Diffusion eqn general solution

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- Diffusion eqn (Find a Fourier series solution)

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = 0 = u(l,t), \quad u(x,0) = f(x)$$

- Separation of variables $u(x,t) = M(x)N(t)$

$$MN_t = DM_{xx}N$$

$$\frac{N_t}{DN} = \frac{M_{xx}}{M} = -\lambda, \quad \lambda \geq 0$$

$$\frac{\partial N}{\partial t} + \lambda DN = 0, \quad \frac{\partial^2}{\partial x^2}M + \lambda M = 0$$

B.C. $M(0)N = 0 = M(l)N \Rightarrow M(0) = 0 = M(l)$

with solution

$$\lambda_k = \left(\frac{k\pi}{l}\right)^2, \quad M_k(x) = \sqrt{\frac{2}{l}}\sin\left(\frac{k\pi x}{l}\right), \quad k = 1,2,3,\ldots$$

(don't want $k=0$ since it generates the trivial soln.)

$\downarrow$

$$N_k(t) = A_k e^{-D\left(\frac{k\pi}{l}\right)^2 t}$$

$$\therefore u_k(x,t) = A_k e^{-D\left(\frac{k\pi}{l}\right)^2 t}\sqrt{\frac{2}{l}}\sin\left(\frac{k\pi x}{l}\right)$$

Now to find the general soln.

$$u(x,t) = \sum_{k=1}^{\infty} A_k e^{-D\left(\frac{k\pi}{l}\right)^2 t}\sqrt{\frac{2}{l}}\sin\left(\frac{k\pi x}{l}\right)$$

impose $u(x,0) = f(x)$

$$\sum_{k=1}^{\infty} A_k \cdot \underbrace{\sqrt{\frac{2}{l}}\sin\left(\frac{k\pi x}{l}\right)}_{M_k(x)} = f(x)$$

$$A_k = (f, M_k)$$

pick out the A_k for each term

note: as $t \to \infty$, $u \to 0$ for all $f(x)$

but the larger k terms decay faster!

- what if $u(0,t) = 0$, $u(l,t) = 1$

we would get $M(l)N = 1$ if we do the same as before!

Consider: steady solution of $\int$ $\left(\frac{\partial}{\partial t} \to 0\right)$

$$D\frac{d^2U}{dx^2} = 0, \quad U(0) = 0, \; U(l) = 1$$

$$\hookrightarrow U(x) = Ax + B \Rightarrow U(x) = \frac{x}{l}$$

If we decompose $u(x,t) = U(x) + \hat{u}(x,t)$, what determines $\hat{u}(x,t)$?

but since $\frac{\partial U}{\partial t} = 0 = \frac{\partial^2 U}{\partial x^2}$, $\quad \underbrace{\frac{\partial \hat{u}}{\partial t} = D\frac{\partial^2 \hat{u}}{\partial x^2}}_{\text{still the same PDE.}}$

$$u(0,t) = 0 = \overset{=0}{U(0)} + \hat{u}(0,t) \Rightarrow \hat{u}(0,t) = 0$$

$$u(l,t) = 1 = \overset{=1}{U(l)} + \hat{u}(l,t) \Rightarrow \hat{u}(l,t) = 0$$

$$\text{IC: } u(x,0) = f(x) = \frac{x}{l} + \hat{u}(x,0) \Rightarrow \hat{u}(x,0) = f - \frac{x}{l} = \hat{f}(x)$$

the solution to $\hat{u}$ is the same as before, just with different IC!

The solution as $t \to \infty$ is just the steady state solution U