To get $E = mc^2$

In relativity, $E = mc^2$ must be the most famous equation. For many of us, we started to see this very equation in our lives, since we have no ides of what does it even means.

There are a larger verities of actual experiments and Gedanken(thought) experiments used by textbooks to reach it. Here, I am going to introduce two examples that most of the textbooks(as far as I know) used, and also two of the easiest, one actual experiment and one Gedanken experiment, to understand.

The following contents assume that you know

1. the conservation laws of both energy and momentum

2. For a photon $E = hf = cp$ where E is the energy of the photon, h is plank constant, c is the speed of light, f is the frequency, p is the momentum of photon

The Einstein's box

Assume there is a box with mass M and length l. In graph 1, an light impulse with energy E is emitted from the LHS.

(We ignore the mass of the light as it is extrmely small compare to M, and the calculation would be tedious if we consider it.)

It have a momentum of $P_1 = \frac{E}{c}$, by conservation of momentum, the box now have a momentum $P_2 = -\frac{E}{c}$ and velocity $v = -\frac{E}{cM}$

assume the impulse reaches B after $\Delta t = \frac{L}{c}$, if $c \gg v$, the box will stop as the impulse gives a impulse that is equal and opposite to the original one. Assume that the box is moved by Δx (which have direction of pointing left) and the light have a mass of m. And the center of mass of the box with light emitted is d

Since there is no external force applied on the system, therefore the center of mass of the system should not move.

We can formulate the following equation if we use A as the original(which is moving, you may choose a fixed point to calculate).

$$
\begin{aligned}
position\ of\ center\ of\ mass\ before &= position\ of\ center\ of\ mass\ after \\
\frac{(M-m)d}{M} &= \Delta x + \frac{mL+(M-m)d}{M} \\
(M-m)d &= \Delta xM + mL + (M-m)d \\
mL + \Delta xM &= 0 \\
\because \Delta x = v\Delta t &= -\frac{E}{cM}\frac{L}{c} = -\frac{EL}{Mc^2} \\
\therefore mL &= \frac{EL}{Mc^2}M \\
\therefore E &= mc^2
\end{aligned}
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reference

French, A.P. *Special relativity*. New York: W·W·NORTON & COMPANY·INC, 1968. Print.

Einstein, Albert, et al. *The Principle of Relativity* New York: Dover, 1958.Print.