Lecture 9
The Uncertainty Principle

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Week of Feb. 2 2026

- We have seen that we cannot simultaneously determine measurement outcomes for two incompatible observable with arbitrary accuracy. If we determine the state of one operator, e.g., S_z with zero uncertainty, then a second measurement in an orthogonal direction, e.g., S_x or S_y is uncertain.

- In this lecture, we will present a general relationship that quantifies this uncertainty.

- We notice that if two observables are incompatible, then the order of measurements produces different outcomes. Recall for example SG-Experiment 3, where switching the order of the second and third x and z analyzers produced different outcomes.

- We know from everyday life that certain types of operations do depend on the order in which they are applied.
- Take for example 3D rotations.
- You may have noticed that if we rotate an object about two different axes, the final configuration of the object depends on the order of the rotations.

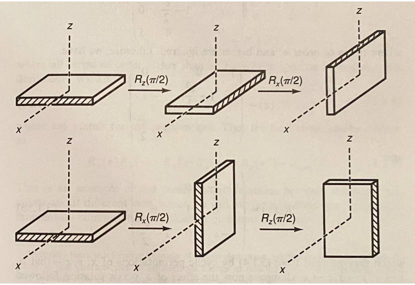


Figure: Modern Quantum Mechanics by J.J. Sakurai

- Consider a rectangular slab which we first rotate by 90° around the $z-$axis followed by a 90° rotation around the $x-$axis.
- Now, compare the final state of the object having first rotated it by 90° around the $x-$axis followed by a 90° rotation around the $z-$axis.

- If the order of operations of two operations produces different outcomes, we say that the operations do not commute.

- Clearly, if we were to rotate around the same axis twice in a row, then we would obtain the same result regardless of their order. For example, if we rotate around the $z-$axis first by 30° then by 60°, we would get the same result as having first rotated by 60° and then by 30°. Both cases produce a 90° rotation around the $z-$axis. Therefore, an operation always commutes with itself.

- We quantify the commutativity of operators $\hat{A}$ and $\hat{B}$ by calculating the commutator. Here, for example, $\hat{A}$ and $\hat{B}$ could refer to operators corresponding to rotations around two different axes.

$$\left[\hat{A}, \hat{B}\right] := \hat{A}\hat{B} - \hat{B}\hat{A} \tag{1}$$

where the square bracket $[\,]$ is the commutator.

- Notice that the commutator for the same operator is zero–$\left[\hat{A}, \hat{A}\right] = 0$.

- Let's calculate the commutation relations for the spin-$1/2$ operators.

$$
\begin{aligned}
[S_x, S_z] &= S_x S_z - S_z S_x \\
&= \frac{\hbar^2}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \frac{\hbar^2}{4}\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] \\
&= -\frac{\hbar^2}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -i\,\frac{\hbar^2}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
&= -i\hbar S_y
\end{aligned}
$$

- Notice $[A, B] = -[B, A]$. Therefore $[S_z, S_x] = i\hbar S_y$

Commutation Relations

- We can likewise derive the commutation relations for all combinations of the spin operators (S_x, S_y, S_z).

$$[S_x, S_y] = i\hbar S_z \tag{2}$$
$$[S_z, S_x] = i\hbar S_y \tag{3}$$
$$[S_y, S_z] = i\hbar S_x \tag{4}$$

- We will use commutation relationship between two operators to express an uncertainty relationship in the measurement outcomes for the two operators for a given state $|\psi\rangle$.

- The uncertainty principle is typically expressed as an inequality

$$\Delta A\,\Delta B \geq \frac{1}{2}|\langle[A,B]\rangle| \tag{5}$$

  where ΔA and ΔB refer the measurement uncertainty obtained for observables A and B for a particular state $|\psi\rangle$. $\langle[A,B]\rangle$ is the expectation value of the commutator $[A,B]$ calculated for the state $|\psi\rangle$.

- Let's see how it works by applying it to S-G Exp. 2.

The Uncertainty Principle in QM

- Suppose the state $|\psi\rangle = |+\rangle$ is prepared to be in an eigenstate of the operator S_z. We then perform a measurement of either S_x or S_y.
- If the particle is in the state $|+\rangle$, then the expectation value of S_z is $\langle +|S_z|+\rangle = \frac{\hbar}{2}$.
- Because the state $|+\rangle$ is an eigenstate of S_z, there is no uncertainty associated with the measurement of S_z, therefore $\Delta S_z = 0$.
- However, for the state $|+\rangle$, the observables S_x and S_y both have uncertainty of measurement outcomes, given by $\Delta S_x = \Delta S_y = \frac{\hbar}{2}$.

The Uncertainty Principle in QM

- Consider the commutation relation for the angular momentum operators

$$[S_x, S_y] = i\hbar S_z$$

- Eq.(5) provides a bound for the uncertainty in our knowledge of all three components of the angular momentum of a spin-1/2 system.

$$\Delta S_x \Delta S_y \geq \frac{\hbar}{2}|\langle S_z \rangle|$$

$$\Delta S_x \Delta S_y \geq \frac{\hbar^2}{4}$$

- $\Delta S_x \Delta S_y = \frac{\hbar^2}{4}$ is the minimum uncertainty allowed by QM. The uncertainty can be larger than this due to limitations in the measurement instrument.

The Uncertainty Principle in QM

- We see that we cannot simultaneously know all three components of the angular momentum. In fact, when we determine one its components, the other two become uncertain.

- Although we can only measure one of the components of $\vec{\mathbf{S}}$ with certainty, we can determine the magnitude of the vector.

$$S^2 = \vec{\mathbf{S}} \cdot \vec{\mathbf{S}} = S_x^2 + S_y^2 + S_z^2 \tag{6}$$

The Uncertainty Principle in QM

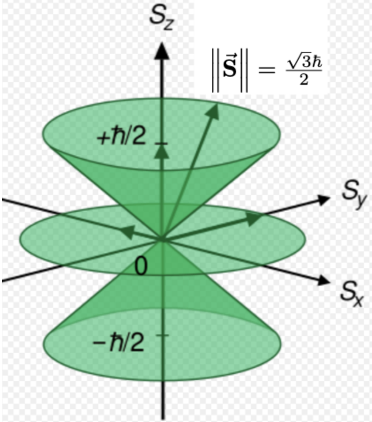
- We can show that the square of any component of the angular momentum operator is proportional to the identity operator.

$$S_i^2 = \frac{\hbar^2}{4}\mathbb{1}, \qquad i = \{x, y, z\}$$

$$\Rightarrow \quad S^2 = \frac{3\hbar^2}{4}\mathbb{1}$$

$$\therefore \quad \left\|\vec{\mathbf{S}}\right\| = \frac{\sqrt{3}\hbar}{2}\mathbb{1}$$

$$\left[S^2, S_i\right] = 0 \tag{7}$$



- The magnitude of the spin angular momentum is longer than its projection that can be measured. We cannot think about the spin-1/2 angular momentum as pointing in a particular direction.