Lecture 6
Generalized Quantum Systems

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Postulate 2

A physical observable is represented mathematically by an operator $\hat{A}$ that acts on kets.

Postulate 3

The only possible result of a measurement of an observable is one of the eigenvalues a_n corresponding to the operator $\hat{A}$.

Note the $(\hat{\ })$ superscript notation is used to signify that the mathematical object is an operator. Most of the time, I will suppress this notation.

Properties of Operators Corresponding to Observables

- Physical observalbes are represented by Hermetian matrices.
- A Hermetian matrix $\hat{A}$ is a square matrix that satisfies the condition

$$\hat{A} = \hat{A}^\dagger \tag{1}$$

$$\text{Definition:} \quad (\hat{A}^\dagger)_{ij} = A^*_{ji}$$

1. Hermetian matrices have real eigenvalues.
   Proof: Consider the quantity

$$\text{Let} \quad \hat{A}\,|a_n\rangle = a_n\,|a_n\rangle$$

$$\langle a_m|\hat{A}|a_n\rangle - \langle a_m|\hat{A}^\dagger|a_n\rangle = 0$$

This statement is true by the definition of a Hermetian matrix (Eq.1).

$$\underbrace{\langle a_m|\hat{A}|a_n\rangle}_{a_n\,\langle a_m|a_n\rangle} - \underbrace{\langle a_m|\,\hat{A}^\dagger}_{(\hat{A}|a_m\rangle)^\dagger = a_m^*\,\langle a_m|} \quad |a_n\rangle = 0$$

$$= (a_n - a_m^*)\,\langle a_m|a_n\rangle = 0$$

Properties of Operators Corresponding to Observables

Consider the case $m = n$:

$$(a_n - a_n^*) \langle a_n | a_n \rangle = 0$$
$$\langle a_n | a_n \rangle \neq 0 \;\therefore (a_n - a_n^*) = 0$$
$$\Rightarrow \quad a_n = a_n^* \therefore a_n \in \mathbb{R}$$

Physically, all measurement outcomes corresponding to an observable must be real. This property of Hermetian matrices ensures real outcomes.

2 All eigenvectors with distinct eigenvalues are orthonormal.

Proof: The eigenvalues $\{a\}$ of a Hermetian matrix are real. We showed that

$$(a_n - a_m^*) \langle a_m | a_n \rangle = 0$$
$$\Rightarrow \quad (a_n - a_m) \langle a_m | a_n \rangle = 0$$

If $a_n \neq a_m$, then $\langle a_m | a_n \rangle = 0$.

Operator Representation

- Let $\{|a\rangle\}$ be the set of eigenvectors (I will also refer to them as eigenkets) corresponding to operator $\hat{A}$ and $\{a\}$ be the corresponding set of eigenvalues.

$$|\psi\rangle = \sum_m c_m |a_m\rangle \tag{2}$$

where $\{c\}$ are the set of probability amplitudes.

- The eigenvectors $\{|a\rangle\}$ form an orthonormal set, with

$$\langle a_n | a_m \rangle = \delta_{nm} \tag{3}$$

where δ_{nm} is the Kronecker delta function

$$\delta_{nm} = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases} \tag{4}$$

- We can use Eq.3 to calculate the probability of being in the state $|a_n\rangle$.

$$P_n = |\langle a_n|\psi\rangle|^2 = \left|\sum_m c_m \langle a_n|a_m\rangle\right|^2 = |c_n|^2 \tag{5}$$

Operator Representation

- Based on Postulate 3, we seek to find the representation of $\hat{A}$ in the eigenbasis $\{|a\rangle\}$.

$$\hat{A}\,|a_n\rangle = a_n\,|a_n\rangle$$

- The operator $\hat{A}$ acting on an eigenvector $|a_n\rangle$ returns the eigenvector $|a_n\rangle$ scaled by the corresponding eigenvalue a_n.
- The matrix elements of $\hat{A}$ are given by

$$A_{nm} := \langle a_n|\hat{A}|a_m\rangle = \langle a_n|\,(\hat{A}\,|a_m\rangle) = a_m\,\langle a_n|a_m\rangle = a_m\delta_{nm} \tag{6}$$

- We see from the form of Eq.6, that the matrix representation of an operator is diagonal when expressed in its eigenbasis, with the diagonal elements being the eigenvalues of the operator.

Operator Representation

- Let $\hat{A}$ be an operator corresponding to an observable in a 2-dimensional Hilbert space.

$$\hat{A}\,|a_1\rangle = a_1\,|a_1\rangle$$
$$\hat{A}\,|a_2\rangle = a_2\,|a_2\rangle$$

$$\hat{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$A_{11} = a_1 \quad A_{12} = A_{21} = 0 \quad A_{22} = a_2$$

$$\hat{A} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$$

Eigenvectors of $\hat{A}$

- To find for the eigenvectors of $\hat{A}$, we solve the eigenvalue equation.

$$\hat{A}\,|\psi\rangle = \lambda\,|\psi\rangle$$

- Let $|\psi\rangle = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$.

$$\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \lambda \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\begin{pmatrix} a_1 c_1 - \lambda c_1 \\ a_2 c_2 - \lambda c_2 \end{pmatrix} = 0$$

- $(\lambda = a_1) \quad \Rightarrow \quad \begin{pmatrix} 0 \\ c_2(a_2 - a_1) \end{pmatrix} = 0$
- In general $a_1 \neq a_2 \;\therefore\; c_2 = 0$

$$\lambda = a_1, \quad |a_1\rangle = \begin{pmatrix} c_1 \\ 0 \end{pmatrix}$$

Eigenvectors of $\hat{A}$

- We require the basis vectors to have a unit norm.

$$\langle a_1|a_1\rangle = |c_1|^2 = 1$$

$$|a_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- $(\lambda = a_2) \quad \Rightarrow \quad \begin{pmatrix} (a_1 - a_2)c_1 \\ 0 \end{pmatrix} = 0$
- In general, $a_1 \neq a_2 \;\therefore\; c_1 = 0$

$$\lambda = a_2, \quad |a_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- Note that the eigenvectors of the operator are orthogonal unit vectors.

Representation of the Operators S_x, S_y and S_z

- As an example, consider the representation of the three components of the angular momentum operator for a spin-1/2 particle.
- In this example, we choose the S_z-basis to represent the three operators.
- By Postulate 3, we know that $\pm\frac{\hbar}{2}$ are the two eigenvalues for each operator.

- Representation of S_z in the S_z-basis:
- We know that the eigenvectors corresponding to S_z are $|\pm\rangle$. We've also seen that eigenvectors are unit vectors if represented in their own basis. Therefore, in the S_z basis, we have

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- We know that the states $|\pm\rangle$ are eigenstates of S_z, with eigenvalues $\pm\frac{\hbar}{2}$.

$$S_z\,|\pm\rangle = \pm\frac{\hbar}{2}\,|\pm\rangle$$

- Using Eq.6, we find the matrix elements of S_z.

$$(S_z)_{11} = \langle + | S_z | + \rangle = \tfrac{\hbar}{2}, \quad (S_z)_{12} = \langle + | S_z | - \rangle = 0$$
$$(S_z)_{21} = \langle - | S_z | + \rangle = 0, \quad (S_z)_{22} = \langle - | S_z | - \rangle = -\tfrac{\hbar}{2}$$

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Representation of S_x in the S_z-basis:
- We need to evaluate the matrix elements $\langle + | S_x | + \rangle$, $\langle + | S_x | - \rangle$, $\langle - | S_x | + \rangle$ and $\langle - | S_x | - \rangle$. This requires that we know how the operator S_x acts on the states $|\pm\rangle$.
- It's not immediately obvious how to do compute this, since $|\pm\rangle$ are not eigenstates of the operator S_x. We do know however how S_x acts on $|\pm\rangle_x$, since these are it's eignestates.

$$S_x \, |\pm\rangle_x = \pm\tfrac{\hbar}{2} \, |\pm\rangle_x$$

Representation of the Operators S_x, S_y and S_z

- We further know how to represent the states $|\pm\rangle_x$ in the S_z-basis.

$$|+\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \qquad |-\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

- To evaluate the matrix elements of S_x, however, we need to know how to represent the S_z-eignestates $|\pm\rangle$ in terms of the eigenstates in the S_x-basis.
- We can apply a change of basis to represent $|\pm\rangle$ in the S_x-basis.
  See Lecture 5 for details.

$$\begin{aligned}|+\rangle &= {}_x\langle+|+\rangle\,|+\rangle_x + {}_x\langle-|+\rangle\,|-\rangle_x\\ |-\rangle &= {}_x\langle+|-\rangle\,|+\rangle_x + {}_x\langle-|-\rangle\,|-\rangle_x\end{aligned}$$

$$\begin{aligned}|+\rangle &= \frac{1}{\sqrt{2}}(|+\rangle_x + |-\rangle_x)\\ |-\rangle &= \frac{1}{\sqrt{2}}(|+\rangle_x - |-\rangle_x)\end{aligned}$$

Representation of the Operators S_x, S_y and S_z

- Calculate the matrix elements of S_x

$$
\begin{aligned}
\langle +|S_x|+\rangle &= \frac{1}{2}\left({}_x\langle +| + {}_x\langle -|\right) S_x \left(|+\rangle_x + |-\rangle_x\right) \\
&= \frac{1}{2}\left({}_x\langle +| + {}_x\langle -|\right)\left(\frac{\hbar}{2}\,|+\rangle_x - \frac{\hbar}{2}\,|-\rangle_x\right) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\langle +|S_x|-\rangle &= \frac{1}{2}\left({}_x\langle +| + {}_x\langle -|\right) S_x \left(|+\rangle_x - |-\rangle_x\right) \\
&= \frac{1}{2}\left({}_x\langle +| + {}_x\langle -|\right)\left(\frac{\hbar}{2}\,|+\rangle_x + \frac{\hbar}{2}\,|-\rangle_x\right) \\
&= \frac{\hbar}{2}
\end{aligned}
$$

$$
\begin{aligned}
\langle -|S_x|+\rangle &= \frac{1}{2}\left({}_x\langle +| - {}_x\langle -|\right) S_x \left(|+\rangle_x + |-\rangle_x\right) \\
&= \frac{1}{2}\left({}_x\langle +| - {}_x\langle -|\right)\left(\frac{\hbar}{2}\,|+\rangle_x - \frac{\hbar}{2}\,|-\rangle_x\right) \\
&= \frac{\hbar}{2}
\end{aligned}
$$

Representation of the Operators S_x, S_y and S_z

$$\begin{aligned}
\langle -|S_x|-\rangle &= \tfrac{1}{2}({}_x\langle +| - {}_x\langle -|)S_x(|+\rangle_x - |-\rangle_x) \\
&= \tfrac{1}{2}({}_x\langle +| - {}_x\langle -|)(\tfrac{\hbar}{2}|+\rangle_x + \tfrac{\hbar}{2}|-\rangle_x) \\
&= 0
\end{aligned}$$

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- Representation of S_y in the S_z-basis:

$$|+\rangle_y = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \qquad |-\rangle_y = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

- To evaluate the matrix elements of S_y, we need to know how to represent the S_z-eignestates $|\pm\rangle$ in terms of the eigenstates in the S_y-basis.

- We can apply a change of basis to represent $|\pm\rangle$ in the S_y-basis.

$$|+\rangle = {}_y\langle +|+\rangle\,|+\rangle_y + {}_y\langle -|+\rangle\,|-\rangle_y$$
$$|-\rangle = {}_y\langle +|-\rangle\,|+\rangle_y + {}_y\langle -|-\rangle\,|-\rangle_y$$

$$|+\rangle = \frac{1}{\sqrt{2}}(|+\rangle_y + |-\rangle_y)$$
$$|-\rangle = -\frac{i}{\sqrt{2}}(|+\rangle_y - |-\rangle_y)$$

- Calculate the matrix elements of S_y

$$\begin{aligned}\langle +|S_y|+\rangle &= \frac{1}{2}({}_y\langle +| + {}_y\langle -|)S_y(|+\rangle_y + |-\rangle_y)\\ &= \frac{1}{2}({}_y\langle +| + {}_y\langle -|)(\frac{\hbar}{2}\,|+\rangle_y - \frac{\hbar}{2}\,|-\rangle_y)\\ &= 0\end{aligned}$$

Representation of the Operators S_x, S_y and S_z

$$\langle +|S_y|-\rangle = -\tfrac{i}{2}({}_y\langle +| + {}_y\langle -|)S_y(|+\rangle_y - |-\rangle_y)$$

$$= -\tfrac{i}{2}({}_y\langle +| + {}_y\langle -|)(\tfrac{\hbar}{2}\,|+\rangle_y + \tfrac{\hbar}{2}\,|-\rangle_y)$$

$$= -i\frac{\hbar}{2}$$

$$\langle -|S_y|+\rangle = \tfrac{i}{2}({}_y\langle +| - {}_y\langle -|)S_y(|+\rangle_y + |-\rangle_y)$$

$$= \tfrac{i}{2}({}_y\langle +| - {}_y\langle -|)(\tfrac{\hbar}{2}\,|+\rangle_y - \tfrac{\hbar}{2}\,|-\rangle_y)$$

$$= i\frac{\hbar}{2}$$

$$\langle -|S_y|-\rangle = \tfrac{1}{2}({}_y\langle +| - {}_y\langle -|)S_y(|+\rangle_y - |-\rangle_y)$$

$$= \tfrac{1}{2}({}_y\langle +| - {}_y\langle -|)(\tfrac{\hbar}{2}\,|+\rangle_y + \tfrac{\hbar}{2}\,|-\rangle_y)$$

$$= 0$$

Representation of the Operators S_x, S_y and S_z

$$S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$