Tutorial 1 Solutions

①

$$M_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Eigenvalues: $\det(M_1 - \lambda \mathbb{1}) = 0 \Rightarrow \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 1 = 0 \Rightarrow \boxed{\lambda = \pm 1}$

Eigenvector for $\lambda = 1$:

$$M_1 |1\rangle = 1 \times |1\rangle \quad , \quad |1\rangle = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow y = x$$

$$\Rightarrow |1\rangle = x \begin{bmatrix} 1 \\ 1 \end{bmatrix} \xrightarrow{\text{Normalize}} \boxed{|1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

Eigenvector for $\lambda = -1$:

$$M_1 |-1\rangle = -1 \times |-1\rangle \quad , \quad |-1\rangle = \begin{bmatrix} x \\ y \end{bmatrix}$$

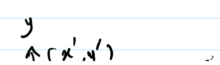
$$\Rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix} \Rightarrow x = -y \Rightarrow |-1\rangle = x \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\xrightarrow{\text{Normalize}} \boxed{|-1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

Geometric interpretation:

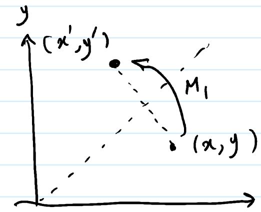
$$\underbrace{\begin{bmatrix} x' \\ y' \end{bmatrix}}_{\text{mapped point}} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\text{linear map}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{arbitrary point}} \Longrightarrow \begin{cases} x' = y \\ y' = x \end{cases}$$

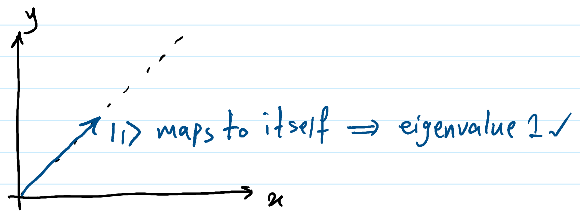
Reflection around $x = y$ line.

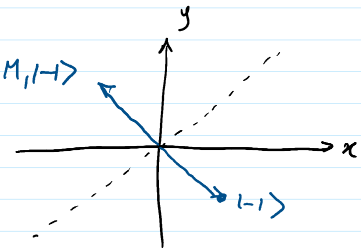


mapped point   linear map   arbitrary point

Geometrically, an eigenvector represents a vector that is mapped to another vector along the same direction. Reflection around the $x=y$ line does the following to the derived eigenvectors:







$|-1\rangle$ maps to $-|-1\rangle \Rightarrow$ eigenvalue -1 ✓

$$M_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Fast method $\Rightarrow$ M_2 is already diagonal $\Rightarrow$ $\left\{ \begin{array}{l} \text{eigenvalues (diagonal elements):} \\ \quad \lambda = 2 \text{ (doubly degenerate)} \\ \text{eigenvectors: } \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \text{(canonical basis)} \end{array} \right.$

Routine method $\Rightarrow$

Eigenvalues: $\det(M_2 - \lambda \mathbb{1}) = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda-2)^2 = 0 \Rightarrow \boxed{\lambda = 2}$

doubly degenerate

Eigenvectors for $\lambda = 2$:

$$M_2 \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

This is true for any $x, y \in \mathbb{R}$, i.e. any non-zero vector is an eigenvector. We can choose any two orthonormal vectors. The simplest ones are:

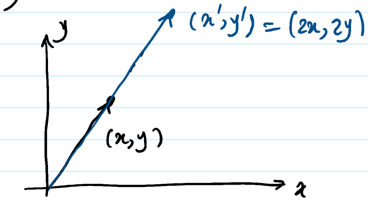
$$\boxed{\underbrace{|2,1\rangle}_{} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \underbrace{|2,2\rangle}_{} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

Notation: 1st eigenvector for eigenvalue $\lambda = 2$ | 2nd eigenvector for eigenvalue $\lambda = 2$

Geometric interpretation:

$$\underbrace{\begin{bmatrix} x' \\ y' \end{bmatrix}}_{\text{mapped point}} = \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}}_{\text{linear map}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{arbitrary point}} \Longrightarrow \begin{cases} x' = 2x \\ y' = 2y \end{cases}$$

scaling by a factor of 2.



Geometrically, an eigenvector represents a vector that is mapped to another vector along the same direction. Since scaling a vector by two doesn't change its direction, any vector is an eigenvector of M_4, and the eigenvalue is the scaling factor, i.e. $\lambda = 2$.

$$M_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Eigenvalues: $\det(M_3 - \lambda \mathbb{1}) = 0 \Rightarrow \begin{vmatrix} \frac{1}{\sqrt{2}} - \lambda & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} - \lambda \end{vmatrix} = 0 \Rightarrow \left(\lambda - \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2} = 0$

$$\Rightarrow \boxed{\lambda = \frac{1}{\sqrt{2}}(1 \pm i)}$$

Eigenvector for $\lambda = \frac{1+i}{\sqrt{2}}$:

$$M_3 \left| \frac{1+i}{\sqrt{2}} \right\rangle = \frac{1+i}{\sqrt{2}} \left| \frac{1+i}{\sqrt{2}} \right\rangle, \quad \left| \frac{1+i}{\sqrt{2}} \right\rangle = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1+i}{\sqrt{2}} \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{cases} \cancel{x} - y = \cancel{x} + ix \\ x + \cancel{y} = \cancel{y} + iy \end{cases} \Rightarrow y = -ix$$

$\Rightarrow \left| \frac{1+i}{\sqrt{2}} \right\rangle = x \begin{bmatrix} 1 & \ldots \end{bmatrix}$

$$\Rightarrow \left|\tfrac{1+i}{\sqrt{2}}\right\rangle = x\begin{bmatrix}1\\-i\end{bmatrix} \xrightarrow{\text{Normalize}} \boxed{\left|\tfrac{1+i}{\sqrt{2}}\right\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}}$$

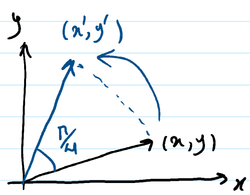
Eigenvector for $\lambda = \frac{1-i}{\sqrt{2}}$:

$$M_3\left|\tfrac{1-i}{\sqrt{2}}\right\rangle = \frac{1-i}{\sqrt{2}}\left|\tfrac{1-i}{\sqrt{2}}\right\rangle \quad , \quad \left|\tfrac{1-i}{\sqrt{2}}\right\rangle = \begin{bmatrix}x\\y\end{bmatrix}$$

$$\Rightarrow \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \frac{1-i}{\sqrt{2}}\begin{bmatrix}x\\y\end{bmatrix} \Rightarrow \begin{cases} x-y = x-ix \\ x+y = y-iy \end{cases} \Rightarrow y = ix$$

$$\Rightarrow \left|\tfrac{1-i}{\sqrt{2}}\right\rangle = x\begin{bmatrix}1\\i\end{bmatrix} \xrightarrow{\text{Normalize}} \boxed{\left|\tfrac{1-i}{\sqrt{2}}\right\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}}$$

Geometric interpretation: M_3 is simply a special case of a rotation matrix $\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$ with $\theta = \frac{\pi}{4}$. Hence, M_3 represents a counter-clockwise $\frac{\pi}{4}$ rotation around the origin. It is easy to see that no non-zero vector in $\mathbb{R}^2$ keeps its direction upon a $\frac{\pi}{4}$ rotation. Hence, M_3 cannot have real eigenvectors, which is consistent with the eigenvectors derived above. ✓



$$M_4 = \begin{bmatrix} i & 1\\ 0 & 0\end{bmatrix}$$

Eigenvalues: $\det(M_4 - \lambda \mathbb{1}) = 0 \Rightarrow \begin{vmatrix} i-\lambda & 1\\ 0 & -\lambda\end{vmatrix} = 0 \Rightarrow \lambda(\lambda - i) = 0 \Rightarrow \boxed{\begin{cases}\lambda = 0\\ \lambda = i\end{cases}}$

Eigenvector for $\lambda = 0$:

$$M_4|0\rangle = 0\,|0\rangle \quad , \quad |0\rangle = \begin{bmatrix}x\\y\end{bmatrix}$$

$$M_4|0\rangle = 0\,|0\rangle \quad , \quad |0\rangle = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow ix + y = 0 \Rightarrow y = -ix$$

$$\Rightarrow |0\rangle = x\begin{bmatrix} 1 \\ -i \end{bmatrix} \xrightarrow{\text{Normalize}} \boxed{|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}}$$

Eigenvector for $\lambda = i$:

$$M_4|i\rangle = i\,|i\rangle, \quad |i\rangle = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ix \\ iy \end{bmatrix} \Rightarrow \begin{cases} ix + y = ix \\ 0 = y \end{cases} \Rightarrow |i\rangle = \begin{bmatrix} x \\ 0 \end{bmatrix}$$

$$\xrightarrow{\text{Normalize}} \boxed{|i\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$

Geometric interpretation: since M_4 is a proper complex matrix, it doesn't allow for a geometric interpretation on $\mathbb{R}^2$.

2

M_1

Inner product: $\langle -1|1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \end{bmatrix} \times \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2}(1-1) = \boxed{0}$

Show that M_1 is normal:

$$M_1^\dagger M_1 - M_1 M_1^\dagger = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \boxed{0} \checkmark$$

M_2

$$\langle 2,1|2,2\rangle = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \boxed{0}$$

Show that M_2 is normal:

$$M_2 M_2^\dagger - M_2^\dagger M_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \boxed{0} \checkmark$$

M_3

$$\left\langle \frac{1+i}{\sqrt{2}} \middle| \frac{1-i}{\sqrt{2}} \right\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \end{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix} = \frac{1}{2}(1-1) = \boxed{0}$$

show that M_3 is normal:

$$M_3 M_3^\dagger - M_3^\dagger M_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} - \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \boxed{0} \checkmark$$

M_4

$$\langle i|0\rangle = \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix} = \boxed{\frac{1}{\sqrt{2}}} \neq 0$$

show that M_4 is not normal:

$$M_4 M_4^\dagger = \begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -i & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$M_4^\dagger M_4 = \begin{bmatrix} -i & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$$

$$\boxed{M_4 M_4^\dagger \neq M_4^\dagger M_4}$$