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(roope.anttila@oulu.fi)

Pointwise Assouad dimension for measures

Roope Anttila
Research Unit of Mathematical Sciences
P.O.Box 8000, FI-90014, University of Oulu, Finland

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1. Introduction

Originally, the Assouad dimension was defined as a means to investigate embedding problems of metric spaces [**?**] and it is still used today as an efficient tool in studying these kinds of problems [**?**]. In some sense, the Assouad dimension quantifies the size of the largest local parts of the space under investigation, which provides a heuristic on why it is so effective in the study of embedding problems: If the space has large local parts, it can not be embedded into a small space. In recent years, the Assouad dimension has gained increasing attention also in fractal geometry and dimension theory. The book by Fraser [**?**] collects these recent developments in one place and provides an introduction to the theory of Assouad dimensions in fractal geometry. As is usual in fractal geometry, the Assouad dimension of a space is closely connected to a dual notion of dimension for the measures supported on the space. This Assouad dimension of a measure has a similar intuition behind it as the Assouad dimension of a space, that is, it gives information on the largest local parts of the measure.

There are also many notions of pointwise dimension for measures, most important being the upper and lower local dimensions. Unlike the different notions of "global" dimension, which are concerned with the average size (e.g. in the case of Hausdorff and packing dimensions) or extremal size (in the case of Assouad and lower dimensions) of the measure on its entire support, these pointwise dimensions quantify the size of the measure around a given point. The upper and lower local dimensions can be thought of as the pointwise analogue of the Hausdorff and packing dimensions of the measure and our aim is to provide a natural pointwise analogue of the Assouad dimension, which captures the size of the largest scales of the measure at a given point. Due to the similarity in flavour to the Assouad dimension of a measure, we call this dimension the *pointwise Assouad dimension* of the measure (see Definition 3.1).

This paper is structured as follows: In Section 2 we establish some notation and recall basic results concerning Assouad dimensions of sets and measures. In Section 3 we introduce the pointwise Assouad dimension and discuss its properties in relation to the existing notions of dimension. In Section 4 we work in a general setting of quasi-Bernoulli measures supported on strongly separated self-conformal sets and prove an exact dimensionality property for the pointwise Assouad dimension of these measures. In Section 5 we compute the Assouad dimension of a class of place dependent invariant measures and use the results of the previous section to obtain the almost sure pointwise Assouad dimension. Finally, in Section 6 we show that the exact dimensionality property holds for doubling self-similar measures, when the strong separation condition is weakened to the open set condition. We also establish the property for self-affine measures on Bedford-McMullen carpets satisfying the very strong separation condition.

2. Preliminaries

Unless stated otherwise, we assume that (X, d) is a complete metric space. Since we assume the metric d to be fixed, we omit it from the notation and refer to (X, d) simply as X. The rest of this section is devoted to introducing some basic notation and results related to Assouad dimensions. Readers familiar with these concepts can easily skip to Section 3.

2.1. Notation

A closed ball centered at $x \in X$ and with radius $r > 0$ is denoted by $B(x, r)$. The space X is said to be *doubling*, if any ball $B(x, r)$ can be covered by N balls $B(x, \frac{r}{2})$, where the constant $N \in \mathbb{N}$ is independent of x and r. Unless stated otherwise, a measure always refers to a finite Borel measure fully supported on X. By the *support of* μ, denoted by $\operatorname{supp}(\mu)$, we mean the smallest closed set with full measure. If $f : X \to Y$ is a map from X to another metric space Y, we denote the *pushforward of the measure* μ under the map f by $f_*\mu := \mu \circ f^{-1}$. For constants C, we sometimes use the convention $C(\cdots)$, if we want to emphasize the dependence of C on the quantities inside the parentheses. To simplify notation in the latter sections, we write $A \lesssim B$ to mean that A is bounded from above by B multiplied by a uniform constant. Similarly, we say that $A \gtrsim B$, if $B \lesssim A$ and $A \approx B$ if $B \lesssim A \lesssim B$.

The *upper and lower local dimensions of a measure* μ *at* $x \in X$ are defined by

$$\overline{\dim}_{\text{loc}}(\mu, x) = \limsup_{r \to 0} \frac{\log \mu(B(x, r))}{\log r},$$

and

$$\underline{\dim}_{\text{loc}}(\mu, x) = \liminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r},$$

respectively. When the upper and lower limits agree, we denote the limit by $\dim_{\text{loc}}(\mu, x)$ and call it the *local dimension of the measure* μ *at* x. If $\dim_{\text{loc}}(\mu, x)$ is a constant for μ-almost every $x \in X$, we say that μ is *exact dimensional*, with the analogous definition for the upper and lower local dimensions.

2.2. Assouad dimension of sets and connection to weak tangents

Our main focus in this paper is dimensions of measures, but to place the results in a wider context, we recall the definition of the Assouad dimension of a set. One defines the *Assouad dimension of* $F \subset X$ by

$$\dim_{\mathrm{A}} F = \inf \Bigg\{ s > 0 \colon \exists C > 0, \text{ s.t. for all } x \in F,\ 0 < r < R, \\ N_r(B(x,R) \cap F) \leqslant \left(\frac{R}{r}\right)^s \Bigg\},$$

where $N_r(E)$ denotes the smallest number of open balls of diameter r needed to cover the set $E \subset X$. A convenient way to bound the Assouad dimension of a set from below is given by the weak tangent approach. Recall that a map $T \colon X \to X$ is a *similarity* if there exists $c > 0$, such that

$$d(T(x), T(y)) = cd(x,y),$$

for all $x, y \in X$. The constant c is called the *similarity ratio (of T)*. The concept of *weak tangent sets* gives an easy way to bound the Assouad dimension from below. For simplicity we give the definition when $X \subset \mathbb{R}^d$ and make a brief remark that weak tangents can be defined in complete metric spaces using pointed convergence in the Gromov-Hausdorff distance [**?** **?**]. A closed set $F \subset \mathbb{R}^d$ is said to be the *weak tangent* of a compact set $X \subset \mathbb{R}^d$ if there is a sequence of similarities $T_n \colon \mathbb{R}^d \to \mathbb{R}^d$, such that

$$T_n(X) \cap B(0,1) \to F,$$

in the Hausdorff distance. The collection of weak tangents of X is denoted by $\mathrm{Tan}(X)$. The following proposition gives an easy way to bound the Assouad dimension from below. For the proof in the general setting see e.g. [**?** , Proposition 6.1.5].

Proposition 2.1. *If $X \subset \mathbb{R}^d$ is compact, then* $\dim_{\mathrm{A}} X \geqslant \dim_{\mathrm{A}} F$, *for all* $F \in \mathrm{Tan}(X)$.

2.3. Assouad dimension of measures

One may define the *Assouad dimension of a measure* μ fully supported on X by

$$\dim_{\mathrm{A}} \mu = \inf \Bigg\{ s > 0 \colon \exists C > 0, \text{ s.t. for all } x \in X,\ 0 < r < R, \\ \frac{\mu(B(x,R))}{\mu(B(x,r))} \leqslant \left(\frac{R}{r}\right)^s \Bigg\}.$$

Originally, this was called the upper regularity dimension in [**?**], but due to the intimate connections between this notion of dimension and the Assouad dimension for sets, nowadays the term Assouad dimension of a measure is widely used. A simple volume argument implies that for a measure μ fully supported on a metric space X, we have the inequality $\dim_{\mathrm{A}} X \leqslant \dim_{\mathrm{A}} \mu$. Moreover, in [**? ?**] it was shown that

$$\dim_{\mathrm{A}} X = \inf\{\dim_{\mathrm{A}} \mu \colon \mu \text{ is a measure fully supported on } X\},$$

which further supports the current terminology.

The notion of the Assouad dimension of a measure is closely linked to the *doubling property*, which the measure μ is said to satisfy if there is a constant $C \geqslant 1$, such that for any $x \in X$, $r > 0$, we have

$$\mu(B(x, 2r)) \leqslant C\mu(B(x, r)). \tag{2.1}$$

Measures that satisfy (2.1) are called *doubling measures.* In fact, it is a simple exercise to show that a measure has finite Assouad dimension if and only if it is doubling [**?** , Lemma 4.1.1]. In practice, the following lemma is sometimes useful when showing that a given measure is doubling.

Lemma 2.2. *Let μ be a finite Borel measure fully supported on a compact set X. If there are constants $C_0 \geqslant 1, r_0 > 0$, such that*

$$\mu(B(x, 2r)) \leqslant C_0\mu(B(x, r)),$$

for all $x \in X$ and $0 < r < r_0$, then μ is doubling.

*Proof.*Let $C_0, r_0 > 0$ be as in the statement of the lemma and let $r \geqslant r_0$. Since X is compact, we may cover it by a finite collection $\mathcal{B}$ of balls of radius $\frac{r_0}{3}$, and since μ is fully supported on X, for any $x \in X$, there is at least one $B_x \in \mathcal{B}$, such that $B_x \subset B(x, r_0)$. Let $C' = \min\{\mu(B) \colon B \in \mathcal{B}\}$. Then

$$\frac{\mu(B(x, 2r))}{\mu(B(x, r))} \leqslant \frac{\mu(X)}{\mu(B(x, r_0))} \leqslant \frac{\mu(X)}{\mu(B_x)} \leqslant \frac{\mu(X)}{C'},$$

so the constant $C = \min\{\frac{\mu(X)}{C'}, C_0\}$ works in (2.1). □

For the pointwise Assouad dimension, which we introduce shortly, we obtain a correspondence between the *pointwise doubling property* of the measure and the measure having finite pointwise Assouad dimension. A measure μ is said to be *pointwise doubling at $x \in X$*, if there is a constant $C(x) \geqslant 1$, such that

$$\mu(B(x, 2r)) \leqslant C(x)\mu(B(x, r)).$$

Clearly all doubling measures are pointwise doubling, but in the following we construct a measure which shows that the converse is not true in general.

Example 2.3. Let $x_n = 2^{-n}$, and let $\mu = \sum_{n=0}^{\infty} 3^{-n}\delta_{-x_n} + 2^{-n}\delta_{x_n}$, where δ_x denotes the point mass centered at x. Clearly the measure is a finite Borel measure fully supported on the set $X = \{0\} \cup \bigcup_{n=0}^{\infty}\{x_n, -x_n\}$. First we observe that μ is not doubling. For this, let $y_k = -x_k$, and $r_k = 2^{-k}$, and note that $r_k \to 0$ as $k \to \infty$. Now

$$\begin{aligned}\frac{\mu(B(y_k, 2r_k))}{\mu(B(y_k, r_k))} &\geqslant \frac{\mu([-x_{k-1}, x_k])}{\mu([-x_{k-1}, 0])} = \frac{\sum_{n=k-1}^{\infty} 3^{-n} + \sum_{n=k}^{\infty} 2^{-n}}{\sum_{n=k-1}^{\infty} 3^{-n}} \\ &= \frac{\frac{9}{2}3^{-k} + 2^{1-k}}{\frac{9}{2}3^{-k}} = 1 + \frac{4}{9}\left(\frac{2}{3}\right)^{-k} \to \infty,\end{aligned}$$

as $k \to \infty$, which shows that μ is not doubling.

To see that μ is pointwise doubling at every point, we first consider the case $x = 0$. Let $r > 0$ and fix an integer k, such that $2^{-k} \leqslant r < 2^{-k+1}$, and note that $B(x, 2r) \subset [-x_{k-2}, x_{k-2}]$, and $[-x_k, x_k] \subset B(x, r)$. In particular

$$\begin{aligned}\frac{\mu(B(x, 2r))}{\mu(B(x, r))} &\leqslant \frac{\mu([-x_{k-2}, x_{k-2}])}{\mu([-x_k, x_k])} \leqslant \frac{\sum_{n=k-2}^{\infty} 3^{-n} + 2^{-n}}{\sum_{n=k}^{\infty} 3^{-n} + 2^{-n}} \\ &\leqslant \frac{2\sum_{n=k-2}^{\infty} 2^{-n}}{\sum_{n=k}^{\infty} 2^{-n}} \leqslant \frac{2^{4-k}}{2^{1-k}} = 8,\end{aligned}$$

which shows that μ is pointwise doubling at 0.

If $x \in X \setminus \{0\}$, then $\mu(\{x\}) > 0$ and clearly

$$\frac{\mu(B(x, 2r))}{\mu(B(x, r))} \leqslant \frac{\mu(X)}{\mu(\{x\})} < \infty.$$

Since the right hand side depends only on x and not on r, this shows that μ is pointwise doubling at x.

3. Pointwise Assouad dimension

Motivated by the definition of the Assouad dimension, define a pointwise variant of the Assouad dimension of a measure.

Definition 3.1. Let μ be a measure fully supported on a metric space X. For $x \in X$ we define the *pointwise Assouad dimension of* μ *at* x as

$$\dim_{\mathrm{A}}(\mu, x) = \inf \Bigg\{ s > 0 \colon \exists C(x) > 0, \text{ s.t. } \forall 0 < r < R, \\ \frac{\mu(B(x, R))}{\mu(B(x, r))} \leqslant C(x) \left(\frac{R}{r}\right)^s \Bigg\}.$$

Crucially, the constant C in Definition 3.1 may depend on the point x, but not on the scales $0 < r < R$. Similarly as the Assouad dimension captures information on the most homogeneous points and scales of the measure, the pointwise Assouad dimension captures information on the most homogeneous scales at each point. For clarity, we may from now on refer to the Assouad dimension of a measure as the global Assouad dimension. Pointwise doubling measures and measures with finite pointwise Assouad dimension are linked by the following proposition. The proof is similar to the analogous result for the global Assouad dimension [**?** , Proposition 3.1].

Proposition 3.2. *Let μ be a Borel measure fully supported on a metric space X. Then for $x \in X$, $\dim_{\mathrm{A}}(\mu, x)$ is finite if and only if μ is pointwise doubling.*

*Proof.*For the rest of the proof we fix $x \in X$. If $\dim_{\mathrm{A}}(\mu, x) < \infty$, then choose any $\dim_{\mathrm{A}}(\mu, x) < s < \infty$, so by definition, there is a constant $C(x)$, such that for all $r > 0$,

$$\frac{\mu(B(x, 2r))}{\mu(B(x, r))} \leqslant C(x) 2^s,$$

so μ is pointwise doubling.

For the other direction, let $0 < r < R$, and let $n \in \mathbb{N}$ be the unique integer, for which $2^n r \leqslant R \leqslant 2^{n+1} r$. Note that in particular, this implies that $n \leqslant \frac{\log \frac{R}{r}}{\log 2}$. Using this and the doubling property inductively, we get

$$\frac{\mu(B(x, R))}{\mu(B(x, r))} \leqslant \frac{\mu(B(x, 2^{n+1} r))}{\mu(B(x, r))} \leqslant C(x)^{n+1} \leqslant C(x) \left(\frac{R}{r}\right)^{\frac{\log C(x)}{\log 2}}.$$

Since this holds for any $0 < r < R$, we have $\dim_{\mathrm{A}}(\mu, x) \leqslant \frac{\log C(x)}{\log 2} < \infty$ proving the claim. □

Let us derive some basic properties of the pointwise Assouad dimension. We start with a relationship between the pointwise Assouad dimension and the classical local dimensions. This can be regarded as an analogue of the fact that

$$\dim_{\mathrm{H}} \mu \leqslant \dim_{\mathrm{P}} \mu \leqslant \dim_{\mathrm{A}} \mu,$$

in the global case [**?**], where the *Hausdorff dimension of* μ and the *packing dimension of* μ are defined as

$$\dim_{\mathrm{H}} \mu := \operatorname*{ess\,inf}_{x \in X} \underline{\dim}_{\mathrm{loc}}(\mu, x), \text{ and } \dim_{\mathrm{P}} \mu := \operatorname*{ess\,inf}_{x \in X} \overline{\dim}_{\mathrm{loc}}(\mu, x),$$

respectively.

Proposition 3.3. *Let μ be a finite measure fully supported on a metric space X. Then*

$$\underline{\dim}_{\mathrm{loc}}(\mu, x) \leqslant \overline{\dim}_{\mathrm{loc}}(\mu, x) \leqslant \dim_{\mathrm{A}}(\mu, x) \leqslant \dim_{\mathrm{A}} \mu,$$

for all $x \in X$.

*Proof.*The first and last inequalities follow straight from the definitions, so it suffices to prove the middle inequality. Fix $x \in X$, and let $s > \dim_{\mathrm{A}}(\mu, x)$ be arbitrary. Then by definition, there is a constant C depending only on x, such that for all $0 < r < R$,

$$\frac{\mu(B(x,r))}{\mu(B(x,R))} \geqslant C\left(\frac{r}{R}\right)^s.$$

In particular, by fixing R we see that

$$\mu(B(x,r)) \geqslant cr^s,$$

where $c = \frac{C\mu(B(x,R))}{R^s}$. This implies that for $r < R$

$$\frac{\log \mu(B(x,r))}{\log r} \leqslant s + \frac{\log c}{\log r},$$

and taking $r \to 0$ shows that $\overline{\dim}_{\mathrm{loc}}(\mu, x) \leqslant s$. Since $s > \dim_{\mathrm{A}}(\mu, x)$ was arbitrary, this finishes the proof. □

Remark 3.4. One can define the *pointwise lower dimension of* μ *at* x analogously to the lower dimension of a measure as

$$\begin{aligned}\dim_{\mathrm{L}}(\mu, x) = \sup\Bigg\{ s > 0 \colon \exists C(x) > 0,\ \text{s.t. } \forall 0 < r < R,& \\ \frac{\mu(B(x,R))}{\mu(B(x,r))} \geqslant C(x)\left(\frac{R}{r}\right)^s \Bigg\}&.\end{aligned}$$

Then the previous proposition is strengthened to

$$\dim_{\mathrm{L}} \mu \leqslant \dim_{\mathrm{L}}(\mu, x) \leqslant \underline{\dim}_{\mathrm{loc}}(\mu, x) \leqslant \overline{\dim}_{\mathrm{loc}}(\mu, x) \leqslant \dim_{\mathrm{A}}(\mu, x) \leqslant \dim_{\mathrm{A}} \mu,$$

for all $x \in X$, where the proofs of the lower bounds are similar to the proof of the previous Proposition. We will not, however, pursue the study of the pointwise lower dimension any further in this paper.

The next lemma is a simple consequence of the definition and will be used numerous times in the examples we provide.

Lemma 3.5. *Let* μ *be a finite measure. If* μ *has an atom at* $x \in X$, *then* $\dim_{\mathrm{A}}(\mu, x) = 0$.

*Proof.*Assume that μ has an atom at $x \in X$. Let $0 < r < R$, and note that

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \leqslant \frac{\mu(X)}{\mu(\{x\})} = \frac{\mu(X)}{\mu(\{x\})}\left(\frac{R}{r}\right)^0,$$

and since the constant $\frac{\mu(X)}{\mu(\{x\})}$ depends only on x, we have $\dim_{\mathrm{A}}(\mu, x) \leqslant 0$, which is enough to prove the claim. □

Next we begin investigating the relationship between the pointwise Assouad dimension and the global one. As Proposition 3.3 shows, the global Assouad dimension provides an upper bound for the pointwise Assouad dimension at every point. The natural question that arrises is if the converse holds at some point, i.e. is it true that $\sup_{x\in X}\dim_{\mathrm{A}}(\mu,x)=\dim_{\mathrm{A}}\mu$. It turns out that generally speaking this is not the case, even in compact spaces. Recall that in Example 2.3 we constructed a measure which is pointwise doubling at all points of its compact support, but fails to be globally doubling. By Proposition 3.2 and the analogous fact for the global Assouad dimension, we see that this is an example of a measure with finite pointwise Assouad dimension at all points, but infinite global Assouad dimension.

After making this observation, it is natural to ask if the Assouad dimension of the supporting space is a lower bound for the pointwise Assouad dimension, as it is for the global one. Our next example shows that this is also not generally the case, in fact, there are measures supported on sets of full Assouad dimension, which have 0 pointwise Assouad dimension at all points. The example is original, but builds on a construction by Le Donne and Rajala [**?** , Example 2.20].

Example 3.6. Let $x_{n,k}=2^{-2^n}+k4^{-2^n}$ and let $X=\{0\}\cup\bigcup_{n=0}^{\infty}\bigcup_{k=0}^{n-1}\{x_{n,k}\}$. Define the measure μ as

$$\mu=\sum_{n=0}^{\infty}\sum_{k=0}^{n-1}\frac{2^{-n}}{n}\delta_{x_{n,k}}.$$

It is straightforward to show that μ is a finite doubling measure fully supported on X. We show that $\dim_{\mathrm{A}}X=1$, and $\dim_{\mathrm{A}}(\mu,x)=0$, for every $x\in X$.

To show that $\dim_{\mathrm{A}}X=1$, by Proposition 2.1 it is enough to show that $[0,1]$ is a weak tangent for the set X. For each $n\in\mathbb{N}$ define a similarity $T_n:\mathbb{R}\to\mathbb{R}$ by

$$T_n(x)=n^{-1}4^{2^n}(x-2^{-2^n}),$$

and note that

$$T_n(X)\cap[0,1]=\bigcup_{k=0}^{n-1}\frac{k}{n}\to[0,1],$$

in the Hausdorff distance as $n\to\infty$. Thus $[0,1]$ is a weak tangent to X and by Proposition 2.1 $\dim_{\mathrm{A}}X\geqslant 1$.

Next we show that $\dim_{\mathrm{A}}(\mu,x)=0$, for every $x\in X$. Note that every point $x\in X\setminus\{0\}$ is an atom so by Lemma 3.5, $\dim_{\mathrm{A}}(\mu,x)=0$, so we only need to consider the case $x=0$. Fix $0<r<R<1$, and choose numbers $L,N\in\mathbb{N}$, such that

$$2^{-2^L}<r\leqslant 2^{-2^{L-1}},\quad 2^{-2^{N+1}}\leqslant R<2^{-2^N},$$

so in particular

$$2^{L-1} \leqslant \frac{\log r}{\log 2}, \quad 2^{N+1} \geqslant \frac{\log R}{\log 2}$$

Clearly $[0, x_{L+1,L}] \subset B(0,r)$ and $B(x,R) \subset [0, x_{N,N-1}]$, so

$$\frac{\mu(B(0,R))}{\mu(B(0,r))} \leqslant \frac{\mu([0,x_{N,N-1}])}{\mu([0,x_{L+1,L}])} \leqslant 2^{L-N+1} \leqslant 2\frac{\log r}{\log R}.$$

Note that for any $s > 0$, the function $\phi(t) = t^s \log t$ is decreasing for $0 < t < e^{-\frac{1}{s}}$, so for all $0 < r < R < e^{-\frac{1}{s}}$ we have

$$\frac{\mu(B(0,R))}{\mu(B(0,r))} \leqslant 2\left(\frac{R}{r}\right)^s.$$

Since this holds for arbitrary $s > 0$, we have $\dim_{\mathrm{A}}(\mu, 0) = 0$.

In [**?**], the authors define the *upper Minkowski dimension of the measure* μ as

$$\begin{aligned}\overline{\dim}_{\mathrm{M}}\mu = \inf\{s > 0 \colon &\text{there exists a constant } c > 0 \text{ such that} \\ &\mu(B(x,r)) \geqslant cr^s, \text{ for all } x \in \operatorname{supp}(\mu) \text{ and } 0 < r < 1\},\end{aligned}$$

and show in their Proposition 4.1 that when μ is compactly supported, this has the property that

$$\dim_{\mathrm{P}} \mu \leqslant \overline{\dim}_{\mathrm{M}}\mu \leqslant \dim_{\mathrm{A}} \mu.$$

Next we give an example showing that in general a measure may have upper Minkowski dimension strictly larger than the pointwise Assouad dimension.

Example 3.7. We give an example of a compactly supported measure μ, such that

$$\overline{\dim}_{\mathrm{M}}\mu > \sup_{x \in \operatorname{supp}(\mu)} \dim_{\mathrm{A}}(\mu, x).$$

Let $\mu = \sum_{n=0}^{\infty} 3^{-n}\delta_{-2^{-n}} + 2^{-n}\delta_{2^{-n}}$ be the measure of Example 2.3, which is a finite measure fully supported on $X = \{0\} \cup \bigcup_{n=0}^{\infty}\{\pm 2^{-n}\}$. Let us first show that $\overline{\dim}_{\mathrm{M}}\mu \geqslant \frac{\log 3}{\log 2}$. Let $s < \frac{\log 3}{\log 2}$ and choose n_0 large enough, such that

$$\frac{n \log 3}{(n+2)\log 2} > s,$$

for all $n \geqslant n_0$. Let $x_n = -2^{-n}$, and $r_n = 2^{-(n+2)}$. Note that now

$$\mu(B(x_n, r_n)) = 3^{-n} = r_n^{\frac{n \log 3}{(n+2)\log 2}} < r_n^s,$$

for all $n \geqslant n_0$. Since $r_n \to 0$ with n, this implies that $\overline{\dim}_{\mathrm{M}}\mu \geqslant s$, and taking $s \to \frac{\log 3}{\log 2}$ gives the claim.

Note that μ has an atom at every $x \in X \setminus \{0\}$, so by Lemma 3.5, $\dim_{\mathrm{A}}(\mu, x) = 0 < \overline{\dim}_{\mathrm{M}}\mu$. To finish off the proof, we show that $\dim_{\mathrm{A}}(\mu, 0) \leqslant 1$. Fix $0 < r < R$, and choose $L, N \in \mathbb{N}$, such that

$$2^{-L} < r \leqslant 2^{-L+1}, \quad 2^{-N-1} \leqslant R < 2^{-N}. \tag{3.1}$$

Now $B(0, R) \subset [-2^{-N}, 2^{-N}]$, and $[-2^{L}, 2^{L}] \subset B(0, r)$, so by equation (3.1)

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \leqslant \frac{\mu([-2^{-N}, 2^{-N}])}{\mu([-2^{-L}, 2^{-L}])} \leqslant \frac{2\sum_{n=N}^{\infty} 2^{-n}}{\sum_{n=L}^{\infty} 2^{-n}} = \frac{2^{2-N}}{2^{1-L}} \leqslant 8\left(\frac{R}{r}\right),$$

and therefore $\dim_{\mathrm{A}}(\mu, 0) \leqslant 1 < \frac{\log 3}{\log 2} \leqslant \overline{\dim}_{\mathrm{M}}\mu$.

Remark 3.8. By definition of $\dim_{\mathrm{P}} \mu$ and Proposition 3.3 we have the general relationship $\dim_{\mathrm{P}} \mu \leqslant \dim_{\mathrm{A}}(\mu, x)$, for μ-almost every x. In general there is no relationship between the upper Minkowski dimension of μ and the pointwise Assouad dimension. A non trivial example of the case where $\overline{\dim}_{\mathrm{M}}\mu < \dim_{\mathrm{A}}(\mu, x)$ is given by some self-affine measures supported on Bedford-McMullen carpets. See [**?** , Claim 5.1] and our Theorem 6.4.

4. Quasi-Bernoulli measures on self-conformal sets

In this section we study the behaviour of the pointwise Assouad dimension for projections of *quasi-Bernoulli* measures on *self-conformal sets*. Our goal is to show that these measures are doubling and that their pointwise Assouad dimensions and global Assouad dimensions agree almost everywhere. Let us start with some definitions.

4.1. Self-conformal sets

We start by defining self-conformal sets, which are a generalisation of self-similar sets. Recall that a IFS $\{\varphi_i\}_{i\in\Lambda}$ on $\mathbb{R}^d$ is called *self-conformal* if it satisfies the following assumptions:

(C1) There is an open set $\Omega \subset \mathbb{R}^d$, which is open, bounded and connected, and a compact set $X \subset \Omega$ with non-empty interior, such that

$$\varphi_i(X) \subset X,$$

for all $i \in \Lambda$.

(C2) For each $i \in \Lambda$, the map $\varphi_i \colon \Omega \to \Omega$ is a conformal $C^{1+\varepsilon}$-diffeomorphism. That is, for all $x \in \Omega$, the linear map $\varphi_i'(x)$ is a similarity, so for every $y \in \Omega$, we have

$$|\varphi_i'(x)y| = |\varphi_i'(x)||y|,$$

where $|\varphi_i'(x)|$ denotes the operator norm of the map $\varphi_i'(x)$.

The use of the open set Ω here is essential since contractive confromal maps defined on whole $\mathbb{R}^d$ are in fact similarities. The attractor F of an IFS satisfying (C1) and (C2) is called a *self-conformal set*. In the following we let $||\varphi'_{\mathbf{i}}|| = \sup_{x\in\Omega} |\varphi'_{\mathbf{i}}(x)|$. It follows from the fact that each φ_i is a contraction, that $||\varphi'_{\mathbf{i}}|| < 1$, for all $\mathbf{i} \in \Sigma_*$, and that for a fixed $\mathbf{i} \in \Sigma$, $||\varphi'_{\mathbf{i}|_n}||$ is strictly decreasing in n. Let us establish some key lemmas for the proof of our main theorem of this section.

Lemma 4.1. *A self-conformal set F satisfies the following.*

(i) *There exists a constant $C_1 \geqslant 1$ such that $|\varphi'_{\mathbf{i}}(x)| \leqslant C_1|\varphi'_{\mathbf{i}}(y)|$, for all $\mathbf{i} \in \Sigma_*$ and $x, y \in \Omega$.*

(ii) *There exists constants C_2 and $\delta > 0$, such that for any $x, y, z \in F$, with $|x - y| \leqslant \delta$, we have*

$$C_2^{-1}|\varphi'_{\mathbf{i}}(z)| \leqslant \frac{|\varphi_{\mathbf{i}}(x) - \varphi_{\mathbf{i}}(y)|}{|x-y|} \leqslant C_2|\varphi'_{\mathbf{i}}(z)|,$$

for all $\mathbf{i} \in \Sigma_*$.

(iii) *There is a constant $C_3 \geqslant 1$, such that*

$$C_3^{-1}\, ||\varphi'_{\mathbf{i}}|| \leqslant \operatorname{diam}(\varphi_{\mathbf{i}}(F)) \leqslant C_3\, ||\varphi'_{\mathbf{i}}||\,,$$

for all $\mathbf{i} \in \Sigma_*$.

*Proof.*The claims (1) and (2) are [**?** , Lemma 2.3]. The proof of (3) is also standard, and can be found for example in [**?**]. □

The first property in Lemma 4.1 is commonly called the *Bounded Distortion Property (BDP)*.

Lemma 4.2. *For all $\mathbf{i}, \mathbf{j} \in \Sigma_*$, we have*

$$C_1^{-1}||\varphi'_{\mathbf{i}}|| \cdot ||\varphi'_{\mathbf{j}}|| \leqslant ||\varphi'_{\mathbf{ij}}|| \leqslant C_1||\varphi'_{\mathbf{i}}|| \cdot ||\varphi'_{\mathbf{j}}||,$$

where $C_1 > 1$ is the constant of Lemma 4.1.

*Proof.*Using the chain rule, and the conformality of the IFS, it is easy to see that for all $x \in F$ we have

$$|\varphi'_{\mathbf{ij}}(x)| = |(\varphi_{\mathbf{i}} \circ \varphi_{\mathbf{j}})'(x)| = |\varphi'_{\mathbf{i}}(\varphi_{\mathbf{j}}(x)) \cdot \varphi'_{\mathbf{j}}(x)| = |\varphi'_{\mathbf{i}}(\varphi_{\mathbf{j}}(x))| \cdot |\varphi'_{\mathbf{j}}(x)|.$$

Applying Lemma 4.1 we get that for all $y \in F$

$$C_1^{-1}|\varphi'_{\mathbf{i}}(y)| \cdot |\varphi'_{\mathbf{j}}(x)| \leqslant |\varphi'_{\mathbf{ij}}(x)| \leqslant C_1|\varphi'_{\mathbf{i}}(y)| \cdot |\varphi'_{\mathbf{j}}(x)|.$$

The result follows by taking suprema. □

Remark 4.3. Let $\mathbf{i} \in \Sigma$ be n-periodic. Notice that, by applying the previous lemma iteratively, we have

$$C_1^{-k}||\varphi'_{\mathbf{i}|_n}||^k \leqslant ||\varphi'_{\mathbf{i}|_{kn}}|| \leqslant C_1^k||\varphi'_{\mathbf{i}|_n}||^k.$$

The exponential growth of the distortion is a problem in Section 5 where we want to establish a lower bound for the Assouad dimension of the measure we investigate. The following lemma provides us with a precise estimate for this purpose.

Lemma 4.4. *If $x = \pi(\mathbf{i})$, where $\mathbf{i}$ is n-periodic for some $n \in \mathbb{N}$, then for any $k \in \mathbb{N}$ we have*

$$|\varphi'_{\mathbf{i}|_{kn}}(x)| = |\varphi'_{\mathbf{i}|_n}(x)|^k.$$

*Proof.*Let $\mathbf{i} \in P(\Sigma)$ be n-periodic, and let $x = \pi(\mathbf{i})$. Note that this implies that

$$\varphi_{\mathbf{i}|_n}(x) = x. \tag{4.1}$$

Let $k \in \mathbb{N}$. Using the chain rule, (4.1) and the conformality of the IFS we find that

$$\begin{aligned}
|\varphi'_{\mathbf{i}|_{kn}}(x)| &= |((\varphi_{i_1} \ldots \circ \varphi_{i_n}) \circ \underset{k \text{ times}}{\ldots} \circ (\varphi_{i_1} \circ \ldots \circ \varphi_{i_n}))'(x)| \\
&= |(\varphi_{i_1} \circ \ldots \circ \varphi'_{i_n}(x)) \cdot \underset{k \text{ times}}{\ldots} \cdot (\varphi_{i_1} \circ \ldots \circ \varphi'_{i_n}(x))| \\
&= |\varphi'_{\mathbf{i}|_n}(x)^k| = |\varphi'_{\mathbf{i}|_n}(x)|^k.
\end{aligned}$$

□

4.2. Quasi-Bernoulli measures

Recall that a probability measure ν on Σ is said to be *quasi-Bernoulli* if there exists a uniform constant $C > 0$, such that for all $\mathbf{i}, \mathbf{j} \in \Sigma_*$, we have

$$C^{-1}\nu([\mathbf{i}])\nu([\mathbf{j}]) \leqslant \nu([\mathbf{ij}]) \leqslant C\nu([\mathbf{i}])\nu([\mathbf{j}]),$$

where $\mathbf{ij}$ denotes the concatenation of the finite words $\mathbf{i}$ and $\mathbf{j}$. Note the similarity to Lemma 4.2. The following lemma is standard, but for completeness we provide a sketch of the proof.

Lemma 4.5. *Every quasi-Bernoulli measure is equivalent to an σ-invariant and ergodic quasi-Bernoulli measure.*

*Proof.*Let ν be a quasi-Bernoulli measure. For all $n \in \mathbb{N}$ define

$$\nu'_n = \frac{1}{n}\sum_{j=0}^{n-1} \nu \circ \sigma^{-j},$$

and let ν' be the measure obtained as the weak-$*$ limit of this sequence of measures. Recall that for any $\mathbf{i} \in \Sigma_*$, the set $[\mathbf{i}]$ is open and closed and thus $\nu'([\mathbf{i}]) = \lim_{n\to\infty} \nu'_n([\mathbf{i}])$. Using the fact that ν is quasi-Bernoulli, we have

$$\nu_n'([\mathbf{i}]) = \frac{1}{n}\sum_{j=0}^{n-1} \nu\circ\sigma^{-j}([\mathbf{i}]) = \frac{1}{n}\sum_{j=0}^{n-1}\sum_{\mathbf{j}\in\Sigma_j} \nu([\mathbf{j}\mathbf{i}]) \approx \frac{1}{n}\sum_{j=0}^{n-1}\underbrace{\sum_{\mathbf{j}\in\Sigma_j}\nu([\mathbf{j}])}_{=1}\nu([\mathbf{i}]) = \nu([\mathbf{i}]),$$

for all $\mathbf{i}\in\Sigma_*$ and $n\in\mathbb{N}$, which shows that ν' is quasi-Bernoulli and equivalent to ν. On the other hand,

$$\begin{aligned}|\nu'([\mathbf{i}]) - \nu'(\sigma^{-1}([\mathbf{i}]))| &= \lim_{n\to\infty} |\nu_n'([\mathbf{i}]) - \nu_n'(\sigma^{-1}([\mathbf{i}]))|\\ &= \lim_{n\to\infty}\frac{1}{n}\left|\sum_{j=0}^{n-1}\nu\circ\sigma^{-j}([\mathbf{i}]) - \sum_{j=0}^{n-1}\nu\circ\sigma^{-(j+1)}([\mathbf{i}])\right|\\ &= \lim_{n\to\infty}\frac{1}{n}\left|\nu([\mathbf{i}]) - \nu\circ\sigma^{-n}([\mathbf{i}])\right| = 0.\end{aligned}$$

By passing to approximating cylinders, we see that this implies that ν' is σ-invariant.

A simple consequence of the quasi-Bernoulli property of ν' is that for all Borel sets $A, B\subset\Sigma$, and $\varepsilon>0$, there exists $n\in\mathbb{N}$, such that

$$\nu'(A\cap\sigma^{-n}(B)) \gtrsim \nu'(A)\nu'(B) - \varepsilon.$$

The ergodicity of ν' follows easily from this: Let $A\subset\Sigma$ be σ-invariant and assume towards contradiction that $0<\nu(A)<1$. Then for some $n\in\mathbb{N}$,

$$0 = \nu'(\Sigma\setminus A\cap A) = \nu'(\Sigma\setminus A\cap\sigma^{-n}(A)) \gtrsim \nu'(\Sigma\setminus A)\nu'(A) - \varepsilon > 0,$$

for some small enough $\varepsilon>0$, which gives a contradiction. □

For the rest of this section, let $\mathcal{N}\subset\Sigma$ denote the set of infinite words, which contain all finite words as a substring, that is

$$\mathcal{N} := \{\mathbf{i}\in\Sigma\colon \mathbf{j}\ll\mathbf{i},\ \text{for all } \mathbf{j}\in\mathbf{\Sigma}_*\}.$$

The following lemma is simple, but crucial to the proof of Theorem 4.8.

Lemma 4.6. *If ν is a quasi-Bernoulli measure, then $\nu(\Sigma\setminus\mathcal{N})=0$.*

*Proof.*By Lemma 4.5, we may assume without loss of generality that ν is σ-invariant and ergodic. Now for every $\mathbf{i}\in\Sigma$, we see by applying Birkhoff's ergodic theorem, that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n}\chi_{[\mathbf{i}]}(\sigma^n\mathbf{j}) = \int_\Sigma \chi_{[\mathbf{i}]}d\nu = \nu([\mathbf{i}]) > 0,$$

for ν-almost every $\mathbf{j}$, where $\chi_{[\mathbf{i}]}$ denotes the indicator function of the set $[\mathbf{i}]$. In particular, this implies that for almost every $\mathbf{j}$, there is $i \in \mathbb{N}$, such that $\chi_{[i]}(\sigma^n\mathbf{j}) = 1$, that is $\mathbf{i} \ll \mathbf{j}$. Let $\Sigma_\mathbf{i} = \{\mathbf{j} \in \Sigma \colon \mathbf{i} \ll \mathbf{j}\}$, so by the previous $\nu(\Sigma \setminus \Sigma_\mathbf{i}) = 0$. By definition of $\mathcal{N}$, we have

$$\nu(\Sigma \setminus \mathcal{N}) = \nu\left(\Sigma \setminus \bigcap_{\mathbf{i}\in\Sigma_*}\Sigma_\mathbf{i}\right) = \nu\left(\bigcup_{\mathbf{i}\in\Sigma_*}\Sigma\setminus\Sigma_\mathbf{i}\right) \leqslant \sum_{\mathbf{i}\in\Sigma_*}\nu(\Sigma\setminus\Sigma_\mathbf{i}) = 0.$$

□

We say that a measure μ supported on a self-conformal set F is quasi-Bernoulli if it is the projection of a quasi-Beronulli measure ν supported on Σ under the natural projection $\pi\colon \Sigma \to F$ defined in the same manner as in Equation (6.2). Next we show that the quasi-Bernoulli measures supported on self-conformal sets satisfying the SSC are doubling, which in particular implies that the Assouad dimensions of these measures are finite. After that, we prove the main theorem of this section, which shows that the pointwise Assouad dimension of these measures agrees with the Assouad dimension almost everywhere.

Proposition 4.7. *If μ is a quasi-Bernoulli measure fully supported on a self-conformal set F satisfying the strong separation condition, then μ is doubling.*

*Proof.*Let $\delta = \min_{i\neq j} d(\varphi_i(F), \varphi_j(F))$, which is positive since F is strongly separated. Fix an integer k satisfying $\max_{\mathbf{i}\in\Sigma_k}||\varphi'_\mathbf{i}|| < \frac{\delta}{2C_3^3}$, where C_3 is the constant given by Lemma 4.1(3). Finally, let $C = \min_{\mathbf{i}\in\Sigma_{k+1}}\nu([\mathbf{i}])$.

Note that since F is compact and μ is fully supported on F by Lemma 2.2 it suffices to consider only uniformly small values of $r > 0$. Therefore let

$$0 < r < \min\{\left|\left|\varphi'_{\mathbf{i}|_k}\right|\right| : \mathbf{i}\in\Sigma_k\}.$$

Note that the right hand side is positive since X is compact so this is possible. Also fix $x \in F$, let $\mathbf{i} \in \Sigma$ be such that $\pi(\mathbf{i}) = x$, and choose $n \in \mathbb{N}$ as the unique integer satisfying $C_3||\varphi'_{\mathbf{i}|_n}|| < r \leqslant C_3||\varphi'_{\mathbf{i}|_{n-1}}||$. This immediately implies that $\varphi_{\mathbf{i}|_n}(F) \subset B(x, r)$. Note that by the assumption on r we have

$$\left|\left|\varphi'_{\mathbf{i}|_n}\right|\right| < \left|\left|\varphi'_{\mathbf{i}|_k}\right|\right|.$$

so in particular $n > k$. For any $l \in \mathbb{N}$ and $i \in \Lambda$ let $\mathbf{i}|_l i$ denote the word $(i_1, i_2, \ldots, i_l, i)$ and notice that by the strong separation condition and Lemma 4.1(3), we have for all $l \in \mathbb{N}$ and $i \neq j$ that

$$d(\varphi_{\mathbf{i}_l i}(F)), \varphi_{\mathbf{i}_l j}(F)) := \inf_{\substack{x\in\varphi_{\mathbf{i}_l i}(F)\\ y\in\varphi_{\mathbf{i}_l j}(F)}} |x-y| \geqslant \delta\cdot\operatorname{diam}(\varphi_{\mathbf{i}|_l}(F)) \geqslant \frac{\delta}{C_3}\left|\left|\varphi'_{\mathbf{i}|_l}\right|\right|. \tag{4.2}$$

Now using Lemma 4.2, we have for every $y\in B(x,2r)$

$$\begin{aligned} d(y,\varphi_{\mathbf{i}|_{n-k-1}}(F)) &\leqslant 2r < \frac{\delta}{C_3^3\max_{\mathbf{i}\in\Sigma_k}||\varphi'_{\mathbf{i}}||}r \leqslant \frac{\delta}{C_3^2\max_{\mathbf{i}\in\Sigma_k}||\varphi'_{\mathbf{i}}||}\left|\left|\varphi'_{\mathbf{i}|_{n-1}}\right|\right| \\ &\leqslant \frac{\delta}{C_3^2}\frac{\left|\left|\varphi'_{\mathbf{i}|_{n-1}}\right|\right|}{\left|\left|\varphi'_{\sigma^{n-k-1}\mathbf{i}|_k}\right|\right|} \leqslant \frac{\delta}{C_3}\left|\left|\varphi'_{\mathbf{i}|_{n-k-1}}\right|\right|, \end{aligned}$$

in particular, combining this with estimate (4.2), we have $B(x,2r)\cap F\subset\varphi_{\mathbf{i}|_{n-k-1}}(F)$. Therefore, using the quasi-Bernoulli property, we have

$$\frac{\mu(B(x,2r))}{\mu(B(x,r))} \leqslant \frac{\mu(\varphi_{\mathbf{i}|_{n-k-1}}(F))}{\mu(\varphi_{\mathbf{i}|_n}(F)))} = \frac{\nu([\mathbf{i}|_{n-k-1}])}{\nu([\mathbf{i}|_n])} \leqslant \frac{1}{C_3\nu([\sigma^{n-k-2}\mathbf{i}|_{k+1}])} \leqslant \frac{1}{C_3C}.$$

Since the upper bound is independent of x and r, the claim follows. □

Our main result of this section is a generalization of Corollary 6.2, with the difference that we do not obtain an explicit value for the pointwise Assouad dimension. However, in Section 5 we provide an example of a measure which satisfies the assumptions of the following theorem, and calculate an explicit value for its Assouad dimension.

Theorem 4.8. *If μ is a quasi-Bernoulli measure fully supported on a self-conformal set F satisfying the strong separation condition, then*

$$\dim_{\mathrm{A}}(\mu,x) = \dim_{\mathrm{A}}\mu < \infty,$$

for μ-almost every $x\in F$.

Proof. It follows from Proposition 4.7 that $\dim_{\mathrm{A}}\mu$ is finite. Let $s<\dim_{\mathrm{A}}\mu$ and $C>0$. Now there is a point $y\in F$ and radii $0<r<R$, satisfying

$$\frac{\mu(B(y,R))}{\mu(B(y,r))} > C\left(\frac{R}{r}\right)^s.$$

Let $\mathbf{i}\in\mathcal{N}$, $x=\pi(\mathbf{i})$ and let $\mathbf{j}\in\Sigma$, such that $\pi(\mathbf{j})=y$. Now choose $k,n\in\mathbb{N}$ as the unique integers which satisfy

$$||\varphi'_{\mathbf{j}|_{n+1}}|| \leqslant R < ||\varphi'_{\mathbf{j}|_n}||, \text{ and } \quad ||\varphi'_{\mathbf{j}|_k}|| < r \leqslant ||\varphi'_{\mathbf{j}|_{k-1}}||.$$

Then $\varphi_{\mathbf{j}|_k}(F)\subset B(y,C_3r)$ and $B(y,\frac{\delta}{2C_3}R)\cap F\subset\varphi_{\mathbf{j}|_n}(F)$, where C_3 is the constant of Lemma 4.1(3). Using the quasi-Bernoulli property and the fact that μ is doubling, we get that

$$C\left(\frac{R}{r}\right)^s < \frac{\mu(B(y,R))}{\mu(B(y,r))} \lesssim \frac{\mu(\varphi_{\mathbf{j}|_n}(F))}{\mu(\varphi_{\mathbf{j}|_k}(F))} = \frac{\nu([\mathbf{j}|_n])}{\nu([\mathbf{j}|_k])} \leqslant C\nu([\sigma^{k-n}\mathbf{j}|_n])^{-1}.$$

Now since $\mathbf{i} \in \mathcal{N}$, there is an index $l \in \mathbb{N}$, such that $\sigma^l \mathbf{i}|_n = \sigma^{k-n}\mathbf{j}|_n$. Let $R' = ||\varphi'_{\mathbf{i}|_l}||$ and $r' = ||\varphi'_{\mathbf{i}|_{l+n}}||$, and observe that by Lemma 4.2,

$$\frac{R'}{r'} = \frac{||\varphi'_{\mathbf{i}|_l}||}{||\varphi'_{\mathbf{i}|_{l+n}}||} \lesssim \frac{1}{||\varphi'_{\sigma^l \mathbf{i}|_n}||} = \frac{1}{||\varphi'_{\sigma^{k-n}\mathbf{j}|_n}||} \lesssim \frac{R}{r}.$$

Again, it is easy to see that $\varphi_{\mathbf{i}|_l}(F) \subset B(x, C_3 R')$ and $B(x, \frac{\delta}{2C_3} r') \cap F \subset \varphi_{\mathbf{i}|_{l+n}}(F)$, so using the doubling and quasi-Bernoulli properties of μ, we see that

$$\begin{aligned}\frac{\mu(B(x,R'))}{\mu(B(x,r'))} &\gtrsim \frac{\mu(\varphi_{\mathbf{i}|_l}(F))}{\mu(\varphi_{\mathbf{i}|_{l+n}}(F))} = \frac{\nu([\mathbf{i}|_l])}{\nu([\mathbf{i}|_{l+n}])} \gtrsim \nu([\sigma^l \mathbf{i}|_n])^{-1} \\ &= \nu([\sigma^{k-n}\mathbf{j}|_n])^{-1} \gtrsim C\left(\frac{R}{r}\right)^s \gtrsim C\left(\frac{R'}{r'}\right)^s.\end{aligned}$$

This shows that $\dim_{\mathrm{A}}(\mu, x) \geqslant s$, and taking $s \to \dim_{\mathrm{A}} \mu$ gives $\dim_{\mathrm{A}}(\mu, x) \geqslant \dim_{\mathrm{A}} \mu$. Since $\mathbf{i} \in \mathcal{N}$, the claim follows from Lemma 4.6. $\square$

5. Invariant measures for iterated function systems with place dependent probabilities

In this section, we study the class of place dependent invariant measures supported on strongly separated self-conformal sets. The results of Section 4 show that these measures are doubling and that their pointwise Assouad dimension coincides with the global Assouad dimension at almost every point. Our main result of this section, gives an explicit formula for the Assouad dimension of these measures. Let us begin by defining our setting.

5.1. Place dependent measures

We assume that our IFS is self-conformal, that is it satisfies (C1) and (C2). In contrast to the case of self-conformal measures where we concentrate a uniform measure p_i on the set $\varphi_i(F)$, now we allow the mass concentration to depend continuously on the point, that is we choose for each $i \in \Lambda$ a Hölder continuous function $p_i \colon X \to (0,1)$, which satisfy $\sum_{i\in\Lambda} p_i(x) \equiv 1$ and consider the probability measures satisfying the equation

$$\int f(x)d\mu(x) = \sum_{i\in\Lambda} \int p_i(x) f\circ\varphi_i(x)d\mu(x), \tag{5.1}$$

for $f \in C(X)$ where here and hereafter $C(X)$ denotes the set of continuous real valued functions on X. We define the *Ruelle operator* $T \colon C(X) \to C(X)$ by

$$(Tf)(x) = \sum_{i \in \Lambda} p_i(x) f \circ \varphi_i(x), \tag{5.2}$$

and let $T^* \colon M(X) \to M(X)$ denote the adjoint operator, where $M(X)$ is the set of Borel probability measures on X. Recall that for $\nu \in M(X)$, $T^*\nu$ is given by

$$T^*\nu(B) = \sum_{i \in \Lambda} \int_{\varphi_i^{-1}(B)} p_i(x) d\nu(x),$$

for all Borel subsets $B \subset X$. Barnsley et al. [**?**] as well as Fan and Lau [**?**] have studied the measures which are invariant under T in a setting which is more general than ours. The next proposition, which is vital to this section, is a special case of [**?** , Theorem 1.1] or [**?** , Theorem 2.1] and we refer to the mentioned papers for the proof.

Proposition 5.1. *Let F be a self-conformal set satisfying the SSC and $p_i \colon X \to (0,1)$ be Hölder continuous for every $i \in \Lambda$. Then there is a unique Borel probability measure μ satisfying*

$$T^*\mu = \mu.$$

Furthermore, for every $f \in C(X)$, $T^n f$ converges uniformly to the constant $\int f(x) d\mu(x)$.

The measure μ is called an *invariant measure with place dependent probabilities*, which we shorten to just *invariant measure* for the remainder of this section. As was the case with self-similar measures and Bernoulli measures on the corresponding code space, there is also a natural correspondence between the invariant measure μ and a Gibbs measure on the code space. Let us define some useful notation for this section. For $\mathbf{i} \in \Sigma$ we slightly abuse notation by writing $p_i(\mathbf{i}) \coloneqq p_i(\pi(\mathbf{i}))$ and $\varphi_i(\mathbf{i}) \coloneqq \varphi_i(\pi(\mathbf{i}))$, where $\pi \colon \Sigma \to F$ is the natural projection given by (6.2). For $\mathbf{i} \in \Sigma$ and $n \in \mathbb{N}$ we let

$$p_{\mathbf{i}|_n}(\mathbf{i}) = \prod_{k=1}^{n} p_{i_k}(\sigma^{k-1}\mathbf{i}).$$

Let $P(\Sigma) \subset \Sigma$ denote the set of periodic points of Σ. For $\mathbf{i} \in P(\Sigma)$ with period of length n, we let

$$\overline{p}_{\mathbf{i}} = p_{\mathbf{i}|_n}(\mathbf{i}),$$

and

$$|\varphi'_{\mathbf{i}}| = |\varphi'_{\mathbf{i}|_n}(\mathbf{i})|.$$

The following lemma is well known [**? ?**], and it follows from the Ruelle-Perron-Frobenius theorem. Following ideas of [**?**], we present a short outline of the proof for the convenience of the reader.

Lemma 5.2. *There exists a unique σ-invariant probability measure ν on Σ, and a constant $C > 1$ such that for any $x \in F$, $\mathbf{i}, \mathbf{j} \in \Sigma$ and $n \geqslant 1$, we have*

$$C^{-1} p_{\mathbf{i}|_n}(\mathbf{j}) \leqslant \nu([\mathbf{i}|_n]) \leqslant C p_{\mathbf{i}|_n}(\mathbf{j}).$$

Furthermore, ν is quasi-Bernoulli and we have $\mu = \pi_* \nu$, where μ is the measure of Proposition 5.1.

*Proof.*It is easy to see that the map $\phi(\mathbf{i}) := \log p_{i_0}(\sigma \mathbf{i})$ is Hölder continuous, so by Theorem 1.4 of [**?**], there exists a unique σ-invariant probability measure ν, and a constants $P > 0$ and $C > 1$, such that

$$C^{-1} \leqslant \frac{\nu([\mathbf{i}|_n])}{\exp\left(-nP + \sum_{k=1}^{n} \phi(\sigma^{k-1}\mathbf{i})\right)} \leqslant C, \tag{5.3}$$

for all $\mathbf{i} \in \Sigma$. This measure is called the Gibbs measures of the potential ϕ. Define the Ruelle operator $\tilde{T} \colon C(\Sigma) \to C(\Sigma)$ of the symbolic representation by

$$(\tilde{T} f)(\mathbf{i}) = \sum_{i \in \Lambda} \exp(\phi(i\mathbf{i})) f(i\mathbf{i}) = \sum_{i \in \Lambda} p_i(\mathbf{i}) f(i\mathbf{i}).$$

Since $\sum_{i \in \Lambda} p_i(\mathbf{i}) = 1$, for all $\mathbf{i} \in \Sigma$, it follows that $\tilde{T} h = h$, when $h \equiv 1$, that is, h is the eigenfunction of $\tilde{T}$ corresponding to the maximal eigenvalue 1. By inspecting the proof of Theorem 1.16 of [**?**], we see that in this case $P = 0$, so equation (5.3) gets simplified to

$$C^{-1} \leqslant \frac{\nu([\mathbf{i}|_n])}{\prod_{k=1}^{n} p_{i_k}(\sigma^{k-1}\mathbf{i})} \leqslant C.$$

The claim then follows easily by the Hölder continuity of the maps p_i. The fact that ν is quasi-Bernoulli is also a simple consequence of this.

The fact that $\pi_* \nu = \mu$ was proved in Proposition 1.3(ii) of [**?**], but for completeness we present a version of their short argument here. Clearly $\pi_* \nu$ is a Borel probability measure. Using Theorem 1.4 of [**?**] again, we see that $\tilde{T}^* \nu = \nu$, so for any $f \in C(X)$, we have

$$\begin{aligned}
\int_X f(x) d\pi_* \nu &= \int_\Sigma f \circ \pi(\mathbf{i}) d\nu = \int_\Sigma (\tilde{T}(f \circ \pi))(\mathbf{i}) d\nu = \int_\Sigma \sum_{i \in \Lambda} p_i(\mathbf{i}) f(\pi(i\mathbf{i})) d\nu \\
&= \int_\Sigma \sum_{i \in \Lambda} p_i(\mathbf{i}) f \circ \varphi_i(\pi(\mathbf{i})) d\nu = \int_X \sum_{i \in \Lambda} p_i(x) f \circ \varphi_i(x) d\pi_* \nu \\
&= \int_X (Tf)(x) d\pi_* \nu,
\end{aligned}$$

so $T^* \pi_* \nu = \pi_* \nu$. Since by Proposition 5.1 μ is the unique measure with this property, this shows that $\mu = \pi_* \nu$. □

We are now ready to compute the Assouad dimension of the place dependent invariant measures. To our knowledge, the value of the dimension is not previously found in the literature.

Theorem 5.3. *Let μ be a place dependent invariant measure fully supported on a self-conformal set F, which satisfies the SSC. Then*

$$\dim_{\mathrm{A}} \mu = \sup_{\mathbf{i} \in P(\Sigma)} \frac{\log \overline{p}_{\mathbf{i}}}{\log |\varphi'_{\mathbf{i}}|}.$$

*Proof.*Let us start with the upper bound. For the rest of the proof let $s = \sup_{\mathbf{i} \in P(\Sigma)} \frac{\log \overline{p}_{\mathbf{i}}}{\log |\varphi'_{\mathbf{i}}|}$. Let $x \in F$ and $\mathbf{i} \in \Sigma$, such that $\pi(\mathbf{i}) = x$. Let $0 < r < R$ and choose integers k and n which satisfy

$$|\varphi'_{\mathbf{i}|_{n+1}}(x)| \leqslant R < |\varphi'_{\mathbf{i}|_n}(x)|, \text{ and } \quad |\varphi'_{\mathbf{i}|_k}(x)| < r \leqslant |\varphi'_{\mathbf{i}|_{k-1}}(x)|.$$

This immediately implies that $\varphi_{\mathbf{i}|_k}(F) \subset B(x, C_3 r)$, where C_3 is the constant of Lemma 4.1(3). As before, let $\delta = \min_{i \neq j} d(\varphi_i(F), \varphi_j(F)) \colon i \neq j\}$. Then it is also easy to see that $B(x, \frac{\delta}{2C_3} R) \cap F \subset \varphi_{\mathbf{i}|_n}(F)$. Let us set $\mathbf{j} = (i_{k-n+1}, i_{k-n+2}, \ldots, i_k, i_{k-n+1}, i_{k-n+2}, \ldots, i_k, \ldots) \in P(\Sigma)$. Note that Lemma 5.2 shows that

$$\prod_{j=k-n+1}^{k} p_{i_j}(\sigma^{j-1}\mathbf{i}) \gtrsim \prod_{l=1}^{n} p_{j_l}(\sigma^{l-1}\mathbf{j}).$$

Using Proposition 4.7 and Lemmas 4.1(1) and 4.2, we get

$$\begin{aligned}
\frac{\mu(B(x,R))}{\mu(B(x,r))} &\lesssim \frac{\mu(B(x, \frac{\delta}{2C_3} R))}{\mu(B(x, C_3 r))} \leqslant \frac{\mu(\varphi_{\mathbf{i}|_n}(F))}{\mu(\varphi_{\mathbf{i}|_k}(F))} \lesssim \frac{p_{\mathbf{i}|_n}(\mathbf{i})}{p_{\mathbf{i}|_k}(\mathbf{i})} \\
&= \frac{\prod_{j=1}^{n} p_{i_j}(\sigma^{j-1}\mathbf{i})}{\prod_{j=1}^{k} p_{i_j}(\sigma^{j-1}\mathbf{i})} = \left(\prod_{j=k-n+1}^{k} p_{i_j}(\sigma^{j-1}\mathbf{i}) \right)^{-1} \\
&\lesssim \left(\prod_{l=1}^{n} p_{j_l}(\sigma^{l-1}\mathbf{j}) \right)^{-1} = |\varphi'_{\mathbf{j}|_n}(\mathbf{j})|^{-\frac{\log \prod_{l=1}^{n} p_{j_l}(\sigma^{l-1}\mathbf{j})}{\log |\varphi'_{\mathbf{j}|_n}(\mathbf{j})|}} \\
&\leqslant |\varphi'_{\mathbf{j}|_n}(\mathbf{j})|^{-s} \lesssim |\varphi'_{\mathbf{i}|_{k-n-1}}(x)|^{-s} \lesssim \left(\frac{|\varphi'_{\mathbf{i}|_n}(x)|}{|\varphi'_{\mathbf{i}|_k}(x)|} \right)^{s} \\
&\lesssim \left(\frac{R}{r} \right)^{s}.
\end{aligned}$$

This shows that $\dim_{\mathrm{A}}(\mu, x) \leqslant s$, for any $x \in F$.

For the lower bound, let $t < s$, and choose $\mathbf{i} \in P(\Sigma)$, such that

$$\frac{\log \overline{p}_{\mathbf{i}}}{\log |\varphi'_{\mathbf{i}|_n}(x)|} \geqslant t,$$

where $x = \pi(\mathbf{i})$ and n is the period of $\mathbf{i}$. For every $k \in \mathbb{N}$ let $r_k = |\varphi'_{\mathbf{i}_{kn}}(x)|$. Using the SSC, Proposition 4.7 and Lemma 4.4 we get

$$\begin{aligned}\mu(B(x,r_k)) &\lesssim \mu(B(x,\tfrac{\delta}{2C_1}r_k)) \leqslant \mu(\varphi_{\mathbf{i}|_{kn}}(F)) \\ &= \prod_{j=1}^{kn} p_{i_j}(\sigma^{j-1}\mathbf{i}) = \left(\prod_{j=1}^{n} p_{i_j}(\sigma^{j-1}\mathbf{i})\right)^k \\ &= \overline{p}_{\mathbf{i}}^k = |\varphi'_{\mathbf{i}_{kn}}(x)|^{\frac{\log \overline{p}_{\mathbf{i}}^k}{\log |\varphi'_{\mathbf{i}_{kn}}(x)|}} = r_k^{\frac{k \log \overline{p}_{\mathbf{i}}}{k \log |\varphi'_{\mathbf{i}|_n}(x)|}} \leqslant r_k^t.\end{aligned}$$

Taking logarithms and limits shows us that

$$\overline{\dim}_{\mathrm{loc}}(\mu, x) \geqslant t,$$

so in particular by Proposition 3.3, $\dim_{\mathrm{A}} \mu \geqslant t$. Taking $t \to s$ finishes the proof. □

Using the fact that the measure μ is quasi-Bernoulli, Theorem 4.8 gives the following immediate corollary.

Corollary 5.4. *If μ is a place dependent invariant measure fully supported on a self-conformal set F satisfying the SSC, then*

$$\dim_{\mathrm{A}}(\mu, x) = \sup_{\mathbf{i} \in P(\Sigma)} \frac{\log \overline{p}_{\mathbf{i}}}{\log |\varphi'_{\mathbf{i}}|},$$

for μ-almost every $x \in F$

Remark 5.5. The dimension formula for the Assouad dimension of self-similar measures is easily recovered as a special case of Theorem 5.3, which can be seen as a generalization of [**?** , Theorem 2.4]. Indeed, by observing that when each φ_i is a similarity with ratio r_i we have $|\varphi'_i(x)| = r_i$, for all $x \in F$ and when each $p_i(x) \equiv p_i$, Theorem 5.3 gives

$$\dim_{\mathrm{A}} \mu = \sup_{\mathbf{i} \in P(\Sigma)} \frac{\log \overline{p}_{\mathbf{i}}}{\log |\varphi'_{\mathbf{i}}|} = \sup_{\mathbf{i} \in \Sigma_*} \frac{\log p_{\mathbf{i}}}{\log r_{\mathbf{i}}} = \max_{i \in \Lambda} \frac{\log p_i}{\log r_i}.$$

Notice, however, that we do not recover Theorem 6.1 as a corollary, since there only the OSC was assumed.

6. Self-similar and self-affine measures

Let Λ be a finite index set, and associate to each $i \in \Lambda$ a contraction map φ_i from a compact subset of $\mathbb{R}^d$ to itself, and a probability $p_i \in (0,1)$, such that $\sum_{i \in \Lambda} p_i =$

1. The collection $\{\varphi_i\}_{i\in\Lambda}$ is known as an iterated function system (IFS). By a foundational result of Hutchinson [?], every IFS has a unique compact and non-empty invariant set F satisfying

$$F = \bigcup_{i\in\Lambda} \varphi_i(F),$$

called the *attractor of the IFS* as well as a unique Borel probability measure μ fully supported on F satisfying

$$\mu = \sum_{i\in\Lambda} p_i \mu \circ \varphi_i^{-1}.$$

When the contractions φ_i are similarities or affine maps, F is called a *self-similar* or a *self-affine set*, respectively, and μ is called a *self-similar* or a *self-affine measure*. If the maps φ_i are similarities, we denote their similarity ratios by $r_i \in (0,1)$. In all of the proofs, we assume that $\operatorname{diam}(F) = 1$, which does not result in loss of generality, since rescaling the set does not affect its geometry.

Self-similar and self-affine sets and measures are perhaps the most important prototypical examples of fractal sets and measures. These classes have been well studied in the past decades, and substantial progress has been made in understanding their dimensional properties. See for example [? ?] for recent developments in the self-similar case and [?] for the self-affine case. To make their study easier, it is usual to impose some sort of separation conditions on the defining maps. The most common separation conditions are the *strong separation condition (SSC)* and the *open set condition (OSC)*. We say that the set F satisfies the strong separation condition, if for any distinct $i, j \in \Lambda$, we have $\varphi_i(F) \cap \varphi_j(F) = \emptyset$. A slightly less restrictive property is the open set condition, which the set F is said to satisfy if there exists an open set $U \subset \mathbb{R}^d$, such that $\varphi_i(U) \subset U$ for all $i \in \Lambda$ and $\varphi_i(U) \cap \varphi_j(U) = \emptyset$ for $i \neq j$. We say that a self-similar measure fully supported on a self-similar set F satisfies the SSC if F does and similarly for the OSC.

6.1. Self-similar measures and the open set condition

The doubling properties of self-similar measures are quite well studied. The fact that self-similar measures satisfying the SSC are Ahlfors regular and therefore certainly doubling goes at least back to [?], and their Assouad dimension was explicitly computed by Fraser and Howroyd [? , Theorem 2.4]. To be specific, they showed that the Assouad dimension of any self-similar measure satisfying the SSC is given by

$$\dim_{\mathrm{A}} \mu = \max_{i\in\Lambda} \frac{\log p_i}{\log r_i}. \tag{6.1}$$

Relaxing the SSC to the OSC changes the situation dramatically. Yung [?] provides examples of self-similar sets satisfying the OSC for which (1) only the canonical self-similar measure is doubling, (2) all self-similar measures are doubling, (3) the measures are doubling for some but not all choices of the weights p_i. In particular this shows that the Assouad dimension of self-similar measures satisfying the OSC can in many cases be infinite. Still, it is an interesting question to study the Assouad dimension of *doubling* self-similar measures which do not satisfy the SSC. In the main theorem of this section, we show that if a self-similar measure satisfying the OSC is doubling, then the Assouad dimension is given by the natural formula (6.1). Furthermore, we show that the pointwise Assouad dimension agrees with the global Assouad dimension almost everywhere. This is analogous to the fact that self-similar measures are exact dimensional.

When studying self-similar sets, it is useful to consider a symbolic representation of the IFS. Let $\Sigma = \{(i_1, i_2, \ldots) \colon i_k \in \Lambda\}$ denote the set of infinite sequences of the symbols in Λ. We call Σ the *symbolic space* and members of Σ *(infinite) words*. For an integer n, let $\Sigma_n = \{(i_1, i_2, \ldots, i_n) \colon i_k \in \Lambda\}$ be the set of *finite words of length* n and let $\Sigma_* = \bigcup_{n\in\mathbb{N}} \Sigma_n \cup \{\emptyset\}$ denote the set of all finite words of any length. Let $\mathbf{i}|_0 = \emptyset$ denote the empty word, for any $\mathbf{i} \in \Sigma$. We use the abbreviation $\mathbf{i} = (i_1, i_2, \ldots)$ for a fixed element of Σ and the same notation $\mathbf{i} = (i_1, \ldots, i_n)$ for elements of Σ_n, but the meaning will be clear from the context. For $\mathbf{i} = (i_1, \ldots, i_n) \in \Sigma_n$, let $\mathbf{i}^- = (i_1, \ldots, i_{n-1})$ denote the finite word obtained by dropping the last element of $\mathbf{i}$. If $\mathbf{i} \in \Sigma$, we write $\mathbf{i}|_n = (i_1, \ldots, i_n) \in \Sigma_n$ for the projection of $\mathbf{i}$ onto the first n coordinates. For $\mathbf{i} \in \Sigma_n$, the cylinder $[\mathbf{i}] \subset \Sigma$ is defined to be the set of all infinite words in Σ whose first n letters are the letters of $\mathbf{i}$. In some proofs, we use for $\mathbf{i} \in \Sigma$ and $\mathbf{j} \in \Sigma_*$ the notation $\mathbf{j} \ll \mathbf{i}$, to mean that the word $\mathbf{i}$ contains the word $\mathbf{j}$ as a substring.

For the similarities φ_i, and any constants c_i we abbreviate

$$\varphi_{\mathbf{i}|_n} = \varphi_{i_1} \circ \ldots \circ \varphi_{i_n},$$

and

$$c_{\mathbf{i}|_n} = \prod_{k=1}^{n} c_{i_k}.$$

Recall that there is a natural correspondence between the symbolic space Σ and the self-similar set F by the coding map $\pi : \Sigma \to F$ defined by

$$\{\pi(\mathbf{i})\} = \bigcap_{n=1}^{\infty} \varphi_{\mathbf{i}|_n}(F). \tag{6.2}$$

When F satisfies the SSC, this map is a bijection. We may construct a Bernoulli measure ν on Σ by setting for all $\mathbf{i} \in \Sigma_n$,

$$\nu([\mathbf{i}]) = p_{\mathbf{i}},$$

and extending this to the whole space Σ in the usual way. There is a natural correspondence between the Bernoulli measure ν and the self-similar measure on F, namely

$$\mu = \pi_* \nu.$$

We are now ready to state the main result of this section. The proof builds on ideas of [**?**] and [**?**].

Theorem 6.1. *Let μ be a self-similar measure satisfying the OSC. If μ is doubling, then*

$$\dim_{\mathrm{A}} \mu = \max_{i \in \Lambda} \frac{\log p_i}{\log r_i},$$

and for μ-almost every $x \in F$, we have

$$\dim_{\mathrm{A}}(\mu, x) = \max_{i \in \Lambda} \frac{\log p_i}{\log r_i}.$$

*Proof.*Let $s = \max_{i \in \Lambda} \frac{\log p_i}{\log r_i}$ be the target dimension. We start by showing that $\dim_A \mu \leqslant s$. Let $x \in F$ and $0 < r < R < 1$. Choose integers k and l, such that

$$r_{\mathbf{i}|_k} \leqslant R < r_{\mathbf{i}|_{k-1}} \text{ and } \quad r_{\mathbf{i}|_{l+1}} < r \leqslant r_{\mathbf{i}|_l}.$$

We may assume from this point on that $l > k$, since otherwise $\frac{R}{r}$ would be bounded from above by a uniform constant, which does not bother us. Now $\varphi_{\mathbf{i}_{l+1}}(F) \subset B(x, r)$, so in particular $\mu(B(x,r)) \geqslant p_{\mathbf{i}|_{l+1}}$. Define $\Lambda_{x,R} = \{\mathbf{j} \in \Sigma_* \colon r_{\mathbf{j}} \leqslant R < r_{\mathbf{j}^-},\, d(x, \varphi_{\mathbf{j}}(F)) \leqslant R\}$. Note that the OSC implies that there is a constant $M \geqslant 1$ independent of x and R, such that

$$\#\Lambda_{x,R} \leqslant M.$$

For the simple proof of this see [**?** , Proposition 1.5.8]. By definition, for every $\mathbf{j} \in \Lambda_{x,R}$ we have $\operatorname{diam}(\varphi_{\mathbf{j}}(F)) = r_{\mathbf{j}} \leqslant R < r_{\mathbf{i}|_{k-1}}$, and since $x \in \varphi_{\mathbf{i}|_{k-1}}(F)$, this implies that $d(\varphi_{\mathbf{i}|_{k-1}}(F), \varphi_{\mathbf{j}}(F)) \leqslant R < r_{\mathbf{i}|_{k-1}}$. Combining these estimates we see that $\varphi_{\mathbf{j}}(F) \subset B(\varphi_{\mathbf{i}|_{k-1}}(F), 2r_{\mathbf{i}|_{k-1}})$, so by Theorem 1.1 of [**?**], there is a constant $C > 0$, such that

$$p_{\mathbf{j}} \leqslant C p_{\mathbf{i}|_{k-1}}$$

holds independently from $\mathbf{j}$ and $\mathbf{i}$. Furthermore, it is clear that

$$B(x, R) \cap F \subset \bigcup_{\mathbf{j} \in \Lambda_{x,R}} \varphi_{\mathbf{j}}(F),$$

so we may estimate

$$
\begin{aligned}
\frac{\mu(B(x,R))}{\mu(B(x,r))} &\leqslant \frac{\sum_{\mathbf{j}\in\Lambda_{x,R}} p_{\mathbf{j}}}{p_{\mathbf{i}|_{l+1}}} \leqslant MC\frac{p_{\mathbf{i}|_{k-1}}}{p_{\mathbf{i}|_{l+1}}} = \frac{MC}{p_{i_k}p_{i_l}}\frac{p_{\mathbf{i}|_k}}{p_{\mathbf{i}|_l}} \\
&\leqslant \frac{MC}{p_{\min}^2}\left(p_{i_{l-k+1}}p_{i_{l-k+2}}\cdots p_{i_l}\right)^{-1} \\
&\leqslant \frac{MC}{p_{\min}^2}\left(r_{i_{l-k+1}}^{\frac{\log p_{i_{k-l+1}}}{\log r_{i_{k-l+1}}}} r_{i_{l-k+2}}^{\frac{\log p_{i_{k-l+2}}}{\log r_{i_{k-l+2}}}}\cdots r_{i_l}^{\frac{\log p_{i_l}}{\log r_{i_l}}}\right)^{-1} \\
&\leqslant \frac{MC}{p_{\min}^2}\left(r_{i_{l-k+1}}r_{i_{l-k+2}}\cdots r_{i_l}\right)^{-s} \\
&= \frac{MC}{p_{\min}^2}\left(\frac{r_{\mathbf{i}|_k}}{r_{\mathbf{i}|_l}}\right)^{s} \leqslant \frac{MC}{p_{\min}^2}\left(\frac{R}{r}\right)^{s},
\end{aligned}
$$

which is enough to show that $\dim_{\mathrm{A}}\mu \leqslant s$.

To finish the proof, it is enough to show that the lower bound holds for the pointwise Assouad dimension at almost every point. For this, let $i \in \Lambda$ be the index maximizing $\frac{\log p_i}{\log r_i}$ and define $\mathcal{N}_n = \{\mathbf{i} \in \Sigma\colon (i,\ldots,i) \ll \mathbf{i}\}$ and subsequently $\mathcal{N} = \bigcap_{n\in\mathbb{N}}\mathcal{N}_n$. Pick $x \in \pi(\mathcal{N})$ and note that as a special case of Lemma 4.6, we have that $\pi(\mathcal{N})$ is a set of full measure. Let $\mathbf{i} \in \mathcal{N}$ be a (not necessarily unique) sequence such that $\pi(\mathbf{i}) = x$. Now for any $n \in \mathbb{N}$ there is an integer k such that

$$
\mathbf{i} = (i_1,\ldots,i_k,\underbrace{i,i,\ldots,i}_{n},i_{k+n+1},\ldots).
$$

Choose $R_n = r_{\mathbf{i}|_k}$ and $r_n = r_{\mathbf{i}|_{k+n}}$, so $\varphi_{\mathtt{i}_k}(F) \subset B(x,R_n)$, and thus

$$
\mu(B(x,R_n)) \geqslant \mu(\varphi_{\mathtt{i}_k}(F)) = p_{\mathbf{i}|_{k-1}},
$$

and by calculations similar to above,

$$
\mu(B(x,r_n)) \leqslant MCp_{\mathbf{i}|_{k+n}}.
$$

Therefore

$$
\frac{\mu(B(x,R_n))}{\mu(B(x,r_n))} \geqslant \frac{1}{MC}p_i^{-n} = \frac{1}{MC}(r_i^{-n})^s = \frac{1}{MC}\left(\frac{R_n}{r_n}\right)^s.
$$

Since $\frac{R_n}{r_n} \to \infty$ as $n \to \infty$, this shows that $\dim_{\mathrm{A}}(\mu,x) \geqslant s$. This finishes the proof, since now at μ-almost every x, we have $s \leqslant \dim_A(\mu,x) \leqslant \dim_A \mu \leqslant s$. $\square$

Since it is known that self-similar measures satisfying the SSC are Ahlfors regular and therefore doubling [**?**], the following corollary is immediate.

Corollary 6.2. *If μ is a self-similar measure satisfying the SSC, then*

$$\dim_{\mathrm{A}}(\mu, x) = \max_{i \in \Lambda} \frac{\log p_i}{\log r_i} = \dim_{\mathrm{A}} \mu,$$

for μ-almost every $x \in F$.

While calculating the lower bound for the Assouad dimension of self-similar measures satifying the SSC in the proof of [**?** , Theorem 2.4], Fraser and Howroyd construct a point where the upper local dimension is bounded from below by $\max_{i \in \Lambda} \frac{\log p_i}{\log r_i}$, which gives the lower bound for the Assouad dimension. Our result generalizes this by showing that, in fact, the pointwise Assouad dimension is maximized at almost every point. It is an interesting question to see if the same formula for the Assouad dimension and the pointwise Assouad dimension works for doubling self-similar measures with less restrictive separation conditions, such as the weak separation condition.

6.2. Self-affine measures on Bedford-McMullen carpets

A result similar to Corollary 6.2 holds for self-affine measures on very strongly separated Bedford-McMullen sponges, which we define as follows. We work in $\mathbb{R}^d$, with $d \geqslant 2$. Start by choosing integers $n_1 < n_2 < \ldots < n_d$, and after that choose a subset $\Lambda \subset \prod_{l=1}^{d} \{0, \ldots, n_l - 1\}$. The set Λ is the code space associated with the Bedford-McMullen sponge. For all $\bar{\imath} = (i_1, i_2, \ldots, i_d) \in \Lambda$, we define an affine transform $\varphi_{\bar{\imath}} : [0,1]^d \to [0,1]^d$ by

$$\varphi_{\bar{\imath}}(x_1, \ldots, x_d) = \left(\frac{x_1 + i_1}{n_1}, \ldots, \frac{x_d + i_d}{n_d} \right).$$

The *Bedford-McMullen sponge* is the self-affine set F, which is the attractor of the IFS $\{\varphi_{\bar{\imath}}\}_{\bar{\imath} \in \Lambda}$. With this we associate a probability vector $(p_{\bar{\imath}})_{\bar{\imath} \in \Lambda}$, and define the self-affine measure on F as usual. To establish bounds for the measures of balls, we need a separation condition which is strictly stronger than the strong separation condition. Following Olsen [**?**], we say that a Bedford-McMullen sponge F satisfies the *very strong separation condition (VSSC)*, if for words $(i_1, \ldots, i_d), (j_1, \ldots, j_d) \in \Lambda$ satisfying $i_k = j_k$, for all $k = 1, \ldots, l-1$, and $i_l \neq j_l$, for some $l = 1, \ldots, d$, we have $|i_l - j_l| > 1$. We also need the following quantity. For $l = 1, \ldots, d$ and $\bar{\imath} = (i_1, \ldots, i_d)$, define

$$p_l(\bar{\imath}) = p(i_l | i_1, \ldots, i_{l-1}) = \frac{\sum\limits_{\substack{\bar{\jmath} \in \Lambda \\ j_k = i_k,\, k=1,\ldots,l}} p_{\bar{\jmath}}}{\sum\limits_{\substack{\bar{\jmath} \in \Lambda \\ j_k = i_k,\, k=1,\ldots,l-1}} p_{\bar{\jmath}}}, \tag{6.3}$$

If $(i_1, \ldots, i_l, i_{l+1}, \ldots, i_d) \in \Lambda$ for some $i_{l+1}, \ldots, i_d$, and 0 otherwise. These numbers can be interpreted as the conditional probabilities that the lth digit of a randomly chosen member of Λ equals the lth digit of $\bar{\imath}$, given that the first $l-1$ coordinates did. Fraser and Howroyd proved the following theorem.

Theorem 6.3. *Let μ be a self-affine measure on a Bedford-McMullen sponge satisfying the VSSC. Then*

$$\dim_{\mathrm{A}} \mu = \sum_{l=1}^{d} \max_{\bar{\imath} \in \Lambda} \frac{-\log p_l(\bar{\imath})}{\log n_l}.$$

Again, we extend this result and prove that the pointwise Assouad dimension coincides with this value at almost every point.

Theorem 6.4. *Let μ be a self-affine measure on a Bedford-McMullen sponge F satisfying the VSSC. Then*

$$\dim_{\mathrm{A}}(\mu, x) = \dim_{\mathrm{A}} \mu,$$

for μ-almost every $x \in F$.

For the proof we need the concept of *approximate cubes* introduced by Olsen [**?**], which we can use to approximate the measures of balls. For clarity, we use ω to represent members of the set Σ instead of $\mathbf{i}$ which we used in the self-similar case. We denote the approximate cube of level $k \in \mathbb{N}$ centered at $\omega = (\bar{\imath}_1, \ldots) = ((i_{1,1}, \ldots, i_{1,d}), \ldots) \in \Sigma$ by $Q_k(\omega)$, and it is defined by

$$Q_k(\omega) = \{\omega' = (\bar{\jmath}_1, \ldots) \in \Sigma \colon j_{t,l} = i_{t,l},\ \forall l = 1, \ldots, d \text{ and } \forall t = 1, \ldots L_l(k)\},$$

where $L_l(k)$ is the unique number that satisfies

$$n_l^{-L_l(k)-1} < n_1^{-k} \leqslant n_l^{-L_l(k)}.$$

In particular, this implies that

$$k\frac{\log n_1}{\log n_l} - 1 < L_l(k) \leqslant k\frac{\log n_1}{\log n_l}.$$

The geometric equivalent of the approximate cube $Q_k(\omega)$ is its image under the projection map $\pi : \Sigma \to \mathbb{R}^d$ defined similarily as in the previous case. The image $\pi(Q_k(\omega))$ is contained in

$$\prod_{l=1}^{d} \left[\frac{i_{1,l}}{n_l} + \ldots + \frac{i_{L_l(k),l}}{n_l^{L_l(k)}}, \frac{i_{1,l}}{n_l} + \ldots + \frac{i_{L_l(k),l}}{n_l^{L_l(k)}} + \frac{1}{n_l^{L_l(k)}} \right],$$

which is a hypercuboid in $\mathbb{R}^d$ with all side lengths comparable to n_1^{-k}.

Olsen [**?**] observed that the measure of an approximate cube is given by

$$\mu(\pi(Q_k(\omega))) = \prod_{l=1}^{d} \prod_{j=0}^{L_l(k)-1} p_l(\sigma^j \omega), \tag{6.4}$$

where $\sigma : \Sigma \to \Sigma$ is the left shift and $p_l(\omega) = p(i_{1,l}|i_{i,1}, \dots, i_{1,l-1})$, where the right hand side is as in equation (6.3). Recall also that a Bernoulli measure on the code space Σ is shift invariant.

The following proposition by Olsen shows that we can closely approximate the measures of balls centered at the Bedford-McMullen carpets by approximate cubes of similar size.

Proposition 6.5. *Let $\omega \in \Sigma$ and $k \in \mathbb{N}$.*

(i) *If the VSSC is satisfied, then $B(\pi(\omega), 2^{-1}n_1^k) \cap F \subset \pi(Q_k(\omega))$.*

(ii) $\pi(Q_k(\omega)) \subset B(\pi(\omega), (n_1 + \ldots + n_d)n_1^k)$.

The proof of the proposition can be found in [**?** , Proposition 6.2.1]. Let us now prove Theorem 6.4. The proof follows ideas of Fraser and Howroyd [**?** , Theorem 2.6], but to establish the result at almost every point, we have to be a little more careful with the spacing of the subsequences which determine the Assouad dimension.

*Proof Of Theorem 6.4.*First we note that $L_l(n)$ increases with n and, since n_l are strictly increasing, decreases with l. It is an elementary exercise to show that for every $k \in \mathbb{N}$, there is an integer n_k, such that for all $n \geqslant n_k$, we have

$$L_d(n) < L_d(n+k) < L_{d-1}(n) < L_{d-1}(n+k) < \ldots < L_1(n) < L_1(n+k).$$

For $l = 1, \dots, d$, let $p_l^{\min} = \min_{\bar{\imath} \in \Lambda} p_l(\bar{\imath})$, and let $\bar{\imath}_l^{\min}$ be some element of Λ which achieves this minimum. Define for every $k \in \mathbb{N}$ the set

$$I_k = \bigcup_{n \geqslant n_k} \bigcap_{l=1}^{d} \sigma^{-L_l(n)}[\underbrace{\bar{\imath}_l^{\min}, \dots, \bar{\imath}_l^{\min}}_{L_l(n+k)-L_l(n) \text{ times}}].$$

Note that an element $\omega \in I_k$ has the form

$$\begin{aligned} \omega = (&\bar{\imath}_1, \dots \bar{\imath}_{L_d(n)}, \bar{\imath}_d^{\min}, \dots, \bar{\imath}_d^{\min}, \bar{\imath}_{L_d(n+k)+1}, \dots, \bar{\imath}_{L_2(n)}, \\ &\bar{\imath}_2^{\min}, \dots, \bar{\imath}_2^{\min}, \bar{\imath}_{L_2(n+k)+1}, \dots, \bar{\imath}_{L_1(n)}, \bar{\imath}_1^{\min}, \dots, \bar{\imath}_1^{\min}, \bar{\imath}_{L_1(n+k)+1}, \dots). \end{aligned} \tag{6.5}$$

Clearly the events $\sigma^{-L_l(n)}[\underbrace{\bar{\imath}_l^{\min}, \dots, \bar{\imath}_l^{\min}}_{L_l(n+k)-L_l(n) \text{ times}}]$, are pairwise independent, and

$$I_k^c = \bigcap_{n \geqslant n_k} \left(\bigcap_{l=1}^{d} \sigma^{-L_l(n)}[\underbrace{\bar{\imath}_l^{\min}, \dots, \bar{\imath}_l^{\min}}_{L_l(n+k)-L_l(n) \text{ times}}] \right)^c.$$

Now we choose $m_1 = n_k$ and then inductively $m_i = L_1(m_{i-1} + k) + 1$, for every $i > 1$. Then the events $A_i := \left(\bigcap_{l=1}^{d} \sigma^{-L_l(m_i)}[\underbrace{\bar{\imath}_l^{\min}, \dots, \bar{\imath}_l^{\min}}_{L_l(m_i+k)-L_l(m_i) \text{ times}}]\right)^c$ are independent, so

$$\begin{aligned}\mu(I_k^c) &\leqslant \mu\left(\bigcap_{i\in\mathbb{N}} A_i\right) = \prod_{i\in\mathbb{N}} \mu(A_i) = \prod_{i\in\mathbb{N}} 1 - \mu\left(\bigcap_{l=1}^{d} \sigma^{-L_l(m_i)}[\underbrace{\bar{\imath}_l^{\min}, \dots, \bar{\imath}_l^{\min}}_{L_l(m_i+k)-L_l(m_i) \text{ times}}]\right) \\ &= \prod_{i\in\mathbb{N}} 1 - \prod_{l=1}^{d} (p_l^{\min})^{L_l(m_i+k)-L_l(m_i)} \leqslant \prod_{i\in\mathbb{N}} \underbrace{1 - (p_l^{\min})^d}_{<1} = 0.\end{aligned}$$

Thus $\mu(I_k) = 1$, and moreover $\mu(I) = 1$, where $I = \bigcap_{k\in\mathbb{N}} I_k$.

Now let $x = \pi(\omega)$, where $\omega \in I$, and let $R_k = (n_1 + \ldots + n_d)n_1^{-n-1}$, and $r_k = 2^{-1}n_1^{-(n+k)-1}$, where k and n are chosen, such that ω is given by equation (6.5). Observe that by proposition 6.5, we have

$$\frac{\mu(B(x, R_k))}{\mu(B(x, r_k))} \geqslant \frac{\mu(\pi(Q_{n+1}(\omega)))}{\mu(\pi(Q_{n+k+1}(\omega)))}.$$

Moreover, equation (6.4) gives

$$\begin{aligned}\frac{\mu(\pi(Q_{n+1}(\omega)))}{\mu(\pi(Q_{n+k+1}(\omega)))} &= \frac{\prod_{l=1}^{d} \prod_{j=0}^{L_l(n)} p_l(\sigma^j\omega)}{\prod_{l=1}^{d} \prod_{j=0}^{L_l(n+k)} p_l(\sigma^j\omega)} = \frac{1}{\prod_{l=1}^{d} \prod_{j=L_l(n)-1}^{L_l(n+k)} p_l(\sigma^j\omega)} \\ &= \prod_{l=1}^{d} \left(\frac{1}{p_l^{\min}}\right)^{L_l(n+k)-L_l(n)+2} \geqslant \prod_{l=1}^{d} \left(\frac{1}{p_l^{\min}}\right)^{(n+k)\frac{\log n_1}{\log n_l} - n\frac{\log n_1}{\log n_l} + 1} \\ &\geqslant (p_l^{\min})^{-d} \prod_{l=1}^{d} \left(\frac{1}{p_l^{\min}}\right)^{k\frac{\log n_1}{\log n_l}} = (p_l^{\min})^{-d} \prod_{l=1}^{d} \left(n_1^k\right)^{\frac{-\log p_l^{\min}}{\log n_l}} \\ &\geqslant (\min_l p_l^{\min})^{-d} \left(n_1^k\right)^{\dim_{\mathrm{A}} \mu} = C\left(\frac{R_k}{r_k}\right)^{\dim_{\mathrm{A}} \mu},\end{aligned}$$

where $C = (\min_l p_l^{\min})^{-d} \cdot (2(n_1 + \ldots + n_d))^{\dim_{\mathrm{A}} \mu} > 0$ is a constant. Taking $k \to \infty$, we see that $\frac{R_k}{r_k} \to \infty$, which is enough to prove that $\dim_{\mathrm{A}}(\mu, x) \geqslant \dim_{\mathrm{A}} \mu$. This holds for all $x = \pi(\omega)$, such that $\omega \in I$, where I has full measure, proving the claim. $\square$

7. Discussion

Most of the results of this paper follow a similar pattern by providing exact dimensionality properties for the pointwise Assouad dimension. A natural follow up to the results of this paper would be to conduct finer analysis of the pointwise Assouad dimension and develop tools for *multifractal analysis* of the pointwise Assouad dimension. Classically, the multifractal spectrum $f(\alpha)$ of a measure is given by the Hausdorff dimension of α-level sets of the local dimension, that is

$$f(\alpha) \coloneqq \dim_{\mathrm{H}}\{x \in X \colon \dim_{\mathrm{loc}}(\mu, x) = \alpha\}.$$

The celebrated multifractal formalism states that, in many cases, this spectrum is given by the Legendre transform of the L^q-spectrum of the measure, see e.g. Chapter 11 of [**?**] for details. Of course, a natural question to ask is if something similar is true for the Hausdorff dimension spectrum of the level sets of the pointwise Assouad dimension.

Question. What is the multifractal Assouad spectrum of a strongly separated self-similar measure μ? By this we mean quantity

$$f_{\mathrm{A}}(\alpha) \coloneqq \dim_{\mathrm{H}}\{x \in X \colon \dim_{\mathrm{A}}(\mu, x) = \alpha\}.$$

Using the Hausdorff dimension instead of the Assouad dimension in the definition is natural, since it is easy to see that each α-level set of the pointwise Assouad dimensions is dense in the support and the Assouad dimension of sets is stable under closures.

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