

Abstract

This paper presents a VIS for 2D scalar RTEs.

1 Section1

In general, RTE can be written as:

$$\hat{\mathbf{s}} \cdot \nabla \psi(\mathbf{r}, \hat{\mathbf{s}}) + \mu_t(\mathbf{r}) \psi(\mathbf{r}, \hat{\mathbf{s}}) - \mu_s(\mathbf{r}) \int d\hat{\mathbf{s}}' f(\hat{\mathbf{s}} \cdot \hat{\mathbf{s}}') \psi(\mathbf{r}, \hat{\mathbf{s}}') = q(\mathbf{r}, \hat{\mathbf{s}})$$

with

\mathbf{r}	$[L]^+1$	position vector
$\hat{\mathbf{s}}$	$[L]^0$	unit direction vector
ψ	$[L]^0$	specific intensity
q	$[L]^{-1}$	source
f	$[L]^0$	phase function
μ_s	$[L]^{-1}$	scattering cross-section
μ_a	$[L]^{-1}$	absorption cross-section
μ_t	$[L]^{-1}$	$\mu_a + \mu_s$, total cross-section

We can construct the volume integral equation (VIE):

$$\begin{aligned} \psi(\mathbf{r}, \hat{\mathbf{s}}) &+ \int d\mathbf{r}' d\hat{\mathbf{s}}' g(\mathbf{r}, \hat{\mathbf{s}}; \mathbf{r}', \hat{\mathbf{s}}') \mu_t(\mathbf{r}') \psi(\mathbf{r}', \hat{\mathbf{s}}') \\ &- \int d\mathbf{r}' d\hat{\mathbf{s}}' g(\mathbf{r}, \hat{\mathbf{s}}; \mathbf{r}', \hat{\mathbf{s}}') \mu_s(\mathbf{r}') \\ &\times \int d\hat{\mathbf{s}}'' f(\hat{\mathbf{s}}' \cdot \hat{\mathbf{s}}'') \psi(\mathbf{r}', \hat{\mathbf{s}}'') \\ &= \int d\mathbf{r}' d\hat{\mathbf{s}}' g(\mathbf{r}, \hat{\mathbf{s}}; \mathbf{r}', \hat{\mathbf{s}}') q(\mathbf{r}', \hat{\mathbf{s}}') \end{aligned} \quad (1)$$

where g is the free space Green's function:

$$g(\mathbf{r}, \hat{\mathbf{s}}; \mathbf{r}', \hat{\mathbf{s}}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} \delta(\hat{\mathbf{s}} - \hat{\mathbf{s}}_{\mathbf{r}-\mathbf{r}'}) \delta(\hat{\mathbf{s}}' - \hat{\mathbf{s}}_{\mathbf{r}-\mathbf{r}'}) \quad (2)$$

In 2D, (1) and (2) become

$$\begin{aligned} \psi(\mathbf{r}, \phi) &+ \int d\mathbf{r}' d\phi' g(\mathbf{r}, \phi; \mathbf{r}', \phi') \mu_t(\mathbf{r}') \psi(\mathbf{r}', \phi') \\ &- \int d\mathbf{r}' d\phi' g(\mathbf{r}, \phi; \mathbf{r}', \phi') \mu_s(\mathbf{r}') \\ &\times \int d\phi'' f(\phi' - \phi'') \psi(\mathbf{r}', \phi'') \\ &= \int d\mathbf{r}' d\phi' g(\mathbf{r}, \phi; \mathbf{r}', \phi') q(\mathbf{r}', \phi') \end{aligned} \quad (3)$$

$$g(\mathbf{r}, \phi; \mathbf{r}', \phi') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} \delta(\phi - \phi_{\mathbf{r}-\mathbf{r}'}) \delta(\phi' - \phi_{\mathbf{r}-\mathbf{r}'}) \quad (4)$$

2 Section2

Expand the specific intensity $\psi(\mathbf{r}, \phi)$:

$$\psi(\mathbf{r}, \phi) = \sum_{n=1}^{N_s} \sum_{m=-N_d}^{N_d} X_{nm} \xi_{nm}(\mathbf{r}, \phi)$$

In this paper, the basis function is chosen as

$$\begin{aligned} \xi_{nm}(\mathbf{r}, \phi) &= S_n(\mathbf{r}) e^{im\phi} \\ S_n(\mathbf{r}, \phi) &= \begin{cases} 1, & \mathbf{r} \in S_n \\ 0, & \mathbf{r} \notin S_n \end{cases} \end{aligned}$$

The latter sections, the following notations are used:

$S_n(\mathbf{r})$	the pulse function
S_n	the n-th triangle
$S(n)$	the area of the n-th triangle
$\sum_{n,m}$	$\sum_{n=1}^{N_s} \sum_{m=-N_d}^{N_d}$

Write

$$\psi(\mathbf{r}, \phi) = \sum_{n', m'} X_{n'm'} \xi_{n'm'}(\mathbf{r}, \phi) \quad (5)$$

Plug (5) into (3), multiply by $\xi_{nm}^*(\mathbf{r}, \phi)$, integrate over $d\mathbf{r}d\phi$, we get a set of linear equations:

$$\sum_{n', m'} Z_{(nm)(n'm')} X_{n'm'} = V_{nm}$$

with

$$\begin{aligned} Z_{(nm)(n'm')} &= A_{(nm)(n'm')} + B_{(nm)(n'm')} \\ A_{(nm)(n'm')} &= \int d\mathbf{r} d\phi \xi_{nm}^*(\mathbf{r}, \phi) \xi_{n'm'}(\mathbf{r}, \phi) \\ &= \int d\mathbf{r} S_n(\mathbf{r}) S_{n'}(\mathbf{r}) \int d\phi e^{-im\phi} e^{im'\phi} \\ &= 2\pi S(n) \delta_{nn'} \delta_{mm'} \\ B_{(nm)(n'm')} &= \int d\mathbf{r} d\phi \xi_{nm}^*(\mathbf{r}, \phi) \int d\mathbf{r}' d\phi' g(\mathbf{r}, \phi; \mathbf{r}', \phi') \\ &\times \mu_t(\mathbf{r}') \xi_{n'm'}(\mathbf{r}', \phi') - \int d\mathbf{r} d\phi \xi_{nm}^*(\mathbf{r}, \phi) \\ &\times \int d\mathbf{r}' d\phi' g(\mathbf{r}, \phi; \mathbf{r}', \phi') \mu_s(\mathbf{r}') \\ &\times \int d\phi'' f(\phi' - \phi'') \xi_{n'm'}(\mathbf{r}', \phi'') \\ V_{nm} &= \int d\mathbf{r} d\phi \xi_{nm}^*(\mathbf{r}, \phi) \psi^I(\mathbf{r}, \phi) \\ \psi^I(\mathbf{r}, \phi) &\equiv \int d\mathbf{r}' d\phi' g(\mathbf{r}, \phi; \mathbf{r}', \phi') q(\mathbf{r}', \phi') \end{aligned}$$

Note that (n, m) represents the filed point, whereas (n', m') represents the source point.

3 Section3

The interaction matrix can be show to be block-wise Toeplitz. Using proper quadrature rules $\{\mathbf{r}_{j_n}, w_{j_n}\}_{j_n=1}^{M_n}$ to compute each elements, we get:

$$\begin{aligned}
B_{(nm)(n'm')} &= C_t(n, n', m - m') \\
&\quad + C_s(n, n', m - m') g^{|m'|} \\
C_t(n, n', \Delta m) &= + \sum_{j_n=1}^{M_n} \sum_{j_{n'}=1}^{M'_{n'}} w_{j_n} w_{j_{n'}} \mu_t(\mathbf{r}_{j_{n'}}) \\
&\quad \times \frac{e^{-i\Delta m \phi_{\mathbf{r}_{j_n} - \mathbf{r}_{j_{n'}}}}}{|\mathbf{r}_{j_n} - \mathbf{r}_{j_{n'}}|} \\
C_s(n, n', \Delta m) &= - \sum_{j_n=1}^{M_n} \sum_{j_{n'}=1}^{M'_{n'}} w_{j_n} w_{j_{n'}} \mu_s(\mathbf{r}_{j_{n'}}) \\
&\quad \times \frac{e^{-i\Delta m \phi_{\mathbf{r}_{j_n} - \mathbf{r}_{j_{n'}}}}}{|\mathbf{r}_{j_n} - \mathbf{r}_{j_{n'}}|}
\end{aligned}$$

We prove the Toeplitzness of each block of B with given (n, n') in two steps. [TODO]

A $2N_d + 1$ by $2N_d + 1$ Toeplitz matrix can be stored in $O(N_d)$ memory and multiplied by a vector in $O(N_d \log N_d)$ time.

4 Section4

Assuming planewave incidence, numerically compute the right hand side (*r.h.s.*), i.e., the input vector:

$$V_{nm} = \sum_{n'}^{N_s} D(n, n', m)$$

where D is a rank-3 tensor:

$$\begin{aligned}
D(n, n', m) &= \sum_{j_n=1}^{M_n} \sum_{j_{n'}=1}^{M'_{n'}} w_{j_n} w_{j_{n'}} \mu_s(\mathbf{r}_{j_{n'}}) \frac{e^{-im\phi_{\mathbf{r}_{j_n} - \mathbf{r}_{j_{n'}}}}}{|\mathbf{r}_{j_n} - \mathbf{r}_{j_{n'}}|} \\
&\quad \times f(\phi_{\mathbf{r}_{j_n} - \mathbf{r}_{j_{n'}}}) e^{-\tau(\mathbf{r}_{j_{n'}}, -\hat{\mathbf{s}}^I)}
\end{aligned}$$

$\tau(\mathbf{r}, -\hat{\mathbf{s}}^I) \equiv \int_{C_{\mathbf{r}, -\hat{\mathbf{s}}^I}} \mu_t dl$, where the integral path $C_{\mathbf{r}, -\hat{\mathbf{s}}^I}$ is the ray that starts from \mathbf{r} and in the direction pointed by $-\hat{\mathbf{s}}^I$. For later use, $\tau(\mathbf{r}' \rightarrow \mathbf{r}) \equiv \int_{C_{\mathbf{r}' \rightarrow \mathbf{r}}}$, where $C_{\mathbf{r}' \rightarrow \mathbf{r}}$ is the line segment from \mathbf{r}' to \mathbf{r} .

In principle, computing τ requires ray tracing. Done with a tree-structure [TODO].

Appendices

A Appendix1

This part documents the quadrature rules employed in computing the matrix elements.

In short, the integral of interest takes on the following form:

$$I = \iint_{\Omega} dx dy \frac{f(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}}$$

where Ω is defined by the three nodes $P_0(x_0, y_0)$, $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $f(x, y)$ is a smooth function.

Use the arcsinh transform method to cancel the singularity at P_0 and transform the triangular domain into a rectangular domain. Let $\{\xi_u^i, w_u^i\}_{i=1}^{M_u}$ and $\{\xi_v^j, w_v^j\}_{j=1}^{M_v}$ be the Legendre quadrature rules over the $(0, 1)$ for the angular part and the radial part, respectively. Then

$$I = |h(u_1 - u_2)| \sum_{i,j}^{M_u, M_v} w_u^i w_v^j f(x, y)|_{(i,j)}$$

with

$$\begin{aligned}
x'_1 &= [(x_2 - x_1)(x_1 - x_0) + (y_2 - y_1)(y_1 - y_0)] / P_1 P_2 \\
x'_2 &= [(x_2 - x_1)(x_2 - x_0) + (y_2 - y_1)(y_2 - y_0)] / P_1 P_2 \\
h &= [-(x_2 - x_1)y_0 + (x_2 - x_0)y_1 - (x_1 - x_0)y_2] / P_1 P_2 \\
u_1 &= \sinh^{-1}(x'_1/h) \quad u_2 = \sinh^{-1}(x'_2/h) \\
u^i &= u_1 + (u_2 - u_1)\xi_u^i \quad v^j = h\xi_v^j
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{A} &= \frac{1}{P_1 P_2} \begin{pmatrix} x_2 - x_1 & -y_2 + y_1 \\ y_2 - y_1 & x_2 - x_1 \end{pmatrix} \\
\begin{pmatrix} x \\ y \end{pmatrix} \Big|_{(i,j)} &= \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \mathbf{A} \cdot \begin{pmatrix} v^j \sinh u^i \\ v^j \end{pmatrix}
\end{aligned}$$

Note that there is no assumption on which one of u_1 and u_2 is larger. The above equations can be coded without modification.

References