Background knowledge

1. moment of inertia
    1. moment of inertia I of a body about a given axis is a measure of its rotational inertia: The greater the value of I, the more difficult it is to change the state of the body's rotation. be expressed as a sum over the particles m_i that make up the body, each of which is at its own perpendicular distance r_i from the axis.
    2. $\mathrm{I} = m_1 r_1^2 + m_2 r_2^2 + \ldots = \sum_i m_i r_i^2$
2. Angular momentum
    1. Angular momentum: The angular momentum of a particle with respect to point O is the vector product of the particle's position vector $\overrightarrow{\boldsymbol{r}}$ relative to O and its momentum $\overrightarrow{\boldsymbol{p}} = m\overrightarrow{\boldsymbol{v}}$.
    2. For particle: $\overrightarrow{\boldsymbol{L}} = \overrightarrow{\boldsymbol{r}} \times \overrightarrow{\boldsymbol{p}} = \overrightarrow{\boldsymbol{r}} \times m\overrightarrow{\boldsymbol{v}}$
    3. For rigid body rotating about axis of symmetry: $\overrightarrow{\boldsymbol{L}} = I\overrightarrow{\boldsymbol{\omega}}$
3. The rotational kinetic energy of a rigid body rotating about a fixed axis depends on the angular speed ω and the moment of inertia I for that rotation axis
    1. $K = \frac{1}{2} I \omega^2$
4. Torsional oscillation
    1. the angular rotation of a object
    2. the relationship between the torsional constant, moment of inertia, and the frequency of oscillation is
        1. $f = \frac{1}{2\pi}\sqrt{\frac{k}{I}}$
    3. you may check this video for more information
5. $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right)$

Ex 1

From Special Relativity by A.P. French page 34 exercise 1-24

Consider a uniform rod of mass M and length $2L$, spins with angular frequency $\omega \ll c/L$ about its center. Calculate its angular momentum and kinetic energy in the form of:

$$
\begin{aligned}
l &= \frac{ML^2\omega}{3}(1+A\frac{\omega^2L^2}{c^2}+\dots) \\
K &= \frac{ML^2\omega}{6}(1+B\frac{\omega^2L^2}{c^2}+\dots)
\end{aligned} \tag{1}
$$

Find the values of A and B.

solution

We need some background knowledge:

What is angular momentum

Similar to linear momentum, it comes with a formula of $L = I\omega$, in which L stands for linear momentum, I stands for moment of inertia, and ω stands for angular velocity.

About symmetry

Quite clear that the rod should be symmetric about its center. Since the rod is rigid, for two tiny parts symmetry about the center of the rod, they should share the same I and ω, hence the same L.

Considering a very small part of the rod which is x away from the centre, since we can ignore the thickness of the rod, this infinitely small part of the rod can be represented by its length - Δx. Therefore its mass is $\Delta m_0 = \frac{\Delta x}{2L}M$. Taking account of relativistic effect, the relativistic mass is $\Delta m = \Delta m_0 \frac{1}{\sqrt{1-(\omega x/c)^2}}$ (where ωl is the velocity of this part relative to the center of the rod), and the angular momentum is $\Delta l = \omega x^2 \Delta m = \omega x^2 \frac{\Delta x}{2L} M \frac{1}{\sqrt{1-(\omega x/c)^2}}$.

Finally we can write the following equation:

$$
l = \frac{\omega M}{2L}\int_{-L}^{L} x^2 \frac{1}{\sqrt{1-(\omega x/c)^2}}dx \tag{2}
$$

By substituting $\omega x/c = \sin\theta$, we can easily integrate eq.(2) :

$$
l = \frac{c^3}{\omega^3}(\arcsin\frac{\omega L}{c} - \frac{\omega L}{c}\cos(\arcsin\frac{\omega L}{c})) \tag{3}
$$

By applying Taylor expansion for eq.(3):

$$
\begin{aligned}
l &= \frac{c^3}{\omega^3}(\frac{2}{3}(\frac{\omega L}{c})^3 + \frac{1}{5}(\frac{\omega L}{c})^5+\dots) \\
\therefore l &= \frac{1}{3}M\omega L^2(1+\frac{3}{10}\frac{\omega^2L^2}{c^2}+\dots) \\
\therefore A &= \frac{3}{10}
\end{aligned} \tag{4}
$$

Recalling that relativistic kinetic energy is given by the following equation:

$$
K = m_0(\frac{1}{\sqrt{1-v^2/c^2}}-1)c^2 \tag{5}
$$

Hence we can list down the following formula:

$$
K = \int_{-L}^{L}\frac{Mc^2}{2L}\frac{1}{\sqrt{1-(\omega x/c)^2}}dx - m_0c^2 \tag{6}
$$

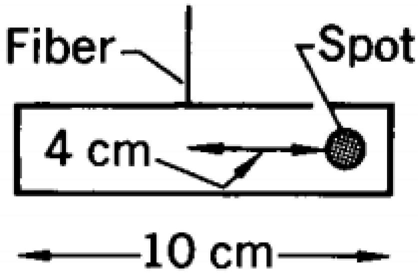
Also by substituting $u = \frac{\omega L}{c}$, then use $\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x$. We can obtain:

$$K = \frac{Mc^3}{L\omega}(\frac{\omega L}{c} + \frac{1}{6}(\frac{\omega L}{c})^3 + \frac{3}{40}(\frac{\omega L}{c})^5) - Mc^2 \tag{7}$$
$$K = \frac{1}{6}M\omega^2 L^2(1 + \frac{3}{40}\frac{\omega^2 L^2}{c^2} + \dots) \tag{8}$$

Therefore $B = \frac{3}{40}$

EX 2

$1 - 4$ A rectangular vane of aluminum foil, 10 cm long and of total mass $100\mathrm{mg}$, hangs vertically in vacuum on a thin fiber (see the figure). The period of torsional oscillation is $40\mathrm{sec}$. What is the static deflection of each end of the vane if 1 watt of radiant energy falls on a spot 4 cm off center? Assume that 60% of the radiation is reflected. The moment of inertia, about an axis through its center, of a rod of mass M and length L is $ML^2/12$. From Special Relativity by A.P. French page 30 exercise 1-4.



Solution

For a torsional oscillation, the relationship between the frequency of the oscillation and the torsional constant is:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{I}} \tag{9}$$

Where I is the moment of inertia of the rod:

$$I = \frac{100 \times 10^{-3} \times 10^{-3} \times (0.1)^2}{12} = \frac{1}{12} \times 10^{-6} \cdot \mathrm{kgm}^2 \tag{10}$$

For photons which have no rest mass:

$$E = cp \quad p = \frac{E}{c} \tag{11}$$
$$\frac{\Delta E}{\Delta t} = P = 1 \times (1 + 60\%) = 1.6\mathrm{W}$$
$$F = \frac{\Delta p}{\Delta t} = \frac{\Delta E}{c\Delta t} = \frac{P}{c} = \frac{1.6}{3 \times 10^8}$$

Therefore the torque is:

$$\tau = F \cdot 4 \times 10^{-2} = 4 \times 10^{-2} \times \frac{1.6}{3 \times 10^8} = \frac{4 \times 1.6}{3} \times 10^{-10}\mathrm{Nm.} \tag{12}$$

By using eq.9 we can deduce that:

$$k = \left(\frac{2\pi}{40}\right)^2 \times \frac{1}{12} \times 10^{-6} = 2.05616 \times 10^{-19}\mathrm{Nm/Rad} \tag{13}$$

The angle of deflection θ simply is:

$$\theta = \frac{J}{k} = 0.10375 \approx 0.104\mathrm{Rad} \tag{14}$$

Hence the deflection is $\theta \times 5\mathrm{cm} = 0.518\mathrm{cm}$

Reference

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