Assignment 2 Solutions

Physics 234

W26

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1. (a) $|\psi_1\rangle = 3|+\rangle - i|-\rangle$,

Normalization $|\psi_1\rangle_N = c|\psi_1\rangle$ with ${}_N\langle\psi_1|\psi_1\rangle_N = 1$

$$\Rightarrow |c|^2 \left(3\langle +| + i\langle -|\right)\left(3|+\rangle - i|-\rangle\right) = 1$$

$$|c|^2 \left(9\underbrace{\langle +|+\rangle}_{=1} - 3i\underbrace{\langle +|-\rangle}_{=0} + 3i\underbrace{\langle -|+\rangle}_{=0} + \underbrace{\langle -|-\rangle}_{=1}\right) = 1$$

$$10|c|^2 = 1 \quad , \quad |c| = \frac{1}{\sqrt{10}}$$

$$\boxed{|\psi_1\rangle_N = \frac{1}{\sqrt{10}}\left(3|+\rangle - i|-\rangle\right)}$$

$$|\psi_2\rangle_N = c|\psi_2\rangle$$

$${}_N\langle\psi_2|\psi_2\rangle_N = |c|^2\left(e^{-i\pi/3}\langle +| + \langle -|\right)\left(e^{i\pi/3}|+\rangle + |-\rangle\right) = 1$$

$$= |c|^2(1+1) = 1 \quad , \quad |c| = \frac{1}{\sqrt{2}}$$

$$\boxed{|\psi_2\rangle_N = \frac{1}{\sqrt{2}}\left(e^{i\pi/3}|+\rangle + |-\rangle\right)}$$

1. (a) $|\psi_3\rangle_N = c|\psi_3\rangle$

$${}_N\langle\psi_3|\psi_3\rangle_N = |c|^2\left(-7i\langle +| - 2\langle -|\right)\left(7i|+\rangle - 2|-\rangle\right) = 1$$

$$= |c|^2 (49 + 4) = 1$$

$$|c| = \frac{1}{\sqrt{53}}$$

$$\boxed{|\psi_3\rangle_N = \frac{1}{\sqrt{53}}\left(7i|+\rangle - 2|-\rangle\right)}$$

(b) $$|\psi_1\rangle_N = \frac{1}{\sqrt{10}}\left(3|+\rangle - i|-\rangle\right)$$

$$|\psi_2\rangle_N = \frac{1}{\sqrt{2}}\left(e^{i\pi/3}|+\rangle + |-\rangle\right)$$

$$|\psi_3\rangle_N = \frac{1}{\sqrt{53}}\left(7i|+\rangle - 2|-\rangle\right)$$

$$\left.\begin{aligned} {}_N\langle\psi_1|\psi_1\rangle_N &= 1 \\ {}_N\langle\psi_2|\psi_2\rangle_N &= 1 \\ {}_N\langle\psi_3|\psi_3\rangle_N &= 1 \end{aligned}\right\} \text{ By construction.}$$

$$ {}_N\langle\psi_1|\psi_2\rangle_N = \frac{1}{\sqrt{20}}\left(3\langle +| + i\langle -|\right)\left(e^{i\pi/3}|+\rangle + |-\rangle\right)$$

$$= \boxed{\frac{1}{\sqrt{20}}\left(3e^{i\pi/3} + i\right)}$$

$$ {}_N\langle\psi_1|\psi_3\rangle_N = \frac{1}{\sqrt{530}}\left(3\langle +| + i\langle -|\right)\left(7i|+\rangle - 2|-\rangle\right)$$

$$= \frac{1}{\sqrt{530}}(21i - 2i) = \boxed{\frac{19i}{\sqrt{530}}}$$

$$ {}_N\langle\psi_2|\psi_3\rangle_N = \frac{1}{\sqrt{106}}\left(e^{-i\pi/3}\langle +| + \langle -|\right)\left(7i|+\rangle - 2|-\rangle\right)$$

$$= \boxed{\frac{1}{\sqrt{106}}\left(7ie^{-i\pi/3} - 2\right)}$$

1. (b) $_N\langle\psi_2|\psi_1\rangle_N = \left(_N\langle\psi_1|\psi_2\rangle_N\right)^* = \boxed{\frac{1}{\sqrt{20}}\left(3e^{-i\pi/3} - i\right)}$

$_N\langle\psi_3|\psi_1\rangle_N = \left(_N\langle\psi_1|\psi_3\rangle_N\right)^* = \boxed{\frac{-19i}{\sqrt{530}}}$

$_N\langle\psi_3|\psi_2\rangle_N = \left(_N\langle\psi_2|\psi_3\rangle_N\right)^* = \boxed{\frac{1}{\sqrt{106}}\left(-7i\, e^{+i\pi/3} - 2\right)}$

(c) (i) Let $|\phi_1\rangle = a|+\rangle + b|-\rangle$ with $\langle\psi_1|\phi_1\rangle = 0$ and $\langle\phi_1|\phi_1\rangle = 1$

$$(3\langle +| + i\langle -|)(a|+\rangle + b|-\rangle) =$$

$$3a + ib = 0 \quad \text{and} \quad |a|^2 + |b|^2 = 1$$

$$b = -3ia \ , \quad |a|^2 + 9|a|^2 = 1 \ , \quad |a| = \frac{1}{\sqrt{10}}$$

$$a = \frac{1}{\sqrt{10}} e^{i\theta} \qquad \theta = \text{arbitrary phase}$$

$$b = -3ia = -\frac{3i}{\sqrt{10}} e^{i\theta}$$

▷ Without loss of generality, we can set $\theta = 0$ as $e^{i\theta}$ is an overall phase factor which does not affect the measured probabilities.

Thus, $\boxed{|\phi_1\rangle = \frac{1}{\sqrt{10}}\left(|+\rangle - 3i\,|-\rangle\right)}$

(ii) Let $|\phi_2\rangle = a|+\rangle + b|-\rangle$ with $\langle\psi_2|\phi_2\rangle = 0$ and $\langle\phi_2|\phi_2\rangle = 1$

$$\left(e^{-i\pi/3}\langle +| + \langle -|\right)\left(a|+\rangle + b|-\rangle\right) = 0$$

$$a e^{-i\pi/3} + b = 0 \ , \quad |a|^2 + |b|^2 = 1$$

$$b = -a e^{-i\pi/3} \ , \quad 2|a|^2 = 1$$

$$|a| = \frac{1}{\sqrt{2}} \implies a = \frac{1}{\sqrt{2}}$$

$$\boxed{|\phi_2\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle - e^{-i\pi/3}|-\rangle\right)}$$

1. (c) Let $|\phi_3\rangle = a|+\rangle + b|-\rangle$ with $\langle\psi_3|\phi_3\rangle = 0$ and $\langle\phi_3|\phi_3\rangle = 1$

$$\left(-7i\langle+| - 2\langle-|\right)\left(a|+\rangle + b|-\rangle\right) = 0$$

$$-7ia - 2b = 0 \quad , \quad |a|^2 + |b|^2 = 1$$

$$b = -\frac{7}{2}ia \quad , \quad |a|^2\left(1 + \frac{49}{4}\right) = 1$$

$$|a| = \frac{2}{\sqrt{53}} \quad \Rightarrow \quad a = \frac{2}{\sqrt{53}}$$

$$\boxed{|\phi_3\rangle = \frac{1}{\sqrt{53}}\left(2|+\rangle - 7i|-\rangle\right)}$$

(d) Let $|n\rangle = a_n|+\rangle + b_n|-\rangle$ and $|\psi_i\rangle = a_i|+\rangle + b_i|-\rangle$

$$|\langle n|\psi_i\rangle|^2 = |\left(a_n^*\langle+| + b_n^*\langle-|\right)\left(a_i|+\rangle + b_i|-\rangle\right)|^2$$

$$= a_n^* a_i + b_n^* b_i$$

$$|1\rangle = |+\rangle \quad \Rightarrow \quad (a_1, b_1) = (1, 0)$$

$$|2\rangle = |-\rangle \quad \Rightarrow \quad (a_2, b_2) = (0, 1)$$

$$|3\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right) \Rightarrow (a_3, b_3) = \frac{1}{\sqrt{2}}(1, 1)$$

$$|4\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right) \Rightarrow (a_4, b_4) = \frac{1}{\sqrt{2}}(1, -1)$$

$$|5\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + i|-\rangle\right) \Rightarrow (a_5, b_5) = \frac{1}{\sqrt{2}}(1, i)$$

$$|6\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle - i|-\rangle\right) \Rightarrow (a_6, b_6) = \frac{1}{\sqrt{2}}(1, -i)$$

$$|\langle 1|\psi_1\rangle_N|^2 = \left|\frac{1}{\sqrt{10}}3\right|^2 = \boxed{\frac{9}{10}} \quad , \quad |\langle 2|\psi_1\rangle_N|^2 = \left|\frac{-i}{\sqrt{10}}\right|^2 = \boxed{\frac{1}{10}}$$

$$|\langle 1|\psi_2\rangle_N|^2 = \left|\frac{1}{\sqrt{2}}e^{i\pi/3}\right|^2 = \boxed{\frac{1}{2}} \quad , \quad |\langle 2|\psi_2\rangle_N|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \boxed{\frac{1}{2}}$$

$$|\langle 1|\psi_3\rangle_N|^2 = \left|\frac{1}{\sqrt{53}}7i\right|^2 = \boxed{\frac{49}{53}} \quad , \quad |\langle 2|\psi_3\rangle_N|^2 = \left|\frac{1}{\sqrt{53}}(-2)\right|^2 = \boxed{\frac{4}{53}}$$

$$|\langle 3|\psi_1\rangle_N|^2 = \left|\frac{1}{\sqrt{20}}(3-i)\right|^2 = \boxed{\frac{1}{2}} \quad , \quad |\langle 4|\psi_1\rangle_N|^2 = \left|\frac{1}{\sqrt{20}}(3+i)\right|^2 = \boxed{\frac{1}{2}}$$

1. (d) $|\langle 3|\psi_2\rangle_N|^2 = \left|\frac{1}{2}(e^{i\pi/3}+1)\right|^2 = \frac{1}{4}\left(1+e^{i\pi/3}\right)\left(1-e^{-i\pi/3}\right)$

$= \frac{1}{4}\left(2+\underbrace{e^{i\pi/3}+e^{-i\pi/3}}_{2\cos\frac{\pi}{3}=1}\right) = \boxed{\frac{3}{4}}$ $\qquad a_n^* a_i + b_n^* b_i$

$|\langle 4|\psi_2\rangle_N|^2 = \left|\frac{1}{2}(e^{i\pi/3}-1)\right|^2 = \frac{1}{4}\left(2-2\cos\frac{\pi}{3}\right) = \boxed{\frac{1}{4}}$

$|\langle 3|\psi_3\rangle|^2 = \left|\frac{1}{\sqrt{106}}(7i-2)\right|^2 = \frac{1}{106}(49+4) = \boxed{\frac{1}{2}}$

$|\langle 4|\psi_3\rangle|^2 = \left|\frac{1}{\sqrt{106}}(7i+2)\right|^2 = \frac{1}{106}(49+4) = \boxed{\frac{1}{2}}$

$|\langle 5|\psi_1\rangle|^2 = \left|\frac{1}{\sqrt{20}}(3+(-i)(-i))\right|^2 = \frac{4}{20} = \boxed{\frac{1}{5}}$

$|\langle 6|\psi_1\rangle|^2 = \left|\frac{1}{\sqrt{20}}(3+(i)(-i))\right|^2 = \frac{16}{20} = \boxed{\frac{4}{5}}$

$|\langle 5|\psi_2\rangle|^2 = \left|\frac{1}{2}(e^{i\pi/3}-i)\right|^2 = \frac{1}{4}\left(e^{i\pi/3}-i\right)\left(e^{-i\pi/3}+i\right) = \frac{1}{4}\left(2+i\left(e^{i\pi/3}-e^{-i\pi/3}\right)\right)$

$= \frac{1}{4}\left(2+i(2i)\sin\frac{\pi}{3}\right) \boxed{=\frac{1}{2}\left(1-\frac{\sqrt{3}}{2}\right)}$

$|\langle 6|\psi_2\rangle|^2 = \left|\frac{1}{2}\left(e^{i\pi/3}+i\right)\right|^2 = \frac{1}{4}\left(e^{i\pi/3}+i\right)\left(e^{-i\pi/3}-i\right) = \frac{1}{4}\left(2+i\left(e^{-i\pi/3}-e^{i\pi/3}\right)\right)$

$= \frac{1}{4}\left(2+i(-2i)\sin\frac{\pi}{3}\right)$

$= \boxed{\frac{1}{2}\left(1+\frac{\sqrt{3}}{2}\right)}$

$|\langle 5|\psi_3\rangle|^2 = \left|\frac{1}{\sqrt{106}}(7i+2i)\right|^2 = \boxed{\frac{81}{106}}$

$|\langle 6|\psi_3\rangle|^2 = \left|\frac{1}{\sqrt{106}}(7i-2i)\right|^2 = \boxed{\frac{25}{106}}$

2.

(a) $|+\rangle_n = \cos\left(\frac{\theta}{2}\right) |+\rangle + \sin\left(\frac{\theta}{2}\right) e^{i\phi} |-\rangle$

for $\theta = \frac{\pi}{2}$, $\phi = 0$ $\quad |+\rangle_n = \frac{1}{\sqrt{2}} \left(|+\rangle + |-\rangle \right) = |+\rangle_x$

for $\theta = \frac{\pi}{2}$, $\phi = \frac{\pi}{2}$ $\quad |+\rangle_n = \frac{1}{\sqrt{2}} \left(|+\rangle + i |-\rangle \right) = |+\rangle_y$

(b) (i) Prob. to measure $S_z = +\frac{\hbar}{2}$: $\quad P_+ = |\langle + | + \rangle_n|^2$

$P_+ = \cos^2\left(\frac{\theta}{2}\right)$

(ii) Prob. to measure $S_z = -\frac{\hbar}{2}$: $\quad P_- = |\langle - | + \rangle_n|^2$

$P_- = \sin^2\left(\frac{\theta}{2}\right)$

$P_+ + P_- = 1$ ✓

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3. $\hat{A} \longrightarrow \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $\hat{B} \longrightarrow \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$

$$(\hat{A}\hat{B})^+ = \left[\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \right]^+ = \begin{pmatrix} 19 & 24 \\ 43 & 50 \end{pmatrix}^+ = \begin{pmatrix} 19 & 43 \\ 24 & 50 \end{pmatrix}$$

$$\hat{B}^+ \hat{A}^+ = \begin{pmatrix} 5 & 7 \\ 6 & 8 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 19 & 43 \\ 24 & 50 \end{pmatrix} \quad ✓$$

$$\therefore (\hat{A}\hat{B})^+ = \hat{B}^+ \hat{A}^+$$

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4. Let $|b_n\rangle = \hat{U}|a_n\rangle$, where $\hat{U}$ is a unitary operator.

If $\{|a_n\rangle\}$ forms an orthonormal basis, then

$$\langle a_n|a_m\rangle = \delta_{nm} \quad \text{where} \quad \delta_{nm} = \begin{cases} 1, & n=m \\ 0, & n\neq m \end{cases}$$

$$\begin{aligned}
\langle b_n|b_m\rangle &= (\hat{U}|a_n\rangle)^{+}(\hat{U}|a_m\rangle) \\
&= (\langle a_n|\hat{U}^{+})(\hat{U}|a_m\rangle) \\
&= \langle a_n|\underbrace{\hat{U}^{+}\hat{U}}_{=\mathbb{1}\ (\text{unitary})}|a_m\rangle
\end{aligned}
\qquad \therefore \boxed{\langle b_n|b_m\rangle = \langle a_n|a_m\rangle}$$

5. Let $\hat{U}$ be a unitary matrix. We can represent $\hat{U}$ in its eigenbasis

$$\hat{U} = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \\ & & & \ddots \end{pmatrix}$$ where $\{\lambda_n\}$ are the eigenvalues of $\hat{U}$.

D for a unitary matrix $\hat{U}\hat{U}^{+} = \mathbb{1}$ $\Rightarrow$

$$\hat{U}\hat{U}^{+} = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \\ & & & \ddots \end{pmatrix}\begin{pmatrix} \lambda_1^* & & 0 \\ & \lambda_2^* & \\ 0 & & \lambda_3^* \\ & & & \ddots \end{pmatrix} = \begin{pmatrix} |\lambda_1|^2 & & 0 \\ & |\lambda_2|^2 & \\ 0 & & |\lambda_3|^2 \\ & & & \ddots \end{pmatrix}$$

5.
$$\begin{pmatrix} |\lambda_1|^2 & & & 0 \\ & |\lambda_2|^2 & & \\ 0 & & |\lambda_3|^2 & \\ & & & \ddots \end{pmatrix} = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ 0 & & 1 & \\ & & & \ddots \end{pmatrix}$$

$$\therefore \quad |\lambda_n|^2 = 1 \quad \forall \lambda_n .$$

$$\Rightarrow \quad \lambda_n = e^{i\theta_n} \quad \text{where } \theta_n = \text{arbitrary phase.}$$

6. Output of

$$SG\,1: |\psi_1\rangle = |+\rangle$$

$$S\text{-}G\,2: |\psi_2\rangle = {}_n\langle +|\psi_1\rangle\, |+\rangle_n$$

$$S\text{-}G\,3: |\psi_3\rangle = \langle +|\psi_2\rangle\, |+\rangle + \langle -|\psi_2\rangle\, |-\rangle$$

(a) Probability to measure $|-\rangle$ in S-G A3.

$$\boxed{\begin{aligned} P\left(-\tfrac{\hbar}{2}\right) &= |\langle -|\psi_3\rangle|^2 = |\langle -|\psi_2\rangle|^2 = |{}_n\langle +|+\rangle \langle -|+\rangle_n|^2 \\ &= \left|\cos\left(\tfrac{\theta}{2}\right) \sin\left(\tfrac{\theta}{2}\right) e^{i\varphi}\right|^2 = \left|\tfrac{1}{2}\sin\theta\, e^{i\varphi}\right|^2 = \tfrac{1}{4}\sin^2\theta \end{aligned}}$$

(b) $\boxed{\text{Max fraction through S-G A3 is } \tfrac{1}{4} \text{ for } \theta = \tfrac{\pi}{2},\ \tfrac{3\pi}{2}.}$

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7. $|\psi\rangle = a|+\rangle + b|-\rangle \qquad |a|^2 + |b|^2 = 1$

$$|\langle +|\psi\rangle|^2 = |a|^2 = 0.9 \qquad \text{Let } a = |a| = \frac{3}{\sqrt{10}},\quad b = \frac{1}{\sqrt{10}} e^{i\beta}$$

$$|{}_y\langle +|\psi\rangle|^2 = \frac{1}{2}\left|(\langle +| - i\langle -|)(a|+\rangle + b|-\rangle)\right|^2 = 0.2$$

$$= \frac{1}{2}|a - ib|^2 = \frac{1}{2}\left|\frac{3}{\sqrt{10}} - \frac{i}{\sqrt{10}} e^{i\beta}\right|^2 = 0.2$$

$$|3 - ie^{i\beta}|^2 = 4$$

$$(3 - ie^{i\beta})(3 + ie^{-i\beta}) = 4$$

$$9 + 3i(e^{-i\beta} - e^{i\beta}) + 1 = 4$$

$$10 + 6\sin\beta = 4 \qquad \sin\beta = -1$$

$$\boxed{|\psi\rangle = \frac{1}{\sqrt{10}}\left\{3|+\rangle + e^{i\beta}|-\rangle\right\} = \frac{1}{\sqrt{10}}\left\{3|+\rangle - i|-\rangle\right\}}$$

$$\boxed{\begin{aligned} P\left(S_x = -\tfrac{\hbar}{2}\right) &= |{}_x\langle -|\psi\rangle|^2 = \left|\frac{1}{\sqrt{20}}\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}3 \\ -i\end{pmatrix}\right|^2 \\ &= \frac{1}{20}|3+i|^2 = \frac{1}{2} \end{aligned}}$$