① (a) We simply have to calculate the eigenvector of $\hat{n}\cdot\vec{S}$ corresponding to eigenvalue $-\hbar/2$:

$$\hat{n}\cdot\vec{S} = \cos\varphi\sin\theta\, S_x + \sin\varphi\sin\theta\, S_y + \cos\theta\, S_z = \frac{\hbar}{2}\left(\cos\varphi\sin\theta\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} + \sin\varphi\sin\theta\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} + \cos\theta\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\right)$$

$$= \frac{\hbar}{2}\begin{bmatrix}\cos\theta & (\cos\varphi - i\sin\varphi)\sin\theta\\ (\cos\varphi + i\sin\varphi)\sin\theta & -\cos\theta\end{bmatrix}$$

$$\Rightarrow \hat{n}\cdot\vec{S} = \frac{\hbar}{2}\begin{bmatrix}\cos\theta & e^{-i\varphi}\sin\theta\\ e^{i\varphi}\sin\theta & -\cos\theta\end{bmatrix} \quad \text{I}$$

Eigenvector $|-\rangle_{\hat{n}} = \begin{bmatrix}x\\ y\end{bmatrix}$, $\quad \hat{n}\cdot\vec{S}\,|+\rangle_{\hat{n}} = -\frac{\hbar}{2}|+\rangle_{\hat{n}}$

$$\overset{\text{I}}{\Rightarrow} \frac{\cancel{\hbar}}{\cancel{2}}\begin{bmatrix}\cos\theta & e^{-i\varphi}\sin\theta\\ e^{i\varphi}\sin\theta & -\cos\theta\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = -\frac{\cancel{\hbar}}{\cancel{2}}\begin{bmatrix}x\\ y\end{bmatrix} \Rightarrow \cos\theta\, x + e^{-i\varphi}\sin\theta\, y = -x$$

$$\Rightarrow \frac{y}{x} = -e^{i\varphi}\frac{1+\cos\theta}{\sin\theta} = -e^{i\varphi}\frac{2\cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)} = -e^{i\varphi}\frac{\cos(\theta/2)}{\sin(\theta/2)} = -e^{i\varphi}\cot(\theta/2) \quad \text{II}$$

Normalization: $\;{}_{\hat{n}}\langle -|-\rangle_{\hat{n}} = 1 \Rightarrow |x|^2 + |y|^2 = 1 \overset{\text{II}}{\Longrightarrow} |x|^2 + \cot^2(\theta/2)\,|x|^2 = 1$

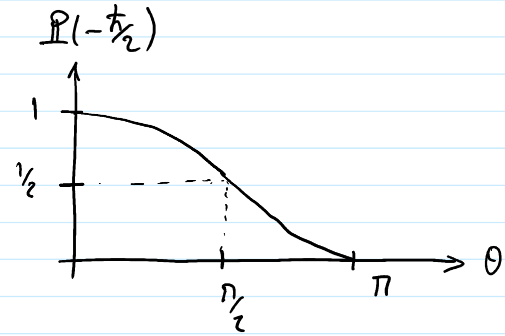
$$\Rightarrow |x|^2 = \frac{1}{1+\cot^2(\theta/2)} = \frac{1}{\frac{1}{\sin^2(\theta/2)}} = \sin^2(\theta/2) \Rightarrow |x| = \sin(\theta/2) \quad \text{III}$$

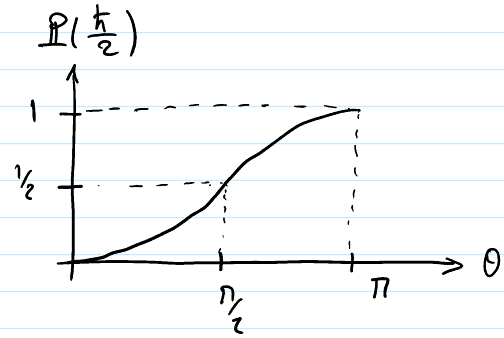
$$\text{II},\ \text{III} \Rightarrow \begin{cases} x = \sin(\theta/2)\\ y = -e^{i\varphi}\cos(\theta/2)\end{cases} \Rightarrow \boxed{|+\rangle_{\hat{n}} = \begin{bmatrix}\sin(\theta/2)\\ -e^{i\varphi}\cos(\theta/2)\end{bmatrix} = \sin(\theta/2)\,|+\rangle - e^{i\varphi}\cos(\theta/2)\,|-\rangle}$$

(b) possible outcomes for measuring S_z are its eigenvalues, which are $\pm \hbar/2$. The corresponding probabilities are:

$$\mathbb{P}(\hbar/2) = |\langle + | - \rangle_{\hat{n}}|^2 = \boxed{\sin^2(\theta/2)}$$

$$\mathbb{P}(-\tfrac{\hbar}{2}) = 1 - \mathbb{P}(\hbar/2) = 1 - \sin^2(\theta/2) = \boxed{\cos^2(\theta/2)}$$





(c) The independence of the probabilities from φ is expected because the statistics of measurement outcomes should not depend on our choice of coordinate system. For example, if we instead chose to analyze the problem with coordinate system $x'y'z'$ (Figure), the azimuthal angle φ' would be different from φ. Therefore, if the measurement statistics depended on φ, we would get different probabilities if we did our analysis using the new coordinates, which wouldn't make sense.