle
\end{aligned}
$$

- Output of $|+\rangle$ port of Analyzer 3b:

$$
\begin{aligned}
|\psi_{\text{out}}\rangle_3 &= |+\rangle\langle+|\,|\psi_{2-}\rangle \\
&= |+\rangle\langle+|\,\left({}_x\langle-|+\rangle\,|-\rangle_x\right) \\
&= {}_x\langle-|+\rangle\,\langle+|-\rangle_x\,|+\rangle \\
&= \left|\langle+|-\rangle_x\right|^2\,|+\rangle = \tfrac{1}{2}\,|+\rangle
\end{aligned}
$$

Analysis of S-G Experiment 3 Using Projection Operators

- Output of $|-\rangle$ port of Analyzer 3b:

$$
\begin{aligned}
|\psi_{\text{out}}\rangle_4 &= |-\rangle\langle -|\,|\psi_{2-}\rangle \\
&= |-\rangle\langle -|\,\left({}_x\langle -|+\rangle\,|-\rangle_x\right) \\
&= {}_x\langle -|+\rangle\,\langle -|-\rangle_x\,|-\rangle = -\tfrac{1}{2}\,|-\rangle
\end{aligned}
$$

Analysis of S-G Experiment 3 Using Projection Operators

Probabilities of measurement outcomes:

1. Probability to exit in the $|+\rangle$ port of A3a:

$$P_1 = \left|\langle + | \psi_{\text{out}}\rangle_1\right|^2 = \frac{1}{4}$$

2. Probability to exit in the $|-\rangle$ port of A3a:

$$P_2 = \left|\langle - | \psi_{\text{out}}\rangle_2\right|^2 = \frac{1}{4}$$

3. Probability to exit in the $|+\rangle$ port of A3b:

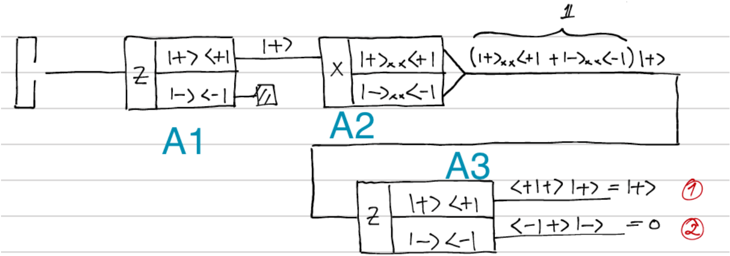
$$P_3 = \left|\langle + | \psi_{\text{out}}\rangle_3\right|^2 = \frac{1}{4}$$

4. Probability to exit in the $|-\rangle$ port of A3b:

$$P_4 = \left|\langle - | \psi_{\text{out}}\rangle_4\right|^2 = \frac{1}{4}$$

Analysis of S-G Experiment 4 Using Projection Operators

- Consider S-G Experiment 4:



- If you recall, in S-G Experiment 4, both outputs of A2 are sent into A3. We found that if we did not measure either output of A2, then the probability amplitudes for each of the two paths through A2 added constructively to produce 100% probability of exiting the $|+\rangle$ port of A3.
- This result comes out naturally using the projection operator formalism.

Analysis of S-G Experiment 4 Using Projection Operators

- Output of $|+\rangle$ port of Analyzer 1:

$$|\psi_1\rangle = |+\rangle$$

- Output of Analyzer 2:

$$\begin{aligned} |\psi_2\rangle &= \left(|+\rangle_{x\ x}\langle+| \quad + \quad |-\rangle_{x\ x}\langle-|\right)|\psi_1\rangle \\ &= \underbrace{\left(|+\rangle_{x\ x}\langle+| \quad + \quad |-\rangle_{x\ x}\langle-|\right)}_{\mathbb{1}}|+\rangle \\ &= |+\rangle \end{aligned}$$

Note, I've used Completeness (Eq.(3)) in simplifying the above expression.

- Output of $|+\rangle$ port of Analyzer 3:

$$\begin{aligned} |\psi_{3+}\rangle &= |+\rangle\langle+|\,|\psi_2\rangle \\ &= \langle+|+\rangle\,|+\rangle \\ &= |+\rangle \end{aligned}$$

Analysis of S-G Experiment 4 Using Projection Operators

- Output of $|-\rangle$ port of Analyzer 3:

$$\begin{aligned} |\psi_{3-}\rangle &= |-\rangle\langle-|\,|\psi_2\rangle \\ &= \langle-|+\rangle\,|-\rangle \\ &= 0 \end{aligned}$$

- Probability of measurement outcomes

1. Probability to exit the $|+\rangle$ port of A3:

$$P_+ = |\langle+|+\rangle|^2 = 1$$

2. Probability to exit the $|-\rangle$ port of A3:

$$P_- = 0$$