AMATH 361

Continuum Mechanics

Lecture Notes

Winter 2026

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Lecture 1

1.1 Einstein Summation Convention

A vector in $\mathbb{R}^3$ can be written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = a_1\hat{\boldsymbol{e}}_1 + a_2\hat{\boldsymbol{e}}_2 + a_3\hat{\boldsymbol{e}}_3 = \sum_{i=1}^{3} a_i\hat{\boldsymbol{e}}_i, \tag{1.1}$$

where each a_i is a coordinate and $\hat{\boldsymbol{e}}_i$ are the basis vectors.

To further simply vector notation, we use Einstein summation convention.

Definition 1.1 *(Einstein Summation Convention).*

$$\sum a_i\hat{\boldsymbol{e}}_i = a_i\hat{\boldsymbol{e}}_i, \tag{1.2}$$

the summation sign is dropped whenever the same index repeated twice in a term.

Important Note

In the context of tensors and Einstein notation, "contracting" and "being summed over" are the same thing. Contracting means we are reducing the rank (the number of indices) of the object.

Difference bewteen contraction and Simplification:

- To sum (contract): You just need the index to appear twice. Order doesn't matter for the ability to sum.
- To substitute (simplify): You need one of the parts to be a Kronecker Delta (or an inner product of orthonormal basis vectors). In this case, the order matters because it determines which index in the final result gets replaced.

Note 1.2 *(Difference between Dummy and Free Index).* A **dummy index** is an index that is summed over; it may be renamed freely and has no independent meaning. It carries no positional information and does not denote the true order of an object. A **free index** is an index that is not summed; it labels a specific component or equation and cannot be renamed arbitrarily. It therefore carries positional information.

In Cartesian coordinates, we do not differentiate between superscripts and subscripts because vectors and co-vectors are equivalent.

1.2 Tensor

Tensor is an object that transforms in a particular way.

1.2.1 Vectors And Covectors

Definition 1.3 *(Contravariant).* Vectors (column vectors) are denoted with an upper index v^j. They are contravariant.

Definition 1.4 *(Covariant).* Covectors (row vectors) are denoted with a lower index v_j. They are covariant.

The following two subsections are in 3-dimensional Cartesian space.

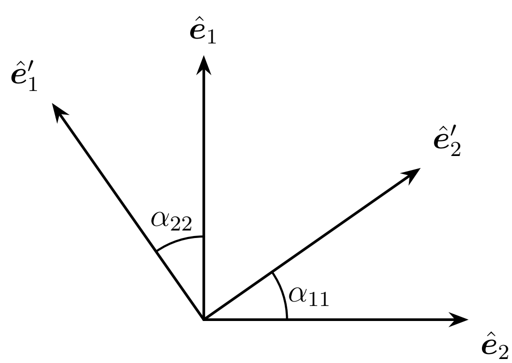
1.2.2 Describing the Same Object with Different Basis (Rotated with Original)

Definition 1.5 *(Inner Product).* The inner product of two vectors is defined as

$$(\boldsymbol{a}, \boldsymbol{b}) = \|\boldsymbol{a}\| \, \|\boldsymbol{b}\| \cos\theta, \tag{1.3}$$

where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. The inner product measures how much one vector points in the other vector's direction.

Consider a rotation of $\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3$ (orthonormal basis) in a horizontal plane,



where α_{ij} is the angle between $\hat{\boldsymbol{e}}_i$ and $\hat{\boldsymbol{e}}'_j$. The non-primed bases span $\mathbb{R}^3$, so

$$\hat{\boldsymbol{e}}'_j = a_k \hat{\boldsymbol{e}}_k \tag{1.4}$$

with a summation over the index k. For each coordinate in eq. (1.4), take the inner product with the i-th original basis from the representation summed over k.

$$(\hat{\boldsymbol{e}}'_j, \hat{\boldsymbol{e}}_i) = (a_k \hat{\boldsymbol{e}}_k, \hat{\boldsymbol{e}}_i) = a_k (\hat{\boldsymbol{e}}_k, \hat{\boldsymbol{e}}_i) = a_k \delta_{ki} = a_i \tag{1.5}$$

Fact 1.6 *(Vector Basis Identity).* Any vectors can be reconstructed by summing its projecting onto basis vectors.

$$\boldsymbol{v} = (\boldsymbol{v}, \hat{\boldsymbol{e}}_k)\, \hat{\boldsymbol{e}}_k. \tag{1.6}$$

Using definition 1.5 and fact 1.6, eq. (1.4) can be rewritten as

$$\hat{\boldsymbol{e}}'_j = (\hat{\boldsymbol{e}}'_j, \hat{\boldsymbol{e}}_k)\, \hat{\boldsymbol{e}}_k \tag{1.7}$$

$$= \cos\alpha_{kj}\, \hat{\boldsymbol{e}}_k. \tag{1.8}$$

with a summation over k.

Index Contraction & The Hidden Transpose

Algebra vs. Structure (The Slots)

The slot of the dummy index determines whether a row or column is contracted. While the summation is algebraically identical in $C_{ij}v_j$ and $C_{ji}v_j$, the structure implies different operations:

- Summing on **Slot 2** (j in C_{ij}) $\to$ Standard Matrix Mult ($[C]$).
- Summing on **Slot 1** (i in C_{ij}) $\to$ Transpose Mult ($[C]^T$).

The "Hidden" Transpose Mechanism

You do **not** need to swap indices to C_{ji} to invert the transformation. The summation index itself acts as the operator.

- **Forward (t_i):** $t_i = C_{ij}t'_j$. Summing on Slot 2 moves across rows (Old $\to$ New).
- **Reverse (t'_j):** $t'_j = C_{ij}t_i$. Summing on Slot 1 moves down columns (New $\to$ Old).

Crucial Note: C_{ij} and C_{ji} are different values. To perform the inverse (transpose) operation, keep the symbol fixed as C_{ij} and simply **move the summation to the first slot**.

1.2.3 Describing the Same Object with Different Basis (Original in Rotated)

Similarly, define

$$\hat{\boldsymbol{e}}_j = b_k \hat{\boldsymbol{e}}'_k \tag{1.9}$$

and project $\hat{\boldsymbol{e}}_j$ onto each $\hat{\boldsymbol{e}}'_i$ to find the coordinate.

$$(\hat{\boldsymbol{e}}_j, \hat{\boldsymbol{e}}'_i) = (b_k \hat{\boldsymbol{e}}'_k, \hat{\boldsymbol{e}}'_i) = b_k \delta_{ki} = b_i. \tag{1.10}$$

Then eq. (1.9) becomes

$$\begin{aligned} \hat{\boldsymbol{e}}_j &= (\hat{\boldsymbol{e}}_j, \hat{\boldsymbol{e}}'_k)\, \hat{\boldsymbol{e}}'_k \\ &= \cos\alpha_{jk}\, \hat{\boldsymbol{e}}'_k. \end{aligned} \tag{1.11}$$

with a summation over k. Eq. (1.8) and eq. (1.11) are dependent, renaming the index, we get

$$\hat{\boldsymbol{e}}_j = \cos\alpha_{jk} \cos\alpha_{\ell k} \hat{\boldsymbol{e}}_\ell. \tag{1.12}$$

Note that j is a free index, so eq. (1.12) is a vector equation. Since $\hat{\boldsymbol{e}}_j$ and $\hat{\boldsymbol{e}}_k$ are the same basis,

$$\cos\alpha_{jk} \cos\alpha_{\ell k} = \delta_{j\ell}. \tag{1.13}$$

We define $C_{jk} = \cos\alpha_{jk}$, $C_{\ell k} = \cos\alpha_{\ell k}$ because each C_{jk} and $C_{\ell k}$ come from the inner product and make up the same matrix C. Then

$$C_{jk} C_{\ell k} = (CC^T)_{j\ell} = \delta_{j\ell}. \tag{1.14}$$

Hence

$$CC^T = I, \tag{1.15}$$

where I is the identity matrix and C is an orthogonal matrix.

Operation Type	**Index Notation**	**Matrix Notation**	**Notes**
Matrix-Vector	$A_{ik}v_k$	$\boldsymbol{Av}$	Summing over columns of A.
Vector-Matrix	$v_i A_{ij}$	$\boldsymbol{v}^T\boldsymbol{A}$	Summing over rows of A.
Standard Product	$A_{ik}B_{kj}$	$\boldsymbol{AB}$	Summing ”inner” indices (cols of A, rows of B).
Row-Row Product	$A_{ik}B_{jk}$	$\boldsymbol{AB}^T$	Both use the 2nd index; equivalent to A times B^T.
Col-Col Product	$A_{ki}B_{kj}$	$\boldsymbol{A}^T\boldsymbol{B}$	Both use the 1st index; equivalent to A^T times B.
Inner Product	$u_i v_i$	$\boldsymbol{u} \cdot \boldsymbol{v}$	Results in a scalar (Rank 0).
Transpose	A_{ji}	$\boldsymbol{A}^T$	Simply swapping the index ”slots”.

Table 1: Tensor Index vs. Matrix Notation Summary

1.2.4 Summary

A vector $\boldsymbol{x}$ in $\mathbb{R}^3$ can be expressed with different bases

$$\boldsymbol{x} = x_k \hat{\boldsymbol{e}}_k = x'_k \hat{\boldsymbol{e}}'_k. \tag{1.16}$$

Project this vector onto $\hat{\boldsymbol{e}}_l$

$$\begin{aligned} (x_k \hat{\boldsymbol{e}}_k, \hat{\boldsymbol{e}}_\ell) &= (x'_k \hat{\boldsymbol{e}}'_k, \hat{\boldsymbol{e}}_\ell) && (1.17) \\ x_k \delta_{kl} &= x'_k \cos \alpha_{lk} && (1.18) \\ \boldsymbol{x} &= C \boldsymbol{x}'. && (1.19) \end{aligned}$$

Project this vector onto $\hat{\boldsymbol{e}}'_l$

$$\begin{aligned} (x_k \hat{\boldsymbol{e}}_k, \hat{\boldsymbol{e}}'_\ell) &= (x'_k \hat{\boldsymbol{e}}'_k, \hat{\boldsymbol{e}}'_\ell) && (1.20) \\ x_k \cos \alpha_{kl} &= x'_k \delta_{lk} && (1.21) \\ C^T \boldsymbol{x} &= \boldsymbol{x}'. && (1.22) \end{aligned}$$

Alternatively,

$$\begin{aligned} \boldsymbol{x} &= C \boldsymbol{x}' && (1.23) \\ C^T \boldsymbol{x} &= C^T C \boldsymbol{x}' && (1.24) \\ C^T \boldsymbol{x} &= \boldsymbol{x}' \quad (\text{since } CC^T = I). && (1.25) \end{aligned}$$

Definition 1.7 *(Order-1 Tensor).* An order-1 tensor is an object that obeys the transformation law $\boldsymbol{x}' = C^T \boldsymbol{x}$, where C^T is the coordinate transformation matrix. In other words, the object is only a tensor if the predicted object with the transformation law aligns with how it actually transforms.

The coordinate transformation matrix is the rotation matrix in 3-dimensional Cartesian space. *This is the default we assume for this course and C is always the Cartesian rotation matrix.*

Lecture 2

2.1 First-Order Tensor

$\boldsymbol{x}$ is a first-order tensor if given to a matrix $C_{jk} = \cos\alpha_{jk}$, then

$$\boldsymbol{x} = C\boldsymbol{x}', \quad \boldsymbol{x}' = C^T\boldsymbol{x}. \tag{2.1}$$

Example 2.1. The position of an object is a first-order tensor.

Example 2.2. Show that velocity is a first-order tensor.
Let $\boldsymbol{x}$ and $\boldsymbol{x}'$ be the positions, then the respective velocities are

$$\boldsymbol{v} = \frac{d\boldsymbol{x}}{dt}, \quad \boldsymbol{v}' = \frac{d\boldsymbol{x}'}{dt}. \tag{2.2}$$

By definition,

$$\boldsymbol{x} = C\boldsymbol{x}' \tag{2.3}$$
$$\frac{d\boldsymbol{x}}{dt} = C\frac{d\boldsymbol{x}'}{dt} \tag{2.4}$$
$$\boldsymbol{v} = C\boldsymbol{v}'. \tag{2.5}$$

Similarly

$$\boldsymbol{x}' = C^T\boldsymbol{x} \tag{2.6}$$
$$\frac{d\boldsymbol{x}'}{dt} = C^T\frac{d\boldsymbol{x}}{dt} \tag{2.7}$$
$$\boldsymbol{v}' = C^T\boldsymbol{v}. \tag{2.8}$$

This method applies to any order of derivative.

Definition 2.3 *(Order-0 Tensor).* A zeroth-order tensor is one that has a single real number that does not change under rotation.

Example 2.4. Mass is a zeroth-order tensor.

Example 2.5. Let $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, any element of a first order tensor is not a zeroth-order tensor.

Definition 2.6 *(Isotropic Tensor).* A tensor whose components are invariant with respect to rotation is isotropic.

2.2 Transformation of Second-Order Tensor

Definition 2.7 *(Second-Order Tensor).* A second-order tensor is a matrix T_{ij} that transforms in a particular way.

In first order, we have

$$x_i = C_{ik}x'_k, \quad x'_i = C_{ki}x_k. \tag{2.9}$$

Not that $C_{ki} = C_{ik}^T$. Then

$$T_{ij} = C_{ik}C_{jl}T'_{kl}, \tag{2.10}$$
$$T'_{ij} = C_{ik}C_{jl}T_{kl}. \tag{2.11}$$

So the rotation happens twice.

Note 2.8 *(Third-Order Tensor).* A third-order tensor satisfies

$$T_{ijk} = C_{il}C_{jm}C_{kn}T'_{lmn}. \tag{2.12}$$

A second-order tensor in matrix form is

$$T_{ij} = C_{ik}T'_{kl}C_{jl} = C_{ik}T'_{kl}C_{lj}^T \tag{2.13}$$

or equivalently

$$T = CT'C^T, \tag{2.14}$$

and

$$T' = C^TTC. \tag{2.15}$$

Definition 2.9. The Levi-Civita symbol is defined as

$$\varepsilon_{ijk} = \begin{cases} 1 & ijk = 123, 231, 312 \\ -1 & ijk = 321, 213, 132 \\ 0 & \text{else.} \end{cases} \tag{2.16}$$

Theorem 2.10 *(Levi-Civita contraction identity).* The Levi-Civita contraction identity is

$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}. \tag{2.17}$$

2.3 Symmetric & Anti-Symmetric Tensor

Definition 2.11 *(Symmetric Tensor).* A tensor T_{ij} is symmetric if

$$T_{ij} = T_{ji}. \tag{2.18}$$

Definition 2.12 *(Anti-Symmetric Tensor).* A tensor T_{ij} is anti-symmetric if

$$T_{ij} = -T_{ji}. \tag{2.19}$$

Any second-order tensor T_{ij} can be decomposed into symmetric and anti-symmetric parts:

$$T_{ij} = \frac{1}{2}T_{ij} + \frac{1}{2}T_{ij} \tag{2.20}$$

$$= \frac{1}{2}(T_{ij} + T_{ji}) + \frac{1}{2}(T_{ij} - T_{ji}). \tag{2.21}$$

We define

$$T_{(ij)} = \frac{1}{2}(T_{ij} + T_{ji}), \tag{2.22}$$

$$T_{[ij]} = \frac{1}{2}(T_{ij} - T_{ji}), \tag{2.23}$$

where $T_{(ij)}$ is the symmetric part and $T_{[ij]}$ is the anti-symmetric part.

Note 2.13. The symmetric tensor $T_{(ij)}$ has 6 free elements, and the anti-symmetric tensor $T_{[ij]}$ has 3 free elements.

2.4 Tensor Products and Contraction

Definition 2.14 *(Contracted Tensor Product (Matrix Product)).* The contracted tensor product is defined as

$$S_{ik}T_{kj} = A_{ij}, \quad ST = A. \tag{2.24}$$

Definition 2.15 *(Tensor Inner Product).* The tensor inner product is defined as

$$S_{ij}T_{ji} = a, \tag{2.25}$$

where a is a scalar.

Definition 2.16 *(Tensor Product (Dyadic Product)).* The tensor (dyadic) product of two vectors is defined as

$$T_{ij} = u_i v_j. \tag{2.26}$$

In matrix form,

$$T = \boldsymbol{u}\boldsymbol{v}^T \neq \boldsymbol{v}^T\boldsymbol{u} \neq \boldsymbol{v}\boldsymbol{u}^T, \tag{2.27}$$

where the order of multiplication matters.

Definition 2.17 *(Contraction).* Given a second-order tensor T_{ij}, its contraction is defined as

$$\mathrm{tr}(T) = T_{ii}, \tag{2.28}$$

which is a scalar and isotropic.

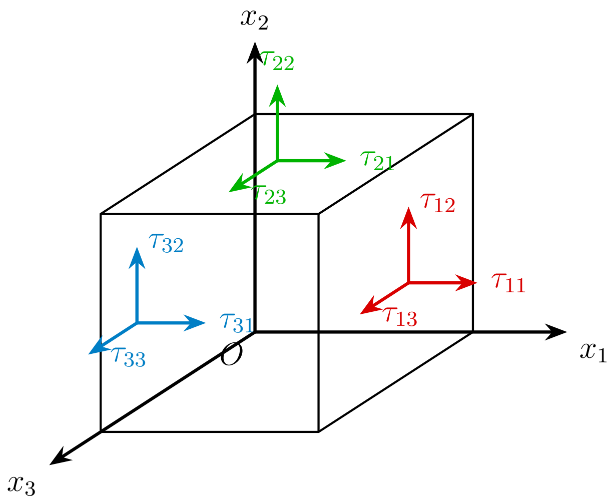
Note 2.18. The stress tensor T_{ij} will be defined later.

Surface tensor force: let $\hat{\boldsymbol{n}}$ be the unit outward normal to the surface. Then the stress (vector) is

$$t_i = n_j T_{ji}, \quad \boldsymbol{t} = \hat{\boldsymbol{n}}^T T. \tag{2.29}$$

2.4.1 Geometry

τ_{ij} will yield the surface force in the i-th direction due to a unit normal in the j-th direction. τ_{1i} force is in different directions and corresponds to the first row of a matrix.



Example 2.19. Given

$$t_i = n_j \tau_{ji}, \tag{2.30}$$

and t_i, n_i are first-order tensors, show that τ_{ij} transforms like a second-order tensor.

Refined Understanding: Tensor Covariance

A physical relationship, such as $t_i = n_j \tau_{ji}$, must be **form-invariant** (covariant) across all coordinate systems.

- **Physical Law:** $t_i = n_j \tau_{ji}$ and $t'_k = n'_\ell \tau'_{\ell k}$ are the same "Truth" in different frames.
- **Tensor Transformation:** If t_i and n_i are first-order tensors, they obey:

$$t_i = C_{ki} t'_k \quad \text{and} \quad n_j = C_{\ell j} n'_\ell \tag{2.31}$$

- **The Result:** Substituting these into the physical law *forces* the coefficients τ_{ji} to transform as a 2nd-order tensor to maintain equality:

$$\tau_{ij} = C_{ik} C_{j\ell} \tau'_{k\ell} \tag{2.32}$$

Conclusion: Changing the frame (priming) is mathematically equivalent to rotating the coordinate system, leaving the physical law unchanged while transforming the components.

We have

$$t_i = n_j \tau_{ji}, \quad t'_i = n'_j \tau'_{ji}. \tag{2.33}$$

Assume $\boldsymbol{t}$ and $\hat{\boldsymbol{n}}$ are first-order tensors. Then

$$t_i = C_{ki} t'_k, \quad t'_i = C_{ik} t_k, \tag{2.34}$$

and

$$n_i = C_{ki} n'_k. \tag{2.35}$$

Claim:

$$\tau_{ij} = C_{ik} C_{jl} \tau'_{kl}. \tag{2.36}$$

Starting from

$$t_i = n_j \tau_{ji}, \tag{2.37}$$

substitute the transformation laws:

$$\begin{aligned} t_i &= C_{ki} t'_k \\ &= C_{ki} n'_j \tau'_{jk}. \end{aligned} \tag{2.38}$$

Using

$$n'_j = C_{mj} n_m, \tag{2.39}$$

we obtain

$$\begin{aligned} C_{ki} t'_k &= C_{mj} n_m \tau'_{jk} \\ &= n_m \left(C_{mj} \tau'_{jk} \right). \end{aligned} \tag{2.40}$$

Since the rotation matrix is the same,

$$C_{mi} C_{ik} = \delta_{mk}, \tag{2.41}$$

we arrive at

$$n_m \tau_{mi} = n_m C_{mj} C_{ik} \tau'_{jk}. \tag{2.42}$$

Because n_m is arbitrary,

$$\tau_{mi} = C_{mj} C_{ik} \tau'_{jk}. \tag{2.43}$$

Renaming dummy indices gives

$$\tau_{ij} = C_{ik} C_{jl} \tau'_{kl}. \tag{2.44}$$

Lecture 3

3.1 Introduction

3.1.1 What is Continuum Mechanics?

Continuum mechanics is the science of how matter deforms and flows at scaled much larger than the intermolecular distances.

3.1.2 Matter

Definition 3.1 *(Four States of Matter).* The four states of matter are:

- Solids: Molecules can vibrate but do not move freely (molecules are tightly packed).
- Liquids: Have a clear shape but they can deform to fill a container (molecules are not as close as in solids).
- Gases: Do not have a fixed shape or volume & expand to fill a container (molecules are further apart).
- Plasmas: High energy state, can be thought of as an electromagnetic fluid (liquid or gas).

Typically, matter can be roughly divided into fluids (liquids and gases).

Definition 3.2 *(Two Properties of Fluids).* The two properties are:

- They tend of fill the containers.
- They don't resist shearing and stretches.

Solids can resist shearing and stretching but tend to return to their original position before the disturbance.

Application of continuum mechanics include: climate & weather, aerodynamics, blood flow, lava flow, solar physics and etc.

3.2 The Continuum Hypothesis

3.2.1 The Continuum Hypothesis

Theorem 3.3 *(The Continuum Hypothesis).* Even though matter is made of discrete molecules, at larger scale it seems continuous. Doing so simplifies the equations for studying molecules. This an approximation that becomes more accurate the more molecules we include.

The continuum hypothesis is a justification where on large enough length scale, we are justified in treating an object as a continuum.

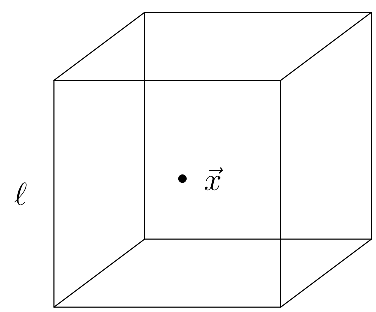
We use Newton's 2nd law and a conservation law to describe the properties of the continuum. For a fluid, we obtain the *Navier-Stokes equations (1822)*.

The following properties of matter will be discussed:

- $\rho(\boldsymbol{x}, t)$: mass density [kg/m^3],
- $\boldsymbol{u}(\boldsymbol{x}, t)$: velocity [m/s],
- $p(\boldsymbol{x}, t)$: pressure [N/m^2].

3.2.2 Continuum Approximation of Density

To build a methodology where we can define the density of a continuum, we consider a cube of length ℓ and center $\boldsymbol{x}$.



The average density over the cube can be defined as

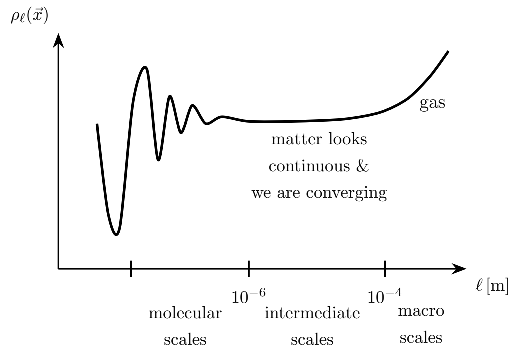
$$\rho_\ell(\boldsymbol{x}) = \frac{M_\ell(\boldsymbol{x})}{\ell^3}, \tag{3.1}$$

which depends on what ℓ we choose. If matter was *truly* continuous, we can take the limit as $\ell \to 0$ and obtain the density at $\boldsymbol{x}$.

Definition 3.4 *(The Continuum Approximation)*. The continuum approximation is

$$\rho(\boldsymbol{x}) = \lim_{\ell \to 0} \frac{M_\ell(\boldsymbol{x})}{\ell^3}. \tag{3.2}$$

Since Matter is not continuous, we define the average density to be definition 3.4 as ℓ gets smaller and starts to converge. We don't go any smaller because our expression will diverge.



The continuum approximation is valid in the intermediate scale and larger. The converging ℓ depends on the matter. Taking ℓ to be small, in the intermediate rate, can allow for convergence. We use this expression definition 3.4 as the average density at $\boldsymbol{x}$.

For the continuum hypothesis (approximation) to be valid, we need that the characterize length scale, L, of the system, is much larger than the mean free path (the average distance molecules are allowed to move), λ, of the particles ($L \gg \lambda$).

Example 3.5 *(λ of Air at Sea Level).* For air at sea level, $\lambda = 70\,\text{nm} \simeq 10^{-7}\,\text{nm}$. If $L \approx 1\,\text{mm}$, then

$$\frac{\lambda}{L} = \frac{10^{-7}\,\text{m}}{10^{-3}\,\text{m}} = 10^{-4} \ll 1. \tag{3.3}$$

If we need $\frac{\lambda}{L} \ll 10^{-2}$, then $L \gg 100 \times \lambda = 10^{-5}\,\text{m}$, or $10\,\mu\text{m}$. λ for liquids and solids are even smaller, so we need even larger L, the length of the system.

Assumption of Continuum Hypothesis

We assume the continuum hypothesis for the rest of the course.

3.3 Kinematics

Definition 3.6 *(Kinematics & Dynamics).* Kinematics is the study of motion without reference to forces. When we include forces, we have *dynamics*.

3.3.1 Fluid Parcel

We introduce the idea of "fluid parcel", which is the fluid at a point. Since we are assuming matter is continuous at our scales of interest, we can suppose we have well-defined properties:

- $\rho(\boldsymbol{x}, t)$: mass density,
- $\boldsymbol{u}(\boldsymbol{x}, t)$: velocity.

Definition 3.7 *(Pathlines)*. If $\boldsymbol{x}(t)$ is the position of a fluid parcel at time t, it must satisfy

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{u}(\boldsymbol{x}(t), t). \tag{3.4}$$

The solutions to the system of DEs are the trajectories and are called pathlines.

Example 3.8. Find the pathlines (trajectories) for the case where $\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{x}$.
The system of DEs becomes,

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{x} \quad \text{or} \quad \frac{dx}{dt} = x,\ \frac{dy}{dt} = y,\ \frac{dz}{dt} = z. \tag{3.5}$$

The solutions are

$$x = x_0 e^t \tag{3.6}$$

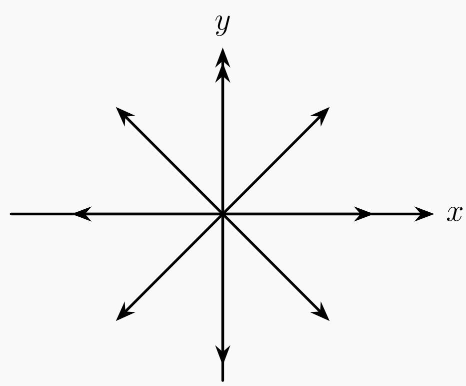
$$y = y_0 e^t \tag{3.7}$$

$$z = z_0 e^t, \tag{3.8}$$

or $\boldsymbol{x} = \boldsymbol{x}(0)e^t$, where $\boldsymbol{x}(0) = (x_0, y_0, z_0)$. Notice that

$$\frac{x}{y} = \frac{x_0 e^t}{y_0 e^t} = \text{constant}, \tag{3.9}$$

so the results are lines.



Example 3.9. Find the pathlines for $\boldsymbol{u} = \left(\frac{x}{t+\alpha}, x + \frac{y}{t+\alpha}\right)$ for $t \geq 0$, $\alpha > 0$.
The system of DEs becomes,

$$\frac{dx}{dt} = \frac{x}{t+\alpha}, \quad \frac{dy}{dt} = x + \frac{y}{t+\alpha}. \tag{3.10}$$

First we solve

$$\frac{dx}{dt} = \frac{x}{t+\alpha} \tag{3.11}$$

$$\int \frac{dx}{x} = \int \frac{dt}{t+\alpha} \tag{3.12}$$

$$\ln\left|\frac{x}{t+\alpha}\right| = C \tag{3.13}$$

$$\left|\frac{x}{t+\alpha}\right| = e^C. \tag{3.14}$$

Given that $t \geq 0$, $\alpha > 0$, if $x \geq 0$, then

$$x = (t+\alpha)e^C. \tag{3.15}$$

After some algebra, we can show that

$$y = \beta(t+\alpha) + e^C(t+\alpha)^2. \tag{3.16}$$

Lecture 4

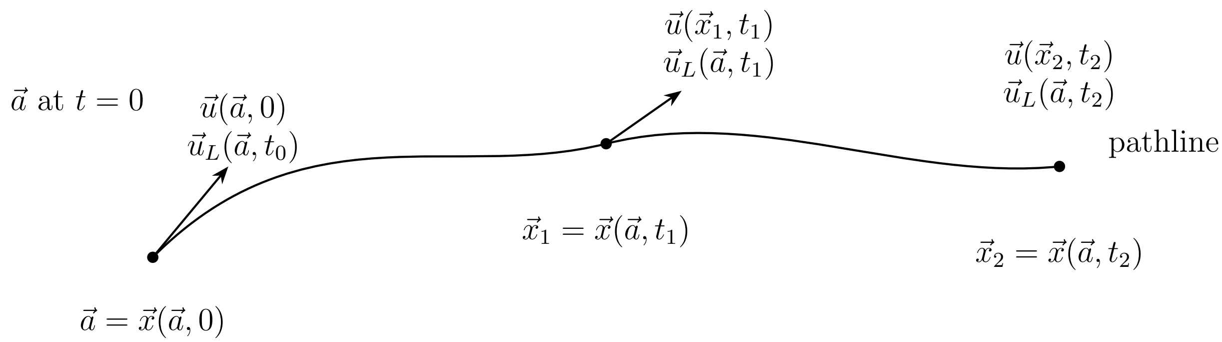
4.1 Eulerian & Lagrangian Descriptions of Flows

There are two ways to describe continuum (two difference frames of reference):

- Lagrangian: following the flow.
- Eulerian: watch the flow at a fixed position.

Recall the path lines (trajectories) solve the equation

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{u}(\boldsymbol{x}, t). \tag{4.1}$$



$\boldsymbol{x}(\boldsymbol{a}, t)$ is an example of a Lagrangian function.

Definition 4.1 *(Velocity at Time t in Lagrangian Frame).* We define

$$\boldsymbol{u}_L(\boldsymbol{a}, t) \tag{4.2}$$

to be the velocity at $\boldsymbol{x}(\boldsymbol{a}, t)$ at time t.

Initially, we are at $\boldsymbol{a}$. Note that the Lagrangian function is always a function of the initial position and time, we are assuming that the fluid parcel "remembers" the initial position.

Definition 4.2 *(Density at Time t in Lagrangian Frame).* We define

$$\rho_L(\boldsymbol{a}, t) \tag{4.3}$$

to be density of the parcel originally at $\boldsymbol{a}$ at time t.

When measuring a property of a fluid, we can *easily* measure at a fixed position (Eulerian). For a Lagrangian frame, we need to move with the fluid. The Eulerian descriptions of a flow is with respect to a fixed position.

Definition 4.3 *(Eulerian Velocity)*. The Eulerian velocity at $\boldsymbol{x}$ and time t is defined as

$$\boldsymbol{u}(\boldsymbol{x}, t). \tag{4.4}$$

Definition 4.4 *(Eulerian Density)*. The Eulerian density at $\boldsymbol{x}$ and time t is defined as

$$\rho(\boldsymbol{x}, t). \tag{4.5}$$

Difference between Lagrangian and Eulerian view

The Lagrangian view uses the initial position $\boldsymbol{a}$ to label each individual particle. In an Eulerian frame, we only care about what is happening at position $\boldsymbol{x}$, we don't need to know what a specific particle is doing. The $\boldsymbol{x}$ is the independent variable.

The confusion between Eulerian and Lagrangian frames often stems from the symbol $\boldsymbol{x}$ representing two fundamentally different mathematical objects depending on context.

- **In the Eulerian Frame ($\boldsymbol{x}$ is an Address):**
  Here, $\boldsymbol{x}$ is an *independent variable*. It represents a fixed coordinate in space (like a mile marker on a highway). It does not move.

  $$\frac{d}{dt}(\boldsymbol{x}_{\text{Eulerian}}) = 0 \tag{4.6}$$

  Therefore, you cannot find velocity by differentiating this $\boldsymbol{x}$. The field $\boldsymbol{u}(\boldsymbol{x}, t)$ records the velocity of the fluid *passing through* this fixed point, not the velocity of the point itself.
- **In the Lagrangian Frame ($\boldsymbol{x}$ is a Trajectory):**
  Here, $\boldsymbol{x}$ is a *dependent variable*, explicitly defined by the mapping $\boldsymbol{x} = \boldsymbol{\Phi}(\boldsymbol{a}, t)$. It tracks the history of a specific particle $\boldsymbol{a}$.

  $$\frac{d}{dt}(\boldsymbol{x}_{\text{Lagrangian}}) = \frac{\partial \boldsymbol{\Phi}}{\partial t} = \boldsymbol{u}(\boldsymbol{x}, t) \tag{4.7}$$

  This derivative gives the physical velocity of the particle.

Key Takeaway: We use the mapping $\boldsymbol{\Phi}$ to momentarily switch from the fixed Eulerian frame (where physics laws like $F = ma$ don't apply directly to space) to the Lagrangian frame (where we can track particles), apply the laws, and then translate the result back to the fixed grid.

4.1.1 Converting between Eulerian and Lagrangian

From Eulerian to Lagrangian:

- Assume we know $\boldsymbol{u}(\boldsymbol{x}, t)$. The pathlines satisfy

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{u}(\boldsymbol{x}, t) \tag{4.8}$$

  and solving this we will obtain an equation $\boldsymbol{x}(\boldsymbol{a}, t)$.
- Substitute the equation $\boldsymbol{x}(\boldsymbol{a}, t)$ into $\boldsymbol{u}(\boldsymbol{x}, t)$ to obtain the Lagrangian velocity

$$\boldsymbol{u}_L(\boldsymbol{a}, t) = \boldsymbol{u}(\boldsymbol{x}(\boldsymbol{a}, t), t). \tag{4.9}$$

- Similarly for the Lagrangian density, we obtain

$$\rho_L(\boldsymbol{a}, t) = \rho(\boldsymbol{x}(\boldsymbol{a}, t), t). \tag{4.10}$$

Another easier method to find the Lagrangian velocity is if we know $\boldsymbol{x}(\boldsymbol{a}, t)$, then

$$\boldsymbol{u}_L = \frac{\partial \boldsymbol{x}(\boldsymbol{a}, t)}{\partial t}. \tag{4.11}$$

From Lagrangian to Eulerian:

- Given $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{a}, t)$, if we can invert this $\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x}, t)$ (implicit function theorem) and given a Lagrangian field $\boldsymbol{u}_L(\boldsymbol{a}, t)$ and $\rho_L(\boldsymbol{a}, t)$.
- We substitute in to get the Lagrangian velocity

$$\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{u}_L(\boldsymbol{a}(\boldsymbol{x}, t), t) \tag{4.12}$$

  and the Lagrangian density

$$\rho(\boldsymbol{x}, t) = \rho_L(\boldsymbol{a}(\boldsymbol{x}, t), t). \tag{4.13}$$

4.2 Streamlines

Definition 4.5 *(Streamlines)*. A streamline is a curve $\boldsymbol{x}(s)$ that everywhere has the velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ as a tangent at a fixed time t. The equation of a streamline is

$$\frac{d\boldsymbol{x}(s)}{ds} = \boldsymbol{u}(\boldsymbol{x}(s), t). \tag{4.14}$$

t as Parameter

Note that the t here is a parameter, it's fixed.

This is similar to pathlines where

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{u}(\boldsymbol{x}(t), t). \tag{4.15}$$

If $\boldsymbol{u}$ is independent of time, then streamlines are pathlines. If $\boldsymbol{u}$ depends on time, in general they will differ. Pathlines are useful to see where the fluid goes. Streamlines are useful for the upcoming lectures.

Example 4.6. For the example from lecture 3, where $\boldsymbol{u} = \boldsymbol{x}$, find $\boldsymbol{x}(\boldsymbol{a}, t)$ and $\boldsymbol{a}(\boldsymbol{x}, t)$.

Example 4.7. Find $\boldsymbol{u}_L(\boldsymbol{a}, t)$ with method 1.

Example 4.8. Find $\boldsymbol{u}_L(\boldsymbol{a}, t)$ with method 2.

Example 4.9. Given pathlines $\boldsymbol{x}(t) = (a_1 e^{-t^2}, a_2 e^{t^2})$, sketch the pathlines, find $\boldsymbol{u}_L(\boldsymbol{a}, t)$ and find $\boldsymbol{u}(\boldsymbol{x}, t)$.

Example 4.10. Find the streamlines for

$$\boldsymbol{u}(\boldsymbol{x}, t) = \left(\frac{x}{t+\alpha}, x + \frac{y}{t+\alpha} \right) \tag{4.16}$$

for $t, \alpha, x \geq 0$.

Lecture 5

From classical mechanics, if we have N particles, each with position $\boldsymbol{X}_i(t)$, then the velocity is

$$\frac{d\boldsymbol{x}_i}{dt} \tag{5.1}$$

and the acceleration is

$$\frac{d^2\boldsymbol{x}_i}{dt^2}. \tag{5.2}$$

In a Lagrangian frame of reference, the position is denoted by

$$\boldsymbol{X}(\boldsymbol{a}, t) \tag{5.3}$$

then the velocity is

$$\boldsymbol{u}_L = \frac{\partial \boldsymbol{x}(\boldsymbol{a}, t)}{\partial t} \tag{5.4}$$

and the acceleration is

$$\boldsymbol{a}_L = \frac{\partial^2 \boldsymbol{x}(\boldsymbol{a}, t)}{\partial t^2}. \tag{5.5}$$

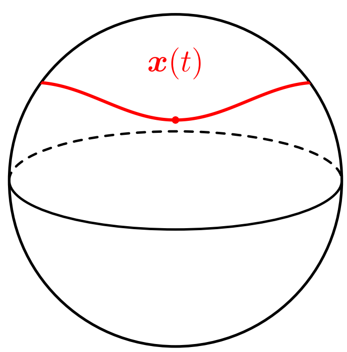
We can find the rate of change of a fluid property following in a Lagrangian frame (surfing frame) with

$$\frac{\partial \rho_L(\boldsymbol{a}, t)}{\partial t}. \tag{5.6}$$

We now study how to find the rate of change of an Eulerian property following the flow ($\rho(\boldsymbol{x}, t)$).

5.1 Material Derivative

Consider a volume of fluid, suppose we have a trajectory following a fluid parcel, $\boldsymbol{x}(\boldsymbol{a}, t)$, we determine how the fluid property $f(\boldsymbol{x}, t)$ changes along the flow.



Recall from lecture 4

$$f_L(\boldsymbol{a}, t) = f(\boldsymbol{x}(\boldsymbol{a}, t), t). \tag{5.7}$$

Or more explicitely,

$$f(x_1(\boldsymbol{a},t), x_2(\boldsymbol{a},t), x_3(\boldsymbol{a},t), t). \tag{5.8}$$

We already know that in the Lagrangian frame, the rate of change following the flow is

$$\frac{\partial f_L(\boldsymbol{a},t)}{\partial t}. \tag{5.9}$$

To find the Eulerian version, we apply $\partial/\partial t$ of the above equations and use the chain rule

$$\frac{\partial f_L(\boldsymbol{a},t)}{\partial t} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t} + \frac{\partial f}{\partial x_3}\frac{\partial x_3}{\partial t} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial t} \tag{5.10}$$

$$= \frac{\partial f}{\partial t} + \left(\frac{\partial x_1}{\partial t}, \frac{\partial x_2}{\partial t}, \frac{\partial x_3}{\partial t}\right) \cdot \boldsymbol{\nabla} f. \tag{5.11}$$

Since

$$\left(\frac{\partial x_1}{\partial t}, \frac{\partial x_2}{\partial t}, \frac{\partial x_3}{\partial t}\right) = \frac{\partial \boldsymbol{x}}{\partial t} = \boldsymbol{u}, \tag{5.12}$$

we obtain the material derivative.

The material derivative of an Eulerian property $f(\boldsymbol{x},t)$, is the change following the flow with the velocity, $\boldsymbol{u}(\boldsymbol{x},t)$.

Definition 5.1 *(Material Derivative).* The material derivative is defined as

$$\frac{Df}{Dt} = \frac{\partial f_L}{\partial t}(\boldsymbol{a},t) = \frac{\partial f}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} f. \tag{5.13}$$

Note that $f(\boldsymbol{x},t)$ and $\boldsymbol{u}(\boldsymbol{x},t)$ are Eulerian fields.

The material derivative have two parts

- $\frac{\partial f}{\partial t}$ is the rate of change of f at a fixed position w.r.t. time.
- The directional derivative is

$$\frac{\boldsymbol{u}}{\|u\|} \cdot \boldsymbol{\nabla} f. \tag{5.14}$$

  This is non-zero if

$$\boldsymbol{u} \cdot \boldsymbol{\nabla} f \neq 0. \tag{5.15}$$

  The second term in the material derivative is the speed times the directional derivative.

Example 5.2. Find the material derivatives of the following:

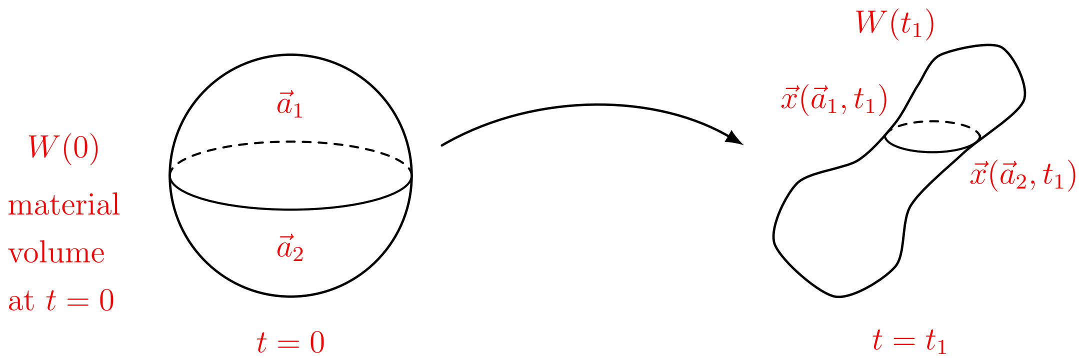
1. $\rho = \rho_0 + \Delta\rho_t t$.
2. $\rho = \rho_0 + \Delta\rho_x x$ with $\boldsymbol{u} = (\pm 1, 0, 0)$.

3. $\rho = \rho_0 + \Delta\rho_t t + \Delta\rho_x x$ with $\boldsymbol{u} = (\pm 1, 0, 0)$.
4. Find the acceleration of $\boldsymbol{u}(\boldsymbol{x}, t) = (x, yt^2, z + t)$.

Solutions:

1. We solve
$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}\rho = \Delta\rho_t. \tag{5.16}$$
2. We solve
$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}\rho = \pm\Delta\rho_x. \tag{5.17}$$
3. It's the superposition of the previous two parts
$$\pm\Delta\rho_x + \Delta\rho_t. \tag{5.18}$$
4. The acceleration is
$$\frac{D\boldsymbol{u}}{Dt} = \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u}, \tag{5.19}$$
where the second term is non-linear. It yields chaos and turbulence. The second term in index notation is
$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial}{\partial x_j} u_i. \tag{5.20}$$
Compute each term
$$\frac{D\boldsymbol{u}}{Dt} = \left(\frac{Du}{Dt}, \frac{Dv}{Dt}, \frac{Dw}{dt}\right). \tag{5.21}$$
Then
$$\frac{Du}{Dt} = \frac{\partial}{\partial t}x + x\frac{\partial}{\partial x}x + yt^2\frac{\partial}{\partial y}x + (z+t)\frac{\partial}{\partial z}x = x \tag{5.22}$$
$$\frac{Dv}{Dt} = \frac{\partial}{\partial t}yt^2 + x\frac{\partial}{\partial x}yt^2 + yt^2\frac{\partial}{\partial y}yt^2 + (z+t)\frac{\partial}{\partial z}yt^2 = 2yt + yt^4 \tag{5.23}$$
$$\frac{Dw}{Dt} = \frac{\partial}{\partial t}(z+t) + x\frac{\partial}{\partial x}(z+t) + yt^2\frac{\partial}{\partial y}(z+t) + (z+t)\frac{\partial}{\partial z}(z+t) \tag{5.24}$$
$$= 1 + z + t. \tag{5.25}$$
This is direct, we couldz also find the Lagrangian velocity first as we did in lecture 4.

5.2 Material Volumes



Definition 5.3 *(Material Volume)*. A material volume is a fixed collection of fluid parcels that can move with the flow. Note that it has all the same fluid parcels for all time.

If we want to find the rate of change of $f_L(\boldsymbol{a}, t)$, then we need to consider all $\boldsymbol{a} \in W(0)$, where $W(0)$ is the material volume at $t = 0$.

For an Eulerian field, $f(\boldsymbol{x}, t)$, we need to find $\boldsymbol{x}(\boldsymbol{a}, t)$, as the flow evolves. In some sense, the Lagrangian frame is easier because we follow along, whereas in Eulerian, the domain is changing.

If we want to find the total of a property in a given material volume in an Eulerian frame we compute

$$I(t) = \iiint_{W(t)} f(\boldsymbol{x}, t)\, dV. \tag{5.26}$$

The changes w.r.t. time are

$$\frac{dI}{dt} = \frac{d}{dt} \iiint_{W(t)} f(\boldsymbol{x}, t)\, dV. \tag{5.27}$$

The Issue with Bringing in d/dt

If the volume is free to change w.r.t. time, we cannot bring the derivative in the integral i.e., the RHS is in general is not

$$\iiint_{W(t)} \frac{\partial f}{\partial t}\, dV \tag{5.28}$$

since the volume can change.

However, using

$$\boldsymbol{x}(\boldsymbol{a}, t) \quad \text{and} \quad \boldsymbol{a}(\boldsymbol{x}, t), \tag{5.29}$$

we can transform between Eulerian and Lagrangian frames (lecture 6). The idea is to change the integral from Eulerian to Lagrangian, assuming we have the mapping. It is useful because in a Lagrangian frame, the bounds of integration are fixed (the initial condition stays the same). For example

$$\frac{d}{dt}\iiint_{W(0)} f_L(\boldsymbol{a},t)\,dV_a = \iiint_{W(0)} \frac{\partial f_L}{\partial t}(\boldsymbol{a},t)\,dV_a. \tag{5.30}$$

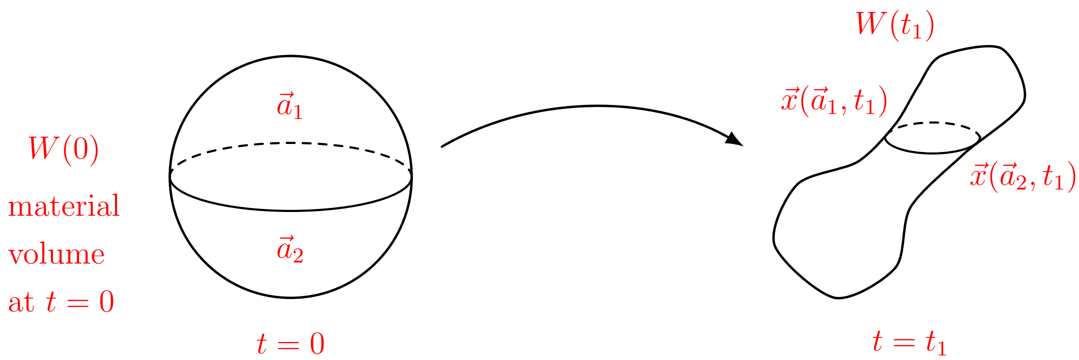
Lecture 6

6.1 Reynolds Transport Theorem

We define a mapping

$$\mathbf{\Phi}(\cdot, t) : \mathbb{R}^3 \to \mathbb{R}^3, \tag{6.1}$$

where $\mathbf{\Phi}(\boldsymbol{a}, t) = \boldsymbol{x}(\boldsymbol{a}, t)$.



The mapping allows us to "transform" the volume. Suppose that there also exists an inverse mapping $\mathbf{\Phi}^{-1}$ between the two. There are several assumptions we are making:

1. $\mathbf{\Phi}$ is invertible and one-to-one.
2. The mapping and its inverse are both C^2 with respect to their arguments.
3. We assume $W(0)$ is simply connected without any holes.

Assumption 1 yields that every $\boldsymbol{x}$ at t comes from a unique $\boldsymbol{a}$ at $t = 0$. If $W(0)$ is a volume, then $W(t) = \mathbf{\Phi}(W(0), t)$ is a material volume. $W(0)$ does not need to be the volume, for instance, if $W(0)$ is a surface, then $W(t) = \mathbf{\Phi}(W(0), t)$ is a material surface; if $W(0)$ is a curve, then $W(t) = \mathbf{\Phi}(W(0), t)$ is a material curve.

Question: We are trying to evaluate

$$\frac{d}{dt} \iiint_{W(t)} f(\boldsymbol{x}, t)\, dV. \tag{6.2}$$

Idea: We will change variables such that the domain of integration is $W(0)$.

6.1.1 Solving the Question in 1D

Consider the one-dimensional analogue to eq. (6.2):

$$\frac{d}{dt} \int_{\beta(t)}^{\alpha(t)} f(s, t)\, ds. \tag{6.3}$$

Using the Leibniz' rule, we obtain

$$\int_{\beta(t)}^{\alpha(t)} \frac{\partial f}{\partial t}\, ds + \frac{d\alpha}{dt} f(\alpha(t), t) - \frac{d\beta}{dt} f(\beta(t), t). \tag{6.4}$$

Leibniz's Rule

Leibniz's Rule (Differentiation under the Integral Sign) is used when you need to differentiate an integral whose limits are also functions of the variable you are differentiating by.

$$\frac{d}{dt}\int_{\beta(t)}^{\alpha(t)} f(x,t)dx = \int_{\beta(t)}^{\alpha(t)} \frac{\partial f}{\partial t} dx + f(\alpha(t),t)\frac{d\alpha}{dt} - f(\beta(t),t)\frac{d\beta}{dt} \tag{6.5}$$

Proof. Let $G(t,\alpha,\beta) = \int_\beta^\alpha f(x,t)dx$. We want to find the total derivative $\frac{dG}{dt}$ where α and β are functions of t. By the **Multivariable Chain Rule**:

$$\frac{dG}{dt} = \frac{\partial G}{\partial t}\frac{dt}{dt} + \frac{\partial G}{\partial \alpha}\frac{d\alpha}{dt} + \frac{\partial G}{\partial \beta}\frac{d\beta}{dt} \tag{6.6}$$

1. **The Partial w.r.t t:** Since the limits are treated as constants for a partial derivative, we can move the derivative inside:

$$\frac{\partial G}{\partial t} = \frac{\partial}{\partial t}\int_\beta^\alpha f(x,t)dx = \int_\beta^\alpha \frac{\partial f}{\partial t} dx \tag{6.7}$$

2. **The Partial w.r.t α:** By the **Fundamental Theorem of Calculus**, the derivative of an integral with respect to its upper limit is simply the integrand evaluated at that limit:

$$\frac{\partial G}{\partial \alpha} = \frac{\partial}{\partial \alpha}\int_\beta^\alpha f(x,t)dx = f(\alpha,t) \tag{6.8}$$

3. **The Partial w.r.t β:** Similarly, for the lower limit (using the property $\int_\beta^\alpha = -\int_\alpha^\beta$):

$$\frac{\partial G}{\partial \beta} = \frac{\partial}{\partial \beta}\int_\beta^\alpha f(x,t)dx = -f(\beta,t) \tag{6.9}$$

Substituting these three parts back into the Chain Rule expression:

$$\frac{dG}{dt} = \int_{\beta(t)}^{\alpha(t)} \frac{\partial f}{\partial t} dx + f(\alpha(t),t)\alpha'(t) - f(\beta(t),t)\beta'(t), \tag{6.10}$$

which completes the proof. □

Using fundamental theorem of calculus, we obtain

$$\int_{\beta(t)}^{\alpha(t)} \frac{df}{dt}\, ds + \int_{\beta(t)}^{\alpha(t)} \frac{\partial}{\partial s}\left(f(s,t)\frac{ds}{dt}\right)\, ds. \tag{6.11}$$

Putting them back together

$$\frac{d}{dt}\int_{\beta(t)}^{\alpha(t)} f(s,t)\, ds = \int_{\beta(t)}^{\alpha(t)} \left(\frac{\partial f}{\partial t} + \frac{\partial}{\partial s}\left(f(s,t)\frac{ds}{dt}\right)\right)\, ds. \tag{6.12}$$

If we consider $\frac{ds}{dt}$ to be the velocity, the three-dimensional version should be something similar to the divergence.

6.1.2 Solving the Question in 3D

We now change variables using

$$\boldsymbol{\Phi}(\boldsymbol{a},t) = \boldsymbol{x}(t). \tag{6.13}$$

Then

$$\iiint_{W(t)} f(\boldsymbol{x},t)\, dV = \iiint_{W(0)} f(\boldsymbol{\Phi}(\boldsymbol{a},t),t)\, J(\boldsymbol{a},t)\, dV_a, \tag{6.14}$$

where the volume element is $dV_a = da_1 da_2 da_3$ and the Jacobian is defined as

$$J(\boldsymbol{a},t) = \det\frac{\partial\boldsymbol{\Phi}}{\partial\boldsymbol{a}} = \det\frac{\partial(\Phi_1,\Phi_2,\Phi_3)}{\partial(a_1,a_2,a_3)} = \det\begin{vmatrix} \partial\Phi_1/\partial a_1 & \partial\Phi_1/\partial a_2 & \partial\Phi_1/\partial a_3 \\ \partial\Phi_2/\partial a_1 & \partial\Phi_2/\partial a_2 & \partial\Phi_2/\partial a_3 \\ \partial\Phi_3/\partial a_1 & \partial\Phi_3/\partial a_2 & \partial\Phi_3/\partial a_3 \end{vmatrix}. \tag{6.15}$$

Note 6.1 *(Jacobian)*. To understand the Jacobian determinant J, consider a change of variables where a region R in (u,v) coordinates is mapped to a region D in (x,y) coordinates. The Jacobian $J = \det\frac{\partial(x,y)}{\partial(u,v)}$ acts as the local scaling factor between the two spaces. Applying this to an area integral:

$$\iint_D 1\, dx\, dy = \iint_R \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\, du\, dv = \iint_R |J|\, du\, dv. \tag{6.16}$$

Physically, $|J|$ represents the ratio of the "real" area to the "map" area at a specific point.

- $|J| > 1$ (expansion) means the new coordinates stretch the area locally.
- $|J| < 1$ (compression) means the new coordinates squish the area locally.
- $J = 0$ (singularity) means the coordinate system collapses (loses a dimension).

Note 6.2 *(Calculation in n-Dimensions).* For a general coordinate transformation $\boldsymbol{T}: \mathbb{R}^n \to \mathbb{R}^n$ mapping inputs $(x_1, \ldots, x_n)$ to outputs $(y_1, \ldots, y_n)$, the Jacobian matrix $\boldsymbol{J}$ is the $n \times n$ matrix where the (i, j)-th entry is $\partial y_i / \partial x_j$:

$$\boldsymbol{J} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_n} \end{bmatrix}. \tag{6.17}$$

To find the volume scaling factor, calculate the determinant of this matrix:

$$dV' = |\det(\boldsymbol{J})| \; dx_1 \cdots dx_n. \tag{6.18}$$

Note 6.3 *(Jacobian Matrix and Determinant).* The "Jacobian" can refer to the matrix or its determinant.

- The Jacobian Matrix ($\boldsymbol{J}$): The fundamental linear map of partial derivatives $\frac{\partial(x,y)}{\partial(u,v)}$. It contains all vector information (rotation, shear, scaling).
- The Jacobian Determinant ($J = \det \boldsymbol{J}$): The scalar value used in integration.

Why the Determinant? Integrals sum up scalar quantities (volumes). Since the determinant measures the *change in volume* of the linear map defined by $\boldsymbol{J}$, it appears as the scaling factor in the Change of Variables formula:

$$dx \, dy = |\det \boldsymbol{J}| \; du \, dv \tag{6.19}$$

The second term (the integral over $W(0)$) is easier to compute because the initial conditions and domain are constant. To differentiate this integral, we need the following lemma regarding the Jacobian.

Lemma 6.4. If the mapping $\boldsymbol{\Phi}$ is invertible and C^2, then

1. $J > 0$ for all time.
2. $\frac{\partial J}{\partial t} = (\nabla \cdot \boldsymbol{u}) J$, where $\boldsymbol{u} = \frac{\partial \boldsymbol{x}}{\partial t}$ is the velocity.

Proof of 1. At $t = 0$, the mapping is the identity, so

$$J(0) = \det \frac{\partial \boldsymbol{\Phi}(\boldsymbol{a}, 0)}{\partial \boldsymbol{a}} = \det \frac{\partial \boldsymbol{a}}{\partial \boldsymbol{a}} = \det(I) = 1. \tag{6.20}$$

Since we are assuming $\boldsymbol{\Phi}$ to be invertible, $J \neq 0$. Furthermore, since $J(\boldsymbol{a}, t)$ is continuous (as $\boldsymbol{\Phi}$ is C^2) and $J(\boldsymbol{a}, 0) > 0$, it follows that $J > 0$ for all time. $\square$

Proof of 2. Let us define the gradient with respect to Lagrangian coordinates as

$$\nabla_a = (\partial/\partial a_1, \partial/\partial a_2, \partial/\partial a_3). \tag{6.21}$$

We can express the Jacobian as the determinant of column vectors:

$$J = \det[\nabla_a \Phi_1, \nabla_a \Phi_2, \nabla_a \Phi_3]^T. \tag{6.22}$$

Using the product rule for determinants, we differentiate term by term:

$$\begin{aligned}
\frac{\partial J}{\partial t} &= \frac{\partial}{\partial t} \det[\nabla_a \Phi_1, \nabla_a \Phi_2, \nabla_a \Phi_3]^T \\
&= \det\left[\frac{\partial}{\partial t}\nabla_a \Phi_1, \nabla_a \Phi_2, \nabla_a \Phi_3\right]^T \qquad \text{(I)} \\
&+ \det\left[\nabla_a \Phi_1, \frac{\partial}{\partial t}\nabla_a \Phi_2, \nabla_a \Phi_3\right]^T \qquad \text{(II)} \\
&+ \det\left[\nabla_a \Phi_1, \nabla_a \Phi_2, \frac{\partial}{\partial t}\nabla_a \Phi_3\right]^T. \qquad \text{(III)}
\end{aligned} \tag{6.23}$$

Consider an element in a column with a time derivative. Since operators commute for C^2 functions:

$$\frac{\partial}{\partial t}\frac{\partial \Phi_i}{\partial a_j} = \frac{\partial}{\partial a_j}\frac{\partial \Phi_i}{\partial t} = \frac{\partial}{\partial a_j}\frac{\partial x_i}{\partial t} = \frac{\partial}{\partial a_j} u_i(\boldsymbol{\Phi}(\boldsymbol{a}, t), t). \tag{6.24}$$

Using the chain rule, we can relate this back to spatial derivatives:

$$\frac{\partial u_i}{\partial a_j} = \frac{\partial u_i}{\partial x_k}\frac{\partial \Phi_k}{\partial a_j}. \tag{6.25}$$

This implies that the time derivative of the gradient vector is a linear combination of the gradients of the mapping:

$$\frac{\partial}{\partial t}\nabla_a \Phi_i = \begin{pmatrix} \dfrac{\partial u_i}{\partial x_k}\dfrac{\partial \Phi_k}{\partial a_1} \\ \dfrac{\partial u_i}{\partial x_k}\dfrac{\partial \Phi_k}{\partial a_2} \\ \dfrac{\partial u_i}{\partial x_k}\dfrac{\partial \Phi_k}{\partial a_3} \end{pmatrix} = \frac{\partial u_i}{\partial x_k}\nabla_a \Phi_k. \tag{6.26}$$

We substitute this back into eq. (6.23). For the first term (I) where $i = 1$:

$$\det\left[\frac{\partial u_1}{\partial x_k}\nabla_a \Phi_k, \nabla_a \Phi_2, \nabla_a \Phi_3\right]. \tag{6.27}$$

The summation over k includes terms involving $\nabla_a \Phi_2$ and $\nabla_a \Phi_3$. Since a determinant vanishes if columns are linearly dependent, only the $k = 1$ term survives:

$$\det\left[\frac{\partial u_1}{\partial x_1}\nabla_a \Phi_1, \nabla_a \Phi_2, \nabla_a \Phi_3\right] = \frac{\partial u_1}{\partial x_1}\det\left[\nabla_a \Phi_1, \nabla_a \Phi_2, \nabla_a \Phi_3\right] = \frac{\partial u_1}{\partial x_1} J. \tag{6.28}$$

Applying similar logic to terms (II) and (III), we obtain

$$\begin{aligned} \frac{\partial J}{\partial t} &= \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right) J && (6.29) \\ &= (\nabla \cdot \boldsymbol{u}) J, && (6.30) \end{aligned}$$

which completes the proof. □

Theorem 6.5 *(Reynolds Transport Theorem).* If $\boldsymbol{\Phi}$ is invertible and C^2, $\boldsymbol{u}$ is C^1, and $f(\boldsymbol{x}, t)$ is C^1, then

$$\frac{d}{dt}\iiint_{W(t)} f(\boldsymbol{x}, t)\, dV = \iiint_{W(t)} \left(\frac{Df}{Dt} + f \nabla \cdot \boldsymbol{u}\right) dV. \tag{6.31}$$

Proof. We begin by changing variables to the fixed reference domain $W(0)$:

$$\frac{d}{dt}\iiint_{W(t)} f(\boldsymbol{x}, t)\, dV = \frac{d}{dt}\iiint_{W(0)} f(\boldsymbol{x}(\boldsymbol{a}, t), t) J(\boldsymbol{a}, t)\, dV_a. \tag{6.32}$$

Since the domain $W(0)$ is fixed, we can move the time derivative inside the integral. Applying the product rule:

$$\begin{aligned} &= \iiint_{W(0)} \left(\left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_i}\frac{\partial \Phi_i}{\partial t}\right) J + f \frac{\partial J}{\partial t}\right) dV_a && (6.33) \\ &= \iiint_{W(0)} \left(\left(\frac{\partial f}{\partial t} + \boldsymbol{u} \cdot \nabla f\right) J + f (\nabla \cdot \boldsymbol{u}) J\right) dV_a. && (6.34) \end{aligned}$$

Recognizing the material derivative $\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \boldsymbol{u} \cdot \nabla f$, we have

$$= \iiint_{W(0)} \left(\frac{Df}{Dt} + f \nabla \cdot \boldsymbol{u}\right) J\, dV_a. \tag{6.35}$$

Finally, we transform the integral back to the current configuration $W(t)$ (using $J dV_a = dV$):

$$= \iiint_{W(t)} \left(\frac{Df}{Dt} + f \nabla \cdot \boldsymbol{u}\right) dV, \tag{6.36}$$

which completes the proof. $\square$

The integrand in the Reynolds Transport Theorem can be rewritten. Note that

$$\frac{Df}{Dt} + f\nabla \cdot \boldsymbol{u} = \frac{\partial f}{\partial t} + (\boldsymbol{u} \cdot \nabla)f + f\nabla \cdot \boldsymbol{u}. \tag{6.37}$$

Using tensor notation (or the product rule for divergence), we see that

$$\frac{\partial f}{\partial t} + u_i \frac{\partial f}{\partial x_i} + f\frac{\partial u_i}{\partial x_i} = \frac{\partial f}{\partial t} + \frac{\partial}{\partial x_i}(u_i f). \tag{6.38}$$

Thus,

$$\frac{Df}{Dt} + f\nabla \cdot \boldsymbol{u} = \frac{\partial f}{\partial t} + \nabla \cdot (f\boldsymbol{u}). \tag{6.39}$$

Then, the Reynolds Transport Theorem becomes

$$\frac{d}{dt}\iiint_{W(t)} f\,dV = \iiint_{W(t)} \left(\frac{Df}{Dt} + f\nabla \cdot \boldsymbol{u}\right) dV \tag{6.40}$$

$$= \iiint_{W(t)} \left(\frac{\partial f}{\partial t} + \nabla \cdot (f\boldsymbol{u})\right) dV. \tag{6.41}$$

By applying Gauss' Theorem (Divergence Theorem) to the second term, we obtain the conservation form:

$$\iiint_{W(t)} \frac{\partial f}{\partial t}\,dV + \iint_{\partial W(t)} f\boldsymbol{u} \cdot \hat{\boldsymbol{n}}\,dS. \tag{6.42}$$

Note 6.6 *(Significance of Reynolds Transport Theorem).* This theorem acts as a bridge between the Lagrangian and Eulerian descriptions. Physical laws (like Newton's Second Law or Conservation of Mass) apply to a specific system (a fixed collection of matter, $W(t)$), which corresponds to the left-hand side of the equation. However, in fluid mechanics, it is often easier to measure properties in a fixed region of space (a control volume). The right-hand side allows us to apply these fundamental physical laws to control volumes. By choosing specific quantities for f (e.g., density ρ or momentum $\rho\boldsymbol{u}$), we will use this theorem to derive the governing equations of fluid mechanics, such as the Continuity Equation and the Navier-Stokes Equations.

Lecture 8

8.1 Derivation of the Governing Equations

We can now use the Reynolds Transport Theorem to derive the fundamental governing equations of fluid mechanics. Recall that the theorem provides a bridge between the rate of change of a property in a material volume $W(t)$ and the Eulerian field representations:

$$\frac{d}{dt}\iiint_{W(t)} f(\boldsymbol{x},t)\,dV = \iiint_{W(t)}\left(\frac{Df}{Dt}+f\nabla\cdot\boldsymbol{u}\right)dV = \iiint_{W(t)}\left(\frac{\partial f}{\partial t}+\nabla\cdot(f\boldsymbol{u})\right)dV. \tag{8.1}$$

8.2 Conservation of Mass

The physical principle of mass conservation states that, in the absence of sources or sinks, the total mass M is conserved following the flow. We pick the property $f=\rho(\boldsymbol{x},t)$, the Eulerian density. Since the mass of a material volume is $M=\iiint_{W(t)}\rho\,dV$, conservation implies:

$$\frac{dM}{dt}=0 \quad \text{or} \quad \frac{d}{dt}\iiint_{W(t)}\rho\,dV \overset{R.T.}{=} \iiint_{W(t)}\left(\frac{D\rho}{Dt}+\rho\nabla\cdot\boldsymbol{u}\right)dV=0. \tag{8.2}$$

Since $W(t)$ is an arbitrary material volume, we apply the Dubois-Reymond lemma (the localization theorem), which states that if the integral over any arbitrary volume is zero, the integrand itself must be zero. This yields the first version of the continuity equation:

$$\frac{D\rho}{Dt}+\rho\nabla\cdot\boldsymbol{u}=0. \tag{8.3}$$

Using the identity $(\boldsymbol{u}\cdot\nabla)\rho+\rho(\nabla\cdot\boldsymbol{u})=\nabla\cdot(\rho\boldsymbol{u})$, we can rewrite eq. (8.3) in its most general conservation form:

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\boldsymbol{u})=0. \tag{8.4}$$

This is our first continuum equation, valid for describing gases, liquids, and plasmas.

Definition 8.1 *(Continuity Equation)*. The continuity equation takes the following forms:

$$\frac{D\rho}{Dt}+\rho\nabla\cdot\boldsymbol{u}=0 \tag{8.5}$$

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\boldsymbol{u})=0. \tag{8.6}$$

Note 8.2 *(Physical Interpretation).* To understand the role of divergence, return to R.T.T. but pick $f = 1$. Then $\iiint_{W(t)} 1\, dV$ is the volume of the material volume at time t. Applying the theorem:

$$\frac{d}{dt}\iiint_{W(t)} 1\, dV = \iiint_{W(t)} \left(\frac{D(1)}{Dt} + 1 \cdot \nabla \cdot \boldsymbol{u}\right) dV = \iiint_{W(t)} \nabla \cdot \boldsymbol{u}\, dV = \frac{dV}{dt}. \tag{8.7}$$

If volume is conserved (as in many liquids), $dV/dt = 0$, implying $\nabla \cdot \boldsymbol{u} = 0$.

- $\nabla \cdot \boldsymbol{u} > 0$ (divergence) means the density ρ decreases.
- $\nabla \cdot \boldsymbol{u} < 0$ (convergence) means the density ρ increases.

Theorem 8.3. If ρ, f, and $\boldsymbol{u}$ are C^1, then

$$\frac{d}{dt}\iiint_{W(t)} \rho f\, dV = \iiint_{W(t)} \rho \frac{Df}{Dt}\, dV. \tag{8.8}$$

Proof. Starting from the L.H.S. and applying the R.T.T.:

$$\frac{d}{dt}\iiint_{W(t)} \rho f\, dV = \iiint_{W(t)} \left(\frac{D}{Dt}(\rho f) + \rho f \nabla \cdot \boldsymbol{u}\right) dV \tag{8.9}$$

$$= \iiint_{W(t)} \left(\rho \frac{Df}{Dt} + f\frac{D\rho}{Dt} + f\rho\nabla \cdot \boldsymbol{u}\right) dV \tag{8.10}$$

$$= \iiint_{W(t)} \left(\rho \frac{Df}{Dt} + f \underbrace{\left(\frac{D\rho}{Dt} + \rho\nabla \cdot \boldsymbol{u}\right)}_{=0 \text{ by continuity}}\right) dV \tag{8.11}$$

$$= \iiint_{W(t)} \rho \frac{Df}{Dt}\, dV, \tag{8.12}$$

which completes the proof. □

8.3 Conservation of Linear Momentum

The "Conservation" vs. "Balance" Naming Quirk

In classical mechanics, "Conservation of Linear Momentum" usually refers to an isolated system where the net external force is zero, meaning the momentum doesn't change ($dP/dt = 0$).

In fluid mechanics, "Conservation of Linear Momentum refers to the **Momentum**

Balance equation, which is a direct application of Newton's Second Law.
Since we aren't looking at an isolated system but a specific "blob" of fluid called a material volume, $W(t)$. The rest of the universe is interacting with it. Because there are external forces acting on our volume, the net force is not zero, meaning the momentum must change.

For a point particle in classical mechanics, Newton's 2nd Law states:

$$\frac{d}{dt}\boldsymbol{p} = \boldsymbol{F}, \tag{8.13}$$

where $\boldsymbol{F}$ is the sum of all the forces and $\boldsymbol{p}$ is the linear momentum, often written as $\boldsymbol{p} = m\boldsymbol{u}$ (m is mass and $\boldsymbol{u}$ is velocity).

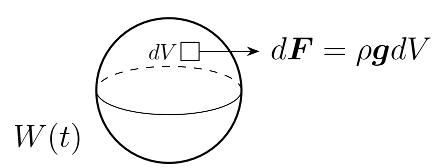
The physical principle for a continuum is that the total force on an object is equal to the rate of change of the linear momentum. For a continuum with density ρ, total force $\boldsymbol{F}_{\text{total}}$, and material volume $W(t)$, we get:

$$\frac{d}{dt}\iiint_{W(t)} \rho\boldsymbol{u}\, dV = \boldsymbol{F}_{\text{total}}. \tag{8.14}$$

This is Newton's 2nd law for a continuum, it is also our equation for conservation of linear momentum.

There are 3 types of forces:

1. **Volume (body) forces**: These act on the whole volume. Examples include gravity and Lorentz force.

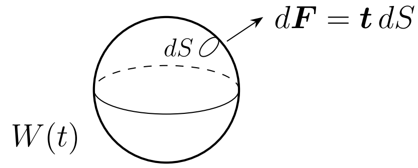


   If $\boldsymbol{g}$ is the acceleration due to gravity, the total gravitational force on $W(t)$ is:

$$\boldsymbol{F}_g = \iiint_{W(t)} \rho\boldsymbol{g}\, dV = -\iiint_{W(t)} \rho\nabla\Pi\, dV, \tag{8.15}$$

   since gravity is a conservative force, $\boldsymbol{g} = -\nabla\Pi$, where Π is the gravitational potential ($\Pi = gz$ as an example).

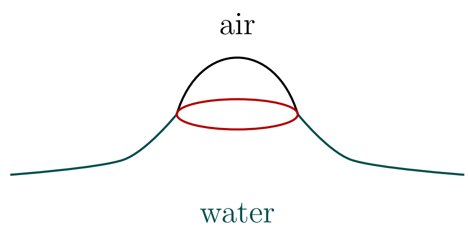
2. **Surface forces**: Matter outside of $W(t)$ exerts a force on $W(t)$. This force acts on $\partial W(t)$ (on the surface), for example, pressure.



If $\boldsymbol{t}(\boldsymbol{x}, t, \hat{\boldsymbol{n}})$, where $\hat{\boldsymbol{n}}$ is defined as the outward unit normal vector, is the stress vector. It has units of force/unit area. The total surface force is:

$$\boldsymbol{F}_s = \oiint_{\partial W(t)} \boldsymbol{t}(\boldsymbol{x}, t, \hat{\boldsymbol{n}})\, dS. \tag{8.16}$$

3. **Line (tensile) force**: These act on the interface between liquids and gases, for example, surface tension.



This can occur between two liquids that do not mix.

We include gravity (body force) & a general surface force in our expression for Newton's law.

$$\frac{d}{dt} \iiint_{W(t)} \rho \boldsymbol{u}\, dV = \boldsymbol{F}_g + \boldsymbol{F}_s = -\iiint_{W(t)} \rho \nabla \Pi\, dV + \oiint_{\partial W(t)} \boldsymbol{t}\, dS \tag{8.17}$$

Instead, consider

$$\boxed{\frac{d}{dt} \iiint_{W(t)} \rho u_i\, dV = -\iiint_{W(t)} \rho \frac{\partial \Pi}{\partial x_i}\, dV + \oiint_{\partial W(t)} t_i\, dS.} \tag{8.18}$$

Applying Reynolds Transport Theorem to the momentum balance (using the flux form $\nabla \cdot (\rho u_i \boldsymbol{u})$):

$$\iiint_{W(t)} \left(\frac{\partial}{\partial t}(\rho u_i) + \nabla_j(\rho u_i u_j) + \rho \frac{\partial \Pi}{\partial x_i} \right) dV = \oiint_{\partial W(t)} t_i\, dS. \tag{8.19}$$

Expanding the product rule for both the time derivative and the divergence term:

$$\iiint_{W(t)} \left(\underbrace{\rho \frac{\partial u_i}{\partial t} + u_i \frac{\partial \rho}{\partial t}}_{\text{time deriv.}} + \underbrace{u_i \nabla_j(\rho u_j) + \rho u_j \nabla_j u_i}_{\text{flux deriv.}} + \rho \frac{\partial \Pi}{\partial x_i} \right) dV = \oiint_{\partial W(t)} t_i\, dS. \tag{8.20}$$

Rearranging terms to isolate the material derivative of velocity ($\frac{Du_i}{Dt}$) and the continuity equation:

$$\iiint_{W(t)} \left(\rho \underbrace{\left(\frac{\partial u_i}{\partial t} + u_j \nabla_j u_i \right)}_{Du_i/Dt} + u_i \underbrace{\left(\frac{\partial \rho}{\partial t} + \nabla_j (\rho u_j) \right)}_{\text{continuity}=0} + \rho \frac{\partial \Pi}{\partial x_i} \right) dV = \oiint_{\partial W(t)} t_i \, dS. \tag{8.21}$$

Thus, since the continuity term is zero, we arrive at the final form for the conservation of linear momentum:

$$\iiint_{W(t)} \left(\rho \frac{Du_i}{Dt} + \rho \frac{\partial \Pi}{\partial x_i} \right) dV = \oiint_{\partial W(t)} t_i \, dS. \tag{8.22}$$

Alternatively, we apply the theorem 8.3 derived earlier:

$$\frac{d}{dt} \iiint_{W(t)} \rho f \, dV = \iiint_{W(t)} \rho \frac{Df}{Dt} \, dV. \tag{8.23}$$

Setting $f = u_i$ and substituting this directly into eq. (8.18), the time derivative term becomes:

$$\frac{d}{dt} \iiint_{W(t)} \rho u_i \, dV = \iiint_{W(t)} \rho \frac{Du_i}{Dt} \, dV. \tag{8.24}$$

Thus, we immediately arrive at the final form for the conservation of linear momentum:

$$\iiint_{W(t)} \left(\rho \frac{Du_i}{Dt} + \rho \frac{\partial \Pi}{\partial x_i} \right) dV = \oiint_{\partial W(t)} t_i \, dS. \tag{8.25}$$

We will rewrite the R.H.S. in the next lecture.

Lecture 9

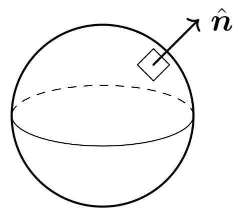
9.1 Surface Forces and Pressure

Last time, using Newton's 2nd Law and including gravity (volume force) and surface forces, we obtained the integral form of the momentum equation:

$$\iiint_{W(t)} \rho \frac{Du_i}{Dt}\, dV = -\iiint_{W(t)} \rho \frac{\partial \Pi}{\partial x_i}\, dV + \oiint_{\partial W(t)} t_i\, dS. \tag{9.1}$$

Today, we focus on the surface force & rewrite it as a triple integral over $W(t)$.

$W(t)$



$\partial W(t)$ = boundary

$d\boldsymbol{f} = \boldsymbol{t}\, ds$

On each subsurface, we have

$$d\boldsymbol{f} = \boldsymbol{t}\, ds \tag{9.2}$$

where $\boldsymbol{t}$ is the stress vector. Let's consider a state of equilibrium where there is no movement

$$\boldsymbol{u} = \vec{0}. \tag{9.3}$$

9.1.1 The Stress Vector in Equilibrium

Consider a state of equilibrium where there is no movement ($\boldsymbol{u} = \vec{0}$). Recall that fluids in equilibrium cannot resist shear forces (stresses). Since we have $\boldsymbol{u} = \vec{0}$, in equilibrium, the surface force must be parallel to the normal vector $\hat{\boldsymbol{n}}$. Then the stress tensor is

$$\boldsymbol{t} = -p(\boldsymbol{x})\hat{\boldsymbol{n}}, \tag{9.4}$$

where we call $p(\boldsymbol{x})$ the pressure. The negative sign indicates that pressure acts *inward* on the volume (compression).

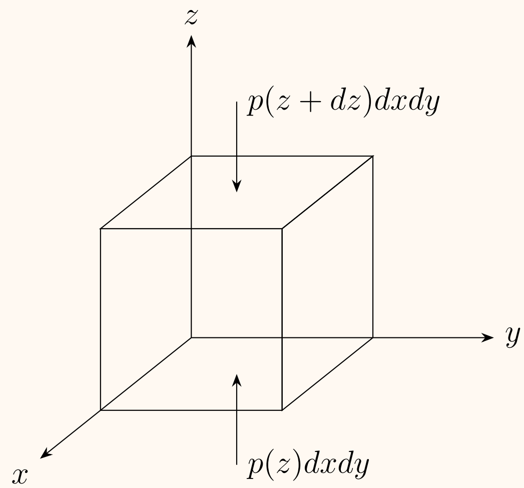
Our global expression for the surface force becomes

$$\boldsymbol{F}_s = \oiint_{\partial W(t)} -p\hat{\boldsymbol{n}}\, dS. \tag{9.5}$$

9.1.2 Derivation of Pressure Force

To rewrite this surface integral as a volume integral, we consider two approaches: a heuristic Taylor expansion and a formal proof using Gauss' Theorem.

Note 9.1 *(Heuristic Derivation using Taylor Expansion)*. Consider the pressure on a small material volume (a cube) with side lengths dx, dy, dz such that $dV = dx\,dy\,dz \ll 1$.



The units of $\boldsymbol{t}$ are $[\boldsymbol{t}] = \mathrm{N/m^2}$. We will look at the balance of forces in the z-direction. For the difference in this force from the two faces, we Taylor expand about z.

$$
\begin{aligned}
dF_z &= p(x, y, z)\,dxdy - p(x, y, z + dz)\,dxdy && (9.6)\\
&= (p(x, y, z) - p(x, y, z + dz))\,dxdy && (9.7)\\
&= (p(x, y, z) - [p(x, y, z) + dz\frac{\partial p}{\partial z}(x, y, z) + \dots])\,dxdy && (9.8)\\
&= -\frac{\partial p}{\partial z}(x, y, z)\,dxdydz && (9.9)\\
&= -\frac{\partial p}{\partial z}\,dV. && (9.10)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
dF_x &= -\frac{\partial p}{\partial x}(x, y, z)\,dxdydz && (9.11)\\
dF_y &= -\frac{\partial p}{\partial y}(x, y, z)\,dxdydz && (9.12)
\end{aligned}
$$

Combining these into a vector equation:

$$d\boldsymbol{F} = (dF_x, dF_y, dF_z) = -\left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}\right) dV = -\nabla p \, dV. \tag{9.13}$$

Integrating over the whole volume, the surface force can be written as:

$$\boldsymbol{F}_s = -\iiint_{W(t)} \nabla p \, dV. \tag{9.14}$$

This very rough calculation suggests the following identity.

Theorem 9.2 *(Gradient Theorem Corollary).*

$$\oiint_{\partial W(t)} p \, \hat{\boldsymbol{n}} \, dS = \iiint_{W(t)} \nabla p \, dV. \tag{9.15}$$

Proof. The idea of the proof is to write the identity in component form using Gauss' Divergence Theorem. Recall Gauss' Divergence Theorem states:

$$\iiint_W \nabla \cdot \boldsymbol{U} \, dV = \oiint_{\partial W} \boldsymbol{U} \cdot \hat{\boldsymbol{n}} \, dS. \tag{9.16}$$

We want to show:

$$\oiint_{\partial W(t)} p(n_x, n_y, n_z) \, dS = \iiint_{W(t)} \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}\right) dV. \tag{9.17}$$

This corresponds to 3 scalar equations.

For the x-direction: Pick the vector field $\boldsymbol{U} = (p, 0, 0)$. Then $\nabla \cdot \boldsymbol{U} = \frac{\partial p}{\partial x}$. Applying Gauss' Theorem:

$$\iiint_{W(t)} \frac{\partial p}{\partial x} \, dV = \oiint_{\partial W(t)} (p, 0, 0) \cdot \hat{\boldsymbol{n}} \, dS = \oiint_{\partial W(t)} p n_x \, dS. \tag{9.18}$$

For the y-direction: Pick $\boldsymbol{U} = (0, p, 0)$. Then $\nabla \cdot \boldsymbol{U} = \frac{\partial p}{\partial y}$.

$$\iiint_{W(t)} \frac{\partial p}{\partial y} \, dV = \oiint_{\partial W(t)} p n_y \, dS. \tag{9.19}$$

For the z-direction: Pick $\boldsymbol{U} = (0, 0, p)$. Then $\nabla \cdot \boldsymbol{U} = \frac{\partial p}{\partial z}$.

$$\iiint_{W(t)} \frac{\partial p}{\partial z}\, dV = \oiint_{\partial W(t)} p n_z \, dS. \tag{9.20}$$

Combining these 3 scalar equations into a vector equation yields the result. □

9.2 Hydrostatics

Given this identity, we return to Newton's 2nd Law in a state of rest ($\boldsymbol{u} = \vec{0}$). The acceleration term is zero.

$$0 = -\iiint_{W(t)} \rho \nabla \Pi \, dV - \oiint_{\partial W(t)} p \hat{\boldsymbol{n}} \, dS \tag{9.21}$$

$$= \iiint_{W(t)} \left[-\rho \nabla \Pi - \nabla p\right] dV \tag{9.22}$$

$$= -\iiint_{W(t)} \left[\rho \nabla \Pi + \nabla p\right] dV. \tag{9.23}$$

Since this holds for any arbitrary volume $W(t)$, we apply the Dubois-Reymond Lemma (localization) to get:

$$\rho \nabla \Pi + \nabla p = 0. \tag{9.24}$$

Assuming gravity acts in the vertical direction, $\Pi = gz$ and $\nabla \Pi = g\hat{\boldsymbol{z}}$.

Definition 9.3 *(Hydrostatic Balance).* The hydrostatic balance equation is

$$\nabla p = -\rho \nabla \Pi = -\rho g \hat{\boldsymbol{z}}, \tag{9.25}$$

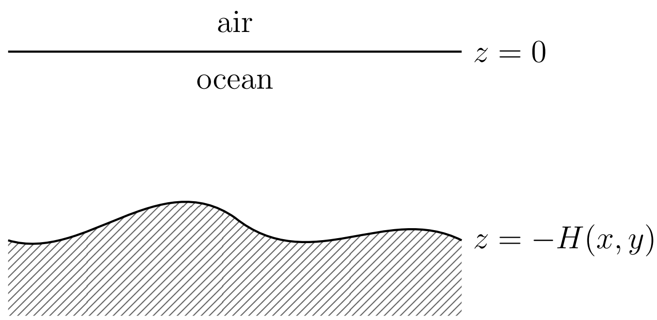
where $\Pi = gz$. In component form:

$$\frac{\partial p}{\partial x} = 0, \quad \frac{\partial p}{\partial y} = 0, \quad \frac{\partial p}{\partial z} = -\rho g. \tag{9.26}$$

This describes the perfect balance between the forces of gravity and pressure.

9.2.1 Hydrostatic Balance of the Ocean

Suppose we consider a simple ocean at rest ($\boldsymbol{u} = \boldsymbol{0}$), and the density is constant $\rho = \rho_0$ (incompressible). Since $\frac{\partial p}{\partial x} = 0$ and $\frac{\partial p}{\partial y} = 0$, pressure depends only on z, i.e., $p(z)$.



The vertical equation is:

$$\frac{\partial p}{\partial z} = -g\rho_0. \tag{9.27}$$

We integrate from depth z to the surface $z = 0$:

$$\int_z^0 \frac{dp}{dz}\, dz = \int_z^0 -g\rho_0\, dz \tag{9.28}$$

$$p(z)\Big|_z^0 = -g\rho_0 z\Big|_z^0 \tag{9.29}$$

$$p(0) - p(z) = 0 - (-g\rho_0 z) = g\rho_0 z \tag{9.30}$$

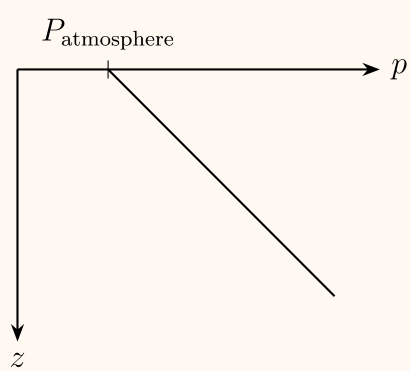
$$p(z) = p(0) - \rho_0 g z. \tag{9.31}$$

If we denote the atmospheric pressure as $P_{\text{atmosphere}} = p(0)$, then

$$p(z) = P_{\text{atmosphere}} - \rho_0 g z. \tag{9.32}$$

Note 9.4. Since z is negative underwater, $-\rho_0 g z$ is a positive term. The pressure at z is equal to the pressure of the atmosphere plus the weight of the fluid column above per unit area:

$$\frac{\rho_0 g(-z)dA}{dA} = \frac{mg}{dA}. \tag{9.33}$$



Example 9.5 *(Numerical Scale)*. It is observed that $P_{\text{atmosphere}} \approx 10^5\,\text{N/m}^2$. The depth of the ocean is $\leq 10\,\text{km} = 10^4\,\text{m}$. The density of water is $\rho_0 \approx 10^3\,\text{kg/m}^3$ and $g \approx 10\,\text{m/s}^2$.
At the top of the ocean: $p \approx 10^5\,\text{N/m}^2$.
At the bottom of the ocean:

$$p \approx 10^5\,\text{N/m}^2 + \left(10^3\frac{\text{kg}}{\text{m}^3}\right)\left(10\frac{\text{m}}{\text{s}^2}\right)\left(10^4\,\text{m}\right) \approx 10^8\,\text{N/m}^2. \tag{9.34}$$

The pressure at the bottom is 1000 times larger than at the surface.

9.2.2 Hydrostatic Balance for the Atmosphere

The density of air changes a lot (it is compressible). To describe air, you need an equation of state. One choice is the Ideal Gas Law:

$$p = \rho R T, \tag{9.35}$$

where

$$R \approx 287\,\frac{\text{J}}{\text{kg}\cdot\text{K}}. \tag{9.36}$$

The temperature is not constant, but if we assume it is for simplicity (say $T = T_0 = \text{const}$), we get simple equations. From the ideal gas law: $\rho = \frac{p}{RT_0}$. Substitute this into the vertical hydrostatic equation:

$$\frac{dp}{dz} = -g\rho = -\frac{gp}{RT_0}. \tag{9.37}$$

This is a separable ODE:

$$\frac{dp}{p} = -\frac{g}{RT_0}dz \tag{9.38}$$

$$\ln p = -\frac{gz}{RT_0} + C. \tag{9.39}$$

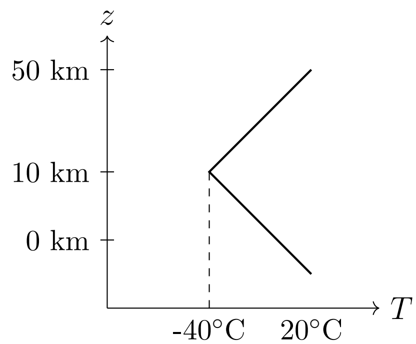
Solving for p:

$$p(z) = p(0)e^{-z/H}, \tag{9.40}$$

where

$$H \equiv \frac{RT_0}{g}. \tag{9.41}$$

H is called the **Scale Height**. For $T_0 \approx 20°\text{C}$, $H \approx 8.4\,\text{km}$.



z
50 km
10 km
0 km
-40°C
20°C
T

Lecture 10

10.1 Introduction

Today we will cover:

- Cauchy's Fundamental Theorem for the Stress Vector.
- Conservation of Linear Momentum.

Previously, we derived the hydrostatic balance equation, which describes a fluid with no motion ($\boldsymbol{u} = \mathbf{0}$). In that specific case, we proposed that the stress vector was normal to the surface:

$$\boldsymbol{t} = -p(\boldsymbol{x})\hat{\boldsymbol{n}}. \tag{10.1}$$

Today, we generalize this result for moving fluids.

10.2 Cauchy's Fundamental Theorem for the Stress Vector

We begin with the integral form of Newton's Second Law for a continuum, as derived previously:

$$\frac{d}{dt}\iiint_{W(t)} \rho\boldsymbol{u}\, dV = -\iiint_{W(t)} \rho\nabla\Pi\, dV + \oint\!\!\oint_{\partial W(t)} \boldsymbol{t}(\boldsymbol{x}, t, \hat{\boldsymbol{n}})\, dS. \tag{10.2}$$

Using the Reynolds Transport Theorem on the left-hand side, we can rewrite this as:

$$\iiint_{W(t)} \left[\rho\frac{D\boldsymbol{u}}{Dt} + \rho\nabla\Pi\right] dV = \oint\!\!\oint_{\partial W(t)} \boldsymbol{t}(\boldsymbol{x}, t, \hat{\boldsymbol{n}})\, dS. \tag{10.3}$$

Theorem 10.1 *(Cauchy's Fundamental Theorem for Stress).* The stress vector

$$\boldsymbol{t}(\boldsymbol{x}, t, \hat{\boldsymbol{n}}) \tag{10.4}$$

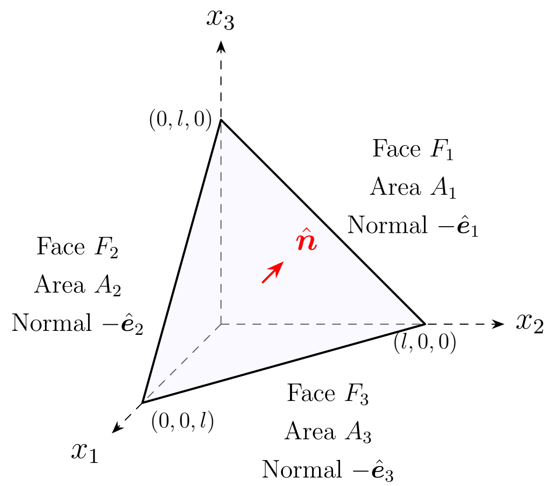
is a linear function of the normal vector $\hat{\boldsymbol{n}}$. This means that

$$t_j = \tau_{ij}n_i \quad \text{or} \quad \boldsymbol{t} = \hat{\boldsymbol{n}} \cdot \boldsymbol{\tau}, \tag{10.5}$$

for a second-order tensor τ_{ij} (or $\boldsymbol{\tau}$), which is called the **stress tensor**.

10.2.1 Proof of Cauchy's Theorem

Consider a material volume that is a small tetrahedron. Three faces are aligned with the coordinate axes with areas A_1, A_2, A_3 and outward normals $-\hat{\boldsymbol{e}}_1, -\hat{\boldsymbol{e}}_2, -\hat{\boldsymbol{e}}_3$. The fourth face F is tilted with area A and outward normal $\hat{\boldsymbol{n}}$.



10.2.2 Proof

Consider a material volume that is a tetrahedron. Three faces are aligned with the coordinate axes, F_1, F_2, F_3, and the 4th is tilted, F, with outward normal $\hat{\boldsymbol{n}}$ and area A.

- F_1: Area A_1, normal $-\hat{\boldsymbol{x}}_1$.
- F_2: Area A_2, normal $-\hat{\boldsymbol{x}}_2$.
- F_3: Area A_3, normal $-\hat{\boldsymbol{x}}_3$.

From geometry, we have the following properties:

1. The volume of the tetrahedron is $\frac{l^3}{6\sqrt{2}}$.
2. The surface area of face F is $A \cong \frac{\sqrt{3}}{4} l^2$ (others are roughly proportional to $\frac{1}{2} l^2$).
3. The projected area of A in the direction of $\hat{\boldsymbol{n}}_p$ is $A_p = |\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}}_p| A$. If we take $\hat{\boldsymbol{n}}_p$ to be $\hat{\boldsymbol{x}}_j$, then $A_j = |\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{x}}_j| A$.

Given $\hat{\boldsymbol{n}} = (n_1, n_2, n_3)$ and $|\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{x}}_j| = n_j$, we have:

$$A_j = n_j \frac{\sqrt{3}}{4} l^2. \tag{10.6}$$

Using the Reynolds Transport Theorem, we can rewrite Newton's 2nd law as:

$$\iiint_{W(t)} \left[\rho \frac{D\boldsymbol{u}}{Dt} + \rho \nabla \Pi \right] dV = \oiint_{\partial W(t)} \boldsymbol{t}(\boldsymbol{x}, t, \hat{\boldsymbol{n}})\, dS. \tag{10.7}$$

We divide the equation by l^2 and take the limit as $l \to 0$:

$$\lim_{l \to 0} \frac{\iiint_{W(t)} [\rho \frac{D\boldsymbol{u}}{Dt} + \rho \nabla \Pi]\, dV}{l^2} = \lim_{l \to 0} \frac{\oiint_{\partial W(t)} \boldsymbol{t}(\boldsymbol{x}, t, \hat{\boldsymbol{n}})\, dS}{l^2} = 0. \tag{10.8}$$

Using the Mean Value Theorem, the Left Hand Side (LHS) scales as:

$$\text{LHS} = \lim_{l \to 0} \frac{Ml^3}{l^2} = 0. \tag{10.9}$$

We can express the Right Hand Side (RHS) in terms of the 4 faces:

$$0 = \lim_{l \to 0} \frac{1}{l^2} \left[\sum_{j=1}^{3} \iint_{F_j} \boldsymbol{t}(\boldsymbol{x}, t, -\hat{\boldsymbol{x}}_j)\, dA + \iint_{F} \boldsymbol{t}(\boldsymbol{x}, t, \hat{\boldsymbol{n}})\, dA \right]. \tag{10.10}$$

Each term in the sum can be written in component form (using k for the index). By the Mean Value Theorem:

$$\frac{1}{l^2} \iint_{F_j} t_k(\boldsymbol{x}, t, -\hat{\boldsymbol{x}}_j)\, dA \approx \frac{1}{l^2} t_k(\boldsymbol{x}_j, t, -\hat{\boldsymbol{x}}_j) n_j \frac{\sqrt{3}}{4} l^2 \quad \text{(where } \boldsymbol{x}_j \in A_j\text{)}. \tag{10.11}$$

The last integral becomes:

$$\frac{1}{l^2} \iint_{F} t_k(\boldsymbol{x}, t, \hat{\boldsymbol{n}})\, dA = \frac{1}{l^2} t_k(\boldsymbol{x}_0, t, \hat{\boldsymbol{n}}) \frac{\sqrt{3}}{4} l^2 \quad \text{(where } \boldsymbol{x}_0 \in A\text{)}. \tag{10.12}$$

Substituting these into our equation:

$$\lim_{l \to 0} \left[\sum_{j=1}^{3} t_k(\boldsymbol{x}_j, t, -\hat{\boldsymbol{x}}_j) n_j \frac{\sqrt{3}}{4} + t_k(\boldsymbol{x}_0, t, \hat{\boldsymbol{n}}) \frac{\sqrt{3}}{4} \right] = 0. \tag{10.13}$$

As $l \to 0$, $\boldsymbol{x}_j, \boldsymbol{x}_0 \to \boldsymbol{x}$. Canceling the constants ($\frac{\sqrt{3}}{4}$):

$$\sum_{j=1}^{3} t_k(\boldsymbol{x}, t, -\hat{\boldsymbol{x}}_j) n_j + t_k(\boldsymbol{x}, t, \hat{\boldsymbol{n}}) = 0. \tag{10.14}$$

We define the stress on the coordinate planes. If we define $\tau_{jk}(\boldsymbol{x}, t) \equiv t_k(\boldsymbol{x}, t, \hat{\boldsymbol{x}}_j)$, then using Newton's 3rd Law ($t_k(\ldots, -\hat{\boldsymbol{x}}_j) = -t_k(\ldots, \hat{\boldsymbol{x}}_j)$), we obtain:

$$t_k(\boldsymbol{x}, t, \hat{\boldsymbol{n}}) = \sum_{j=1}^{3} t_k(\boldsymbol{x}, t, \hat{\boldsymbol{x}}_j) n_j. \tag{10.15}$$

This shows $\boldsymbol{t}$ is a linear function of $\hat{\boldsymbol{n}}$.

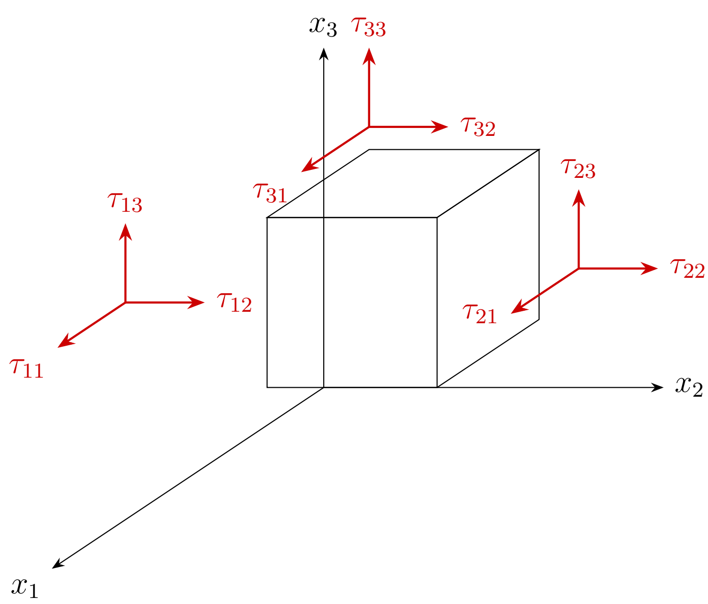
Definition 10.2. The relationship is given by:

$$t_k = \tau_{jk} n_j \quad \text{or} \quad \boldsymbol{t} = \hat{\boldsymbol{n}} \cdot \underline{\underline{\tau}}. \tag{10.16}$$

Here, τ_{jk} forms the components of the stress tensor $\underline{\underline{\tau}}$:

$$\underline{\underline{\tau}} = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix}. \tag{10.17}$$

τ_{jk} is the k-th component of the force per unit area acting on a surface with unit outward normal $\hat{\boldsymbol{x}}_j$.



10.3 Conservation of Linear Momentum

We return to Newton's 2nd Law:

$$\iiint_{W(t)} \left[\rho \frac{D\boldsymbol{u}}{Dt} + \rho \nabla \Pi \right] dV = \oiint_{\partial W(t)} \boldsymbol{t}(\boldsymbol{x}, t, \hat{\boldsymbol{n}})\, dS. \tag{10.18}$$

Cauchy's theorem tells us $\boldsymbol{t} = \hat{\boldsymbol{n}} \cdot \underline{\underline{\tau}}$ or $t_k = n_j \tau_{jk}$. Thus:

$$\oiint_{\partial W(t)} \boldsymbol{t}\, dS = \oiint_{\partial W(t)} \hat{\boldsymbol{n}} \cdot \underline{\underline{\tau}}\, dS. \tag{10.19}$$

In indicial notation:

$$\iiint_{W(t)} \left[\rho \frac{Du_i}{Dt} + \rho \frac{\partial \Pi}{\partial x_i}\right] dV = \oiint_{\partial W(t)} n_j \tau_{ji} \, dS. \tag{10.20}$$

Gauss' Divergence Theorem states

$$\iiint_W \frac{\partial U_j}{\partial x_j} dV = \oiint_{\partial W} U_j n_j dS. \tag{10.21}$$

Applying this for each index i:

$$\oiint_{\partial W(t)} n_j \tau_{ji} \, dS = \iiint_{W(t)} \frac{\partial \tau_{ji}}{\partial x_j} \, dV. \tag{10.22}$$

Hence, combining all terms under one volume integral:

$$\iiint_{W(t)} \left[\rho \frac{D\boldsymbol{u}}{Dt} + \rho \nabla \Pi - \nabla \cdot \underline{\underline{\tau}}\right] dV = 0. \tag{10.23}$$

Using our lemma (localization theorem), since the volume is arbitrary, the integrand must be zero:

$$\rho \frac{D\boldsymbol{u}}{Dt} = -\rho \nabla \Pi + \nabla \cdot \underline{\underline{\tau}} \quad \text{or} \quad \rho \frac{Du_i}{Dt} = -\rho \frac{\partial \Pi}{\partial x_i} + \frac{\partial \tau_{ji}}{\partial x_j}. \tag{10.24}$$

This is the **Conservation of Linear Momentum**.

Theorem 10.3 *(Conservation of Linear Momentum).* Mathematically, conservation of linear momentum is stated as:

$$-\rho \frac{\partial \Pi}{\partial x_i} + \frac{\partial \tau_{ji}}{\partial x_j}. \tag{10.25}$$

10.3.1 The Closure Problem

Recall the Conservation of Mass equation:

$$\boxed{\frac{D\rho}{Dt} + \rho \nabla \cdot \boldsymbol{u} = 0.} \tag{10.26}$$

This gives us a total of 4 equations. However, if we count the variables:

- ρ (1 component)
- u_1, u_2, u_3 (3 components)
- $\underline{\underline{\tau}}$ (9 components)

This is not a closed system. We must specify a form for the stress tensor.

For the case of no motion, we had $\boldsymbol{t} = -p\hat{\boldsymbol{n}}$. Since $\boldsymbol{t} = \hat{\boldsymbol{n}} \cdot \underline{\underline{\tau}}$, this implies:

$$\underline{\underline{\tau}} = -p\underline{\underline{I}} \quad \text{or} \quad \underline{\underline{\tau}} = -p \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{10.27}$$

Lecture 11

11.1 Symmetry of the Stress Tensor

11.1.1 Introduction

Recall that using the principles of conservation of mass and linear momentum, we obtained the following equations:

$$\frac{D\rho}{Dt} + \rho\nabla \cdot \boldsymbol{u} = 0 \tag{11.1}$$

$$\rho\frac{D\boldsymbol{u}}{Dt} = -\rho\nabla\Pi + \nabla \cdot \boldsymbol{\tau}. \tag{11.2}$$

There are 4 equations (1 for mass, 3 for momentum) but many more unknowns (density ρ, velocity $\boldsymbol{u}$, and the 9 components of the stress tensor $\boldsymbol{\tau}$). Today, we reduce the number of unknowns by showing that the stress tensor is symmetric.

11.1.2 Conservation of Angular Momentum

Newton's 2nd law yielded the equation stating that linear momentum is conserved:

$$\frac{d}{dt}\iiint_{W(t)} \rho\boldsymbol{u}\, dV = \boldsymbol{F}_{\text{total}}. \tag{11.3}$$

This is the analogue of $\frac{d}{dt}(m\boldsymbol{v}) = \boldsymbol{F}$ in classical mechanics. In classical mechanics, we also have an equation for the conservation of angular momentum:

$$\frac{d}{dt}(\boldsymbol{x} \times m\boldsymbol{v}) = \boldsymbol{x} \times \boldsymbol{F}, \tag{11.4}$$

which relates the rate of change of angular momentum to the torque.

In continuum mechanics, we can obtain a new equation for angular momentum conservation.

Definition 11.1 *(Conservation of Angular Momentum).* The conservation of angular momentum for a material volume $W(t)$ is given by:

$$\frac{d}{dt}\iiint_{W(t)} \rho(\boldsymbol{x} \times \boldsymbol{u})\, dV = \iiint_{W(t)} \rho(\boldsymbol{x} \times \boldsymbol{g})\, dV + \oiint_{\partial W(t)} \boldsymbol{x} \times \boldsymbol{t}\, dS. \tag{11.5}$$

The first term on the RHS represents the torque due to body forces (gravity), and the second term represents the torque due to surface forces.

Theorem 11.2 *(Symmetry of the Stress Tensor)*. The principle of Conservation of Angular Momentum implies that the stress tensor is symmetric:

$$\boldsymbol{\tau} = \boldsymbol{\tau}^T \quad \text{or} \quad \tau_{ij} = \tau_{ji}. \tag{11.6}$$

Proof. We apply the Reynolds Transport Theorem to the Left Hand Side (LHS) of eq. (11.5):

$$\begin{aligned} \text{LHS} &= \frac{d}{dt} \iiint_{W(t)} \rho(\boldsymbol{x} \times \boldsymbol{u})\, dV && (11.7) \\ &= \iiint_{W(t)} \rho \frac{D}{Dt}(\boldsymbol{x} \times \boldsymbol{u})\, dV && (11.8) \\ &= \iiint_{W(t)} \rho \left[\frac{D\boldsymbol{x}}{Dt} \times \boldsymbol{u} + \boldsymbol{x} \times \frac{D\boldsymbol{u}}{Dt} \right] dV. && (11.9) \end{aligned}$$

Note that $\frac{D\boldsymbol{x}}{Dt} = \boldsymbol{u}$, so the first term in the bracket is $\boldsymbol{u} \times \boldsymbol{u} = \boldsymbol{0}$. Thus,

$$\text{LHS} = \iiint_{W(t)} \rho \left(\boldsymbol{x} \times \frac{D\boldsymbol{u}}{Dt} \right) dV. \tag{11.10}$$

Now consider the Right Hand Side (RHS). We defined $\boldsymbol{g} = -\nabla\Pi$, so the body force term is $-\iiint \boldsymbol{x} \times (\rho\nabla\Pi) dV$. The surface term requires more manipulation:

$$\text{Surface Term} = \oiint_{\partial W(t)} \boldsymbol{x} \times \boldsymbol{t}\, dS. \tag{11.11}$$

In index notation, the cross product is $(\boldsymbol{x} \times \boldsymbol{t})_i = \epsilon_{ijk} x_j t_k$. Recall Cauchy's formula $t_k = n_l \tau_{lk}$ (where l is the dummy index for the normal contraction). Substituting this:

$$\oiint_{\partial W(t)} \epsilon_{ijk} x_j (n_l \tau_{lk})\, dS = \oiint_{\partial W(t)} n_l (\epsilon_{ijk} x_j \tau_{lk})\, dS. \tag{11.12}$$

This form allows us to use Gauss' Divergence Theorem:

$$= \iiint_{W(t)} \frac{\partial}{\partial x_l} (\epsilon_{ijk} x_j \tau_{lk})\, dV. \tag{11.13}$$

Expanding the derivative using the product rule:

$$\frac{\partial}{\partial x_l} (\epsilon_{ijk} x_j \tau_{lk}) = \epsilon_{ijk} \frac{\partial x_j}{\partial x_l} \tau_{lk} + \epsilon_{ijk} x_j \frac{\partial \tau_{lk}}{\partial x_l}. \tag{11.14}$$

Since $\frac{\partial x_j}{\partial x_l} = \delta_{jl}$, the first term becomes $\epsilon_{ijk}\delta_{jl}\tau_{lk} = \epsilon_{ijk}\tau_{jk}$. Thus, the integral becomes:

$$\iiint_{W(t)} \epsilon_{ijk}\tau_{jk}\, dV + \iiint_{W(t)} \epsilon_{ijk} x_j \frac{\partial \tau_{lk}}{\partial x_l}\, dV. \tag{11.15}$$

Converting back to vector notation, this is:

$$\iiint_{W(t)} \epsilon_{ijk}\tau_{jk}\, dV + \iiint_{W(t)} \boldsymbol{x} \times (\nabla \cdot \boldsymbol{\tau})\, dV. \tag{11.16}$$

We now equate the simplified LHS (eq. (11.10)) and the expanded RHS. From the Conservation of Linear Momentum, we know:

$$\nabla \cdot \boldsymbol{\tau} = \rho \frac{D\boldsymbol{u}}{Dt} + \rho \nabla \Pi. \tag{11.17}$$

Substitute this into the second integral of eq. (11.16):

$$\begin{aligned} \text{RHS} &= \iiint_{W(t)} \epsilon_{ijk}\tau_{jk}\, dV + \iiint_{W(t)} \boldsymbol{x} \times \left(\rho \frac{D\boldsymbol{u}}{Dt} + \rho \nabla \Pi\right) dV \qquad (11.18) \\ &= \iiint_{W(t)} \epsilon_{ijk}\tau_{jk}\, dV + \iiint_{W(t)} \rho \left(\boldsymbol{x} \times \frac{D\boldsymbol{u}}{Dt}\right) dV + \iiint_{W(t)} \boldsymbol{x} \times (\rho \nabla \Pi)\, dV. \qquad (11.19) \end{aligned}$$

Putting all pieces together in the angular momentum equation:

$$\underbrace{\iiint_{W(t)} \rho \left(\boldsymbol{x} \times \frac{D\boldsymbol{u}}{Dt}\right) dV}_{\text{LHS}} = \underbrace{- \iiint_{W(t)} \boldsymbol{x} \times (\rho \nabla \Pi)\, dV}_{\text{Body Torque}} + \underbrace{\iiint_{W(t)} \epsilon_{ijk}\tau_{jk}\, dV}_{\text{Surface Torque}} + \dots. \tag{11.20}$$

The term $\iiint \rho(\boldsymbol{x} \times \frac{D\boldsymbol{u}}{Dt})$ cancels on both sides. The term involving $\nabla \Pi$ also cancels (note the sign change in the momentum substitution vs body force definition). We are left with:

$$\iiint_{W(t)} \epsilon_{ijk}\tau_{jk}\, dV = 0. \tag{11.21}$$

By the localization lemma (since $W(t)$ is arbitrary), the integrand must be zero:

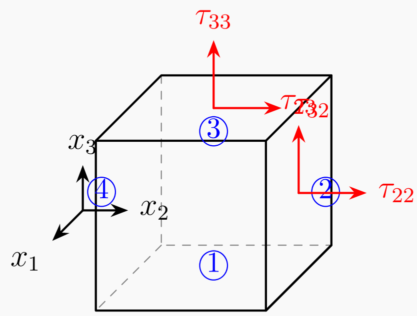
$$\epsilon_{ijk}\tau_{jk} = 0. \tag{11.22}$$

The term $\epsilon_{ijk}\tau_{jk}$ represents the contraction between an antisymmetric tensor (ϵ) and the tensor τ. For this contraction to be zero everywhere, the antisymmetric part of τ must be zero. Hence, $\tau_{jk} = \tau_{kj}$. The stress tensor is symmetric. $\square$

11.2 Geometric Interpretation

To visualize why the stress tensor must be symmetric, we can consider the torque on a small fluid element.

Example 11.3 *(Torque on a Cube)*. **Geometric Idea:** Consider a small cubic fluid parcel centered at the origin $(0,0,0)$ with side lengths $\Delta x_1, \Delta x_2, \Delta x_3$. We analyze the rotation about the x_1-axis.



The torque due to the surface forces in general can be written as:

$$\oiint_{\partial W(t)} \boldsymbol{x} \times \boldsymbol{t}\, dS. \tag{11.23}$$

Using linear algebra, we expand the cross product $\boldsymbol{x} \times \boldsymbol{t}$:

$$\boldsymbol{x} \times \boldsymbol{t} = \begin{vmatrix} \hat{\boldsymbol{e}}_1 & \hat{\boldsymbol{e}}_2 & \hat{\boldsymbol{e}}_3 \\ x_1 & x_2 & x_3 \\ t_1 & t_2 & t_3 \end{vmatrix} = \begin{bmatrix} x_2t_3 - x_3t_2 \\ x_3t_1 - x_1t_3 \\ x_1t_2 - x_2t_1 \end{bmatrix}. \tag{11.24}$$

Because we are studying rotation in the x_1 direction, we consider the first component only:

$$\oiint_{\partial W(t)} (x_2t_3 - x_3t_2)\, dS. \tag{11.25}$$

Consider a state of rest without gravity. We require that the contribution from the surface force is 0. If the torques did not balance, we would have infinite angular acceleration (motion), which is a contradiction.

The integral in eq. (11.25) can be decomposed into the sum of integrals over the four relevant faces: ① Bottom, ② Right, ③ Top, and ④ Left. (The front and back faces do not contribute to the moment about the x_1 axis).

$$\oiint = \iint_{\text{① Bottom}} + \iint_{\text{② Right}} + \iint_{\text{③ Top}} + \iint_{\text{④ Left}} . \tag{11.26}$$

We evaluate the stress vector $\boldsymbol{t}$ on each surface using the relation $t_i = n_j \tau_{ji}$. Specifically for the components we need:

$$t_2 = n_j \tau_{j2} \quad \text{and} \quad t_3 = n_j \tau_{j3}. \tag{11.27}$$

1. Substitution of Normals and Stresses:

- **Bottom (①):** $\hat{\boldsymbol{n}} = -\hat{\boldsymbol{e}}_3$. Thus $t_2 = -\tau_{32}$ and $t_3 = -\tau_{33}$.

$$\iint_{\text{Bottom}} [x_2(-\tau_{33}) - x_3(-\tau_{32})] \, dx_1 dx_2. \tag{11.28}$$

- **Right (②):** $\hat{\boldsymbol{n}} = +\hat{\boldsymbol{e}}_2$. Thus $t_2 = \tau_{22}$ and $t_3 = \tau_{23}$.

$$\iint_{\text{Right}} [x_2(\tau_{23}) - x_3(\tau_{22})] \, dx_1 dx_3. \tag{11.29}$$

- **Top (③):** $\hat{\boldsymbol{n}} = +\hat{\boldsymbol{e}}_3$. Thus $t_2 = \tau_{32}$ and $t_3 = \tau_{33}$.

$$\iint_{\text{Top}} [x_2(\tau_{33}) - x_3(\tau_{32})] \, dx_1 dx_2. \tag{11.30}$$

- **Left (④):** $\hat{\boldsymbol{n}} = -\hat{\boldsymbol{e}}_2$. Thus $t_2 = -\tau_{22}$ and $t_3 = -\tau_{23}$.

$$\iint_{\text{Left}} [x_2(-\tau_{23}) - x_3(-\tau_{22})] \, dx_1 dx_3. \tag{11.31}$$

2. Approximation: If our surfaces are small, then the components of $\boldsymbol{\tau}$ are nearly constant on each face. We can approximate the coordinates x_2 and x_3 (the moment arms) based on the position of the face relative to the center:

$$\begin{aligned} x_3 \text{ on Bottom} &\approx -\frac{1}{2}\Delta x_3, & x_3 \text{ on Top} &\approx +\frac{1}{2}\Delta x_3, \\ x_2 \text{ on Right} &\approx +\frac{1}{2}\Delta x_2, & x_2 \text{ on Left} &\approx -\frac{1}{2}\Delta x_2. \end{aligned}$$

3. Summation: Substituting these approximations into the integrals (and noting that $\iint dx_i dx_j = \Delta x_i \Delta x_j$), we sum the terms. The normal stress terms involving τ_{33} and τ_{22} cancel out due to symmetry (or because their moment arm x is 0 on average over the face, or they act through the axis). We focus on the shear terms:

$$\text{Bottom Term:} \quad -\left[-\frac{1}{2}\Delta x_3\right](-\tau_{32})\Delta x_1\Delta x_2 = -\frac{1}{2}\tau_{32}\Delta x_1\Delta x_2\Delta x_3 \tag{11.32}$$

$$\text{Right Term:} \quad +\left[+\frac{1}{2}\Delta x_2\right](\tau_{23})\Delta x_1\Delta x_3 = +\frac{1}{2}\tau_{23}\Delta x_1\Delta x_2\Delta x_3 \tag{11.33}$$

$$\text{Top Term:} \quad -\left[+\frac{1}{2}\Delta x_3\right](\tau_{32})\Delta x_1\Delta x_2 = -\frac{1}{2}\tau_{32}\Delta x_1\Delta x_2\Delta x_3 \tag{11.34}$$

$$\text{Left Term:} \quad +\left[-\frac{1}{2}\Delta x_2\right](-\tau_{23})\Delta x_1\Delta x_3 = +\frac{1}{2}\tau_{23}\Delta x_1\Delta x_2\Delta x_3 \tag{11.35}$$

Summing these four contributions:

$$\sum = (\tau_{23} - \tau_{32})\,\Delta x_1\Delta x_2\Delta x_3. \tag{11.36}$$

Since the total torque must be zero for equilibrium:

$$(\tau_{23} - \tau_{32})\Delta V = 0 \implies \tau_{23} = \tau_{32}. \tag{11.37}$$

A similar argument applies to the other off-diagonal entries, confirming the symmetry of the stress tensor.

Lecture 12

12.1 Introduction to Elasticity Theory

Today we begin our discussion on solid mechanics. The motivation for this section is to find a constitutive relation for the stress tensor $\underline{\underline{\tau}}$ for solids. Recall the conservation of linear momentum equation:

$$\rho \frac{D\boldsymbol{u}}{Dt} = -\rho \nabla \Pi + \nabla \cdot \underline{\underline{\tau}}. \tag{12.1}$$

To close this system for solids, we need to describe how the material deforms and how that deformation relates to stress.

> **Definition 12.1** *(Types of Solids).* Solids can generally be categorized into three types based on their deformation behavior:
>
> 1. **Elastic:** Deformations are reversible; the material "bounces back" to its original shape (e.g., rubber, steel).
> 2. **Plastic:** Deformations are permanent; the material stays deformed (e.g., clay).
> 3. **Viscoelastic:** The material exhibits both elastic and viscous characteristics (e.g., cornstarch, memory foam).

12.2 Strain Tensors

We study elastic solids. Before we can state a suitable form for the stress tensor $\underline{\underline{\tau}}$, we need to study deformations. These are described by **strain tensors**.

Consider a material volume before and after deformation. We define two coordinate systems:

- **Lagrangian (Material) Coordinates:** $\boldsymbol{a}$ denotes the initial position of a particle.
- **Eulerian (Spatial) Coordinates:** $\boldsymbol{x}$ denotes the final position of the particle.

Let P and Q be two neighboring points in the material. Initially, the segment PQ has a distance dS_0. In the deformed state, the points move to P' and Q', separated by a distance dS. Using index notation, the squared distances are:

$$dS_0^2 = da_i da_i \quad \text{and} \quad dS^2 = dx_k dx_k. \tag{12.2}$$

To measure the deformation, we examine the difference between the squares of the distances:

$$dS^2 - dS_0^2 = dx_k dx_k - da_i da_i. \tag{12.3}$$

This involves x's and a's, hence Eulerian and Lagrangian. Next, we find an expression that is purely Lagrangian, then purely Eulerian.

12.2.1 Lagrangian Strain Tensor

To find an expression that is purely Lagrangian (in terms of $\boldsymbol{a}$), we consider the mapping $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{a})$. If $|d\boldsymbol{a}|$ is small, we can Taylor expand the position of Q' relative to P':

$$x_k(\boldsymbol{a} + d\boldsymbol{a}) \approx x_k(\boldsymbol{a}) + \frac{\partial x_k}{\partial a_j} da_j. \tag{12.4}$$

Thus, the differential vector is

$$dx_k = \frac{\partial x_k}{\partial a_j} da_j. \tag{12.5}$$

Substituting this into eq. (12.3):

$$dS^2 - dS_0^2 = \left(\frac{\partial x_k}{\partial a_j} da_j\right)\left(\frac{\partial x_k}{\partial a_l} da_l\right) - \delta_{jl} da_j da_l \tag{12.6}$$

$$= \left(\frac{\partial x_k}{\partial a_j}\frac{\partial x_k}{\partial a_l} - \delta_{jl}\right) da_j da_l. \tag{12.7}$$

Because j and l are dummy indices, we can symmetrize and rename them to obtain the definition of the strain tensor.

Definition 12.2 *(Green-St. Venant Strain Tensor).* The Green-St. Venant strain tensor E_{jk} (Lagrangian) is defined such that $dS^2 - dS_0^2 \approx 2E_{jk} da_j da_k$:

$$E_{jk} \equiv \frac{1}{2}\left(\frac{\partial x_i}{\partial a_j}\frac{\partial x_i}{\partial a_k} - \delta_{jk}\right). \tag{12.8}$$

12.2.2 Eulerian Strain Tensor

Conversely, to find an expression that is purely Eulerian (in terms of $\boldsymbol{x}$), we consider the inverse mapping $\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x})$. We Taylor expand the initial position:

$$a_i(\boldsymbol{x} + d\boldsymbol{x}) \approx a_i(\boldsymbol{x}) + \frac{\partial a_i}{\partial x_j} dx_j. \tag{12.9}$$

Thus,

$$da_i = \frac{\partial a_i}{\partial x_j} dx_j. \tag{12.10}$$

Substituting this into eq. (12.3):

$$dS^2 - dS_0^2 = \delta_{jk} dx_j dx_k - \left(\frac{\partial a_i}{\partial x_j} dx_j\right)\left(\frac{\partial a_i}{\partial x_k} dx_k\right) \tag{12.11}$$

$$= \left(\delta_{jk} - \frac{\partial a_i}{\partial x_j}\frac{\partial a_i}{\partial x_k}\right) dx_j dx_k. \tag{12.12}$$

Definition 12.3 *(Almansi-Hamel Strain Tensor)*. The Almansi-Hamel strain tensor e_{jk} (Eulerian) is defined such that $dS^2 - dS_0^2 \approx 2e_{jk} dx_j dx_k$:

$$e_{jk} \equiv \frac{1}{2}\left(\delta_{jk} - \frac{\partial a_i}{\partial x_j}\frac{\partial a_i}{\partial x_k}\right). \tag{12.13}$$

To summarize:

$$dS^2 - dS_0^2 \approx 2E_{jk} da_j da_k \approx 2e_{jk} dx_j dx_k. \tag{12.14}$$

12.3 Strain in Terms of Displacements

Sometimes, it is useful to express this in terms of the displacements.

$$\boxed{q_i = x_i - a_i} \tag{12.15}$$

Next, we rewrite E_{jk} and e_{jk} in terms of the displacements.

For E_{jk}: Since $x_i = q_i + a_i$, we have $\frac{\partial x_i}{\partial a_j} = \frac{\partial}{\partial a_j}(q_i + a_i) = \frac{\partial q_i}{\partial a_j} + \delta_{ij}$. Then the term in the tensor becomes:

$$\frac{\partial x_i}{\partial a_j}\frac{\partial x_i}{\partial a_k} = \left(\frac{\partial q_i}{\partial a_j} + \delta_{ij}\right)\left(\frac{\partial q_i}{\partial a_k} + \delta_{ik}\right) \tag{12.16}$$

$$= \frac{\partial q_i}{\partial a_j}\frac{\partial q_i}{\partial a_k} + \frac{\partial q_i}{\partial a_j}\delta_{ik} + \delta_{ij}\frac{\partial q_i}{\partial a_k} + \delta_{ij}\delta_{ik} \tag{12.17}$$

$$= \frac{\partial q_i}{\partial a_j}\frac{\partial q_i}{\partial a_k} + \frac{\partial q_k}{\partial a_j} + \frac{\partial q_j}{\partial a_k} + \delta_{jk}. \tag{12.18}$$

Substituting this back into the definition of E_{jk} (where the δ_{jk} cancels out):

$$\boxed{E_{jk} = \frac{1}{2}\left[\frac{\partial q_j}{\partial a_k} + \frac{\partial q_k}{\partial a_j} + \frac{\partial q_i}{\partial a_j}\frac{\partial q_i}{\partial a_k}\right] \quad \text{(Lagrangian)}} \tag{12.19}$$

Similarly for e_{jk}: Using $\frac{\partial a_i}{\partial x_j} = \frac{\partial}{\partial x_j}(x_i - q_i) = \delta_{ij} - \frac{\partial q_i}{\partial x_j}$. Then,

$$\begin{aligned}\frac{\partial a_i}{\partial x_j}\frac{\partial a_i}{\partial x_k} &= \left(\delta_{ij} - \frac{\partial q_i}{\partial x_j}\right)\left(\delta_{ik} - \frac{\partial q_i}{\partial x_k}\right) \\ &= \delta_{jk} - \frac{\partial q_j}{\partial x_k} - \frac{\partial q_k}{\partial x_j} + \frac{\partial q_i}{\partial x_j}\frac{\partial q_i}{\partial x_k}.\end{aligned}$$

Substituting into the definition for e_{jk}:

$$\boxed{e_{jk} = \frac{1}{2}\left[\frac{\partial q_j}{\partial x_k} + \frac{\partial q_k}{\partial x_j} - \frac{\partial q_i}{\partial x_j}\frac{\partial q_i}{\partial x_k}\right] \quad \text{(Eulerian)}} \tag{12.20}$$

12.4 Linear Elasticity Theory

As we can see, both E_{jk} and e_{jk} are nonlinear functions of the displacement gradients. To simplify the theory, we assume **small deformations** (or small displacements). This implies that the displacement gradients are small: $|\frac{\partial q_i}{\partial a_j}| \ll 1$.

Consider the relationship between derivatives in the two frames using the chain rule:

$$\frac{\partial q_j}{\partial a_k} = \frac{\partial q_j}{\partial x_l}\frac{\partial x_l}{\partial a_k} = \frac{\partial q_j}{\partial x_l}\left(\delta_{lk} + \frac{\partial q_l}{\partial a_k}\right) = \frac{\partial q_j}{\partial x_k} + \frac{\partial q_j}{\partial x_l}\frac{\partial q_l}{\partial a_k}. \tag{12.21}$$

Similarly,

$$\frac{\partial q_k}{\partial a_j} = \frac{\partial q_k}{\partial x_j} + \frac{\partial q_k}{\partial x_l}\frac{\partial q_l}{\partial a_j}. \tag{12.22}$$

We can substitute these expansions into the full expression for E_{jk}. The result is a series of linear and higher-order terms:

$$E_{jk} = \frac{1}{2}\left[\underline{\frac{\partial q_j}{\partial x_k}} + \frac{\partial q_j}{\partial x_l}\frac{\partial q_l}{\partial a_k} + \underline{\frac{\partial q_k}{\partial x_j}} + \frac{\partial q_k}{\partial x_l}\frac{\partial q_l}{\partial a_j} + \left(\frac{\partial q_i}{\partial x_k} + \dots\right)\left(\frac{\partial q_i}{\partial x_j} + \dots\right)\right]. \tag{12.23}$$

If we assume small displacements, we proceed by **ignoring quadratics and higher-order terms** in q. This effectively removes the nonlinear products of derivatives.

Definition 12.4 *(Infinitesimal Strain Tensor)*. In this limit, the Lagrangian and Eulerian strains converge to the same linear form:

$$\boxed{E_{jk} \approx e_{jk} \approx \frac{1}{2}\left(\frac{\partial q_j}{\partial x_k} + \frac{\partial q_k}{\partial x_j}\right)} \tag{12.24}$$

These are the strain tensors we consider for linear elastic solids.

Note 12.5. If we compute the derivative with respect to time, we obtain the strain rate tensor used in fluid mechanics:

$$\frac{dE_{jk}}{dt} \approx \frac{de_{jk}}{dt} \approx \frac{1}{2}\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right) \quad \text{(fluids).} \tag{12.25}$$

Lecture 14

14.1 Linear Elastic Theory (Continued)

Last time, we showed that

$$dS^2 - dS_0^2 = 2E_{jk}\, da_j\, da_k, \tag{14.1}$$

where the strain tensor in terms of displacements is

$$E_{jk} = \frac{1}{2}\left(\frac{\partial q_j}{\partial x_k} + \frac{\partial q_k}{\partial x_j}\right), \tag{14.2}$$

where $q_i = x_i - a_i$. The strain tensor is a symmetric 3×3 matrix:

$$\underline{\underline{E}} = \begin{pmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{pmatrix}. \tag{14.3}$$

Note 14.1. The strain tensor is symmetric by construction: since $E_{jk} = \frac{1}{2}(\partial q_j/\partial x_k + \partial q_k/\partial x_j)$, we have $E_{jk} = E_{kj}$. This means there are only 6 independent components. Each diagonal element E_{ii} tells us how much stretching or compression occurs in the x_i direction, while the off-diagonal elements E_{ij} $(i \neq j)$ describe shear deformations.

14.1.1 Interpretation of the Strain Tensor

Example 14.2 *(Stretching in the x_1 Direction).* Consider a material element that is stretched in the x_1 direction only. The initial element has length $dS_0 = da_1$ and the deformed element has length dS.

The mapping is

$$x_1 = a_1 + \alpha(a_1), \quad x_2 = a_2, \quad x_3 = a_3, \tag{14.4}$$

so the displacement is

$$q_1 = x_1 - a_1 = \alpha(a_1), \quad q_2 = x_2 - a_2 = 0, \quad q_3 = x_3 - a_3 = 0. \tag{14.5}$$

Since $E_{jk} = 0$ for all j, k not both equal to 1, the only nonzero component is

$$E_{11} = \frac{\partial q_1}{\partial a_1} = \frac{d\alpha}{da_1}. \tag{14.6}$$

If $\frac{d\alpha}{da_1} > 0$, then $E_{11} > 0$ (stretching). If $\frac{d\alpha}{da_1} < 0$, then $E_{11} < 0$ (compression). If $\frac{d\alpha}{da_1} = 0$, then $E_{11} = 0$ (no deformation).

Note 14.3. More generally, we can diagonalize the strain tensor. The eigenvalues give the principal strains, and the eigenvectors give the principal directions. Each element along the diagonal of the diagonalized strain tensor tells how much stretching or compression occurs in each principal direction. This is analogous to diagonalizing the stress tensor to find principal stresses.

To generalize, consider displacements of the form

$$q_1 = \alpha(a_1), \quad q_2 = \beta(a_2), \quad q_3 = \gamma(a_3). \tag{14.7}$$

Then the strain tensor is diagonal:

$$\underline{\underline{E}} = \begin{pmatrix} \dfrac{d\alpha}{da_1} & 0 & 0 \\ 0 & \dfrac{d\beta}{da_2} & 0 \\ 0 & 0 & \dfrac{d\gamma}{da_3} \end{pmatrix}. \tag{14.8}$$

Each element along the diagonal tells how much stretching or compression occurs in each principal direction. More generally, we can always diagonalize the strain tensor; the eigenvalues give the principal strains, and the eigenvectors give the principal directions.

Example 14.4 *(Shear Deformation in the x_1 Direction).* Consider a shear deformation where material in the x_1 direction is displaced proportionally to a_2 (or equivalently x_2): This can be written as

$$\boldsymbol{x}(\boldsymbol{a}) = \boldsymbol{a} + \alpha(a_2, 0, 0), \tag{14.9}$$

$$\boldsymbol{a}(\boldsymbol{x}) = \boldsymbol{x} - \alpha(x_2, 0, 0). \tag{14.10}$$

The displacement is

$$\boldsymbol{q} = \boldsymbol{x} - \boldsymbol{a} = \alpha(a_2, 0, 0) = \alpha(x_2, 0, 0). \tag{14.11}$$

The strain tensor is (assuming small strain and neglecting quadratic terms):

$$\underline{\underline{E}} = \begin{pmatrix} 0 & \frac{\alpha}{2} & 0 \\ \frac{\alpha}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad \text{(small strain, } \alpha \ll 1) \tag{14.12}$$

To verify this: since $q_1 = \alpha a_2$ and $q_2 = q_3 = 0$, the only nonzero displacement gradient is $\frac{\partial q_1}{\partial a_2} = \alpha$. Then:

- $E_{12} = E_{21} = \frac{1}{2}\left(\frac{\partial q_1}{\partial x_2} + \frac{\partial q_2}{\partial x_1}\right) = \frac{1}{2}(\alpha + 0) = \frac{\alpha}{2}$.
- All diagonal entries are zero since $\frac{\partial q_i}{\partial x_i} = 0$ for each i.

If we use the full (nonlinear) Lagrangian strain tensor, we get $E_{22} = \frac{\alpha^2}{2}$, but this is second-order in α and is neglected in the linear theory.

For fun, we can also compute the full strain tensor (including quadratic terms):

$$E_{jk}^{\text{full}} = \frac{1}{2}\left[\frac{\partial q_j}{\partial a_k} + \frac{\partial q_k}{\partial a_j} + \frac{\partial q_i}{\partial a_j}\frac{\partial q_i}{\partial a_k}\right] \implies \underline{\underline{E}}^{\text{full}} = \begin{pmatrix} 0 & \frac{\alpha}{2} & 0 \\ \frac{\alpha}{2} & \frac{\alpha^2}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{14.13}$$

Similarly, from the Eulerian strain tensor:

$$\underline{\underline{e}} = \begin{pmatrix} 0 & \frac{\alpha}{2} & 0 \\ \frac{\alpha}{2} & -\frac{\alpha^2}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{14.14}$$

The only difference between the two tensors is the sign of the quadratic term. The terms linear in α are identical, confirming that $E_{jk} \approx e_{jk}$ for small strains.

14.2 Equations of Motion for Linear Elasticity

14.2.1 Governing Equations

Recall the governing equations from the conservation laws:

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\boldsymbol{u} = 0 \tag{14.15}$$

$$\rho\frac{D\boldsymbol{u}}{Dt} = -\rho\nabla\Pi + \nabla\cdot\underline{\underline{\tau}} \tag{14.16}$$

We specialize these equations for a **linear elastic solid** with the following assumptions:

1. **Small strain.**

2. **Constant density:** $\rho = \rho_0$, which yields $\nabla \cdot \boldsymbol{u} \approx 0$ from eq. (14.15).
3. **Small velocity** ($|\boldsymbol{u}| \ll 1$): Since

$$\frac{D\boldsymbol{u}}{Dt} = \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \approx \frac{\partial \boldsymbol{u}}{\partial t}, \tag{14.17}$$

the advective term $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$ is negligible.
Furthermore, $\boldsymbol{u} = \frac{\partial \boldsymbol{q}}{\partial t}$, so

$$\frac{D\boldsymbol{u}}{Dt} \approx \frac{\partial \boldsymbol{u}}{\partial t} = \frac{\partial^2 \boldsymbol{q}}{\partial t^2}. \tag{14.18}$$

4. **Specify** $\underline{\underline{\tau}}$, the stress tensor (constitutive relation).

Note 14.5. More precisely, this follows from a scaling argument. If U is a typical velocity, L a typical length, and T a typical time, then

$$\frac{\partial \boldsymbol{u}}{\partial t} \sim \frac{U}{T} \tag{14.19}$$

and

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \sim \frac{U^2}{L} = \frac{U}{T} \cdot \frac{UT}{L} \tag{14.20}$$

We want U to be small enough that $\frac{UT}{L} \ll 1$, i.e., the displacement over one period is much smaller than the length scale.

With these assumptions, the momentum equation becomes:

$$\boxed{\rho_0 \frac{\partial^2 \boldsymbol{q}}{\partial t^2} = -\rho_0 \nabla \Pi + \nabla \cdot \underline{\underline{\tau}}} \tag{14.21}$$

14.2.2 Constitutive Theory

Experiments on solids show that the stress tensor for a solid, $\underline{\underline{\tau}}$, depends on the **strain tensor**. This is because the restoring force depends on the deformation itself (not on the rate of deformation, as it does for Newtonian fluids).

The simplest relation possible is a generalization of **Hooke's law**. Assuming the medium is isotropic and using the symmetry of $\underline{\underline{\tau}}$:

$$\boxed{\tau_{ij} = 2\mu\, e_{ij} + \lambda\, e_{kk}\, \delta_{ij}} \tag{14.22}$$

where μ and λ are the **Lamé constants**.

Note 14.6. This constitutive relation has the same form as the one for a Newtonian fluid, except that here e_{ij} is the *strain tensor* (measuring deformation), not the strain *rate* tensor (measuring rate of deformation). For a Newtonian fluid, $\tau_{ij} = 2\mu\,\dot{e}_{ij} + \lambda\,\dot{e}_{kk}\,\delta_{ij}$, where the dot denotes a time derivative. The physical difference is fundamental: in a fluid, stress depends on how fast the material is deforming, while in a solid, stress depends on how much it has deformed.

Alternatively, we can use the **Young's modulus** E and **Poisson's ratio** ν:

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \quad \text{(Young's modulus)}, \tag{14.23}$$

$$\nu = \frac{\lambda}{2(\lambda + \mu)} \quad \text{(Poisson's ratio)}. \tag{14.24}$$

On Assignment 3, we will show that:

$$\tau_{ij} = \frac{E}{1+\nu}\left(e_{ij} + \frac{\nu}{1-2\nu}\,e_{kk}\,\delta_{ij}\right), \tag{14.25}$$

which can be solved to express the strain in terms of the stress:

$$e_{ij} = \frac{1+\nu}{E}\,\tau_{ij} - \frac{\nu}{E}\,\tau_{kk}\,\delta_{ij}. \tag{14.26}$$

14.2.3 Derivation of the Navier Equation

With these assumptions, the momentum equation eq. (14.21) becomes:

$$\rho_0 \frac{\partial^2 q_i}{\partial t^2} = -\rho_0 \frac{\partial \Pi}{\partial x_i} + \frac{\partial}{\partial x_j}\tau_{ji}, \tag{14.27}$$

but

$$\frac{\partial}{\partial x_j}\tau_{ji} = \frac{\partial}{\partial x_j}\left(2\mu\,e_{ji} + \lambda\,e_{kk}\,\delta_{ji}\right), \tag{14.28}$$

where

$$e_{ji} = \frac{1}{2}\left(\frac{\partial q_j}{\partial x_i} + \frac{\partial q_i}{\partial x_j}\right) \tag{14.29}$$

and

$$e_{kk} = \nabla \cdot \vec{q}. \tag{14.30}$$

Expanding:

$$\rho_0 \frac{\partial^2 q_i}{\partial t^2} = -\rho_0 \frac{\partial \Pi}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial q_j}{\partial x_i} + \frac{\partial q_i}{\partial x_j}\right) + \lambda(\nabla \cdot \vec{q})\,\delta_{ji}\right]. \tag{14.31}$$

We differentiate each of the three terms separately:

$$\mu \frac{\partial}{\partial x_j} \frac{\partial q_j}{\partial x_i} = \mu \frac{\partial}{\partial x_i} \left(\frac{\partial q_j}{\partial x_j} \right) = \mu \frac{\partial}{\partial x_i} (\nabla \cdot \vec{q}), \tag{14.32}$$

$$\mu \frac{\partial}{\partial x_j} \frac{\partial q_i}{\partial x_j} = \mu \nabla^2 q_i, \tag{14.33}$$

$$\lambda \frac{\partial}{\partial x_j} \left[(\nabla \cdot \vec{q})\, \delta_{ji} \right] = \lambda \frac{\partial}{\partial x_i} (\nabla \cdot \vec{q}). \tag{14.34}$$

Combining all three contributions gives us the **Navier equation**.

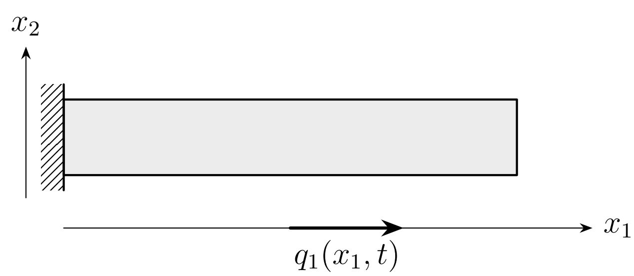
Definition 14.7 *(Navier-Cauchy Equation)*. The Navier–Cauchy equation for linear elasticity is:

$$\rho_0 \frac{\partial^2 \boldsymbol{q}}{\partial t^2} = -\rho_0 \nabla \Pi + \mu \nabla^2 \boldsymbol{q} + (\mu + \lambda) \nabla (\nabla \cdot \boldsymbol{q}) \tag{14.35}$$

This is a system of 3 coupled PDEs for the 3 unknowns q_1, q_2, q_3.

14.3 Waves in Elastic Solids

14.3.1 Longitudinal Vibrations in a Bar



Consider longitudinal vibrations in a bar. The displacement has the form

$$\boldsymbol{q} = (q_1(x_1, t), 0, 0). \tag{14.36}$$

Ignoring gravity ($\Pi = 0$), the Navier equation eq. (14.35) becomes:

$$\rho_0 \frac{\partial^2 q_1}{\partial t^2} = \mu \frac{\partial^2 q_1}{\partial x_1^2} + (\mu + \lambda) \frac{\partial^2 q_1}{\partial x_1^2}. \tag{14.37}$$

This simplifies to the longitudinal wave equation:

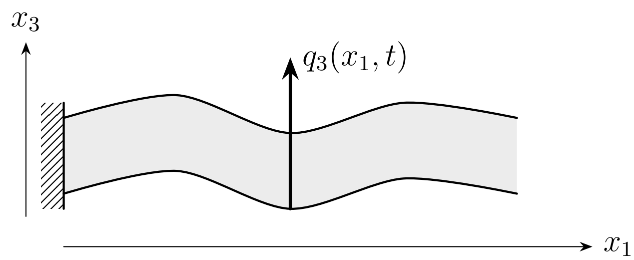
$$\frac{\partial^2 q_1}{\partial t^2} = \frac{2\mu + \lambda}{\rho_0} \frac{\partial^2 q_1}{\partial x_1^2} \tag{14.38}$$

where the longitudinal wave speed is

$$C_D^2 = \frac{2\mu + \lambda}{\rho_0}. \tag{14.39}$$

Note 14.8. This is a standard 1D wave equation. Longitudinal waves are also called **compression waves** or **P-waves** (primary waves) in seismology. The particle motion is in the same direction as the wave propagation. These are the sound waves in the solid; for steel, $C_D \approx 6 \times 10^3$ m/s.

14.3.2 Transverse Waves on a Bar



Now imagine the displacement in the bar has the form $\boldsymbol{q} = (0, 0, q_3(x_1, t))$. Our equation for q_3 is:

$$\rho_0 \frac{\partial^2 q_3}{\partial t^2} = \mu \frac{\partial^2 q_3}{\partial x_1^2}, \tag{14.40}$$

since $\nabla \cdot \boldsymbol{q} = 0$ (the displacement is perpendicular to the direction of variation). This gives the transverse wave equation:

$$\frac{\partial^2 q_3}{\partial t^2} = \frac{\mu}{\rho_0} \frac{\partial^2 q_3}{\partial x_1^2} \tag{14.41}$$

where the transverse wave speed is

$$C_T^2 = \frac{\mu}{\rho_0}. \tag{14.42}$$

Since $\mu > 0$ and $\lambda > 0$ for typical materials, we have

$$C_T^2 = \frac{\mu}{\rho_0} < \frac{2\mu + \lambda}{\rho_0} = C_D^2, \tag{14.43}$$

so **transverse waves travel slower than longitudinal waves**. Transverse waves are also called **shear waves** or **S-waves** (secondary waves) in seismology.

Note 14.9. This is why, during an earthquake, the P-wave (longitudinal) arrives first and the S-wave (transverse) arrives later. The time difference between the two arrivals can be used to estimate the distance to the earthquake epicenter.

14.3.3 General Elastic Waves in Unbounded Media

Starting from the Navier equation eq. (14.35) (ignoring body forces):

$$\rho_0 \frac{\partial^2 \boldsymbol{q}}{\partial t^2} = \mu \nabla^2 \boldsymbol{q} + (\mu + \lambda)\, \nabla(\nabla \cdot \boldsymbol{q}), \tag{$\star$}$$

we can derive wave equations for two types of waves.

Dilatational (P) waves: Taking the divergence of eq. ($\star$):

$$\rho_0 \frac{\partial^2}{\partial t^2}(\nabla \cdot \boldsymbol{q}) = \mu \nabla^2(\nabla \cdot \boldsymbol{q}) + (\mu + \lambda)\nabla^2(\nabla \cdot \boldsymbol{q}), \tag{14.44}$$

which gives:

$$\boxed{\frac{\partial^2}{\partial t^2}(\nabla \cdot \boldsymbol{q}) = \frac{2\mu + \lambda}{\rho_0}\, \nabla^2(\nabla \cdot \boldsymbol{q})} \tag{14.45}$$

This is a 3D wave equation for the **dilation** $\nabla \cdot \boldsymbol{q}$, propagating at speed C_D.

Note 14.10. To see this, note that $\nabla \cdot (\nabla^2 \boldsymbol{q}) = \nabla^2(\nabla \cdot \boldsymbol{q})$ (since the divergence and Laplacian commute for sufficiently smooth fields), and $\nabla \cdot [\nabla(\nabla \cdot \boldsymbol{q})] = \nabla^2(\nabla \cdot \boldsymbol{q})$.

Rotational (S) waves: Computing the curl of eq. ($\star$): since $\nabla \times \nabla(\nabla \cdot \boldsymbol{q}) = \mathbf{0}$ (the curl of a gradient is always zero), we get:

$$\boxed{\frac{\partial^2 \omega_{ij}}{\partial t^2} = \frac{\mu}{\rho_0}\, \nabla^2 \omega_{ij}} \quad \text{where } \omega_{ij} = \frac{\partial q_i}{\partial x_j} - \frac{\partial q_j}{\partial x_i}. \tag{14.46}$$

This is a 3D wave equation for the **rotation tensor** ω_{ij}, propagating at speed C_T.

Note 14.11. To derive this more explicitly: take the partial derivative of the i-th component of eq. ($\star$) with respect to x_j, then take the partial derivative of the j-th component with respect to x_i, and subtract. The $(\mu + \lambda)$ terms cancel because

$$\frac{\partial^2}{\partial x_i \partial x_j}(\nabla \cdot \boldsymbol{q}) - \frac{\partial^2}{\partial x_j \partial x_i}(\nabla \cdot \boldsymbol{q}) = 0 \tag{14.47}$$

by equality of mixed partials.