Precalculus

Sets

Foundationality axiom:
we never have xεx, or (xεy and yεx), etc.

Extensionality axiom:
S=T iff:
S (= T and T(= S

Union:
For each set S and T, there is a set S U T such that xεS U T
iff (xεS and xεT)

Pairing:
For every x and y there is a set {x,y} that z {x,y} iff z=x ∨ z=y

Abstraction:
If S is any set and "…x…" is a one-variable sentence, then there is a set {xεS| "…x…"} such that yε{xεS| "…x…"} iff yεS ^ (…y…)

S intersect T := {xεSUT| xεS ^ xεT}

Empty Set
There is a set {} such that ~(xε{}) for all x. This set is unique, there can't be two different empty sets

Ordered Pairs and Functions

Ordered Pairs: (x,y)
- not the same as set {x,y} ({x,y}={y,x}, {x,x}={x})

Axioms:
For any x and y, there is an ordered pair (x,y)
(x,y)=(u,v) iff x=u, y=v

For any two sets S and T, there is a set SxT consisting of exactly all ordered pairs (x,y) with xεS, yεT

SxT is called the Cartesian product of S with T

- for any sets A and B, there is a set AxB ={x|x=(a,b), for some aεA and bεB}

Functions

Let A and B be two nonempty sets, a function f from A into B (f:A→B), where A is the domain of the function(A = dom f) and B is some codomain.

Iff not unique, is a subset f (= AxB such that

If (a,b) ε f and (a,b') ε f, then b = b'

Every a in A has some b εB with (a,b) ε f

If B(=B', then B' is also a valid codomain for the function

Identity functions

For every set A != {}, there is a special function idA:A→A defined by idA(a):= a

** To define a function f, 1) specify domain A !={}; 2) for each aεA give a rule for computing f(a)

e.g. A=|R; for each aε|R, f(x) = x2+x

if f:A→ b, then f o idA = f, idB o f = f

Inverse functions:

For any function f:A→ B, there is precisely one function g: R→A(R= ran f) such that g o f = idA, f o g = idR (g = f^-1)

If f:A→ B is one-to-one and g:C→ D is one-to-one, and f is composite with g, then g o f is also one-to-one, and (g o f)^-1 = f-1 o g-1

Functional Notation

If f:A→ B and if aεA, we write f(a) for the unique element of B such that (a,f(a))εf

f:A→ B, g:C→ D, f=g iff A=C^(f(x)=g(x) for all xεA)

Range

Range of f:A→ B is the set of all values taken on by f(a) as a varies through A

ran f := {bεB| b=f(a), for some aεA} (ran f (= B, ran f != {})

Composition

Let f:A→B and g:C→ D and assume (ran f) intersect (dom g) != {} is a partial composition

Define Ap := {aεA|f(a) ε dom g}

For any aεA#, define h(a):g(f(a))

Then h is a function h:Ap->D
We write g o f and call it the composition of f then g
Dom(g o f) = {aε dom f| f(a) dom g}

Partial Associative Model:

If g o f, h o (g o f), h o g, (h o g) o f are all well defined,

h o (g o f) = (h o g) o f

(whenever both side exists)

Power Sets

For any set S, there is another set P(S) whose members are all the subsets of S (including {} and S)

Size of a finite set

#S is the number of elements in S, #Sε |Z, #S >= 0 (#S = 0 iff S = {})

#P(S) = 2^#S, when S is finite

Numerical Math

Axioms

1. real numbers form a set ℝ ["real numbers" undefined]
2. for any x,y in ℝ, there are numbers $x + y \in \mathbb{R}$ and $x \cdot y \in \mathbb{R}$, which depend on x and y [+,* undefined]
3. for any x,y,z in R, x+(y+z)=(x+y)+z and x*(y*z)=(x*y)*z
4. for any x,y in R, x+y=y+x; xy=yx
5. for any x,y,z in R, x*(y+z)=y*x+z*x
6. there are numbers 0,1 in R such that x+0=x and x*1=x for all x in R (0,1 undefined)
7. for any x in R with x!=0, there are numbers -x in R and 1/x in R such that x+(-x)=0 and x*1/x=1
    a. -x (additive inverse is undefined, but x-y is defined, x-y=x+(-y), similarly, x/y=x*1/y
8. If x<y and u<v, then x+u<y+v ('<' undefined)
9. If x<y and z>0, then xz<yz
    a. Def: x>y if y<x, x<=y if (x<y or x=y), x>=y if y<=x
10. x!<x and x+1>x for all x in R
11. if x+y=0 and x!=0, then x<0 or y<0
12. Completeness Axiom: If S is a subset of R, S is nonempty, and if S is bounded $[S \subseteq [a, b]\ a, b \in \mathbb{R}\ [a < b] \rightarrow\ [a, b] \coloneqq \{x \in \mathbb{R} | a \leq x \leq b\}]$, then there is a

tightest closed interval containing S. That is, there are numbers $a_0 < b_0 \in \mathbb{R}$ such that $S \subseteq [a_0, b_0]$ and if $S \subseteq [c, d]$ then $c < a_0, d > b_0$

a. A0=glb(S), b0=lub(S)

The Archimedean Property of R: for any number a in R, there are some n in N such that n>a.

- Proof: suppose this wasn't true. Then some a in R would be >= all natural numbers n. Then N is a nonempty set in [1,a]. So it has a highest container [1,b], where b>=1. Since b-1<b b-1 is NOT an upper bound for N. There must be some N in **N** such that N>b-1. So N+1>b, but N+1 is in **N**. However, b is an upper bound for **N**. So we have a contradiction.

Def. A set S in **R** is called *inductiv*e if 1 is in S and S is closed under succession (adding 1): if x in S, then x+1 in S.

Def. $\mathbb{N} \coloneqq \{x \in \mathbb{R} |$for every inductive set S, x is in S (the intersection of all possible inductive sets)$\}$

Thm: **N** is itself an inductive set. i.e. 1 is in **N**, and if x is in **N**, then x+1 is in **N**

Def $\mathbb{Z} \coloneqq \mathbb{N} \cup \{0\} \cup (-\mathbb{N} \coloneqq \{-n | n \in \mathbb{N}\})$

$\mathbb{Q} \coloneqq \left\{\frac{n}{m} \middle| n, m \in \mathbb{Z}, m \neq 0\right\}$

Mathematical Induction Theorem. If $S \subseteq \mathbb{N}$ and S is inductive, then S= $\mathbb{N}$

- Proof: S= $\mathbb{N}$ iff $S \subseteq \mathbb{N} \wedge \mathbb{N} \subseteq S$. All we need is that $\mathbb{N} \subseteq S$. Then by extensionality, it follows that S=**N**. We must show that if n is in **N**, then n is in S. Because n is in N, we know that n belongs to all inductive sets. So n is in S.

Induction Proof Technique.

- "say we have a one-variable sentence '…n…' we wish to prove 'for all n in **N**, (…n…)'. To do that, establish the base case (x=1, or else). Verify that (…1…).
- Next is the "jump step": Assume (…n…), use that to prove (…n+1…)

Given a nonempty set S, a sequence in S is a function $\sigma: \mathbb{N} \to S$

- We write σ in the term of an infinite list: $\sigma = (\sigma(1), \sigma(2), \sigma(3) \dots) = \left(\sigma(n)\right)_{n=1}^{\infty}$
- Binet's Formula: $f_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$

Recursion Theorem

Let S be a nonempty set, and let a be in S.

Consider a function F:Sx**N**→S, F(s,n)→s' in S

- There is exactly one sequence $\sigma: \mathbb{N} \to S$ in S satisfying

- $\sigma(1) = a$
- $\sigma(n+1) = F(\sigma(n), n)$ for all n in **N**

- We write this unique sequence as seq_a^F

Proof Sketch:

- dom σ = **N**
- $\sigma(n) = n^2 = nn$
- Uniqueness:
  - σ, τ both work, then they are equal:
  - dom σ = **N** = dom τ
  - $\sigma(n) = \tau(n)\ \forall n \in \mathbb{N}$
    - Base: $\sigma(1) = \tau(1)$
    - Jump: $\sigma(n) = \tau(n)$; $\sigma(n+1) = F(\sigma(n), n+1) = F(\tau(n), n+1) = \tau(n+1)$

Fix some universal nonempty set U

Define a finite subset of U, and its size $\mathcal{s}_n(U)$ {collection of all sets of size n in U}

$\mathcal{s}_0(U) = \{\emptyset\}\mathcal{s}_{n+1}(U) := \{S \in \mathcal{P}(U) | S = T \cup \{x\}\ x \in U, T \in \mathcal{s}_n(U), x \notin T\}$

- P is the powerset of U
- A subset T of U is finite of T in Sn(U) for at least an n in **N**U{0}

Limits

Preliminaries

1. Triangle Inequality |x+y|<=|x|+|y| (equality when x,y>0); more generally $|\sum_{k=1}^{n} x_k| \leq \sum_{k=1}^{n} |x_k|$
   a. Proof: for all x in **R**, -|x|<=x<=|x|, same for y, so –(|x|+|y|)<=x+y<=|x|+|y|, so |x+y|<=|x|+|y|
2. Reverse Triangle Inequality |x-y|>=||x|-|y||
   a. Let x-y=z, so x=z+y. |x|=|z+y|<=|z|+|y|, |x|-|y|<=|x-y|, |y|-|x|<=|y-x|=|x-y|, so |x-y|>=max(|x|-|y|,|y|-|x|), |x-y|>=||x|-|y||
3. Floor and Ceiling
   a. max{k in Z|k<=x}, otherwise min, >=

Given a real sequence *an*, we say that *an* approaches the number *l* in **R**, write a*n*→*l*, if

Every p>0 has a corresponding Np in **N** such that

{an|n>=Np} subseteq (*l*-p,*l*+p)

Equivalently, iff each p>0 has Np in **N** so n>=Np implies |an-*l*|<p

Uniqueness of Limits: [an→*l*, an→*k*] implies *l*=*k*

- Proof: assume *l*!=*k*. p=1/2|*l*-*k*|>0
  - $(l-p, l+p) \cap (k-p, k+p) = \emptyset$
  - An when n>=N, Am when m>=M respectively
  - a(N+M)= $(l-p, l+p)$=$(k-p, k+p)$, but this is a contradiction

Monotone Convergence Theorem (MCT)

1. if $a_n \leq a_{n+1} \forall n \in \mathbb{N}$, and if $a_n \leq b \forall n \in \mathbb{N}, b \in \mathbb{R}$, then there is some l in **R** such that $a_n \to l$
   a. by completeness axiom, l := least S in **R**
2. A decreasing bounded sequence must converge in **R**

Some Limit Laws (an converge to a, bn to b)

1. $a_n + b_n \to a + b$
   a. $\left||a_n + b_n| - (a + b)\right| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b| < p(a + b)$
      i. $|a_n - a| < \frac{p}{2}\ when\ n \geq N, |b_n - b| < \frac{p}{2}\ when\ n \geq M, take\ n \geq N + M$
2. $a_n b_n \to ab$
   a. $|a_n b_n - ab| = |(a_n - a)b_n + (b_n - b)a_n - (a_n - a)(b_n - b)|$
      i. $|a_n - a|, |b_n - b| < q\ for\ n$ large enough
   b. $\leq |a_n - a||b_n| + |b_n - b||a_n| + |a_n - a||b_n - b| \leq B|a_n - a| + A|b_n - b| + |a_n - a||b_n - b|$
   c. $\leq (A + B)q + q^2$
      i. $(A + B)q < \frac{p}{2} \to q < \frac{p}{2(A+B)}$
      ii. $q^2 < \frac{p}{2} \to q < \frac{\sqrt{p}}{2}$
      iii. $q = \frac{1}{2}\min(\frac{p}{2(A+B)}, \frac{\sqrt{p}}{2})$
   d. $< p$

Compound Interest

- Existence:
  - X=0→exp0=1
  - X>0→ Strictly increasing, bonded
    - Use binomial theorem to prove strictly increasing

AM-GM-HM proof:

- AM-GM: case 2, case n→ case 2n, case n→ case n-1

Geometry

Lemmas:

- $(ac + bd)^2 + (ad - bc)^2 = (a^2 + b^2)(c^2 + d^2)$
- Wedge product $(a, b) \wedge (c, d) \coloneqq ac - bd$
- Dot product
- Order Lemma: $-1 \leq x \leq u \leq 1 \Rightarrow u\hat{x} - \hat{u}x \geq 0 \;\; or \; (u,\, \hat{u}) \wedge (x, \hat{x}) \geq 0$
  - $\hat{x} \coloneqq \sqrt{1 - x^2} \; for \; x \in [-1,1]$
  - Proof: three cases, 0<=x<=u, x<0<=u, x<=u<0,

Find length x such that (x,x,a) fits into the upper semi-unit circle ("chordal halving function"), where 0<=a<=2

- $x = \sqrt{2 - \sqrt{4 - a^2}} \; = H(a)$ (Archimedes)
  - Use α_j to approximate length of arc
  - $\alpha_n = 2^n H^{\circ n}(a)$
  - By MCT, $\lim\limits_{n \to \infty} \alpha_n$ exists (strictly increasing, smaller than 4)
  - $arc\; a \coloneqq \lim\limits_{n \to \infty} 2^n H^{\circ n}(a)\,, arc\; 2 \coloneqq \pi$

Two chords add up to chord joining them: $a \oplus b = \frac{1}{2} a\sqrt{4 - b^2} + \frac{1}{2} b\sqrt{4 - a^2}$

- 0<=a,b<=2
- $arc(a \oplus b) = arc\; a + arc\; b$
  - $H(a) \oplus H(a) = a; a \oplus a = D(a) = a\sqrt{4 - a^2}; D\big(H(a)\big) = a$
  - $a \oplus b = H(a) \oplus H(a) \oplus b = \big(H(a) \oplus H(b)\big) \oplus \big(H(a) \oplus H(b)\big) =$ $D(H(a) \oplus H(b))$
  - $H^{\circ n}(a \oplus b) = H^{\circ n}(a) \oplus H^{\circ n}(b)$ from n=1 and recursion
- $\frac{arc(a \oplus b)}{arc(a) + arc(b)} = \frac{H^{\circ n}(a \oplus b)}{H^{\circ n}(a) + H^{\circ n}(b)} = \frac{H^{\circ n}(a) \oplus H^{\circ n}(b)}{H^{\circ n}(a) + H^{\circ n}(b)} = \frac{\alpha_n \oplus \beta_n}{\alpha_n + \beta_n}$
- $\left(\frac{\alpha_n \oplus \beta_n}{\alpha_n + \beta_n}\right)^2 = \frac{\alpha_n^2 + \beta_n^2 - \frac{1}{2}\beta_n^2\alpha_n^2 + \frac{1}{2}\alpha_n\beta_n\sqrt{4 - \alpha_n^2}\sqrt{4 - \beta_n^2}}{\alpha_n^2 + \beta_n^2 + 2\alpha_n\beta_n} = \frac{k^2 + 1 - \frac{1}{2}\alpha_n^2 + \frac{1}{2}k\sqrt{4 - \alpha_n^2}\sqrt{4 - \beta_n^2}}{k^2 + 1 + 2k} =^? 1$
  - Where $\frac{\alpha_n}{\beta_n} = k$
  - Does $\alpha_n^2 = 0, \beta_n^2 = 0$???
- chord$\theta \coloneqq \lim\limits_{n \to \infty} D^{\circ n}(\frac{\theta}{2^n})$, where D is the chordal doubling function, decreasing

- $rep\ \theta := \theta - 2\pi n, |rep\theta| \leq \pi$

Trig Functions

- $\arccos x \coloneqq arc\sqrt{2-2x}$
- $\arcsin y \coloneqq sign(y) * \arccos\sqrt{1-y^2}$
- $\arctan m \coloneqq \arcsin\frac{m}{\sqrt{1+m^2}}$
- Arg(x,y):= $\begin{Bmatrix} \pi, x < 0, y = 0 \\ sign(y)\arccos\frac{x}{\sqrt{x^2+y^2}} \end{Bmatrix}$

BC Calculus

- chunnel (tunnel in English channel)
- puddles and brooklyn puddle (removable discont)
  - puddles and asymptotes: remove bad xs when dividing
- pointy and honorary pointy

Limits

- $\forall\varepsilon\exists\delta[|f(x)-L|<\varepsilon \Leftarrow 0<|x-a|<\delta] \Rightarrow \lim_{\mathrm{x\to a}} f(x) = L$
  - Does not care about x=a
  - Solve for delta in terms of epsilon, plug back in
    - Beware of negative numbers
  - For infinity limits, replace delta by x>N for some N (point of no return)

- Unit Circle sinx/x:
  - Draw P (cos, sin) & extend, extend tangent at I=(1,0) to intersection Q. IQ is tan. OIQ is greater than OIP, area of section of circle is $\theta/2$, which is in between.
  - So $\frac{\sin\theta}{2} \le \frac{\theta}{2} \le \frac{\tan\theta}{2} \Rightarrow \boldsymbol{\sin\theta \le \theta} \le \tan\theta \Rightarrow 1 \le \frac{x}{\sin x} \le \sec x \Rightarrow 1 \ge \frac{\sin x}{x} \ge \cos x$
  - Take limit as $x\to 0$: $\lim_{\mathrm{x\to 0}} \frac{\sin x}{x} = 1$ by Squeeze Theorem
- $\lim_{x\to 0}\frac{1-\cos x}{x} = \lim_{x\to 0}\frac{1-\cos^2 x}{x(1+\cos x)} = \lim_{0}\frac{\sin^2 x}{x(1+\cos x)} = 0$

Continuity

- $\lim_{\mathrm{a}^-} f(x) = \lim_{\mathrm{a}^+} f(x) = \mathrm{f(a)}$
- Continuity of Sine curve
  - $\sin(p+q) - \sin(p-q) = 2\cos p \sin q$
    - Let A=p+q, B=p-q
    - $\sin(A) - \sin(B) = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$
  - $0 \le |\sin(x) - \sin B| = \left|2\sin\frac{x-B}{2}\cos\frac{x+B}{2}\right| = 2\left|\sin\frac{x-B}{2}\right|\left|\cos\frac{x+B}{2}\right| \le 2\left|\sin\frac{x-B}{2}\right| \le |x-B|$
  - Take limit: $0 \le \lim_{x\to B}|\sin x - \sin B| \le 0 \Rightarrow \lim_{x\to B}\sin x = \sin B$

Balzano's Theorem:

- *f* cont [a,b], f(a)<0, f(b)>0, there is at least one value c, a<c<b, such that f(c)=0
- Intermediate Value Theorem: f(x) is continuous [a,b], f(a)=j, f(b)=h, for any m between h and j, there exists at least one c such that f(c)=m

Extreme Value Theorem:

- There exists a min/max on a **closed** interval
- Test boundaries & derivatives=0 for min/max
  - f'(x)=0 → roundy min/max
  - f'(x) DNE → pointy min/max

Chain Rule

- $\left(a, f(a)\right)\begin{Bmatrix} \left(a+\Delta x, f(a+\Delta x)\right), \Delta y_f = f(a+\Delta x) - f(a) = m_{sec}\Delta x \\ (a+\Delta x, f(a)+f'(a)\Delta x), \Delta y_t = f'(a)\Delta x \end{Bmatrix}$
- So $\Delta y_{f+t} = \Delta x(m_s - m_t) = \Delta x(\varepsilon)$; epsilon is the "slope error", secant slope-tangent slope
- $\Delta y_f = f'(a)\Delta x + \varepsilon\Delta x$
- Assume f'(a) exists. $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(a),\ \varepsilon \to 0$
- Now, x→u=g(x)→y=f(u)
  - $\Delta u = \Delta x(g'(a) + \varepsilon)$
  - $\Delta y = \Delta u(f'(b) + \hat{\varepsilon})$, with g(a)=b
- $\Delta y = \Delta x(g'(a) + \varepsilon)(f'(b) + \hat{\varepsilon})$
- $\frac{\Delta y}{\Delta x} = g'(a)f(b) + \varepsilon f'(b) + \hat{\varepsilon}g'(a) + \varepsilon\hat{\varepsilon}$
  - As $\Delta x \to 0$, $\frac{\Delta y}{\Delta x} \to f'\left(g(a)\right)g'(a) \to \frac{dy}{dx} = f'(u)u'$

Zones of indifference: fxn behaves differently on either side

Mechanics of Solids/Statics

Law of Cosines Proof:

- (a+c)(a-c)=b(2ac*cost-b) by power of a point (c is opposite)
- Inscribe rectangle in circle r=a, draw another diameter from O crossing a non-hypotenuse side at Q. Take power of that point, assign c and b accordingly.

F' and F''

Fermat's Theorem:

- If f(x) has a local min/max at x=c and f'(c) exists, then f'(c)=0
- Proof: $f'(c) = \lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$, a local max has $f(c+h) \leq f(c)$ for small h
    - Either h>0, or h<0
        - If h<0, f'(c)=lim((negative or zero)/negative)>=0
        - If h>0, f'(c)=lim((negative or zero)/positive)<=0
    - Since f'(c) exists, the left and right limit must yield the same answer. The only option is 0
- Concessions on a definitional front(…? Notes/understanding deficiency)
    - Exceptions of rules
        - f(x)=0 has infinitely many roots
        - increasing/decreasing; not strictly, to prove Fermat's Theorem (extrema & f'(x) exists)
        - 0 has no multiplicative inverse

Rolle's Theorem

- f is continuous [a,b], differentiable (a,b). If f(a)=f(b), then f'(c)=0 for at least one c, a<c<b
- either horiz. line; min/max at end; end is min, max is in middle; end is max, min is in middle; both in middle; by EVT

Mean Value Theorem

- f is continuous [a,b], differentiable (a,b), there exists at least one c where the secant slope = tangent slope, i.e. $f'(c) = \frac{f(b)-f(a)}{b-a}$
- Proof:
    - Let R(x)=mx+n be the secant line, where R(a)=f(a), R(b)=f(b)
    - Define h(x)=f(x)-R(x), so h(a)=h(b)=0
    - h'(x)=f'(x)-R'(x), which is continuous [a,b] and differentiable (a,b)
    - by Rolle's Theorem, h'(c)=0 for some c, so f'(c)=R'(c)=[f(b)-f(a)]/[b-a]

Tangent line to a concave up (y''>0) curve only hits it once

- Proof: by contradiction. Assume it hits it twice, at the other point (b,f(b))s
    - Slope = $\frac{f(b)-f(a)}{b-a} = f'(a)$, and f is cont in [a,b], diff. in (a,b)
    - MVT: f'(c)=secant slope=f'(a) for some c
    - But f''>0, so f'(c)>f'(a) when c>a

Aside: Basis of Inequalities

- $0 \leq (x-y)^2 = x^2 + y^2 - 2xy$
- $4xy \leq x^2 + y^2 + 2xy = (x+y)^2 \Rightarrow \sqrt{xy} \leq \frac{x+y}{2}$, Geometric

mean<=arithmetic mean

Jensen's Inequality (Chinese Olympiad Day)

- $f\left(\sum_{j=1}^{n}\frac{x_j}{n}\right) \leq \sum_{j=1}^{n}\frac{f(x_j)}{n}$ for convex function
  - Equality if xj=xk for all j,k
- Let $\hat{y} = (b-a)f'(a) + f(a), \hat{y} - f(a) = \Delta y$
- Secant slope=$\frac{f(b)-f(a)}{b-a} = m$
  - By Mean Value Theorem, $m = f'(c), c \in (a,b)$
  - $f(b) - f(a) = f'(c)(\mathrm{b}-\mathrm{a})$
  - $f'(c) > f'(a)$ since f''>0
    - In other direction, $f'(\hat{c}) < f'(a)$ since f''>0, but (b-a)<0.
  - $f(b) > \hat{y}$
- Let $\frac{\sum_{j=1}^{n}x_j}{n} = \alpha, x_1 < \alpha < x_n$
  - $f'(\alpha)(x_j - \alpha) + f(\alpha) = \hat{y}_j < f(y_j)$
  - $f'(\alpha)\left(\sum_{j=1}^{n}x_j - n\alpha\right) + nf(\alpha) \leq \sum_{j=1}^{n}f(x_j)$
  - The first part is zero, divide by n, get Jensen's Inequality. ■

- AM>=GM: $\ln\frac{a+b}{2} \geq \frac{\ln a + \ln b}{2} = \ln\sqrt{ab}$

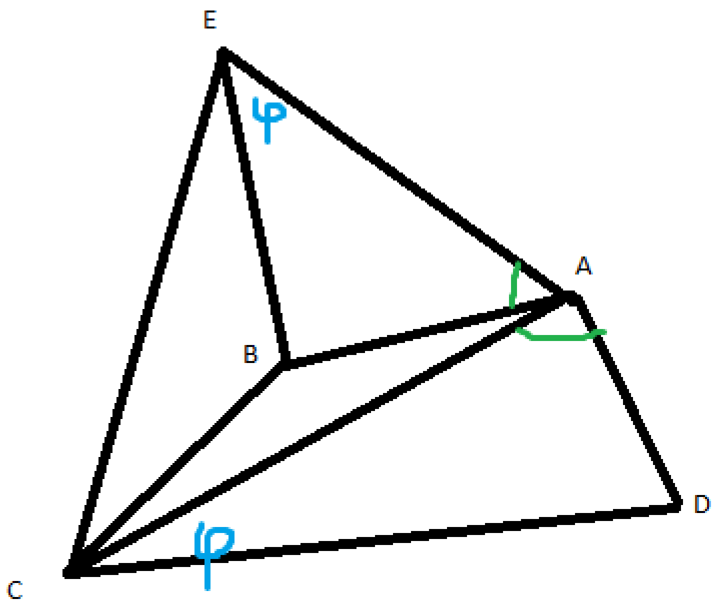
- Protip: $x^{\frac{odd}{even}}$ →honorary pointy, $x^{\frac{even}{odd}}$ → pointy

Optimization

- Deformed trapezoid (ABCE form a rectangle, BEC a triangle, DEC collinear), AD=x+y=BE, DE=x=AB, DC=y, so BC= $\sqrt{(y-x)^2 + (x+y)^2} = \sqrt{2(x^2+y^2)} \geq x + y = BE$
  - So $\sqrt{xy} \leq \frac{x+y}{2} \leq \sqrt{\frac{x^2+y^2}{2}}$, GM<=AM<=Quadratic mean/root mean squared
- Five ways to Square in Circle
  - $x^2 + y^2 = 4r^2, xy = A \leq \sqrt{4r^2/2}$ by GM<=AM inequality with x,y squared

- $A = x\sqrt{4r^2 - x^2} \rightarrow A^2 = 4x^2r^2 - x^4, \alpha = x^2 \rightarrow x = \pm\sqrt{2}r$ at vertex, or else find zero of derivative of A. get $x, y = \sqrt{2}r$
  - Area of Triangle=2(1/2 base*height)=2(1/2)(2r)(h), since 2r is fixed, maximize h=r, get sides of rsqrt2.
  - Area of Rectangle=xy=$4r^2 \cos\theta \sin\theta = 2r^2 \sin 2\theta$, maximum at pi/4
  - Area of a Triangle=1/2r^2sint, area total is r^2sint, its derivative is zero at pi/2

Ptolemy's Theorem
- Cyclic Quadrilateral ABCD, (AC)(BD)=(AB)(CD)+(BC)(AD)



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- $\Delta ABE \sim \Delta ADC, \frac{AE}{AC} = \frac{\boldsymbol{AB}}{\boldsymbol{AD}} = \frac{BE}{\boldsymbol{DC}}$, want BC and BD.
- By SAS, $\Delta EAC \sim \Delta BAD \rightarrow \frac{EA}{BA} = \frac{AC}{AD} = \frac{EC}{BD} \rightarrow EC = \frac{AC}{AD}BD$, and from above, $BE = \frac{AB}{AD}DC$
- BC+BE>EC, equal only when CBE are collinear. BC+AB*DC/AD>AC*BD/AD, so (BC)(AD)+(AB)(DC)>(AC)(BD), equal when CBE are collinear, then <CBA+<ABE=180=<CBA+<CDA, so ABCD is a cyclic quadrangle

Integration

- f(x) exists in [a,b], then there is as minimum m and maximum M. $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$
- FTC: f continuous [a,b], $g(x) = \int_a^x f(t)dt$, $\frac{g(x+h)-g(x)}{h} = \frac{\int_x^{x+h} f(t)dt}{h}$, $m \leq \frac{\int_x^{x+h} f(t)dt}{h} \leq M$, take limit as h goes to zero, $f(x) \leq g'(x) \leq f(x)$

The Genius of the System

- $y = b^x, \lim_{h \to 0} \frac{b^x(b^h-1)}{h} = b^x C$
  - $b^x \ln b$
  - If C=1, can derive b=e
- Power rule proof: $y = x^n, \ln y = n \ln x, \frac{1}{y} y' = \frac{n}{x}, y' = nx^{n-1}$
- y=arctan x, take derivative, use triangle, y'=1/(x^2+1)
  - area under curve from -Inf to Inf of 1/(x^2+1) is pi
- $y = \operatorname{arcsec} x, y' = \frac{1}{\sec y \tan y} = \frac{1}{x\sqrt{x^2-1}}$, uses QIII for arcsec; most books has |x| for the x below, suing QII

u-substitution: undoing chain rule

Lazy American vs Crazy Japanese way of doing int(sec)
- Multiply top and bottom by tanx+secx
- Multiply top and bottom by cosx, Trig Id, fraction decomposition, u substitution, ln property, simplify

Power Reduction Formula
- $\tan^n x = \tan^{n-2} x (\sec^2 x - 1)$ integration by parts,
- $\int \sec^3 x\, dx$: int by parts, trig id, go back.
- Sin^2 → by parts or semiangle

L'Hospital's Rule
- If get 0/0 of Inf/Inf, can take derivative and limit will be that of derivative
- Quick proof: f(x)/g(x), f(a)=g(a)=0
      - (need to deal with hole, DNE at a)
  - Assume there's no hole; if there's a hole, fill it in
  - Tangent approximation: y-0=m(x-a)

- $\frac{f(x)}{g(x)} \approx \frac{m_f(x-a)}{m_g(x-a)} = \frac{m_f}{m_g} = \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)} * \frac{x-a}{x-a} = \lim_{x\to a} \frac{f(x)}{g(x)}$

- If hole, f, g and f', g' DNE
  - $G(x) = \begin{Bmatrix} g(x), x \neq a \\ 0, x = a \end{Bmatrix}$
  - Similar to previous application of definition of a limit; may use Cauchy MVT and use the fact that they are zero at c, and similarly when they approach it
- Cauchy MVT: $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$
  - f,g cont. [a,b], diff (a,b) g'(x)!=0
  - let $h(x) = f(x) - f(a) - \frac{f(b)-f(a)}{g(b)-g(a)}(g(x) - g(a))$ (dummy equation)
  - h(a)=0, h(b)=0, $h'(x) = f'(x) - g'(x)\frac{f(b)-f(a)}{g(b)-g(a)}$
  - Rolle's: $h'(c) = 0$ for some c in (a,b)
    - So $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

Sequences

- Limit comparison test: can use epsilon definition as upper/lower bond; limit mustn't be zero
- Center "a", radius of convergence
- Ratio test: |*|<1, endpoints must test

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n(x)^n \to \frac{1}{x} = \sum_{n=0}^{\infty}(-1)^n(x-1)^n \to \ln x = C + \sum_{n=0}^{\infty}\frac{(x-1)^{n+1}}{(-1)^n(n+1)}$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n} \to \arctan x = C + \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$$

Lagrange Error (at best ties A.H. series)

- $\left\|f^{(n)}(x)\right\|_{[a,x]}\frac{(x-a)^n}{n!}$

Taylor series by integration by parts

- 2 A.M. trick to get x-a in there, get Cauchy form of the remainder which is smaller than or equal to the Lagrange error

$$\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

- McLaurin series for sin x, for sin sqrt(x), and for sin (sqrt(x))/ sqrt(x)
  - $\frac{\sin\sqrt{x}}{\sqrt{x}} = 1 - \frac{x}{3!} + \frac{x^2}{5!} - \frac{x^3}{7!} + \cdots$ has roots at pi^2k^2 for integers k!=0
  - $= \left(1 - \frac{x}{\pi^2}\right)\left(1 - \frac{x}{4\pi^2}\right) \ldots$
  - $-\frac{x}{3!} = -\frac{x}{\pi^2} - \frac{x}{4\pi^2} - \frac{x}{9\pi^2} - \cdots$
  - $\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots$

Spin a circle on the x axis about the y axis, form a torus. Volume by integration is the same as that of a cylinder of height 2piR

- $y^2 + (x - R)^2 = r^2, x = R \pm \sqrt{r^2 - y^2}$
- $\int_{-r}^{r} \pi\mathcal{R}^2 - \pi r^2 dy$ (big circle minus small circle)
- $\pi \int_{-r}^{r} X^2 - x^2 dy = \pi \int_{-r}^{r} (X - x)(X + x) dy = \pi \int_{-r}^{r} 2R\left(2\sqrt{r^2 - y^2}\right) dy$
- $= 4\pi R \int_{\pi}^{0} -r^2 \sin^2\theta \, d\theta = 4\pi R \left(\frac{\pi r^2}{2}\right)$

e converge to something less than 4

- $b^{n+1} - a^{n+1} = (b - a)\left(\sum_{k=0}^{n} b^{n-k} a^k\right) = (b - a)b^n$
- $b > a > 0 \Rightarrow b^{n+1} - a^{n+1} < (n + 1)b^n(b - a)$
- $b^n[a(n + 1) - nb] < a^{n+1}$
- $c_n = \left(1 + \frac{1}{n}\right)^n, let\ a = 1 + \frac{1}{n+1}, b = 1 + \frac{1}{n}$
- $\left(1 + \frac{1}{n}\right)^n [n + 1 + 1 - n - 1] < \left(1 + \frac{1}{n+1}\right)^{n+1}$, so sequence c is monotonic increasing
- $Let\ a = 1, b = 1 + \frac{1}{2n} \rightarrow \left(1 + \frac{1}{2n}\right)^n \left[n + 1 - n - \frac{1}{2}\right] < 1$
- $\left(1 + \frac{1}{2n}\right)^n < 2 \rightarrow \left(1 + \frac{1}{u}\right)^{u/2} < 2 \rightarrow \left(1 + \frac{1}{2n}\right)^{2n} < 4$, so every term is bonded by 4. Even terms converge, so odd terms also converge. Sequence is monotonic, so converges to something less than 4.