Homework 7 (Due 6/5)

Question 1 Suppose the population mean and covariance matrix of $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ are

$$\vec{\mu} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}, \quad \mathbf{\Sigma} = \begin{bmatrix} 2 & \sqrt{3} \\ \sqrt{3} & 4 \end{bmatrix}.$$

(a) Determine the first and second principal components Y_1 and Y_2, and find their variances, respectively.
(b) Determine the proportion of total variance due to the Y_1.
(c) Compare the contributions of X_1 and X_2 to the determination of Y_1 based on loadings and correlations, respectively.

Question 2 Suppose that the random vector $\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$ has the following population mean and covariance matrix:

$$\boldsymbol{\mu} = \mathbf{0}, \quad \mathbf{\Sigma} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}.$$

(a) Determine the first two principal components Y_1 and Y_2 and the proportion of total variance due to them. For $i = 1, 2$, compare the contributions of X_1, X_2 and X_3 to the determination of Y_i based on loadings and correlations, respectively.
(b) Let Z_1, Z_2, Z_3 be the standardized variables of X_1, X_2, X_3, respectively. Find the first two principal components W_1 and W_2 of (Z_1, Z_2, Z_3). For $i = 1, 2$, compare the contributions of Z_1, Z_2 and Z_3 to the determination of W_i based on loadings and correlations, respectively.

Question 3 Suppose a sample $\vec{x}_1, \ldots, \vec{x}_n$ has the sample mean $\overline{\vec{x}}$ and sample covariance $\boldsymbol{S}$. Let $\hat{y}_i$ be the i-th sample principal component, where $i = 1, \ldots, p$. Then we transform the data matrix

$$\boldsymbol{X} = \begin{bmatrix} \vec{x}_1^\top \\ \vec{x}_2^\top \\ \cdots \\ \vec{x}_n^\top \end{bmatrix}$$

into the data matrix of the first $r(< p)$ sample principal components

$$\widehat{\boldsymbol{Y}} = \begin{bmatrix} \hat{y}_{11} & \hat{y}_{12} & \cdots & \hat{y}_{1r} \\ \hat{y}_{21} & \hat{y}_{22} & \cdots & \hat{y}_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{y}_{n1} & \hat{y}_{n2} & \cdots & \hat{y}_{nr} \end{bmatrix}.$$

Please find the relationship between $\boldsymbol{X}$ and $\widehat{\boldsymbol{Y}}$ with the spectral decomposition of $\boldsymbol{S}$.

Question 4 You are given a sample $\vec{x}_1, \ldots, \vec{x}_6$ from a 2-dimension population. Moreover, the sample principal components are

$$\hat{y}_{i1} = \frac{\sqrt{2}}{2} x_{i1} + \frac{\sqrt{2}}{2} x_{i2}, \quad \hat{y}_{i2} = \frac{\sqrt{2}}{2} x_{i1} - \frac{\sqrt{2}}{2} x_{i2}, \ i = 1 \ldots, 6.$$

Assume the sample variances of $(\hat{y}_{i1})_{i=1}^6$ and $(\hat{y}_{i2})_{i=1}^6$ are 3 and 2, respectively. Fin the sample covariance matrix of the original data matrix $\boldsymbol{X}$.

Question 5 Consider a sample $\vec{x}_{11}, \ldots, \vec{x}_{1n_1}$ of size $n_1 = 10$ from population 1 (corresponding to class π_1) and a sample $\vec{x}_{21}, \ldots, \vec{x}_{2n_2}$ of size $n_2 = 10$ from population 2 (corresponding to class π_2). The summary statistics for these two samples are

$$\bar{\vec{x}}_1 = \begin{bmatrix} 6 \\ 0 \end{bmatrix}, \quad \boldsymbol{S}_1 = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix},$$

$$\bar{\vec{x}}_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \quad \boldsymbol{S}_2 = \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}.$$

For some $\vec{x}_0 = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}$, derive the following classifiers:

1. Classifier 1: Fisher's rule based on $\vec{x}_0$, and give the D^2 distance denoted as D_1^2;
2. Classifier 2: Fisher's rule based on x_{01}, and give the D^2 distance denoted as D_2^2;
3. Classifier 3: Fisher's rule based on x_{02}, and give the D^2 distance denoted as D_3^2;
4. Classifier 4: Fisher's rule based on $\vec{x}_0^\top (\bar{\vec{x}}_1 - \bar{\vec{x}}_2)$, and give the D^2 distance denoted as D_4^2;
5. Classifier 5: Fisher's rule based on $\vec{x}_0^\top \boldsymbol{S}_{pooled}^{-1} (\bar{\vec{x}}_1 - \bar{\vec{x}}_2)$, and give the D^2 distance denoted as D_5^2;
6. Compare the above D^2 distances.

Question 6 Suppose we have n_1 p-variate observations from π_1 and n_2 p-variate observations from π_2. The respective data matrices are

$$\boldsymbol{X}_1 = \begin{bmatrix} \vec{x}_{11}^\top \\ \vec{x}_{12}^\top \\ \vdots \\ \vec{x}_{1n_1}^\top \end{bmatrix}, \quad \boldsymbol{X}_2 = \begin{bmatrix} \vec{x}_{21}^\top \\ \vec{x}_{22}^\top \\ \vdots \\ \vec{x}_{2n_2}^\top \end{bmatrix}.$$

Suppose $\bar{\vec{x}}$ is the overall sample mean of these two samples. Consider the data matrices

$$\boldsymbol{Z}_1 = \begin{bmatrix} \vec{z}_{11}^\top \\ \vec{z}_{12}^\top \\ \vdots \\ \vec{z}_{1n_1}^\top \end{bmatrix}, \quad \boldsymbol{Z}_2 = \begin{bmatrix} \vec{z}_{21}^\top \\ \vec{z}_{22}^\top \\ \vdots \\ \vec{z}_{2n_2}^\top \end{bmatrix},$$

where $\vec{z}_{lj} = \vec{x}_{lj} - \bar{\vec{x}}$, $l = 1, 2$, $j = 1, \ldots, n_l$. Show that by Fisher's linear discriminant, $\vec{x}_0$ is allocated to the first population based on $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ if and only if $\vec{x}_0 - \bar{\vec{x}}$ is allocated to the first population based on $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$.

Question 7 Consider three independent samples from three classes:
π_1: distribution $\mathcal{N}_p(\vec{\mu}_1, \mathbf{\Sigma})$, sample size n_1, sample mean $\bar{\vec{x}}_1$, sample covariance $\boldsymbol{S}_1$;
π_2: distribution $\mathcal{N}_p(\vec{\mu}_2, \mathbf{\Sigma})$, sample size n_2, sample mean $\bar{\vec{x}}_2$, sample covariance $\boldsymbol{S}_2$;
π_3: distribution $\mathcal{N}_p(\vec{\mu}_3, \mathbf{\Sigma})$, sample size n_3, sample mean $\bar{\vec{x}}_3$, sample covariance $\boldsymbol{S}_3$.

Assume that these three populations have equal prior probabilities. For a new observation $\vec{x}_0$, we aim to classify it to one of the three classes with pairwise linear discriminant analyses based on Fisher's rule. Given the three population covariance matrices are assumed to be the same, in all linear discriminant analyses we use

$$\boldsymbol{S}_{pooled} = \frac{1}{n_1 + n_2 + n_3 - 3} \left((n_1 - 1)\boldsymbol{S}_1 + (n_2 - 1)\boldsymbol{S}_2 + (n_3 - 1)\boldsymbol{S}_3 \right).$$

Suppose that in the comparison between π_1 and π_2, $\vec{x}_0$ is allocated to π_2, while in the comparison between π_2 and π_3, $\vec{x}_0$ is allocated to π_3. Show that in the comparison between π_1 and π_3, $\vec{x}_0$ is allocated to π_3.