

# Parameterization of copula functions for bivariate survival data in the `surrosurv` package

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Let define the joint survival function of  $S$  and  $T$  via a copula function:

$$S(s, t) = P(S \geq s, T \geq t) = C(u, v)|_{u=S_S(s), v=S_T(t)}, \quad (1)$$

where  $S_S(\cdot) = P(S \geq s)$  and  $S_T(\cdot) = P(T \geq t)$  are the marginal survival functions of  $S$  and  $T$ .

In the case of possibly right-censored data, the individual contribution to the likelihood is

- $S(s, t) = C(u, v)|_{S_S(s), S_T(t)}$  if  $S$  is censored at time  $s$  and  $T$  is censored at time  $t$ ,
- $-\frac{\partial}{\partial t} S(s, t) = \frac{\partial}{\partial v} C(u, v)|_{S_S(s), S_T(t)} f_S(s)$  if  $S$  is censored at time  $s$  and  $T = t$ ,
- $-\frac{\partial}{\partial s} S(s, t) = \frac{\partial}{\partial u} C(u, v)|_{S_S(s), S_T(t)} f_T(t)$  if  $S = s$  and  $T$  is censored at time  $t$ ,
- $\frac{\partial^2}{\partial s \partial t} S(s, t) = \frac{\partial^2}{\partial u \partial v} C(u, v)|_{S_S(s), S_T(t)} f_S(s) f_T(t)$  if  $S = s$  and  $T = t$ .

## Clayton copula

The bivariate Clayton [1978] copula is defined as

$$C(u, v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}, \quad \theta > 0. \quad (2)$$

The first derivative with respect to  $u$  is

$$\begin{aligned} \frac{\partial}{\partial u} C(u, v) &= (u^{-\theta} + v^{-\theta} - 1)^{-\frac{1+\theta}{\theta}} u^{-(1+\theta)} \\ &= \left[ \frac{C(u, v)}{u} \right]^{1+\theta}. \end{aligned} \quad (3)$$

The second derivative with respect to  $u$  and  $v$  is

$$\frac{\partial^2}{\partial u \partial v} C(u, v) = (1 + \theta) \frac{C(u, v)^{1+2\theta}}{(uv)^{1+\theta}}. \quad (4)$$

The Kendall [1938]'s tau for the Clayton copula is

$$\tau = \frac{\theta}{\theta + 2}. \quad (5)$$

## Plackett copula

The bivariate Plackett [1965] copula is defined as

$$C(u, v) = \frac{[Q - R^{1/2}]}{2(\theta - 1)}, \quad \theta > 0, \quad (6)$$

with

$$\begin{aligned} Q &= 1 + (\theta - 1)(u + v), \\ R &= Q^2 - 4\theta(\theta - 1)uv. \end{aligned} \quad (7)$$

Given that

$$\frac{\partial}{\partial u} Q = \theta - 1, \quad (8)$$

$$\frac{\partial}{\partial u} R = 2(\theta - 1)(1 - (\theta + 1)v + (\theta - 1)u), \quad (9)$$

the first derivative of  $C(u, v)$  with respect to  $u$  is

$$\begin{aligned} \frac{\partial}{\partial u} C(u, v) &= \frac{1}{2} \left[ 1 - \frac{1 - (\theta + 1)v + (\theta - 1)u}{R^{1/2}} \right] \\ &= \frac{1}{2} \left[ 1 - \frac{Q - 2\theta v}{R^{1/2}} \right]. \end{aligned} \quad (10)$$

By defining

$$f = 1 - (\theta + 1)v + (\theta - 1)u, \quad (11)$$

$$g = R^{1/2} \quad (12)$$

and given that

$$f' = \frac{\partial}{\partial u} f = -(\theta + 1), \quad (13)$$

$$g' = \frac{\partial}{\partial u} g = \frac{\theta - 1}{R^{1/2}} (1 - (\theta + 1)u + (\theta - 1)v), \quad (14)$$

then, the second derivative with respect to  $u$  and  $v$  is

$$\begin{aligned} \frac{\partial^2}{\partial u \partial v} C(u, v) &= -\frac{f'g - fg'}{2g^2} \\ &= -\frac{\theta}{R^{3/2}} [1 + (\theta - 1)(u + v - 2uv)] \\ &= -\frac{\theta}{R^{3/2}} [Q - 2(\theta - 1)uv]. \end{aligned} \quad (15)$$

The Kendall's tau for the Plackett copula cannot be computed analytically and is obtained numerically.

## Gumbel–Hougaard copula

The bivariate Gumbel [1960]–Hougaard [1986] copula is defined as

$$C(u, v) = \exp(-Q^\theta), \quad \theta \in (0, 1), \quad (16)$$

with

$$Q = (-\ln u)^{1/\theta} + (-\ln v)^{1/\theta}. \quad (17)$$

Given that

$$\frac{\partial}{\partial u} Q = -\frac{(-\ln u)^{1/\theta-1}}{\theta u}, \quad (18)$$

then, the first derivative with respect to  $u$  is

$$\frac{\partial}{\partial u} C(u, v) = \frac{(-\ln u)^{1/\theta-1}}{u} C(u, v) Q^{\theta-1} \quad (19)$$

and the second derivative with respect to  $u$  and  $v$  is

$$\frac{\partial^2}{\partial u \partial v} C(u, v) = \frac{[(-\ln u)(-\ln v)]^{1/\theta-1}}{uv} C(u, v) Q^{\theta-2} \left[ \frac{1}{\theta} - 1 + Q^\theta \right]. \quad (20)$$

The Kendall's tau for the Gumbel–Hougaard copula is

$$\tau = 1 - \theta. \quad (21)$$

## References

- DG Clayton. A model for association in bivariate life tables and its application in epidemiological studies of familial tendency in chronic disease incidence. *Biometrika*, 65:141–151, 1978. doi:10.1093/biomet/65.1.141.
- EJ Gumbel. Distributions des valeurs extrêmes en plusieurs dimensions. *Publ Inst Statist Univ Paris*, 9:171–173, 1960.
- P Hougaard. A class of multivariate failure time distributions. *Biometrika*, 73:671–678, 1986. doi:10.1093/biomet/73.3.671.
- MG Kendall. A new measure of rank correlation. *Biometrika*, 30(1/2):81–93, 1938. URL <http://www.jstor.org/stable/2332226>.
- RL Plackett. A class of bivariate distributions. *J Am Statist Assoc*, 60:516–522, 1965. doi:10.1080/01621459.1965.10480807.