Homework 3 (Due 4/24)

Question 1 Consider the following sample of $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$:

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \end{bmatrix}.$$

Find its sample mean vector and sample covariance matrix. Furthermore, find the sample mean and sample variance of $X_1 + 2X_2$.

Question 2 A p-variate sample $\vec{x}_1, \dots, \vec{x}_n$ is transformed into $\vec{y}_1, \dots, \vec{y}_n$ by

$$y_{ij} = c_j x_{ij} + d_j, \; j = 1, \dots, p, \; i = 1, \dots, n.$$

Here $c_j > 0$ for $j = 1, \dots, p$. In other words, the p variates $X_1, \dots, X_p$ are transformed into $Y_1, \dots, Y_p$ in that $Y_j = c_j X_j + d_j$. Denote by r^x_{jk} the sample correlation between X_j and X_k, and denote by r^y_{jk} the sample correlation between Y_j and Y_k. Prove that $r^x_{jk} = r^y_{jk}$.

Question 3 For a sample of $\overrightarrow{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, the sample mean and sample covariance matrix are

$$\bar{\boldsymbol{x}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \boldsymbol{S} = \begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix},$$

respectively.
(a) Find the spectral decompositions of $\boldsymbol{S}$ and $\boldsymbol{S}^{-1}$, respectively.
(b) Sketch the mean-centered ellipse

$$(\boldsymbol{x} - \bar{\boldsymbol{x}})^\top \boldsymbol{S}^{-1} (\boldsymbol{x} - \bar{\boldsymbol{x}}) \le c^2.$$

(c) Determine the sample correlation matrix $\boldsymbol{R}$. Find the spectral decompositions of $\boldsymbol{R}$ and $\boldsymbol{R}^{-1}$, respectively.
(d) Sketch the mean-centered ellipse

$$\boldsymbol{x}^\top \boldsymbol{R}^{-1} \boldsymbol{x} \le c^2.$$

Question 4 For a sample of $\overrightarrow{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}$ with sample covariance matrix

$$\boldsymbol{S} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Find the sample cross covariance matrix between $\begin{bmatrix} X_1 + X_2 \\ X_1 - X_2 \end{bmatrix}$ and $\begin{bmatrix} X_3 + X_4 \\ X_3 - X_4 \end{bmatrix}$.

Question 5 Coming back to the random sample

$$X_1, \ldots, X_n \overset{i.i.d.}{\sim} \mathcal{N}(\mu, \sigma^2).$$

with sample mean $\bar{X}$ and sample variance S^2. Show that $\frac{\sqrt{n}(\overline{X}-\mu)}{\sigma} \sim \mathcal{N}(0,1)$ and $\mathbb{E}(S^2) = \sigma^2$.