

# Lecture 8

## Expectation Value of Operators

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## 1 The Expectation Value

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# Introduction to State Tomography

- We wish to consider the average value of an observable corresponding to the operator  $\hat{A}$  when performed on a particular state  $|\psi\rangle$ .
- We know from Postulate 3 of QM that the measurement will yield one of the possible eigenvalues of the observable with probability  $P_n = |\langle a_n | \psi \rangle|^2$  (Postulate 4), where  $\{|a\rangle\}$  represent the eigenbasis of  $\hat{A}$ . The new state after measurement will be  $|\psi'\rangle = |a_n\rangle$  (Postulate 5).
- Each measurement projects or “collapses” the initial quantum state  $|\psi\rangle$  to one of the eigenstates of  $\hat{A}$ .
- Therefore, to determine the original  $|\psi\rangle$ , we must repeat the measurement many times and determine the probabilities for each outcome. This means we need many identical copies of  $|\psi\rangle$  in order to reconstruct it.
- From the distribution of measurement outcomes, we can reconstruct the original  $|\psi\rangle$  up to some overall phase factor.

# The Expectation Value

- We express the state  $|\psi\rangle$  in the eigenbasis of operator  $\hat{A}$ .

$$|\psi\rangle = \sum_n c_n |a_n\rangle$$

- We denote the average or “expected” value of the measurement of  $\hat{A}$  for the state  $|\psi\rangle$  as  $\langle\hat{A}\rangle$ .

$$\langle\hat{A}\rangle = \sum_n P_n a_n \quad (1)$$

- The average value of  $\hat{A}$  for the state  $|\psi\rangle$  is equal to the sum over all eigenvalues  $\{a\}$  of  $\hat{A}$  multiplied by the corresponding probability of measurement.

$$\langle\hat{A}\rangle = \sum_n a_n |\langle a_n|\psi\rangle|^2 \quad (2)$$

$$= \sum_n a_n |c_n|^2 = \sum_n a_n \langle\psi|\hat{P}_n|\psi\rangle \quad (3)$$

where  $\hat{P}_n = |a_n\rangle\langle a_n|$  is the projection operator for state  $|a_n\rangle$  (See Lecture 7).

# The Expectation Value

- Using spectral decomposition (Lecture 7), we express operator  $\hat{A}$  using projection operators.

$$\hat{A} = \sum_n^{\text{all states}} a_n |a_n\rangle\langle a_n| \quad (4)$$

- Inserting eq.(4) into eq.(3), we find that the expectation value of operator  $\hat{A}$  for state  $|\psi\rangle$  is

$$\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle \quad (5)$$

- The expectation value of an operator for a particular state is a physical quantity. Therefore, it cannot depend on our choice of basis for expressing  $|\psi\rangle$ .

# Root-Mean-Square (RMS) Variation of Measurements

- Consider the expectation or average values  $\langle S_z \rangle$  and  $\langle S_x \rangle$  in the  $|\pm\rangle$  basis.

$$\langle \pm | S_z | \pm \rangle = \pm \frac{\hbar}{2} \langle \pm | \pm \rangle = \pm \frac{\hbar}{2}$$

$$\begin{aligned} \langle + | S_x | + \rangle &= \langle + | \left( \frac{\hbar}{2} | + x \rangle \langle + x | - \frac{\hbar}{2} | - x \rangle \langle - x | \right) | + \rangle \\ &= \frac{1}{\sqrt{2}} \left( \frac{\hbar}{2} \right) \langle + | (| + x \rangle - | - x \rangle) \rangle = 0 \end{aligned}$$

$$\begin{aligned} \langle - | S_x | - \rangle &= \langle - | \left( \frac{\hbar}{2} | + x \rangle \langle + x | - \frac{\hbar}{2} | - x \rangle \langle - x | \right) | - \rangle \\ &= \frac{1}{\sqrt{2}} \left( \frac{\hbar}{2} \right) \langle - | (| + x \rangle + | - x \rangle) \rangle = 0 \end{aligned}$$

- We see that the expectation value of  $S_x$  for the states  $|\pm\rangle$  is zero. We know from S-G exp. 2 that if we prepare a state in a eigenstate of one component of the angular momentum operator, then make a second measurement along an orthogonal direction, we obtain a 50%/50% probability of measuring spin up or down. Therefore, the average value of the second measurements will yield zero.

# Root-Mean-Square (RMS) Variation of Measurements

- The RMS value of a set of measurements of operator  $A$  is defined as

$$\Delta A = \sqrt{\langle (A - \langle A \rangle)^2 \rangle}$$

$$\begin{aligned}\Delta A &= \sqrt{\langle A^2 - 2A\langle A \rangle + \langle A \rangle^2 \rangle} \\ &= \sqrt{\langle A^2 \rangle - 2\langle A \rangle \langle A \rangle + \langle A \rangle^2} \\ \Delta A &= \sqrt{\langle A^2 \rangle - \langle A \rangle^2}\end{aligned}$$

where  $\langle A^2 \rangle$  = expectation value of  $A^2$ .

# Root-Mean-Square (RMS) Variation of Measurements

- Consider the RMS deviation for measurements of  $S_z$  for the  $S_z$  eigenkets  $|\pm\rangle$ .

$$\langle \pm | S_z^2 | \pm \rangle = \frac{\hbar^2}{4}$$

$$\Rightarrow \Delta S_z = \sqrt{\langle \pm | S_z^2 | \pm \rangle - (\langle \pm | S_z | \pm \rangle)^2} = 0$$

- Recall that  $\Delta S_z$  is the average distance between a particular measurement of  $S_z$  and the average value of  $S_z$  for a collection of measurements performed on the states  $|\pm\rangle$ . Because, each measurement yields the same result, either  $+\frac{\hbar}{2}$  or  $-\frac{\hbar}{2}$ , the variation between measurements is zero.

$$\langle + | S_z | + \rangle \pm \Delta S_z \Rightarrow \frac{\hbar}{2} \pm 0$$

$$\langle - | S_z | - \rangle \pm \Delta S_z \Rightarrow -\frac{\hbar}{2} \pm 0$$

# Root-Mean-Square (RMS) Variation of Measurements

- Consider the RMS deviation of the operators  $S_x$  and  $S_y$  for the states  $|\pm\rangle$ .
- We have found that  $\langle \pm | S_x | \pm \rangle = 0$ . Likewise, you can also show that  $\langle \pm | S_y | \pm \rangle = 0$ .
- To calculate  $\Delta S_x$  and  $\Delta S_y$ , we need to calculate  $\langle S_x^2 \rangle$  and  $\langle S_y^2 \rangle$ .
- There are at least two equivalent ways of calculating expectation values. Previously, I used the spectral decomposition of  $S_x$  to calculate  $\langle S_x \rangle$ . We can use this approach again to calculate  $S_x^2$ .

$$S_x = \frac{\hbar}{2} | +x \rangle \langle +x | - \frac{\hbar}{2} | -x \rangle \langle -x |$$

$$S_x^2 = \left( \frac{\hbar}{2} | +x \rangle \langle +x | - \frac{\hbar}{2} | -x \rangle \langle -x | \right) \left( \frac{\hbar}{2} | +x \rangle \langle +x | - \frac{\hbar}{2} | -x \rangle \langle -x | \right)$$

$$= \frac{\hbar^2}{4} | +x \rangle \langle +x | +x \rangle \langle +x | - \frac{\hbar^2}{4} | +x \rangle \langle +x | -x \rangle \langle -x | - \frac{\hbar^2}{4} | -x \rangle \langle -x | +x \rangle \langle +x | \\ + \frac{\hbar^2}{4} | -x \rangle \langle -x | -x \rangle \langle -x |$$

$$= \frac{\hbar^2}{4} (| +x \rangle \langle +x | + | -x \rangle \langle -x |) = \frac{\hbar^2}{4} \mathbb{1} \quad (\text{Completeness})$$

# Root-Mean-Square (RMS) Variation of Measurements

- We can also calculate  $S_x^2$  using the matrix representation of the operator

$$S_x^2 = \frac{\hbar^2}{4} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\hbar^2}{4} \mathbb{1}$$

- The expectation value  $\langle \pm | S_x^2 | \pm \rangle = \frac{\hbar^2}{4}$ .

$$\Delta S_x = \sqrt{\langle \pm | S_x^2 | \pm \rangle - (\langle \pm | S_x | \pm \rangle)^2} = \frac{\hbar}{2}$$

$$\langle + | S_x | + \rangle \pm \Delta S_x \Rightarrow 0 \pm \frac{\hbar}{2}$$

$$\langle - | S_x | - \rangle \pm \Delta S_x \Rightarrow 0 \pm \frac{\hbar}{2}$$

# Root-Mean-Square (RMS) Variation of Measurements

- We likewise have for the operator  $S_y$

$$S_y^2 = \frac{\hbar^2}{4} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\hbar^2}{4} \mathbb{1}$$

- The expectation value  $\langle \pm | S_y^2 | \pm \rangle = \frac{\hbar^2}{4}$ .

$$\Delta S_y = \sqrt{\langle \pm | S_y^2 | \pm \rangle - (\langle \pm | S_y | \pm \rangle)^2} = \frac{\hbar}{2}$$

$$\langle + | S_y | + \rangle \pm \Delta S_y \Rightarrow 0 \pm \frac{\hbar}{2}$$

$$\langle - | S_y | - \rangle \pm \Delta S_y \Rightarrow 0 \pm \frac{\hbar}{2}$$

- These set of results demonstrate something very basic and profound about QM. Namely, we cannot simultaneously determine measurement outcomes for two non-commuting observable with arbitrary accuracy. If we determine the state of one operator, e.g.,  $S_z$  with zero uncertainty, then a second measurement in an orthogonal direction, e.g.,  $S_x$  or  $S_y$  is uncertain. In the next lecture, we will derive a rigorous relationship that quantifies this uncertainty relationship.