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STA 135 HW 6

1. Given a sample $\bar{x}_1, \cdots, \bar{x}_{10}$ from $N_2(\vec{\mu}, \mathbf{\Sigma})$, the summary statistics are

$$\bar{\boldsymbol{x}} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \boldsymbol{S} = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}.$$

   a. One-at-a-time 95% confidence interval for μ_1 and μ_2:

$$\bar{x}_1 - \frac{s_1}{\sqrt{10}} t_{10-1}\left(\frac{0.05}{2}\right) \le \mu_1 \le \bar{x}_1 + \frac{s_1}{\sqrt{10}} t_{n-1}\left(\frac{0.05}{2}\right)$$

$$2 - \frac{4}{\sqrt{10}} t_9(0.025) \le \mu_1 \le 2 + \frac{4}{\sqrt{10}} t_9(0.025)$$

$$\boxed{0.8614276 \le \mu_1 \le 4.861428}$$

$$\bar{x}_2 - \frac{s_2}{\sqrt{10}} t_{n-1}\left(\frac{\alpha}{2}\right) \le \mu_2 \le \bar{x}_2 + \frac{s_2}{\sqrt{10}} t_{n-1}\left(\frac{\alpha}{2}\right)$$

$$2 - \frac{4}{\sqrt{10}} t_9(0.025) \le \mu_2 \le 2 + \frac{4}{\sqrt{10}} t_9(0.025)$$

$$\boxed{0.8614276 \le \mu_2 \le 4.861428}$$

   b. Simultaneous $\ge$ 95% confidence interval for $\mu_1, \mu_2, \mu_1 + \mu_2, \mu_1 - \mu_2$ based on T^2:
   Using the following formula,

$$\left(\vec{a}^T \bar{\boldsymbol{x}} - \sqrt{\frac{p(n-1)}{n(n-p)} F_{p,n-p}(\alpha) \vec{a}^T \boldsymbol{S} \vec{a}}, \qquad \vec{a}^T \bar{\boldsymbol{x}} + \sqrt{\frac{p(n-1)}{n(n-p)} F_{p,n-p}(\alpha) \vec{a}^T \boldsymbol{S} \vec{a}} \right)$$

$$\mu_1 : \vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{a}^T \bar{\boldsymbol{x}} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2$$

$$\vec{a}^T \boldsymbol{S} \vec{a} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 4$$

$$\sqrt{\frac{2(10-1)}{10(10-2)} F_{2,10-2}(0.05) 4} = \sqrt{\frac{9}{10} F_{2,8}(0.05)} = 2.003266$$

$$(2 - 2.003266, \qquad 2 + 2.003266\ ) = \boxed{(-0.003266, \qquad 4.003266\ )}$$

$$\mu_2 : \vec{a} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\vec{a}^T \overline{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2$$

$$\vec{a}^T \boldsymbol{S} \vec{a} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 4$$

$$\sqrt{\frac{2(10-1)}{10(10-2)} F_{2,10-2}(0.05) 4} = \sqrt{\frac{9}{10} F_{2,8}(0.05)} = 2.003266$$

$$(2 - 2.003266, \quad 2 + 2.003266\ ) = \boxed{(-0.003266, \quad 4.003266\ )}$$

$$\mu_1 + \mu_2 : \vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\vec{a}^T \overline{\boldsymbol{x}} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 4$$

$$\vec{a}^T \boldsymbol{S} \vec{a} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 14$$

$$\sqrt{\frac{2(10-1)}{10(10-2)} F_{2,10-2}(0.05) 14} = \sqrt{\frac{18 * 14}{80} F_{2,8}(0.05)} = 3.747767$$

$$(4 - 3.747767, 4 + 3.747767) = \boxed{(0.2522332, 7.747767\ )}$$

$$\mu_1 - \mu_2 : \vec{a} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\vec{a}^T \overline{\boldsymbol{x}} = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 0$$

$$\vec{a}^T \boldsymbol{S} \vec{a} = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2$$

$$\sqrt{\frac{2(10-1)}{10(10-2)} F_{2,10-2}(0.05) 2} = \sqrt{\frac{36}{80} F_{2,8}(0.05)} = 1.416523$$

$$(0 - 1.416523, 0 + 1.416523) = \boxed{(-1.416523, 1.416523)}$$

c. Simultaneous $\geq 95\%$ confidence interval for μ_1 and μ_2 based on Bonferroni correction:

$$\bar{x}_1 - t_{n-1}\left(\frac{\alpha}{2p}\right)\sqrt{\frac{s_{11}}{n}} \leq \mu_1 \leq \bar{x}_1 + t_{n-1}\left(\frac{\alpha}{2p}\right)\sqrt{\frac{s_{11}}{n}}$$

$$2 - t_9\left(\frac{0.05}{4}\right)\sqrt{\frac{4}{10}} \leq \mu_1 \leq 2 + t_9\left(\frac{0.05}{4}\right)\sqrt{\frac{4}{10}}$$

$$\boxed{0.30185 \leq \mu_1 \leq 3.69815}$$

$$\bar{x}_2 - t_{n-1}\left(\frac{\alpha}{2p}\right)\sqrt{\frac{s_{22}}{n}} \leq \mu_2 \leq \bar{x}_2 + t_{n-1}\left(\frac{\alpha}{2p}\right)\sqrt{\frac{s_{22}}{n}}$$

$$2 - t_9\left(\frac{0.05}{4}\right)\sqrt{\frac{4}{10}} \leq \mu_2 \leq 2 + t_9\left(\frac{0.05}{4}\right)\sqrt{\frac{4}{10}}$$

$$\boxed{0.30185 \leq \mu_2 \leq 3.69815}$$

2. Let $\vec{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \sim N_3(\vec{\mu}, \mathbf{\Sigma})$ with $\vec{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}$. Let $\vec{x}_1, \cdots, \vec{x}_{10}$ be an observed sample from the above population, for which the sample mean and sample covariance matrix are

$$\bar{\vec{x}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \boldsymbol{S}_x = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix},$$

respectively.

a. Simultaneous $\geq 95\%$ confidence interval for $\mu_1 - \mu_2$ and $\mu_1 - \mu_3$ with the T^2 based on the sample $\vec{x}_1, \cdots, \vec{x}_{10}$:

$$\left(\vec{a}^T\bar{\boldsymbol{x}} - \sqrt{\frac{p(n-1)}{n(n-p)}F_{p,n-p}(\alpha)\vec{a}^T\boldsymbol{S}\vec{a}}, \qquad \vec{a}^T\bar{\boldsymbol{x}} + \sqrt{\frac{p(n-1)}{n(n-p)}F_{p,n-p}(\alpha)\vec{a}^T\boldsymbol{S}\vec{a}}\right)$$

$$\mu_1 - \mu_2 : \vec{a} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\vec{a}^T\bar{\boldsymbol{x}} = [1 \quad -1 \quad 0]\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 2$$

$$\vec{a}^T\boldsymbol{S}\vec{a} = [1 \quad -1 \quad 0]\begin{bmatrix}4 & 1 & 1\\1 & 4 & 1\\1 & 1 & 4\end{bmatrix}\begin{bmatrix}1\\-1\\0\end{bmatrix} = [3 \quad -3 \quad 0]\begin{bmatrix}1\\-1\\0\end{bmatrix} = 6$$

$$\sqrt{\frac{3(10-1)}{10(10-3)}F_{3,10-3}(0.05)6} = \sqrt{\frac{27\times 6}{70}F_{3,10}(0.05)} = 2.929502$$

$$(2-2.929502, 2+2.929502) = \boxed{(-0.9295024, 4.929502)}$$

$$\mu_1 - \mu_3: \vec{a} = \begin{bmatrix}1\\0\\-1\end{bmatrix}$$

$$\vec{a}^T\overline{\boldsymbol{x}} = [1 \quad 0 \quad -1]\begin{bmatrix}1\\0\\1\end{bmatrix} = 0$$

$$\vec{a}^T\boldsymbol{S}\vec{a} = [1 \quad 0 \quad -1]\begin{bmatrix}4 & 1 & 1\\1 & 4 & 1\\1 & 1 & 4\end{bmatrix}\begin{bmatrix}1\\0\\-1\end{bmatrix} = [3 \quad 0 \quad -3]\begin{bmatrix}1\\0\\-1\end{bmatrix} = 6$$

$$\sqrt{\frac{3(10-1)}{10(10-3)}F_{3,10-3}(0.05)6} = \sqrt{\frac{27\times 6}{70}F_{3,7}(0.05)} = 2.929502$$

$$(2-3.17172, 2+3.17172) = \boxed{(-1.17172, 5.17172)}$$

b. The original sample is transformed to the 2-variate sample $\vec{y}_1, \cdots, \vec{y}_{10}$ by the linear transformation $Y_1 = X_1 - X_2$ and $Y_2 = X_1 - X_3$. The $\geq 95\%$ simultaneous confidence intervals for $\mu_1 - \mu_2$ and $\mu_1 - \mu_3$ with the T^2 based on the new sample:

$$\vec{C} = \begin{bmatrix}1 & -1 & 0\\1 & 0 & -1\end{bmatrix}$$

$$\vec{Y} = \begin{bmatrix}Y_1\\Y_2\end{bmatrix} = \begin{bmatrix}X_1 - X_2\\X_1 - X_3\end{bmatrix} \sim N_2\left(\vec{C}\vec{\mu}, \vec{C}\boldsymbol{\Sigma}\vec{C}^T\right)$$

$$\overline{\boldsymbol{y}} = \vec{C}\bar{\vec{x}} = \begin{bmatrix}1 & -1 & 0\\1 & 0 & -1\end{bmatrix}\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$$

$$\boldsymbol{S}_y = \vec{C}\boldsymbol{S}_x\vec{C}^T = \begin{bmatrix}1 & -1 & 0\\1 & 0 & -1\end{bmatrix}\begin{bmatrix}4 & 1 & 1\\1 & 4 & 1\\1 & 1 & 4\end{bmatrix}\begin{bmatrix}1 & 1\\-1 & 0\\0 & -1\end{bmatrix} = \begin{bmatrix}3 & -3 & 0\\3 & 0 & -3\end{bmatrix}\begin{bmatrix}1 & 1\\-1 & 0\\0 & -1\end{bmatrix} = \begin{bmatrix}6 & 3\\3 & 6\end{bmatrix}$$

$$\left(\vec{a}^T\overline{\boldsymbol{y}} - \sqrt{\frac{q(n-1)}{n(n-q)}F_{q,n-q}(\alpha)\vec{a}^T\boldsymbol{S}_y\vec{a}}, \qquad \vec{a}^T\overline{\boldsymbol{y}} + \sqrt{\frac{q(n-1)}{n(n-q)}F_{q,n-q}(\alpha)\vec{a}^T\boldsymbol{S}_y\vec{a}}\right)$$

$$\mu_1 - \mu_2 : \vec{a} = \begin{bmatrix}1\\0\end{bmatrix}$$

$$\vec{a}^T\overline{\boldsymbol{y}} = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = 1$$

$$\vec{a}^T\boldsymbol{S}_y\vec{a} = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}6 & 3\\3 & 6\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}6 & 3\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = 3$$

$$\sqrt{\frac{2(10-1)}{10(10-2)}F_{2,10-2}(0.05)3} = \sqrt{\frac{18\times 3}{80}F_{2,8}(0.05)} = 1.734879$$

$$(1 - 1.734879, 1 + 1.734879) = \boxed{(-0.7348789, 2.734879)}$$

$$\mu_1 - \mu_3 : \vec{a} = \begin{bmatrix}0\\1\end{bmatrix}$$

$$\vec{a}^T\overline{\boldsymbol{y}} = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = 0$$

$$\vec{a}^T\boldsymbol{S}_y\vec{a} = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}6 & 3\\3 & 6\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}3 & 6\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = 6$$

$$\sqrt{\frac{2(10-1)}{10(10-2)}F_{2,10-2}(0.05)6} = \sqrt{\frac{18\times 6}{80}F_{2,8}(0.05)} = 2.453489$$

$$(0 - 2.453489, 0 + 2.453489) = \boxed{(-2.453489, 2.453489)}$$

c. Simultaneous $\geq$ 95% confidence intervals for $\mu_1 - \mu_2$ and $\mu_1 - \mu_3$ with Bonferroni correction:

$$\bar{y}_1 - t_{n-1}\left(\frac{\alpha}{2q}\right)\sqrt{\frac{s_{11}}{n}} \leq \mu_1 - \mu_2 \leq \bar{y}_1 + t_{n-1}\left(\frac{\alpha}{2q}\right)\sqrt{\frac{s_{11}}{n}}$$

$$1 - t_9\left(\frac{0.05}{4}\right)\sqrt{\frac{6}{10}} \leq \mu_1 - \mu_2 \leq 1 + t_9\left(\frac{0.05}{4}\right)\sqrt{\frac{6}{10}}$$

$$\boxed{-1.0798 \leq \mu_1 - \mu_2 \leq 3.0798}$$

$$\bar{y}_2 - t_{n-1}\left(\frac{\alpha}{2q}\right)\sqrt{\frac{s_{22}}{n}} \le \mu_1 - \mu_3 \le \bar{y}_2 + t_{n-1}\left(\frac{\alpha}{2q}\right)\sqrt{\frac{s_{22}}{n}}$$

$$0 - t_9\left(\frac{0.05}{4}\right)\sqrt{\frac{6}{10}} \le \mu_1 - \mu_3 \le 0 + t_9\left(\frac{0.05}{4}\right)\sqrt{\frac{6}{10}}$$

$$\boxed{-2.0798 \le \mu_1 - \mu_3 \le 2.0798}$$

d. Comparing the above results, the transformed variables from part 2 have a smaller width, making them less conservative than the untransformed variables from part 1.

3. Bonferroni-corrected two-sample test: $H_0: \vec{\mu}_1 - \vec{\mu}_2 = \vec{\delta}_0$ is rejected if

$$\max_{1\le j\le p}\left|\frac{(\bar{x}_{1j} - \bar{x}_{2j}) - \delta_{0j}}{s_{pooled,j}\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}\right| \ge t_{n_1+n_2-2}\left(\frac{\alpha}{2p}\right).$$

Next it will be proven that the following Type I error control:

$$\mathbb{P}_{null}\left(\max_{1\le j\le p}\left|\frac{(\bar{x}_{1j} - \bar{x}_{2j}) - \delta_{0j}}{s_{pooled,j}\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}\right| \ge t_{n_1+n_2-2}\left(\frac{\alpha}{2p}\right)\right) \le \alpha.$$

Assuming that the variables are independent…

$$\mathbb{P}_{null}\left(\max_{1\le j\le p}\left|\frac{(\bar{x}_{1j} - \bar{x}_{2j}) - \delta_{0j}}{s_{pooled,j}\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}\right| \ge t_{n_1+n_2-2}\left(\frac{\alpha}{2p}\right)\right)$$

$$= 1 - \mathbb{P}_{null}\left(\max_{1\le j\le p}\left|\frac{(\bar{x}_{1j} - \bar{x}_{2j}) - \delta_{0j}}{s_{pooled,j}\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}\right| < t_{n_1+n_2-2}\left(\frac{\alpha}{2p}\right)\right)$$

$$= 1 - \mathbb{P}_{null}\left(\left|\frac{(\bar{x}_{1j} - \bar{x}_{2j}) - \delta_{0j}}{s_{pooled,j}\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}\right| < t_{n_1+n_2-2}\left(\frac{\alpha}{2p}\right), \text{for any } j \text{ in } 1,2,\cdots,p\right)$$

$$= 1 - \prod_{j=1}^{p} \mathbb{P}_{null}\left(\left|\frac{(\bar{x}_{1j} - \bar{x}_{2j}) - \delta_{0j}}{s_{pooled,j}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}\right| < t_{n_1+n_2-2}\left(\frac{\alpha}{2p}\right)\right)$$

$$= 1 - \prod_{j=1}^{p}\left(1 - \frac{\alpha}{p}\right)$$

$$= 1 - \left(1 - \frac{\alpha}{p}\right)^p$$

$$= 1 - \left(1 - \frac{p\alpha}{p} + \frac{p(p-1)}{2}\left(\frac{\alpha}{p}\right)^2 - \frac{p(p-1)(p-2)}{3!}\left(\frac{\alpha}{p}\right)^3\right)$$

$$= \alpha - \frac{p(p-1)}{2}\left(\frac{\alpha}{p}\right)^2 + \frac{p(p-1)(p-2)}{3!}\left(\frac{\alpha}{p}\right)^3$$

$$\leq \alpha$$

Then it can be said that $\mathbb{P}_{null}\left(\max_{1\leq j\leq p}\left|\frac{(\bar{x}_{1j}-\bar{x}_{2j})-\delta_{0j}}{s_{pooled,j}\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}\right| \geq t_{n_1+n_2-2}\left(\frac{\alpha}{2p}\right)\right) \leq \alpha$. ■

4. Given a sample of size $n_1 = 14$ from $N_2(\vec{\mu}_1, \mathbf{\Sigma}_1)$ and a sample size of $n_2 = 14$ from $N_2(\vec{\mu}_2, \mathbf{\Sigma}_2)$. Assume also that $\mathbf{\Sigma}_1 = \mathbf{\Sigma}_2$. The summary statistics for the two samples are

$$\bar{\vec{x}}_1 = \begin{bmatrix} 8 \\ 5 \end{bmatrix}, \boldsymbol{S}_1 = \begin{bmatrix} 3 & -1 \\ -1 & 6 \end{bmatrix}, \bar{\vec{x}}_2 = \begin{bmatrix} 10 \\ 4 \end{bmatrix}, \boldsymbol{S}_2 = \begin{bmatrix} 3 & -1 \\ -1 & 4 \end{bmatrix}.$$

a. Test $H_0: \vec{\mu}_1 = \vec{\mu}_2$ at the level $\alpha = 0.05$ with Hotelling's T^2.
We reject the null hypothesis if the following is true,

$$T^2 = \left(\bar{\vec{x}}_1 - \bar{\vec{x}}_2\right)^T \left[\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\boldsymbol{S}_{pooled}\right]^{-1} \left(\bar{\vec{x}}_1 - \bar{\vec{x}}_2\right) > c^2,$$

where $c^2 = \frac{(n_1+n_2-2)p}{(n_1+n_2-p-1)} F_{p,n_1+n_2-p-1}(\alpha)$.

$$\boldsymbol{S}_{pooled} = \frac{\Sigma_{j=1}^{n_1}\left(\vec{x}_{1j} - \bar{\vec{x}}_1\right)\left(\vec{x}_{1j} - \bar{x}_1\right)^T + \Sigma_{j=1}^{n_2}\left(\vec{x}_{2j} - \bar{\vec{x}}_2\right)\left(\vec{x}_{2j} - \bar{x}_2\right)^T}{n_1 + n_2 - 2}$$

$$= \frac{n_1 - 1}{n_1 + n_2 - 2}\boldsymbol{S}_1 + \frac{n_2 - 1}{n_1 + n_2 - 2}\boldsymbol{S}_2 = \frac{1}{2}\begin{bmatrix} 3 & -1 \\ -1 & 6 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 3 & -1 \\ -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 3 \end{bmatrix} + \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix}$$

$$\left[\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\boldsymbol{S}_{pooled}\right]^{-1} = \left[\frac{1}{7}\begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix}\right]^{-1} = 7\begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{7}{3} & 0 \\ \frac{7}{15} & \frac{7}{5} \end{bmatrix}$$

$$Note\colon \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix}^{-1} = \frac{1}{15}\begin{bmatrix} 5 & 0 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ \frac{1}{15} & \frac{1}{5} \end{bmatrix}$$

$$T^2 = \left(\bar{\vec{x}}_1 - \bar{\vec{x}}_2\right)^T \left[\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\boldsymbol{S}_{pooled}\right]^{-1} \left(\bar{\vec{x}}_1 - \bar{\vec{x}}_2\right)$$

$$= \left(\begin{bmatrix} 8 \\ 5 \end{bmatrix} - \begin{bmatrix} 10 \\ 4 \end{bmatrix}\right)^T \begin{bmatrix} \frac{7}{3} & 0 \\ \frac{7}{15} & \frac{7}{5} \end{bmatrix} \left(\begin{bmatrix} 8 \\ 5 \end{bmatrix} - \begin{bmatrix} 10 \\ 4 \end{bmatrix}\right)$$

$$= \begin{bmatrix} -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{7}{3} & 0 \\ \frac{7}{15} & \frac{7}{5} \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{63}{15} & \frac{7}{5} \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \frac{126}{15} + \frac{7}{5} = \frac{147}{15} = \frac{49}{5}$$

$$c^2 = \frac{(n_1 + n_2 - 2)p}{(n_1 + n_2 - p - 1)} F_{p, n_1 + n_2 - p - 1}(\alpha)$$

$$= \frac{52}{25} F_{2,25}(0.05) = 7.041195$$

So, the test statistic is larger than the critical value and so we reject the null hypothesis that $\vec{\mu}_1 = \vec{\mu}_2$ at the level $\alpha = 0.05$ with Hotelling's T^2.

b. Find ≥ 95% simultaneous confidence intervals for $\mu_{1j} - \mu_{2j}, j = 1, \cdots, p$ with Bonferroni corrected t-tests.
The general formula is,

$$\mu_{1j} - \mu_{2j}\colon \left(\bar{x}_{1j} - \bar{x}_{2j}\right) \pm t_{n_1 + n_2 - 2}\left(\frac{\alpha}{2p}\right) \sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right) s_{jj,pooled}}.$$

$$\mu_{11} - \mu_{21} : (\bar{x}_{11} - \bar{x}_{21}) \pm t_{n_1+n_2-2}\left(\frac{\alpha}{2p}\right)\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)s_{11,pooled}}$$

$$-2 \pm t_{26}\left(\frac{0.05}{4}\right)\sqrt{\frac{3}{7}}$$

$$(-3.557281, -0.4427188)$$

$$\mu_{12} - \mu_{22} : (\bar{x}_{12} - \bar{x}_{22}) \pm t_{n_1+n_2-2}\left(\frac{\alpha}{2p}\right)\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)s_{22,pooled}}$$

$$1 \pm t_{26}\left(\frac{0.05}{4}\right)\sqrt{\frac{5}{7}}$$

$$(-1.010441, 3.010441)$$

c. Test $H_0: \vec{\mu}_1 = \vec{\mu}_2$ at the level $\alpha = 0.05$ with Bonferroni corrected t-tests. Using the following formula,

$$\max_{1\le j\le p}\left|\frac{(\bar{x}_{1j} - \bar{x}_{2j}) - \delta_{0j}}{s_{pooled,j}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}\right| \ge t_{n_1+n_2-2}\left(\frac{\alpha}{2p}\right)$$

the test will be calculated for $j = 1,2$.

$$\left|\frac{(\bar{x}_{11} - \bar{x}_{21})}{s_{pooled,1}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}\right| = \left|\frac{-2}{3\sqrt{\frac{1}{7}}}\right| = 1.763834$$

$$\left|\frac{(\bar{x}_{12} - \bar{x}_{22})}{s_{pooled,2}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}\right| = \left|\frac{1}{5\sqrt{\frac{1}{7}}}\right| = 0.5291503$$

$$t_{n_1+n_2-2}\left(\frac{\alpha}{2p}\right) = t_{26}\left(\frac{0.05}{4}\right) = 2.378786$$

The max value 1.763834 is less than the critical value, so we fail to reject the null hypothesis that $\vec{\mu}_1 = \vec{\mu}_2$ at the level $\alpha = 0.05$ with Bonferroni corrected t-tests.

5. Given the following two samples:

$$\text{Sample 1: } \vec{x}_{11}, \cdots, \vec{x}_{1n_1} \text{ from } N_p(\vec{\mu}_1, \mathbf{\Sigma}_1)$$

$$\text{Sample 2: } \vec{x}_{21}, \cdots, \vec{x}_{2n_2} \text{ from } N_p(\vec{\mu}_2, \mathbf{\Sigma}_2)$$

Two new samples are defined through the linear transformations $\vec{y}_{lj} = \boldsymbol{C}\vec{x}_{lj} + \vec{d}$ for all $l = 1,2$ and $j = 1,2, \cdots, n_l$, where $\boldsymbol{C}$ is a $p \times p$ nonsingular matrix and $\vec{d}$ is a $p \times 1$ vector. Based on Samples 1 and 2, the T^2-statistic for testing $\vec{\mu}_1 = \vec{\mu}_2$ is denoted as T_x^2. On the other hand, based on the two samples $\vec{y}_{11}, \cdots, \vec{y}_{1n_1}$ and $\vec{y}_{21}, \cdots, \vec{y}_{2n_2}$, the T^2-statistic for testing the equality of vector means is denoted as T_y^2. Next it will be shown that $T_x^2 = T_y^2$.

$$T_y^2 = \left(\bar{\vec{y}}_1 - \bar{\vec{y}}_2\right)^T \left[\frac{1}{n_1}\boldsymbol{S}_{y_1} + \frac{1}{n_2}\boldsymbol{S}_{y_2}\right]^{-1} \left(\bar{\vec{y}}_1 - \bar{\vec{y}}_2\right)$$

$$= \left(\boldsymbol{C}\bar{\vec{x}}_1 + \vec{d} - \left(\boldsymbol{C}\bar{\vec{x}}_2 + \vec{d}\right)\right)^T \left[\frac{1}{n_1}\boldsymbol{C}\boldsymbol{S}_{x_1}\boldsymbol{C}^T + \frac{1}{n_2}\boldsymbol{C}\boldsymbol{S}_{x_2}\boldsymbol{C}^T\right]^{-1} \left(\boldsymbol{C}\bar{\vec{x}}_1 + \vec{d} - \left(\boldsymbol{C}\bar{\vec{x}}_2 + \vec{d}\right)\right)$$

$$= \left(\bar{\vec{x}}_1 - \bar{\vec{x}}_2\right)^T \boldsymbol{C}^T \left[\boldsymbol{C}\left(\frac{1}{n_1}\boldsymbol{S}_{x_1} + \frac{1}{n_2}\boldsymbol{S}_{x_2}\right)\boldsymbol{C}^T\right]^{-1} \boldsymbol{C}\left(\bar{\vec{x}}_1 - \bar{\vec{x}}_2\right)$$

$$= \left(\bar{\vec{x}}_1 - \bar{\vec{x}}_2\right)^T \boldsymbol{C}^T (\boldsymbol{C}^T)^{-1} \left(\frac{1}{n_1}\boldsymbol{S}_{x_1} + \frac{1}{n_2}\boldsymbol{S}_{x_2}\right)^{-1} \boldsymbol{C}^{-1}\boldsymbol{C}\left(\bar{\vec{x}}_1 - \bar{\vec{x}}_2\right)$$

$$= \left(\bar{\vec{x}}_1 - \bar{\vec{x}}_2\right)^T \left[\frac{1}{n_1}\boldsymbol{S}_{x_1} + \frac{1}{n_2}\boldsymbol{S}_{x_2}\right]^{-1} \left(\bar{\vec{x}}_1 - \bar{\vec{x}}_2\right) = T_x^2 \blacksquare$$

6. Consider two independent samples from 3-variate multivariate normal populations:

Population 1 with $\vec{\mu}_1 = \begin{bmatrix}\mu_{11}\\ \mu_{12}\\ \mu_{13}\end{bmatrix}$: sample size $n_1 = 10, \bar{\vec{x}}_1 = \begin{bmatrix}3\\4\\5\end{bmatrix}, \boldsymbol{S}_1 = \begin{bmatrix}3 & 2 & 2\\ 2 & 3 & 2\\ 2 & 2 & 2\end{bmatrix}$;

Population 2 with $\vec{\mu}_2 = \begin{bmatrix}\mu_{21}\\ \mu_{22}\\ \mu_{23}\end{bmatrix}$: sample size $n_2 = 10, \bar{\vec{x}}_2 = \begin{bmatrix}4\\6\\6\end{bmatrix}, \boldsymbol{S}_2 = \begin{bmatrix}3 & 2 & 2\\ 2 & 3 & 2\\ 2 & 2 & 4\end{bmatrix}$.

Assume also that the population covariance matrices of the two populations are the same. We aim to test:

$$H_0: \mu_{11} - \mu_{21} = \mu_{12} - \mu_{22} = \mu_{13} - \mu_{23}.$$

First the null hypothesis will be written as two systems of equations,

$$\mu_{11} - \mu_{21} = \mu_{12} - \mu_{22}$$

and

$$\mu_{11} - \mu_{21} = \mu_{13} - \mu_{23}.$$

These can be rewritten as,

$$\mu_{11} - \mu_{12} = \mu_{21} - \mu_{22}$$

and

$$\mu_{11} - \mu_{13} = \mu_{21} - \mu_{23}.$$

Then a matrix $\boldsymbol{C}$ can be found to represent this as

$$\boldsymbol{C}\vec{\mu}_1 = \boldsymbol{C}\vec{\mu}_2$$

where $\boldsymbol{C} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$.

a. Test H_0 with $\alpha = 0.05$ by Hotelling's T^2

$$T^2 = \left(\boldsymbol{C}\bar{\vec{x}}_1 - \left(\boldsymbol{C}\bar{\vec{x}}_2\right)\right)^T \left[\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\boldsymbol{C}\boldsymbol{S}_{pooled}\boldsymbol{C}^T\right]^{-1} \left(\boldsymbol{C}\bar{\vec{x}}_1 - \left(\boldsymbol{C}\bar{\vec{x}}_2\right)\right) > c^2,$$

where $c^2 = \frac{(n_1+n_2-2)q}{(n_1+n_2-q-1)} F_{q,n_1+n_2-q-1}(\alpha)$.

$$\boldsymbol{C}\bar{\vec{x}}_1 = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$\boldsymbol{C}\bar{\vec{x}}_2 = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

$$\boldsymbol{S}_{pooled} = \frac{n_1 - 1}{n_1 + n_2 - 2}\boldsymbol{S}_1 + \frac{n_2 - 1}{n_1 + n_2 - 2}\boldsymbol{S}_2 = \frac{1}{2}\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2} & 1 & 1 \\ 1 & \frac{3}{2} & 1 \\ 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} \frac{3}{2} & 1 & 1 \\ 1 & \frac{3}{2} & 1 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}$$

$$\boldsymbol{C}\boldsymbol{S}_{pooled}\boldsymbol{C}^T = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\left[\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\boldsymbol{C}\boldsymbol{S}_{pooled}\boldsymbol{C}^T\right]^{-1} = \left[\frac{1}{5}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right]^{-1} = 5\begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{5}{3} & 0 \\ \frac{1}{3} & 1 \end{bmatrix}$$

$$Note: \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix}^{-1} = \frac{1}{15}\begin{bmatrix} 5 & 0 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ \frac{1}{15} & \frac{1}{5} \end{bmatrix}$$

$$= \left(\begin{bmatrix} -1 \\ -2 \end{bmatrix} - \begin{bmatrix} -2 \\ -2 \end{bmatrix}\right)^T \begin{bmatrix} \frac{5}{3} & 0 \\ \frac{1}{3} & 1 \end{bmatrix} \left(\begin{bmatrix} -1 \\ -2 \end{bmatrix} - \begin{bmatrix} -2 \\ -2 \end{bmatrix}\right)$$

$$= \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)^T \begin{bmatrix} \frac{5}{3} & 0 \\ \frac{1}{3} & 1 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

$$= \left(\begin{bmatrix} 1 & 0 \end{bmatrix}\right) \begin{bmatrix} \frac{5}{3} & 0 \\ \frac{1}{3} & 1 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

$$= \left(\begin{bmatrix} \frac{5}{3} & 0 \end{bmatrix}\right) \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \frac{5}{3} = 1.\bar{6}$$

$$c^2 = \frac{(n_1 + n_2 - 2)q}{(n_1 + n_2 - q - 1)} F_{q, n_1 + n_2 - q - 1}(\alpha)$$

$$= \frac{36}{17} F_{2,17}(0.05) = 7.605594$$

The rejection rule is $T^2 > 7.605594$ at $\alpha = 0.05$. Therefore, we fail to reject the null hypothesis that $\mu_{11} - \mu_{21} = \mu_{12} - \mu_{22} = \mu_{13} - \mu_{23}$.

b. Test H_0 with $\alpha \le 0.05$ by Bonferroni correction.
Using the following formula,

$$\max_{1 \le j \le p} \left| \frac{(\boldsymbol{C}\bar{x}_{1j} - \boldsymbol{C}\bar{x}_{2j})}{a_j^T \boldsymbol{C}\boldsymbol{S}_{pooled,j}\boldsymbol{C}^T a_j \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \right| \ge t_{n_1 + n_2 - 2}\left(\frac{\alpha}{2p}\right)$$

where $a_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $a_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and the test will be calculated for $j = 1,2$.

$$a_1^T \boldsymbol{C}\boldsymbol{S}_{pooled}\boldsymbol{C}^T a_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2$$

$$\left| \frac{(\boldsymbol{C}\bar{x}_{11} - \boldsymbol{C}\bar{x}_{21})}{a_1^T \boldsymbol{C}\boldsymbol{S}_{pooled,2}\boldsymbol{C}^T a_1 \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \right| = \left| \frac{1}{2\sqrt{\frac{1}{5}}} \right| = 1.118034$$

$$a_2^T \boldsymbol{C}\boldsymbol{S}_{pooled}\boldsymbol{C}^T a_2 = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1$$

$$\left| \frac{(\boldsymbol{C}\bar{x}_{12} - \boldsymbol{C}\bar{x}_{22})}{a_2^T \boldsymbol{C}\boldsymbol{S}_{pooled,2}\boldsymbol{C}^T a_2 \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \right| = \left| \frac{0}{1\sqrt{\frac{1}{5}}} \right| = 0$$

$$t_{n_1+n_2-2}\left(\frac{\alpha}{2p}\right) = t_{18}\left(\frac{0.05}{4}\right) = 2.445006$$

The max value 1.118034 is less than the critical value, so we fail to reject the null hypothesis that $\vec{\mu}_1 = \vec{\mu}_2$ at the level $\alpha = 0.05$ with Bonferroni corrected t-tests.