Clayton–Weibull model

Clayton copula

The joint survival function is

$$S_{J_0}(\mathbf{t}_{J_0}) = \left(1 + \sum_{j\in J_0} \left[S_j(t_j)^{-\theta} - 1\right]\right)^{-1/\theta}, \qquad \mathbf{t}_{J_0} = (1,2,3), \tag{1}$$

with $\theta > 0$ and one can easily show that the derivatives are of the form

$$\frac{\partial^i}{\partial t_{(1)} \cdots \partial t_{(i)}} S_{J_0}(\mathbf{t}_{J_0}) = (-1)^i \prod_{h=1}^{i} \left[(1+(h-1)\theta) f_h(t_h) \left(S_h(t_h)\right)^{-\theta-1}\right] \left(S_{J_0}(\mathbf{t}_{J_0})\right)^{1+i\theta}.$$

Therefore the conditional survival distributions are

$$\begin{aligned} S_{(k)|(1),\ldots,(k-1)}\left(t_{(k)}|t_{(1)},\ldots,t_{(k-1)}\right) =& \frac{(-1)^{k-1}\prod_{h=1}^{k-1}\left[(1+(h-1)\theta) f_h(t_h)\left(S_h(t_h)\right)^{-\theta-1}\right]}{(-1)^{k-1}\prod_{h=1}^{k-1}\left[(1+(h-1)\theta) f_h(t_h)\left(S_h(t_h)\right)^{-\theta-1}\right]} \\ &\times \frac{\left(S_{J_0}\left(t_{(1)},\ldots,t_{(k)},0,\ldots,0\right)\right)^{1+(k-1)\theta}}{\left(S_{J_0}\left(t_{(1)},\ldots,t_{(k-1)},0,\ldots,0\right)\right)^{1+(k-1)\theta}} \\ =& \left(1 + \frac{S_k(t_k)^{-\theta} - 1}{1+\sum_{j=1}^{k-1}(S_j(t_j)^{-\theta}-1)}\right)^{1-k-\frac{1}{\theta}} \qquad k = 2,3. \end{aligned} \tag{2}$$

The times can be simulated as

$$T_k|T_1,\ldots,T_{k-1} = S_k^{-1}\left(\left\{1 + \left[U_k^{\theta/(\theta(1-k)-1)} - 1\right]\left[1 + \sum_{j=1}^{k-1}\left(S_j(t_j)^{-\theta} - 1\right)\right]\right\}^{-\frac{1}{\theta}}\right), \tag{3}$$

with $U_k \sim \mathrm{U}(0,1)$.

Weibull marginals

In case of Weibull marginal survival functions we have

$$S_k(t_k) = \exp(-\lambda_k t_k^{\rho_k}) \qquad \text{and} \qquad S_k^{-1}(u_k) = \left(-\log(u_k)/\lambda_k\right)^{1/\rho_k}.$$

Therefore equation (3) is in this case

$$T_k|T_1,\ldots,T_{k-1} = \left(\frac{1}{\theta\lambda_k}\log\left\{1 + \left[U_k^{-\theta/((k-1)\theta+1)} - 1\right]\left[1 + \sum_{j=1}^{k-1}\left(\exp(\theta\lambda_j t_j^{\rho_j}) - 1\right)\right]\right\}\right)^{1/\rho_k},$$

which is

$$T_2|T_1 = \left(\frac{1}{\theta\lambda_2}\log\left\{1 + \left[U_2^{-\theta/(\theta+1)} - 1\right]\exp(\theta\lambda_1 t_1^{\rho_1})\right\}\right)^{1/\rho_2} \qquad \text{for } k=2 \text{ and}$$

$$T_3|T_1,T_2 = \left(\frac{1}{\theta\lambda_3}\log\left\{1 + \left[U_3^{-\theta/(2\theta+1)} - 1\right]\left[\exp(\theta\lambda_1 t_1^{\rho_1}) + \exp(\theta\lambda_2 t_2^{\rho_2}) - 1\right]\right\}\right)^{1/\rho_3} \qquad \text{for } k=3.$$