

# Lecture 4

## Quantum State Vectors

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Week of January 12, 2026

# Table of Contents

1 Hilbert Space

2 Inner Product

3 Postulate 4 for a Spin-1/2 System

4 Normalization

5 Matrix Notation

# Hilbert Space

- The ket  $|\psi\rangle$ , that mathematically represents the quantum state of the system, belongs to a Hilbert space.
- A Hilbert space is a generalization of Euclidean space, having finite or infinite number of dimensions, and can be real or complex.
- A Hilbert space is **complete**, by which we mean that we can define a distance using an inner product.
- For the spin-1/2 system, there are two states, represented by the kets  $|+\rangle$  and  $|-\rangle$ , that define two independent directions.
- We can construct **any** state vector  $|\psi\rangle$  as a linear superposition of states  $|+\rangle$  and  $|-\rangle$ .

$$|\psi\rangle = a|+\rangle + b|-\rangle$$

where,  $a$  and  $b$  are complex constants.

- The dimension of the Hilbert space for a spin-1/2 particle is 2.

## Inner Product

- We define the inner product as  $\langle \phi | \psi \rangle$ , which measures the projection of the quantum state  $|\phi\rangle$  on the state  $|\psi\rangle$ .
- We refer to the object  $\langle \cdots |$  as a “bra”.

$$\begin{aligned} (\text{"bra"}) + (\text{"ket"}) &= \langle \cdots | \cdots \rangle \\ &= \text{"braket"} \end{aligned}$$

- Mathematically, the bra is defined as

$$\langle \psi | = (| \psi \rangle)^\dagger$$

where the “dagger”  $^\dagger$  represents the conjugate transpose operator, sometimes also called the Hermitian transpose.

Likewise,

$$| \psi \rangle = (\langle \psi |)^\dagger$$

# Inner Product

- We will see the significance of the conjugate transpose operator when using a matrix representation of the state vectors.
- For a linear superposition, we calculate the bra  $\langle \psi |$  corresponding to the ket  $|\psi\rangle$  in the following way

$$\begin{aligned} |\psi\rangle &= a|+\rangle + b|-\rangle \\ (\langle\psi|)^\dagger &= (a|+\rangle + b|-\rangle)^\dagger \\ \langle\psi| &= a^*(|+)\rangle^\dagger + b^*(|-\rangle)^\dagger \\ \langle\psi| &= a^* \langle+| + b^* \langle-| \end{aligned}$$

- ① The basis vectors are orthogonal.

$$\langle +|-\rangle = \langle -|+ \rangle = 0$$

- ② The basis vectors have a norm = 1.

$$\langle +|+ \rangle = \langle -|-\rangle = 1$$

## Inner Product

- **Completeness:** Any state vector in the Hilbert space of a spin-1/2 particle can be expressed as a linear superposition of the basis vectors.

$$|\psi\rangle = a|+\rangle + b|-\rangle$$

- From the properties of the inner product, and using the orthogonality and norm properties we find

$$\begin{aligned}\langle +|\psi\rangle &= \langle +|(a|+\rangle + b|-\rangle) \\ &= a\underbrace{\langle +|+ \rangle}_{=1} + b\underbrace{\langle +|- \rangle}_{=0}\end{aligned}$$

- We define the **probability amplitudes**:

$$a = \langle +|\psi\rangle$$

$$b = \langle -|\psi\rangle$$

- We also find

$$\begin{aligned}\langle\psi|+ \rangle &= (\langle +|\psi\rangle)^\dagger = (a^* \langle +| + b^* \langle -|)|+\rangle \\ &= a^* \langle +|+ \rangle + b^* \langle -|+ \rangle = a^*\end{aligned}$$

## Postulate 4

### Postulate 4 (Spin-1/2 System)

The probability of obtaining the value  $\pm \frac{\hbar}{2}$  in a measurement of the observable  $S_z$  of a system in the state  $|\psi\rangle$  is

$$P_{\pm} = |\langle \pm | \psi \rangle|^2$$

- If you think back to Young's double slit experiment, we can make a correspondence with the probability amplitude and the complex field amplitude of an electromagnetic wave.

$$E_0 e^{i\phi} \iff a$$

- The intensity of an electromagnetic wave is proportional to the square of the magnitude of the electric field amplitude. If you recall, we said that the **probability** of a photon hitting the screen at a given position is proportional to the intensity of light at that position.

$$I = |E|^2 = E_0^2 \iff |a|^2$$

## Normalization

- There are only two measurement outcomes that are possible for a spin-1/2 particle. Therefore, we require that the sum of the probabilities of these two outcomes equal 1.

$$|\langle +|\psi \rangle|^2 = |a|^2 \Rightarrow \text{Probability that the state is } |+\rangle$$

$$|\langle -|\psi \rangle|^2 = |b|^2 \Rightarrow \text{Probability that the state is } |-\rangle$$

$$P_+ + P_- = 1$$

$$|\langle +|\psi \rangle|^2 + |\langle -|\psi \rangle|^2 = 1$$

$$\therefore |a|^2 + |b|^2 = 1$$

# Normalization

- **Normalization:** The preceding statement requires that  $\langle \psi | \psi \rangle = 1$ .

$$\begin{aligned}\langle \psi | \psi \rangle &= (a^* \langle +| + b^* \langle -|) (a |+ \rangle + b |- \rangle) \\ &= aa^* \underbrace{\langle +| +}_{=1} + a^* b \underbrace{\langle +| -}_{=0} + b^* a \underbrace{\langle -| +}_{=0} + bb^* \underbrace{\langle -| -}_{=1} \\ &= |a|^2 + |b|^2 = 1\end{aligned}$$

## Example: Normalization of a State Vector

**Problem:** Find the normalized state vector corresponding to the state

$$|\psi\rangle = 3|+\rangle + 2i|-\rangle$$

We define the normalized state vector  $|\psi\rangle_N$  corresponding to the state  $|\psi\rangle$  as

$$|\psi\rangle_N := C|\psi\rangle$$

where  $C$  is the normalization constant. To find  $C$ , apply the normalization condition to  $|\psi\rangle_N$ .

$$_N\langle\psi|\psi\rangle_N = 1$$

$$|C|^2(3\langle+|-2i\langle-|)(3|+>+2i|-\rangle) = 1$$

$$|C|^2(9+4) = 1$$

$$|C| = \frac{1}{\sqrt{13}}$$

$$C = \frac{1}{\sqrt{13}}e^{i\theta}$$

## Important Note on Normalization

Multiplying a state vector by an overall phase factor  $e^{i\theta}$  does not change the probability of a measurement.

$$\text{Let } |\psi_1\rangle = e^{i\theta} |\psi\rangle$$

Probability to observe state  $|a\rangle$  :  $P = |\langle a|\psi_1\rangle|^2$

$$\begin{aligned} P &= \left| e^{i\theta} \langle a|\psi \rangle \right|^2 \\ &= \left( e^{i\theta} \langle a|\psi \rangle \right) \left( e^{-i\theta} \langle a|\psi \rangle^* \right) \\ &= |\langle a|\psi \rangle|^2 \\ \therefore P &= |\langle a|\psi_1\rangle|^2 = |\langle a|\psi \rangle|^2 \end{aligned}$$

Therefore, without loss of generality, we can set  $\theta = 0$  and the overall phase factor of any quantum state  $e^{i\theta} = 1$  without changing the outcome of measurement probabilities.

## Example: Normalization of a State Vector

Continued ...

$$C = \frac{1}{\sqrt{13}} e^{i\theta}$$

Set  $\theta = 0$

$$C = \frac{1}{\sqrt{13}}$$

$$|\psi\rangle_N = \frac{1}{\sqrt{13}}(3|+\rangle + 2i|-\rangle)$$

## Matrix Notation

We represent a ket as a column matrix and a bra as a row matrix.

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\langle +| = (1 \quad 0), \quad \langle -| = (0 \quad 1)$$

Unit Norm:

$$\langle +|+ \rangle = (1 \quad 0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1, \quad \langle -|- \rangle = (0 \quad 1) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1$$

Orthogonality:

$$\langle +|- \rangle = (1 \quad 0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0, \quad \langle -|+ \rangle = (0 \quad 1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

## Matrix Notation

A general state  $|\psi\rangle = a|+\rangle + b|-\rangle$  can be represented in matrix form as

$$|\psi\rangle = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

The corresponding bra  $\langle\psi|$  is

$$\langle\psi| = (|\psi\rangle)^\dagger = (a^* \quad b^*)$$

The inner product in matrix form is thus

$$\langle\psi|\psi\rangle = (a^* \quad b^*) \begin{pmatrix} a \\ b \end{pmatrix} = |a|^2 + |b|^2$$

The inner product  $\langle\psi|\psi\rangle$  is a measure of the square of the magnitude of the quantum state vector  $|\psi\rangle$ .