Homework 2 (Due on 4/17)

Question 1 Suppose that $\boldsymbol{A} \in \mathbb{R}^{k \times k}$ can be written as

$$\boldsymbol{A} = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^\top$$

where

$$\boldsymbol{P} = \begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_k \end{bmatrix}$$

is an orthogonal matrix, and

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix}$$

is a diagonal matrix. Show that

- $\boldsymbol{A}$ is a symmetric matrix;
- $\lambda_1, \dots, \lambda_k$ are eigenvalues of $\boldsymbol{A}$, and $\vec{v}_1, \dots, \vec{v}_k$ are corresponding eigenvectors, respectively.

Question 2 Suppose $\boldsymbol{A} \in \mathbb{R}^{k \times k}$ is symmetric matrix. Let

$$\boldsymbol{A} = \lambda_1 \vec{v}_1 \vec{v}_1^\intercal + \dots + \lambda_k \vec{v}_k \vec{v}_k^\intercal = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^\intercal$$

be the spectral decomposition. If $\lambda_1, \dots \lambda_k$ are all nonzero, show that

$$\boldsymbol{A}^{-1} = \frac{1}{\lambda_1} \vec{v}_1 \vec{v}_1^\intercal + \dots + \frac{1}{\lambda_k} \vec{v}_k \vec{v}_k^\intercal = \boldsymbol{P}\boldsymbol{\Lambda}^{-1}\boldsymbol{P}^\intercal,$$

where

$$\boldsymbol{\Lambda}^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{\lambda_k} \end{bmatrix}.$$

Question 3 Let $\boldsymbol{A} = \sum_{i=1}^k \lambda_i \vec{v}_i \vec{v}_i^\top$ be the spectral decomposition with positive eigenvalues $\lambda_1, \dots, \lambda_k > 0$. Set

$$\boldsymbol{P} = \begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_k \end{bmatrix}, \quad \boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix}.$$

Prove the following properties:

1. $\boldsymbol{A}^{\frac{1}{2}}$ is symmetric and $\boldsymbol{A}^{\frac{1}{2}} = \boldsymbol{P}\boldsymbol{\Lambda}^{\frac{1}{2}}\boldsymbol{P}^\top$ is its spectral decomposition;
2. $\boldsymbol{A}^{\frac{1}{2}}\boldsymbol{A}^{\frac{1}{2}} = \boldsymbol{A}$;

3. Denote $\boldsymbol{A}^{-\frac{1}{2}} = \left(\boldsymbol{A}^{\frac{1}{2}}\right)^{-1}$. Then

$$\boldsymbol{A}^{-\frac{1}{2}} = \sum_{i=1}^{k} \frac{1}{\sqrt{\lambda_i}} \vec{v}_i \vec{v}_i^\top = \boldsymbol{P} \boldsymbol{\Lambda}^{-\frac{1}{2}} \boldsymbol{P}^\top,$$

where

$$\boldsymbol{\Lambda}^{-\frac{1}{2}} = \begin{bmatrix} \frac{1}{\sqrt{\lambda_1}} & & \\ & \ddots & \\ & & \frac{1}{\sqrt{\lambda_k}} \end{bmatrix}.$$

4. $\boldsymbol{A}^{-\frac{1}{2}} \boldsymbol{A}^{-\frac{1}{2}} = \boldsymbol{A}^{-1}$.

Question 4 Consider a p-variate sample with size n:

$$\vec{x}_1, \ldots, \vec{x}_n.$$

For some $\boldsymbol{C} \in \mathbb{R}^{q \times p}$ and $\vec{a} \in \mathbb{R}^q$, consider the linear transformation

$$\vec{y}_i = \boldsymbol{C} \vec{x}_i + \vec{a}, \quad i = 1, \ldots, n.$$

Furthermore, for some $\boldsymbol{D} \in \mathbb{R}^{r \times p}$ and $\vec{b} \in \mathbb{R}^r$, consider the linear transformation

$$\vec{z}_i = \boldsymbol{D} \vec{x}_i + \vec{b}, \quad i = 1, \ldots, n.$$

For any $j = 1, \ldots, q$ and $k = 1, \ldots, r$, denote by s_{Y_j, Z_k} the sample covariance between $\{y_{ij}\}_{i=1}^n$ and $\{z_{ik}\}_{i=1}^n$. Define the matrix

$$\boldsymbol{S}_{\vec{Y}, \vec{Z}} = \begin{bmatrix} s_{Y_1,Z_1} & s_{Y_1,Z_2} & \cdots & s_{Y_1,Z_r} \\ s_{Y_2,Z_1} & s_{Y_2,Z_2} & \cdots & s_{Y_2,Z_r} \\ \vdots & \vdots & \ddots & \vdots \\ s_{Y_q,Z_1} & s_{Y_q,Z_2} & \cdots & s_{Y_q,Z_r} \end{bmatrix}.$$

Show that

$$\boldsymbol{S}_{\vec{Y}, \vec{Z}} = \boldsymbol{C} \boldsymbol{S}_{\vec{X}} \boldsymbol{D}^\top,$$

where $\boldsymbol{S}_{\vec{X}}$ is the sample covariance matrix of $\vec{x}_1, \ldots, \vec{x}_n$.

Question 5 Let the observed sample covariance matrix of two variates $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ be

$$\boldsymbol{S} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}.$$

(a) Does there exist a scalar α, such that $X_2 + \alpha X_1$ has zero sample variance? Explain.
(b) Find α such that X_1 and $X_2 + \alpha X_1$ have zero sample correlation.