JHU Engineering for Professionals
Applied and Computational Mathematics
Data Mining: 625.740

Module 4 Homework Solutions

1. Assuming x and y are random variables with zero mean,

$$
\begin{aligned}
f_Y(y|x) &= \frac{f_{XY}(x,y)}{f_X(x)} \\
&= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\alpha^2}} \exp\left\{-\frac{1}{2(1-\alpha^2)}\left(\frac{x^2}{\sigma_x^2} - \frac{2\alpha xy}{\sigma_x\sigma_y} + \frac{y^2}{\sigma_y^2}\right)\right\} \div \frac{1}{\sigma_x\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\frac{x^2}{\sigma_x^2}\right\} \\
&= \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\alpha^2}} \exp\left\{-\frac{1}{2(1-\alpha^2)}\left(\frac{\alpha^2 x^2}{\sigma_x^2} - \frac{2\alpha xy}{\sigma_x\sigma_y} + \frac{y^2}{\sigma_y^2}\right)\right\} \\
&= \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\alpha^2}} \exp\left\{-\frac{1}{2\sigma_y^2(1-\alpha^2)}\left(y - \frac{\alpha x\sigma_y}{\sigma_x}\right)^2\right\}.
\end{aligned}
$$

Now, replace x and y by $x-\mu_x$ and $y-\mu_y$, respectively,

$$
f_Y(y|x) = \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\alpha^2}} \exp\left\{-\frac{1}{2\sigma_y^2(1-\alpha^2)}\left[y-\mu_y-\frac{\alpha(x-\mu_x)\sigma_y}{\sigma_x}\right]^2\right\}.
$$

So, y conditioned on x is a Gaussian random variable with mean $\frac{\alpha\sigma_y}{\sigma_x}x+\mu_y-\frac{\alpha\sigma_y\mu_x}{\sigma_x}$ and standard deviation $\sigma_y\sqrt{1-\alpha^2}$ and the result follows immediately. We can also check our result by computing $\mathbf{E}(y|x)$.

$$
\begin{aligned}
\mathbf{E}(y|x) &= \int_{-\infty}^{\infty} f_Y(y|x)\, y\, dy \\
&= \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\alpha^2}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\sigma_y^2(1-\alpha^2)}\left[y-\mu_y-\frac{\alpha(x-\mu_x)\sigma_y}{\sigma_x}\right]^2\right\} y\, dy \\
&= \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\alpha^2}} \int_{-\infty}^{\infty} \exp\left\{-\frac{z^2}{2\sigma_y^2(1-\alpha^2)}\right\}\left[z+\frac{\alpha\sigma_y}{\sigma_x}x+\mu_y-\frac{\alpha\sigma_y\mu_x}{\sigma_x}\right] dz \\
&= \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\alpha^2}} \int_{-\infty}^{\infty} \exp\left\{-\frac{z^2}{2\sigma_y^2(1-\alpha^2)}\right\} z\, dz \\
&\qquad + \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\alpha^2}} \int_{-\infty}^{\infty} \exp\left\{-\frac{z^2}{2\sigma_y^2(1-\alpha^2)}\right\}\left[\frac{\alpha\sigma_y}{\sigma_x}x+\mu_y-\frac{\alpha\sigma_y\mu_x}{\sigma_x}\right] dz
\end{aligned}
$$

The first integral is odd and *converges absolutely*, therefore it converges to zero. The second integral is seen to be a Gaussian pdf times a constant (with respect to z).

$$
\mathbf{E}(y|x) = \frac{\alpha\sigma_y}{\sigma_x}x+\mu_y-\frac{\alpha\sigma_y\mu_x}{\sigma_x}.
$$

We also note that $E(y|x)$ is the regression of y on x since it minimizes the mean sqare error.

$$\mathbf{E}(y|x) = \arg\min_{\eta} \mathbf{E}(|y(x) - \eta(x)|^2).$$

See, for example, [1].

2. (a)

$$\begin{aligned} L &= (Y - \beta X)^T (Y - \beta X) \\ 0 = -\frac{\partial L}{\partial \beta} &= X^T (Y - \beta X) \\ X^T Y &= \hat{\beta} X^T X \\ \hat{\beta} &= \frac{X^T Y}{X^T X} \end{aligned}$$

(b)

$$\begin{aligned} \mathrm{Var}(\hat{\beta}) &= \mathrm{Var}(\frac{X^T Y}{X^T X}) = \frac{X^T X}{(X^T X)^2} \mathrm{Var}(Y) = \frac{\sigma^2}{X^T X} \\ \sqrt{\mathrm{Var}(\hat{\beta})} &= \frac{\sigma}{\sqrt{X^T X}} \end{aligned}$$

(c) By Chebyshev's inequality,

$$P(|\hat{\beta} - \beta| > \epsilon) < \frac{\sigma^2}{\epsilon^2 X^T X}.$$

For $\hat{\beta}$ to be a consistent estimate, we require that $X^T X = \sum_{j=1}^{n} X_j^2 \to \infty$ as $n \to \infty$.

[1] Theodoridis, S. and Koutroumbas, K., **Pattern Recognition**, Fourth edition, Academic Press, pp. 110—111.