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HW 3 STA 135

1. Given below is a sample of $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$:

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \end{bmatrix}.$$

First, I will begin with writing out the $\boldsymbol{X}$ matrix for the sample data,

$$\boldsymbol{X} = \begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \\ x_{3,1} & x_{3,2} \\ x_{4,1} & x_{4,2} \\ x_{5,1} & x_{5,2} \\ x_{6,1} & x_{6,2} \\ x_{7,1} & x_{7,2} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & -2 \\ 2 & 1 \\ 0 & 0 \\ -1 & 1 \\ -2 & -2 \\ -3 & 1 \end{bmatrix}.$$

The sample mean vector is as follows,

$$\vec{\bar{x}} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{n}\sum_{i=1}^{n} x_{i,1} \\ \frac{1}{n}\sum_{i=1}^{n} x_{i,2} \end{bmatrix} = \frac{1}{n}\sum_{i=1}^{n} \begin{bmatrix} x_{i,1} \\ x_{i,2} \end{bmatrix} = \boxed{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}.$$

The sample covariance matrix is as follows,

$$\boldsymbol{S}_{\bar{X}} = \frac{1}{n-1}\left(\boldsymbol{X} - 1_n\vec{\bar{x}}^T\right)^T\left(\boldsymbol{X} - 1_n\vec{\bar{x}}^T\right) = \begin{bmatrix} s_{1,1} & s_{1,2} \\ s_{2,1} & s_{2,2} \end{bmatrix}$$

where

$$s_{j,k} = s_{k,j} = \frac{1}{n-1}\sum_{i=1}^{n}\left(x_{i,j} - \bar{x}_j\right)\left(x_{i,k} - \bar{x}_k\right),$$

then,

$$\boxed{\boldsymbol{S} = \begin{bmatrix} \frac{13}{3} & 0 \\ 0 & 2 \end{bmatrix}}.$$

In the case of $X_1 + 2X_2$, the sample mean can be calculated doing the following:
Rewrite $X_1 + 2X_2$ as $y_i = x_{i,1} + 2x_{i,2}$, for $i = 1, \cdots, 7$. Then, $\bar{y} = \frac{\sum_{i=1}^{n} y_i}{n} = \frac{\sum_{i=1}^{n} x_{i,1} + 2\sum_{i=1}^{n} x_{i,2}}{n}$. So, the sample mean of $X_1 + 2X_2$ is $\bar{y} = \bar{x}_1 + 2\bar{x} = 0 + 2 \times 0 = \boxed{0}$.

The sample variance can be found by doing the following:

$$s_y^2 = \frac{\sum_{i=1}^n (y_i - \bar{y})^2}{n-1} = \frac{\sum_{i=1}^n \left((x_{i,1} + 2x_{i,2}) - (\bar{x}_1 + 2\bar{x}_2) \right)^2}{n-1} = \frac{\sum_{i=1}^n \left((x_{i,1} - \bar{x}_1) + 2(x_{i,2} - \bar{x}_2) \right)^2}{n-1}$$

$$= \frac{\sum_{i=1}^n (x_{i,1} - \bar{x}_1)^2 + 4\sum_{i=1}^n (x_{i,1} - \bar{x}_1)(x_{i,2} - \bar{x}_2) + 4\sum_{i=1}^n (x_{i,2} - \bar{x}_2)^2}{n-1}$$

$$= s_{x_1}^2 + 4s_{x_1x_2} + 4s_{x_2}^2 = \frac{13}{3} + 4 \times 2 + 4 \times 0 = \boxed{\frac{37}{3}}.$$

2. It is given that a p-variate sample $\vec{x}_1, \cdots, \vec{x}_n$ is transformed into $\vec{y}_1, \cdots, \vec{y}_n$ by

$$y_{ij} = c_j x_{ij} + d_j, j = 1, \cdots, p, i = 1, \cdots, n,$$

where $c_j > 0$ for $j = 1, \cdots, p$. i.e., the p variates $X_1, \cdots, X_p$ are transformed into $Y_1, \cdots, Y_p$ in that $Y_j = c_j X_j + d_j$. Then, denoting r_{jk}^x as the sample correlation between X_j and X_k, and denote by r_{jk}^y the sample correlation between Y_j and Y_k. Below it will be shown that $r_{jk}^x = r_{jk}^y$:
First note that,

$$r_{jk}^y = \frac{s_{jk}^y}{\sqrt{s_{jj}^y s_{kk}^y}},$$

where,

$$s_{jk}^y = \frac{1}{n-1}\sum_{i=1}^n (y_{ij} - \bar{y}_j)(y_{ik} - \bar{y}_k),$$

$$s_{jj}^y = \frac{1}{n-1}\sum_{i=1}^n (y_{ij} - \bar{y}_j)^2,$$

$$s_{kk}^y = \frac{1}{n-1}\sum_{i=1}^n (y_{ik} - \bar{y}_k)^2.$$

Then it follows that,

$$r_{jk}^y = \frac{s_{jk}^y}{\sqrt{s_{jj}^y s_{kk}^y}} = \frac{\frac{1}{n-1}\sum_{i=1}^n (y_{ij} - \bar{y}_j)(y_{ik} - \bar{y}_k)}{\sqrt{\frac{1}{n-1}\sum_{i=1}^n (y_{ij} - \bar{y}_j)^2 \frac{1}{n-1}\sum_{i=1}^n (y_{ik} - \bar{y}_k)^2}}.$$

Note that,

$$\bar{y}_j = \frac{\sum_{i=1}^n y_{ij}}{n} = \frac{\sum_{i=1}^n (c_j x_{ij} + d_j)}{n} = \frac{\sum_{i=1}^n c_j x_{ij} + \sum_{i=1}^n d_j}{n} = c_j \bar{x}_j + d_j.$$

Then s_{jj}^y can be rewritten as,

$$s_{jj}^y = \frac{1}{n-1}\sum_{i=1}^n (y_{ij} - \bar{y}_j)^2 = \frac{1}{n-1}\sum_{i=1}^n \left(c_j x_{ij} + d_j - (c_j \bar{x}_j + d_j) \right)^2 = \frac{1}{n-1}\sum_{i=1}^n c_j^2 (x_{ij} - \bar{x}_j)^2$$
$$= c_j^2 s_{jj}^x.$$

Similarly, for s_{kk}^y,

$$s_{kk}^{y} = \frac{1}{n-1}\sum\nolimits_{i=1}^{n} (y_{ik} - \bar{y}_k)^2 = \cdots = c_k^2 s_{kk}^{x}.$$

In the case of s_{jk}^{y},

$$\begin{aligned} s_{jk}^{y} &= \frac{1}{n-1}\sum\nolimits_{i=1}^{n} \left(y_{ij} - \bar{y}_j\right)(y_{ik} - \bar{y}_k) \\ &= \frac{1}{n-1}\sum\nolimits_{i=1}^{n} \left(c_j x_{ij} + d_j - \left(c_j \bar{x}_j + d_j\right)\right)\left(c_k x_{ik} + d_k - (c_k \bar{x}_k + d_k)\right) = \\ &= \frac{1}{n-1}\sum\nolimits_{i=1}^{n} c_j c_k \left(x_{ij} - \bar{x}_j\right)(x_{ik} - \bar{x}_k) = c_j c_k s_{jk}^{x}. \end{aligned}$$

Then returning to r_{jk}^{y},

$$r_{jk}^{y} = \frac{s_{jk}^{y}}{\sqrt{s_{jj}^{y} s_{kk}^{y}}} = \frac{c_j c_k s_{jk}^{x}}{\sqrt{c_j^2 s_{jj}^{x} c_k^2 s_{kk}^{x}}} = \frac{s_{jk}^{x}}{\sqrt{s_{jj}^{x} s_{kk}^{x}}} = r_{jk}^{x}. \blacksquare$$

3. In a sample of $\vec{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, the sample mean, and sample covariance matrix are

$$\vec{\boldsymbol{x}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \boldsymbol{S} = \begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix},$$

respectively.

a) The spectral decomposition of $\boldsymbol{S}$ is as follows:

$$|\boldsymbol{S} - \lambda \boldsymbol{I}| = 0$$

$$\begin{vmatrix} 6-\lambda & 2 \\ 2 & 6-\lambda \end{vmatrix} = 0$$

$$(6-\lambda)^2 - (2)^2 = 0$$

Using $a^2 - b^2 = (a+b)(a-b)$.

$$(6 - \lambda + 2)(6 - \lambda - 2) = 0$$

$$(8-\lambda)(4-\lambda) = 0$$

$$\lambda_1 = 8, \lambda_2 = 4$$

In the case of $\lambda_1 = 8$,

$$\begin{vmatrix} 6-\lambda & 2 \\ 2 & 6-\lambda \end{vmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

NOTE: This is a linear equation in the form of $\boldsymbol{Ax} = \boldsymbol{b}$.

$$\begin{vmatrix} -2 & 2 \\ 2 & -2 \end{vmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$R_1 + R_2 \to R_2, \qquad \begin{vmatrix} -2 & 2 \\ 0 & 0 \end{vmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-\frac{1}{2}R_1 \rightarrow R_1, \qquad \begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix} \binom{x}{y} = \binom{0}{0}$$

This produces the following two linear equations:

$$x - y = 0$$

$$0 + 0 = 0$$

Next, showing the normalized eigenvector,

$$\frac{x}{y} = 1$$

$$\boldsymbol{x} = \binom{1}{1}$$

$$e_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

In the case of $\lambda_2 = 4$,

$$\begin{vmatrix} 6-\lambda & 2 \\ 2 & 6-\lambda \end{vmatrix} \binom{x}{y} = \binom{0}{0}$$

$$\begin{vmatrix} 2 & 2 \\ 2 & 2 \end{vmatrix} \binom{x}{y} = \binom{0}{0}$$

$$R_1 - R_2 \rightarrow R_2, \qquad \begin{vmatrix} 2 & 2 \\ 0 & 0 \end{vmatrix} \binom{x}{y} = \binom{0}{0}$$

$$\frac{1}{2}R_1 \rightarrow R_1, \qquad \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} \binom{x}{y} = \binom{0}{0}$$

This produces the following two linear equations:

$$x + y = 0$$

$$0 + 0 = 0$$

Next, showing the normalized eigenvector,

$$\frac{x}{y} = -1$$

$$\boldsymbol{x} = \binom{-1}{1}$$

$$e_2 = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Then the spectral decomposition of $\boldsymbol{S}$ can be written as,

$$\boldsymbol{S} = \lambda_1 e_1 e_1' + \lambda_2 e_2 e_2'$$

$$= 8\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} + 4\begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Then in the case of the spectral decomposition for $\boldsymbol{S}^{-1}$, first $\boldsymbol{S}^{-1}$ will be found using the rules of an inverse for a 2×2 matrix.

$$\boldsymbol{S}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{6^2 - 2^2}\begin{bmatrix} 6 & -2 \\ -2 & 6 \end{bmatrix} = \begin{bmatrix} \frac{3}{16} & -\frac{1}{16} \\ -\frac{1}{16} & \frac{3}{16} \end{bmatrix}$$

The eigenvalues will be found first.

$$|\boldsymbol{S}^{-1} - \lambda \boldsymbol{I}| = 0$$

$$\begin{vmatrix} \frac{3}{16} - \lambda & -\frac{1}{16} \\ -\frac{1}{16} & \frac{3}{16} - \lambda \end{vmatrix} = 0$$

$$\left(\frac{3}{16} - \lambda\right)^2 - \left(-\frac{1}{16}\right)^2 = 0$$

Using $a^2 - b^2 = (a + b)(a - b)$.

$$\left(\frac{3}{16} - \lambda - \frac{1}{16}\right)\left(\frac{3}{16} - \lambda + \frac{1}{16}\right) = 0$$

$$\left(\frac{2}{16} - \lambda\right)\left(\frac{4}{16} - \lambda\right) = 0$$

$$\lambda_1 = \frac{1}{4}\ \lambda_2 = \frac{1}{8}$$

In the case of $\lambda_1 = \frac{1}{4}$,

$$\begin{vmatrix} \frac{3}{16} - \lambda & -\frac{1}{16} \\ -\frac{1}{16} & \frac{3}{16} - \lambda \end{vmatrix}\binom{x}{y} = \binom{0}{0}$$

$$\begin{vmatrix} \frac{3}{16} - \frac{1}{4} & -\frac{1}{16} \\ -\frac{1}{16} & \frac{3}{16} - \frac{1}{4} \end{vmatrix}\binom{x}{y} = \binom{0}{0}$$

$$\begin{vmatrix} \frac{-1}{16} & -\frac{1}{16} \\ -\frac{1}{16} & \frac{-1}{16} \end{vmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$R_1 - R_2 \to R_2, \qquad \begin{vmatrix} \frac{-1}{16} & -\frac{1}{16} \\ 0 & 0 \end{vmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-16R_1 \to R_1, \qquad \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This produces the following two linear equations:

$$x + y = 0$$

$$0 + 0 = 0$$

Next, showing the normalized eigenvector,

$$\frac{x}{y} = -1$$

$$\boldsymbol{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$e_1 = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

In the case of $\lambda_2 = \frac{1}{8}$,

$$\begin{vmatrix} \frac{3}{16} - \frac{1}{8} & -\frac{1}{16} \\ -\frac{1}{16} & \frac{3}{16} - \frac{1}{8} \end{vmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{vmatrix} \frac{1}{16} & -\frac{1}{16} \\ -\frac{1}{16} & \frac{1}{16} \end{vmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$R_1 + R_2 \to R_2, \qquad \begin{vmatrix} \frac{1}{16} & -\frac{1}{16} \\ 0 & 0 \end{vmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$16R_1 \to R_1, \qquad \begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This produces the following two linear equations:

$$x - y = 0$$

$$0 + 0 = 0$$

Next, showing the normalized eigenvector,

$$\frac{x}{y} = 1$$

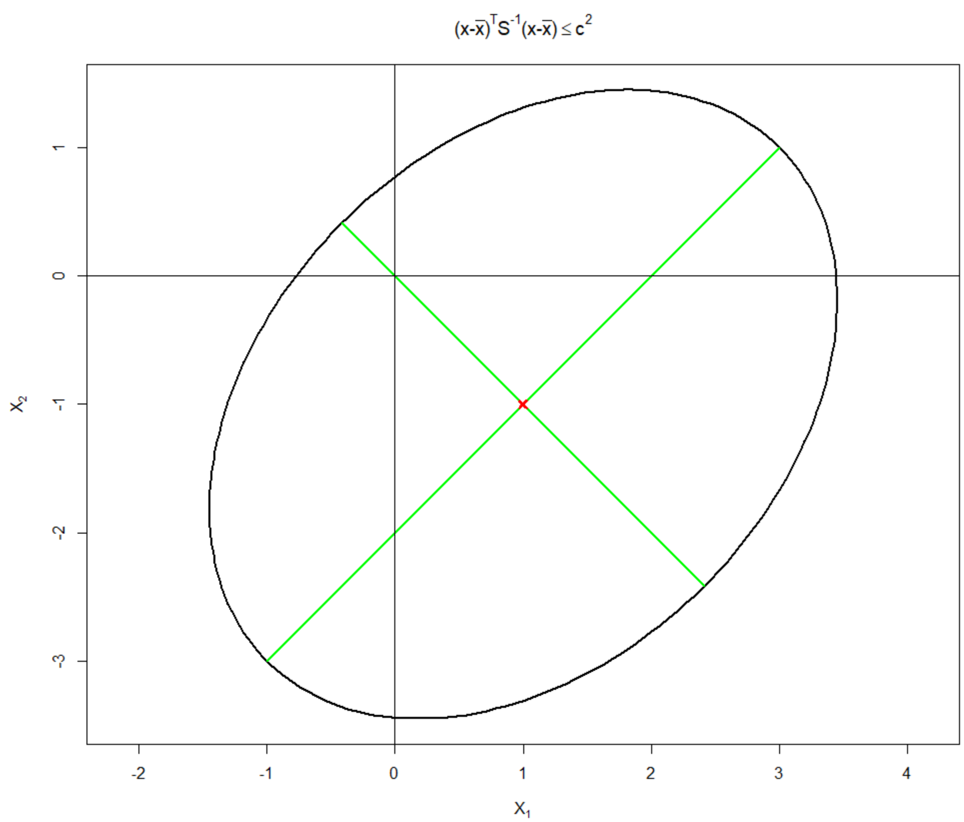
$$\boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$e_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Then using the formula for the spectral decomposition,

$$\boldsymbol{S}^{-1} = \lambda_1 e_1 e_1' + \lambda_2 e_2 e_2' = \frac{1}{4}\begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} + \frac{1}{8}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

b) Below is a sketch of the mean-centered ellipse, $(\boldsymbol{x} - \overline{\boldsymbol{x}})^T \boldsymbol{S}^{-1} (\boldsymbol{x} - \overline{\boldsymbol{x}}) \leq c^2$:



c) The sample correlation matrix $\boldsymbol{R}$ will be shown below:

$$\boldsymbol{R} = \begin{bmatrix} 1 & r_{1,2} \\ r_{2,1} & 1 \end{bmatrix}$$

where $r_{jk} = \frac{s_{jk}}{\sqrt{s_{jj}s_{kk}}}$. Then,

$$\boldsymbol{R} = \begin{bmatrix} 1 & \frac{s_{1,2}}{\sqrt{s_{1,1}s_{2,2}}} \\ \frac{s_{2,1}}{\sqrt{s_{2,2}s_{1,1}}} & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{2}{\sqrt{6 \times 6}} \\ \frac{2}{\sqrt{6 \times 6}} & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{3} \\ \frac{1}{3} & 1 \end{bmatrix}.$$

Then the spectral decomposition of $\boldsymbol{R}$ is as follows:

$$|\boldsymbol{R} - \lambda \boldsymbol{I}| = 0$$

$$\begin{vmatrix} 1 - \lambda & \frac{1}{3} \\ \frac{1}{3} & 1 - \lambda \end{vmatrix} = 0$$

$$(1 - \lambda)^2 - \left(\frac{1}{3}\right)^2 = 0$$

Using $a^2 - b^2 = (a + b)(a - b)$.

$$\left(1 - \lambda + \frac{1}{3}\right)\left(1 - \lambda - \frac{1}{3}\right) = 0$$

$$\left(\frac{4}{3} - \lambda\right)\left(\frac{2}{3} - \lambda\right) = 0$$

$$\lambda_1 = \frac{4}{3}, \lambda_2 = \frac{2}{3}$$

In the case of $\lambda_1 = \frac{4}{3}$,

$$\begin{vmatrix} 1 - \lambda & \frac{1}{3} \\ \frac{1}{3} & 1 - \lambda \end{vmatrix} \binom{x}{y} = \binom{0}{0}$$

$$\begin{vmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix} \binom{x}{y} = \binom{0}{0}$$

$$R_1 + R_2 \to R_2, \qquad \begin{vmatrix} -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 \end{vmatrix} \binom{x}{y} = \binom{0}{0}$$

$$-3R_1 \to R_1, \qquad \begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix} \binom{x}{y} = \binom{0}{0}$$

This produces the following two linear equations:

$$x - y = 0$$

$$0 + 0 = 0$$

Next, showing the normalized eigenvector,

$$x = y$$

$$\frac{x}{y} = 1$$

$$\boldsymbol{x} = \binom{1}{1}$$

$$e_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

In the case of $\lambda_2 = \frac{2}{3}$,

$$\begin{vmatrix} 1-\lambda & \frac{1}{3} \\ \frac{1}{3} & 1-\lambda \end{vmatrix} \binom{x}{y} = \binom{0}{0}$$

$$\begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{vmatrix} \binom{x}{y} = \binom{0}{0}$$

$$R_1 - R_2 \to R_2, \qquad \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ 0 & 0 \end{vmatrix} \binom{x}{y} = \binom{0}{0}$$

$$3R_1 \to R_1, \qquad \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} \binom{x}{y} = \binom{0}{0}$$

This produces the following two linear equations:

$$x + y = 0$$

$$0 + 0 = 0$$

Next, showing the normalized eigenvector,

$$x = -y$$

$$\frac{x}{y} = -1$$

$$\boldsymbol{x} = \binom{-1}{1}$$

$$e_2 = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Then the spectral decomposition of $\boldsymbol{R}$ can be written as,

$$\boldsymbol{R} = \lambda_1 e_1 e_1' + \lambda_2 e_2 e_2'$$

$$= \frac{4}{3}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} + \frac{2}{3}\begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

To find the spectral decomposition of $\boldsymbol{R}$ it is necessary to first find $\boldsymbol{R}^{-1}$.

$$\boldsymbol{R}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{1^2 - \frac{1}{3}^2}\begin{bmatrix} 1 & -\frac{1}{3} \\ -\frac{1}{3} & 1 \end{bmatrix} = \begin{bmatrix} \frac{9}{8} & -\frac{3}{8} \\ -\frac{3}{8} & \frac{9}{8} \end{bmatrix}$$

$$|\boldsymbol{R}^{-1} - \lambda \boldsymbol{I}| = 0$$

$$\begin{vmatrix} \frac{9}{8} - \lambda & -\frac{3}{8} \\ -\frac{3}{8} & \frac{9}{8} - \lambda \end{vmatrix} = 0$$

$$\left(\frac{9}{8} - \lambda\right)^2 - \left(-\frac{3}{8}\right)^2 = 0$$

Using $a^2 - b^2 = (a + b)(a - b)$.

$$\left(\frac{9}{8} - \lambda - \frac{3}{8}\right)\left(\frac{9}{8} - \lambda + \frac{3}{8}\right) = 0$$

$$\left(\frac{3}{4} - \lambda\right)\left(\frac{3}{2} - \lambda\right) = 0$$

$$\lambda_1 = \frac{3}{2}, \lambda_2 = \frac{3}{4}$$

In the case of $\lambda_1 = \frac{3}{2}$,

$$\begin{vmatrix} \frac{9}{8} - \lambda & -\frac{3}{8} \\ -\frac{3}{8} & \frac{9}{8} - \lambda \end{vmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{vmatrix} \frac{9}{8} - \frac{3}{2} & -\frac{3}{8} \\ -\frac{3}{8} & \frac{9}{8} - \frac{3}{2} \end{vmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{vmatrix} \frac{-3}{8} & -\frac{3}{8} \\ -\frac{3}{8} & \frac{-3}{8} \end{vmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$R_1 - R_2 \to R_2, \qquad \begin{vmatrix} \frac{-3}{8} & -\frac{3}{8} \\ 0 & 0 \end{vmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-\frac{8}{3} R_1 \to R_1, \qquad \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This produces the following two linear equations:

$$x + y = 0$$

$$0 + 0 = 0$$

Next, showing the normalized eigenvector,

$$x = -y$$

$$\frac{x}{y} = -1$$

$$\boldsymbol{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$e_1 = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

In the case of $\lambda_2 = \frac{3}{4}$,

$$\begin{vmatrix} \frac{9}{8} - \lambda & -\frac{3}{8} \\ -\frac{3}{8} & \frac{9}{8} - \lambda \end{vmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{vmatrix} \frac{9}{8} - \frac{3}{4} & -\frac{3}{8} \\ -\frac{3}{8} & \frac{9}{8} - \frac{3}{4} \end{vmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{vmatrix} \frac{3}{8} & -\frac{3}{8} \\ -\frac{3}{8} & \frac{3}{8} \end{vmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$R_1 + R_2 \to R_2, \qquad \begin{vmatrix} \frac{3}{8} & -\frac{3}{8} \\ 0 & 0 \end{vmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\frac{8}{3} R_1 \to R_1, \qquad \begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This produces the following two linear equations:

$$x - y = 0$$

$$0 + 0 = 0$$

Next, showing the normalized eigenvector,

$$x = y$$

$$\frac{x}{y} = 1$$

$$\boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

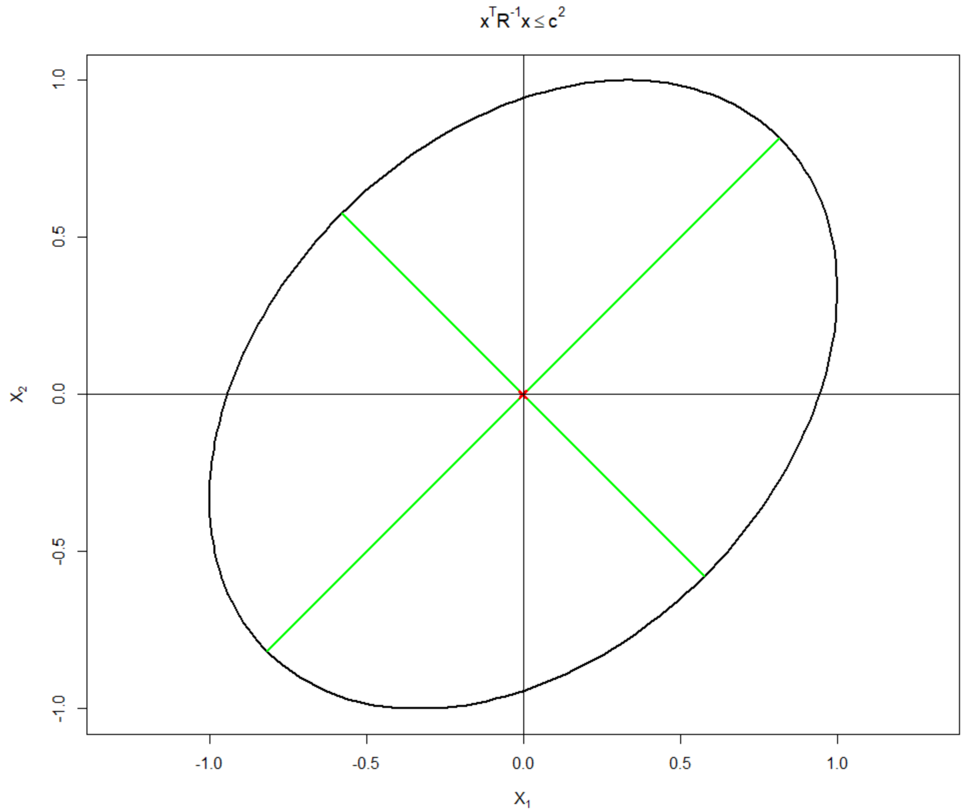
$$e_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Then the spectral decomposition of $\boldsymbol{R}^{-1}$ can be written as,

$$\boldsymbol{R}^{-1} = \lambda_1 e_1 e_1' + \lambda_2 e_2 e_2'$$

$$= \frac{3}{2} \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} + \frac{3}{4} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

d) Below is a sketch of the mean-centered ellipse, $\boldsymbol{x}^T \boldsymbol{R}^{-1} \boldsymbol{x} \leq c^2$:



4. A sample of $\vec{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}$ with sample covariance matrix

$$S = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Then to find the sample cross covariance matrix between $\begin{bmatrix} X_1 + X_2 \\ X_1 - X_2 \end{bmatrix}$ and $\begin{bmatrix} X_3 + X_4 \\ X_3 - X_4 \end{bmatrix}$, the following steps will be taken:

First let,

$$\begin{bmatrix} X^{(1)} \\ X^{(2)} \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}.$$

Then it can be shown that,

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}}_{\vec{X}} = \underbrace{\begin{pmatrix} X_1 + X_2 \\ X_1 - X_2 \\ X_3 + X_4 \\ X_3 - X_4 \end{pmatrix}}_{\vec{Y}}$$

where $\vec{Y} = C\vec{X}$. Then to find the variance covariance matrix, the following will be done,

$$S_{\vec{Y}} = Var(\vec{Y}) = Var(C\vec{X}) = CVar(\vec{X})C^T$$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & -2 & 1 & 1 \\ 1 & -1 & 3 & 1 \\ 0 & 0 & 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 1 & 1 \\ 0 & 4 & -1 & -1 \\ 1 & -1 & 6 & 0 \\ 1 & -1 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} Var\begin{pmatrix} X_1 + X_2 \\ X_1 - X_2 \end{pmatrix} & Cov\left(\begin{bmatrix} X_1 + X_2 \\ X_1 - X_2 \end{bmatrix}, \begin{bmatrix} X_3 + X_4 \\ X_3 - X_4 \end{bmatrix}\right) \\ Cov\left(\begin{bmatrix} X_3 + X_4 \\ X_3 - X_4 \end{bmatrix}, \begin{bmatrix} X_1 + X_2 \\ X_1 - X_2 \end{bmatrix}\right) & Var\begin{pmatrix} X_3 + X_4 \\ X_3 - X_4 \end{pmatrix} \end{bmatrix}.$$

Therefore,

$$Cov\left(\begin{bmatrix} X_1 + X_2 \\ X_1 - X_2 \end{bmatrix}, \begin{bmatrix} X_3 + X_4 \\ X_3 - X_4 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}.$$

5. The random sample

$$X_1, \cdots, X_n \sim i.i.d.\mathcal{N}(\mu, \sigma^2)$$

with sample mean $\bar{X}$ and sample variance S^2.

The sample mean $\bar{X}$ is defined as $\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$. The sample variance S^2 is defined as $S^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}$. To show that $\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} \sim \mathcal{N}(0,1)$, first I will take the expectation of the left-hand side.

$$E\left(\frac{\sqrt{n}(\bar{X} - \mu)}{\sigma}\right) = \frac{\sqrt{n}}{\sigma}E(\bar{X} - \mu) = \frac{\sqrt{n}}{\sigma}[E(\bar{X}) - E(\mu)] = \frac{\sqrt{n}}{\sigma}[E(\bar{X}) - \mu] = \frac{\sqrt{n}}{\sigma}\left[E\left(\frac{\sum_{i=1}^{n} X_i}{n}\right) - \mu\right]$$

$$= \frac{\sqrt{n}}{\sigma}\left[\frac{1}{n}E\left(\sum_{i=1}^{n} X_i\right) - \mu\right] = \frac{\sqrt{n}}{\sigma}\left[\frac{1}{n}\sum_{i=1}^{n} E(X_i) - \mu\right] = \frac{\sqrt{n}}{\sigma}\left[\frac{1}{n}n\mu - \mu\right] = 0$$

Then I will take the variance of the left-hand side.

$$Var\left(\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma}\right)=\frac{n}{\sigma^2}Var(\bar{X}-\mu)=\frac{n}{\sigma^2}Var(\bar{X})=\frac{n}{\sigma^2}Var\left(\frac{\sum_{i=1}^n X_i}{n}\right)=\frac{n}{\sigma^2}\frac{1}{n^2}Var\left(\sum_{i=1}^n X_i\right)$$
$$=\frac{1}{n\sigma^2}\sum_{i=1}^n Var(X_i)=\frac{n\sigma^2}{n\sigma^2}=1$$

The linear combination of independent normal random variables will also follow a normal distribution. In this case variable $\bar{X}$ is the linear combination of an independent normal random variable. This concludes that $\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma}\sim\mathcal{N}(0,1)$ with mean 0 and variance 1.■

Next, I will show that $E(S^2)=\sigma^2$.

$$E(S^2)=E\left(\frac{\sum_{i=1}^n(X_i-\bar{X})^2}{n-1}\right)=\frac{1}{n-1}E\left(\sum_{i=1}^n(X_i-\bar{X})^2\right)=\frac{1}{n-1}E\left(\sum_{i=1}^n\left((X_i-\mu)-(\bar{X}-\mu)\right)^2\right)$$
$$=\frac{1}{n-1}E\left(\sum_{i=1}^n(X_i-\mu)^2-2(X_i-\mu)(\bar{X}-\mu)+(\bar{X}-\mu)^2\right)$$
$$=\frac{1}{n-1}\sum_{i=1}^n\left(E((X_i-\mu)^2)-2E\left((X_i-\mu)(\bar{X}-\mu)\right)+E((\bar{X}-\mu)^2)\right)$$
$$=\frac{1}{n-1}\sum_{i=1}^n\left(Var(X_i)-2Cov(X_i,\bar{X})+Var(\bar{X})\right)$$

In the case of $Cov(X_i,\bar{X})$,

$$Cov(X_i,\bar{X})=Cov\left(X_i,\frac{\sum_{j=1}^n X_j}{n}\right)=\frac{1}{n}Cov\left(X_i,\sum_{j=1}^n X_j\right)$$

$$\because Cov\left(X_i,X_j\right)=0 \text{ if } i\neq j.$$

Therefore, $Cov(X_i,\bar{X})=\frac{1}{n}Cov(X_i,X_i)=\frac{1}{n}Var(X_i)=\frac{\sigma^2}{n}$.

$$=\frac{1}{n-1}\sum_{i=1}^n\left(\sigma^2-\frac{2\sigma^2}{n}+\frac{\sigma^2}{n}\right)=\frac{1}{n-1}\sum_{i=1}^n\left(\frac{n\sigma^2-\sigma^2}{n}\right)=\frac{\sigma^2}{n-1}\frac{n(n-1)}{n}=\sigma^2$$■