Homework 4 (Due 5/1)

Question 1 For jointly distributed random vectors $\vec{X}$ and $\vec{Y}$, show that

$$\text{Cov}(\boldsymbol{C}\vec{X}, \boldsymbol{D}\vec{Y}) = \boldsymbol{C}\text{Cov}(\vec{X}, \vec{Y})\boldsymbol{D}^\top.$$

Question 2 For mutually independent random vectors $\vec{X}_1, \ldots, \vec{X}_n \in \mathbb{R}^p$, show that

$$\text{Cov}(a_1\vec{X}_1 + \ldots + a_n\vec{X}_n + \vec{c}) = a_1^2\text{Cov}(\vec{X}_1) + \ldots + a_n^2\text{Cov}(\vec{X}_n).$$

Question 3 If $\vec{Z} = \begin{bmatrix} Z_1 \\ \vdots \\ Z_p \end{bmatrix} \sim \mathcal{N}_p(\vec{0}, \boldsymbol{I}_p)$, then

$$\vec{Z}^\top\vec{Z} = Z_1^2 + \ldots + Z_p^2 \sim \chi_p^2.$$

Question 4 Suppose the joint p.d.f. of X and Y is

$$f(x, y) = \frac{1}{\sqrt{3}\pi}\exp\left(-\frac{2}{3}\left[(x-3)^2 + (y+2)^2 + (x-3)(y+2)\right]\right),$$

Find $Z = h(X, Y)$, such that $Z \sim \chi_2^2$.

Question 5 Let $\vec{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$ be a random vector with the population covariance $\boldsymbol{\Sigma}$. If

$$\boldsymbol{\Sigma} = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix},$$

find (α, β), such that

$$\text{Cov}(X_1, X_3 - (\alpha X_1 + \beta X_2)) = \text{Cov}(X_2, X_3 - (\alpha X_1 + \beta X_2)) = 0.$$

Question 6 Let $\vec{X} \sim \mathcal{N}_p(\vec{\mu}, \boldsymbol{\Sigma})$. Let

$$\boldsymbol{\Sigma} = \sum_{j=1}^{p} \lambda_j \vec{v}_j \vec{v}_j^\top$$

be the spectral decomposition. Let $Y_j = \vec{v}_j^\top\vec{X}$ for all $j = 1, \ldots, p$. Show that $Y_1, \ldots, Y_p$ are mutually independent.