Lecture 7: Projection Operators - Summary & Formula Sheet

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1 Lecture Overview

- **Topic:** Projection Operators, Measurement (Postulate 5), and Stern-Gerlach Analysis.
- **Goal:** Develop an operator formalism for calculating measurement outcomes and analyzing quantum systems.

2 1. The Projection Operator

The lecture introduces the projection operator to formalize how we describe quantum states and measurements.

2.1 Definition

For a state $|\psi\rangle$ expanded in a basis of eigenvectors $|a_n\rangle$ (where $|\psi\rangle = \sum c_n |a_n\rangle$), the projection operator $\hat{P}_n$ is defined as the outer product of the eigenket with its own bra:

$$\hat{P}_n \equiv |a_n\rangle \langle a_n| \tag{1}$$

2.2 Action on a State

When $\hat{P}_n$ acts on an arbitrary state $|\psi\rangle$, it "projects" out the component of $|\psi\rangle$ corresponding to $|a_n\rangle$:

$$\hat{P}_n |\psi\rangle = |a_n\rangle \langle a_n|\psi\rangle = c_n |a_n\rangle \tag{2}$$

2.3 Matrix Representation

While the inner product $\langle\psi|\psi\rangle$ is a scalar, the outer product $|\psi\rangle\langle\psi|$ creates an $n \times n$ matrix (or tensor).

- **Example (S_z Operator):**

$$\hat{P}_+ = |+\rangle \langle +| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{3}$$

$$\hat{P}_- = |-\rangle \langle -| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{4}$$

3 2. Key Properties & Theorems

3.1 Completeness Relation

The sum of all projection operators for a complete basis equals the identity operator. This implies that the probability of finding the system in *any* valid state sums to 1:

$$\sum_n \hat{P}_n = \sum_n |a_n\rangle \langle a_n| = \mathbb{1} \tag{5}$$

3.2 Spectral Decomposition

Any observable operator $\hat{A}$ can be written as a sum of its eigenvalues weighted by their corresponding projection operators:

$$\hat{A} = \sum_n a_n |a_n\rangle \langle a_n| \tag{6}$$

4 3. Measurement & Postulate 5

This section formally defines the effect of measurement on a quantum state.

4.1 The Collapse

A measurement of observable $\hat{A}$ yielding result a_n "projects" or "collapses" the initial state $|\psi_i\rangle$ onto the eigenstate $|a_n\rangle$.

4.2 Postulate 5 Formula

The state after measurement ($|\psi_f\rangle$) is the projection of the initial state, normalized by the probability amplitude:

$$|\psi_f\rangle = \frac{\hat{P}_n |\psi_i\rangle}{\sqrt{\langle\psi_i| \hat{P}_n |\psi_i\rangle}} \tag{7}$$

Note: The denominator ensures normalization. Any overall phase factor $e^{i\alpha}$ is physically irrelevant.

5 4. Expectation Value

The expectation value represents the average value of an operator for a given state. It is the sum of eigenvalues weighted by their probabilities:

$$\langle\hat{A}\rangle = \langle\psi| \hat{A} |\psi\rangle = \sum_n a_n |c_n|^2 \tag{8}$$

6 5. Analysis of S-G Experiments

6.1 Experiment 3: Measurement Interruption

Setup: Oven $\rightarrow$ Analyzer 1 (z) $\rightarrow$ Analyzer 2 (x) $\rightarrow$ Analyzer 3 (z).

- The state enters Analyzer 2 (x-basis) and is projected into superpositions of z-basis states.
- **Result:** Because the x-basis measurement "scrambles" the z-information, the probability for each final outcome (Ports 3a+, 3a-, 3b+, 3b-) is exactly 1/4 (25%).

6.2 Experiment 4: Recombination (No Measurement)

Setup: Similar to Exp 3, but beams from Analyzer 2 are recombined without detection.

- **Analysis:** The action of splitting and recombining is equivalent to applying the sum of projections, which is Identity:

$$\hat{P}_{total} = |+\rangle_x \langle+|_x + |-\rangle_x \langle-|_x = \mathbb{1} \tag{9}$$

- **Result:** The state passes through unchanged ($|+\rangle \rightarrow |+\rangle$). Probability of exiting $|+\rangle$ port of A3 is 100%.

Midterm Formula Sheet

1. **Projection Operator Definition**

$$\hat{P}_n = |a_n\rangle \langle a_n| \tag{10}$$

2. **Completeness Relation (Resolution of Identity)**

$$\sum_n \hat{P}_n = \nVdash \tag{11}$$

3. **Spectral Decomposition**

$$\hat{A} = \sum_n a_n \hat{P}_n = \sum_n a_n |a_n\rangle \langle a_n| \tag{12}$$

4. **Post-Measurement State (Postulate 5)**

$$|\psi_{final}\rangle = \frac{\hat{P}_n |\psi_{initial}\rangle}{\sqrt{\langle \psi_{initial}| \hat{P}_n |\psi_{initial}\rangle}} \tag{13}$$

5. **Expectation Value**

$$\langle \hat{A} \rangle = \langle \psi | \hat{A} |\psi\rangle \tag{14}$$

6. **Probability of Measuring a_n**

$$P(a_n) = \langle \psi | \hat{P}_n |\psi\rangle = |\langle a_n|\psi\rangle|^2 \tag{15}$$

Core Takeaway

Measurement is an active mathematical operation (projection) that fundamentally alters the state.

- **Measure/Block:** Apply a single $\hat{P}_n$ $\rightarrow$ Collapse state.
- **No Measurement (Recombine):** Apply $\sum \hat{P}_n = \nVdash$ $\rightarrow$ State is unchanged.