一、初等数学

1.1 因式分解

$(a \pm b)^2 = a^2 \pm 2ab + b^2$

$a^2 - b^2 = (a+b)(a-b)$

$a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$

1.2 三角函数

$\sin 2x = 2\sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

二、高等数学

2.1 函数、极限、连续

2.1.1 连续

$\lim\limits_{x \to x_0} f(x) = f(x_0)$

2.1.2 极限

$\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$

$\lim\limits_{x \to 0}(1+x)^{\frac{1}{x}} = e$

$\lim\limits_{x \to \infty} \sqrt[n]{n} = 1$

$x^y = e^{y \ln x}$

2.1.3 $x \to 0$时的等价无穷小

$\sin x \sim \tan x \sim \arcsin x \sim \arctan x \sim \ln(1+x) \sim (e^x - 1) \sim x$

$1 - \cos x \sim \frac{1}{2}x^2$

$(1+x)^a - 1 \sim ax$

2.2 一元函数微分学

2.2.1 导数

$f'(x_0) = \lim\limits_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$

2.2.2 常用微分公式：

- $dC = 0$
- $dx^{\alpha} = \alpha x^{\alpha-1}dx$
- $da^{x} = a^{x}\ln a\,dx$
- $d\log_a^x = \frac{dx}{x\ln a}$
- $d\sin x = \cos x dx$
- $d\cos x = -\sin x\,dx$
- $d\tan x = \sec^2 x\,dx$
- $d\cot x = -\csc^2 x\,dx$
- $d\sec x = \sec x\tan x\,dx$
- $d\csc x = -\csc x\cot x\,dx$
- $d\arcsin x = \frac{dx}{\sqrt{1-x^2}}$
- $d\arctan x = \frac{dx}{1+x^2}$
- $duv = udv + vdu$
- $d\frac{v}{u} = \frac{udv-vdu}{u^2}$

2.3 一元函数积分学

2.3.1 常用不定积分公式

- $\int x^{\alpha}dx = \frac{1}{\alpha+1}x^{\alpha+1} + C$
- $\int \frac{1}{x}dx = \ln x + C$
- $\int \alpha^{x}dx = \frac{1}{\ln a}a^{x} + C$
- $\int \cos x dx = \sin x + C$
- $\int \sin x dx = -\cos x + C$
- $\int \sec x dx = \ln|\sec x + \tan x| + C$
- $\int \csc x dx = \ln|\csc x - \cot x| + C$
- $\int \tan x dx = -\ln|\cos x| + C$
- $\int \cot x dx = \ln|\sin x| + C$
- $\int \sec^2 x dx = \tan x + C$
- $\int \csc^2 x = -\cot x + C$
- $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\arctan\frac{x}{a} + C$
- $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$
- $\int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin\frac{x}{a} + C$
- $\int \frac{1}{\sqrt{x^2\pm a^2}}dx = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C$
- $\int udv = uv - \int vdu$
- $\int [f(x) + f'(x)]e^{x}dx = f(x)e^{x} + C$

2.3.2 万能代换公式

$\sin x = \frac{2t}{1+t^2}$

$\cos x = \frac{1-t^2}{1+t^2}$

$\tan x = \frac{2t}{1+t^2}$

$dx = \frac{2}{1+t^2}dt$

2.3.3 Γ函数

2.3.3.1 定义

$\Gamma(\alpha)=\int_0^{+\infty} x^{\alpha-1}e^{-x}dx$

2.3.3.2 计算

$\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$
$\Gamma(n+1)=n!$
$\Gamma(\frac{1}{2})=\sqrt{\pi}$

2.3.4 华莱士公式

$$\int_0^{\frac{\pi}{2}}\sin^n xdx=\int_0^{\frac{\pi}{2}}\cos^n xdx=\begin{cases}\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{1}{2}\cdot\frac{\pi}{2} & \text{n为偶数,}\\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{2}{3}\cdot 1 & \text{n为奇数}\end{cases}$$

2.3.5 定积分公式

$\int_0^{\pi} xf(\sin x)dx=\frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$

$f(b)-f(a)=\int_a^b f'(x)dx$

$f(b)-f(a)=f'(\xi)(b-a)\quad(a<\xi<b)$

2.3.6 不等式

$(\int_a^b f(x)g(x)dx)^2\leq\int_a^b f^2(x)dx\int_a^b g^2(x)dx$

$\left|\int_a^b f(x)dx\right|\leq\int_a^b|f(x)|\,dx$

2.3.7 反常积分

$\int_a^b f(x)dx$

$$\lim_{x\to b^-}(b-x)^p f(x)=A\Rightarrow\begin{cases}p<1 & \text{收敛}\\ p\geq 1 & \text{发散}\end{cases}$$

$$\lim_{x\to a^+}(x-a)^p f(x)=A\Rightarrow\begin{cases}p<1 & \text{收敛}\\ p\geq 1 & \text{发散}\end{cases}$$

$$\lim_{x\to\pm\infty}x^p f(x)=A\Rightarrow\begin{cases}p>1 & \text{收敛}\\ p\leq 1 & \text{发散}\end{cases}$$

2.3.8 定积分应用

2.3.8.1 弧长

$s=\int_\alpha^\beta\sqrt{x'^2(t)+y'^2(t)}dt$

$s=\int_\alpha^\beta\sqrt{1+y'^2(x)}dx$

$s=\int_\alpha^\beta\sqrt{r^2(\theta)+r'^2(\theta)}d\theta$

2.3.8.2 旋转体体积

$V=\pi\int_\alpha^\beta[y_2^2(x)-y_1^2(x)]dx\quad$ 绕 x 轴

$V=2\pi\int_\alpha^\beta x(y_2(x)-y_1(x))dx\quad$ 绕 y 轴

2.3.8.3 旋转曲面面积

$S=2\pi\int_\alpha^\beta|y|\sqrt{1+f'^2(x)}dx$

$S=2\pi\int_{\alpha}^{\beta}|y(t)|\sqrt{x'^2(t)+y'^2(t)}dt$

2.4 向量代数与空间解析几何

2.5 多元函数微分学

2.6 多元函数积分学

2.7 无穷级数

2.7.1 级数审敛准则

2.7.1.1 必要条件

$\lim\limits_{n\to\infty}u_n=0$

2.7.1.2 正项级数

$$\lim_{n\to\infty}\frac{u_n}{v_n}=l\Rightarrow\begin{cases}0<l<+\infty & \sum\limits_{n=1}^{\infty}u_n\text{与}\sum\limits_{n=1}^{\infty}v_n\text{同敛散}\\ l=0 & \sum\limits_{n=1}^{\infty}v_n\text{收敛}\rightarrow\sum\limits_{n=1}^{\infty}u_n\text{收敛}\\ l=+\infty & \sum\limits_{n=1}^{\infty}v_n\text{发散}\rightarrow\sum\limits_{n=1}^{\infty}u_n\text{发散}\end{cases}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\rho\Rightarrow\sum_{n=1}^{\infty}u_n\begin{cases}\text{收敛} & \rho<1\\ \text{发散} & \rho>1\\ \text{不确定} & \rho=1\end{cases}$$

$$\lim_{n\to\infty}\sqrt[n]{u_n}=\rho\Rightarrow\sum_{n=1}^{\infty}u_n\begin{cases}\text{收敛} & \rho<1\\ \text{发散} & \rho>1\\ \text{不确定} & \rho=1\end{cases}$$

2.7.1.3 交错级数

$$\left.\begin{matrix}u_n\geq u_{n+1}\\ \lim\limits_{n=1}^{\infty}u_n=0\end{matrix}\right\}\Rightarrow\sum_{n=1}^{\infty}(-1)^{n-1}u_n\text{收敛}$$

2.7.2 常用幂级数

$e^x=\sum\limits_{n=0}^{\infty}\frac{x^n}{n!}\quad x\in(-\infty,+\infty)$

$\sin x=\sum\limits_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}\quad x\in(-\infty,+\infty)$

$\cos x=\sum\limits_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}\quad x\in(-\infty,+\infty)$

$\ln(1+x)=\sum\limits_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}\quad x\in(-1,1]$

$\frac{1}{1-x}=\sum\limits_{n=0}^{\infty}x^n\quad x\in(-1,1)$

$\frac{1}{1+x}=\sum\limits_{n=0}^{\infty}(-1)^n x^n\quad x\in(-1,1)$

$(1+x)^{\alpha}=\sum\limits_{n=0}^{\infty}\frac{\alpha!}{(\alpha-n)!n!}x^{n} \quad R=1$

2.7.3 傅里叶级数

$a_n=\frac{1}{l}\int_{-l}^{l}f(x)\cos\frac{n\pi x}{l}dx$

$b_n=\frac{1}{l}\int_{-l}^{l}f(x)\sin\frac{n\pi x}{l}dx$

2.8 常微分方程

三、线性代数

3.1 行列式

3.1.1 拉普拉斯展开

$\begin{vmatrix}A & * \\ O & B\end{vmatrix}=\begin{vmatrix}A & O \\ * & B\end{vmatrix}=|\,A\,|\cdot|\,B\,|$

$\begin{vmatrix}O & A \\ B & *\end{vmatrix}=\begin{vmatrix}* & A \\ B & O\end{vmatrix}=(-1)^{mn}\,|\,A\,|\cdot|\,B\,|$

3.1.2 范德蒙行列式

$\begin{vmatrix}1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_3^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1}\end{vmatrix}=\prod\limits_{1\le i<j\le n}(x_i-x_j)$

3.1.3 行列式公式

$|\,A^T\,|=|\,A\,|$

$|\,kA\,|=k^n\,|\,A\,|$

$|\,AB\,|=|\,A\,|\,|\,B\,|$

$|\,A^*\,|=|\,A\,|^{n-1}$

$|\,A^{-1}\,|=|\,A\,|^{-1}$

$|\,A\,|=\prod\limits_{i=1}^{n}\lambda_i$

$A\sim B\Rightarrow|\,A\,|=|\,B\,|$

3.1.4 代数余子式

$A_{ij}=(-1)^{i+j}M_{ij}$

3.2 矩阵

3.2.1 矩阵公式

$(A+B)^T = A^T + B^T$

$(kA)^T = kA^T$

$(AB)^T = B^T A^T$

3.2.2 伴随矩阵

$AA^* = A^* A = |\,A\,| E$

$(A^*)^{-1} = (A^{-1})^* = \frac{1}{|\,A\,|} A$

$(A^*)^T = (A^T)^*$

$(kA)^* = k^{n-1} A^*$

$|\,A^*\,| = |\,A\,|^{n-1}$

$r(A^*) = \begin{cases} n & r(A) = n, \\ 1 & r(A) = n-1, \\ 0 & r(A) < n-1 \end{cases}$

3.2.3 逆矩阵

$A^{-1} = \frac{1}{|\,A\,|} A^*$

$\begin{bmatrix} B & O \\ O & C \end{bmatrix}^{-1} = \begin{bmatrix} B^{-1} & O \\ O & C^{-1} \end{bmatrix}$

$\begin{bmatrix} O & B \\ C & O \end{bmatrix}^{-1} = \begin{bmatrix} O & C^{-1} \\ B^{-1} & O \end{bmatrix}$

3.2.4 秩

$r\begin{pmatrix} A & O \\ O & B \end{pmatrix} = r\,(\,A\,) + r\,(\,B\,)$

$A_{\text{可逆}} \Rightarrow r\,(\,AB\,) = r\,(\,BA\,) = r\,(\,B\,)$

3.3 向量

3.3.1 Schmidt正交化

$\alpha_1, \alpha_2, \alpha_3$线性无关

1. $\beta_1 = \alpha_1$
   $\beta_2 = \alpha_2 - \frac{(\alpha_2,\beta_1)}{(\beta_1,\beta_1)}\beta_1$
   $\beta_3 = \alpha_3 - \frac{(\alpha_3,\beta_1)}{(\beta_1,\beta_1)}\beta_1 - \frac{(\alpha_3,\beta_2)}{(\beta_2,\beta_2)}\beta_2$
2. $\gamma_1 = \frac{\beta_1}{|\beta_1|} \quad \gamma_2 = \frac{\beta_2}{|\beta_2|} \quad \gamma_3 = \frac{\beta_3}{|\beta_3|}$

3.4 线性方程组

3.5 特征值

3.5.1 定义

$A\alpha = \lambda\alpha$

特征值：λ

特征向量：α

特征矩阵：$\lambda E - A$

特征方程：$|\lambda E - A| = 0$

3.5.2 性质

$A = [a_{ij}]_{n\times n}$

$\sum\limits_{i=1}^{n} \lambda_i = \sum\limits_{i=1}^{n} a_{ii}$

$\prod\limits_{i=1}^{n} \lambda_i = |A|$

3.5.3 相似

3.5.4 实对称矩阵的相似对角化

1. 解特征方程$|\lambda E - A| = 0$
2. $\forall \lambda_i$，解$(\lambda_i E - A)x = 0$
3. 施密特正交化特征向量
4. 令$Q = [\gamma_{11}, \gamma_{12} \cdots]$

3.6 二次型

四、概率论与数理统计