STA 137

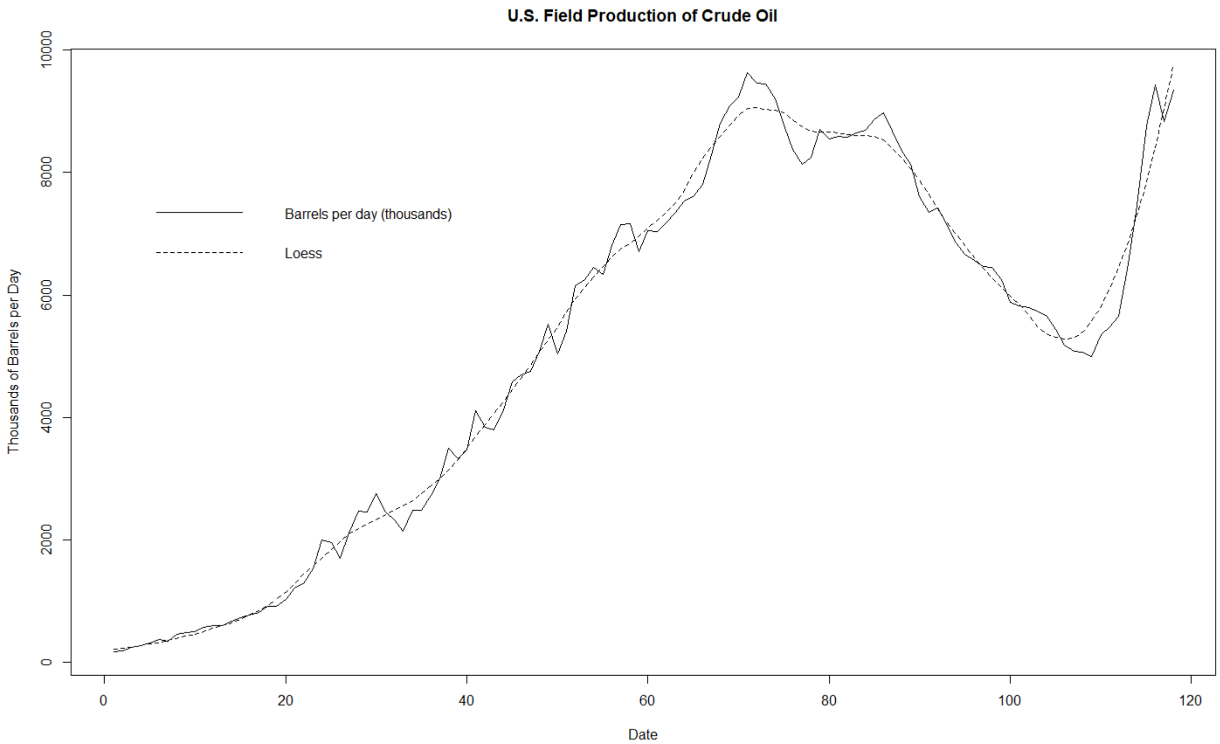
Homework 2

Time Series

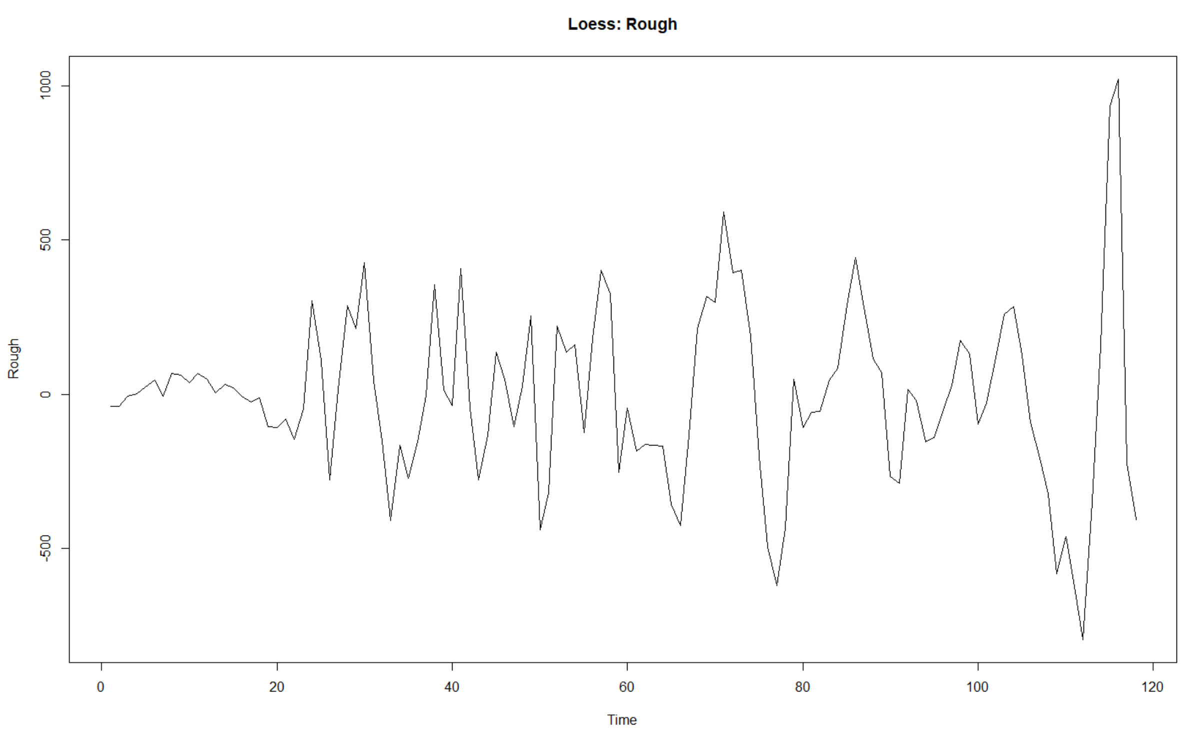
Jared Yu, Danli Zheng

1-23-2019

1. A) Below is a plot of the loess fit for the U.S. Crude Oil data:

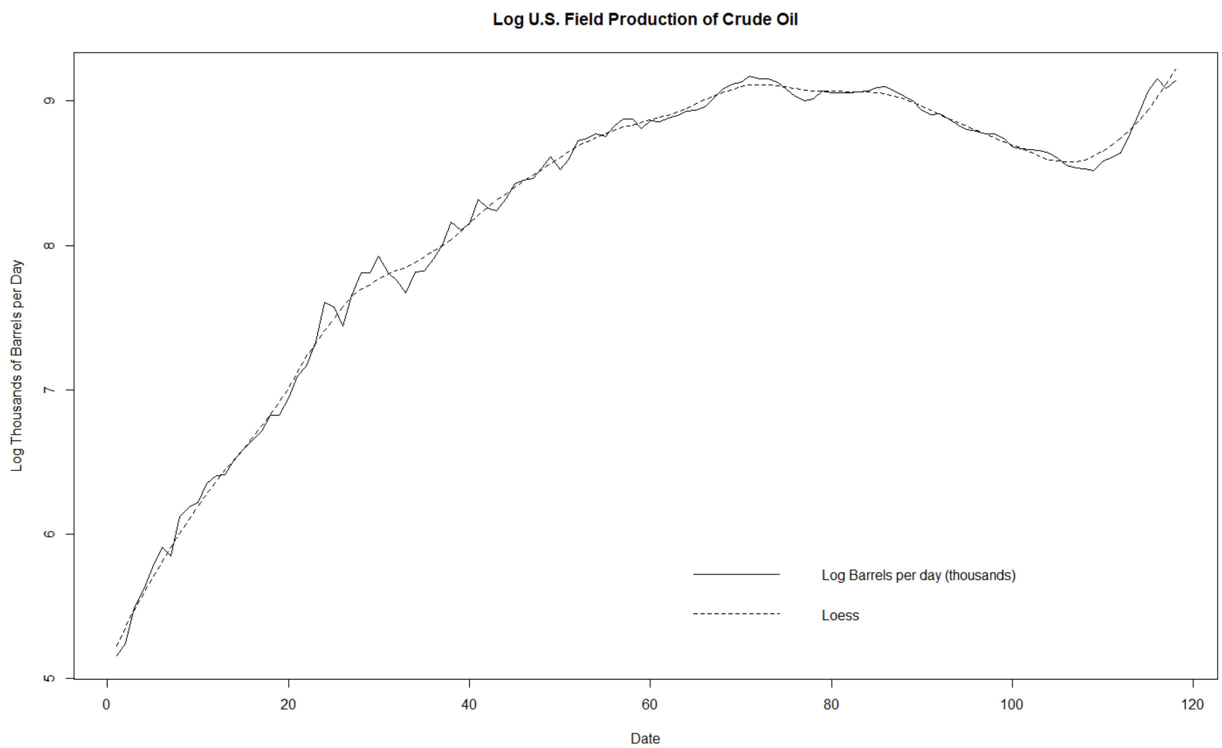


The fitted Loess line is shown as the dashed line. The span for the fit is 0.25. Below is another plot showing the Rough against time:

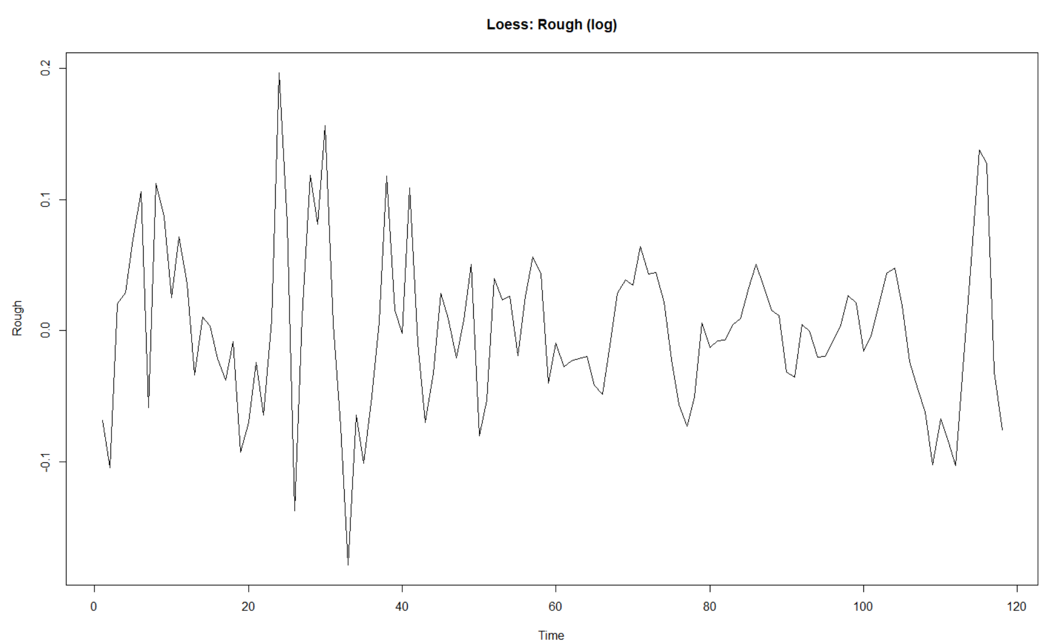


The $R^2 = 1 - \frac{\sum(Y_t - \widehat{m}_t)^2}{\sum(Y_t - \bar{Y})^2} = 0.9912359.$

1. B) Below is a plot with the fitted Loess line like before, except the number of barrels has been transformed using logarithm:



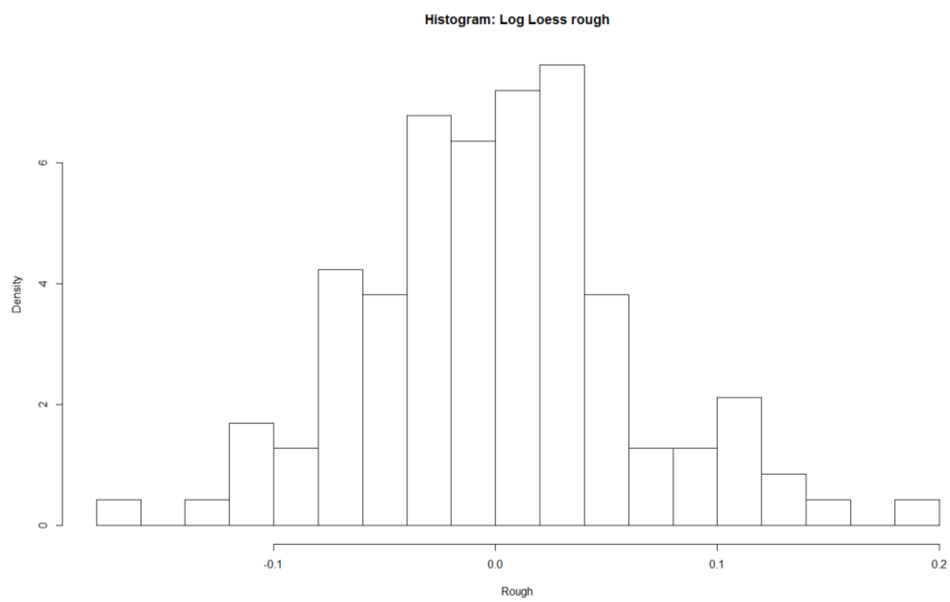
Below is a plot of the Rough using the same span of 0.25.



The $R^2 = 1 - \frac{\sum(Y_t - \hat{m}_t)^2}{\sum(Y_t - \bar{Y})^2} = 0.9966076.$

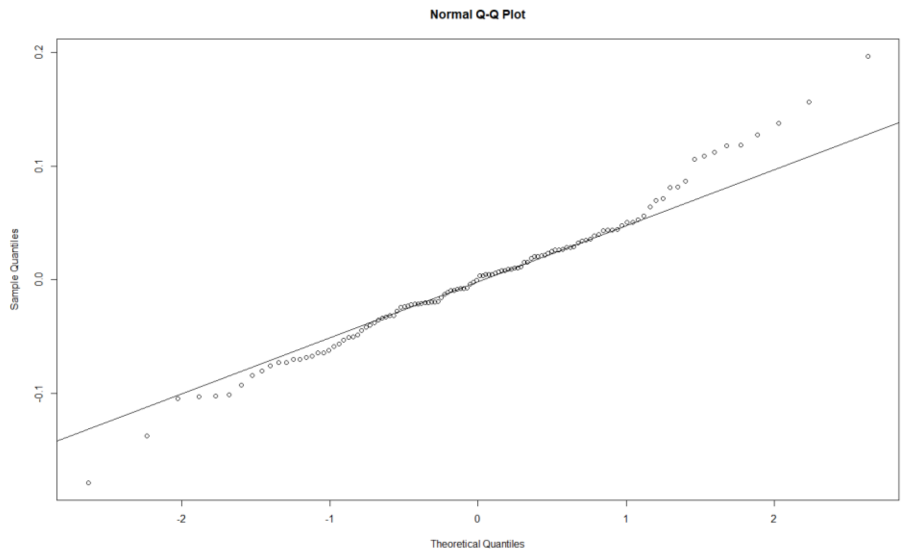
After using the transformation of logarithm, it is apparent that the R^2 has improved slightly. In the transformed data, the fluctuations about the trend have been reduced. Previously, the rough would gradually develop a higher variance. In the transformed data, the rough has a much more even spread, with a horizontal band becoming more noticeable.

1. C) Below is a histogram of the rough $\{\hat{X}_t\}$:



This histogram appears mostly normal, with a roughly bell-shaped curve. There is a peak in the center of the distribution, with a sloping density to the left and right. There may be a slightly more steep drop on the right side, but it does not appear too severe. There are also a large enough number of observations to not worry that there are too few observations to get a good idea of what the distribution of the rough looks like through a histogram.

Below is a normal probability plot of $\{\hat{X}_t\}$:

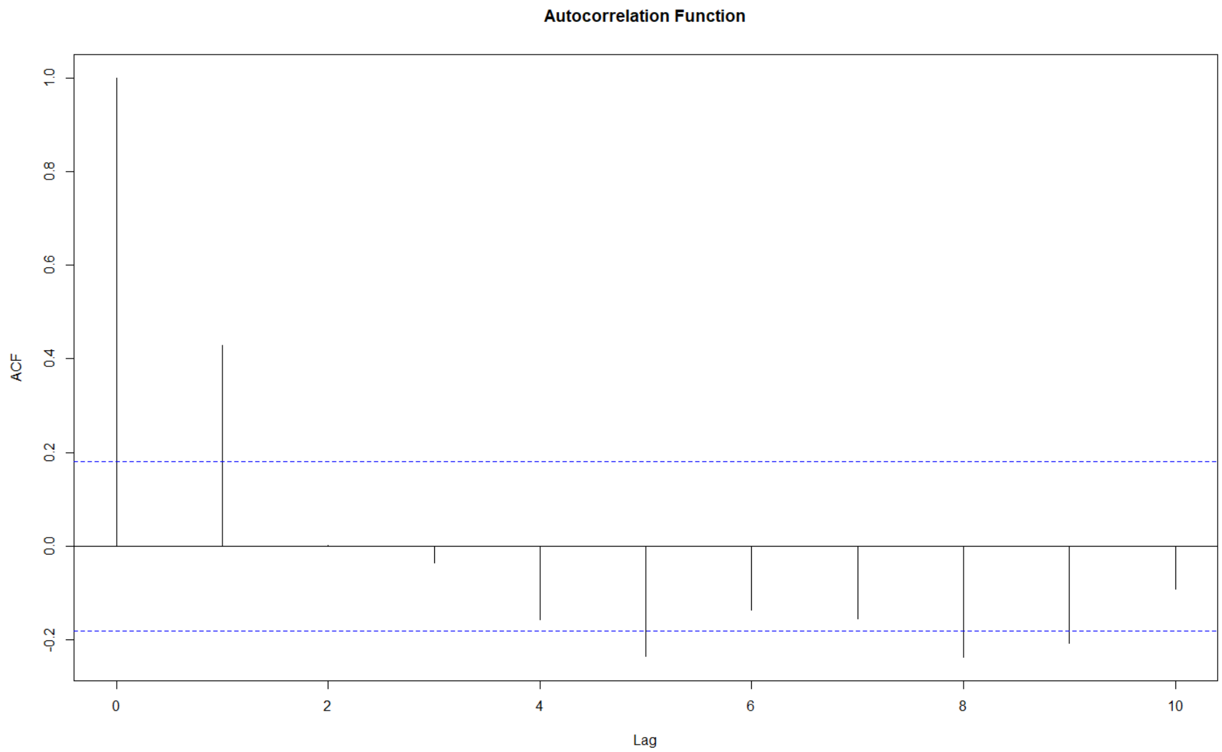


The normal probability plot appears linear in the center; however, the tails seem slightly heavy on the right side. The tails altogether don't seem to show too great of a deviation from the line, so there shouldn't be too much of a worry about the rough not being normal through the Normal Q-Q plot. The correlation of the points is quite high at 0.9912319. However, it is possible also that the heaviness of the tails can be balancing the correlation such that the number will be higher.

When observing the rough through the histogram and the Normal Q-Q plot, it is apparent that there may be some slight deviations in the plots. However, they do not appear significant enough to say that the rough is no longer normally distributed.

To double check the doubt regarding any departure from normality, a Shapiro-Wilk test was done which tested the null hypothesis $H_0: x_1, \cdots, x_n$ come from a normally distributed population. The results are that the test statistic $W = 0.9846$ with $p - value = 0.1981$. Therefore, we fail to reject the null hypothesis at the 95% confidence level. The conclusion is that the rough is from a normally distributed population.

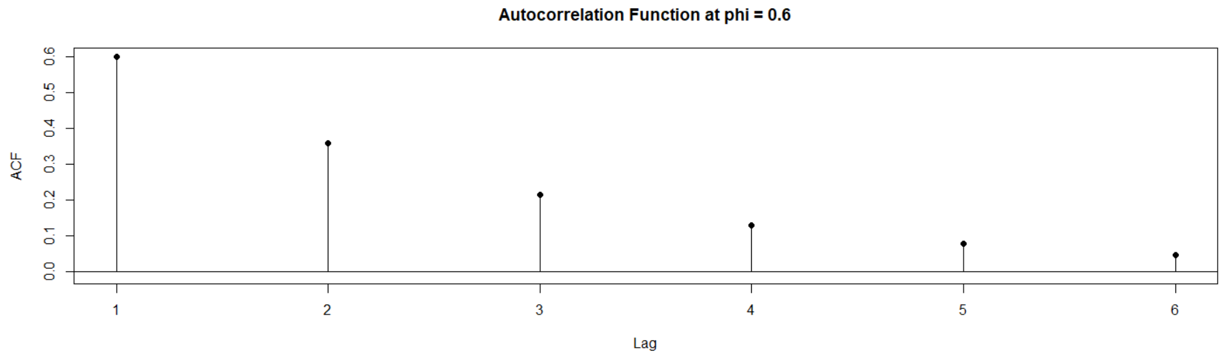
1. D) Below is an ACF plot of the Rough up to order 10 along with the confidence interval.

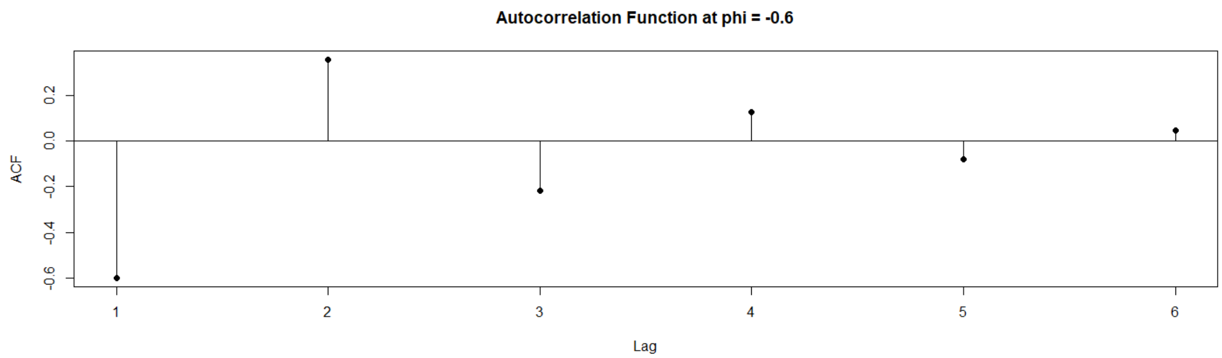


The plot shows the transformed rough of the log U.S. Oil purchases, and this gives the residuals which are taken from the Loess model. Given up to order 10 of the ACF plot, there seems to be significant correlation up to lag 1. This would indicate that the rough is positively correlated with the previous rough at time X_{t-1}, and they would move in sync with each other to some extent. However, there are other possibly significant points that show the ACF crossing the bounds at lag 5 and 8. The only issue is that these departures are not as severe.

1. E) The following is the null an alternative hypothesis of the Box-Ljeung test. $H_0: \rho(1) = \cdots = \rho(10) = 0$ vs. $H_1: at\ least\ one\ of\ \rho(1) = \cdots = \rho(10)\ is\ nonzero$
   The test statistic is equal to 51.682, with 10 degrees of freedom. The p-value from this is less than 1.307×10^{-7}. The decision then is to reject the null hypothesis in favor of the alternative since the p-value is far less than 0.05. The conclusion then is that the sequence of the rough are not independent and identically distributed. Instead, they exhibit correlation.

2. A) Given the following $AR(1)$ sequence:
   $$X_t - \mu = \phi(X_{t-1} - \mu) + \varepsilon_t,$$
   where ε_t is mean zero iid with variance σ^2, here is a plot of the autocorrelations $\rho(j)$ against $j = 0,1,\cdots,6$ when $\phi = 0.6$ and when $\phi = -0.6$.

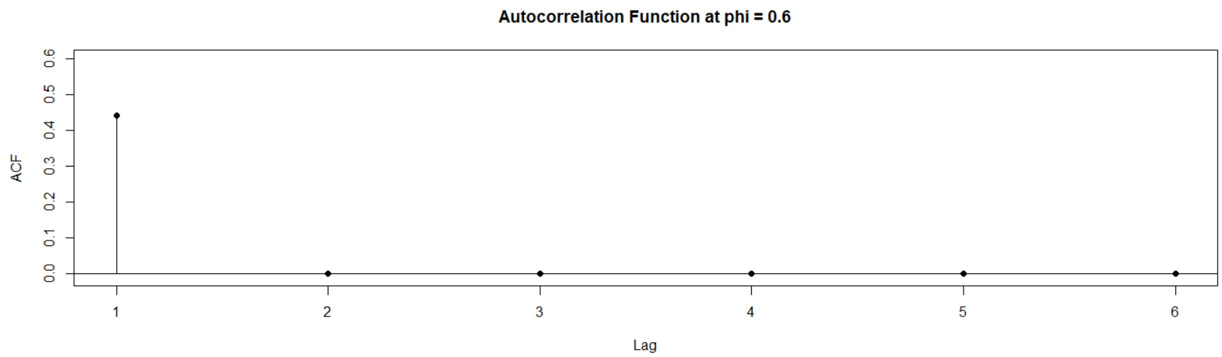


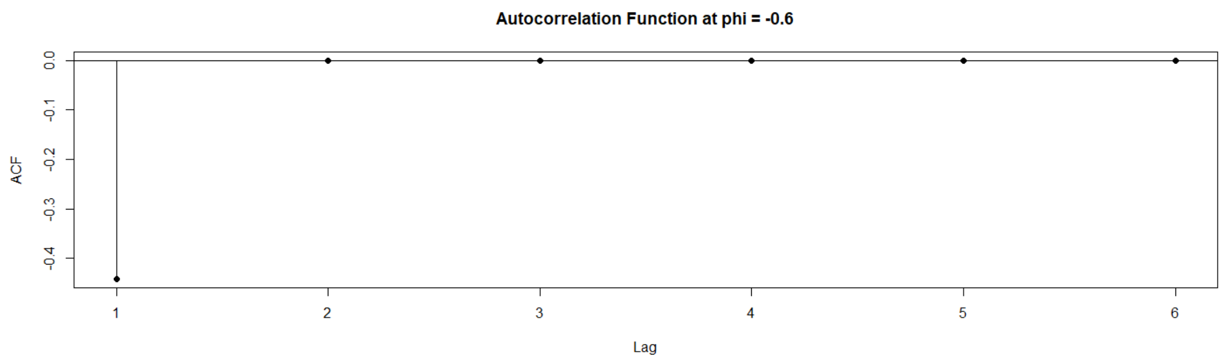


2. B) Given the following $MA(1)$ sequence:

$$X_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1},$$

where ε_t are mean zero iid with variance σ^2, here is a plot of the autocorrelations $\rho(j)$ against $j = 0,1,\cdots,6$ when $\phi = 0.6$ and when $\phi = -0.6$.





3. A) The given $MA(2)$ sequence has the form:

$$X_t = \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2},$$

where ε_t are mean zero and iid with variance σ^2.

The $Var(X_t)$ can be written as such:

$$\begin{aligned} Var(X_t) &= Var(\varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2}) \\ &= Var(\varepsilon_t) + \theta_1^2 Var(\varepsilon_{t-1}) + \theta_2^2 Var(\varepsilon_{t-2}) \\ &= \sigma^2 + \theta_1^2\sigma^2 + \theta_2^2\sigma^2 = \sigma^2(1 + \theta_1^2 + \theta_2^2) \end{aligned}$$

The autocovariance $Cov(X_t, X_{t+j})$ can be written as such:

$$\begin{aligned} Cov(X_t, X_{t+j}) &= Cov(\varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2}, \varepsilon_{t+j} + \theta_1\varepsilon_{t+j-1} + \theta_2\varepsilon_{t+j-2}) \\ &= Cov(\varepsilon_t, \varepsilon_{t+j}) + Cov(\varepsilon_t, \theta_1\varepsilon_{t+j-1}) + Cov(\varepsilon_t, \theta_2\varepsilon_{t+j-2}) + Cov(\theta_1\varepsilon_{t-1}, \varepsilon_{t+j}) \\ &\quad + Cov(\theta_1\varepsilon_{t-1}, \theta_1\varepsilon_{t+j-1}) + Cov(\theta_1\varepsilon_{t-1}, \theta_2\varepsilon_{t+j-2}) + Cov(\theta_2\varepsilon_{t-2}, \varepsilon_{t+j}) \\ &\quad + Cov(\theta_2\varepsilon_{t-2}, \theta_1\varepsilon_{t+j-1}) + Cov(\theta_2\varepsilon_{t-2}, \theta_2\varepsilon_{t+j-2}) \\ &= Cov(\varepsilon_t, \varepsilon_{t+j}) + \theta_1 Cov(\varepsilon_t, \varepsilon_{t+j-1}) + \theta_2 Cov(\varepsilon_t, \varepsilon_{t+j-2}) + \theta_1 Cov(\varepsilon_{t-1}, \varepsilon_{t+j}) \\ &\quad + \theta_1^2 Cov(\varepsilon_{t-1}, \varepsilon_{t+j-1}) + \theta_1\theta_2 Cov(\varepsilon_{t-1}, \varepsilon_{t+j-2}) + \theta_2 Cov(\varepsilon_{t-2}, \varepsilon_{t+j}) \\ &\quad + \theta_1\theta_2 Cov(\varepsilon_{t-2}, \varepsilon_{t+j-1}) + \theta_2^2 Cov(\varepsilon_{t-2}, \varepsilon_{t+j-2}) \\ &= Cov(\varepsilon_t, \varepsilon_{t+j}) + \theta_1 Cov(\varepsilon_t, \varepsilon_{t+j-1}) + \theta_2 Cov(\varepsilon_t, \varepsilon_{t+j-2}) + \theta_1 Cov(\varepsilon_{t-1}, \varepsilon_{t+j}) \\ &\quad + \theta_1^2 Cov(\varepsilon_{t-1}, \varepsilon_{t+j-1}) + \theta_1\theta_2 Cov(\varepsilon_{t-1}, \varepsilon_{t+j-2}) + \theta_2 Cov(\varepsilon_{t-2}, \varepsilon_{t+j}) \\ &\quad + \theta_1\theta_2 Cov(\varepsilon_{t-2}, \varepsilon_{t+j-1}) + \theta_2^2 Cov(\varepsilon_{t-2}, \varepsilon_{t+j-2}) \end{aligned}$$

When $j = 0, \Rightarrow\ Cov(X_t, X_t) = Cov(\varepsilon_t, \varepsilon_t) + \theta_1 Cov(\varepsilon_t, \varepsilon_{t-1}) + \theta_2 Cov(\varepsilon_t, \varepsilon_{t-2}) +$
$\theta_1 Cov(\varepsilon_{t-1}, \varepsilon_t) + \theta_1^2 Cov(\varepsilon_{t-1}, \varepsilon_{t-1}) + \theta_1\theta_2 Cov(\varepsilon_{t-1}, \varepsilon_{t-2}) + \theta_2 Cov(\varepsilon_{t-2}, \varepsilon_t) +$
$\theta_1\theta_2 Cov(\varepsilon_{t-2}, \varepsilon_{t-1}) + \theta_2^2 Cov(\varepsilon_{t-2}, \varepsilon_{t-2})$

$$= \sigma^2 + \theta_1 \times 0 + \theta_2 \times 0 + \theta_1 \times 0 + \theta_1^2\sigma^2 + \theta_1\theta_2 \times 0 + \theta_2 \times 0 + \theta_1\theta_2 \times 0 + \theta_2^2\sigma^2$$
$$= \sigma^2 + \theta_1^2\sigma^2 + \theta_2^2\sigma^2 = \sigma^2(1 + \theta_1^2 + \theta_2^2)$$

When $j = 1 \Rightarrow Cov(X_t, X_{t+1}) = Cov(\varepsilon_t, \varepsilon_{t+1}) + \theta_1 Cov(\varepsilon_t, \varepsilon_t) + \theta_2 Cov(\varepsilon_t, \varepsilon_{t-1}) +$
$\theta_1 Cov(\varepsilon_{t-1}, \varepsilon_{t+1}) + \theta_1^2 Cov(\varepsilon_{t-1}, \varepsilon_t) + \theta_1\theta_2 Cov(\varepsilon_{t-1}, \varepsilon_{t-1}) + \theta_2 Cov(\varepsilon_{t-2}, \varepsilon_{t+1}) +$
$\theta_1\theta_2 Cov(\varepsilon_{t-2}, \varepsilon_t) + \theta_2^2 Cov(\varepsilon_{t-2}, \varepsilon_{t-1})$

$$= 0 + \theta_1\sigma^2 + \theta_2 \times 0 + \theta_1 \times 0 + \theta_1^2 \times 0 + \theta_1\theta_2\sigma^2 + \theta_2 \times 0 + \theta_1\theta_2 \times 0 + \theta_2^2 \times 0$$
$$= \theta_1\sigma^2 + \theta_1\theta_2\sigma^2 = \sigma^2\theta_1(1 + \theta_2)$$

When $j = 2 \Rightarrow Cov(X_t, X_{t+2}) = Cov(\varepsilon_t, \varepsilon_{t+2}) + \theta_1 Cov(\varepsilon_t, \varepsilon_{t+1}) + \theta_2 Cov(\varepsilon_t, \varepsilon_t) +$
$\theta_1 Cov(\varepsilon_{t-1}, \varepsilon_{t+2}) + \theta_1^2 Cov(\varepsilon_{t-1}, \varepsilon_{t+1}) + \theta_1\theta_2 Cov(\varepsilon_{t-1}, \varepsilon_t) + \theta_2 Cov(\varepsilon_{t-2}, \varepsilon_{t+2}) +$
$\theta_1\theta_2 Cov(\varepsilon_{t-2}, \varepsilon_{t+1}) + \theta_2^2 Cov(\varepsilon_{t-2}, \varepsilon_t)$

$$= 0 + \theta_1 \times 0 + \theta_2\sigma^2 + \theta_1 \times 0 + \theta_1^2 \times 0 + \theta_1\theta_2 \times 0 + \theta_2 \times 0 + \theta_1\theta_2 \times 0 + \theta_2^2 \times 0$$
$$= \theta_2\sigma^2$$

When $j = 3 \Rightarrow Cov(X_t, X_{t+3}) = Cov(\varepsilon_t, \varepsilon_{t+3}) + \theta_1 Cov(\varepsilon_t, \varepsilon_{t-2}) + \theta_2 Cov(\varepsilon_t, \varepsilon_{t+1}) +$
$\theta_1 Cov(\varepsilon_{t-1}, \varepsilon_{t+3}) + \theta_1^2 Cov(\varepsilon_{t-1}, \varepsilon_{t+2}) + \theta_1\theta_2 Cov(\varepsilon_{t-1}, \varepsilon_{t+1}) + \theta_2 Cov(\varepsilon_{t-2}, \varepsilon_{t+3}) +$
$\theta_1\theta_2 Cov(\varepsilon_{t-2}, \varepsilon_{t+2}) + \theta_2^2 Cov(\varepsilon_{t-2}, \varepsilon_{t+1})$

$$= 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 0$$

When $|j| \geq 3 \Rightarrow Cov(X_t, X_{t+j}) = 0$

So, for any t,

$$\gamma(j) = Cov(X_t, X_{t+j}) = \begin{cases} \sigma^2(1 + \theta_1^2 + \theta_2^2) & j = 0 \\ \sigma^2(\theta_1 + \theta_1\theta_2) & j = \pm 1 \\ \sigma^2\theta_2 & j = \pm 2 \\ 0 & j = \pm 3 \end{cases}$$

Hence $\{X_t\}$ is stationary. Note that

$$\rho(j) = \begin{cases} 1 & j = 0 \\ \dfrac{\theta_1(1 + \theta_2)}{1 + \theta_1^2 + \theta_2^2} & j = \pm 1 \\ \dfrac{\theta_2}{1 + \theta_1^2 + \theta_2^2} & j = \pm 2 \\ 0 & j = \pm 3 \end{cases}$$

3. B) The given $AR(2)$ sequence has the form:

$$X_t - \mu = \phi_1(X_{t-1} - \mu) + \phi_2(X_{t-2} - \mu) + \varepsilon_t,$$

where ε_t are mean zero iid with variance σ^2.
The above equation can be rewritten as:

$$X_t - \mu = \phi_1 X_{t-1} - \phi_1\mu + \phi_2 X_{t-2} - \phi_2\mu + \varepsilon_t$$
$$\Rightarrow X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} - \phi_1\mu - \phi_2\mu + \mu + \varepsilon_t$$
$$\Rightarrow X_t = \beta_0 + \beta_1 X_{t-1} + \beta_2 X_{t-2} + \varepsilon_t$$

where $\beta_0 = \mu(1 - \phi_1 - \phi_2), \beta_1 = \phi_1, \beta_2 = \phi_2$

References:

http://www.sthda.com/english/wiki/normality-test-in-r
https://en.wikipedia.org/wiki/Shapiro%E2%80%93Wilk_test
https://www.burns-stat.com/plot-ranges-of-data-in-r/