Quantum Mechanics Study Guide

Lectures 6 – 9: Operators, Projections, and Uncertainty

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1 Lecture 6: Generalized Quantum Systems

1.1 Core Concepts

- **Postulate 2:** Physical observables are represented by **operators** acting on kets.
- **Postulate 3:** Measurement outcomes are the **eigenvalues** $\{a_n\}$ of that operator.
- **Hermitian Operators:** Observables must be Hermitian ($\hat{A} = \hat{A}^\dagger$) because:
  1. They have **real eigenvalues** (measurement results must be real).
  2. Eigenvectors with distinct eigenvalues are **orthogonal**.

1.2 Matrix Representation (How to Build Operators)

To represent an operator $\hat{A}$ as a matrix in a specific basis $\{|a_n\rangle\}$, calculate the matrix elements A_{nm}:

$$A_{nm} = \langle a_n|\hat{A}|a_m\rangle \tag{1}$$

If the basis consists of eigenvectors of $\hat{A}$, the matrix is **diagonal** with eigenvalues on the diagonal.

Memorize: Spin Matrices (in Z-basis)

For a spin-1/2 particle, the operators in the S_z basis are:

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

2 Lecture 7: Projection Operators

2.1 The Projection Operator

A projector isolates the component of a vector along a specific direction (eigenstate).

$$\hat{P}_n \equiv |a_n\rangle\langle a_n| \tag{2}$$

2.2 Important Identities

- **Completeness Relation:** Summing all projectors returns the identity matrix. Use this to insert "1" into equations to simplify them.

$$\sum_n \hat{P}_n = \sum_n |a_n\rangle\langle a_n| = \mathbb{I} \tag{3}$$

- **Spectral Decomposition:** Any operator can be written as a sum of its eigenvalues weighted by its projectors.

$$\hat{A} = \sum_n a_n \hat{P}_n = \sum_n a_n |a_n\rangle\langle a_n| \tag{4}$$

2.3 Postulate 5: Measurement Collapse

If you measure observable $\hat{A}$ and get result a_n, the state collapses.

Collapse Formula

$$|\psi_f\rangle = \frac{\hat{P}_n|\psi_i\rangle}{\sqrt{\langle\psi_i|\hat{P}_n|\psi_i\rangle}} \tag{5}$$

2.4 Solving Stern-Gerlach (S-G) Problems

Problem Type: "What is the probability of exiting the final port?" or "What is the final state?"
Step-by-Step Method:

1. **Identify the Input:** Start with the initial state vector $|\psi_{in}\rangle$.
2. **Apply Projectors:** For every S-G analyzer the particle passes through, apply the corresponding projection operator.
   - Example: Passing through an S_z+ filter means applying $\hat{P}_{z+} = |+\rangle\langle+|$.
   - $|\psi_{after}\rangle = \hat{P}_{filter}|\psi_{before}\rangle$.
3. **Normalize (if asking for state):** If asking for the state after measurement, divide by the norm (length) of the vector.
4. **Calculate Probability (if asking for prob):**

$$P = |\langle \text{final state}|\text{current state}\rangle|^2 \quad \text{OR} \quad P = \langle\psi_{current}|\hat{P}_{final}|\psi_{current}\rangle \tag{6}$$

3 Lecture 8: Expectation Values

3.1 Definitions

The **Expectation Value** is the average result of a large number of measurements on identical systems.

$$\left\langle \hat{A} \right\rangle = \langle\psi|\hat{A}|\psi\rangle \tag{7}$$

The **RMS Deviation** (Uncertainty) is:

$$\Delta A = \sqrt{\langle A^2\rangle - \langle A\rangle^2} \tag{8}$$

3.2 Calculation Strategy

To calculate ΔA:

1. Calculate $\langle A\rangle$:
$$\langle A\rangle = \langle\psi|\hat{A}|\psi\rangle$$
(Matrix multiplication: Row Vector × Matrix × Column Vector).
2. Calculate $\hat{A}^2$: Square the matrix $\hat{A}$.
3. Calculate $\left\langle A^2\right\rangle$:
$$\left\langle A^2\right\rangle = \langle\psi|\hat{A}^2|\psi\rangle$$
4. Plug into the RMS formula.

4 Lecture 9: The Uncertainty Principle

4.1 Commutators

The commutator measures if two operators commute (order doesn't matter) or not.

$$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A} \tag{9}$$

- If $[\hat{A}, \hat{B}] = 0$: The observables are **compatible**. They share a common basis and can be measured simultaneously with arbitrary precision.
- If $[\hat{A}, \hat{B}] \neq 0$: The observables are **incompatible**.

4.2 Spin Commutation Relations

Remember the cyclic permutation ($x \to y \to z \to x$):

$$[S_x, S_y] = i\hbar S_z$$
$$[S_y, S_z] = i\hbar S_x$$
$$[S_z, S_x] = i\hbar S_y$$

4.3 Heisenberg Uncertainty Principle

The general form for any two operators:

General Uncertainty Relation

$$\Delta A \Delta B \geq \frac{1}{2} \left| \left\langle [\hat{A}, \hat{B}] \right\rangle \right| \tag{10}$$

4.4 Useful Identity: Magnitude of Spin

While individual components are uncertain, the *magnitude* of the spin vector is constant:

$$\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 = \frac{3\hbar^2}{4}\mathbb{I} \tag{11}$$