

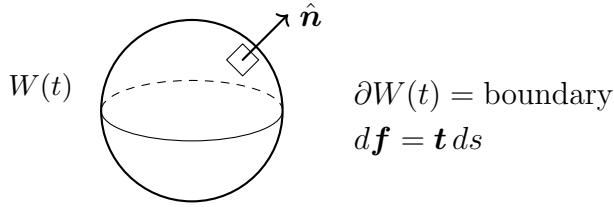
Lecture 9

9.1 Surface Forces and Pressure

Last time, using Newton's 2nd Law and including gravity (volume force) and surface forces, we obtained the integral form of the momentum equation:

$$\iiint_{W(t)} \rho \frac{Du_i}{Dt} dV = - \iiint_{W(t)} \rho \frac{\partial \Pi}{\partial x_i} dV + \iint_{\partial W(t)} t_i dS.$$

Today, we focus on the surface force & rewrite it as a triple integral over $W(t)$.



On each subsurface, we have

$$d\mathbf{f} = \mathbf{t} ds$$

where \mathbf{t} is the stress vector. Let's consider a state of equilibrium where there is no movement

$$\mathbf{u} = \vec{0}.$$

9.1.1 The Stress Vector in Equilibrium

Consider a state of equilibrium where there is no movement ($\mathbf{u} = \vec{0}$). Recall that fluids in equilibrium cannot resist shear forces (stresses). Since we have $\mathbf{u} = \vec{0}$, in equilibrium, the surface force must be parallel to the normal vector \hat{n} . Then the stress tensor is

$$\mathbf{t} = -p(\mathbf{x})\hat{n},$$

where we call $p(\mathbf{x})$ the pressure. The negative sign indicates that pressure acts *inward* on the volume (compression).

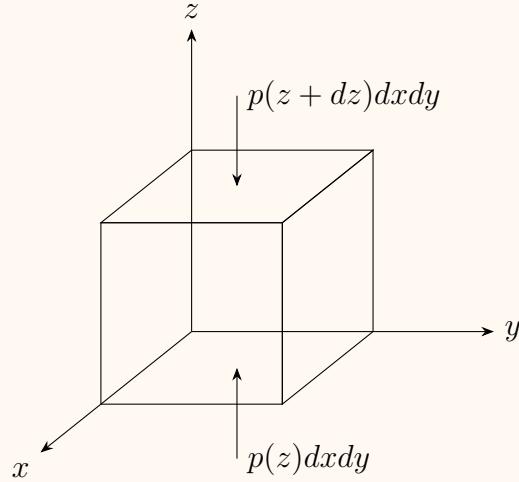
Our global expression for the surface force becomes

$$\mathbf{F}_s = \iint_{\partial W(t)} -p\hat{n} dS.$$

9.1.2 Derivation of Pressure Force

To rewrite this surface integral as a volume integral, we consider two approaches: a heuristic Taylor expansion and a formal proof using Gauss' Theorem.

Note 9.1 (Heuristic Derivation using Taylor Expansion). Consider the pressure on a small material volume (a cube) with side lengths dx, dy, dz such that $dV = dx dy dz \ll 1$.



The units of \mathbf{t} are $[t] = \text{N/m}^2$. We will look at the balance of forces in the z -direction. For the difference in this force from the two faces, we Taylor expand about z .

$$\begin{aligned} dF_z &= p(x, y, z) dx dy - p(x, y, z + dz) dx dy \\ &= (p(x, y, z) - p(x, y, z + dz)) dx dy \\ &= (p(x, y, z) - [p(x, y, z) + dz \frac{\partial p}{\partial z}(x, y, z) + \dots]) dx dy \\ &= -\frac{\partial p}{\partial z}(x, y, z) dx dy dz \\ &= -\frac{\partial p}{\partial z} dV. \end{aligned}$$

Similarly,

$$dF_x = -\frac{\partial p}{\partial x}(x, y, z) dx dy dz \quad \& \quad dF_y = -\frac{\partial p}{\partial y}(x, y, z) dx dy dz$$

Combining these into a vector equation:

$$d\mathbf{F} = (dF_x, dF_y, dF_z) = -\left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}\right) dV = -\nabla p dV.$$

Integrating over the whole volume, the surface force can be written as:

$$\mathbf{F}_s = - \iiint_{W(t)} \nabla p dV.$$

This very rough calculation suggests the following identity.

Theorem 9.2 (*Gradient Theorem Corollary*).

$$\oint\!\!\!\oint_{\partial W(t)} p \hat{\mathbf{n}} dS = \iiint_{W(t)} \nabla p dV.$$

Proof. The idea of the proof is to write the identity in component form using Gauss' Divergence Theorem. Recall Gauss' Divergence Theorem states:

$$\iiint_W \nabla \cdot \mathbf{U} dV = \oint\!\!\!\oint_{\partial W} \mathbf{U} \cdot \hat{\mathbf{n}} dS.$$

We want to show:

$$\oint\!\!\!\oint_{\partial W(t)} p(n_x, n_y, n_z) dS = \iiint_{W(t)} \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z} \right) dV.$$

This corresponds to 3 scalar equations.

For the x-direction: Pick the vector field $\mathbf{U} = (p, 0, 0)$. Then $\nabla \cdot \mathbf{U} = \frac{\partial p}{\partial x}$. Applying Gauss' Theorem:

$$\iiint_{W(t)} \frac{\partial p}{\partial x} dV = \oint\!\!\!\oint_{\partial W(t)} (p, 0, 0) \cdot \hat{\mathbf{n}} dS = \oint\!\!\!\oint_{\partial W(t)} p n_x dS.$$

For the y-direction: Pick $\mathbf{U} = (0, p, 0)$. Then $\nabla \cdot \mathbf{U} = \frac{\partial p}{\partial y}$.

$$\iiint_{W(t)} \frac{\partial p}{\partial y} dV = \oint\!\!\!\oint_{\partial W(t)} p n_y dS.$$

For the z-direction: Pick $\mathbf{U} = (0, 0, p)$. Then $\nabla \cdot \mathbf{U} = \frac{\partial p}{\partial z}$.

$$\iiint_{W(t)} \frac{\partial p}{\partial z} dV = \oint\!\!\!\oint_{\partial W(t)} p n_z dS.$$

Combining these 3 scalar equations into a vector equation yields the result. \square

9.2 Hydrostatics

Given this identity, we return to Newton's 2nd Law in a state of rest ($\mathbf{u} = \vec{0}$). The acceleration term is zero.

$$\begin{aligned} 0 &= - \iiint_{W(t)} \rho \nabla \Pi dV - \oint\!\!\!\oint_{\partial W(t)} p \hat{\mathbf{n}} dS \\ &= \iiint_{W(t)} [-\rho \nabla \Pi - \nabla p] dV \\ &= - \iiint_{W(t)} [\rho \nabla \Pi + \nabla p] dV. \end{aligned}$$

Since this holds for any arbitrary volume $W(t)$, we apply the Dubois-Reymond Lemma (localization) to get:

$$\rho \nabla \Pi + \nabla p = 0.$$

Assuming gravity acts in the vertical direction, $\Pi = gz$ and $\nabla \Pi = g\hat{z}$.

Definition 9.3 (Hydrostatic Balance). The hydrostatic balance equation is

$$\nabla p = -\rho \nabla \Pi = -\rho g \hat{z},$$

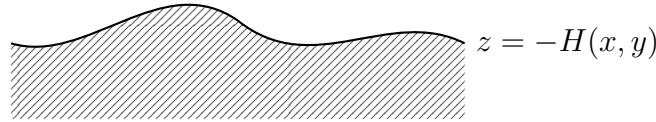
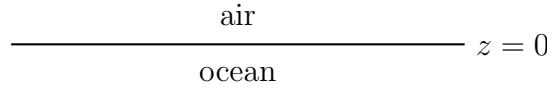
where $\Pi = gz$. In component form:

$$\frac{\partial p}{\partial x} = 0, \quad \frac{\partial p}{\partial y} = 0, \quad \frac{\partial p}{\partial z} = -\rho g.$$

This describes the perfect balance between the forces of gravity and pressure.

9.2.1 Hydrostatic Balance of the Ocean

Suppose we consider a simple ocean at rest ($\mathbf{u} = \mathbf{0}$), and the density is constant $\rho = \rho_0$ (incompressible). Since $\frac{\partial p}{\partial x} = 0$ and $\frac{\partial p}{\partial y} = 0$, pressure depends only on z , i.e., $p(z)$.



The vertical equation is:

$$\frac{\partial p}{\partial z} = -g\rho_0.$$

We integrate from depth z to the surface $z = 0$:

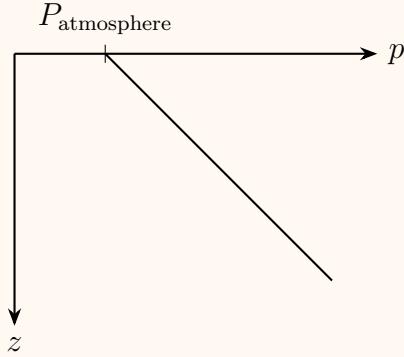
$$\begin{aligned} \int_z^0 \frac{dp}{dz} dz &= \int_z^0 -g\rho_0 dz \\ p(z) \Big|_z^0 &= -g\rho_0 z \Big|_z^0 \\ p(0) - p(z) &= 0 - (-g\rho_0 z) = g\rho_0 z \\ p(z) &= p(0) - \rho_0 g z. \end{aligned}$$

If we denote the atmospheric pressure as $P_{\text{atmosphere}} = p(0)$, then

$$p(z) = P_{\text{atmosphere}} - \rho_0 g z.$$

Note 9.4. Since z is negative underwater, $-\rho_0 g z$ is a positive term. The pressure at z is equal to the pressure of the atmosphere plus the weight of the fluid column above per unit area:

$$\frac{\rho_0 g(-z)dA}{dA} = \frac{mg}{dA}.$$



Example 9.5 (Numerical Scale). It is observed that $P_{\text{atmosphere}} \approx 10^5 \text{ N/m}^2$. The depth of the ocean is $\leq 10 \text{ km} = 10^4 \text{ m}$. The density of water is $\rho_0 \approx 10^3 \text{ kg/m}^3$ and $g \approx 10 \text{ m/s}^2$.

At the top of the ocean: $p \approx 10^5 \text{ N/m}^2$.

At the bottom of the ocean:

$$p \approx 10^5 \text{ N/m}^2 + \left(10^3 \frac{\text{kg}}{\text{m}^3}\right) \left(10 \frac{\text{m}}{\text{s}^2}\right) (10^4 \text{ m}) \approx 10^8 \text{ N/m}^2.$$

The pressure at the bottom is 1000 times larger than at the surface.

9.2.2 Hydrostatic Balance for the Atmosphere

The density of air changes a lot (it is compressible). To describe air, you need an equation of state. One choice is the Ideal Gas Law:

$$p = \rho R T, \quad R \approx 287 \frac{\text{J}}{\text{kg} \cdot \text{K}}.$$

The temperature is not constant, but if we assume it is for simplicity (say $T = T_0 = \text{const}$), we get simple equations. From the ideal gas law: $\rho = \frac{p}{RT_0}$. Substitute this into the vertical hydrostatic equation:

$$\frac{dp}{dz} = -g\rho = -\frac{gp}{RT_0}.$$

This is a separable ODE:

$$\frac{dp}{p} = -\frac{g}{RT_0} dz \quad \Rightarrow \quad \ln p = -\frac{gz}{RT_0} + C.$$

Solving for p :

$$p(z) = p(0)e^{-z/H}, \quad \text{where } H \equiv \frac{RT_0}{g}.$$

H is called the **Scale Height**. For $T_0 \approx 20^\circ\text{C}$, $H \approx 8.4 \text{ km}$.

