Econometrics by Prof. Ju

- 前半学期都在讲高级概率论的知识；然后再讲微观计量，不讲时间序列与financial econometrics

经济学是科学吗?

经济学的研究过程：先看到observation/data，回去找economic theory去解释现实。

Chapter 1: Introduction to Probability

Probability Space

$(\Omega, \mathcal{F}, \mathbf{P})$ is the triple.

Ω: sample space; $\Omega = \{\omega | \omega \textit{ is a potential outcome of a random experiment}\}$.

Event E is a collection of possible outcomes of an experiment Ω.

Occurrence: An event E occurs $\Leftrightarrow$ The realized outcome ω belongs to E.

$\sigma - field$

1. $\Omega \in \mathcal{F}$
2. $E \in \mathcal{F}$ implies $E^C \in \mathcal{F}$. That is, $\mathcal{F}$ is closed under complement.
3. $E_1, E_2, ... \in \mathcal{F}$ implies $\bigcup_{n=1}^{\infty} E_n \in \mathcal{F}$.

$\sigma - field$ generation

If $\pi \subset 2^\Omega$, $\sigma(\pi)$ is the smallest $\sigma - field$ that contains π.

$B(R) = \sigma(\{open\ sets\})$: *Borel*

$\{(-\infty, x] | x \in R\}$ is not closed under countable union, but we can generate a $\sigma - field$ based on it $\Rightarrow$ $\sigma(\{(-\infty, x] | x \in R\})$

Theorem 1.1

$B(R) = \sigma(\{(-\infty, x] | x \in R\})$... to be proved as homework.

Hint:

- If π_1 and π_2 are $\sigma - field$, then $\pi_1 \cap \pi_2$ is also a $\sigma - field$;
- If $\pi_1 \subset \pi_2$, then $\sigma(\pi_1) \subset \sigma(\pi_2)$.

Probability Measure

A set function $\mathbf{P}$ on the $\sigma - field\, \mathcal{F}$ (domain) is a *probability* or *probability measure* if it satisfies the following conditions:

1. $\mathbf{P}(E) \geq 0$ for all $E \in \mathcal{F}$;
2. $\mathbf{P}(\Omega) = 1$;
3. if $E_1, E_2, \ldots \in \mathcal{F}$ are disjoint, then $\mathbf{P}(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mathbf{P}(E_n)$.

Conditional Probability

For an event F such that $P(F) > 0$, we define the conditional probability of E given F by

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

Limit concepts in probability space

Theorem: When $\{E_n\}$ is monotone, $P(\lim_{n\to\infty} E_n) = \lim_{n\to\infty} P(E_n)$.

从数列的上/下极限引入集合的上/下极限：

$$\lim_{n\to\infty} \inf a_n = \inf_{n\geq 1} \sup_{k\geq n} a_k, \quad \lim_{n\to\infty} \sup a_n = \sup_{n\geq 1} \inf_{k\geq n} a_k.$$

If $\lim_{n\to\infty} \inf a_n = \lim_{n\to\infty} \sup a_n$, we say $\lim_{n\to\infty} a_n$ exists.

Likely in probability space,

$$\lim_{n\to\infty} \sup E_n = \cap_{n\geq 1}^{\infty} \cup_{k\geq n}^{\infty} E_k, \quad \lim_{n\to\infty} \inf E_n = \cup_{n\geq 1}^{\infty} \cap_{k\geq n}^{\infty} E_k.$$

Explanation:

$$\omega \in \lim_{n\to\infty} \sup E_n \quad \Leftrightarrow \quad \forall n, \omega \in B_n (= \cup_{k=n}^{\infty} E_k) \quad \Leftrightarrow \quad \forall n, \exists k \geq n, s.t. \omega \in E_k,$$
$$\lim_{n\to\infty} \sup E_n = \{\omega | \omega \in E_n, \quad i.o.\}$$

$$\omega \in \lim_{n\to\infty} \inf E_n \quad \Leftrightarrow \quad \exists n, \omega \in A_n (= \cap_{k=n}^{\infty} E_k) \quad \Leftrightarrow \quad \exists n, \forall k \geq n, s.t. \omega \in E_k,$$
$$\lim_{n\to\infty} \inf E_n = \{\omega | \omega \in E_n, \quad ev.\}$$

Obviously, $\lim_{n\to\infty} \inf E_n \subset \lim_{n\to\infty} \sup E_n$.

If $\lim_{n\to\infty} \inf E_n = \lim_{n\to\infty} \sup E_n$, we say $\lim_{n\to\infty} E_n$ exists.

E.g. $\{E_n\} : A, B, A, B, ...$ The limit of $\{E_n\}$ does not exist.

$$\limsup E_n = A \cup B, \quad \liminf E_n = A \cap B.$$

Borel-Cantelli lemma

$$\sum_{n=1}^{\infty} \mathbf{P}(E_n) < \infty \quad \Rightarrow \quad \mathbf{P}(\limsup E_n) = 0$$

Analogy: $\forall n, a_n \geq 0, \sum_{n=1}^{\infty} a_n < \infty \quad \Rightarrow \quad \lim a_n = 0$.

$\pi-$ and $\lambda-$ systems

$\pi-$system: $E, F \in \mathcal{P}$ implies $E \cap F \in \mathcal{P}$.

$\lambda-$ system:

- $\Omega \in \mathcal{L}$;
- If $E, F \in \mathcal{L}$ and $E \subset F$, then $F - E \in \mathcal{L}$;
- If $E_1, E_2, ... \in \mathcal{L}$ and $E_n \uparrow E$, then $E \in \mathcal{L}$.

Theorem: A class $\mathcal{F}$ of subsets of Ω is a $\sigma - field \Leftrightarrow \mathcal{F}$ is both a $\pi-$ and $\lambda-$ system.

Definition 1.1

$\sigma(S)$: the smallest $\sigma - field$ that contains S;

$\pi(S)$: the smallest $\pi-$ system that contains S;

$\lambda(S)$: the smallest $\lambda-$ system that contains S.

Dynkin's Lemma: Let $\mathcal{P}$ be a $\pi-$system. Then $\lambda(\mathcal{P}) = \sigma(\mathcal{P})$

Chapter 2: Random Variables, Distributions and Densities

$$(\Omega, \mathcal{F}, \mathcal{P}) \quad \rightarrow \quad (R, \mathcal{B}(R), P_X) \quad or \quad (R^n, \mathcal{B}(R^n), P_X)$$

Definition 2.1: A *random variable* X is a *measurable function* from Ω to $\mathbf{R}$, i.e., it assigns a number to each outcome: $X = X(\omega)$.

Transformation of a random variable: ($Y = f(X(\omega))$)

$$(\Omega, \mathcal{F}, \mathcal{P}) \quad \rightarrow^X \quad (R, \mathcal{B}(R), P_X) \quad \rightarrow^f \ (R, \mathcal{B}(R), P_Y)$$

Define the distribution of Y to be

$$\forall A \in \mathcal{B}(R), \quad P_Y(A) \equiv P_X[f^{-1}(A)] = P[X^{-1}(f^{-1}(A))]$$

Chapter 3: Expectations

$$E(X) = \int X dP = \int X(\omega) dP$$

where X here is called *"integrand"*, and P is called *"integrator"*. ---- Lebesque Integral

Lebesque Integral

直观理解：按照纵轴来积分，

$$\int f d\mu = \lim \sum_{i=1}^{n} \xi_i \cdot \mu\{x : y_{i-1} \leq f(x) \leq y_i\}$$

where μ is the measure (length), $\xi \in [y_{i-1}, y_i]$.

Lebesque Integral is a generalization of Riemann Integral, in the sense that Lebesque Integral == Riemann Integral *whenever the latter exists*.

E.g. (Dirichlet $f(\cdot)$)

$$\begin{aligned} \int_0^1 f(x)dx &= 1 \cdot \mu(\{x | f(x) = 1\}) + 0 \cdot \mu(\{x | f(x) = 0\}) \\ &= 1 \cdot 0 + 0 \cdot 1 = 0 \end{aligned}$$

- Note: 有理数是可列的，测度为零。

计算方法：

When $f(\cdot) \geq 0$ is continuous, we construct a sequence $\{f_n\} \uparrow f$, so that

$$\int f dP \equiv \lim_{n\to\infty} \int f_n dP,$$

where $\{f_n\}$ is a *simple function*.

More generally, when $f(\cdot)$ is continuous (not necessarily nonnegative), construct

$$f = f^+ - f^- = \max(f(x), 0) - \max(-f(x), 0)$$

to divide $f(\cdot)$ into positive & negative parts, so that f^+ & f^- are both nonnegative.

$$\Rightarrow \quad \int f dP = \int f^+ dP - \int f^- dP$$

但一般情况下，当对应的黎曼积分存在时，仍然用黎曼积分方法计算更为简单，否则按照上述方法也可计算。

Transformation of a random variable

$$Y = f(X(\omega))$$

$$E(Y) = E[g(X)] = \int g(X(\omega))dP = \int g(X)dP_X$$

所以没必要用$\int g(X(\omega))dP$去计算，完全可以用$\int g(X)dP_X$计算，即进入到X的概率空间算，不需要回到ω的概率空间。

$$dP_X = p(x) \cdot d\mu = p(x) \cdot \mu(dx) = p(x) \cdot dx = dF(x)$$
$$\Rightarrow \quad \frac{dP_X}{d\mu} = p(x) : local\ ratio$$

$$E[Y] = E[g(X)] = \int g(X(\omega))dP = \int g(x)dP_X = \int g(x)p(x)d\mu = \int g(x)dF(x)$$

Theorem 3.1: (Random-Nikodym Theorem)

Let μ and ν be two nonnegative measures on a measure space $(M, \mathcal{M})$. If ν is *absolutely continuous* with respect to (w.r.t.) μ. That is, $\forall A \in \mathcal{M}, \nu(A) = 0$, whenever $\mu(A) = 0$.

Then, ν can be represented as

$$\nu(A) = \int_A f d\mu$$

for a measurable nonnegative function f and any set $A \in \mathcal{M} \quad \Rightarrow \quad f d\mu$

Some properties

$$Cov(X,Y) = E[(X-EX)(Y-EY)^T] = E[XY^T] - E[X](E[Y])^T$$

$$Var(X) = Cov(X,X) = E[XX^T] - E[X](E[X])^T$$

$$\Rightarrow \quad Cov(AX,BY) = ACov(X,Y)B^T$$

$$Var(AX) = AVar(X)A^T$$

$$\rho_{X,Y} \equiv corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)\cdot Var(Y)}}$$

Properties of Expectation

1. Linearity
2. Monotonicity: $g_1(X) \geq g_2(X) \quad \Rightarrow \quad E(g_1(X)) \geq E(g_2(X))$
3. Chebyshev Inequalities: $P\{|X| \geq \varepsilon\} \leq \frac{E|X|^k}{\varepsilon^k}, \forall \varepsilon > 0$
   - The Inequality gives the upper bound of the probability measure of the tails;
   - $$Proof: \qquad \mathbf{1}\{|X| \geq \varepsilon\} \leq \left|\frac{X}{\varepsilon}\right|^k$$ $$\Rightarrow \quad \mathbf{E}[1\{|X| \geq \varepsilon\}] = P(\{|X| \geq \varepsilon\}) \leq \frac{E|X|^k}{\varepsilon^k}$$
4. Cauchy-Schwartz Inequality
5. Jensen's Inequality
6. Define k-th moment of X to be $\mu_k \equiv E[X^k]$, k-th central moment $\mu_k^* \equiv E[(X-E[X])^k]$.
   - If $0 < p < q, \mu_q < \infty \Rightarrow \mu_p < \infty$
   - $\mu_k < \infty \Leftrightarrow \mu_k^* < \infty$

Independence

Definition 3.1: Independence between 2 $\sigma - fields$

Suppose F and G are 2 $\sigma - fields$. We say F and G are independent iff $\forall A \in F$ and $\forall B \in G$, A and B are independent.

Definition 3.2: X and Y are independent if $\sigma(X)$ and $\sigma(Y)$ are independent

Intuitively, $P(\{X \in A\}|\{Y \in B\}) = P(\{X \in A\})$. The information that Y provides no new information for inferring X, and vice versa.

Theorem 3.2: X and Y are independent $\Leftrightarrow p(x, y) = p(x) \cdot p(y)$

Proof:

$$\begin{gathered} X \ and \ Y \ are \ independent \\ \Leftrightarrow \sigma(X) \ and \ \sigma(Y) \ are \ independent \\ \Leftrightarrow \forall A \in \mathcal{B}(R), \forall B \in \mathcal{B}(R), P(X^{-1}(A) \cap P(Y^{-1}(B)) = P(X^{-1}(A)) \cdot P(Y^{-1}(B)) \end{gathered}$$

Let $Z = (X, Y)'$

$$\begin{gathered} LHS = P(\{\omega | \omega \in X^{-1}(A) \ and \ \omega \in Y^{-1}(B)\}) \\ = P(\{\omega | X(\omega) \in A \ and \ Y(\omega) \in B\}) \\ = P(\{\omega | Z(\omega) \in A \times B\}) \\ = P(Z^{-1}(A \times B)) = P_Z(A \times B) \end{gathered}$$

$$RHS = P_X(A) \cdot P_Y(B)$$

$$\Leftrightarrow \quad \forall A, B \in \mathcal{B}(R), P_Z(A \times B) = P_X(A) \cdot P_Y(B)$$

Using the Random-Nikodym Theorem, we have

$$\begin{gathered} P_Z(A \times B) = \int\int_{A \times B} p(x, y) \mu(dxdy), \\ P_X(A) = \int_A p(x) \mu(dx), P_Y(B) = \int_B p(y) \mu(dy), \end{gathered}$$

so

$$\begin{gathered} \forall A, B \in \mathcal{B}(R), \int\int_{A \times B} p(x, y) \mu(dxdy) = \int_A p(x) \mu(dx) \cdot \int_B p(y) \mu(dy) \\ = \int\int_{A \times B} p(x) p(y) \mu(dxdy) \\ \Rightarrow \quad p(x, y) = p(x) \cdot p(y). \end{gathered}$$

- Note: Fubini's Theorem: a double Integral can be computed using iterated integral

Conditional Expectation

$$E(Y|X) = E(Y|\sigma(X))$$

E.g. $y = g(X) + u$. g and u are unknown, while y and X are observable. We can estimate it using the identification condition $E(u|X) = 0$. Then, we have

$$E(y|X) = E(g(X)|X) + E(u|X) = E(g(X)|\sigma(X)) + 0 = g(X).$$

Chapter 6: Asymptotic Theory

Convergence of a R.V. sequence

- From frequency to probability...
- R.V.本身也是ω的函数，考虑R.V.的收敛很复杂。

数学概念：连续与一致连续

Convergence Modes

Definition 6.1 (uniformly continuous)

$$\forall \varepsilon > 0, \exists \delta \quad s.t. \quad \forall x, x_0, |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \varepsilon$$

- Remark: δ is uniform for any choice of x_0

Definition 6.2 (point-wise convergence)

Let $\{x_n\}$ be defined on a common probability space $(\Omega, \mathcal{F}, \mathcal{P})$. We say that $\{x_n\}$ converges point wise(逐点) to x ($x_n \to_{p.w.} x$) if

$$\forall \omega \in \Omega, x_n(\omega) \to x(\omega).$$

Definition 6.3 (almost sure convergence, strong convergence)

Let $\{x_n\}$ be defined on a common probability space $(\Omega, \mathcal{F}, \mathcal{P})$. We say that $\{x_n\}$ converges almost surely to x if

$$P(\{\omega | x_n(\omega) \to x(\omega)\}) = 1,$$

which is equivalent to

$$\forall \varepsilon > 0, P[\cap_{N=1}^{\infty} \cup_{n \geq N} \{\omega : |x_n(\omega) - x(\omega)| > \varepsilon\}] = 0.$$

Obviously,

$$\to_{p.w.} \quad \Rightarrow \quad \to_{a.s.}$$

Definition 6.4 (convergence in probability, weak convergence)

Let $\{x_n\}$ be defined on a common probability space $(\Omega, \mathcal{F}, \mathcal{P})$. We say that $\{x_n\}$ converges in probability to x ($x_n \to_p x$) if

$$\forall \varepsilon > 0, P\{\omega : |x_n(\omega) - x(\omega)| > \varepsilon\} \to 0.$$

Compare that

- $P\{\omega : ...\} \to 0$ is the probability of a number ω;
- $P(\{\omega|...\}) = 1$ is the probability of an event $\{\omega|...\}$.

Definition 6.5 (convergence in L^p)

Let $\{x_n\}$ be defined on a common probability space $(\Omega, \mathcal{F}, \mathcal{P})$. We say that $\{x_n\}$ converges in L^p to x ($x_n \to_{L^p} x$) if

$$E|x_n - x|^p \to 0.$$

Note:

L^p space is a space of functions for which the p-th power of the absolute value is Lebesque Integrable. That is,

$$||f||_p \equiv \left[\int |f|^P d\mu\right]^{1/p} < \infty. \qquad 1 \leq p < \infty$$

Relation among modes of convergence

- $\to_{a.s.}$ does not imply $\to_{L^p}$, and vice versa;
- $\to_{a.s.} \quad \Rightarrow \quad \to_P$, while the reverse is false;
- $\to_{L^p} \quad \Rightarrow \quad \to_P$ (easily proved through Chebyshev Inequality), while the reverse is false;
- why consider $\to_{L^p} \quad \Rightarrow \quad \to_P$? Sometimes, in order to prove $\to_P$, we need to first prove $\to_{L^p}$, which is often easier.
  - E.g. if $T_n \to_{L^2} \theta_0$, i.e.
  - $$E[T_n - \theta_0]^2 = (E[T_n] - \theta_0)^2 + Var[T_n], or\ MSE = bias^2 + variance.$$

  if $bias^2 \to 0$ & $variance \to 0$, then $MSE \to 0$.

Definition 6.6 (convergence in distribution)

Let $\{x_n\}$ be defined on a common probability space $(\Omega, \mathcal{F}, \mathcal{P})$. We say that $\{x_n\}$ converges in distribution to x ($x_n \to_d x$) if

$$E[f(x_n)] \to E[f(x)]$$

for all f that is bounded and continuous almost surely in P_X

Remark:

- $\to_d$ is a weak convergence mode, so that it does not imply $\to_{a.s.\ or\ p}$;
- $P_{X_1} = P_{X_2} \quad \Leftrightarrow \quad \int f dP_{X_1} = \int f dP_{X_2}, \forall f \quad \Leftrightarrow \quad E[f(X_1)] = E[f(X_2)], \forall f$

Lemma 6.1

Let F_{X_n} and F_X denote the distribution functions for X_n and X. Let $\varphi_n(t) = E[e^{itX_n}]$ and $\varphi(t) = E[e^{itX}]$ denote the characteristic functions of X_n and X respectively. Then, the following are equivalent:

- $X_n \to_d X$;
- $E[f(X_n)] \to E[f(X_n)]$ for all f that is bounded and ***uniformly continuous***;
- $F_{X_n}(t) \to F_X(t)$ for every continuity point of F_X;
- $\varphi_{X_n}(t) \to \varphi_X(t)\ \forall t$. ($f_X$ 与 $\varphi_X(t) = \int e^{itx} \cdot f_X(x)dx$ 是傅里叶变换与反变换的关系，因此等价)

Theorem 6.1

$$X_n \to_P X \quad \Leftrightarrow \quad X_n \to_d X$$

Theorem 6.2

If $X_n \to_P X$, then there exists $\{X_{n_k}\}$ s.t. $X_{n_k} \to_{a.s} X$.

Theorem 6.3

Let $\{X_n\}$ be defined on a common probability space $(\Omega, \mathcal{F}, \mathcal{P})$, and c be a **constant**. If $X_n \to_d c$, then $X_n \to_P c$.

Theorem 6.4

Let f be a continuous function. We have

- $X_n \to_{a.s} X \Rightarrow f(X_n) \to_{a.s} f(X)$
- $X_n \to_P X \Rightarrow f(X_n) \to_P f(X)$
- $X_n \to_d X \Rightarrow f(X_n) \to_d f(X)$

即，连续变换不改变收敛性质。

Convergence rate

收敛速度慢的叫leading term，快的叫smaller term。E.g. When calculating $\frac{1}{\sqrt{n}} + \frac{1}{n^2}(n \to \infty)$, $\frac{1}{\sqrt{n}}$ is the leading term while $\frac{1}{n^2}$ is the smaller term.

Definition 6.7

$$\begin{aligned} x_n = o(a_n) \quad &\Leftrightarrow \quad \frac{x_n}{a_n} \to 0, \\ y_n = O(b_n) \quad &\Leftrightarrow \quad \left|\frac{y_n}{b_n}\right| < M. \end{aligned}$$

- Pronounce the above as: x_n is small o, y_n is big O.
- Big O means “bounded by”.

Remark 6.1

- $x_n = o(1) : \frac{x_n}{1} \to 0, i.e. x_n \to 0$
- $y_n = O(1) : \left|\frac{y_n}{1}\right| < M, i.e. |y_n| < M$, or y_n is bounded.
- $o(a_n) = a_n o(1)$, where the $=$ here represents “equal”, not “is”
- Similarly, $O(b_n) = b_n O(1)$
- $o(1) \subset O(1)$, or $o(1)$ is $O(1)$; the reverse is not true

Lemma 6.2

- $o(1) \cdot O(1) = o(1)$
- $O(o(1)) = o(1)$
- $o(O(1)) = o(1)$

Definition 6.8

- If $\frac{x_n}{a_n} \to_P 0$, we write $x_n = o_p(a_n)$ (smaller term)
- If $\forall \varepsilon > 0, \exists M > 0$, s.t. $P\left\{\left|\frac{y_n}{b_n}\right| > M\right\} < \varepsilon$ eventually ($\exists N, \forall n \geq N$), we write $y_n = O_p(b_n)$ (bounded in probability by b_n up to constant M)

Theorem 6.5

Let $X_n \to_d X$. Then

- $X_n = O_p(1)$
- $X_n + o_p(1) \to_d X$ (i.e. Noise不影响收敛性)

Corollary 6.1

If $X_n \to_d X$ and $Y_n \to_p c$, then $X_n Y_n \to_d cX$.

$\Delta - method$

Chapter 7: Law of Large Numbers (LLN) and Central Limit Theorem (CLT)

Law of Large Numbers

- Strong law of large numbers (SLLN): convergence almost surely
- Weak law of large numbers (WLLN): convergence in probability

Remark 7.1

- Regularity condition requires some degree of independence of the random variables.

Kolmogorov LLN

If $\{\xi_i\}$ is $i.i.d.$, and $E\xi_i < \infty$, then

$$\frac{1}{n}\sum_{i=1}^{n}\xi_i \to_{a.s.} E\xi_i.$$

- 即，只需要一阶矩有限，不需要二阶四阶矩的有限性。主要原因在于$i.i.d.$的条件已经非常强了。

Central Limit Theorem

A sufficient condition (Lindeberg condition)

$$\sum_{i=1}^{n} E\xi_{ni}^2 1\{|\xi_{ni}| > \varepsilon\} \to 0, \forall \varepsilon > 0.$$

- The tails of all the small random variables disappear; each random variable (or shock) is not dominant.

A much stronger condition (Liapounov condition)

$$\sum_{i=1}^{n} E|\xi_{ni}|^{2+\delta} \to 0, \delta > 0.$$

Note that

$$\sum_{i=1}^{n} E\xi_{ni}^2 1\{|\xi_{ni}| > \varepsilon\} \leq \sum_{i=1}^{n} E\xi_{ni}^2 1\{|\xi_{ni}| > \varepsilon\} \cdot \left|\frac{\xi_{ni}}{\varepsilon}\right|^{\delta} \leq \frac{\sum_{i=1}^{n} E|\xi_{ni}|^{2+\delta}}{\varepsilon^{\delta}} \to 0.$$

Corollary 7.1

Let $\{X_i\}$ be $i.i.d$ random variables with $EX_i = 0, Var(X_i) = \sigma^2$. Then

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i = \sqrt{n}\bar{X} \to_d N(0, \sigma^2).$$

Dominated convergence theorem (DCT)

If $f_n \to_{a.s./p} f$ and $|f_n(x)| \leq g(x)$ almost surely $\forall n$, then

$$\lim_{n\to\infty} Ef_x = E \lim_{n\to\infty} f_n = Ef.$$

Chapter 8: Asymptotics for MLE

DGP

$y = g(z, \theta_0) + u$. θ_0 is unknown; g is known. $X_1, ..., X_n$ is i.i.d. random variables, with density $\mathcal{P} = \{p(\cdot, \theta)|\theta \in \Theta\}$.

Likelihood

$p(X, \theta)$ is interpreted is a function of θ

$$p(X, \theta) = \prod_{i=1}^{n} p(X_i, \theta)$$

Definition 8.1: (MLE)

$$\hat{\theta}_n \equiv \arg\max_{\theta\in\Theta} p(X, \theta) = \arg\max_{\theta\in\Theta} \prod_{i=1}^{n} p(X_i, \theta)$$

- How to know such an estimator is a good one? Consistency, ...

$$L(X, \theta) = ln(p(X, \theta)) \Rightarrow$$
$$\hat{\theta}_n \equiv \arg\max_{\theta\in\Theta} L(X, \theta)/n = \arg\max_{\theta\in\Theta} \sum_{i=1}^{n} L(X_i, \theta)/n$$

- Why calculate log?

Lemma 8.1

$$\forall\theta \in \Theta, \quad E_0 l(X_i, \theta_0) \geq E_0 l(X_i, \theta).$$

Theorem 8.1

Under suitable regularit conditions, we have $\hat{\theta}_n \rightarrow_{a.s.\ or\ p} \theta_0$.

Score Function

$$s(X,\theta) \equiv \frac{\partial}{\partial\theta} l(X,\theta) = (l_1, ..., l_m)'$$
$$s(X_i,\theta) \equiv l(X_i,\theta)$$
$$\Rightarrow s(X,\theta) = \sum_{i=1}^{n} s(X_i,\theta)$$

Then, we have

$$s(X, \hat{\theta}_n) = 0.$$

Proposition 8.1

$$E_\theta s(X,\theta) = 0.$$

Fisher Information

$$I(\theta) \equiv E_\theta[s(X,\theta) \cdot s(X,\theta)'] = Var_\theta s(X,\theta)$$

$$\iota_i(\theta) \equiv E_\theta[s(X_i,\theta) \cdot s(X_i,\theta)']$$

$$\Rightarrow I(\theta) = E_\theta[(\sum_{i=1}^{n} s(X_i,\theta)) \cdot (\sum_{i=1}^{n} s(X_i,\theta)')] = \sum_{i=1}^{n} \iota_i(\theta) =^{i.i.d.} n\iota(\theta)$$

- 直观上理解，ι 越大，说明s 越大，即$loglikelihood$ 的曲度越大。

Hessian Matrix

$$h(X,\theta) = \frac{\partial^2}{\partial\partial'} L(X,\theta) = \begin{pmatrix} l_{11} & \cdots & l_{1m} \\ \vdots & \ddots & \vdots \\ l_{m1} & \cdots & l_{mm} \end{pmatrix}$$

$$h(X_i,\theta)=\frac{\partial^2}{\partial\partial'}L(X_i,\theta)=\begin{pmatrix} l_{11} & \cdots & l_{1m} \\ \vdots & \ddots & \vdots \\ l_{m1} & \cdots & l_{mm} \end{pmatrix}$$

$$h(X,\theta)=\sum_{i=1}^{n}h(X_i,\theta)$$

Expected Hessian

$$H(\theta)\equiv E_\theta h(X,\theta)$$

$H_i(\theta)$类似定义。

Proposition 8.2

$$I(\theta)=-H(\theta)$$

- 上式将Variance 与 Curvature 联系起来!

Asymptotic Normality of MLE

Theorem 8.2

Under suitable regularity conditions,

$$\sqrt{n}(\hat{\theta}_n-\theta_0)\rightarrow_d X(0,\iota(\theta_0)^{-1})$$

Regularity conditions are:

1. $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}s(X_i,\theta_0)=\sqrt{n}\bar{s}(X,\theta_0)=\rightarrow_d N(0,\iota(\theta_0))$ ----from CLT
2. $\bar{h}(X,\theta_0)=\frac{1}{n}\sum_{i=1}^{n}h(X_i,\theta_0)\rightarrow_P H(\theta_0)=-I(\theta_0)$ ----from LLN
3. $\bar{s}(X,\theta)$ is differentiable at θ_0 for all X

- 这里$\iota(\cdot)\equiv I(\cdot)$，记号通用。

Asymptotic Test of MLE (LR, LM, Wald)

$$H_0:\theta=\theta_0$$

$$H_1:\theta\neq\theta_0$$

Definition 8.1

$$LR = 2\left[\sum_{i=1}^{n} l(X_i, \hat{\theta}_0) - \sum_{i=1}^{n} l(X_i, \theta_0)\right]$$

$$W = \sqrt{n}(\hat{\theta}_n - \theta_0)'\iota(\hat{\theta}_n)\sqrt{n}(\hat{\theta}_n - \theta_0)$$

$$LM = \left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} s(X_i, \theta_0)\right)' \iota(\theta_0)^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} s(X_i, \theta_0)\right)$$

Theorem 8.3

Under suitable conditions,

$$LR, W, LM \to_d \chi_m^2.$$

Chapter 4: Matrix Algebra and Normal Distribution

4.1 Matrix Algebra

Transpose and inverse

Idempotent matrix: 幂等矩阵

$$A^2 = A \cdot A = A \Rightarrow A^k = A$$

(orthogonal) projection matrix

column space(列空间): $b = (b_1, b_2, ..., b_k) \in R^k$

正交投影要求满足的特性

$(Y_{n\times 1} - Xb) \perp X_i,\ i = 1, ..., k \quad \Rightarrow \quad X'(Y - Xb) = 0 \quad \Rightarrow \quad b = (X'X)^{-1}X'Y \quad \Rightarrow \quad projection\ is\ Xb$

Let $P_X = X(X'X)^{-1}X'$. Then projection residual is

$$Y - Xb = Y - P_X Y = (I - P_X)Y.$$

Two idempotent matrices P_X and $I - P_X$:

1. $P_X Y$: projection of Y onto the space spanned by columns of X. So obviously, $P_X(P_X Y)$ means projected twice, which is not needed; and $P_X(P_X Y) = P_X Y$.
2. $(I - P_X)Y$: projection of Y onto the orthogonal of space spanned by columns of X.

Trace

$$Tr(AD) = Tr(DA)$$

E.g.

$$Tr(P_X) = Tr(X(X'X)^{-1}X') = Tr((X'X)^{-1}X'X) = Tr(I_k) = k$$

$$Tr(I_n - P_X) = Tr(I_n) - Tr(P_X) = n - k$$

Positve/Negative (semi) definite

Rank

Definition: dimension of the space generated by row/column vectors

$$r(A) = r(A'A)$$

How to prove it? Two ways:

- **rank-nullity theorem**: Consider a mapping $T : V \to W$. Then the dimension of domain = dim of Image + dim of $NULL(T)$. For the 2 mapping $T_1 : V \to AV$ and $T_2 : V \to A'AV$, the dim of domain and $NULL(T)$ are the same. So the dim of Image are also the same, which means $r(A) = r(A'A)$.
- **Lemma**: $NULL(A) = [R(A')]^{\perp}$ where $R(A') = \{y|y = A'X : X \in R^m\}$.
So, $NULL(A)^{\perp} = R(A')$ and $NULL(A'A)^{\perp} = R(A'A) \quad \Rightarrow \quad R(A') = R(A'A) \quad \Rightarrow \quad r(A') = r(A'A)$

Partitioned matrices

Characteristic vectors (eigen vectors) and characteristic roots (eigen values)

$$AC = \lambda C$$

Theorem 4.1

For a symmetric matrix $A_{n\times n}$, it has n distinct eigen vectors $c_1, c_2, ..., c_n$ s.t. $C'C = I,\ C = (c_1, c_2, ..., c_n)$.

- This means $c_i' \cdot c_j = 0, \forall i \neq j$. 每一个特征向量都是标准正交向量。

Eigen decomposition

$$Ac_i = \lambda_i c_i, i = 1, ..., n$$

$$A(c_1, ..., c_n) = (c_1, ..., c_n) \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \Rightarrow AC = C\Lambda.$$

We have (eigen decomposition)

$$A = C\Lambda C' = (c_1, ..., c_n) \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \sum_{i=1}^{n} \lambda_i c_i c_i'$$

$r(A) = r(C\Lambda C') = r(\Lambda)$. Using this way to calculate the rank of A is much easier.

What if A is not symmetric? Note that $r(A) = r(A'A)$ and do as the above.

Theorem 4.2

$$A^2 = A \quad \Rightarrow \quad \lambda = 0 \; or \; 1.$$

It is easy to find that

$$A^k = C\Lambda^k C',$$

and we can define that

$$A^{1/2} = C\Lambda^{1/2} C'.$$

Decomposition of orthogonal projection

Theorem 4.3

An orthogonal projection $P_{n\times n}$ of m-dimension ($r(P_{n\times n}) = m$) can be written as $P_{n\times n} = H_{n\times m} \cdot H'_{n\times m}$, where $H'_{n\times m} \cdot H_{n\times m} = I_{m\times m}$.

4.2 Multivariate Normal Distribution

Definition 4.1

Multivariate normal distribution $X \sim N(\mu, \sum)$ if

$$p(x) = \frac{1}{(2\pi)^{n/2}} (det\Sigma)^{-1/2} \exp\left(-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\right).$$

Lemma 4.1

Let $Z \sim N_n(0, I)$ and $X = \mu + \Sigma^{1/2} Z$. Then $X \sim N(\mu, \Sigma)$.

Corollary 4.1

$$X \sim N(\mu, \Sigma), Y = AX + b \quad \Rightarrow \quad Y \sim N(A\mu + b, A\Sigma A').$$

4.3 Quadratic Forms

Theorem 4.4

Let $X \sim N_n(0, \Sigma)$. Then

$$X'\Sigma^{-1}X \sim \chi_n^2.$$

Theorem 4.5

Let $Z \sim N_n(0, I)$ and P be an *m*-dimensional orthogonal projection in R^n. Then we have

$$Z'PZ \sim \chi_m^2.$$

Theorem 4.6

Let $Z \sim N(0, I)$ and let A and B be non-random matrices. Then $A'Z$ and $B'Z$ are independent iff $A'B = 0$.

Corollary 4.2

Let $Z \sim N(0, I)$, and let P and Q are orthogonal projections s.t. $PQ = 0$. Then $Z'PZ$ and $Z'QZ$ are independent.

Corollary 4.3

Let $Z \sim N(0, I)$, and let P and Q are orthogonal projections of dimension p and q, s.t. $PQ = 0$. Then

$$\frac{(Z'PZ)/p}{(Z'QZ)/q} \sim F_{p,q}$$

Chapter 9: Regression Models

9.1 The Model

Matrix form:

$$y = X\beta + \epsilon$$

It is impossible to identify the model without assumptions, because the number of equations (n) is smaller than that of unknowns (n+m)

Identification conditions:

1. X has a full column rank: $rank(x) = m$. 列满秩（x不完全共线），不是行满秩!
2. $E(X_i u_i) = 0$ (m restrictions) and $E(u_i) = 0$. Actually, we can specify a stronger assumption: $E(u_i|X_i) = 0 \Rightarrow E(X_i u_i) = E[X_i E(u_i|X_i)] = 0$.

According to LLN, we can use sample mean to estimate population mean.

9.2 Sample Analogue Estimator (SAE)

$$\beta = [E(X'X)]^{-1} E(X'Y)$$
$$\Rightarrow \hat{\beta}_{SAE} = (X'X)^{-1} X'Y = \beta + (X'X)^{-1} X' u_i \rightarrow_p \beta$$

where (according to LLN)

$$(X'X)^{-1} X' u_i = (E(X'X) + o_p(1)) o_p(1) = o_p(1).$$

- Consistency is satisfied.

$$\sqrt{n}(\hat{\beta} - \beta) \rightarrow_d N[0, (EX'X)^{-1} E(u^2 X'X)(EX'X)^{-1}]$$
$$\Rightarrow \hat{\beta} = \beta + O_p(\frac{1}{\sqrt{n}})$$

$$\hat{u} = y - X\hat{\beta}, u = y - X\beta$$
$$\Rightarrow \hat{u} = u - X(\hat{\beta} - \beta) = (I - P_X)u$$
$$= u - O_p(1) \cdot O_p(\frac{1}{\sqrt{n}}) = u + O_p(\frac{1}{\sqrt{n}})$$

That is, convergence speed of $\hat{\beta}$ and $\hat{u}$ are the same, $O_p(\frac{1}{\sqrt{n}})$.

Let

$$V = (EX'X)^{-1} E(u^2 X'X)(EX'X)^{-1}$$

- Case 1: Homoskedasticity $E(u_i^2|x_i) = \sigma^2$. Then $E(u^2 X'X) = \sigma^2 E(X'X)$, so that $V = \sigma^2 (EX'X)^{-1}$

Now the question is: how to estimate σ^2

$$\hat{\sigma}^2 = \frac{1}{n-m}\sum_{i=1}^{n}\hat{u}_i^2,$$

We can show that $\hat{\sigma}^2 \to_p \sigma^2$ and $E\hat{\sigma}^2 = \sigma^2$. Proof for the latter:

$$\begin{aligned}
E\hat{\sigma}^2 &= \frac{1}{n-m}E[\hat{u}'\hat{u}] = \frac{1}{n-m}E[u'(I-P_X)u] \\
&= \frac{1}{n-m}E[Tr(u'(I-P_X)u)] = \frac{1}{n-m}E[Tr((I-P_X)uu')] \\
&= \frac{1}{n-m}Tr(E[(I-P_X)uu']) = \frac{1}{n-m}Tr(E[(I-P_X)E(uu'|X)]) \\
&= \frac{1}{n-m}Tr(E[(I-P_X)\sigma^2 I]) = \frac{\sigma^2}{n-m}Tr(E(I-P_X)) \\
&= \frac{\sigma^2}{n-m}E(Tr(I-P_X)) \\
&= \frac{\sigma^2}{n-m}E(n-m) = \sigma^2
\end{aligned}$$

- Case 2: Heteroskedasticity i.e. $E(u_i^2|X_i) = \sigma^2(X_i)$

$$V = (E(X'X))^{-1}E(\sigma^2(X)X'X)(E(X'X))^{-1}$$

We want to show that

$$\frac{1}{n}\hat{u}^2X'X \to_p E(u^2X'X) = E(\sigma^2(X)X'X)$$

is a good estimator, which is true.

9.3 Least Square Estimator (LSE)

$$y_i = g(X_i, \theta_0) + u_i, i = 1, 2, ..., n, \quad E(u_i|X_i) = 0$$

where $g(\cdot)$ is known. $E(u_i|X_i) = 0$ is the **identification condition**.

We need to optimize

$$\hat{\theta}_{LSE} = \arg\min_{\theta}\frac{1}{n}\sum_{i=1}^{n}[y_i - g(X_i, \theta)]^2$$

The above is a **parametric** model. We can even estimate an **unparametric** one as the following:

$$y_i = g(X_i) + u_i, i = 1, 2, ..., n, \quad E(u_i|X_i) = 0$$

where $g(\cdot)$ is unknown. $E(u_i|X_i) = 0$ is the identification condition.

Calculate the conditional mean

$$E(y_i|X_i=x)=E[g(X_i)|X_i=x]+E(u_i|X_i=x)=g(x)+0$$
$$\Rightarrow g(x)=E(y_i|X_i=x)$$

With sample mean, we can create a **mapping relation** to estimate the function of $g(\cdot)$.

Go back to linear model

$$y_i=X_i'\beta+u_i, i=1,2,...,n,$$

we have

$$\hat{\beta}_{SAE}=(X'X)^{-1}X'Y,$$
$$\hat{\beta}_{LSE}=\arg\min_{\beta}\frac{1}{n}\sum_{i=1}^{n}(y_i-X_i'\beta)^2=(X'X)^{-1}X'Y$$

Nonlinear Least Squares (NLS)

$$y_i=m(X_i,\beta)+u_i, i=1,2,...,n, \quad E(u_i|X_i)=0$$

$$\hat{\theta}_{NLS}=\arg\min_{\beta}\frac{1}{n}\sum_{i=1}^{n}[y_i-m(X_i,\beta)]^2$$

Asymptotic Normality of NLS estimator

$$\sqrt{n}(\hat{\beta}-\beta)\rightarrow_d [E(\nabla_\beta m\nabla_\beta m')]^{-1}N[0,E(u_i^2\nabla_\beta m\nabla_\beta m')]$$

Supplement for derivative calculation

$$\frac{\partial}{\partial\beta}\alpha'\beta=\alpha, \quad \frac{\partial}{\partial\beta}\beta'\alpha=\alpha$$
$$\frac{\partial}{\partial\beta}\beta'A\beta=(A+A')\beta$$

9.4 MLE of Linear Model

$$y_i=m(X_i,\beta)+u_i, i=1,2,...,n, \quad u_i|X_i\sim N(0,\sigma^2)$$
$$\Rightarrow y|X\sim N(X\beta,\sigma^2 I)$$
$$\Rightarrow l(\beta,\sigma^2)=-\frac{n}{2}ln(2\pi)-\frac{n}{2}ln(\sigma^2)-\frac{(y-X\beta')(y-X\beta)}{2\sigma^2}$$
$$\Rightarrow \hat{\beta}_{MLE}=(X'X)^{-1}X'Y, \hat{\sigma}^2=\frac{1}{n}(y-X\hat{\beta})'(y-X\hat{\beta})=\frac{1}{n}\hat{u}'\hat{u}$$

MLE要求知道的信息比之前的都要多（要求知道扰动项的分布，而不只是零条件均值），所以肯定能识别出来。

9.5 Optimality

$$\hat{\beta}_{SAE} = \hat{\beta}_{OLS} = \hat{\beta}_{MLE} = (X'X)^{-1}X'y$$

$$s^2 = \frac{1}{n-k}\hat{u}'\hat{u}, E(s^2) = \sigma^2$$

Under MLE,

$$\hat{\sigma}^2 = \frac{1}{n}\hat{u}'\hat{u},$$

which is biased (finite sample).

Desirable properties for a good estimator:

1. Consistency;
2. Unbiasedness;

Definition 9.1: Mean Square Error $T = \tau(X)$

$$\begin{aligned} E_\theta(T-\theta)^2 &= E_\theta[(T-E_\theta T)^2] + (E_\theta T - \theta)^2 \\ &= Var_\theta T + (bias_\theta T)^2 \end{aligned}$$

If mean square error converges to 0, then (according to **convergence in L^p**) we get **convergence in probability**, so that **consistency** is satisfied.

We need to balance between variance and bias.

Definition 9.2: Uniformly Minimum Variance Unbiased (UMVU) Estimator $T_* = \tau^*(X)$

- $E_\theta T^* = \theta$;
- $E_\theta(T_* - \theta)^2 \leq E_\theta(T-\theta)^2$ for any **unbiased** estimator T and θ.
- UMVU finds the one with **minimum variance** among **unbiased estimators**.

Definition 9.3

$E_\theta T_1 = E_\theta T_2 = \theta$, we say T_1 is more **efficient** than T_2 if $Var_\theta T_1 \leq Var_\theta T_2$.

9.6 Finite sample properties of OLS

Theorem 9.1

$$
E\hat{\beta} = \beta, \quad Var(\hat{\beta}) = \sigma^2(X'X)^{-1}
$$
$$
Es^2 = \sigma^2
$$

Under normality assumption, $u_i|X_i \sim N(0, \sigma^2)$,$\hat{\beta} = \beta + (X'X)^{-1}X'u$

- $\hat{\beta} \sim N[\beta, \sigma^2(X'X)^{-1}]$
- $\frac{(n-k)s^2}{\sigma^2} \sim \chi^2(n-k)$
- $\hat{\beta}$ and s^2 are statistically independent.

Theorem 9.2: Gauss-Markov

The OLS estimator $\hat{\beta}$ of the model $y_i = X_i'\beta + u_i$ ($E(u_i|X_i) = 0$ and $E(u_i^2|X_i) = \sigma^2$) is the best linear unbiased estimator (BLUE).

Theorem 9.3

Let $T = \tau(X)$ be unbiased estimator of θ, then

$$
Var_\theta \tau(X) \geq [I(\theta)]^{-1},
$$

which means $[I(\theta)]^{-1}$ is the lower bound, as is called **Cramer-Rao Bound**.

Lemma 9.1

$\tau(X)$ is unbiased. We have

$$
E_\theta[\tau(X) \cdot s(X, \theta)'] = I.
$$

Proof:

$$
\begin{aligned}
LHS &= \int \tau(x) \frac{\frac{\partial}{\partial \theta'} p(x, \theta)}{p(x, \theta)} p(x, \theta) dx \\
&= \frac{\partial}{\partial \theta'} \int \tau(x) p(x, \theta) dx \\
&= \frac{\partial}{\partial \theta'} E_\theta \tau(x) = \frac{\partial}{\partial \theta'} \theta = I
\end{aligned}
$$

Corollary 9.1

$$
\begin{aligned}
Cov(\tau(X), s(X, \theta)) &= E_\theta(\tau(X) s(X, \theta)') - E_\theta \tau(X)[E_\theta s(X, \theta)'] \\
&= I - E_\theta \tau(X) \cdot 0 = I
\end{aligned}
$$

Conclusion: s^2 estimated from OLS is **unbiased**, but $\hat{\sigma}^2$ estimated from MLE has the least variance (even lower than Cramer-Rao Bound $\frac{2\sigma^4}{n}$ since it is a **biased** estimator).

$$\begin{aligned} Var(s^2) &= Var(\frac{(n-k)s^2}{\sigma^2} \cdot \frac{\sigma^2}{n-k}) = 2(n-k)\frac{\sigma^4}{(n-k)^2} \\ &= \frac{2\sigma^4}{n-k} > \frac{2\sigma^4}{n} \end{aligned}$$

$$\begin{aligned} Var(\sigma^2_{MLE}) &= Var(\frac{n-k}{n}s^2) = \frac{(n-k)^2}{n^2}\frac{2\sigma^4}{n-k} \\ &= \frac{n-k}{n}\frac{2\sigma^4}{n} < \frac{2\sigma^4}{n} \end{aligned}$$

9.7 F-test

$$H_0 : R\beta = r, \quad H_1 : R\beta \neq r$$

Definition 9.4: Constrained Least Squares (CLS)

$$y = X\beta_u, \quad R\beta = r$$

We wanna minimize

$$\tilde{\beta} = \arg\min_{\beta}(y - X\beta)'(y - X\beta), \quad s.t. R\beta = r.$$

Note that

$$\hat{\beta} = \arg\min_{\beta}(y - X\beta)'(y - X\beta) = (X'X)^{-1}X'y$$

Theorem 9.4

- $\tilde{\beta} = \hat{\beta} - (X'X)^{-1}R'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)$
- $\tilde{u}'\tilde{u} - \hat{u}'\hat{u} = (R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)$

$$\min \alpha'(X'X)\alpha \quad s.t.\ R\alpha = R\hat{\beta} - r, \alpha \equiv \hat{\beta} - \beta$$

Definition 9.5

$$\tilde{\sigma}^2 = \frac{1}{n}\tilde{u}'\tilde{u}, \quad \tilde{u} = y - X\tilde{\beta}$$

Note: The **CLS estimators** $\tilde{\beta}$ and $\tilde{\sigma}^2$ are also the **constrained ML estimator under normality**.

Hypothesis Testing under Normality

$$H_0 : R\beta = r, \quad H_1 : R\beta \neq r$$

Theorem 9.5

Assume normality. The LR test rejects the null hypothesis $H_0 : R\beta = r$ v.s. $H_1 : R\beta = r$ for large values of

$$F = \frac{(\tilde{u}'\tilde{u} - \hat{u}'\hat{u})/q}{s^2} = \frac{\frac{1}{q}(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)}{s^2}.$$

Asymptotic Test

Robust Variance-Covariance Matrix

$$\sqrt{n}(\hat{\beta} - \beta) \sim N(0, [E(X_iX_i')]^{-1}E(u_i^2X_iX_i')[E(X_iX_i')]^{-1})$$

Wald, LR, LM tests

$$R\hat{\beta} - r \sim_a N(0, 0, \sigma^2 R(X'X)^{-1}R')$$
$$W = \frac{(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)}{\hat{\sigma}^2} \to_d \chi_q^2$$

If $W \gg 0$, then we can reject $H_0 : R\beta = r$.

$$LR = ... = W + O_p(\frac{1}{n}) \to_d \chi_q^2$$

For LM test, define

$$\mathcal{L} = \mathcal{L}(\beta, \sigma^2) + \lambda'(R\beta - r)$$
$$= -\frac{n}{2}ln(2\pi) - \frac{n}{2}ln(\sigma^2) - \frac{1}{2\sigma^2}(y - X\beta)'(y - X\beta) + \lambda'(R\beta - r)$$
$$F.O.C \Rightarrow \tilde{\lambda} = \frac{1}{\tilde{\sigma}^2}[R(X'X)^{-1}R']^{-1}(R\hat{\beta} - r)$$

Notice that $\tilde{\lambda}$ is a linear transformation of random variable $\hat{\beta}$, so that we can construct quadratic form

$$\tilde{\lambda}'\square^{-1}\tilde{\lambda} \sim \chi^2$$

$$LM = \tilde{\lambda}'(\tilde{\sigma}^2 R(X'X)^{-1}R')\tilde{\lambda} = \frac{(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)}{\tilde{\sigma}^2}$$
$$= \frac{(\tilde{u}'\tilde{u} - \hat{u}'\hat{u})}{\tilde{\sigma}^2} = \frac{W}{1 + W/n} = W + o_p(1) \to_d \chi_q^2$$

9.8 GLS

$$y = X\beta + u, \quad E(u|X) = 0, E(u^2|X) = \Omega$$

If we know the structure of Ω, we can rewrite

$$\Omega^{-1/2}y = \Omega^{-1/2}X\beta + \Omega^{-1/2}u$$
$$\Rightarrow y^* = X^*\beta + u^*$$

$$\min(y^* - x^*\beta)'(y^* - x^*\beta)$$
$$= (\Omega^{-1/2}y - \Omega^{-1/2}X\beta)'(\Omega^{-1/2}y - \Omega^{-1/2}X\beta)$$
$$= (y - X\beta)'\Omega^{-1}(y - X\beta)$$

$F.O.C$

$$\hat{\beta}_{GLS} = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y$$

But in reality, we need to first estimate the structure of Ω, $\hat{\Omega}$.

Feasible GLS: $\hat{\Omega} \to \Omega$

$$\hat{\beta}_{FGLS} = (X'\hat{\Omega}^{-1}X)^{-1}X'\hat{\Omega}^{-1}y.$$

The problem is: the num needed to estimate is $\frac{n(n+1)}{2}$, too big! So we need to first assume some properties/structure (E.g. within-group homogeneity, between-group irrelavence)

Theorem 9.6 Gauss-Markov of GLS

The GLS estimator $\hat{\beta}_{GLS}$ of $y = X\beta + u$ ($E(u|X = 0), Var(u|X) = \Omega$ is known) is **BLUE**.

Proof:

Define an unbiased estimator $\bar{\beta} = T'y$.

$$E\bar{\beta} = \beta \Rightarrow \quad T'X = I$$
$$\Rightarrow Var(\bar{\beta}|X) = T'\Omega T.$$

$$\hat{\beta}_{GLS} = \beta + (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}u \Rightarrow$$
$$Var(\hat{\beta}_{GLS}|X) = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}\Omega\Omega^{-1}X(X'\Omega^{-1}X)^{-1}$$
$$= (X'\Omega^{-1}X)^{-1}$$

We now need to prove that $Var(\bar{\beta}|X) - Var(\hat{\beta}_{GLS}|X) \geq 0$.

$$Var(\bar{\beta}|X) - Var(\hat{\beta}_{GLS}|X) = T'\Omega T - (X'\Omega^{-1}X)^{-1}$$
$$= T'\Omega^{1/2}\Omega^{1/2}T - (X'\Omega^{-1/2}\Omega^{-1/2}X)^{-1}$$
$$= T'\Omega^{1/2}\Omega^{1/2}T - T'\Omega^{1/2}P_{\Omega^{-1/2}X}\Omega^{1/2}T$$
$$= T'\Omega^{1/2}(I - P_{\Omega^{-1/2}X})\Omega^{1/2}T$$

is p.s.d. .

where

$$P_{\Omega^{-1/2}X} = \Omega^{-1/2}X(X'\Omega^{-1/2}\Omega^{-1/2}X)^{-1}X'\Omega^{-1/2}$$

is the projection matrix of $\Omega^{-1/2}X$.

- Intuition: 通过$\Omega^{-1/2}$的映射，我们把扰动项u还原为球形扰动项，将已有数据中的GLS结构信息用上了。信息越多，variance越小。

9.9 R-Squared

- $\hat{u}'\hat{u}$: RSS, residual sum of squares
- $y'y$: TSS, total sum of squares
- $\hat{y}'\hat{y}$: ESS, explained sum of squares

Define

$$R^2 = \frac{\hat{y}'\hat{y}}{y'y} = \frac{ESS}{TSS} = 1 - \frac{RSS}{TSS} \in [0,1]$$
$$\bar{R}^2 = 1 - \frac{\frac{1}{n-k}y'(I-P_X)y}{\frac{1}{n-1}y'(I-P_\iota)y}, \quad \iota = (1,1,...,1)'_{n\times 1}$$

9.10 Endogeneity

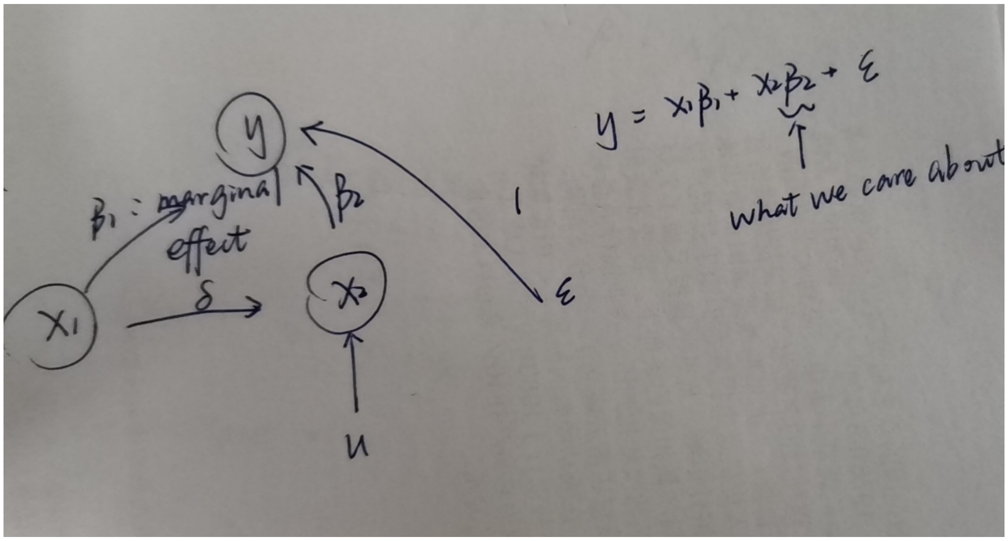
What if $E(u_i|X_i) \neq 0$? The estimation will be **inconsistent** and **biased**.

$$\hat{\beta}_{SAE} = [\frac{1}{n}\sum_{i=1}^{n} X_i X_i']^{-1}(\frac{1}{n}\sum_{i=1}^{n} X_i u_i) + \beta \neq \beta$$

Source of endogeneity:

- omitted variable
- measurement error: especially survey data
- simultaneity problem
- selection

Two-step regression



Idea:

- Step 1: 剔除X_1的影响
- Step 2: $y^* = X_2^* \beta_2 + \varepsilon$

One-step regression:

$$y = X_1 \hat{\beta}_1 + X_2 \hat{\beta}_2 + \hat{\varepsilon}$$

Proposition 9.1

One-step estimator equals two-step estimator.

$$\hat{\beta}_2 = [X_2'(I - P_{X_1})X_2]^{-1} X_2' \cdot (I - P_{X_1})y$$
$$\hat{\beta}_1 = (X_1'X_1)^{-1} X_1'(y - X_2\hat{\beta}_2)$$

- Question: is $\hat{\beta}_1$ consistent???

Proof:

$$y = X_1 \hat{\beta}_1 + X_2 \hat{\beta}_2 + \hat{\varepsilon}$$

Premultiply $I - P_{X_1}$

$$(I - P_{X_1})y = (I - P_{X_1})X_2\hat{\beta}_2 + (I - P_{X_1})\hat{\varepsilon}$$
$$= (I - P_{X_1})X_2\hat{\beta}_2 + \hat{\varepsilon}$$

- Intuitively, $\hat{\varepsilon}$ 已经是P_X投影后的残差项了，再对P_{X_1}投影残差不改变。从数学上看，

$$P_{X_1}P_X = X_1(X_1'X_1)^{-1}X_1'P_X = X_1(X_1'X_1)^{-1}(P_X X_1)'$$
$$= X_1(X_1'X_1)^{-1}(X_1)' = P_{X_1}, \Rightarrow$$
$$(I - P_{X_1})\hat{\varepsilon} = (I - P_{X_1})(I - P_X)\hat{\varepsilon} = (I - P_X)\hat{\varepsilon} = \hat{\varepsilon}.$$

Premultiply X_2'

$$\begin{gathered}X_2'(I-P_{X_1})y = X_2'(I-P_{X_1})X_2\hat{\beta}_2 + X_2'\hat{\varepsilon} = X_2'(I-P_{X_1})X_2\hat{\beta}_2 \\ \Rightarrow \hat{\beta}_2 = [X_2'(I-P_{X_1})X_2]^{-1}X_2' \cdot (I-P_{X_1})y\end{gathered}$$