A Fast Volume Integral Solver for 2D Scalar Radiative Transport Equation

LaTeX by Zhongming Qu January 23, 2014

Abstract

This paper presents a VIS for 2D scalar RTEs.

1 Section1

In general, RTE can be written as:

$$\begin{aligned}\hat{\mathbf{s}}\cdot\boldsymbol{\nabla}\psi(\mathbf{r},\hat{\mathbf{s}})+\mu_t(\mathbf{r})\psi(\mathbf{r},\hat{\mathbf{s}})\\ -\mu_s(\mathbf{r})\int d\hat{\mathbf{s}}' f(\hat{\mathbf{s}}\cdot\hat{\mathbf{s}}')\psi(\mathbf{r},\hat{\mathbf{s}}')=q(\mathbf{r},\hat{\mathbf{s}})\end{aligned}$$

with

$\mathbf{r}$	$[L]^{+1}$	position vector
$\hat{\mathbf{s}}$	$[L]^0$	unit direction vector
ψ	$[L]^0$	specific intensity
q	$[L]^{-1}$	source
f	$[L]^0$	phase function
μ_s	$[L]^{-1}$	scattering cross-section
μ_a	$[L]^{-1}$	absorption cross-section
μ_t	$[L]^{-1}$	$\mu_a+\mu_s$, total cross-section

We can construct the volume integral equation (VIE):

$$\begin{aligned}\psi(\mathbf{r},\hat{\mathbf{s}})+&\int d\mathbf{r}'d\hat{\mathbf{s}}' g(\mathbf{r},\hat{\mathbf{s}};\mathbf{r}',\hat{\mathbf{s}}')\mu_t(\mathbf{r}')\psi(\mathbf{r}',\hat{\mathbf{s}}')\\ &-\int d\mathbf{r}'d\hat{\mathbf{s}}' g(\mathbf{r},\hat{\mathbf{s}};\mathbf{r}',\hat{\mathbf{s}}')\mu_s(\mathbf{r}')\\ &\times\int d\hat{\mathbf{s}}'' f(\hat{\mathbf{s}}'\cdot\hat{\mathbf{s}}'')\psi(\mathbf{r}',\hat{\mathbf{s}}'')\\ &=\int d\mathbf{r}'d\hat{\mathbf{s}}' g(\mathbf{r},\hat{\mathbf{s}};\mathbf{r}',\hat{\mathbf{s}}')q(\mathbf{r}',\hat{\mathbf{s}}')\end{aligned}\tag{1}$$

where g is the free space Green's function:

$$g(\mathbf{r},\hat{\mathbf{s}};\mathbf{r}',\hat{\mathbf{s}}')=\frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta(\hat{\mathbf{s}}-\hat{\mathbf{s}}_{\mathbf{r}-\mathbf{r}'})\delta(\hat{\mathbf{s}}'-\hat{\mathbf{s}}_{\mathbf{r}-\mathbf{r}'})\tag{2}$$

In 2D, (1) and (2) become

$$\begin{aligned}\psi(\mathbf{r},\phi)+&\int d\mathbf{r}'d\phi' g(\mathbf{r},\phi;\mathbf{r}',\phi')\mu_t(\mathbf{r}')\psi(\mathbf{r}',\phi')\\ &-\int d\mathbf{r}'d\phi' g(\mathbf{r},\phi;\mathbf{r}',\phi')\mu_s(\mathbf{r}')\\ &\times\int d\phi'' f(\phi'-\phi'')\psi(\mathbf{r}',\phi'')\\ &=\int d\mathbf{r}'d\phi' g(\mathbf{r},\phi;\mathbf{r}',\phi')q(\mathbf{r}',\phi')\end{aligned}\tag{3}$$

$$g(\mathbf{r},\phi;\mathbf{r}',\phi')=\frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta(\phi-\phi_{\mathbf{r}-\mathbf{r}'})\delta(\phi'-\phi_{\mathbf{r}-\mathbf{r}'}\tag{4}$$

2 Section2

Expand the specific intensity $\psi(\mathbf{r},\phi)$:

$$\psi(\mathbf{r},\phi)=\sum_{n=1}^{N_s}\sum_{m=-N_d}^{N_d}X_{nm}\xi_{nm}(\mathbf{r},\phi)$$

In this paper, the basis function is chosen as

$$\begin{aligned}\xi_{nm}(\mathbf{r},\phi)=&S_n(\mathbf{r})e^{im\phi}\\ S_n(\mathbf{r},\phi)=&\begin{cases}1, & \mathbf{r}\in S_n\\ 0, & \mathbf{r}\notin S_n\end{cases}\end{aligned}$$

The latter sections, the following notations are used:

$S_n(\mathbf{r})$	the pulse function
$S_n)$	the n-th triangle
$S(n)$	the area of the n-th triangle
$\sum_{n,m}$	$\sum_{n=1}^{N_s}\sum_{m=-N_d}^{N_d}$

Write

$$\psi(\mathbf{r},\phi)=\sum_{n',m'}X_{n'm'}\xi_{n'm'}(\mathbf{r},\phi)\tag{5}$$

Plug (5) into (3), multiply by $\xi^*_{nm}(\mathbf{r},\phi)$, integrate over $d\mathbf{r}d\phi$, we get a set of linear equations:

$$\sum_{n',m'}Z_{(nm)(n'm')}X_{n'm'}=V_{nm}$$

with

$$\begin{aligned}Z_{(nm)(n'm')}=&A_{(nm)(n'm')}+B_{(nm)(n'm')}\\ A_{(nm)(n'm')}=&\int d\mathbf{r}d\phi\xi^*_{nm}(\mathbf{r},\phi)\xi_{n'm'}(\mathbf{r},\phi)\\ =&\int d\mathbf{r}S_n(\mathbf{r})S_{n'}(\mathbf{r})\int d\phi e^{-im\phi}e^{+im'\phi}\\ =&2\pi S(n)\delta_{nn'}\delta_{mm'}\\ B_{(nm)(n'm')}=&\int d\mathbf{r}d\phi\xi^*_{nm}(\mathbf{r},\phi)\int d\mathbf{r}'d\phi' g(\mathbf{r},\phi;\mathbf{r}',\phi')\\ &\times\mu_t(\mathbf{r}')\xi_{n'm'}(\mathbf{r}',\phi')-\int d\mathbf{r}d\phi\xi^*_{nm}(\mathbf{r},\phi)\\ &\times\int d\mathbf{r}'d\phi' g(\mathbf{r},\phi;\mathbf{r}',\phi')\mu_s(\mathbf{r}')\\ &\times\int d\phi'' f(\phi'-\phi'')\xi_{n'm'}(\mathbf{r}',\phi'')\\ V_{nm}=&\int d\mathbf{r}d\phi\xi^*_{nm}(\mathbf{r},\phi)\psi^I(\mathbf{r},\phi)\\ \psi^I(\mathbf{r},\phi)\equiv&\int d\mathbf{r}'d\phi' g(\mathbf{r},\phi;\mathbf{r}',\phi')q(\mathbf{r}',\phi')\end{aligned}$$

Note that (n,m) represents the filed point, whereas (n',m') represents the source point.

3 Section3

The interaction matrix can be show to be block-wise Toeplitz. Using proper quadrature rules $\{\mathbf{r}_{j_n}, w_{j_n}\}_{j_n=1}^{M_n}$ to compute each elements, we get:

$$
\begin{aligned}
B_{(nm)(n'm')} =& C_t(n,n',m-m') \\
&+ C_s(n,n',m-m')g^{|m'|} \\
C_t(n,n',\Delta m) =& + \sum_{j_n=1}^{M_n} \sum_{j_{n'}=1}^{M_n'} w_{j_n} w_{j_{n'}} \mu_t(\mathbf{r}_{j_{n'}}) \\
&\times \frac{e^{-i\Delta m \phi_{\mathbf{r}_{j_n}-\mathbf{r}_{j_{n'}}}}}{|\mathbf{r}_{j_n}-\mathbf{r}_{j_{n'}}|} \\
C_s(n,n',\Delta m) =& - \sum_{j_n=1}^{M_n} \sum_{j_{n'}=1}^{M_n'} w_{j_n} w_{j_{n'}} \mu_s(\mathbf{r}_{j_{n'}}) \\
&\times \frac{e^{-i\Delta m \phi_{\mathbf{r}_{j_n}-\mathbf{r}_{j_{n'}}}}}{|\mathbf{r}_{j_n}-\mathbf{r}_{j_{n'}}|}
\end{aligned}
$$

We prove the Toeplitzness of each block of B with given (n, n') in two steps. [TODO]

A $2N_d + 1$ by $2N_d + 1$ Toeplitz matrix can be stored in $O(N_d)$ memory and multiplied by a vector in $O(N_d \log N_d)$ time.

4 Section4

Assuming planewave incidence, numerically compute the right hand side ($r.h.s.$), i.e., the input vector:

$$
V_{nm} = \sum_{n'}^{N_s} D(n,n',m)
$$

where D is a rank-3 tensor:

$$
\begin{aligned}
D(n,n',m) =& \sum_{j_n=1}^{M_n} \sum_{j_{n'}=1}^{M_{n'}} w_{j_n} w_{j_{j'}} \mu_s(\mathbf{r}_{j_{n'}}) \frac{e^{-im\phi_{\mathbf{r}_{j_n}-\mathbf{r}_{j_{n'}}}}}{|\mathbf{r}_{j_n}-\mathbf{r}_{j_{n'}}|} \\
&\times f(\phi_{\mathbf{r}_{j_n}-\mathbf{r}_{j_{n'}}}) e^{-\tau(\mathbf{r}_{j_{n'}}, -\hat{\mathbf{s}}^I)}
\end{aligned}
$$

$\tau(\mathbf{r}, -\hat{\mathbf{s}}^I) \equiv \int_{C_{\mathbf{r},-\hat{\mathbf{s}}^I}} \mu_t dl$, where the integral path $C_{\mathbf{r},-\hat{\mathbf{s}}^I}$ is the ray that starts from $\mathbf{r}$ and in the direction pointed by $-\hat{\mathbf{s}}^I$. For later use, $\tau(\mathbf{r}' \to \mathbf{r}) \equiv \int_{C_{\mathbf{r}'\to\mathbf{r}}}$, where $C_{\mathbf{r}'\to\mathbf{r}}$ is the line segment from $\mathbf{r}'$ to $\mathbf{r}$.

In principle, computing τ requires ray tracing. Done with a tree-structure [TODO].

Appendices

A Appendix1

This part documents the quadrature rules employed in computing the matrix elements.

In short, the integral of interest takes on the following form:

$$
I = \iint_{\Omega} dxdy \frac{f(x,y)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}}
$$

where Ω is defined by the three nodes $P_0(x_0, y_0)$, $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $f(x, y)$ is a smooth function.

Use the arcsinh transfrom method to cancel the singularity at P_0 and transform the triangular domain into a rectangular domain. Let $\{\xi_u^i, w_u^i\}_{i=1}^{M_u}$ and $\left\{\xi_v^j, w_v^j\right\}_{j=1}^{M_v}$ be the Legendre quadrature rules over the $(0, 1)$ for the angular part and the radial part, respectively. Then

$$
I = |h(u_1-u_2)| \sum\nolimits_{i,j}^{M_u,M_v} w_u^i w_v^j f(x,y)\big|_{(i,j)}
$$

with

$$
\begin{aligned}
x_1' =& \left[(x_2-x_1)(x_1-x_0) + (y_2-y_1)(y_1-y_0)\right]/P_1P_2 \\
x_2' =& \left[(x_2-x_1)(x_2-x_0) + (y_2-y_1)(y_2-y_0)\right]/P_1P_2 \\
h =& \left[-(x_2-x_1)y_0 + (x_2-x_0)y_1 - (x_1-x_0)y_2\right]/P_1P_2 \\
u_1 =& \sinh^{-1}(x_1'/h) \qquad u_2 = \sinh^{-1}(x_2'/h) \\
u^i =& u_1 + (u_2-u_1)\xi_u^i \qquad v^j = h\xi_v^j
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{A} =& \frac{1}{P_1P_2}\begin{pmatrix} x_2-x_1 & -y_2+y_1 \\ y_2-y_1 & x_2-x_1 \end{pmatrix} \\
\begin{pmatrix} x \\ y \end{pmatrix}\Bigg|_{(i,j)} =& \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \mathbf{A} \cdot \begin{pmatrix} v^j \sinh u^i \\ v^j \end{pmatrix}
\end{aligned}
$$

Note that there is no assumption on which one of u_1 and u_2 is larger. The above equations can be coded without modification.

References