Tutorial 3

① $|\psi\rangle = \frac{3}{5}|+\rangle + i\frac{4}{5}|-\rangle$

(a) i)

$$\boxed{|\psi\rangle \doteq \begin{pmatrix} 3/5 \\ i4/5 \end{pmatrix}}$$

S_z eigenbasis

Temporarily use $\doteq$ notation to mean "represented as".

will drop this notation later on.

ii) $|\psi\rangle = |+\rangle_y \, {}_y\langle+|\psi\rangle + |-\rangle_y \, {}_y\langle-|\psi\rangle \doteq \begin{pmatrix} {}_y\langle+|\psi\rangle \\ {}_y\langle-|\psi\rangle \end{pmatrix}$ ①

S_y eigenbasis

$$\begin{cases} {}_y\langle+|\psi\rangle = \frac{1}{\sqrt{2}}\left(\langle+| - i\langle-|\right)\left(\frac{3}{5}|+\rangle + \frac{4}{5}i|-\rangle\right) = \frac{1}{5\sqrt{2}}(3+4) = \frac{7}{5\sqrt{2}} \\ {}_y\langle-|\psi\rangle = \frac{1}{\sqrt{2}}\left(\langle+| + i\langle-|\right)\left(\frac{3}{5}|+\rangle + \frac{4}{5}i|-\rangle\right) = \frac{1}{5\sqrt{2}}(3-4) = -\frac{1}{5\sqrt{2}} \end{cases}$$

$$\xRightarrow{①} \boxed{|\psi\rangle \doteq \frac{1}{5\sqrt{2}}\begin{pmatrix} 7 \\ -1 \end{pmatrix}}$$

S_y eigenbasis

iii) $$\begin{cases} |\phi_1\rangle = \frac{3}{5}|+\rangle + i\frac{4}{5}|-\rangle \\ |\phi_2\rangle = -\frac{4}{5}|+\rangle + i\frac{3}{5}|-\rangle \end{cases}$$

No need to calculate, $|\psi\rangle = |\phi_1\rangle$

so:

$$\boxed{|\psi\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

$\{|\phi_1\rangle, |\phi_2\rangle\}$ basis

(b)

i) $\mathcal{P}\left(S_y = \frac{\hbar}{2}\right) = |{}_y\langle+|\psi\rangle|^2 = \left|\frac{1}{\sqrt{2}}(1 \;\; -i)\begin{pmatrix} 3/5 \\ i4/5 \end{pmatrix}\right|^2$

i) $\mathbb{P}\left(S_y = \frac{\hbar}{2}\right) = \left|{}_y\langle + | \psi \rangle\right|^2 = \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \end{pmatrix} \begin{pmatrix} 3/5 \\ i\,4/5 \end{pmatrix} \right|^2$

$= \frac{1}{2} \left| \frac{3}{5} + \frac{4}{5} \right|^2 = \boxed{\frac{49}{50}}$

ii) $\mathbb{P}\left(S_y = \frac{\hbar}{2}\right) = \left|{}_y\langle + | \psi \rangle\right|^2 = \left| \begin{pmatrix} 1 & 0 \end{pmatrix} \frac{1}{5\sqrt{2}} \begin{pmatrix} 7 \\ -1 \end{pmatrix} \right|^2$

$= \frac{1}{50} |7|^2 = \boxed{\frac{49}{50}}$

iii) For this, we need to calculate $|+\rangle_y$ in the $\{|\phi_1\rangle, |\phi_2\rangle\}$ basis:

$$|+\rangle_y = |\phi_1\rangle\langle\phi_1|+\rangle_y + |\phi_2\rangle\langle\phi_2|+\rangle_y \underset{\{|\phi_1\rangle, |\phi_2\rangle\}\text{ basis}}{\doteq} \begin{pmatrix} \langle\phi_1|+\rangle_y \\ \langle\phi_2|+\rangle_y \end{pmatrix} \quad \text{II}$$

$$\langle\phi_1|+\rangle_y = \left(\frac{3}{5}\langle+| - i\frac{4}{5}\langle-|\right)\left(\frac{1}{\sqrt{2}}|+\rangle + \frac{i}{\sqrt{2}}|-\rangle\right) = \boxed{\frac{7}{5\sqrt{2}}}$$

$$\langle\phi_2|+\rangle_y = \left(-\frac{4}{5}\langle+| - i\frac{3}{5}\langle-|\right)\left(\frac{1}{\sqrt{2}}|+\rangle + \frac{i}{\sqrt{2}}|-\rangle\right) = \boxed{\frac{-1}{5\sqrt{2}}}$$

$$\xrightarrow{\text{II}} |+\rangle_y \underset{\{|\phi_1\rangle, |\phi_2\rangle\}\text{ basis}}{\doteq} \frac{1}{5\sqrt{2}} \begin{pmatrix} 7 \\ -1 \end{pmatrix}$$

$$\Rightarrow \mathbb{P}\left(S_y = \frac{\hbar}{2}\right) = \left|{}_y\langle + | \psi \rangle\right|^2 = \left| \frac{1}{5\sqrt{2}} \begin{pmatrix} 7 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right|^2 = \boxed{\frac{49}{50}} \checkmark$$

(2) $A \doteq \begin{pmatrix} 1 & 2i \\ -2i & 1 \end{pmatrix}$ S_z eigenbasis

(a) To be a valid observable, we need A to be Hermitian:

$$A^\dagger = \begin{pmatrix} 1 & (-2i)^* \\ (2i)^* & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2i \\ -2i & 1 \end{pmatrix} \Rightarrow \boxed{A = A^\dagger}$$ A can be a valid observable

(b) Find the eigenvalues of A:

$$\det(A - \lambda \mathbb{1}) = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 2i \\ -2i & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)^2 - 4 = 0 \Rightarrow \boxed{\begin{cases} \lambda = 3 \\ \lambda = -1 \end{cases} \text{ Possible measurement values}}$$

(c) $A = \begin{pmatrix} {}_y\langle +|A|+\rangle_y & {}_y\langle -|A|+\rangle_y \\ {}_y\langle +|A|-\rangle_y & {}_y\langle -|A|-\rangle_y \end{pmatrix}$

$${}_y\langle +|A|+\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \end{pmatrix}\begin{pmatrix} 1 & 2i \\ -2i & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -1 & i \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2}(-1-1) = \boxed{-1}$$

$${}_y\langle -|A|+\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \end{pmatrix}\begin{pmatrix} 1 & 2i \\ -2i & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3 & 3i \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$= \frac{1}{2}(3-3) = \boxed{0}$$

$$ {}_y\langle + | A | - \rangle_y = {}_y\langle - | A^\dagger | + \rangle_y^* = {}_y\langle - | A | + \rangle_y^* = \boxed{0} $$

$$ {}_y\langle - | A | - \rangle_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \end{pmatrix} \begin{pmatrix} 1 & 2i \\ -2i & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} $$

$$ = \frac{1}{2} \begin{pmatrix} 3 & 3i \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{2}(3+3) = \boxed{3} $$

$$ \Rightarrow \boxed{A \doteq \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}} $$

S_y eigenbasis

(d) From (c), we see that A is diagonal in the S_y basis, meaning that its eigenvectors are simply $|+\rangle_y$, $|-\rangle_y$.

$$ \Rightarrow \mathbb{P}(A = -1) = |{}_y\langle + | + \rangle_y|^2 = \boxed{1} $$

$$ \mathbb{P}(A = 3) = |{}_y\langle + | - \rangle_y|^2 = \boxed{0} $$