一、初等数学

1.1 因式分解

$(a \pm b)^2 = a^2 \pm 2ab + b^2$

$a^2 - b^2 = (a+b)(a-b)$

$a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$

1.2 三角函数

$\sin 2x = 2 \sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

二、高等数学

2.1 函数、极限、连续

2.1.1 连续

$\lim\limits_{x \to x_0} f(x) = f(x_0)$

2.1.2 极限

$\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$

$\lim\limits_{x \to 0} (1+x)^{\frac{1}{x}} = e$

$\lim\limits_{x \to \infty} \sqrt[n]{n} = 1$

$x^y = e^{y \ln x}$

2.1.3 $x \to 0$时的等价无穷小

$\sin x \sim \tan x \sim \arcsin x \sim \arctan x \sim \ln(1+x) \sim (e^x - 1) \sim x$

$1 - \cos x \sim \frac{1}{2} x^2$

$(1+x)^a - 1 \sim ax$

2.2 一元函数微分学

2.2.1 导数

$f'(x_0) = \lim\limits_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$

2.2.2 常用微分公式：

- $dC = 0$
- $dx^{\alpha} = \alpha x^{\alpha-1} dx$
- $da^{x} = a^{x} \ln a \, dx$
- $d \log_a^x = \frac{dx}{x \ln a}$
- $d \sin x = \cos x dx$
- $d \cos x = -\sin x \, dx$
- $d \tan x = \sec^2 x \, dx$
- $d \cot x = -\csc^2 x \, dx$
- $d \sec x = \sec x \tan x \, dx$
- $d \csc x = -\csc x \cot x \, dx$
- $d \arcsin x = \frac{dx}{\sqrt{1-x^2}}$
- $d \arctan x = \frac{dx}{1+x^2}$
- $duv = udv + vdu$
- $d \frac{v}{u} = \frac{udv - vdu}{u^2}$

2.3 一元函数积分学

2.3.1 常用不定积分公式

- $\int x^{\alpha} dx = \frac{1}{\alpha+1} x^{\alpha+1} + C$
- $\int \frac{1}{x} dx = \ln x + C$
- $\int \alpha^{x} dx = \frac{1}{\ln a} a^{x} + C$
- $\int \cos x dx = \sin x + C$
- $\int \sin x dx = -\cos x + C$
- $\int \sec x dx = \ln|\sec x + \tan x| + C$
- $\int \csc x dx = \ln|\csc x - \cot x| + C$
- $\int \tan x dx = -\ln|\cos x| + C$
- $\int \cot x dx = \ln|\sin x| + C$
- $\int \sec^2 x dx = \tan x + C$
- $\int \csc^2 x = -\cot x + C$
- $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \arctan \frac{x}{a} + C$
- $\int \frac{1}{a^2-x^2} dx = \frac{1}{2a} \ln\left|\frac{a+x}{a-x}\right| + C$
- $\int \frac{1}{\sqrt{a^2-x^2}} dx = \arcsin \frac{x}{a} + C$
- $\int \frac{1}{\sqrt{x^2 \pm a^2}} dx = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C$
- $\int udv = uv - \int vdu$
- $\int [f(x) + f'(x)] e^{x} dx = f(x) e^{x} + C$

2.3.2 万能代换公式

$\sin x = \frac{2t}{1+t^2}$

$\cos x = \frac{1-t^2}{1+t^2}$

$\tan x = \frac{2t}{1+t^2}$

$dx = \frac{2}{1+t^2} dt$

2.3.3 Γ函数

2.3.3.1 定义

$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1} e^{-x} dx$

2.3.3.2 计算

$\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$
$\Gamma(n+1) = n!$
$\Gamma(\frac{1}{2}) = \sqrt{\pi}$

2.3.4 华莱士公式

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \cos^n x dx = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2} & \text{n为偶数,} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3} \cdot 1 & \text{n为奇数} \end{cases}$$

2.3.5 定积分公式

$\int_0^{\pi} x f(\sin x) dx = \frac{\pi}{2} \int_0^{\pi} f(\sin x) dx$

$f(b) - f(a) = \int_a^b f'(x) dx$

$f(b) - f(a) = f'(\xi)(b-a) \quad (a < \xi < b)$

2.3.6 不等式

$(\int_a^b f(x) g(x) dx)^2 \le \int_a^b f^2(x) dx \int_a^b g^2(x) dx$

$\left|\int_a^b f(x) dx\right| \le \int_a^b |f(x)|\, dx$

2.3.7 反常积分

$\int_a^b f(x) dx$

$$\lim_{x \to b^-} (b-x)^p f(x) = A \Rightarrow \begin{cases} p < 1 & \text{收敛} \\ p \ge 1 & \text{发散} \end{cases}$$

$$\lim_{x \to a^+} (x-a)^p f(x) = A \Rightarrow \begin{cases} p < 1 & \text{收敛} \\ p \ge 1 & \text{发散} \end{cases}$$

$$\lim_{x \to \pm\infty} x^p f(x) = A \Rightarrow \begin{cases} p > 1 & \text{收敛} \\ p \le 1 & \text{发散} \end{cases}$$

2.3.8 定积分应用

2.3.8.1 弧长

$s = \int_\alpha^\beta \sqrt{x'^2(t) + y'^2(t)} dt$

$s = \int_\alpha^\beta \sqrt{1 + y'^2(x)} dx$

$s = \int_\alpha^\beta \sqrt{r^2(\theta) + r'^2(\theta)} d\theta$

2.3.8.2 旋转体体积

$V = \pi \int_\alpha^\beta [y_2^2(x) - y_1^2(x)] dx \quad$ 绕 x 轴

$V = 2\pi \int_\alpha^\beta x(y_2(x) - y_1(x)) dx \quad$ 绕 y 轴

2.3.8.3 旋转曲面面积

$S = 2\pi \int_\alpha^\beta |y| \sqrt{1 + f'^2(x)} dx$

$S=2\pi\int_\alpha^\beta|y(t)|\sqrt{x'^2(t)+y'^2(t)}dt$

2.4 向量代数与空间解析几何

2.5 多元函数微分学

2.6 多元函数积分学

2.7 无穷级数

2.7.1 级数审敛准则

2.7.1.1 必要条件

$\lim\limits_{n\to\infty}u_n=0$

2.7.1.2 正项级数

$$\lim_{n\to\infty}\frac{u_n}{v_n}=l\Rightarrow\begin{cases}0<l<+\infty & \sum\limits_{n=1}^{\infty}u_n\text{与}\sum\limits_{n=1}^{\infty}v_n\text{同敛散}\\ l=0 & \sum\limits_{n=1}^{\infty}v_n\text{收敛}\rightarrow\sum\limits_{n=1}^{\infty}u_n\text{收敛}\\ l=+\infty & \sum\limits_{n=1}^{\infty}v_n\text{发散}\rightarrow\sum\limits_{n=1}^{\infty}u_n\text{发散}\end{cases}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\rho\Rightarrow\sum_{n=1}^{\infty}u_n\begin{cases}\text{收敛} & \rho<1\\ \text{发散} & \rho>1\\ \text{不确定} & \rho=1\end{cases}$$

$$\lim_{n\to\infty}\sqrt[n]{u_n}=\rho\Rightarrow\sum_{n=1}^{\infty}u_n\begin{cases}\text{收敛} & \rho<1\\ \text{发散} & \rho>1\\ \text{不确定} & \rho=1\end{cases}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^p}\begin{cases}\text{收敛} & p>1\\ \text{发散} & p\le 1\end{cases}$$

$$\sum_{n=1}^{\infty}aq^n\begin{cases}\text{收敛},\frac{a}{1-q} & q<1\\ \text{发散} & q\ge 1\end{cases}$$

2.7.1.3 交错级数（牛顿-莱布尼茨准则）

$$\left.\begin{matrix}u_n\ge u_{n+1}\\ \lim\limits_{n\to\infty}u_n=0\end{matrix}\right\}\Rightarrow\sum_{n=1}^{\infty}(-1)^{n-1}u_n\text{收敛}$$

2.7.2 常用幂级数

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}\quad x\in(-\infty,+\infty)$$

$$\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}\quad x\in(-\infty,+\infty)$$

$$\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}\quad x\in(-\infty,+\infty)$$

$$\ln(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}\quad x\in(-1,1]$$

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n \quad x\in(-1,1)$$

$$\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n \quad x\in(-1,1)$$

$$(1+x)^{\alpha}=\sum_{n=0}^{\infty}\frac{\alpha!}{(\alpha-n)!n!}x^n \quad R=1$$

2.7.3 收敛半径

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\rho\Rightarrow\begin{cases}R=0 & \rho=+\infty\\ R=+\infty & \rho=0\\ R=\frac{1}{\rho} & 0<\rho<+\infty\end{cases}$$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\rho\Rightarrow\begin{cases}R=0 & \rho=+\infty\\ R=+\infty & \rho=0\\ R=\frac{1}{\rho} & 0<\rho<+\infty\end{cases}$$

2.7.4 傅里叶级数

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

$$a_0=\frac{1}{l}\int_{-l}^{l}f(x)dx$$

$$a_n=\frac{1}{l}\int_{-l}^{l}f(x)\cos\frac{n\pi x}{l}dx$$

$$b_n=\frac{1}{l}\int_{-l}^{l}f(x)\sin\frac{n\pi x}{l}dx$$

2.8 常微分方程

三、线性代数

3.1 行列式

3.1.1 拉普拉斯展开

$$\begin{vmatrix}A & *\\ O & B\end{vmatrix}=\begin{vmatrix}A & O\\ * & B\end{vmatrix}=|A|\cdot|B|$$

$$\begin{vmatrix}O & A\\ B & *\end{vmatrix}=\begin{vmatrix}* & A\\ B & O\end{vmatrix}=(-1)^{mn}|A|\cdot|B|$$

3.1.2 范德蒙行列式

$$\begin{vmatrix}1 & 1 & \cdots & 1\\ x_1 & x_2 & \cdots & x_n\\ x_1^2 & x_2^2 & \cdots & x_3^2\\ \vdots & \vdots & & \vdots\\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1}\end{vmatrix}=\prod_{1\le i<j\le n}(x_i-x_j)$$

3.1.3 行列式公式

$$|A^T|=|A|$$

$|kA| = k^n |A|$

$|AB| = |A||B|$

$|A^*| = |A|^{n-1}$

$|A^{-1}| = |A|^{-1}$

$|A| = \prod_{i=1}^{n} \lambda_i$

$A \sim B \Rightarrow |A| = |B|$

3.1.4 代数余子式

$A_{ij} = (-1)^{i+j} M_{ij}$

3.2 矩阵

3.2.1 矩阵公式

$(A+B)^T = A^T + B^T$

$(kA)^T = kA^T$

$(AB)^T = B^T A^T$

3.2.2 伴随矩阵

$AA^* = A^*A = |A|E$

$(A^*)^{-1} = (A^{-1})^* = \frac{1}{|A|}A$

$(A^*)^T = (A^T)^*$

$(kA)^* = k^{n-1}A^*$

$|A^*| = |A|^{n-1}$

$$r(A^*) = \begin{cases} n & r(A) = n, \\ 1 & r(A) = n-1, \\ 0 & r(A) < n-1 \end{cases}$$

3.2.3 逆矩阵

$A^{-1} = \frac{1}{|A|}A^*$

$$\begin{bmatrix} B & O \\ O & C \end{bmatrix}^{-1} = \begin{bmatrix} B^{-1} & O \\ O & C^{-1} \end{bmatrix}$$

$$\begin{bmatrix} O & B \\ C & O \end{bmatrix}^{-1} = \begin{bmatrix} O & C^{-1} \\ B^{-1} & O \end{bmatrix}$$

3.2.4 秩

$$r\begin{pmatrix} A & O \\ O & B \end{pmatrix} = r(A) + r(B)$$

$A_{\text{可逆}} \Rightarrow r(AB) = r(BA) = r(B)$

3.3 向量

3.3.1 *Schmidt*正交化

$\alpha_1, \alpha_2, \alpha_3$线性无关

1. $\beta_1 = \alpha_1$
   $\beta_2 = \alpha_2 - \frac{(\alpha_2,\beta_1)}{(\beta_1,\beta_1)}\beta_1$
   $\beta_3 = \alpha_3 - \frac{(\alpha_3,\beta_1)}{(\beta_1,\beta_1)}\beta_1 - \frac{(\alpha_3,\beta_2)}{(\beta_2,\beta_2)}\beta_2$
2. $\gamma_1 = \frac{\beta_1}{|\beta_1|} \quad \gamma_2 = \frac{\beta_2}{|\beta_2|} \quad \gamma_3 = \frac{\beta_3}{|\beta_3|}$

3.4 线性方程组

解向量个数$= n - r(A)$

3.4.1 齐次方程组$Ax = 0$

$|\,A\,| = 0 \Rightarrow$方程有非零解

通解：$x = k_1\eta_1 + k_2\eta_2 + \cdots + k_{n-r(A)}\eta_{n-r(A)}$

3.4.2 非齐次方程组$Ax = b$

$r(A) \neq r(\overline{A}) \Rightarrow$方程无解

$r(A) = r(B) = n \Rightarrow$方程有唯一解

$r(A) = r(B) < n \Rightarrow$方程有无穷多解

通解：$x = \alpha + k_1\eta_1 + \cdots + k_{n-r(A)}\eta_{n-r(A)}$

3.5 特征值

3.5.1 定义

$A\alpha = \lambda\alpha$

特征值：λ

特征向量：α

特征矩阵：$\lambda E - A$

特征方程：$|\,\lambda E - A\,| = 0$

3.5.2 性质

$A = [a_{ij}]_{n\times n}$

$\sum\limits_{i=1}^{n}\lambda_i = \sum\limits_{i=1}^{n}a_{ii}$

$\prod\limits_{i=1}^{n}\lambda_i = |\,A\,|$

3.5.3 相似

$P^{-1}AP = B \Rightarrow A \sim B$

3.5.4 实对称矩阵的相似对角化

1. 解特征方程$|\lambda E - A| = 0$
2. $\forall \lambda_i$，解$(\lambda_i E - A)x = 0$
3. 施密特正交化特征向量
4. 令$Q = [\gamma_{11}, \gamma_{12} \cdots]$

3.6 二次型

四、概率论与数理统计