Vibrações Mecânicas

Aula 02 – Molas

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Molas

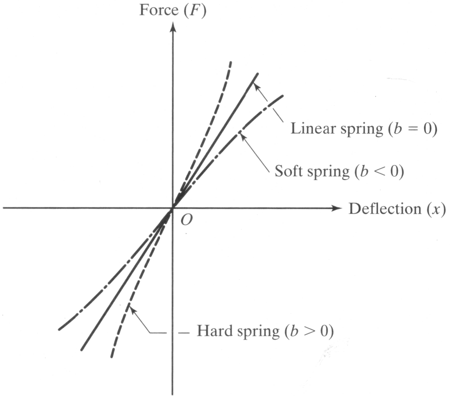
Leitura: Rao 1.7

Características:

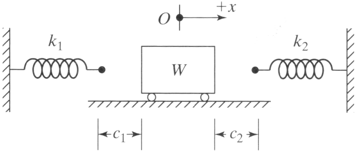
- Elemento mecânico que produz uma força em reação a um deslocamento;
- Mecânicas (helicoidais, torcionais, pneumáticas, etc.);
- Para molas lineares, $F = \kappa x$;
- Energia de deformação: $U = \int_0^x F \, \mathrm{d}x$;
- Para molas lineares, $U = \frac{1}{2}\kappa x^2$;

Molas Não Lineares

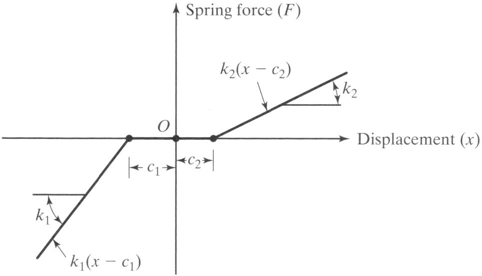
- Qualquer relação diferente de $F = \kappa x$;
- Normalmente, $F(x) = \kappa(x)x$;
- Pequenas não linearidades usualmente são representadas por molas cúbicas: $F(x) = ax + bx^3$, isto é, $\kappa(x) = a + bx^2$;



Molas Não Lineares



(a)



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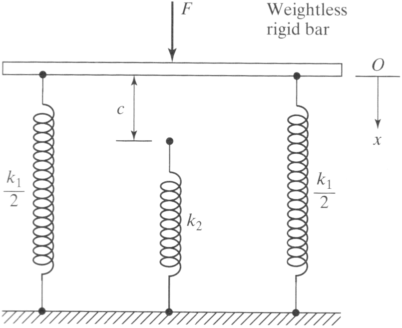

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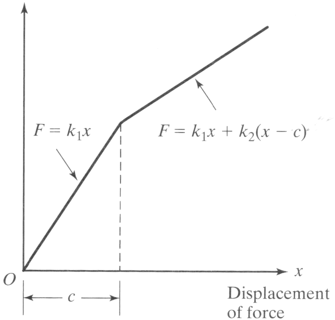
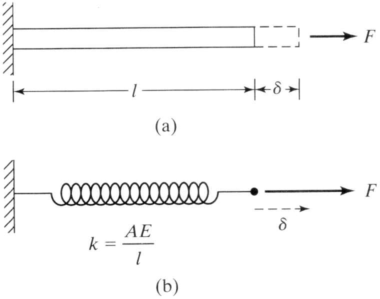


(b)

Constante de Mola de Barras

Barra homogênea de seção uniforme



$$\delta = \varepsilon l$$

$$\delta = \frac{\sigma}{E} l$$

$$\delta = \frac{Fl}{AE}$$

$$\kappa = \frac{F}{\delta} = \frac{AE}{l}$$

Constante de Mola de Barras

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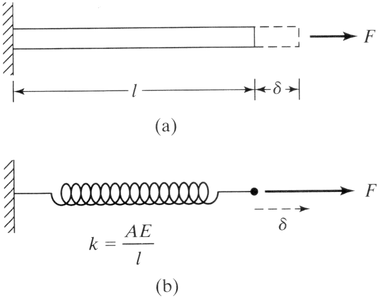

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Constante de Mola para Vigas em Balanço

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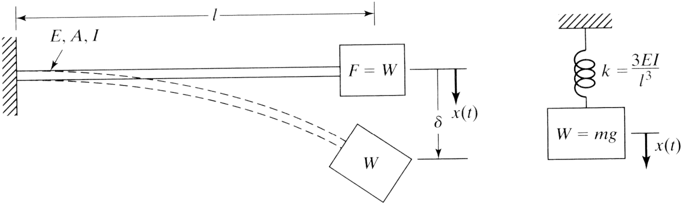

(a) Cantilever with end force

(b) Equivalent spring

$$\delta = \frac{Wl^3}{3EI} \qquad \kappa = \frac{W}{\delta} = \frac{3EI}{l^3}$$

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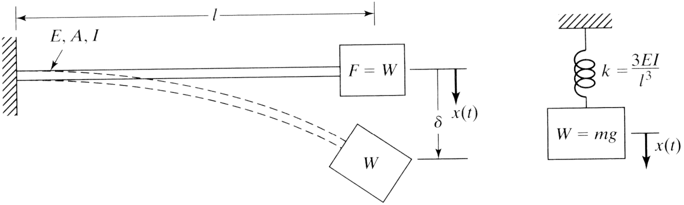

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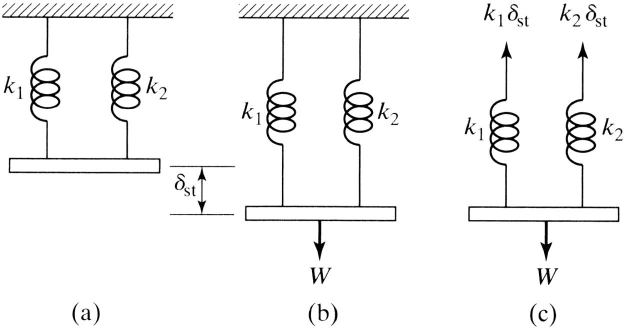
Combinação de Molas

Molas em paralelo



$W = \kappa_1\delta_{st} + \kappa_2\delta_{st}$ $\quad W = (\kappa_1 + \kappa_2)\delta_{st}$ $\quad \kappa_{eq} = \kappa_1 + \kappa_2$

Generalizando: $\kappa_{eq} = \kappa_1 + \kappa_2 + \cdots + \kappa_n$.

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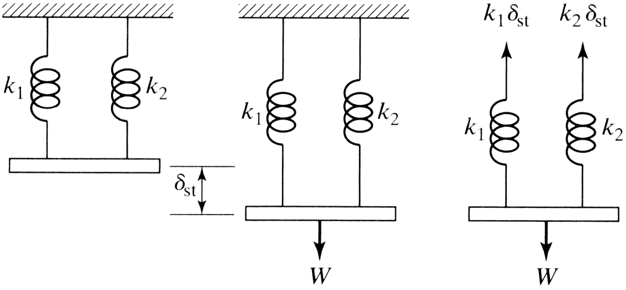

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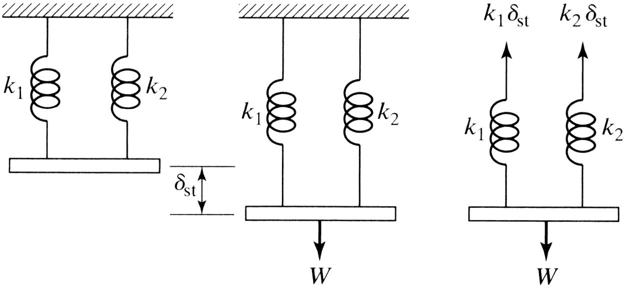

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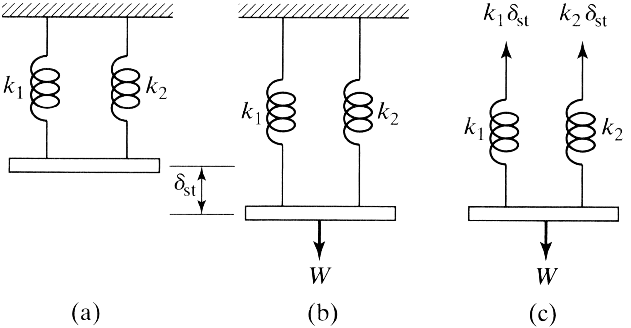

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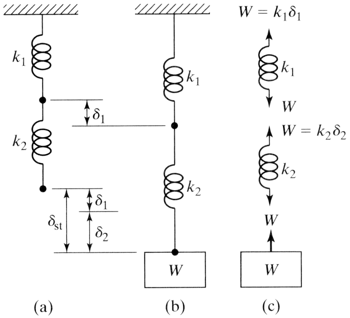
Combinação de Molas

Molas em série



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$$\frac{\kappa_{eq}\delta_{st}}{\kappa_1} + \frac{\kappa_{eq}\delta_{st}}{\kappa_2} = \delta_{st}$$

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Combinação de Molas

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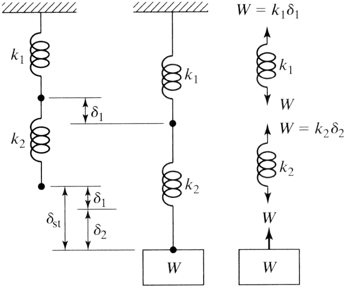

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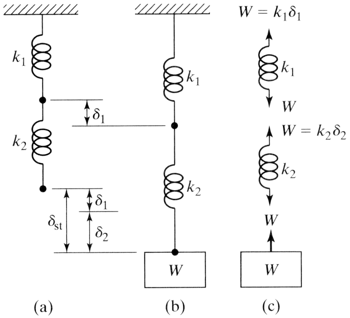

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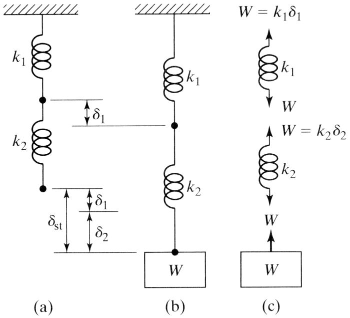

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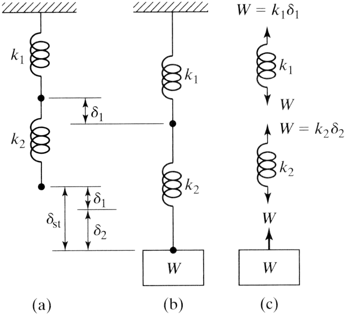

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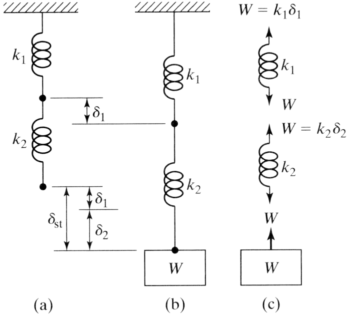

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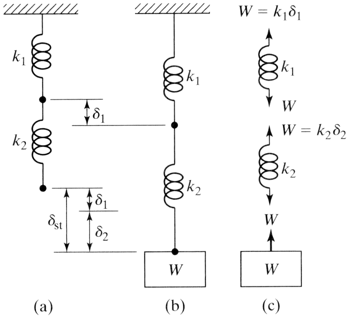
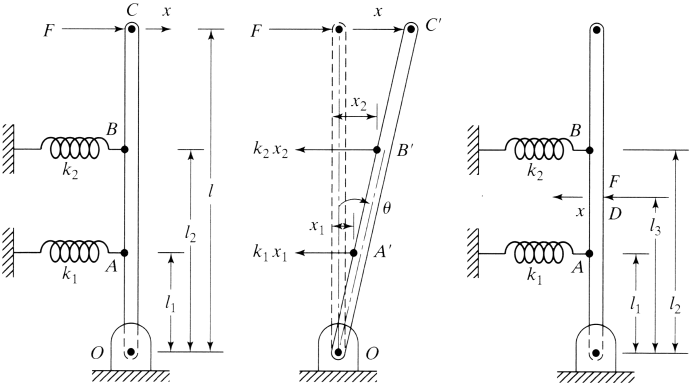


Generalizando:

$$\frac{1}{\kappa_{eq}} = \frac{1}{\kappa_1} + \frac{1}{\kappa_2} + \cdots + \frac{1}{\kappa_n}$$

Combinação de Molas

Princípio Geral: sistema equivalente com mesma energia potencial.
Exemplo (pequenos deslocamentos):



(a) (b) (c)

Combinação de Molas

Por equilíbrio:

$$x_1 = l_1 \sin\theta \qquad x_2 = l_2 \sin\theta,$$

pequenos deslocamentos

$$x_1 = l_1\theta \qquad x_2 = l_2\theta,$$

equilíbrio de momentos

$$\kappa_1 x_1 l_1 + \kappa_2 x_2 l_2 = Fl,$$

ou

$$F = \kappa_1 \frac{x_1 l_1}{l} + \kappa_2 \frac{x_2 l_2}{l}.$$

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Combinação de Molas

Continuando...

$$F = \kappa_{\mathrm{eq}} x = \kappa_1 \frac{x_1 l_1}{l} + \kappa_2 \frac{x_2 l_2}{l}$$

e

$$x = l\theta, \quad x_1 = l_1\theta, \quad x_2 = l_2\theta,$$

assim

$$\kappa_{\mathrm{eq}} l\theta = \kappa_1 \frac{l_1^2\theta}{l} + \kappa_2 \frac{l_2^2\theta}{l}$$

portanto

$$\kappa_{\mathrm{eq}} = \kappa_1 \left(\frac{l_1}{l}\right)^2 + \kappa_2 \left(\frac{l_2}{l}\right)^2$$

Combinação de Molas

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Combinação de Molas

Por energia:

Trabalho da força F = Energia armazenada nas molas κ_1 e κ_2

Para pequenos deslocamentos:

$$\frac{1}{2}Fx = \frac{1}{2}\kappa_{\mathrm{eq}}x^2 = \frac{1}{2}\kappa_1 x_1^2 + \frac{1}{2}\kappa_2 x_2^2,$$

mas como

$$\frac{x}{l} = \frac{x_1}{l_1} = \frac{x_2}{l_2},$$

temos

$$x_1 = \frac{xl_1}{l}, \quad x_2 = \frac{xl_2}{l},$$

portanto

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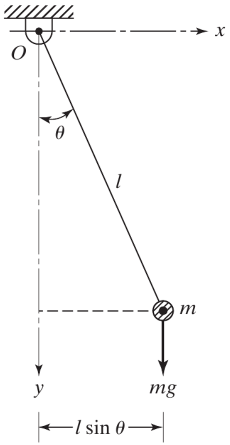
Mola Associada à gravidade



Para um deslocamento θ,

$$T = mg(l \sin\theta),$$

mas se θ é pequeno,

$$T = mgl\theta.$$

Colocando na forma $T = k_t\theta$, é óbvio que

$$k_t = mgl.$$