Homework 3

Jared Yu

April 23, 2018

Problem 1.

(a) MLE of θ :

$l'(\theta) = x_1/(2+\theta) - (x_2+x_3)/(1-\theta) + x_4/\theta = 0$

$x_1\theta(1-\theta) - (x_2+x_3)(2+\theta)\theta + x_4(2+\theta)(1-\theta) = 0$

$x_1(\theta-\theta^2) - (x_2+x_3)(2\theta+\theta^2) + x_4(2-\theta-\theta^2) = 0$

$x_1\theta - x_1\theta^2 - [2\theta x_2 + x_2\theta^2 + 2\theta x_3 + \theta^2 x_3] + 2x_4 - \theta x_4 - \theta^2 x_4 = 0$

$x_1\theta - x_1\theta^2 - 2x_2\theta - x_2\theta^2 - 2x_3\theta - x_3\theta^2 + 2x_4 - \theta x_4 - \theta^2 x_4 = 0$

$\theta(x_1 - 2x_2 - 2x_3 - x_4) + \theta^2(-x_1 - x_2 - x_3 - x_4) + 2x_4 = 0$

(from the quadratic formula)

$$\frac{-x_1 + 2x_2 + 2x_3 + x_4 \pm \sqrt{(x_1 - x_2 - x_3 - x_4)^2 - 4(-x_1 - x_2 - x_3 - x_4)(2x_4)}}{2(-x_1 - x_2 - x_3 - x_4)}$$

Let $x_1 = 125, x_2 = 21, x_3 = 20, x_4 = 33,$

then $\dfrac{-10 \pm \sqrt{52,636}}{-398}$

0.60157 or -0.55132

However, if $\theta = -0.55132$, $p_4 = \dfrac{1}{4}\theta < 0$ which doesn't make sense. So it is rejected. $\therefore \hat{\theta} = 0.60157$

(b) E-Step:

$Q(\theta, \theta^{(k)}) = E_{\theta^{(k)}}(l_c(\theta)|\vec{x})$

$= E_{\theta^{(k)}}((x_{12} + x_4)log\theta + (x_2 + x_3)log(1-\theta)|\vec{x})$

$= [E_{\theta^{(k)}} + x_4]log\theta + (x_2 + x_3)log(1-\theta)$

$$P(x_{12}|x_1) = \frac{P(x_{12} \cap x_1)}{P(x_1)} = \frac{\frac{\theta}{4}}{\frac{1}{2} + \frac{\theta}{4}} = \frac{\theta}{2+\theta}$$

$E_{\theta^{(k)}}(x_{12}|x_1) = \dfrac{x_1\theta^{(k)}}{2+\theta^{(k)}} \because x_{12}|x_1 \sim Bin(x_1, \dfrac{\theta}{2+\theta})$

So, $Q(\theta, \theta^{(k)}) = [\dfrac{x_1\theta^{(k)}}{2+\theta^{(k)}} + x_4]log\theta + (x_2 + x_3)log(1-\theta)$

M-Step:

$$\frac{\partial}{\partial\theta}Q(\theta,\theta^{(k)}) = [\frac{x_1\theta^{(k)}}{2+\theta^{(k)}} + x_4]\frac{1}{\theta} - \frac{(x_2+x_3)}{1-\theta}$$

Set $\frac{\partial}{\partial\theta}Q(\theta,\theta^{(k)}) = 0$

$$[\frac{x_1\theta^{(k)}}{2+\theta^{(k)}} + x_4](1-\theta) = \theta(x_2+x_3)$$

$$[\frac{x_1\theta^{(k)}}{2+\theta^{(k)}} + x_2 + x_3 + x_4] = \frac{x_1\theta^{(k)}}{2+\theta^{(k)}} + x_4]$$

$$\theta^{(k+1)} = \frac{\frac{x_1\theta^{(k)}}{2+\theta^{(k)}} + x_4}{\frac{x_1\theta^{(k)}}{2+\theta^{(k)}} + x_2 + x_3 + x_4}$$

When $k \to \infty$, let $\theta^{(k+1)} = \theta^{(k)} = \theta^{(*)}$

$$\theta^{(*)} = \frac{\frac{x_1\theta^{(*)}}{2+\theta^{(*)}} + x_4}{\frac{x_1\theta^{(*)}}{2+\theta^{(*)}} + x_2 + x_3 + x_4}$$

$$\theta^{(*)}[2(x_2+x_3+x_4) + (x_1+x_2+x_3+x_4)\theta^{(*)}] = (x_1+x_4)\theta^{(*)} + 2x_4$$

$$\theta^{(*)}[2(n-x_1) + n\theta^{(*)}] = (x_1+x_4)\theta^{(*)} + 2x_4$$

$$n(\theta^{(*)})^2 + (2n - 2x_1 - x_1 - x_4)\theta^{(*)} - 2x_4 = 0$$

(from the quadratic formula)

$$\frac{-(2n-3x_1-x_4) \pm \sqrt{(2n-3x_1-x_4)^2 - 4n(-2x_4)}}{2n}$$

Given $(x_1,x_2,x_3,x_4) = (125, 21, 20, 33), n = 199$

$$\theta^{(*)} = \frac{-10 \pm \sqrt{52,636}}{-398}$$

$\theta^{(*)} = 0.60157$ or $\theta^{(*)} = -0.55132$

Since $p_4 = \frac{1}{4}\theta$ has to be positive, we reject $\theta^{(*)} = -0.5513$. The answer is $\theta^{(*)} = 0.60157$.

(c) The two answers are the same.

Problem 2.

$$\vec{x}_i = (\frac{x_{1i}}{x_{2i}}) \overset{\text{i.i.d.}}{\sim} N((\frac{\mu_1}{\mu_2}), \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}) = N(\vec{\mu}, \vec{\Sigma})$$

$$f(\vec{x}_i) = |2\pi\vec{\Sigma}|^{-\frac{1}{2}} exp[-\frac{1}{2}(\vec{x}_i - \vec{\mu})^T\vec{\Sigma}^{-1}(\vec{x}_i - \vec{\mu})]$$

$f(\vec{x}) = f(\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n)$
$= \Pi_{i=1}^{n} f(\vec{x}_i)$
$= \Pi_{i=1}^{n}[|2\pi\vec{\Sigma}|^{-1/2} exp[-1/2(\vec{x}_i - \vec{\mu})^T \vec{\Sigma}^{-1}(\vec{x}_i - \vec{\mu})]]$
$= |2\pi\vec{\Sigma}|^{-n/2} exp[-1/2\Sigma_{i=1}^{n}(\vec{x}_i - \vec{\mu})\vec{\Sigma}^{-1}(\vec{x}_i - \vec{\mu})]$
$l_c(\vec{x}) = (-n/2)ln|2\pi\vec{\Sigma}| - (1/2)\Sigma_{i=1}^{n}(\vec{x}_i - \vec{\mu})^T \vec{\Sigma}^{-1}(\vec{x}_i - \vec{\mu})$
Let $n = p + q + r$
$= \dfrac{-(p+q+r)}{2} ln|2\pi\vec{\Sigma}| - (1/2)\Sigma_{i=1}^{n}(\vec{x}_i - \vec{\mu})^T \vec{\Sigma}^{-1}(\vec{x}_i - \vec{\mu})$
$-(1/2)\Sigma_{i=p+1}^{p+q}(\vec{x}_i - \vec{\mu})^T \vec{\Sigma}^{-1}(\vec{x}_i - \vec{\mu})$
$-(1/2)\Sigma_{i=p+1}^{p+q}(\vec{x}_i - \vec{\mu})^T \vec{\Sigma}^{-1}(\vec{x}_i - \vec{\mu})$

E-Step:
Let $\vec{\theta} = (\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \sigma_{12})^T$ and
$\theta^{(\vec{k})} = (\mu_1^{(k)}, \mu_2^{(k)}, \sigma_1^{2(k)}, \sigma_2^{2(k)}, \sigma_{12}^{(k)})^T$

$Q(\vec{\theta}, \vec{\theta}^{(k)}) = E_{\vec{\theta}^{(k)}}(l_c(\vec{x})|\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n)$, where $n = p + q + r$
$= -\dfrac{p+q+r}{2} ln|2\pi\vec{\Sigma}| - (1/2)\Sigma_{i=1}^{p} E_{\theta}^{(k)}[(\vec{x}_i - \vec{\mu})^T \vec{\Sigma}^{-1}(\vec{x}_i - \vec{\mu})|\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n]$
$-(1/2)\Sigma_{i=p+1}^{p+k} E_{\theta}^{(k)}[(\vec{x}_i - \vec{\mu})^T \vec{\Sigma}^{-1}(\vec{x}_i - \vec{\mu})|\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n]$
$-(1/2)\Sigma_{i=p+q+1}^{p+q+r}(\vec{x}_i - \vec{\mu})^T \vec{\Sigma}^{-1}(\vec{x}_i - \vec{\mu})$

Since $\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$,
$|\Sigma| = \sigma_1^2\sigma_2^2 - \sigma_{12}^2$
$\Sigma^{-1} = \begin{pmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{12} & \sigma_1^2 \end{pmatrix} * \dfrac{1}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}$

So, $(\vec{x}_i - \vec{\mu})^T \vec{\Sigma}^{-1}(\vec{x}_i - \vec{\mu}) = (x_{1i} - \mu_1 x_{2i} - \mu_2) \begin{pmatrix} \dfrac{\sigma_2^2}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2} & -\dfrac{\sigma_{12}^2}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2} \\ -\dfrac{\sigma_{12}^2}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2} & \dfrac{\sigma_1^2}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2} \end{pmatrix} \begin{pmatrix} x_{1i} - \mu_1 \\ x_{2i} - \mu_2 \end{pmatrix}$
$= \dfrac{1}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}[\sigma_2^2(x_{1i} - \mu_1) - \sigma_{12}(x_{2i} - \mu_2) - \sigma_{12}(x_{1i} - \mu_1) + \sigma_1^2(x_{2i} - \mu_2)] \begin{pmatrix} x_{1i} - \mu_1 \\ x_{2i} - \mu_2 \end{pmatrix}$
$= \dfrac{1}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}[\sigma_2^2(x_{1i} - \mu_1)^2 - 2\sigma_{12}(x_{2i} - \mu_2)(x_{1i} - \mu_1) + \sigma_1^2(x_{2i} - \mu_2)^2]$

And we know $x_{1i} \sim N(\mu_1, \sigma_1^2)$, $x_{2i} \sim N(\mu_2, \sigma_2^2)$
If $1 \le i \le p$, we have x_{1i} is missing and x_{2i} is observed.
$E[(\vec{x}_i - \vec{\mu})^T \vec{\Sigma}^{-1}(\vec{x}_i - \vec{\mu})|\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n]$
$= \frac{1}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}\{\sigma_2^2 E_{\theta^{(k)}}[(x_{1i} - \mu_1)^2|\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n]$
$- 2\sigma_{12}E_{\theta^{(k)}}[x_{1i} - \mu_1|\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n](x_{2i} - \mu 2)$
$+ \sigma_1^2(x_{2i} - \mu_2)^2)\}$

where $E_{\theta^{(k)}}[(x_{1i} - \mu_1)^2|\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n]$
$= E_{\theta^{(k)}}[x_{1i}^2 - 2x_{1i}\mu_1 + \mu_1^2|\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n]$
$= E_{\theta^{(k)}}[x_{1i}^2|\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n] - 2\mu_1 E(x_{1i}|\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n) + \mu_1^2$
$= \sigma_1^{2(k)} + (\mu_1^(k))^2 - 2\mu_1\mu_1^{(k)} + \mu_1^2$

and $E(x_{1i} - \mu_1|\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n)$
$= E(x_{1i}|\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n) - \mu_1$
$= \mu_1^(k) - \mu_1$

Therefore,
if $1 \le i \le p$,
$E[(\vec{x}_i - \vec{\mu})^T \vec{\Sigma}^{-1}(\vec{x}_i - \vec{\mu})]$
$= \frac{1}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}\{\sigma_2^2[\sigma_1^{2(k)} + (\mu_1^{(k)})^2 - 2\mu_1\mu_1^{(k)} + \mu_1^2]$
$- 2\sigma_{12}(\mu_1^{(k)} - \mu_1)(x_{1i} - \mu_2) + \sigma_2^2(x_{2i} - \mu_2)^2\}$

Similarly, if $p + 1 \le i \le p + q$,
$E[(\vec{x}_i - \vec{\mu})^T \vec{\Sigma}^{-1}(\vec{x}_i - \vec{\mu})]$
$= \frac{1}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}\{\sigma_2^2(x_{1i} - \mu_1)^2 - 2\sigma_{12}(x_{1i} - \mu_1)(\mu_2^{(k)} - \mu_2)$
$+ \sigma_1^2[\sigma_2^{2(k)} + (\mu_2^{(k)})^2 - 2\mu_2\mu_2^{(k)} + \mu_2^2]\}$

If we put all the results together,
$Q(\vec{\theta}, \vec{\theta}^{(k)})$
$= -\frac{p+q+r}{2} ln[(2\pi)^2(\sigma_1^2\sigma_2^2 - \sigma_{12}^2)]$
$-\frac{1}{2(\sigma_1^2\sigma_2^2 - \sigma_{12}^2)}\{p\sigma_2^2[\sigma_1^{2(k)} + (\mu_1^{(k)} - \mu_1)^2] - 2\sigma_{12}(\mu_1^{(k)} - \mu_1)\Sigma_{i=1}^p(x_{2i} -$

$\mu_1) + \sigma_2^2 \Sigma_{i=1}^{p} (x_{2i} - \mu_2)^2\}$
$-\dfrac{1}{2(\sigma_1^2\sigma_2^2 - \sigma_{12}^2)}\{\sigma_2^2 \Sigma_{i=p+1}^{p+q} (x_{1i} - \mu_1)^2 - 2\sigma_{12} \Sigma_{i=p+1}^{p+q} (x_{1i} - \mu_1)(\mu_2^{(k)} -$
$\mu_2) + q\sigma_1^2 [\sigma_2^{2(k)} + (\mu_2^{(k)} - \mu_2)^2]\}$
$-(1/2)\Sigma_{i=p+q+1}^{p+q+r} [(\vec{x}_i - \vec{\mu})^T \vec{\Sigma}^{-1} (\vec{x}_i - \vec{\mu})]$

M-Step:
(1) Set an initial value $\theta^{(0)} = (\mu_1^{(0)}, \mu_2^{(0)}, \sigma_1^{2(0)}, \sigma_2^{2(0)}, \sigma_{12}^{(0)})$
(2) Use algorithm to find θ that maximizes $Q(\vec{\theta}, \theta^{(k)})$ and call it $\theta^{(k+1)}$, i.e., $\theta^{(k+1)} = \arg\max_\theta Q(\theta, \theta^{(k)})$

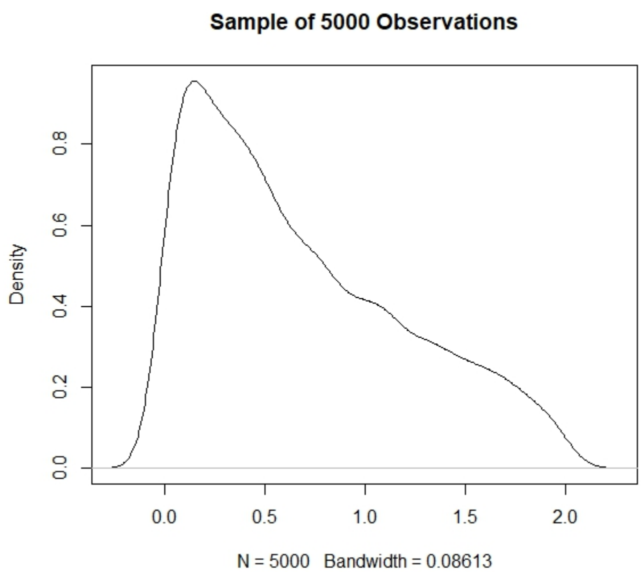
Problem 3.

$f(x) \propto e^{-x}$, $0 < x < 2$.
So, $f(x) = ke^{-x}$
$\int_0^2 f(x)dx = k \int_0^2 e^{-x} dx$
$= k[-e^{-x}]_0^2$
$= k(1 - e^{-2})$
And $\int_0^2 f(x)dx = 1$
$k(1 - e^{-2}) = 1$
$k = \dfrac{1}{1 - e^{-2}}$
So, $f(x) = \dfrac{e^{-x}}{1 - e^{-2}}$
$F(x) = \int_0^x \dfrac{e^{-t}}{1 - e^{-2}} dt$
$\dfrac{1}{1 - e^{-2}}[-e^{-t}]_0^x$
$\dfrac{1}{1 - e^{-2}}(1 - e^{-x})$

We have $F(x) = \dfrac{1 - e^{-x}}{1 - e^{-2}}$
$F(x)(1 - e^{-2}) = 1 - e^{-x}$
$e^{-x} = 1 - F(x)(1 - e^{-2})$
$-x = ln(1 - F(x)(1 - e^{-2}))$
$x = -ln(1 - F(x)(1 - e^{-2}))$ or
$x = ln[\dfrac{1}{1 - F(x)(1 - e^{-2})}]$

So, $F^{-1}(x) = ln[\frac{1}{1 - F(x)(1 - e^{-2})}]$

So, $F^{-1}(u) = ln[\frac{1}{1 - F(x)(1 - e^{-2})}] \sim$ the distribution where $U \sim U(0, 1)$



Problem 4.

(a) $f(x) \propto g(x) = \frac{e^{-x}}{1 + x^2}$, $x > 0$

$g_1(x) = e^{-x}$ $(exp(1))$

$g_2(x) = \frac{2}{\pi(1 + x^2)}$, $x > 0$

plot for $0 < x < 5$

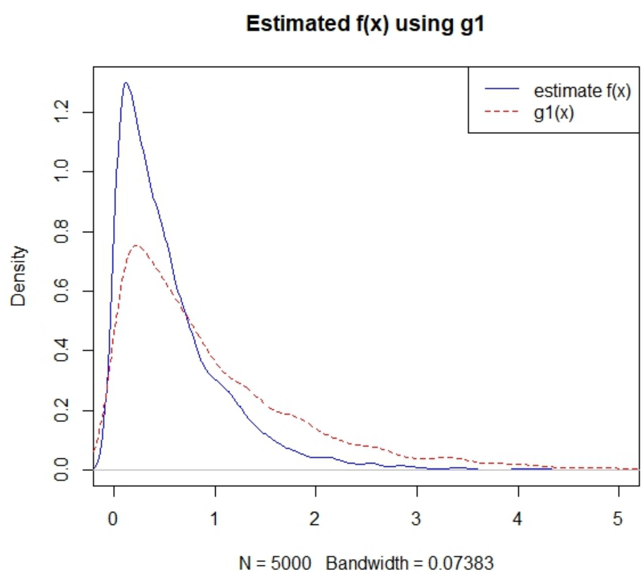
If we use $g_1(x)$ to simulate $f(x)$,

first, $q(x) = \frac{e^{-x}}{1 + x^2}$ for $x > 0$

$< \frac{e^{-x}}{1 + 0^2}$

$= e^{-x}$
$= g_1(x)$

So, the envelope is $\propto g_1(x) = e^{-x}$ with $\alpha = 1$.
Then, the algorithm is
(1) Sample $X \sim g_1(i.e., exp(1))$ and $U \sim U(0,1)$
(2) if $U > \dfrac{q(x)}{\propto g_1(x)} = \dfrac{q(x)}{g_1(x)}$, then go to step 1, otherwise return x
(3) Repeat until we get 5000 samples.



If we use $g_2(x)$ to simulate $f(x)$,

first, $\dfrac{q(x)}{g_2(x)} = \dfrac{\dfrac{e^{-x}}{1+x^2}}{\dfrac{2}{\pi(1+x^2)}} = \dfrac{\pi e^{-x}}{2} < \dfrac{\pi}{2}$ for $x > 0$

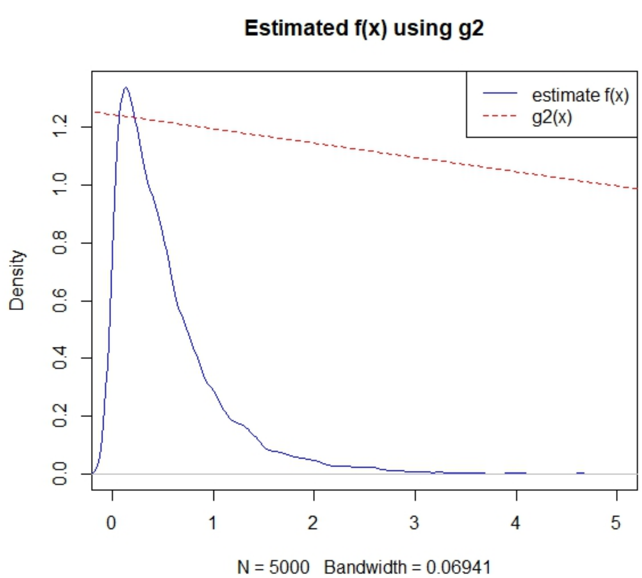
So, $q(x) < \dfrac{\pi}{2} g_2(x)$

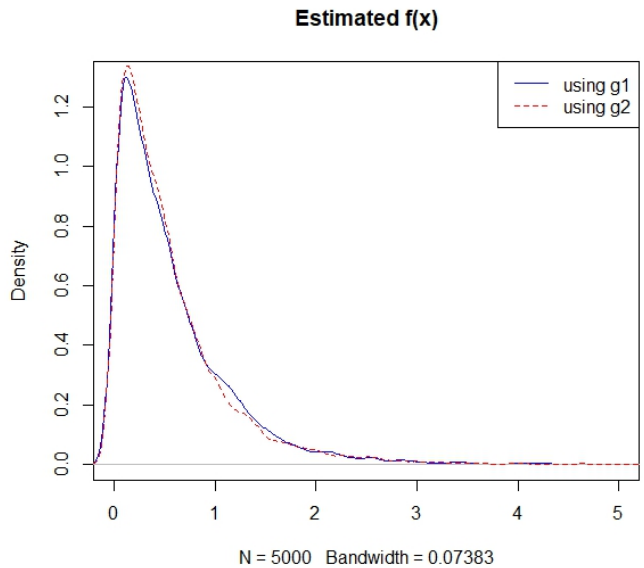
the envelope is $\propto g_2(x) = \dfrac{\pi}{2} g_2(x)$ with $\alpha = \dfrac{\pi}{2}$

Also, for $Y \sim Cauchy(location = 0, scale = 1)$,

$f_y(y) = \dfrac{1}{\pi(1+y^2)}$ for $-\infty < y < \infty$

if we consider the transformation $X = |Y|$
$F_x(x) = P(X \leq x)$
$= P(|Y| \leq x)$
$= P(-x \leq Y \leq x)$
$= \int_{-x}^{x} f_y(y)dy$
$= \int_{c}^{x} f_y(y)dy + \int_{-x}^{c} f_y(y)dy$ for some constant c
$f_x(x) = \dfrac{d}{dx}\int_{c}^{x} f_y(y)dy - \dfrac{d}{dx}\int_{c}^{-x} f_y(y)dy$
$= f_y(x) + f_y(-x)$
$= \dfrac{1}{\pi(1+x^2)} + \dfrac{1}{\pi(1+(-x)^2)}$
$= \dfrac{2}{\pi(1+x^2)}$ for $x > 0$
So, to sample from $g_2(x)$, we can sample from $Cauchy(location = 0, scale = 1)$ and take absolute.
Then, the algorithm is
(1) Sample $X \sim g_2$(i.e., absolute $Cauchy(location = 0, scale = 1)$) and $U \sim U(0,1)$
(2) if $U > \dfrac{q(x)}{\propto g_2(x)} = \dfrac{q(x)}{(\pi/2)g_2(x)}$, then go to step 1, otherwise return x
(3) repeat until we get 5000 samples





(b) $g_1(x)$ converged in 0.31 seconds, while $g_2(x)$ converged in 0.62 seconds. Therefore, the speed of $g_1(x)$ is faster. The ratio of accepted values using g_1 is around 62% and the ratio of accepted values using g_2 is around 40%.

Problem 5.

(a) $g(x) \propto (2x^{\theta-1} + x^{\theta-(1/2)})e^{-x}$, $x > 0$
i.e., $g(x) = k(2x^{\theta-1} + x^{\theta-(1/2)})e^{-x}$
$\int_0^\infty g(x)dx = k[2\int_0^\infty x^{\theta-1}e^{-x}dx + \int_0^\infty x^{\theta-(1/2)}e^{-x}dx]$
We know if $y \sim Gamma(\alpha, \beta)$,
$f(y) = \dfrac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1}e^{-\beta y}$

Consider $\int_0^\infty x^{\theta-1}e^{-x}dx = \dfrac{\Gamma(\theta)}{(1)^\theta} \int_0^\infty \underbrace{\dfrac{(1)^\theta}{\Gamma(\theta)} x^{\theta-1}e^{-(1)x}dx}_{=1}$
$= \Gamma(\theta)$
$\int_0^\infty x^{\theta-(1/2)e^{-x}}dx = \int_0^\infty x^{\theta+(1/2)-1}e^{-x}dx$

$$= \frac{\Gamma(\theta+(1/2))}{(1)^{\theta+(1/2)}} \int_0^\infty \underbrace{\frac{(1)^{\theta+(1/2)}}{\Gamma(\theta+(1/2))} x^{\theta+(1/2)-1} e^{-(1)x} dx}_{=1}$$

$$= \Gamma(\theta+(1/2))$$

Therefore,
$\int_0^\infty g(x)dx = k[2\int_0^\infty x^{\theta-1}e^{-x}dx + \int_0^\infty x^{\theta-(1/2)}e^{-x}dx]$
$= k[2\Gamma(\theta) + \Gamma(\theta+(1/2))]$
and $\int_0^\infty g(x)dx = 1$
$\implies k = \dfrac{1}{2\Gamma(\theta)+\Gamma(\theta+(1/2))}$ (normalizing constant for $g(x)$) and
$g(x) = \dfrac{1}{2\Gamma(\theta)+\Gamma(\theta+(1/2))}(2x^{\theta-1} + x^{\theta-(1/2)})e^{-x}$, $x > 0$

(b) From what was done in part (a),

$$g(x) = \frac{2\Gamma(\theta)}{2\Gamma(\theta)+\Gamma(\theta+(1/2))}[\underbrace{\frac{(1)^\theta}{\Gamma(\theta)} x^{\theta-1} e^{-(1)x}}_{\sim Gamma(\theta,1)}]$$

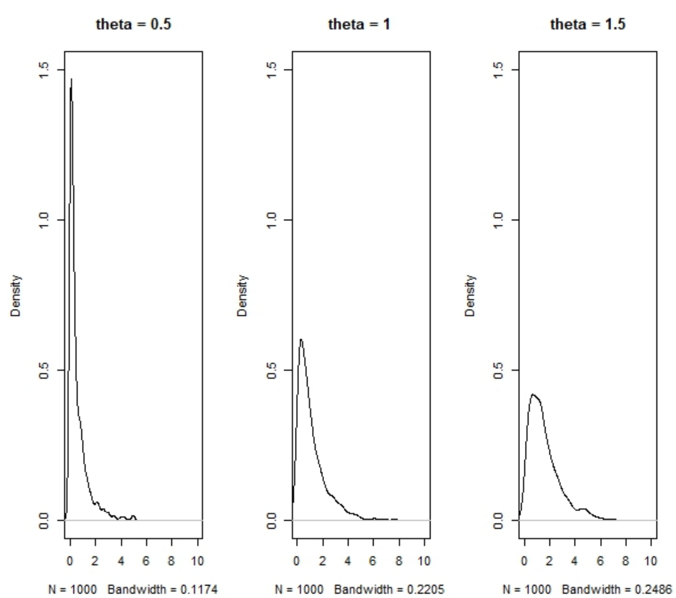
$$+ \frac{\Gamma(\theta+(1/2))}{2\Gamma(\theta)+\Gamma(\theta+(1/2))}[\underbrace{\frac{(1)^{\theta+(1/2)}}{\Gamma(\theta+(1/2))} x^{\theta+(1/2)-1} e^{-(1)x}}_{\sim Gamma(\theta+(1/2),1)}]$$

So, $g(x)$ is a mixture of $Gamma(\theta,1)$ and $Gamma(\theta+(1/2),1)$ with weight $\dfrac{2\Gamma(\theta)}{2\Gamma(\theta)+\Gamma(\theta+(1/2))}$ and $\dfrac{\Gamma(\theta+(1/2))}{2\Gamma(\theta)+\Gamma(\theta+(1/2))}$ respectively

(c) Assume θ is given,
(1) sample $U \sim U(0,1)$
(2) if $U < \dfrac{2\Gamma(\theta)}{2\Gamma(\theta)+\Gamma(\theta+(1/2))}$, we sample a value from $Gamma(\theta,1)$, otherwise, sample a value from $Gamma(\theta+(1/2),1)$
(3) repeat until we get desired sample size, and this sample represents the sample from $g(x)$

(d) $f(x) \propto \sqrt{4+x}x^{\theta-1}e^{-x}$, $x > 0$
let $g(x) = \sqrt{4+x}x^{\theta-1}e^{-x}$ and we have
$g(x) = k(2x^{\theta-1} + x^{\theta-(1/2)})e^{-x}$ where k is the normalizing constant from part (a)
$\dfrac{q(x)}{g(x)} = \dfrac{1}{k(2+\sqrt{x})} < \dfrac{1}{2k}$, $x > 0$
$q(x) < \dfrac{1}{2k}g(x)$

Therefore $\alpha g(x)$ is the envelope with $\alpha = \dfrac{1}{2k}$
$= \Gamma(\theta) + (1/2)\Gamma(\theta + (1/2))$
(1) using procedure in (c) to generate value x from $g(x)$
(2) sample $U \sim U(0,1)$
(3) if $U > \dfrac{q(x)}{\alpha g(x)}$, go back to step 1
otherwise return x
(where $q(x) = \sqrt{4+x}x^{\theta-1}e^{-x}$
$\alpha g(x) = [\Gamma(\theta) + (1/2)\Gamma(\theta + (1/2))](2x^{\theta-1} + x^{\theta-(1/2)})e^{-x})$



Problem 6.

$f(x,y) \propto x^\alpha y \implies$ let $q(x,y) = x^\alpha y$
Consider $g(x,y) = k$ for $x, y > 0$ and $x^2 + y^2 \leq 1$.
So, $k = \dfrac{1}{\text{(Area of unit circle in upper right quarter)}} = \dfrac{1}{(\pi/4)} = \dfrac{4}{\pi}$

Next we try to find constant β such that $q(x,y) \leq \beta g(x,y)\ \forall x, y$
$\dfrac{q(x,y)}{g(x,y)} = \dfrac{x^\alpha y}{(4/\pi)} = (\pi/4)x^\alpha y,\ x > 0,\ y > 0,\ x^2 + y^2 \leq 1.$

To find x, y to minimize $(\pi/4)x^\alpha y$, subject to $x^2 + y^2 \leq 1$,
we can find x, y to minimize the following objective function.

$h(x,y) = x^\alpha y + \lambda(x^2 + y^2 - 1)$
$\dfrac{\partial h}{\partial x} = \alpha x^{\alpha-1}y + 2x\lambda \implies \lambda = -(\alpha/2)x^{\alpha-2}y$ (1)
$\dfrac{\partial h}{\partial y} = x^\alpha + 2y\lambda \implies \lambda = -\dfrac{x^\alpha}{2y}$ (2)
$\dfrac{\partial h}{\partial \lambda} = x^2 + y^2 - 1 \implies \lambda = x^2 + y^2 = 1$ (3)

Set (1) = (2), $-(\alpha/2)x^{\alpha-2}y = -\dfrac{x^\alpha}{2y}$
$\alpha y^2 = x^2$ (4)
put (4) into (3), $y^2(1+\alpha) = 1$
$y = \dfrac{1}{\sqrt{1+\alpha}}$ (reject $-\dfrac{1}{\sqrt{1+\alpha}}$ $\because y > 0$)
by (4) $x = \sqrt{\alpha}y$
$= \dfrac{\sqrt{\alpha}}{\sqrt{1+\alpha}}$

So, we select $x = \sqrt{\dfrac{\alpha}{1+\alpha}}$ and $y = \sqrt{\dfrac{1}{1+\alpha}}$
to maximize $(\pi/4)x^\alpha y$, we have
$\dfrac{q(x,y)}{g(x,y)} = (\pi/4)x^\alpha y$
$\leq (\pi/4)(\sqrt{\dfrac{\alpha}{1+\alpha}})^\alpha \sqrt{\dfrac{1}{1+\alpha}}$

$$= (\pi/4)\frac{\alpha^{\alpha/2}}{(1+\alpha)^{(\alpha+1)/2}}$$

So, set $\beta = (\pi/4)\alpha^{\alpha/2}(1+\alpha)^{-(\alpha+1)/2}$

Algorithm:
(1) simulate $X \sim U(0,1)$, $Y \sim (0,1)$
(2) if $x^2 + y^2 > 1$, go back to (1) otherwise go to next step
(3) sample $U \sim Unif(0,1)$
(4) if $U > \dfrac{q(x,y)}{\beta g(x,y)}$, then go to step (1) otherwise return to (x,y)