Lecture 8: Expectation Value & RMS Deviation - Summary

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1 Lecture Overview

- **Topic:** State Tomography, Expectation Values, and Uncertainty (RMS Deviation).
- **Goal:** To quantify the "average" outcome of a quantum measurement and the "spread" or uncertainty associated with it.

2 1. Introduction to State Tomography

- A single measurement collapses the state, destroying the information we wanted to measure.
- To determine an unknown state $|\psi\rangle$, we need many identical copies.
- By measuring these copies repeatedly, we build a probability distribution from which we can reconstruct (tomography) the original state.

3 2. The Expectation Value

The expectation value $\langle \hat{A} \rangle$ is the theoretical average of many measurements.

3.1 Definition

It is defined as the sum of eigenvalues weighted by their probabilities. In the operator formalism:

$$\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle \tag{1}$$

This quantity is physically observable and independent of the basis used to represent the vectors.

4 3. RMS Deviation (Uncertainty)

While the expectation value gives the mean, the Root-Mean-Square (RMS) deviation gives the spread (standard deviation) of the measurement results.

4.1 Formula

$$\Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2} \tag{2}$$

Where $\langle A^2 \rangle$ is the expectation value of the operator squared ($\hat{A} \cdot \hat{A}$).

5 4. Example: Spin Measurements

The lecture analyzes the uncertainty relations using Spin-1/2 operators.

5.1 Properties of Squared Spin Operators

For spin-1/2 matrices, the square of any Pauli matrix is the Identity.

$$S_x^2 = S_y^2 = S_z^2 = \frac{\hbar^2}{4}\nVdash \tag{3}$$

Consequently, the expectation value of any squared spin component is always:

$$\langle S_i^2 \rangle = \frac{\hbar^2}{4} \tag{4}$$

5.2 Case Study: Measuring S_z vs S_x

Assume the system is in the state $|+\rangle$ (an eigenstate of S_z).

1. **Measuring S_z (The "Same" Direction):**
   - $\langle S_z \rangle = \frac{\hbar}{2}$
   - $\Delta S_z = \sqrt{\frac{\hbar^2}{4} - (\frac{\hbar}{2})^2} = 0$
   - **Result:** Zero uncertainty. The result is deterministic.
2. **Measuring S_x (The "Orthogonal" Direction):**
   - $\langle S_x \rangle = 0$ (Average of $+\hbar/2$ and $-\hbar/2$)
   - $\Delta S_x = \sqrt{\frac{\hbar^2}{4} - (0)^2} = \frac{\hbar}{2}$
   - **Result:** Maximum uncertainty. The result is completely random.

Midterm Formula Sheet

1. **Expectation Value (Operator Form)**

$$\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle \tag{5}$$

2. **Expectation Value (Summation Form)**

$$\langle \hat{A} \rangle = \sum_n a_n P_n \tag{6}$$

3. **RMS Deviation (Uncertainty)**

$$\Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2} \tag{7}$$

4. **Squared Spin Operator Identity**

$$S_i^2 = \frac{\hbar^2}{4} \mathbb{1} \quad \text{for } i = x, y, z \tag{8}$$

Core Takeaway

The Uncertainty Principle: We cannot simultaneously know the values of non-commuting observables.

- Precise knowledge of S_z ($\Delta S_z = 0$) forces maximum uncertainty in S_x and S_y ($\Delta S_{x,y} = \hbar/2$).