A Fast Volume Integral Solver for 2D Scalar Radiative Transport Equation

LaTeX by Zhongming Qu January 20, 2014

1 Section1

In general, RTE can be written as:

$$\begin{aligned}\hat{\mathbf{s}} \cdot \boldsymbol{\nabla} \psi(\mathbf{r}, \hat{\mathbf{s}}) + \mu_t(\mathbf{r}) \psi(\mathbf{r}, \hat{\mathbf{s}}) \\ -\mu_s(\mathbf{r}) \int d\hat{\mathbf{s}}' f(\hat{\mathbf{s}} \cdot \hat{\mathbf{s}}') \psi(\mathbf{r}, \hat{\mathbf{s}}') = q(\mathbf{r}, \hat{\mathbf{s}})\end{aligned}$$

with

$\mathbf{r}$	$[L]^{+1}$	position vector
$\hat{\mathbf{s}}$	$[L]^0$	unit direction vector
ψ	$[L]^0$	specific intensity
q	$[L]^{-1}$	source
f	$[L]^0$	phase function
μ_s	$[L]^{-1}$	scattering cross-section
μ_a	$[L]^{-1}$	absorption cross-section
μ_t	$[L]^{-1}$	$\mu_a + \mu_s$, total cross-section

We can construct the volume integral equation (VIE):

$$\begin{aligned}\psi(\mathbf{r}, \hat{\mathbf{s}}) + &\int d\mathbf{r}' d\hat{\mathbf{s}}' g(\mathbf{r}, \hat{\mathbf{s}}; \mathbf{r}', \hat{\mathbf{s}}') \mu_t(\mathbf{r}') \psi(\mathbf{r}', \hat{\mathbf{s}}') \\ &- \int d\mathbf{r}' d\hat{\mathbf{s}}' g(\mathbf{r}, \hat{\mathbf{s}}; \mathbf{r}', \hat{\mathbf{s}}') \mu_s(\mathbf{r}') \int d\hat{\mathbf{s}}'' f(\hat{\mathbf{s}}' \cdot \hat{\mathbf{s}}'') \psi(\mathbf{r}', \hat{\mathbf{s}}'') \\ &= \int d\mathbf{r}' d\hat{\mathbf{s}}' g(\mathbf{r}, \hat{\mathbf{s}}; \mathbf{r}', \hat{\mathbf{s}}') q(\mathbf{r}', \hat{\mathbf{s}}')\end{aligned} \tag{1.1}$$

where g is the free space Green's function:

$$g_t(\mathbf{r}, \hat{\mathbf{s}}; \mathbf{r}', \hat{\mathbf{s}}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} \delta(\hat{\mathbf{s}} - \hat{\mathbf{s}}_{\mathbf{r}-\mathbf{r}'}) \delta(\hat{\mathbf{s}}' - \hat{\mathbf{s}}_{\mathbf{r}-\mathbf{r}'}) \tag{1.2}$$

In 2D, (1.1) and (1.2) are

$$\begin{aligned}\psi(\mathbf{r}, \phi) + &\int d\mathbf{r}' d\phi' g(\mathbf{r}, \phi; \mathbf{r}', \phi') \mu_t(\mathbf{r}') \psi(\mathbf{r}', \phi') \\ &- \int d\mathbf{r}' d\phi' g(\mathbf{r}, \phi; \mathbf{r}', \phi') \mu_s(\mathbf{r}') \int d\phi'' f(\phi' - \phi'') \psi(\mathbf{r}', \phi'') \\ &= \int d\mathbf{r}' d\phi' g(\mathbf{r}, \phi; \mathbf{r}', \phi') q(\mathbf{r}', \phi')\end{aligned} \tag{1.3}$$

and

$$g_t(\mathbf{r}, \phi; \mathbf{r}', \phi') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} \delta(\phi - \phi_{\mathbf{r}-\mathbf{r}'}) \delta(\phi' - \phi_{\mathbf{r}-\mathbf{r}'}) \tag{1.4}$$

respectively.

2 Section2

Expand the specific intensity $\psi(\mathbf{r}, \phi)$:

$$\psi(\mathbf{r}, \phi) = \sum_{n=1}^{N_s} \sum_{m=-N_d}^{N_d} X_{nm} \xi_{nm}(\mathbf{r}, \phi)$$

In this paper, the basis function is chosen as

$$\begin{aligned}\xi_{nm}(\mathbf{r}, \phi) =& S_n(\mathbf{r}) e^{im\phi} \\ S_n(\mathbf{r}, \phi) =& \begin{cases} 1, & \mathbf{r} \in S_n \\ 0, & \mathbf{r} \notin S_n \end{cases}\end{aligned}$$

The latter sections, the following notations are used:

$S_n(\mathbf{r})$	the pulse function
$S_n)$	the n-th triangle
$S(n)$	the area of the n-th triangle
$\sum_{n,m}$	$\sum_{n=1}^{N_s} \sum_{m=-N_d}^{N_d}$