Lecture 12
Spin Precession

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Classical spin precession

- In this lecture, we will treat the dynamics of a spin in a static magnetic field using the quantum mechanical formalism.
- Before we delve into the quantum problem, it is instructive to consider the dynamics of a classical magnetic moment $\vec{\boldsymbol{\mu}}$ in the presence of a uniform external field $\vec{\boldsymbol{B}}$ to help build our intuition.
- Recall the relationship from classical mechanics between angular momentum $\vec{\boldsymbol{L}}$ of a rotating body and the sum of all the torques $\vec{\boldsymbol{\tau}}$ acting on the body.

$$\frac{d\vec{\boldsymbol{L}}}{dt} = \sum \vec{\boldsymbol{\tau}}$$

- This relationship is analogous to the relationship $\frac{d\vec{\boldsymbol{p}}}{dt} = \sum \vec{\boldsymbol{F}}$ for linear motion.
- Recall that a uniform magnetic field produces a torque $\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{\mu}} \times \vec{\boldsymbol{B}}$.
- Also recall that $\vec{\boldsymbol{\mu}}$ and is related to the spin angular momentum $\vec{\boldsymbol{S}}$

$$\vec{\boldsymbol{\mu}} = g\left(\frac{q}{2m}\right)\vec{\mathbf{S}}$$

- The Landé g-factor is a dimensionless number. For an electron spin $g \approx 2$.
- q is the charge of the particle, and m is the particle mass.

Classical spin precession

- We can use the relationship between $\vec{\boldsymbol{\mu}}$ and $\vec{\boldsymbol{S}}$ to express the torque equation only in terms of a single physical observable

$$\frac{d\vec{\boldsymbol{S}}}{dt} = \vec{\boldsymbol{\tau}} = \vec{\boldsymbol{\mu}} \times \vec{\boldsymbol{B}} = \underbrace{g\left(\frac{q}{2m}\right)}_{\gamma} \vec{\boldsymbol{S}} \times \vec{\boldsymbol{B}}$$

- where γ is called the gyromagnetic ratio. For an electron spin,

$$\gamma_e \approx 2\left(\frac{-1.6 \times 10^{-19}\,\mathsf{C}}{2 \cdot 9.11 \times 10^{-31}\,\mathsf{kg}}\right) = 1.76 \times 10^{11}\ (\mathsf{s{\cdot}T})^{-1}$$

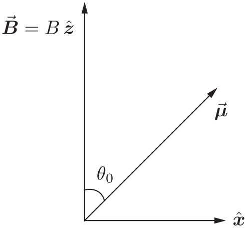
$$\frac{\gamma_e}{2\pi} \approx -28\,\mathsf{GHz/T}$$

- The classical torque equations are

$$\boxed{\begin{aligned} \frac{d\vec{\boldsymbol{S}}}{dt} &= \gamma\,(\vec{\boldsymbol{S}} \times \vec{\boldsymbol{B}}) \\ \frac{d\vec{\boldsymbol{\mu}}}{dt} &= \gamma\,(\vec{\boldsymbol{\mu}} \times \vec{\boldsymbol{B}}) \end{aligned}}$$

Classical spin precession

- To see the dynamics generated by the magnetic torque, consider a magnetic moment $\vec{\boldsymbol{\mu}}$ that is initially canted by an angle θ_0 with respect to $\vec{\boldsymbol{B}}$.



- Initial state at $t = 0$:

$$\vec{\boldsymbol{\mu}}(t=0) = \mu\left[\sin\theta_0\,\hat{\boldsymbol{x}} + \cos\theta_0\,\hat{\boldsymbol{z}}\right]$$

- For $t > 0$:

$$\vec{\boldsymbol{\mu}}(t) = \mu\left[\sin\theta(t)\cos\varphi(t)\,\hat{\boldsymbol{x}} + \sin\theta(t)\sin\varphi(t)\,\hat{\boldsymbol{y}} + \cos\theta(t)\,\hat{\boldsymbol{z}}\right]$$

- where in principle, $\theta(t)$ and $\varphi(t)$ can be time dependent.

$$\frac{d\vec{\boldsymbol{\mu}}}{dt} = \gamma\left(\vec{\boldsymbol{\mu}} \times \vec{\boldsymbol{B}}\right) = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \mu_x & \mu_y & \mu_z \\ 0 & 0 & B \end{vmatrix}$$

$$\begin{pmatrix} \dot{\mu}_x \\ \dot{\mu}_y \\ \dot{\mu}_z \end{pmatrix} = \begin{pmatrix} \gamma\, B\, \mu_y \\ -\gamma\, B\, \mu_x \\ 0 \end{pmatrix}$$

- We have three (coupled) first order linear differential equations. Note I have used the compact dot notation $\dot{f}(t) := \frac{df}{dt}$ to indicate the time derivative.
- If we first consider the equation for μ_z

$$\dot{\mu}_z = 0 \qquad \Rightarrow \qquad \dot{\theta}\sin\theta = 0$$

- We must have $\dot{\theta} = 0$, therefore $\theta(t) = \text{const.} = \theta_0$.
- The only time dependence is threfore through $\varphi(t)$.

- Next, let's consider the equation for μ_x:

$$\dot{\mu}_x = \gamma\, B\, \mu_y(t)$$

$$-\dot{\varphi}\, \mu\, \sin\theta_0 \sin\varphi(t) = \gamma\, B\, \mu\, \sin\theta_0 \sin\varphi(t) \quad \Rightarrow$$

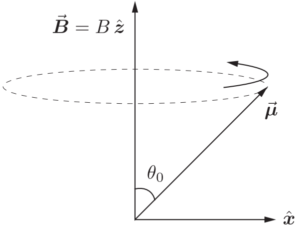
$$\dot{\varphi} = -\gamma\, B$$

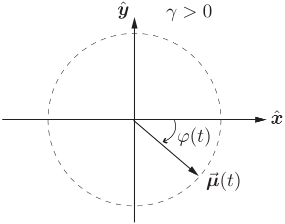
$$\varphi(t) = -\underbrace{\gamma\, B}_{\omega_0}\, t + \varphi(t=0)$$

- $\omega_0 = \gamma\, B$ is the Larmor frequency.
- With $\varphi(t=0) = 0$, we express the time dependent $\vec{\boldsymbol{\mu}}(t)$.

$$\boxed{\vec{\boldsymbol{\mu}}(t) = \mu\left[\sin\theta_0 \cos(\omega_0 t)\hat{\boldsymbol{x}} - \sin\theta_0 \sin(\omega_0 t)\hat{\boldsymbol{y}} + \cos\theta_0\, \hat{\boldsymbol{z}}\right]}$$

Classical spin precession





- A classical spin precesses around the static external field at the Larmor frequency.
- Note that γ can be positive or negative depending on the sign of the charge of the spin.
- For positively charged spins, like protons, which are also spin-$1/2$, $\gamma > 0$ and the precession is clockwise.
- For electrons, that carry negative charge, the precession is counterclockwise.

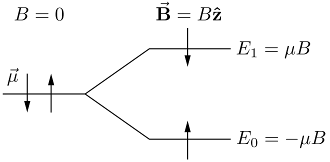
Quantum mechanical spin precession

- We start by considering the Hamiltonian of a spin-1/2 particle in a uniform magnetic field.

$$\mathcal{H} = -\vec{\boldsymbol{\mu}} \cdot \vec{\mathbf{B}} = -\left(\frac{gq}{2m}\right) \vec{\mathbf{S}} \cdot \vec{\mathbf{B}} = -\gamma \, \vec{\mathbf{S}} \cdot \vec{\mathbf{B}}$$

- We know that there are only two projections of $\vec{\mathbf{S}} \cdot \hat{\mathbf{n}}$, $\left(\pm \frac{\hbar}{2}\right)$ that can be observed for any direction $\hat{\mathbf{n}}$ in space.
- Therefore, there are only two values of energy, corresponding to the two spin projections in the direction of the applied magnetic field $\vec{\mathbf{B}} = B\hat{\mathbf{n}}$.

$$E_0 = -\frac{\hbar\omega_0}{2}, \qquad E_1 = \frac{\hbar\omega_0}{2} \tag{1}$$



Quantum mechanical spin precession

- The energy difference between the two states is

$$\Delta E = E_1 - E_0 = \left(\frac{gqB}{2m}\right)\hbar = 2\mu B = \omega_0 \hbar$$

$$\mu = \frac{gq\hbar}{4m} = \frac{\gamma\hbar}{2}$$

- Let's consider the situation of a time-independent, or static, external field $\vec{\mathbf{B}} = B\hat{\mathbf{z}}$ applied in the $z-$direction.
- The Hamiltonian for this problem is

$$\mathcal{H} = -\omega_0 S_z$$

- Note that the energy basis is the same as the S_z basis since the Hamiltonian is just proportional to S_z.

$$\mathcal{H} = -\frac{\hbar\omega_0}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Quantum mechanical spin precession

$$\mathcal{H}\,|+\rangle = E_0\,|+\rangle$$

$$\mathcal{H}\,|-\rangle = E_1\,|-\rangle$$

- We can express any state that is a linear superposition of the energy eigenkets $|\pm\rangle$ as

$$|\psi(t)\rangle = c_1 \exp\{-iE_0 t/\hbar\}\,|+\rangle + c_2 \exp\{-iE_1 t/\hbar\}\,|-\rangle$$
$$= c_1 e^{i\omega_0 t/2}\,|+\rangle + c_2 e^{-i\omega_0 t/2}\,|-\rangle$$

- Let's determine the dynamics of a spin in a uniform external field.
- We do so by calculating expectation values of $\vec{\mathbf{S}}$ for an arbitrary superposition $|\psi(t)\rangle$.

$$|\psi(t)\rangle = \begin{pmatrix} c_1 e^{i\omega_0 t/2} \\ c_2 e^{-i\omega_0 t/2} \end{pmatrix} := \begin{pmatrix} a \\ b \end{pmatrix}$$

Quantum mechanical spin precession

- Expectation value $\langle S_z \rangle$:

$$\langle \psi(t)|S_z|\psi(t)\rangle = \begin{pmatrix} a^* & b^* \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} a^* & b^* \end{pmatrix} \begin{pmatrix} a \\ -b \end{pmatrix}$$

$$= \frac{\hbar}{2}(|a|^2 - |b|^2)$$

$$= \boxed{\frac{\hbar}{2}(|c_1|^2 - |c_2|^2)} \tag{2}$$

- Because the energy is conserved ($\mathcal{H}$ is time independent), the average energy, i.e., $\langle \mathcal{H} \rangle$ is time independent (conserved).
- Because $[S_z, \mathcal{H}] = 0$, it means that $\langle S_z \rangle$ must also be conserved.

- Expectation value $\langle S_x \rangle$:

$$\begin{aligned}
\langle \psi(t)|S_x|\psi(t)\rangle &= \begin{pmatrix} a^* & b^* \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} a^* & b^* \end{pmatrix} \begin{pmatrix} b \\ a \end{pmatrix} \\
&= \frac{\hbar}{2}(a^* b + b^* a) \\
&= \hbar \,\mathrm{Re}\{a^* b\} \\
&= \hbar \,\mathrm{Re}\left\{ c_1^* e^{-i\omega_0 t/2} c_2 e^{-i\omega_0 t/2} \right\} \\
&= \boxed{\hbar \,\mathrm{Re}\left\{ c_1^* c_2 e^{-i\omega_0 t} \right\}}
\end{aligned} \tag{3}$$

Quantum mechanical spin precession

- Expectation value $\langle S_y \rangle$:

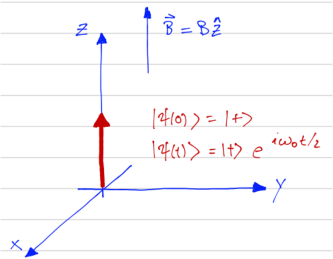
$$\begin{aligned}\langle \psi(t)|S_y|\psi(t)\rangle &= \begin{pmatrix} a^* & b^* \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \frac{i\hbar}{2} \begin{pmatrix} a^* & b^* \end{pmatrix} \begin{pmatrix} -b \\ a \end{pmatrix} \\ &= \frac{i\hbar}{2}(-a^* b + b^* a) \\ &= \hbar \, \mathrm{Im}\{a^* b\} \\ &= \hbar \, \mathrm{Im}\left\{ c_1^* e^{-i\omega_0 t/2} c_2 e^{-i\omega_0 t/2} \right\} \\ &= \boxed{\hbar \, \mathrm{Im}\left\{ c_1^* c_2 e^{-i\omega_0 t} \right\}} \end{aligned} \tag{4}$$

Example 1

- Consider a spin initially polarized in the z-direction: $(c_1 = 1, c_2 = 0)$.

$$|\psi(0)\rangle = |+\rangle$$

$$|\psi(t)\rangle = e^{i\omega_0 t/2}\,|+\rangle$$



- Since $|\psi(0)\rangle$ is an eignestate of the Hamiltonian, it corresponds to a stationary state.

eqns.(2),(3),(4) $\boxed{\langle S_z \rangle = \frac{\hbar}{2},\ \langle S_x \rangle = \langle S_y \rangle = 0}$

Example 2

- Consider a spin initially polarized in the x-direction: $(c_1 = c_2 = \frac{1}{\sqrt{2}})$.

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}(e^{i\omega_0 t/2}\,|+\rangle + e^{-i\omega_0 t/2}\,|-\rangle)$$

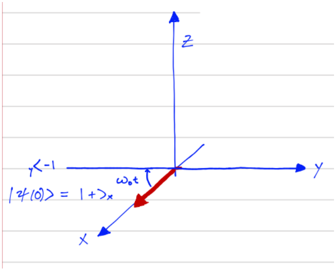
$$|\psi(t)\rangle = \frac{e^{i\omega_0 t/2}}{\sqrt{2}}(|+\rangle + e^{-i\omega_0 t}\,|-\rangle)$$

eq.(2) $\boxed{\langle S_z\rangle = 0}$

eq.(3) $\boxed{\langle S_x\rangle = \frac{\hbar}{2}\cos(\omega_0 t)}$

eq.(4) $\boxed{\langle S_y\rangle = -\frac{\hbar}{2}\sin(\omega_0 t)}$

Example 2



Dropping the overall phase factor in $|\psi(t)\rangle$:

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}(|+\rangle + e^{-i\omega_0 t}\,|-\rangle)$$

$$|\psi(0)\rangle = |+\rangle_x$$

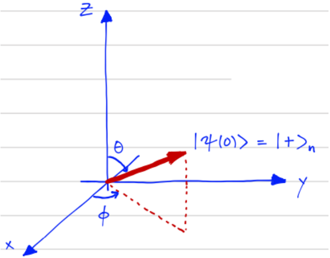
$$\left|\psi(t = \tfrac{\pi}{2\omega_0})\right\rangle = \tfrac{1}{\sqrt{2}}(|+\rangle + e^{-i\pi/2}\,|-\rangle) = |-\rangle_y$$

If the spin is initially polarized along the $x-$direction, then in the presence of the static magnetic field $\vec{\mathbf{B}} = B\hat{\mathbf{z}}$, the spin will precess clockwise in the $xy-$plane at the Larmor frequency ω_0.

Example 3

- Consider the dynamics of a spin that has an arbitrary initial orientation.

$$|+\rangle_n = \cos\left(\frac{\theta}{2}\right)|+\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}|-\rangle$$



$$|\psi(0)\rangle = |+\rangle_n$$

$$|\psi(t)\rangle = \cos\left(\frac{\theta}{2}\right)e^{i\omega_0 t/2}|+\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}e^{-i\omega_0 t/2}|-\rangle$$

Example 3

- What is the probability to find the spin in states $|\pm\rangle$ at time t?

$$P_+ = |\langle + | \psi(t)\rangle|^2 = \cos\left(\frac{\theta}{2}\right)^2$$

$$P_- = |\langle - | \psi(t)\rangle|^2 = \sin\left(\frac{\theta}{2}\right)^2$$

- What are the expectation values $\langle S_x\rangle$, $\langle S_y\rangle$, and $\langle S_z\rangle$?

$$\text{eq.(3)} \quad \langle S_x\rangle = \hbar\,\mathrm{Re}\left\{\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)e^{-i(\omega_0 t-\phi)}\right\}$$

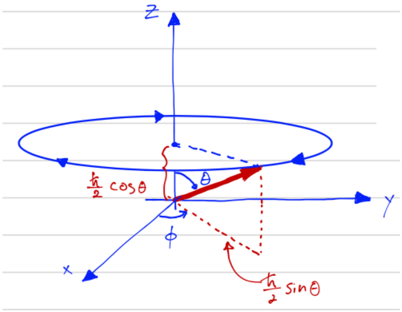
$$= \boxed{\frac{\hbar}{2}\sin\theta\cos(\omega_0 t-\phi)}$$

$$\text{eq.(4)} \quad \langle S_y\rangle = \hbar\,\mathrm{Im}\left\{\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)e^{-i(\omega_0 t-\phi)}\right\}$$

$$= \boxed{-\frac{\hbar}{2}\sin\theta\sin(\omega_0 t-\phi)}$$

$$\text{eq.(2)} \quad \langle S_z\rangle = \frac{\hbar}{2}\left(\cos\left(\frac{\theta}{2}\right)^2 - \sin\left(\frac{\theta}{2}\right)^2\right)$$

$$= \boxed{\frac{\hbar}{2}\cos\theta}$$



Ehrenfest's Theorem

The expectation value of quantum observables obey classical laws.

- The dynamics that the expectation value of $\vec{\mathbf{S}}$ exhibits is the same as that of a classical magnetic moment precessing around a uniform magnetic field.