Chapter 5: Inference for a single population

Introductory Statistics for Engineering Experimentation

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Outline

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The Central Limit Theorem

- The central limit theorem is one of the most important results in mathematical statistics. It says that the sample means from a *random sample* (meaning independent samples from a stable process) will be normally distributed, regardless of what the original distribution was, when n is sufficiently large.
- Formally, if $\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_n$ is a random sample from a distribution with $\sigma^2 < \infty$ then for large samples, $\bar{\mathcal{Y}}$ is approximately normally distributed.
- This is a remarkably powerful result; first, because it is very general and secondly because it is a description of the asymptotic or "limiting" distribution but it holds for quite small values of n.

Other properties of the distribution of the sample mean

- If the random variables $\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_n$ are a random sample (sometimes also described as a "independent and identically distributed" or i.i.d. sample) from a distribution with mean μ and variance σ^2 then $\mathrm{E}(\bar{\mathcal{Y}}) = \mu$ and $\mathrm{Var}(\bar{\mathcal{Y}}) = \sigma^2/n$.
- So the central limit theorem states that, for large n,

$$\bar{\mathcal{Y}} \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$$

- Exactly how large n must be depends on the form of the original distribution. If it is continuous and reasonably symmetric then $n = 15$ may be large enough. If it is skewed but continuous we may need $n = 30$ or more. For discrete and skewed we may need as much as $n = 100$.
- Although in practice we only have one sample and one average, $\bar{y}$ we can use computer simulation to consider the sorts of samples we could have gotten and the distribution of the statistic.

Conducting a simulation study (not part of the course)

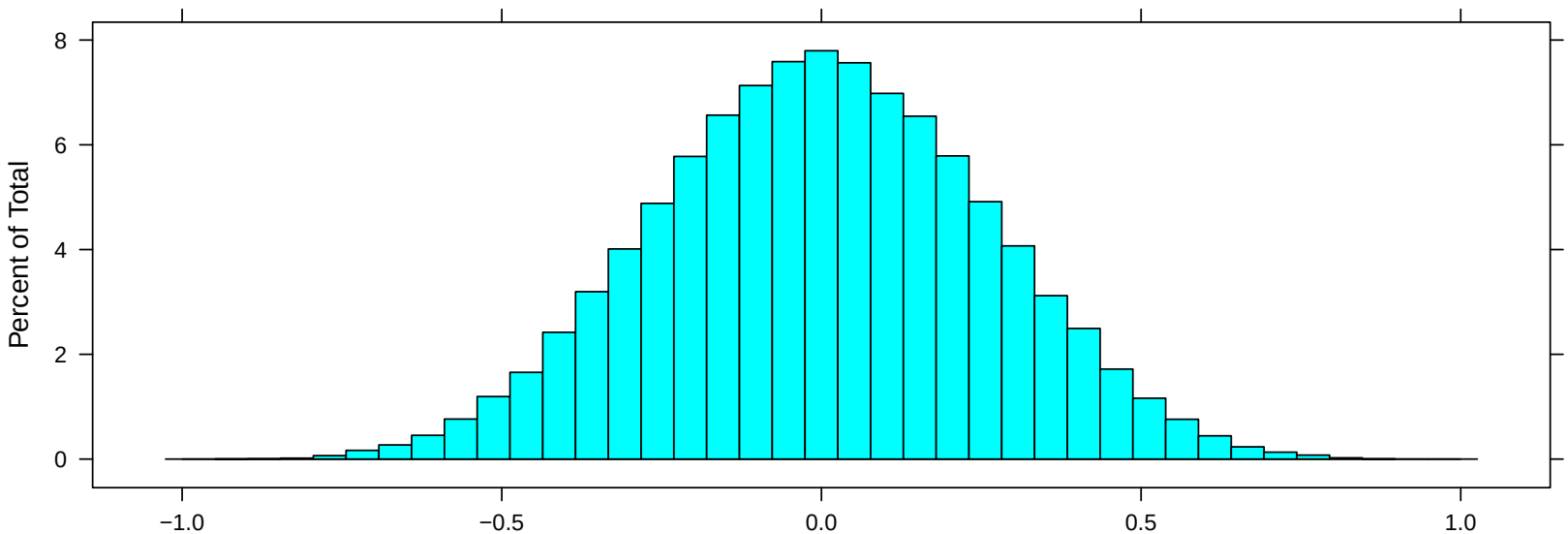
- Suppose we wish to simulate the value of a statistic (e.g. mean or median or variance or standard deviation) from samples of size n drawn from a certain distribution. Let K be the number of replicates we want to obtain.
- The *sample size*, n, is typically small. The number of replicates, K, can be very large. The larger the value of K, the more accurately we can determine the distribution of the statistic. With modern computers we can afford to use values of K in the hundreds of thousands or more.

- First determine how to evaluate the statistic from a single sample of size n then use the `replicate` function to repeat the process K times.

Mean of samples of size 5 from U(-1,1)

What is the shape of the distribution of the mean of a sample of size $n = 5$ from a $U(-1, 1)$ distribution?

```
> mns5 <- replicate(50000, mean(runif(5, min = -1, max = 1)))
> histogram(~mns5,breaks = seq(-1, 1, len = 40))
```



Sampling densities of statistics

- The idiom

  `replicate(K, <statfn>(r<distab>(n, <pars>)))`

  produces `K` replicates of the statistic calculated by `<statfn>` (examples are `mean`, `median`, `var` and `sd`) on samples of size `n` from distribution `<distab>` with parameter(s) `<pars>`.
- Typically `K` is large and `n` is small. Values of 10,000 or 100,000 are used for `K` on modern computers. The larger the value of `K` the smoother the approximation to the sampling density. `n` is the size of the actual sample you can afford to collect.

Effect of changing the sample size, n

- Performing multiple simulations allows us to see how characteristics of the distribution of $\bar{\mathcal{Y}}$ depends on n.

```
> mns1 <- runif(50000, -1, 1)
> mns10 <- replicate(50000, mean(runif(10, -1, 1)))
> mns20 <- replicate(50000, mean(runif(20, -1, 1)))
> sapply(list(mns1, mns5, mns10, mns20), mean)
```

```
[1] -3.073299e-03 -3.762864e-04 -3.569190e-04  5.290012e-05
```

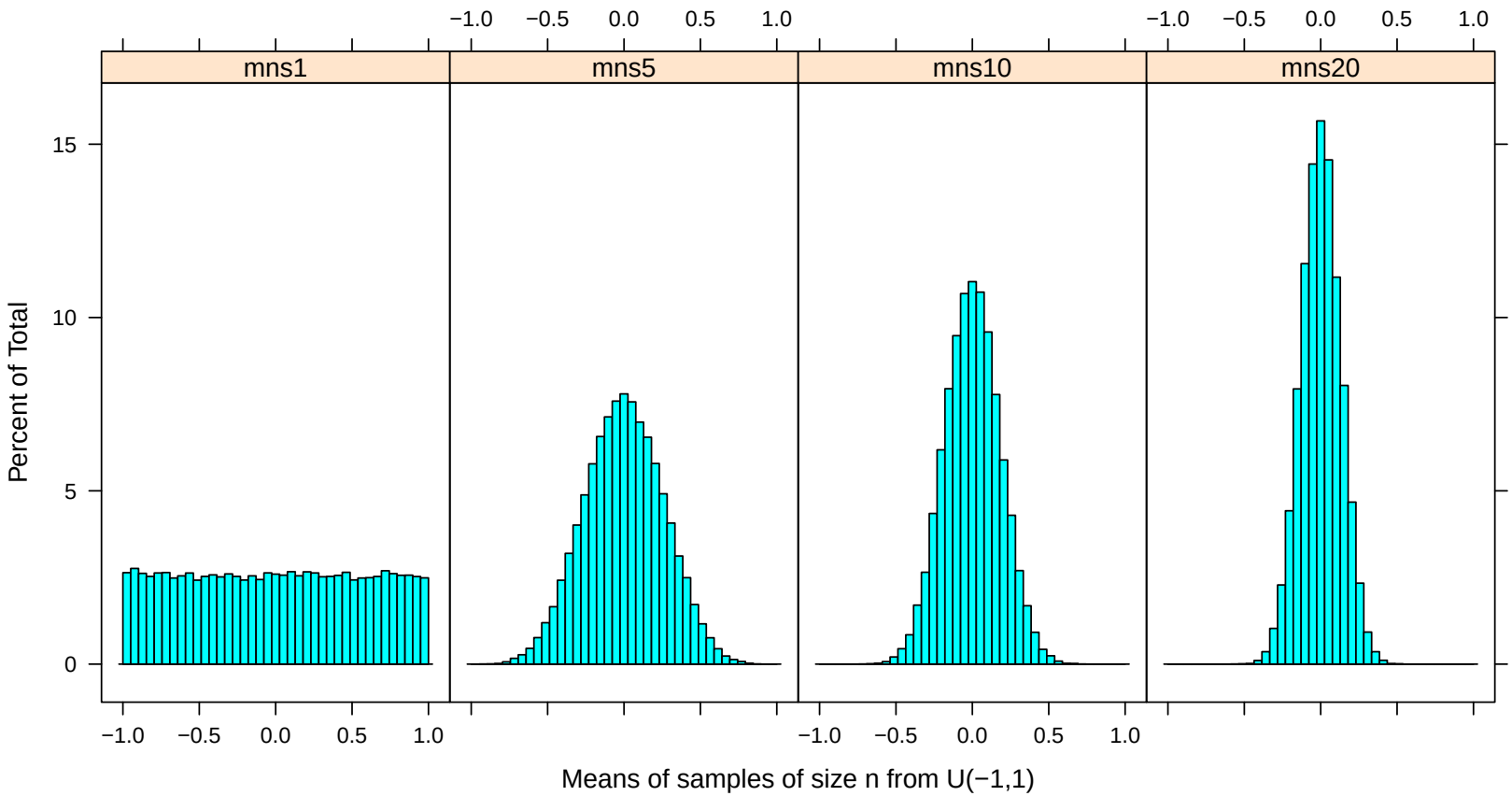
```
> sapply(list(mns1, mns5, mns10, mns20), var)
```

```
[1] 0.33492466 0.06679307 0.03308384 0.01680129
```

- As n increases the expected value of the sample mean stays near 0.
- As n increases the variance of the sample mean decreases. Roughly, $V(\bar{X}_n) = \frac{1}{3} \cdot \frac{1}{n}$

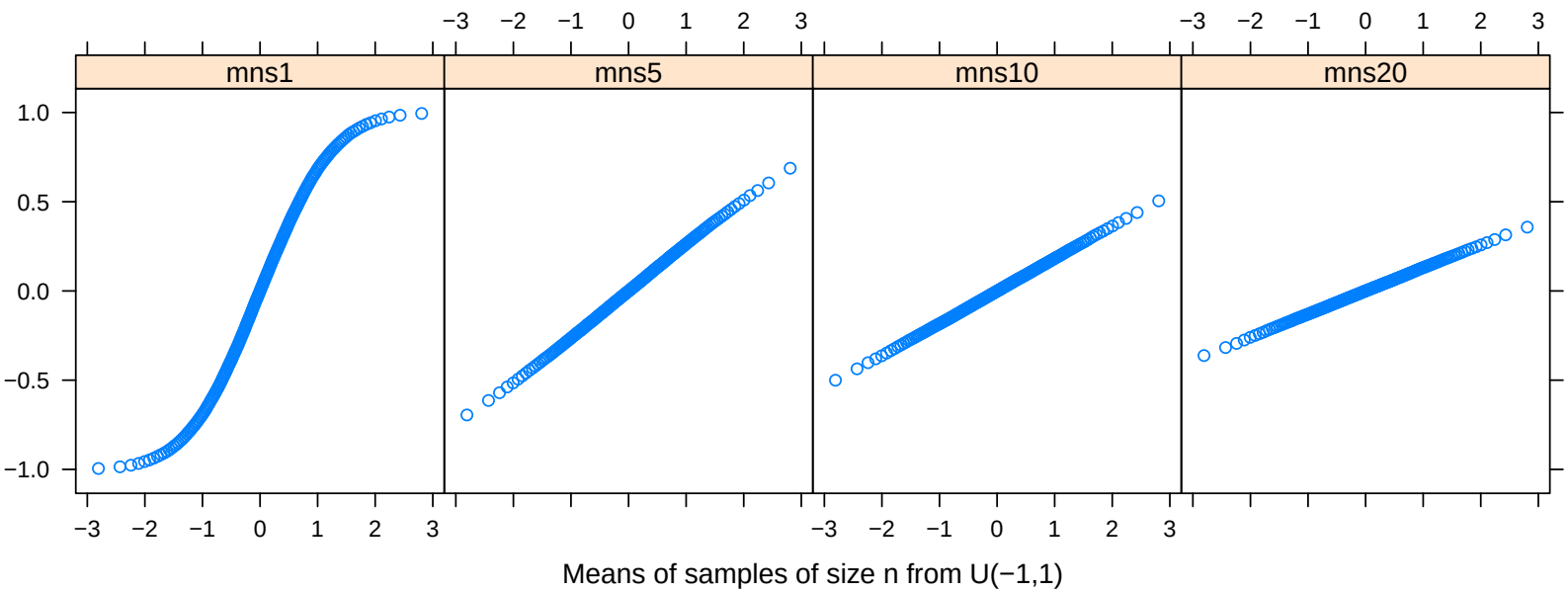
Shape of distribution of $\bar{X}_n$

- As n increases, the shape of the distribution of $\bar{X}_n$ tends to the "bell-curve" or Gaussian shape and it has less variability. That is, it tends to a "central limit".

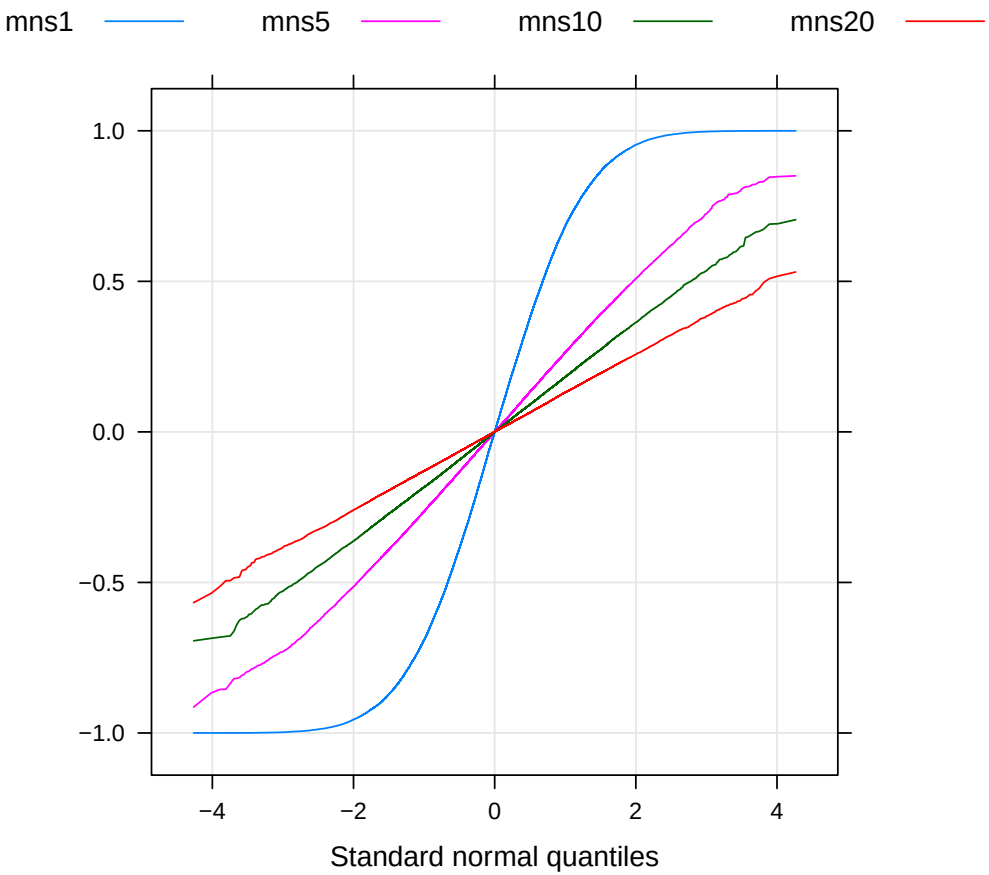


More detail on the shape of the distribution of $\bar{y}$

- In addition to the histogram we can use normal probability plots to evaluate the deviations of the distribution of $\bar{y}$ from normality.



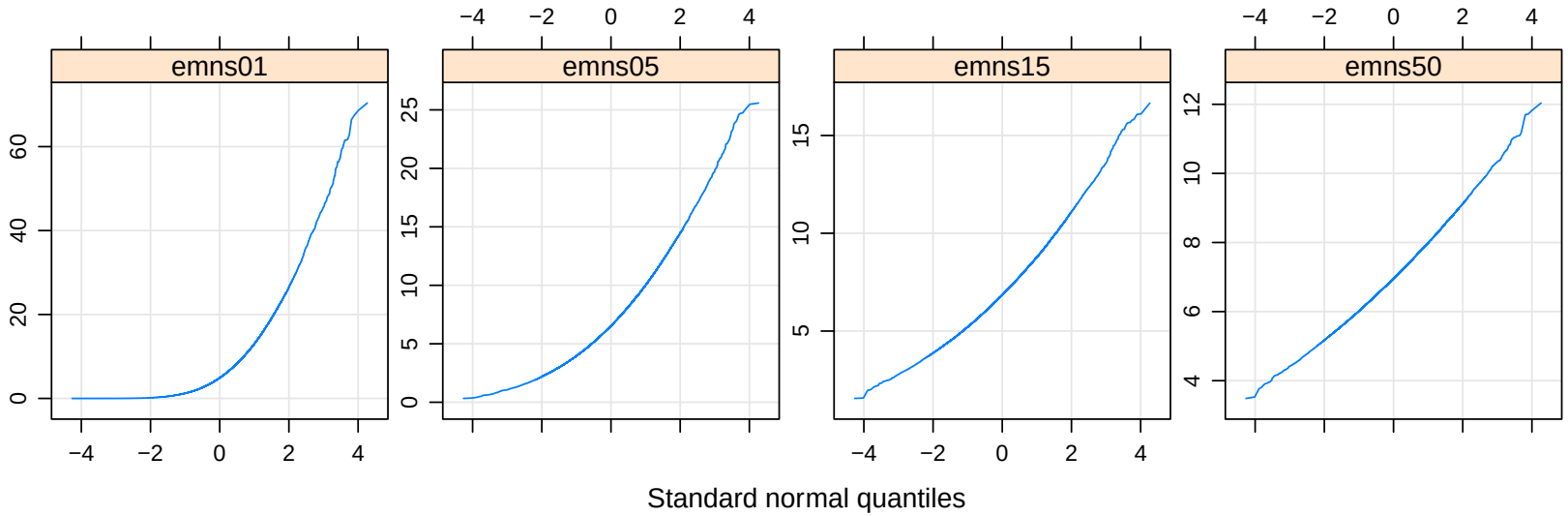
Overlaid normal probability plots for $\bar{\mathcal{Y}}_n$



The conclusion is that the distribution of means from an i.i.d. sample of a uniform distribution is very close to a normal, even for $n = 5$.

Sample means from an exponential distribution

```
> emns01 <- replicate(50000, mean(rexp(1, rate = 1/7)))
> emns05 <- replicate(50000, mean(rexp(5, rate = 1/7)))
> emns15 <- replicate(50000, mean(rexp(15, rate = 1/7)))
> emns50 <- replicate(50000, mean(rexp(50, rate = 1/7)))
```



Even for $n = 50$ there is noticeable skewness in the distribution (althought we would not be far wrong in assuming normality at $n = 50$).

Elementary uses of the C.L.T.

- If we have plausible values of the variance of our process, perhaps from a pilot study, we can use the normal distribution and the Central Limit Theorem (C.L.T.) to evaluate probabilities regarding the sample mean.
- Example 5.1.3 discusses product lifetimes that have an unknown mean and a variance of approximately 8 years. The number of products to sample so that we are 95% certain that $\bar{y}$ will be within 1 year of the true mean is derived from

$$0.95 = P(|\bar{\mathcal{Y}} - \mu| < 1)$$

The distribution of $\bar{\mathcal{Y}}$ will be approximately normal with mean μ and standard deviation $\sigma/\sqrt{n}$. For a standard normal, 95% of the probability is within "2" standard deviations of the mean (the actual multiple is `qnorm(0.025)= -1.95996`) so we want $1 = \texttt{qnorm}(0.025)^2 \frac{8}{n}$. That is, $n >$

```
> 8 * qnorm(0.025)^2
```

```
[1] 30.73167
```

Approximations for binomial or Poisson distributons

- The text describes approximations of the probabilities for a binomial or Poisson distribution based on the normal distributon.
- These are interesting from the point of view of understanding that these distributions will tend to have a "bell-curve" shape when n is large and p is moderate for the binomial or $\lambda\, t$ is large for the Poisson.
- In practice, though, you can evaluate probabilities for such distributions exactly so there is no need to use approximations.

Confidence intervals

- Our "best guess" at a parameter is called a *point estimate*. For example, we usually use the sample mean, $\bar{y}$, as the point estimate of μ.
- An *interval estimate* or *confidence interval* is an interval of plausible values for the parameter. Values outside the interval are "unreasonable" and values inside are "not unreasonable".
- To calibrate the meaning of "unreasonable" we assign a value α to the probability of getting data like we did or even more extreme when the parameter is outside. This corresponds to the "p-value" in a hypothesis test.
- The *coverage probability* or *confidence level* is $1 - \alpha$. Typically we set $\alpha = 0.05$ or $\alpha = 0.01$ resulting in 95% or 99% confidence intervals.
- Formally, the coverage probability is the probability that an interval constructed in this way will cover the true parameter value.

A confidence interval on μ

- In the unlikely event that someone were to tell us what the standard deviation, σ, of the population was but somehow not know much about the mean, μ, we could create a $(1-\alpha)$ confidence interval as

$$\bar{y} \pm z(\alpha/2)\frac{\sigma}{n}$$

where $z(\alpha/2)$ is the **upper** $\alpha/2$ quantile of the standard normal distribution.

- For example, the upper 0.025 quantile of the standard normal is

```
> qnorm(0.025, low = FALSE)
```

```
[1] 1.959964
```

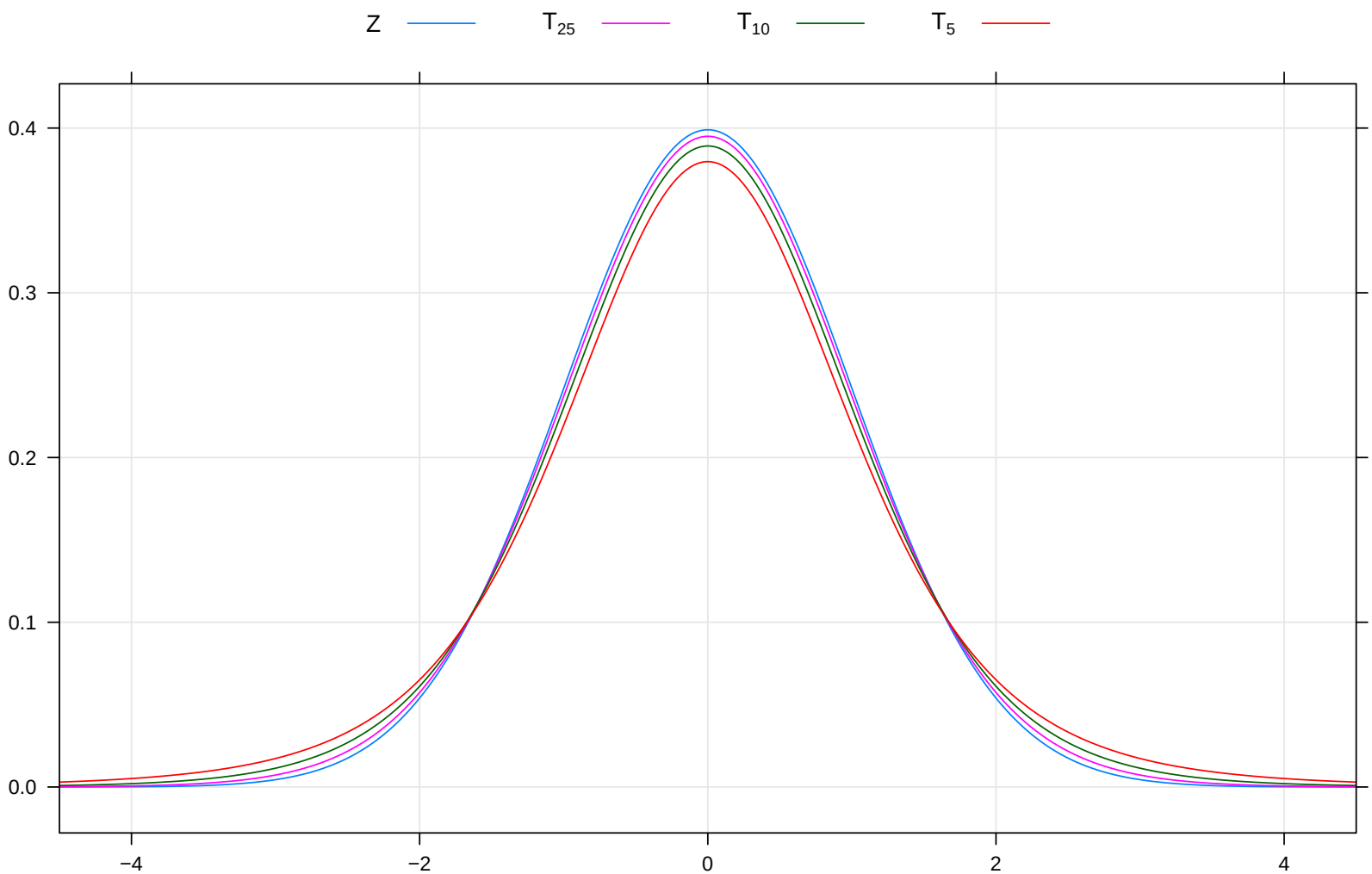
so a 95% confidence interval on μ for this artificial, "known sigma" case is

$$\bar{y} \pm 1.959964\frac{\sigma}{\sqrt{n}}$$

Use of Student's T distribution

- In the real world no one tells us what σ is and we must estimate it as s. A statistician named William Gossett, who wrote under the pseudonym "A Student", derived the distribution of the shifted, scaled sample mean when the scale is based on the estimate, s, not the theoretical value σ.
- This distribution is called the "Student's t distribution". It is similar to the standard normal distribution but a bit more spread out. The spreading depends on the number of "degrees of freedom" in the estimate of σ^2. The degrees of freedom are written as ν. For a single sample $\nu = n - 1$.
- As ν increases the T distribution approaches the standard normal. If we were using tables we would call anything with $\nu > 30$ a standard normal. When using a computer we don't bother.
- Notation: the t distribution with ν degrees of freedom is written $t(\nu)$. The corresponding *R* functions are `dt`, `pt`, `qt` and `rt`. The upper α quantile is written $t(\alpha; \nu)$.

Graphical comparison of $t(\nu)$ and $\mathcal{Z}$



General form of the confidence interval

- The general form of the confidence interval on μ is

$$\bar{y} \pm t\left(\frac{\alpha}{2}, n-1\right)\frac{s}{\sqrt{n}}$$

- We can use this formula for any values of n. If n is large we don't need strong assumptions on the shape of the original distribution. If n is small we must assume that the original distribution is close to the normal (but, of course, we can't check this with a small sample - a "Catch 22" situation).
- The *R* function to create this interval is `t.test`. The name comes from the corresponding hypothesis test, which we will discuss later.

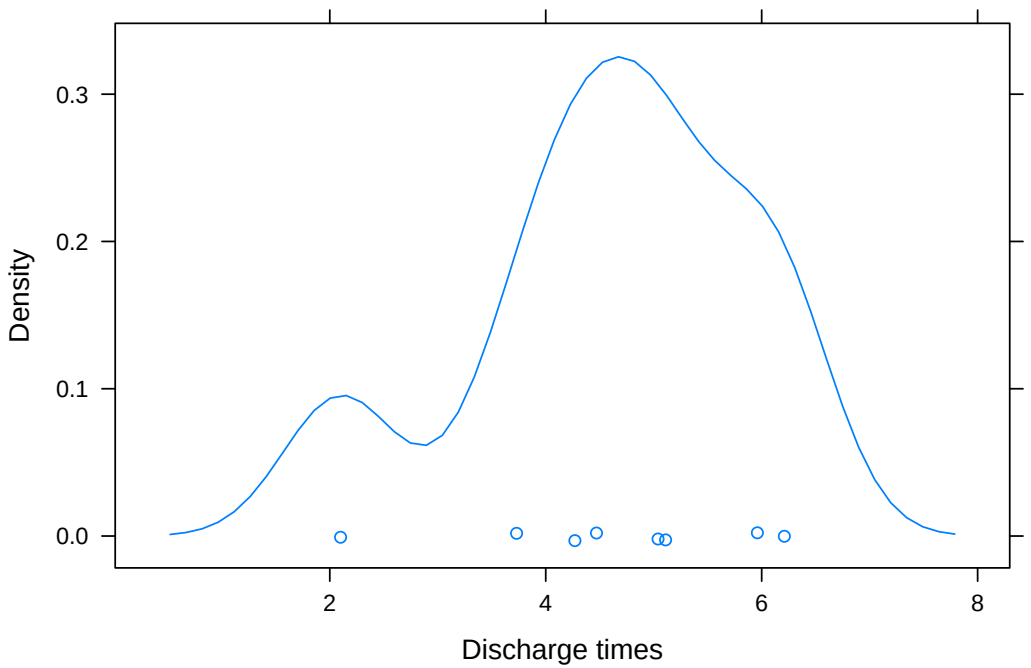
Example 5.2.2

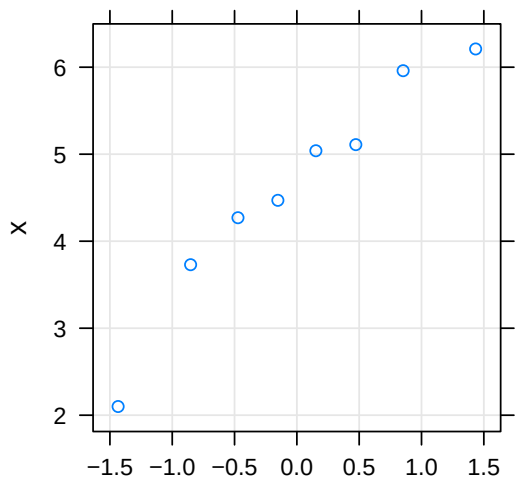
The example provides (probably fictitious) discharge times for a particular electric vehicle

```
> sd(charge <- c(5.11,2.1,4.27,5.04,4.47,3.73,5.96,6.21))

[1] 1.3108

> summary(charge)

   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
   2.10    4.14    4.76    4.61    5.32    6.21

> t.test(charge)

One Sample t-test
data:  charge
t = 9.9502, df = 7, p-value = 2.211e-05
alternative hypothesis: true mean is not equal to 0
95 percent confidence interval:
 3.5154 5.7071
sample estimates:
mean of x
   4.6113
```

Example 5.2.2 (cont'd)

Because the degrees of freedom, $\nu = 7$, are quite small we should check for normality.





Another *R* evaluation of confidence intervals

- Another way of evaluating a confidence intervals on μ is with the `confint` function, which provides confidence intervals on the parameters in a fitted model.
- To use `confint` we fit what we sometimes call the "trivial" model

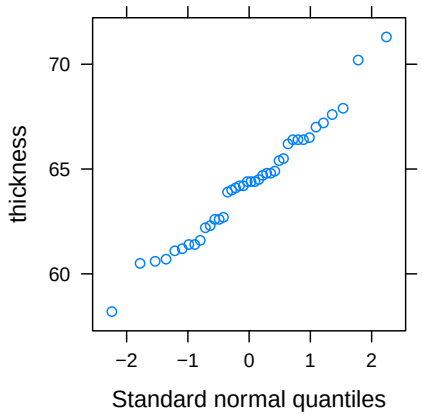
$$\mathcal{Y}_i = \mu + \epsilon_i, \quad i = 1, \ldots, n$$

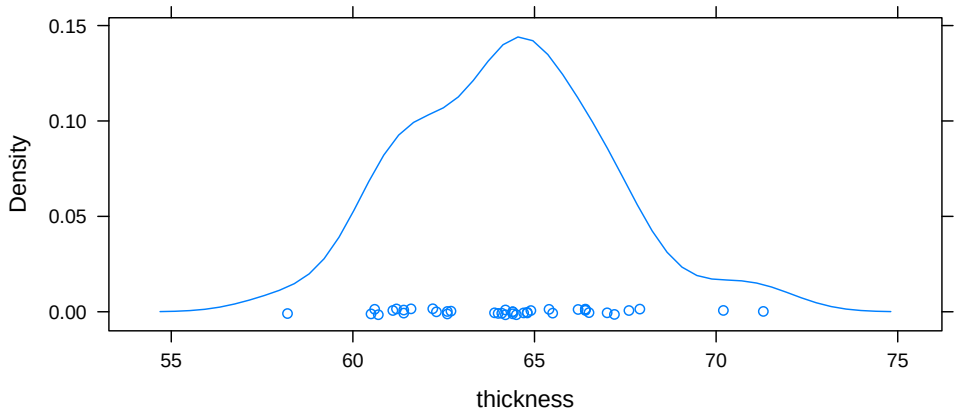
The estimate of μ, $\widehat{\mu} = \bar{y}$ will be called `(Intercept)` in the output. The formula for the model contains the constant term, `1`, as the only predictor.

```
> confint(fm1 <- lm(charge ~ 1))
```

```
             2.5 % 97.5 %
(Intercept) 3.5154 5.7071
```

Clear-coat thickness (example 5.2.4)






```
> with(ccthickn, summary(thickness))

   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
   58.2    62.3    64.4    64.3    66.2    71.3

> sd(ccthickn$thickness)

[1] 2.7176

> confint(fm2 <- lm(thickness ~ 1, ccthickn))

             2.5 % 97.5 %
(Intercept) 63.391 65.129
```

Clear-coat thickness (cont'd)

The summary of the fit of the "trivial" model includes many of the statistics from the data.

```
> summary(fm2)
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  64.2600     0.4297   149.5   <2e-16
Residual standard error: 2.718 on 39 degrees of freedom
```

The values in this summary include

$\bar{y} = 64.26$ The parameter estimate, $\hat{\mu}$.

$s = 2.718$ The sample standard deviation, $\hat{\sigma}$

$n - 1 = 39$ The degrees of freedom, ν, for the variance estimate, s^2.

$\frac{s}{\sqrt{n}} = 0.4297$ The standard error of the mean, $\sqrt{\mathrm{Var}(\bar{\mathcal{Y}})}$

Sample sizes

- The half-width of a confidence interval, also called the *margin of error* depends on

  The confidence level Higher confidence levels require wider intervals

  The standard deviation More variability in the original distribution results in wider intervals.

  The sample size Larger samples produce narrower intervals.
- Given a working value for σ we can determine the sample size needed to attain a given margin of error.
- If we are willing to assume that n is large we can use $z(\alpha/2)$ in the calculation. For small n it gets tricky because $\nu = n - 1$ determines the multiplier which, in turn, affects the sample size. We must solve a nonlinear equation but computers are good at that.

Sample size calculations

- Example 5.2.5 shows calculations for the sample size from the formula $n = \left[\frac{t(\alpha/2;\infty)s}{d}\right]^2$ when the desired margin of error, d, is 0.2, the working value of s is 0.4 and α is 5% and we round the answer to the next largest integer.

```
> ceiling((qnorm(0.025)*0.4/0.2)^2)
```

```
[1] 16
```

- Because this is a small value of n we should instead solve for n in $n = \left[\frac{t(\alpha/2;n-1)s}{d}\right]^2$

```
> ceiling(uniroot(function(x) x-(qt(.025,x-1)*0.4/0.2)^2,
+                 c(2,100))$root)
```

```
[1] 18
```

Section 5.3: Prediction and tolerance intervals

- A confidence interval on μ provides a measure of the precision of the information regarding the unknown population parameter. It does not directly tell us about bounds on where we expect a future observation to fall.
- A *prediction interval* indicates where a single future observation is likely to be.
- A *tolerance interval* indicates where a large proportion of the population is likely to be.
- Unlike the confidence interval on μ, prediction intervals and tolerance intervals depend strongly on the shape of the distribution of the data.
- In theory one can make a confidence interval arbitrarily narrow by taking a sufficiently large sample. You can't do this for a prediction interval.

Prediction intervals on a future observation

- If it is reasonable to assume that the *data* (i.e. $\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_n$) are from normal distribution then we could say that a model for the data is

$$\mathcal{Y}_i = \mu + \epsilon_i, \quad \epsilon_i \sim \mathcal{N}(0, \sigma^2)$$

- Our estimate $\widehat{\mu} = \bar{\mathcal{Y}}_n$ is independent of ϵ_{n+1}. The variability in the difference between $\mathcal{Y}_{n+1}$ and $\bar{\mathcal{Y}}_n$ is the sum of the variability in $\bar{\mathcal{Y}}_n - \mu$ $(\frac{\sigma^2}{n})$ and the variability in ϵ_{n+1} (σ^2).
- Because we estimate σ^2 the $(1-\alpha)$ prediction interval becomes

$$\bar{y} \pm t(\alpha/2; n-1) s \sqrt{1 + \frac{1}{n}}$$

Evaluating a prediction interval

- The prediction interval could be evaluated according to the formula. For the clear-coat thickness data the 95% prediction interval on a future thickness measurement is

```
> with(ccthickn, mean(thickness) + c(-1,1) * qt(0.975, 39) *
+       sd(thickness) * sqrt(1 + 1/40))
```

```
[1] 58.69478 69.82522
```

- An alternative is to use the `predict` function applied to the trivial model and with the optional argument `interval = "pred"`. This produces a matrix with n rows that are identical so I just look at the first row.

```
> predict(fm2, int = "pred")[1,]
```

```
     fit      lwr      upr
64.26000 58.69478 69.82522
```

Tolerance intervals

- A tolerance interval is more difficult to describe and to calculate than is a prediction interval.
- Methods for tolerance intervals are given in the text but we will not cover this topic in this course.

Section 5.4 Hypothesis tests

- A hypothesis test is a procedure for deciding if a particular value of a parameter is reasonable, given the observed data.
- We have a probability model (e.g. our sample is a random sample from a normal distribution with mean μ and variance σ^2), the observed data, $y_1, y_2, \ldots, y_n$ and a particular value of the parameter in mind (e.g. the mean clear-coat thickness should be 65 microns).
- We consider two competing claims called the *null hypothesis*, written H_0, and the *alternative hypothesis*, written H_a.
- H_0 is the "no change" assumption. For our example, it is $\mu = 65$. H_a is the result we are trying to establish. It is also the result indicated by the data. In our example $\bar{y} = 64.26$ microns. If we are interested only in whether we are "on target" then $H_a : \mu \neq 65$. If we are interested in whether the clear coats are systematically too thin then $H_a : \mu < 65$.
- These are called "two-tailed" and "one-tailed" alternatives, respectively.