A Fast Volume Integral Solver for 2D Scalar Radiative Transport Equation

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Abstract

This paper presents a VIS for 2D scalar RTEs.

1   Section1

In general, RTE can be written as:

$$\hat{\mathbf{s}} \cdot \boldsymbol{\nabla}\psi(\mathbf{r}, \hat{\mathbf{s}}) + \mu_t(\mathbf{r})\psi(\mathbf{r}, \hat{\mathbf{s}})$$

$$-\mu_s(\mathbf{r}) \int d\hat{\mathbf{s}}' f(\hat{\mathbf{s}} \cdot \hat{\mathbf{s}}')\psi(\mathbf{r}, \hat{\mathbf{s}}') = q(\mathbf{r}, \hat{\mathbf{s}})$$

with

$\mathbf{r}$	$[L]^{+1}$	position vector
$\hat{\mathbf{s}}$	$[L]^0$	unit direction vector
ψ	$[L]^0$	specific intensity
q	$[L]^{-1}$	source
f	$[L]^0$	phase function
μ_s	$[L]^{-1}$	scattering cross-section
μ_a	$[L]^{-1}$	absorption cross-section
μ_t	$[L]^{-1}$	$\mu_a + \mu_s$, total cross-section

We can construct the volume integral equation (VIE):

$$\psi(\mathbf{r}, \hat{\mathbf{s}}) + \int d\mathbf{r}' d\hat{\mathbf{s}}' g(\mathbf{r}, \hat{\mathbf{s}}; \mathbf{r}', \hat{\mathbf{s}}')\mu_t(\mathbf{r}')\psi(\mathbf{r}', \hat{\mathbf{s}}')$$

$$- \int d\mathbf{r}' d\hat{\mathbf{s}}' g(\mathbf{r}, \hat{\mathbf{s}}; \mathbf{r}', \hat{\mathbf{s}}')\mu_s(\mathbf{r}')$$

$$\times \int d\hat{\mathbf{s}}'' f(\hat{\mathbf{s}}' \cdot \hat{\mathbf{s}}'')\psi(\mathbf{r}', \hat{\mathbf{s}}'') \tag{1}$$

$$= \int d\mathbf{r}' d\hat{\mathbf{s}}' g(\mathbf{r}, \hat{\mathbf{s}}; \mathbf{r}', \hat{\mathbf{s}}')q(\mathbf{r}', \hat{\mathbf{s}}')$$

where g is the free space Green's function:

$$g(\mathbf{r}, \hat{\mathbf{s}}; \mathbf{r}', \hat{\mathbf{s}}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|}\delta(\hat{\mathbf{s}} - \hat{\mathbf{s}}_{\mathbf{r}-\mathbf{r}'})\delta(\hat{\mathbf{s}}' - \hat{\mathbf{s}}_{\mathbf{r}-\mathbf{r}'}) \tag{2}$$

In 2D, (1) and (2) become

$$\psi(\mathbf{r}, \phi) + \int d\mathbf{r}' d\phi' g(\mathbf{r}, \phi; \mathbf{r}', \phi')\mu_t(\mathbf{r}')\psi(\mathbf{r}', \phi')$$

$$- \int d\mathbf{r}' d\phi' g(\mathbf{r}, \phi; \mathbf{r}', \phi')\mu_s(\mathbf{r}')$$

$$\times \int d\phi'' f(\phi' - \phi'')\psi(\mathbf{r}', \phi'') \tag{3}$$

$$= \int d\mathbf{r}' d\phi' g(\mathbf{r}, \phi; \mathbf{r}', \phi')q(\mathbf{r}', \phi')$$

$$g(\mathbf{r}, \phi; \mathbf{r}', \phi') = \frac{1}{|\mathbf{r} - \mathbf{r}'|}\delta(\phi - \phi_{\mathbf{r}-\mathbf{r}'})\delta(\phi' - \phi_{\mathbf{r}-\mathbf{r}'}) \tag{4}$$

2   Section2

Expand the specific intensity $\psi(\mathbf{r}, \phi)$:

$$\psi(\mathbf{r}, \phi) = \sum_{n=1}^{N_s} \sum_{m=-N_d}^{N_d} X_{nm}\xi_{nm}(\mathbf{r}, \phi)$$

In this paper, the basis function is chosen as

$$\xi_{nm}(\mathbf{r}, \phi) = S_n(\mathbf{r})e^{im\phi}$$

$$S_n(\mathbf{r}, \phi) = \begin{cases} 1, & \mathbf{r} \in S_n \\ 0, & \mathbf{r} \notin S_n \end{cases}$$

The latter sections, the following notations are used:

$S_n(\mathbf{r})$	the pulse function
$S_n)$	the n-th triangle
$S(n)$	the area of the n-th triangle
$\sum_{n,m}$	$\sum_{n=1}^{N_s}\sum_{m=-N_d}^{N_d}$

Write

$$\psi(\mathbf{r}, \phi) = \sum_{n',m'} X_{n'm'}\xi_{n'm'}(\mathbf{r}, \phi) \tag{5}$$

Plug (5) into (3), multiply by $\xi_{nm}^*(\mathbf{r}, \phi)$, integrate over $d\mathbf{r}d\phi$, we get a set of linear equations:

$$\sum_{n',m'} Z_{(nm)(n'm')}X_{n'm'} = V_{nm}$$

with

$$Z_{(nm)(n'm')} = A_{(nm)(n'm')} + B_{(nm)(n'm')}$$

$$A_{(nm)(n'm')} = \int d\mathbf{r}d\phi \xi_{nm}^*(\mathbf{r}, \phi)\xi_{n'm'}(\mathbf{r}, \phi)$$

$$= \int d\mathbf{r}S_n(\mathbf{r})S_{n'}(\mathbf{r}) \int d\phi e^{-im\phi}e^{+im'\phi}$$

$$= 2\pi S(n)\delta_{nn'}\delta_{mm'}$$

$$B_{(nm)(n'm')} = \int d\mathbf{r}d\phi \xi_{nm}^*(\mathbf{r}, \phi) \int d\mathbf{r}'d\phi' g(\mathbf{r}, \phi; \mathbf{r}', \phi')$$

$$\times \mu_t(\mathbf{r}')\xi_{n'm'}(\mathbf{r}', \phi') - \int d\mathbf{r}d\phi \xi_{nm}^*(\mathbf{r}, \phi)$$

$$\times \int d\mathbf{r}'d\phi' g(\mathbf{r}, \phi; \mathbf{r}', \phi')\mu_s(\mathbf{r}')$$

$$\times \int d\phi'' f(\phi' - \phi'')\xi_{n'm'}(\mathbf{r}', \phi'')$$

$$V_{nm} = \int d\mathbf{r}d\phi \xi_{nm}^*(\mathbf{r}, \phi)\psi^I(\mathbf{r}, \phi)$$

$$\psi^I(\mathbf{r}, \phi) \equiv \int d\mathbf{r}'d\phi' g(\mathbf{r}, \phi; \mathbf{r}', \phi')q(\mathbf{r}', \phi')$$

Note that (n, m) represents the filed point, whereas (n', m') represents the source point.

3 Section3

The interaction matrix can be show to be block-wise Toeplitz. Using proper quadrature rules $\{\mathbf{r}_{j_n}, w_{j_n}\}_{j_n=1}^{M_n}$ to compute each elements, we get:

$$B_{(nm)(n'm')} = C_t(n, n', m - m')$$
$$+ C_s(n, n', m - m')g^{|m'|}$$

$$C_t(n, n', \Delta m) = +\sum_{j_n=1}^{M_n} \sum_{j_{n'}=1}^{M'_n} w_{j_n} w_{j_{n'}} \mu_t(\mathbf{r}_{j_{n'}})$$
$$\times \frac{e^{-i\Delta m \phi_{\mathbf{r}_{j_n} - \mathbf{r}_{j_{n'}}}}}{\left|\mathbf{r}_{j_n} - \mathbf{r}_{j_{n'}}\right|}$$

$$C_s(n, n', \Delta m) = -\sum_{j_n=1}^{M_n} \sum_{j_{n'}=1}^{M'_n} w_{j_n} w_{j_{n'}} \mu_s(\mathbf{r}_{j_{n'}})$$
$$\times \frac{e^{-i\Delta m \phi_{\mathbf{r}_{j_n} - \mathbf{r}_{j_{n'}}}}}{\left|\mathbf{r}_{j_n} - \mathbf{r}_{j_{n'}}\right|}$$

We prove the Toeplitzness of each block of B with given (n, n') in two steps. [TODO]

A $2N_d + 1$ by $2N_d + 1$ Toeplitz matrix can be stored in $O(N_d)$ memory and multiplied by a vector in $O(N_d \log N_d)$ time.

4 Section4

Assuming planewave incidence, numerically compute the right hand side (*r.h.s.*), i.e., the input vector:

$$V_{nm} = \sum_{n'}^{N_s} D(n, n', m)$$

where D is a rank-3 tensor:

$$D(n, n', m) = \sum_{j_n=1}^{M_n} \sum_{j_{n'}=1}^{M_{n'}} w_{j_n} w_{j_{j'}} \mu_s(\mathbf{r}_{j_{n'}}) \frac{e^{-im\phi_{\mathbf{r}_{j_n} - \mathbf{r}_{j_{n'}}}}}{\left|\mathbf{r}_{j_n} - \mathbf{r}_{j_{n'}}\right|}$$
$$\times f(\phi_{\mathbf{r}_{j_n} - \mathbf{r}_{j_{n'}}}) e^{-\tau(\mathbf{r}_{j_{n'}}, -\hat{\mathbf{s}}^I)}$$

$\tau(\mathbf{r}, -\hat{\mathbf{s}}^I) \equiv \int_{C_{\mathbf{r}, -\hat{\mathbf{s}}^I}} \mu_t dl$, where the integral path $C_{\mathbf{r}, -\hat{\mathbf{s}}^I}$ is the ray that starts from $\mathbf{r}$ and in the direction pointed by $-\hat{\mathbf{s}}^I$. For later use, $\tau(\mathbf{r}' \to \mathbf{r}) \equiv \int_{C_{\mathbf{r}' \to \mathbf{r}}}$, where $C_{\mathbf{r}' \to \mathbf{r}}$ is the line segment from $\mathbf{r}'$ to $\mathbf{r}$.

In principle, computing τ requires ray tracing. Done with a tree-structure [TODO].

Appendices

A Appendix1

This part documents the quadrature rules employed in computing the matrix elements.

In short, the integral of interest takes on the following form:

$$I = \iint_{\Omega} dxdy \frac{f(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}}$$

where Ω is defined by the three nodes $P_0(x_0, y_0)$, $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $f(x, y)$ is a smooth function.

Use the arcsinh transfrom method to cancel the singularity at P_0 and transform the triangular domain into a rectangular domain. Let $\left\{\xi_u^i, w_u^i\right\}_{i=1}^{M_u}$ and $\left\{\xi_v^j, w_v^j\right\}_{j=1}^{M_v}$ be the Legendre quadrature rules over the $(0, 1)$ for the angular part and the radial part, respectively. Then

$$I = |h(u_1 - u_2)| \sum_{i,j}^{M_u, M_v} w_u^i w_v^j f(x, y)\big|_{(i,j)}$$

with

$$x'_1 = [(x_2 - x_1)(x_1 - x_0) + (y_2 - y_1)(y_1 - y_0)] / P_1 P_2$$
$$x'_2 = [(x_2 - x_1)(x_2 - x_0) + (y_2 - y_1)(y_2 - y_0)] / P_1 P_2$$
$$h = [-(x_2 - x_1)y_0 + (x_2 - x_0)y_1 - (x_1 - x_0)y_2] / P_1 P_2$$
$$u_1 = \sinh^{-1}(x'_1/h) \qquad u_2 = \sinh^{-1}(x'_2/h)$$
$$u^i = u_1 + (u_2 - u_1)\xi_u^i \qquad v^j = h\xi_v^j$$

and

$$\mathbf{A} = \frac{1}{P_1 P_2} \begin{pmatrix} x_2 - x_1 & -y_2 + y_1 \\ y_2 - y_1 & x_2 - x_1 \end{pmatrix}$$
$$\begin{pmatrix} x \\ y \end{pmatrix}\bigg|_{(i,j)} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \mathbf{A} \cdot \begin{pmatrix} v^j \sinh u^i \\ v^j \end{pmatrix}$$

Note that there is no assumption on which one of u_1 and u_2 is larger. The above equations can be coded without modification.

References