Lecture 9: The Uncertainty Principle - Summary & Formula Sheet

Based on Lecture by Raffi Budakian (Univ. of Waterloo)

Week of Feb. 2, 2026

1 Lecture Overview

- **Topic:** Commutators, The Generalized Uncertainty Principle, and Total Spin Magnitude.
- **Goal:** Quantify the limit of simultaneous knowledge for incompatible observables.

2 1. The Commutator

The order in which operators are applied matters. This is analogous to 3D rotations, where rotating around Z then X produces a different result than rotating X then Z. We quantify this difference using the commutator.

2.1 Definition

$$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A} \tag{1}$$

- If $[\hat{A}, \hat{B}] = 0$: Operators commute (independent, compatible).
- If $[\hat{A}, \hat{B}] \neq 0$: Operators do not commute (incompatible).

3 2. Spin Commutation Relations

Using the matrix representations of the spin operators, we can derive their commutation relations.

3.1 Derivation for S_x, S_z

$$\begin{aligned}
[S_x, S_z] &= S_x S_z - S_z S_x \\
&= \frac{\hbar^2}{4}\left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} - \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right] \\
&= -i\hbar S_y
\end{aligned} \tag{2}$$

Note: $[A, B] = -[B, A]$, so $[S_z, S_x] = i\hbar S_y$.

3.2 Cyclic Relations

The relations follow a cyclic pattern ($x \to y \to z \to x$):

$$[S_x, S_y] = i\hbar S_z \tag{3}$$

$$[S_y, S_z] = i\hbar S_x \tag{4}$$

$$[S_z, S_x] = i\hbar S_y \tag{5}$$

4 3. The Generalized Uncertainty Principle

The uncertainty principle is not just about position and momentum; it applies to any two non-commuting observables.

4.1 The Inequality

$$\Delta A \Delta B \geq \frac{1}{2}|\langle[\hat{A}, \hat{B}]\rangle| \tag{6}$$

Where ΔA is the RMS deviation (uncertainty) of the measurement.

4.2 Application to Spin

For a Spin-1/2 system, the uncertainty relationship between S_x and S_y is:

$$\Delta S_x \Delta S_y \geq \frac{\hbar}{2}|\langle S_z\rangle| \tag{7}$$

If the particle is in the eigenstate $|+\rangle$:

- $\langle S_z\rangle = \hbar/2$
- Minimum uncertainty: $\Delta S_x \Delta S_y \geq \hbar^2/4$.
- Since $\Delta S_x = \Delta S_y = \hbar/2$, the product is exactly $\hbar^2/4$. The uncertainty is minimized but non-zero.

5 4. Magnitude of Spin Vector

While we cannot know the components (S_x, S_y, S_z) simultaneously, we can determine the total magnitude of the spin vector.

5.1 Total Spin Operator S^2

$$S^2 = \vec{S} \cdot \vec{S} = S_x^2 + S_y^2 + S_z^2 \tag{8}$$

Since $S_i^2 = \frac{\hbar^2}{4}\nVdash$ for all i:

$$S^2 = \frac{3\hbar^2}{4}\nVdash \tag{9}$$

This operator commutes with all components: $[S^2, S_i] = 0$.

5.2 Length of the Vector

The magnitude (norm) of the spin vector is the square root of the eigenvalue of S^2:

$$||\vec{S}|| = \sqrt{\frac{3\hbar^2}{4}} = \frac{\sqrt{3}}{2}\hbar \tag{10}$$

Midterm Formula Sheet

1. **Commutator Definition**

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} \tag{11}$$

2. **Spin Commutation Relations**

$$[S_x, S_y] = i\hbar S_z \quad \text{(cyclic)} \tag{12}$$

3. **Generalized Uncertainty Principle**

$$\Delta A \Delta B \geq \frac{1}{2}|\langle[\hat{A}, \hat{B}]\rangle| \tag{13}$$

4. **Spin Uncertainty Limit**

$$\Delta S_x \Delta S_y \geq \frac{\hbar}{2}|\langle S_z \rangle| \tag{14}$$

5. **Total Spin Operator**

$$S^2 = \frac{3\hbar^2}{4}\mathbb{1} \tag{15}$$

6. **Spin Vector Magnitude**

$$||\vec{S}|| = \frac{\sqrt{3}\hbar}{2} \tag{16}$$

Core Takeaway

The Quantum Cone: The magnitude of the spin vector ($\approx 0.866\hbar$) is strictly larger than the maximum measurable projection on any axis ($0.5\hbar$).

- The spin vector can never perfectly align with the Z-axis.
- It must precess in a "cone" to satisfy the uncertainty relations for S_x and S_y.