STA 137

Homework 7

Prof. Burman

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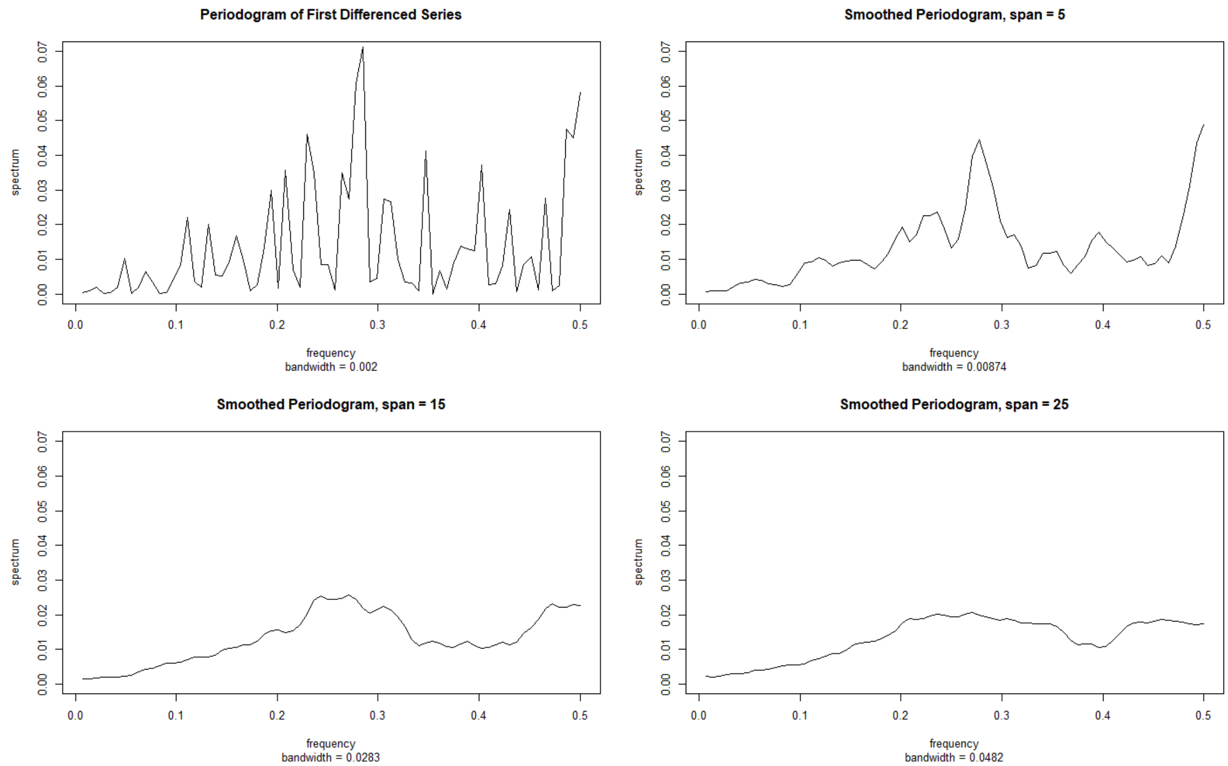
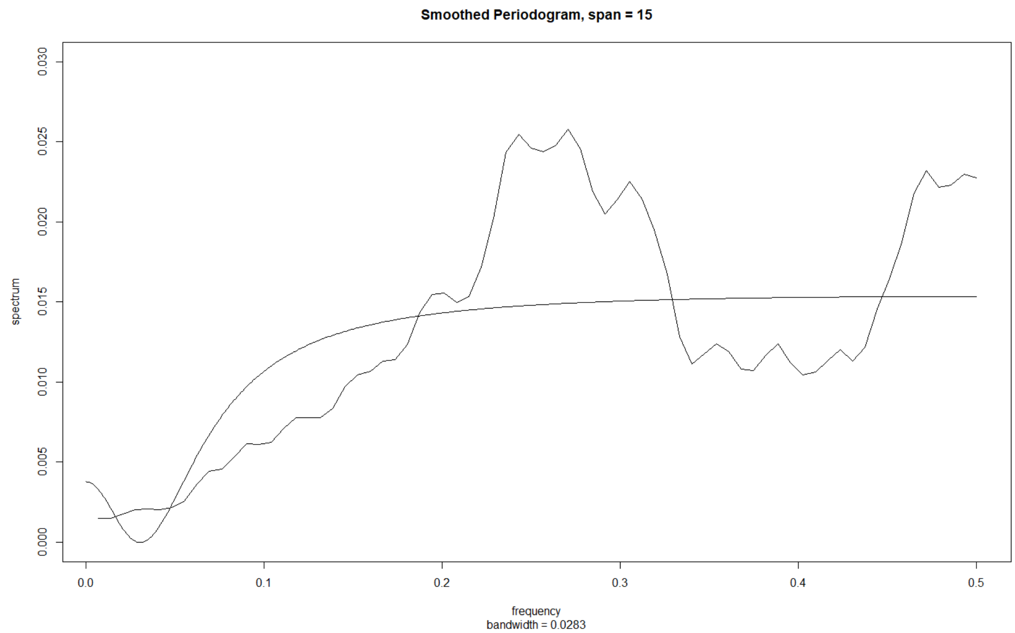
1. a) The data comes from ‘GlobalTemp_NASA’, and the first difference has been taken of the data. This series will be referred to as X_t. Below is a periodogram of the series X_t, also there are three more smoothed periodograms using the modified Daniell’s method. The different smoothed periodograms have spans 5, 15, and 25, respectively.



It seems that the best periodogram is the one with $span = 15$. This periodogram does not look to be too rough, while at the same time accounting for both the peak in the middle, and the rise towards the right side. The periodogram with $span = 5$, seems to be too rough, despite having a nice similarity with the original periodogram. The periodogram with $span = 25$ however is too smooth, and makes it seem like the data is no longer being properly represented.

1. b) The optimal amount of smoothing when using the `specselect()` function shows that we should use 7 as the value for k.

1. c) Below is a plot of the periodogram with $span = 2 \times 7 + 1 = 15$, along with the spectral density function for an $ARMA(2,2)$ model.



The spectral density function seems to describe quite well an overall pattern emerging from the data that is also evident in the smoothed periodogram. The pattern exhibits a beginning which seems rather small, and a slowly rising trend. The trend seems to overall be going upwards, but it is uncertain of the true pattern, which is evident in the slow but rising trajectory.

2. a) X_t is an $MA(1)$ sequence with $\theta = 0.7$ and $\sigma^2 = 2$.

$$X_t = \theta\varepsilon_{t-1} + \varepsilon_t,\ \text{where } \varepsilon_t \sim WN(0, \sigma^2), t = 1,2,\cdots$$

We will now take the first difference of the series $\nabla X_t = X_t - X_{t-1}$.

$$\nabla X_t = X_t - X_{t-1}$$

$$= \theta\varepsilon_{t-1} + \varepsilon_t - (\theta\varepsilon_{t-2} + \varepsilon_{t-1})$$

$$= -\theta\varepsilon_{t-2} + (\theta - 1)\varepsilon_{t-1} + \varepsilon_t$$

$$= -0.7 \times \varepsilon_{t-2} + (0.7 - 1\ ) \times \varepsilon_{t-1} + \varepsilon_t$$

$$= -0.7 \times \varepsilon_{t-2} - 0.3 \times \varepsilon_{t-1} + \varepsilon_t$$

The $MA(1)$ model has now been written as an $MA(2)$ model after taking the first difference.

2. b) The general formula for spectral density of an $MA(q)$ series is as follows:

$$f(w) = \sigma^2|\theta(z)|^2, \text{where } z = \exp(-2\pi i w) \tag{1}$$

So, for the current model with $MA(2)$, we are interested in finding the spectral density. Then we will first look at $\theta(z)$, the polynomial for an $MA(q)$ process that is related to stationarity, etc.

$$\theta(z) = 1 + (-0.3) \times z + (-0.7) \times z^2$$

$$= 1 - 0.3 \times z - 0.7 \times z^2$$

Now using the formula for the spectral density function (1) we can replace z with $\exp(-2\pi iw)$.

$$= 1 - 0.3 \times \exp(-2\pi iw) - 0.7 \times \exp(-4\pi iw)$$

Then using the formula $\exp(-iA) = \cos(A) - isin(A)$,

$$= 1 - 0.3 \times (\cos(2\pi w) - isin(2\pi w)) - 0.7 \times (\cos(4\pi w) - isin(4\pi w))$$

$$= 1 - 0.3 \times \cos(2\pi w) - 0.7 \times \cos(4\pi w) + [0.3 \times sin(2\pi w) + 0.7 \times sin(4\pi w)]i$$

Then using the formula for $|\theta(z)|^2$, we can drop the i:

$$= (1 - 0.3 \times \cos(2\pi w) - 0.7 \times \cos(4\pi w))^2 + \left(0.3 \times sin(2\pi w) + 0.7 \times sin(4\pi w)\right)^2$$

$$\begin{aligned} = 1^2 &+ (-0.3 \times \cos(2\pi w))^2 + (-0.7 \times \cos(4\pi w))^2 + 2 \times 1 \times (-0.3 \times \cos(2\pi w)) \\ &+ 2 \times (-0.3 \times \cos(2\pi w)) \times (-0.7 \times \cos(4\pi w)) + 2 \times 1 \times (-0.7 \times \cos(4\pi w)) \\ &+ \left(0.3 \times sin(2\pi w)\right)^2 + 2 \times \left(0.3 \times sin(2\pi w)\right) \times \left(0.7 \times sin(4\pi w)\right) \\ &+ \left(0.7 \times sin(4\pi w)\right)^2 \end{aligned}$$

$$\begin{aligned} = 1 &+ 0.09\cos^2(2\pi w) + 0.49\cos^2(4\pi w) - 0.6\cos(2\pi w) + 0.42\cos(2\pi w)\, cos(4\pi w) \\ &- 1.4\cos(4\pi w) + 0.09\sin^2(2\pi w) + 0.42\sin(2\pi w)\sin(4\pi w) + 0.49\sin^2(4\pi w) \end{aligned}$$

$$\begin{aligned} = 1 &+ 0.09 + 0.49 - 0.6\cos(2\pi w) + 0.42(\cos(2\pi w)\cos(4\pi w) + \sin(2\pi w)\sin(4\pi w)) \\ &- 1.4\cos(4\pi w) \end{aligned}$$

$$= 1.58 - 0.6\cos(2\pi w) + 0.42\cos(2\pi w - 4\pi w) - 0.14\cos(4\pi w)$$

$$= 1.58 - 0.18\cos(2\pi w) - 1.4\cos(4\pi w)$$

Now referring to the formula (1) for $f(w)$,

$$f(w) = \sigma^2(1.58 - 0.18\cos(2\pi w) - 1.4\cos(4\pi w))$$

$$= 2 \times (1.58 - 0.18\cos(2\pi w) - 1.4\cos(4\pi w))$$

$$= 3.16 - 0.36\cos(2\pi w) - 2.8\cos(4\pi w)$$

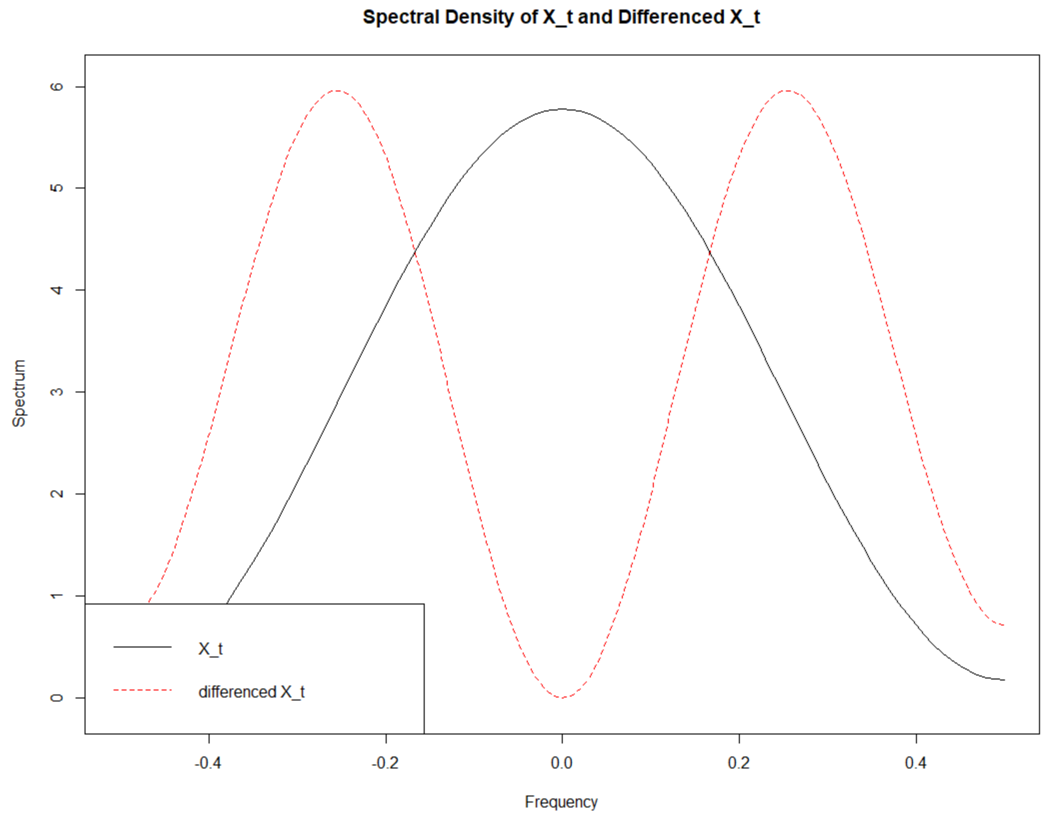
2. c) The formula for the spectral density of an $MA(1)$ is given in the handout, and it is $f(w) = \sigma^2(1 + \theta^2 + 2\theta\cos(2\pi w))$. Then plugging in the values that are already given, we find that,

$$f(w) = 2 \times (1 + 0.7^2 + 2 \times 0.7 \times \cos(2\pi w))$$

$$= 2 \times (1.49 + 1.4\cos(2\pi w))$$

$$= 2.98 + 2.8\cos(2\pi w)$$

Below is a plot of both spectral density functions for X_t and ∇X_t:



It seems that ∇X_t is the flip of the original X_t when looking at their spectral density functions. The differenced X_t also looks somewhat narrower in terms of shape when being flipped. The original X_t looks much wider in terms of its shape. The narrowness seems to allow more of the cosine pattern to also be visible in the range of the plot.

3. a) Given that X_t is white noise with $\sigma^2 = 2$, we are given the additional sequence:

$$Z_t = \frac{(X_{t+1} + X_t + X_{t-1})}{\sqrt{3}}$$

Then using the formula from handout 11, we have $Z_t = \sum_{-\infty<s<\infty} \psi_s X_{t-s}$, we have in our case that $s = -1, 0$, and 1. Also, we have that $\psi_s = \frac{1}{\sqrt{3}}$ for all s. The constants also satisfy the condition $\sum_{-\infty<s<\infty} |\psi_s| < \infty$.

Since we have white noise X_t, $\gamma(0) = \sigma^2$, and $\gamma(h) = 0$ when $h \neq 0$. The reason is that white noise is independent, and so any covariance with $h \neq 0$, results in a covariance equal to 0. So, when looking at the formula for a spectral density function, we get the following result:

$$f_X(w) = \sum_{h=-\infty}^{\infty} \gamma(h)\exp(-2\pi i w h) = \gamma(0) + 2\sum_{h=1}^{\infty} \gamma(h)\cos(2\pi w h) = \gamma(0) = \sigma^2 = 2.$$

So, we want to find the spectral density function for Z_t, which we can write as follows:

$$f_Z(w) = |\Psi(w)|^2 f_X(w), \text{ where } i = \sqrt{-1}$$

Since we have already found $f_X(w)$, we are interested next in the formula for the frequency response function $\Psi(w)$.

$$\Psi(w) = \sum\nolimits_{-\infty<s<\infty} \psi_s \exp(-2\pi i w s)$$

Then using the constraints of $s = -1, 0$, and 1,

$$\Psi(w) = \psi_{-1} \exp(2\pi i w) + \psi_0 + \psi_1 \exp(-2\pi i w)$$

$$= 1/\sqrt{3} \exp(2\pi i w) + 1/\sqrt{3} + 1/\sqrt{3} \exp(-2\pi i w)$$

$$= \frac{1}{\sqrt{3}}(\exp(2\pi i w) + \exp(-2\pi i w)) + \frac{1}{\sqrt{3}}$$

Then using the formula that $\exp(iA) + \exp(-iA) = 2\cos(A)$,

$$= \frac{1}{\sqrt{3}} 2\cos(2\pi w) + \frac{1}{\sqrt{3}}$$

Then the continuing the formula for $f_Z(w)$:

$$f_Z(w) = |\Psi(w)|^2 f_X(w), i = \sqrt{-1}$$

$$f_Z(w) = \left(\frac{1}{\sqrt{3}} 2\cos(2\pi w) + \frac{1}{\sqrt{3}}\right)^2 \times 2$$

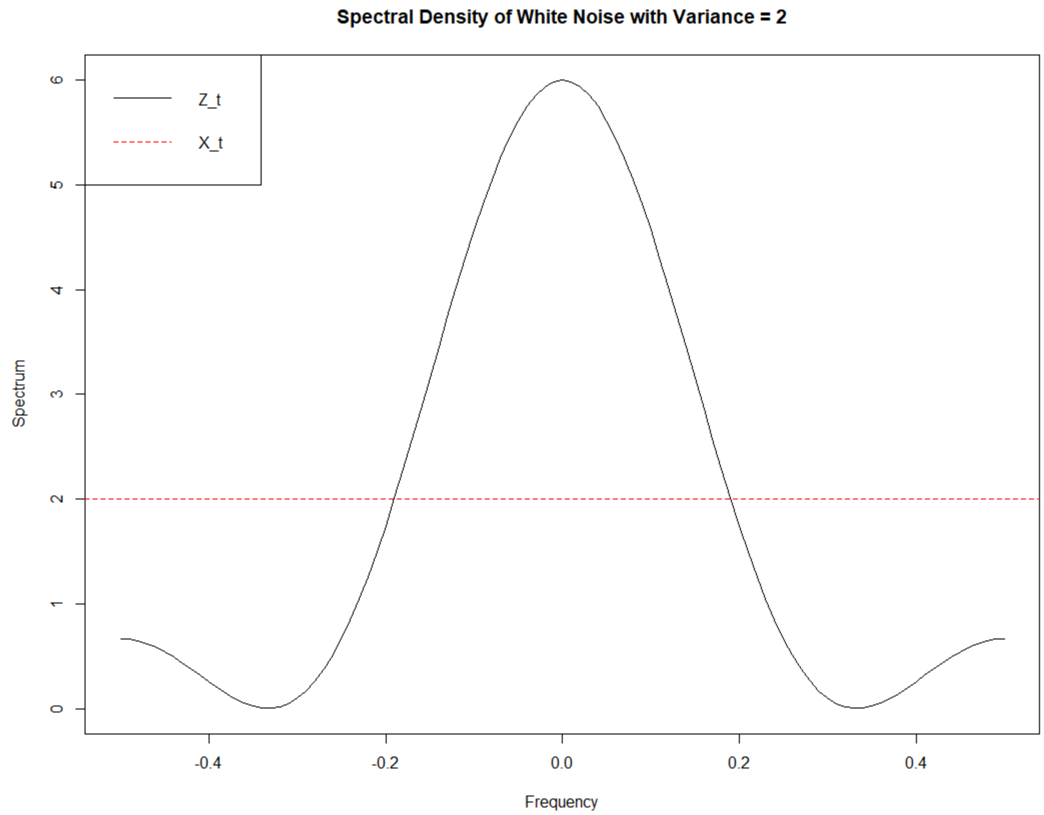
Because when taking $|\Psi(w)|^2$, the value $\Psi(w)$ is itself squared, since there are no imaginary numbers i.

$$= \left(\frac{1}{\sqrt{3}}(2\cos(2\pi w) + 1)\right)^2 \times 2$$

$$= \frac{2}{3}(2\cos(2\pi w) + 1)^2$$

$$= \frac{2}{3}(1 + 4\cos(2\pi w) + 4\cos^2(2\pi w))$$

3. b) Below is a plot of the spectral function of X_t and Z_t.



The difference is quite high, as the spectral density function of Z_t shows the bell curve which has been seen previously, while the spectral density of X_t is simply a flat line.