Jared Yu

STA 135

HW 1

1. $\boldsymbol{D} = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_p \end{bmatrix} \in \mathbb{R}^{p\times p}, \boldsymbol{A} = \begin{bmatrix} \vec{a}_1^T \\ \vdots \\ \vec{a}_p^T \end{bmatrix} \in \mathbb{R}^{p\times q}$

$$\boldsymbol{DA} = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_p \end{bmatrix}\begin{bmatrix} \vec{a}_1^T \\ \vdots \\ \vec{a}_p^T \end{bmatrix} = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_p \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1q} \\ a_{21} & a_{22} & \cdots & a_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pq} \end{bmatrix} =$$
$$\begin{bmatrix} d_1\times a_{11} & d_1\times a_{12} & \cdots & d_1\times a_{1q} \\ d_2\times a_{21} & d_2\times a_{22} & \cdots & d_2\times a_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ d_p\times a_{p1} & d_p\times a_{p2} & \cdots & d_p\times a_{pq} \end{bmatrix} = \begin{bmatrix} d_1\vec{a}_1^T \\ d_2\vec{a}_2^T \\ \vdots \\ d_p\vec{a}_p^T \end{bmatrix}$$

2. $\boldsymbol{D} = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_q \end{bmatrix} \in \mathbb{R}^{q\times q}, \boldsymbol{A} = \left[\vec{a}_1 \cdots \vec{a}_q\right] \in \mathbb{R}^{p\times q}$

$$\boldsymbol{AD} = \left[\vec{a}_1 \cdots \vec{a}_q\right]\begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_q \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{q1} \\ a_{12} & a_{22} & \cdots & a_{q2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1p} & a_{2p} & \cdots & a_{qp} \end{bmatrix}\begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_q \end{bmatrix}$$
$$= \begin{bmatrix} d_1\times a_{11} & d_2\times a_{21} & \cdots & d_q\times a_{q1} \\ d_1\times a_{12} & d_2\times a_{22} & \cdots & d_q\times a_{q2} \\ \vdots & \vdots & \ddots & \vdots \\ d_1\times a_{1p} & d_2\times a_{2p} & \cdots & d_q\times a_{qp} \end{bmatrix} = [d_1\vec{a}_1\ d_2\vec{a}_2 \cdots d_q\vec{a}_q]$$

3. $\boldsymbol{A} = [\vec{a}_1 \cdots \vec{a}_k] \in \mathbb{R}^{n\times k}, \boldsymbol{B} = \begin{bmatrix} \vec{b}_1^T \\ \vdots \\ \vec{b}_k^T \end{bmatrix} \in \mathbb{R}^{k\times p}$, here we have the case where:

$$\vec{a}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix}, \text{and } \vec{b}_j = \begin{bmatrix} b_{j1} \\ b_{j2} \\ \vdots \\ b_{jp} \end{bmatrix}.$$

Then we can write out the product of the two matrices as follows:

$$\boldsymbol{AB} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nk} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k1} & b_{k2} & \cdots & b_{kp} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{k} a_{1i}b_{i1} & \sum_{i=1}^{k} a_{1i}b_{i2} & \cdots & \sum_{i=1}^{k} a_{1i}b_{ip} \\ \sum_{i=1}^{k} a_{2i}b_{i1} & \sum_{i=1}^{k} a_{2i}b_{i2} & \cdots & \sum_{i=1}^{k} a_{2i}b_{ip} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{k} a_{ni}b_{i1} & \sum_{i=1}^{k} a_{ni}b_{i2} & \cdots & \sum_{i=1}^{k} a_{ni}b_{ip} \end{bmatrix}$$

$$= \sum_{i=1}^{k} a_i b_i^T = \vec{a}_1\vec{b}_1^T + \vec{a}_2\vec{b}_2^T + \cdots + \vec{a}_k\vec{b}_k^T$$

4. The question is in the form of an if and only if statement. So, to prove it, I will first show that if row vectors are unit and pairwise perpendicular, then the column vectors are unit and pairwise perpendicular. Afterwards, I will prove the other way, showing that if column vectors are unit and pairwise perpendicular, then the row vectors are unit and pairwise perpendicular. The main assumption is that the matrix is orthogonal.

   Given that $\boldsymbol{Q}$ is a $q \times q$ matrix we will next show the following:

$$\boldsymbol{QQ}^T = \boldsymbol{I}$$

   The result from this is that the row vectors are unit length and pairwise perpendicular. The row rank of $\boldsymbol{Q} = q$, so it is full rank. This means that $\boldsymbol{Q}^{-1}$ exists. I will then show that:

$$\boldsymbol{Q}^{-1}\boldsymbol{QQ}^T = \boldsymbol{Q}^{-1}$$

$$\boldsymbol{IQ}^T = \boldsymbol{Q}^{-1}$$

$$\boldsymbol{Q}^T = \boldsymbol{Q}^{-1}$$

$$\boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{Q}^{-1}\boldsymbol{Q} = \boldsymbol{I}$$

   The last line shows also that the columns are also then unit and pairwise perpendicular. This concludes the first part showing that if the row vectors are unit and pairwise perpendicular, then the column vectors are unit and pairwise perpendicular.

   I must next also show the other way of the proof for the if and only if statement. So, I will begin with showing that the column vectors are unit and pairwise perpendicular,

$$\boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{I}.$$

   The column rank of $\boldsymbol{Q} = q$, also $\boldsymbol{Q}$ is full rank. Therefore, $\boldsymbol{Q}^{-1}$ exists. Next, I will show the following:

$$\boldsymbol{Q}^T\boldsymbol{QQ}^{-1} = \boldsymbol{Q}^{-1}$$

$$\boldsymbol{Q}^T\boldsymbol{I} = \boldsymbol{Q}^{-1}$$

$$\boldsymbol{Q}^T = \boldsymbol{Q}^{-1}$$

$$QQ^T = QQ^{-1} = I$$

The last line shows that the row vector is also unit and pairwise perpendicular. This concludes the entire proof, since we have shown both ways that if the row vectors or column vectors are unit and pairwise perpendicular, then the vice versa is true.

5. a) Let $\boldsymbol{A} = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$, then the eigenvalues can be found using the following:

$$|\boldsymbol{A} - \boldsymbol{\lambda I}| = 0$$

$$\begin{bmatrix} a-\lambda & b \\ b & a-\lambda \end{bmatrix} = 0$$

$$(a-\lambda)^2 - b^2 = 0$$

Then using the rule that $A^2 - B^2 = (A-B)(A+B)$,

$$(a-\lambda-b)(a-\lambda+b) = 0$$

So, the possible values for λ are $\lambda_1: a-b$ and $\lambda_2: a+b$. The order is arbitrary, as the actual values are unknown. Then in the case of $\lambda_1: a-b$,

$$\begin{bmatrix} b & b \\ b & b \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

NOTE: This is a linear equation in the form of $\boldsymbol{Ax} = \boldsymbol{b}$.
This produces the two linear equations,

$$bx + by = 0,$$

$$bx + by = 0.$$

Next, I will show the normalized eigenvector.

$$bx = -by$$

$$\frac{x}{y} = -1$$

$$\boldsymbol{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$e_1 = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Then in the case of $\lambda_2: a+b$,

$$\begin{bmatrix} -b & b \\ b & -b \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Next, it is necessary to do put the matrix on the left in reduced row form. It will be done using the following steps:

$$R_1 + R_2 \to R_2, \begin{bmatrix} -b & b \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$R_1 \times -\frac{1}{b} \to R_1, \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This produces the two linear equations,

$$x - y = 0,$$

$$0 + 0 = 0.$$

Next, I will show the normalized eigenvector,

$$x = y$$

$$\frac{x}{y} = 1$$

$$\boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$e_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Lastly, to put it in the form of the spectral decomposition:

$$\boldsymbol{A} = \lambda_1 e_1 e_1' + \lambda_2 e_2 e_2' = (a-b) \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} + (a+b) \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= (a-b) \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix} + (a+b) \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

b) To show the relationship between a and b for invertibility, we will use the rule $\det(\boldsymbol{A}) \neq 0$. The determinant is $a^2 - b^2$, so we must show:

$$a^2 - b^2 \neq 0$$

$$a^2 \neq b^2$$

$$a \neq b \text{ or } a \neq -b.$$

To find $\boldsymbol{A}^{-1}$, the rules for finding the inverse of a 2×2 matrix will be used:

$$\frac{1}{a^2 - b^2} \begin{pmatrix} a & -b \\ -b & a \end{pmatrix} = \begin{pmatrix} \frac{a}{a^2 - b^2} & -\frac{b}{a^2 - b^2} \\ -\frac{b}{a^2 - b^2} & \frac{a}{a^2 - b^2} \end{pmatrix}$$

Then to find the eigenvalues, the following must be done,

$$\begin{bmatrix} \frac{a}{a^2-b^2}-\lambda & -\frac{b}{a^2-b^2} \\ -\frac{b}{a^2-b^2} & \frac{a}{a^2-b^2}-\lambda \end{bmatrix} = 0$$

$$\left(\frac{a}{a^2-b^2}-\lambda\right)^2 - \left(-\frac{b}{a^2-b^2}\right)^2 = 0$$

Then again using the rule that $A^2 - B^2 = (A-B)(A+B)$,

$$\left(\frac{a+b}{a^2-b^2}-\lambda\right)\left(\frac{a-b}{a^2-b^2}-\lambda\right) = 0$$

$$\left(\frac{1}{a-b}-\lambda\right)\left(\frac{1}{a+b}-\lambda\right) = 0$$

$$\lambda_1 = \frac{1}{a-b}, \lambda_2 = \frac{1}{a+b}.$$

These eigenvalues are the inverse of what they were previously, before taking the inverse $\boldsymbol{A}^{-1}$. In the case of λ_1:

$$\begin{bmatrix} \frac{a}{a^2-b^2}-\frac{1}{a-b} & -\frac{b}{a^2-b^2} \\ -\frac{b}{a^2-b^2} & \frac{a}{a^2-b^2}-\frac{1}{a-b} \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{bmatrix} \frac{-b}{a^2-b^2} & -\frac{b}{a^2-b^2} \\ -\frac{b}{a^2-b^2} & \frac{-b}{a^2-b^2} \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$R_1 - R_2 \to R_2, \begin{bmatrix} \frac{-b}{a^2-b^2} & -\frac{b}{a^2-b^2} \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$R_1 \times -\frac{a^2-b^2}{b} \to R_1, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

This produces the two linear equations,

$$x + y = 0$$

$$0 + 0 = 0.$$

Next, I will show the normalized eigenvector,

$$x = -y$$

$$\frac{x}{y} = -1$$

$$\boldsymbol{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$e_1 = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

In the case of λ_2:

$$\begin{bmatrix} \frac{a}{a^2-b^2} - \frac{1}{a+b} & -\frac{b}{a^2-b^2} \\ -\frac{b}{a^2-b^2} & \frac{a}{a^2-b^2} - \frac{1}{a+b} \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{bmatrix} \frac{b}{a^2-b^2} & -\frac{b}{a^2-b^2} \\ -\frac{b}{a^2-b^2} & \frac{b}{a^2-b^2} \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$R_1 + R_2 \to R_2, \begin{bmatrix} \frac{b}{a^2-b^2} & -\frac{b}{a^2-b^2} \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$R_1 \times \frac{a^2-b^2}{b} \to R_1, \begin{bmatrix} \frac{b}{a^2-b^2} & -\frac{b}{a^2-b^2} \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

This produces the two linear equations,

$$x - y = 0$$

$$0 + 0 = 0.$$

Next, I will show the normalized eigenvector,

$$x = y$$

$$\frac{x}{y} = 1$$

$$\boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$e_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

These eigen vectors are in the same direction as what they were previously, before taking the inverse $\boldsymbol{A}^{-1}$. Lastly, to put it in the form of the spectral decomposition:

$$A^{-1} = \frac{1}{\lambda_1} e_1 e_1' + \frac{1}{\lambda_2} e_2 e_2' = \frac{1}{(a-b)} \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} + \frac{1}{(a+b)} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \frac{1}{(a-b)} \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix} + \frac{1}{(a+b)} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

6. a) Let $\boldsymbol{A} = \begin{bmatrix} 2 & 2 \\ -3 & 5 \\ 5 & -3 \\ -4 & -4 \end{bmatrix}$, then $\boldsymbol{A}^T = \begin{bmatrix} 2 & -3 & 5 & -4 \\ 2 & 5 & -3 & -4 \end{bmatrix}$. To calculate $\boldsymbol{A}^T\boldsymbol{A}$, we have:

$$\begin{bmatrix} 2 & -3 & 5 & -4 \\ 2 & 5 & -3 & -4 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -3 & 5 \\ 5 & -3 \\ -4 & -4 \end{bmatrix} = \begin{bmatrix} 4+9+25+16 & 4-15-15+16 \\ 4-15-15+16 & 4+25+9+16 \end{bmatrix} = \begin{bmatrix} 54 & -10 \\ -10 & 54 \end{bmatrix}.$$

Then, using the rules for symmetric matrices from problem 5, the possible values for λ are $\lambda_1 \colon a-b$ and $\lambda_2 \colon a+b$. So, $\lambda_1 = 54 - (-10) = 64$, and $\lambda_2 = 54 + (-10) = 44$. The corresponding eigenvectors are:

$$e_1 = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, e_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

The spectral decomposition for the matrix is,

$$\boldsymbol{A}^T\boldsymbol{A} = \lambda_1 e_1 e_1' + \lambda_2 e_2 e_2' = 64 \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix} + 44 \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

b) Using R, the eigenvalues for $\boldsymbol{A}\boldsymbol{A}^T$ are $\lambda_1 = 64, \lambda_2 = 44, \lambda_3 = 2.84217 \times 10^{-14}, \lambda_4 = 1.421085 \times 10^{-14}$. The first two eigenvalues are the same as the previous eigenvalues for $\boldsymbol{A}^T\boldsymbol{A}$ and the last two eigenvalues are close to 0.

7. a) Let $\boldsymbol{S}$ be a $k \times k$ invertible symmetric matrix, and $\boldsymbol{C}$ be a $k \times k$ invertible matrix. Also, $\vec{x}$ is a k-dimensional vector. Then I will show the following,

$$(\boldsymbol{C}\vec{x})^T (\boldsymbol{CSC}^T)^{-1} (\boldsymbol{C}\vec{x}) = \vec{x}^T \boldsymbol{S}^{-1} \vec{x}$$

Starting from the left-hand side,

$$(\boldsymbol{C}\vec{x})^T (\boldsymbol{CSC}^T)^{-1} (\boldsymbol{C}\vec{x}) = \vec{x}^T \boldsymbol{C}^T (\boldsymbol{C}^T)^{-1} \boldsymbol{S}^{-1} \boldsymbol{C}^{-1} \boldsymbol{C}\vec{x}$$

$$= \vec{x}^T \boldsymbol{I} \boldsymbol{S}^{-1} \boldsymbol{I} \vec{x}$$

$$= \vec{x}^T \boldsymbol{S}^{-1} \vec{x}$$

b) Given that $\boldsymbol{S} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \vec{x} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$.

First, I will calculate $(\boldsymbol{C}\vec{x})^T(\boldsymbol{CSC}^T)^{-1}(\boldsymbol{C}\vec{x})$:

$$(\boldsymbol{C}\vec{x})^T(\boldsymbol{CSC}^T)^{-1}(\boldsymbol{C}\vec{x}) = \left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}\right)^T \left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^T\right)^{-1} \left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 2+0+1 \\ 0+3+1 \end{bmatrix}^T \left(\begin{bmatrix} 3+0+0 & 0+0+0 & 0+0+5 \\ 0+0+0 & 0+2+0 & 0+0+5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}\right)^{-1} \begin{bmatrix} 2+0+1 \\ 0+3+1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 4 \end{bmatrix}\left(\begin{bmatrix} 3 & 0 & 5 \\ 0 & 2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}\right)^{-1} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 4 \end{bmatrix}\left(\begin{bmatrix} 3+0+5 & 0+0+5 \\ 0+0+5 & 0+2+5 \end{bmatrix}\right)^{-1} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 4 \end{bmatrix}\left(\begin{bmatrix} 8 & 5 \\ 5 & 7 \end{bmatrix}\right)^{-1} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 4 \end{bmatrix}\left(\frac{1}{56-25}\right)\begin{bmatrix} 7 & -5 \\ -5 & 8 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 4 \end{bmatrix}\begin{bmatrix} \frac{7}{31} & -\frac{5}{31} \\ -\frac{5}{31} & \frac{8}{31} \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{21}{31} + \left(-\frac{20}{31}\right) & -\frac{15}{31} + \frac{32}{31} \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{31} & \frac{17}{31} \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{31} + \frac{68}{31} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{71}{31} \end{bmatrix}$$

Next, I will calculate $\vec{x}^T\boldsymbol{S}^{-1}\vec{x}$:

$$\vec{x}^T\boldsymbol{S}^{-1}\vec{x} = \begin{bmatrix} 2 & 3 & 1 \end{bmatrix}\left(\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}\right)^{-1}\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

I will begin by calculating $\boldsymbol{S}^{-1}$:

$$= \left(\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}\right)^{-1}$$

To find the inverse, I will do the following:

$$[\boldsymbol{S} \quad | \quad \boldsymbol{I}] \rightarrow [\boldsymbol{I} \quad | \quad \boldsymbol{S}]$$

$$= \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\frac{1}{3}R_1 \rightarrow R_1 = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\frac{1}{2}R_2 \rightarrow R_2 = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\frac{1}{5}R_3 \rightarrow R_3 = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{5} \end{array}\right]$$

This shows that $\boldsymbol{S}^{-1} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$. Then following from before,

$$\vec{x}^T \boldsymbol{S}^{-1} \vec{x} = [2 \quad 3 \quad 1] \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} & \frac{3}{2} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$$= \left[\frac{4}{3} + \frac{9}{2} + \frac{1}{5}\right]$$

$$= \left[\frac{181}{30}\right]$$

There is somewhat of a contradiction, because $\boldsymbol{C}$ given in the example is a 2×3 matrix rather than a $k \times k$ matrix. Therefore, the result from part a does not necessarily apply to part b. However, if $\boldsymbol{C}$ was a 3×3 matrix and the answers were different, then there would be a contradiction.

R Code:

```
A <- matrix(c(2,-3, 5, -4, 2, 5, -3, -4), ncol = 2)

ATA <- A%*%t(A)

eigen(ATA)
```

References:

https://math.stackexchange.com/questions/688339/product-of-inverse-matrices-ab-1

https://www.mathsisfun.com/algebra/matrix-inverse.html

https://math.stackexchange.com/questions/1676135/proving-an-if-and-only-if-statement

http://www-cs-students.stanford.edu/~csilvers/proof/node4.html