# Homework 1 (Due on 4/10)

**Question 1** Let

$$\boldsymbol{D} = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_p \end{bmatrix} \in \mathbb{R}^{p\times p} \quad \text{and} \quad \boldsymbol{A} = \begin{bmatrix} \vec{a}_1^\top \\ \vdots \\ \vec{a}_p^\top \end{bmatrix} \in \mathbb{R}^{p\times q}.$$

Show that

$$\boldsymbol{D}\boldsymbol{A} = \begin{bmatrix} d_1\vec{a}_1^\top \\ \vdots \\ d_p\vec{a}_p^\top \end{bmatrix}.$$

**Question 2** Let

$$\boldsymbol{D} = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_q \end{bmatrix} \in \mathbb{R}^{q\times q} \quad \text{and} \quad \boldsymbol{A} = \begin{bmatrix} \vec{a}_1 & \dots & \vec{a}_q \end{bmatrix} \in \mathbb{R}^{p\times q}.$$

Show that

$$\boldsymbol{A}\boldsymbol{D} = \begin{bmatrix} d_1\vec{a}_1 & \dots & d_q\vec{a}_q \end{bmatrix}.$$

**Question 3** Let

$$\boldsymbol{A} = \begin{bmatrix} \vec{a}_1 & \dots & \vec{a}_k \end{bmatrix} \in \mathbb{R}^{n\times k} \quad \text{and} \quad \boldsymbol{B} = \begin{bmatrix} \vec{b}_1^\top \\ \vdots \\ \vec{b}_k^\top \end{bmatrix} \in \mathbb{R}^{k\times p}.$$

Show that

$$\boldsymbol{A}\boldsymbol{B} = \vec{a}_1\vec{b}_1^\top + \vec{a}_2\vec{b}_2^\top + \ldots + \vec{a}_k\vec{b}_k^\top.$$

**Question 4** Let $\boldsymbol{Q}$ be a $q \times q$ matrix. Show that the row vectors are unit and pairwise perpendicular if and only if the column vectors are unit and pairwise perpendicular.

**Question 5** Let $\boldsymbol{A} = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$.

(a) Find the spectral decomposition of $\boldsymbol{A}$.

(b) Characterize the relationship between $a$ and $b$ such that $\boldsymbol{A}$ is invertible, and then find $\boldsymbol{A}^{-1}$ and its spectral decomposition.

**Question 6** Let $\boldsymbol{A} = \begin{bmatrix} 2 & 2 \\ -3 & 5 \\ 5 & -3 \\ -4 & -4 \end{bmatrix}$.

(a) Calculate $\boldsymbol{A}^\top\boldsymbol{A}$ and find its spectral decomposition.

(b) Compute the eigenvalues of $\boldsymbol{A}\boldsymbol{A}^\top$ with R and compare them with those of $\boldsymbol{A}^\top\boldsymbol{A}$.

**Question 7**

(a) Let $\boldsymbol{S}$ be a $k \times k$ invertible symmetric matrix, and $\boldsymbol{C}$ be a $k \times k$ invertible matrix. Moreover, let $\vec{x}$ be a $k$-dimensional vector. Show the following equality

$$(\boldsymbol{C}\vec{x})^\top (\boldsymbol{C}\boldsymbol{S}\boldsymbol{C}^\top)^{-1}(\boldsymbol{C}\vec{x}) = \vec{x}^\top \boldsymbol{S}^{-1}\vec{x}.$$

(b) Set

$$\boldsymbol{S} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}.$$

Calculate $(\boldsymbol{C}\vec{x})^\top (\boldsymbol{C}\boldsymbol{S}\boldsymbol{C}^\top)^{-1}(\boldsymbol{C}\vec{x})$ and $\vec{x}^\top \boldsymbol{S}^{-1}\vec{x}$. Does your answer contradict the claim in part (a)? Explain.