Assignment 3

Physics 234

W26

1. (a) $[\hat{A},\hat{B}+\hat{C}] = \hat{A}(\hat{B}+\hat{C}) - (\hat{B}+\hat{C})\hat{A} = \hat{A}\hat{B} + \hat{A}\hat{C} - \hat{B}\hat{A} - \hat{C}\hat{A} = \boxed{[\hat{A},\hat{B}] + [\hat{A},\hat{C}]}$

(b) $[\hat{A},\hat{B}\hat{C}] = \hat{A}\hat{B}\hat{C} - \hat{B}\hat{C}\hat{A}$

$$[\hat{A},\hat{B}]\hat{C} = \hat{A}\hat{B}\hat{C} - \underline{\hat{B}\hat{A}\hat{C}}$$

$$\hat{B}[\hat{A},\hat{C}] = \underline{\hat{B}\hat{A}\hat{C}} - \hat{B}\hat{C}\hat{A}$$

$$[\hat{A},\hat{B}]\hat{C} + \hat{B}[\hat{A},\hat{C}] = \hat{A}\hat{B}\hat{C} - \hat{B}\hat{C}\hat{A}$$

$\therefore \boxed{[\hat{A},\hat{B}\hat{C}] = [\hat{A},\hat{B}]\hat{C} + \hat{B}[\hat{A},\hat{C}]}$

2. Show $\hat{C}$ is Hermitian $\quad i\hat{C} := [\hat{A},\hat{B}]$ with $\hat{A},\hat{B}$ Hermitian.

Show $\hat{C} = \hat{C}^{+}$

$$i\hat{C} = \hat{A}\hat{B} - \hat{B}\hat{A}$$

$$-i\hat{C}^{+} = (\hat{A}\hat{B})^{+} - (\hat{B}\hat{A})^{+} = \hat{B}\hat{A} - \hat{A}\hat{B}$$

$$\Rightarrow i\hat{C}^{+} = \hat{A}\hat{B} - \hat{B}\hat{A}$$

$\therefore \boxed{\hat{C} = \hat{C}^{+}}$

3. $\Delta S_x = \sqrt{\langle S_x^2\rangle - \langle S_x\rangle^2}$

$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$S_x^2 = \frac{\hbar^2}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar^2}{4}\mathbb{1}$, $S_y^2 = \frac{\hbar^2}{4}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\hbar^2}{4}\mathbb{1}$

$$\langle +|S_x|+\rangle = \frac{\hbar}{2}(1 \quad 0)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}(1 \quad 0)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

$$\langle -|S_x|-\rangle = \frac{\hbar}{2}(0 \quad 1)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\hbar}{2}(0 \quad 1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

$$\langle \pm|S_x^2|\pm\rangle = \frac{\hbar^2}{4}$$

$$\Delta S_x = \sqrt{\frac{\hbar^2}{4} - 0} = \frac{\hbar}{2}$$

$$\langle +|S_y|+\rangle = \frac{\hbar}{2}(1 \quad 0)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}(1 \quad 0)\begin{pmatrix} 0 \\ i \end{pmatrix} = 0$$

3. $\langle - | S_y | - \rangle = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} -i \\ 0 \end{pmatrix} = 0$

$\langle - | S_y^2 | - \rangle = \frac{\hbar^2}{4}$

$$\Delta S_y = \frac{\hbar}{2} .$$

- We find $\Delta S_x \Delta S_y = \frac{\hbar^2}{4}$. This is the minimum allowed uncertainty by the uncertainty relationship $\Delta S_x \Delta S_y \geq \frac{\hbar}{2} |\langle S_z \rangle|$ for the states $|\pm\rangle$. For these states $\langle S_z \rangle = \pm \frac{\hbar}{2}$ and $\Delta S_x \Delta S_y = \frac{\hbar^2}{4}$.

- For an eigenstate of S_x, $\Delta S_x = 0$ and $\langle S_z \rangle = 0$

$$\therefore \quad \Delta S_x \Delta S_y = \frac{\hbar}{2} |\langle S_z \rangle| = 0 .$$

4. Spin-1 particle : $|\psi\rangle \xrightarrow[S_z\text{-basis}]{} \frac{1}{\sqrt{14}}\begin{pmatrix}1\\2\\3i\end{pmatrix}$

(a) Measurements of S_z :

$$\boxed{\begin{aligned} P(+\hbar) &= |\langle 1|\psi\rangle|^2 = \frac{1}{14} \\ P(0) &= |\langle 0|\psi\rangle|^2 = \frac{4}{14} \\ P(-\hbar) &= |\langle -1|\psi\rangle|^2 = \frac{9}{14} \end{aligned}}$$

$$\langle S_z\rangle = \sum_{n}^{1} a_n P_n =$$

$$\hbar\left(\frac{1}{14}\right) + 0 - \hbar\left(\frac{9}{14}\right)$$

$$\boxed{\langle S_z\rangle = -\hbar\frac{8}{14} = -\frac{4\hbar}{7}}$$

(b) Find $\langle\psi|S_x|\psi\rangle$:

$$\underset{S_z\text{-basis}}{\underset{\sim}{S_x}} = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix} \text{ eq. (2.115) McIntyre}$$

$$\langle\psi|S_x|\psi\rangle = \frac{\hbar}{\sqrt{2}}\frac{1}{14}\begin{pmatrix}1 & 2 & -3i\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}1\\2\\3i\end{pmatrix} = \frac{1}{14}\begin{pmatrix}1 & 2 & -3i\end{pmatrix}\begin{pmatrix}2\\1+3i\\2\end{pmatrix}\frac{\hbar}{\sqrt{2}}$$

$$\boxed{\langle S_x\rangle = \frac{1}{14}\frac{\hbar}{\sqrt{2}}\left(2 + 2(1+3i) - 6i\right) = \frac{\sqrt{2}\hbar}{7}}$$

(c) Probability that a measurement of S_x yields a value $+\hbar$:

$$P(+\hbar) = |{}_x\langle 1|\psi\rangle|^2 \qquad |1\rangle_x = \frac{1}{2}|1\rangle + \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|-1\rangle \quad \text{eq. (2.113)}$$

$$P(+\hbar) = \left|\frac{1}{\sqrt{14}}\begin{pmatrix}\frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}1\\2\\3i\end{pmatrix}\right|^2 = \frac{1}{14}\left|\frac{1}{2} + \sqrt{2} + \frac{3i}{2}\right|^2 = \frac{1}{56}\left|1 + 2\sqrt{2} + 3i\right|^2$$

$$\boxed{P(+\hbar) = \frac{1}{56}\left\{(1+2\sqrt{2})^2 + 9\right\} \approx 0.42}$$

5. $|\psi_i\rangle = \frac{1}{\sqrt{6}}|1\rangle - \sqrt{\frac{2}{6}}|0\rangle + i\sqrt{\frac{3}{6}}|-1\rangle$

$$|\psi_f\rangle = \frac{1+i}{\sqrt{7}}|1\rangle_y + \frac{2}{\sqrt{7}}|0\rangle_y - \frac{i}{\sqrt{7}}|-1\rangle_y$$

Find $P = |\langle\psi_f|\psi_i\rangle|^2$

- I will calculate $U_{z\to y}$ and use it to calculate $|\psi_i\rangle$ in the S_y basis.

$$|1\rangle_y = \frac{1}{2}|1\rangle + \frac{i}{\sqrt{2}}|0\rangle - \frac{1}{2}|-1\rangle$$

$$|0\rangle_y = \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|-1\rangle$$

$$|-1\rangle_y = \frac{1}{2}|1\rangle - \frac{i}{\sqrt{2}}|0\rangle - \frac{1}{2}|-1\rangle$$

eq. (2.114)

$$U_{z\to y} = \begin{pmatrix} {}_y\langle 1|1\rangle & {}_y\langle 1|0\rangle & {}_y\langle 1|-1\rangle \\ {}_y\langle 0|1\rangle & {}_y\langle 0|0\rangle & {}_y\langle 0|-1\rangle \\ {}_y\langle -1|1\rangle & {}_y\langle -1|0\rangle & {}_y\langle -1|-1\rangle \end{pmatrix}$$

$$U_{z\to y} = \begin{pmatrix} \frac{1}{2} & -\frac{i}{\sqrt{2}} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{i}{\sqrt{2}} & -\frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -i\sqrt{2} & -1 \\ \sqrt{2} & 0 & \sqrt{2} \\ 1 & i\sqrt{2} & -1 \end{pmatrix}$$

$$U_{z\to y}|\psi_i\rangle = \frac{1}{2}\frac{1}{\sqrt{6}}\begin{pmatrix} 1 & -i\sqrt{2} & -1 \\ \sqrt{2} & 0 & \sqrt{2} \\ 1 & i\sqrt{2} & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -\sqrt{2} \\ i\sqrt{3} \end{pmatrix} = \frac{1}{2}\frac{1}{\sqrt{6}}\begin{pmatrix} 1+2i-i\sqrt{3} \\ \sqrt{2}+i\sqrt{6} \\ 1-2i-i\sqrt{3} \end{pmatrix}$$

$$\boxed{\begin{aligned} |\langle\psi_f|\psi_i\rangle|^2 &= \frac{1}{24}\frac{1}{7}\left|\begin{pmatrix}(1-i) & 2 & i\end{pmatrix}\begin{pmatrix} 1+(2-\sqrt{3})i \\ \sqrt{2}+i\sqrt{6} \\ 1-(2+\sqrt{3})i \end{pmatrix}\right|^2 \\ &= \frac{1}{168}\left((5+2\sqrt{2})^2 + (2-\sqrt{3}+2\sqrt{6})^2\right) \approx 0.524 \end{aligned}}$$

6. Determine the matrix representations of the spin-$\frac{1}{2}$ angular momentum operators S_x, S_y, & S_z using the S_y basis.

→ We know the representations of S_x, S_y and S_z in the S_z-basis. Thus, calculate the transformation matrix $U = U_{z\to y}$ and calculate

$$\underbrace{S_x}_{S_y\text{-basis}} = U \cdot \underbrace{S_x}_{S_z\text{-basis}} \cdot U^{\dagger}$$

$$\underbrace{S_y}_{S_y\text{-basis}} = U \cdot \underbrace{S_y}_{S_y\text{-basis}} \cdot U^{\dagger}$$

$$|\pm\rangle_y = \frac{1}{\sqrt{2}}\left(|+\rangle \pm i\,|-\rangle\right)$$

$$\underbrace{S_z}_{S_y\text{-basis}} = U \cdot \underbrace{S_z}_{S_z\text{-basis}} \cdot U^{\dagger}$$

$$U_{z\to y} = \begin{pmatrix} {}_y\langle+|+\rangle & {}_y\langle+|-\rangle \\ {}_y\langle-|+\rangle & {}_y\langle-|-\rangle \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

$$\underbrace{S_x}_{S_y\text{-basis}} = \frac{1}{2}\frac{\hbar}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \frac{\hbar}{4}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} = \frac{\hbar}{4}\begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix}$$

$$\boxed{\underbrace{S_x}_{S_y\text{-basis}} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}$$

$$\underbrace{S_y}_{S_y\text{-basis}} = \frac{1}{2}\frac{\hbar}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \frac{\hbar}{4}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix} = \frac{\hbar}{4}\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

$$\boxed{\underbrace{S_y}_{S_y\text{-basis}} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

$$\underbrace{S_z}_{S_y\text{-basis}} = \frac{1}{2}\frac{\hbar}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \frac{\hbar}{4}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} = \frac{\hbar}{4}\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

$$\boxed{\underbrace{S_z}_{S_y\text{-basis}} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

7. $\vec{S}^2 := S_x^2 + S_y^2 + S_z^2$, we note $S_i^2 = \frac{\hbar^2}{4} \mathbb{1}$ $i \in \{x, y, z\}$

$\therefore \vec{S}^2 = \frac{3}{4} \hbar^2 \mathbb{1}$

$$\boxed{[\vec{S}^2, S_i] = \frac{3}{4}\hbar^2 [\mathbb{1}, S_i] = 0}$$

$$[\vec{S}^2, S_x] = [S_x^2 + S_y^2 + S_z^2, S_x] = \underbrace{[S_x^2, S_x]}_{0} + [S_y^2, S_x] + [S_z^2, S_x]$$

$$[S_y^2, S_x] = -[S_x, S_y^2] = -S_y[S_x, S_y] - [S_x, S_y]S_y \quad \text{(see prob. 1)}$$

$$[S_x, S_y] = i\hbar S_z$$

$$\Rightarrow [S_y^2, S_x] = -i\hbar \{S_y S_z + S_z S_y\}$$

$$[S_z^2, S_x] = -[S_x, S_z^2] = -S_z[S_x, S_z] - [S_x, S_z]S_z$$

$$[S_z, S_x] = i\hbar S_y$$

$$\Rightarrow [S_z^2, S_x] = i\hbar\{S_z S_y + S_y S_z\}$$

$$\boxed{[\vec{S}^2, S_x] = -i\hbar\{S_y S_z + S_z S_y\} + i\hbar\{S_z S_y + S_y S_z\} = 0}$$

$$[\vec{S}^2, S_y] = [S_x^2, S_y] + \underbrace{[S_y^2, S_y]}_{0} + [S_z^2, S_y]$$

$$[S_z^2, S_y] = -[S_y, S_z^2] = -S_z[S_y, S_z] - [S_y, S_z]S_z$$

$$[S_y, S_z] = i\hbar S_x$$

$$\Rightarrow [S_z^2, S_y] = -i\hbar\{S_z S_x + S_x S_z\}$$

$$[S_x^2, S_y] = -[S_y, S_x^2] = -S_x[S_y, S_x] - [S_y, S_x]S_x$$

$$= i\hbar\{S_x S_z + S_z S_x\}$$

$$\boxed{[\vec{S}^2, S_y] = -i\hbar\{S_z S_x + S_x S_z\} + i\hbar\{S_x S_z + S_z S_x\} = 0}$$

$$\boxed{\therefore \quad [\vec{S}^2, S_i] = 0 \quad i \in \{x, y, z\}}$$