Calibration of the Svensson model to simulated yield curves

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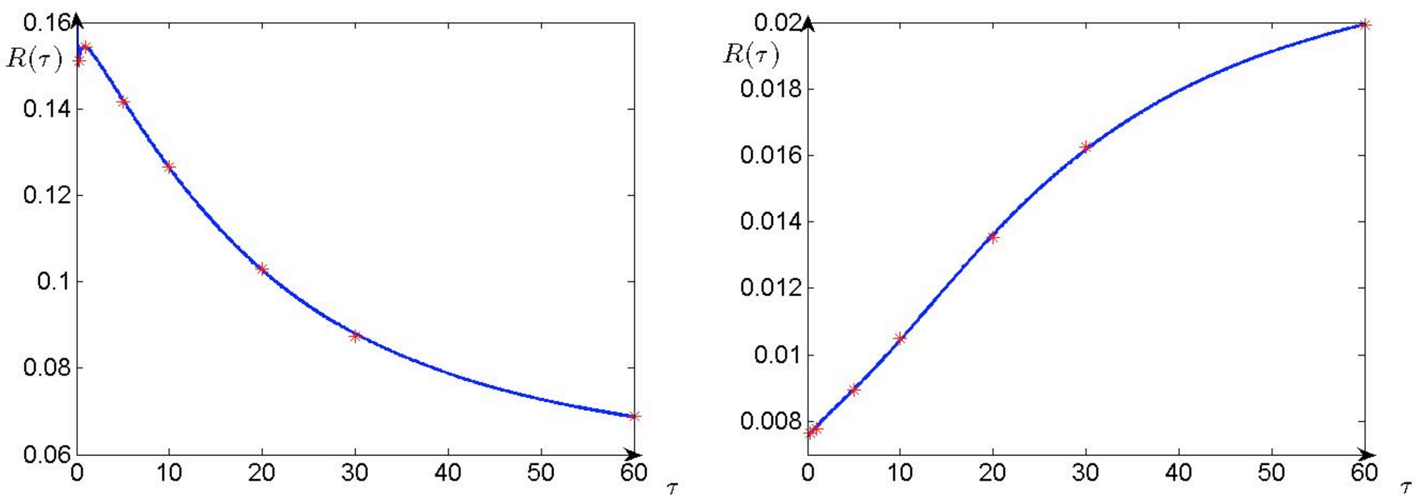
Introduction
Notation



- T is the time of maturity, t is the current time, $\tau = T - t$ the time to maturity.
- $P(t, T)$ is the price of a zero coupon bond at time t with maturity T.
- $R(t, T) = -\frac{1}{\tau} \ln(P(t, T))$ is the zero rate.
- $f(t, T) = -\frac{\partial}{\partial T} \ln(P(t, T))$ is the instantaneous forward rate.

Introduction
Problem formulation

Aim: approximate the given discrete yield curve by a continuous function, which is defined over the entire maturity domain ($\tau \geq 0$).

$\Rightarrow$ Parametric models, which assume for the instantaneous forward rate:

$$\hat{f}(\tau) = \sum_{i=1}^{K} \alpha_i \varphi_i(\beta_i, \tau).$$

By integration this gives the approximation of the zero rates:

$$\hat{R}(\tau) = \frac{1}{\tau} \left(\sum_{i=1}^{K} \alpha_i \int_t^T \varphi_i(\beta_i, s) ds \right).$$

The corresponding parameters α_i and β_i can be determined by solving the following minimization problem:

$$\sum_{j=1}^{J} \left(R(\tau_j) - \hat{R}(\tau_j) \right)^2 \longrightarrow \min .$$

Where $R(\tau_j)$ are the given zero rates for τ_j, with $j = \{1, ..., J\}$.

Fitting of yield curves
The Nelson and Siegel method (1987)

Assume the following functional form for the instantaneous forward rate:

$$\hat{f}(\tau) = \alpha_0 + \alpha_1 \exp\left(-\frac{\tau}{\beta_1}\right) + \alpha_2 \left(\frac{\tau}{\beta_1} \exp\left(-\frac{\tau}{\beta_1}\right)\right).$$

Integration leads to:

$$\hat{R}(\tau) = \alpha_0 + \alpha_1 \left(\frac{1 - \exp\left(-\frac{\tau}{\beta_1}\right)}{\frac{\tau}{\beta_1}}\right) + \alpha_2 \left(\frac{1 - \exp\left(-\frac{\tau}{\beta_1}\right)}{\frac{\tau}{\beta_1}} - \exp\left(-\frac{\tau}{\beta_1}\right)\right).$$

- $\alpha_0 > 0$ is the long term zero rate.
- $\alpha_0 + \alpha_1 > 0$ is the short rate.
  Hence α_1 represents the deviation from the asymptote α_0.
- α_2 determines the height and direction of the hump.
- $\beta_1 > 0$ determines the position of the hump.

Fitting of yield curves
The Svensson method (1994)

Extention of the Nelson and Siegel model, which improves the flexibility of the curves.

$$\hat{f}(\tau) = \alpha_0 + \alpha_1 \exp\left(-\frac{\tau}{\beta_1}\right) + \alpha_2 \left(\frac{\tau}{\beta_1} \exp\left(-\frac{\tau}{\beta_1}\right)\right) + \alpha_3 \left(\frac{\tau}{\beta_2} \exp\left(-\frac{\tau}{\beta_2}\right)\right).$$

$$\hat{R}(\tau) = \alpha_0 + \alpha_1 \left(\frac{1 - \exp\left(-\frac{\tau}{\beta_1}\right)}{\frac{\tau}{\beta_1}}\right) +$$

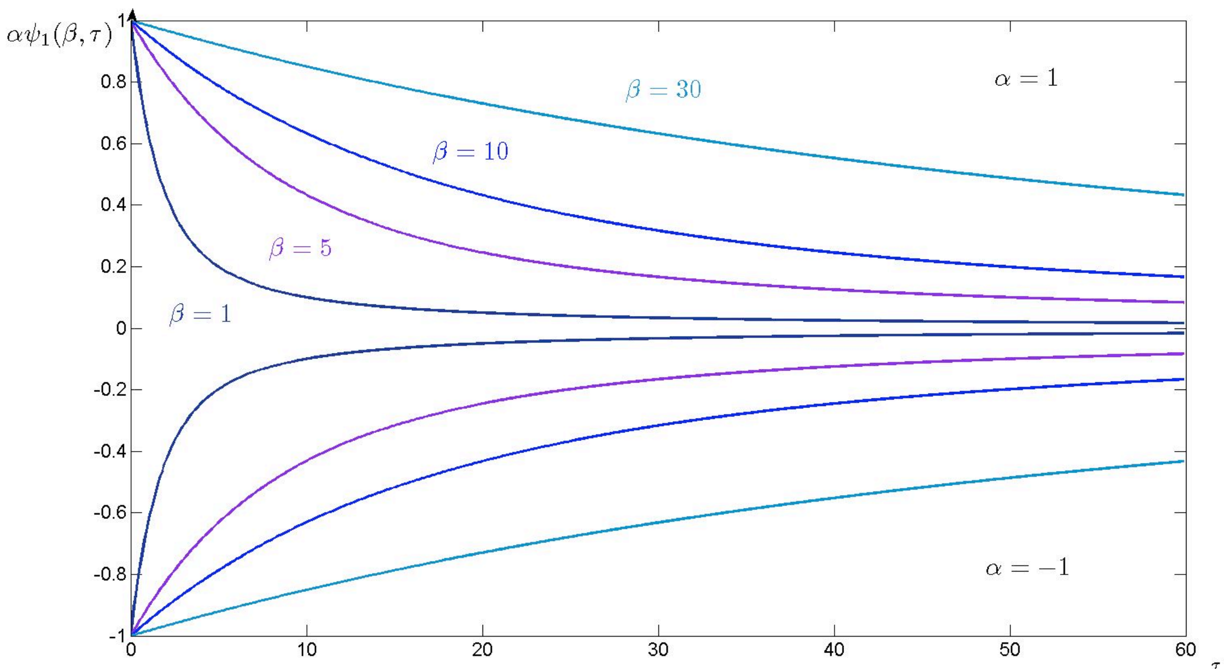
$$\alpha_2 \left(\frac{1 - \exp\left(-\frac{\tau}{\beta_1}\right)}{\frac{\tau}{\beta_1}} - \exp\left(-\frac{\tau}{\beta_1}\right)\right) + \alpha_3 \left(\frac{1 - \exp\left(-\frac{\tau}{\beta_2}\right)}{\frac{\tau}{\beta_2}} - \exp\left(-\frac{\tau}{\beta_2}\right)\right).$$

The new parameters α_3 and $\beta_2 > 0$ determine the height, direction and position of the second hump.

The Svensson method

The different components of $\hat{R}(\tau) = \sum_{i=0}^{4} \alpha_i \psi_i(\beta_i, \tau)$ are:

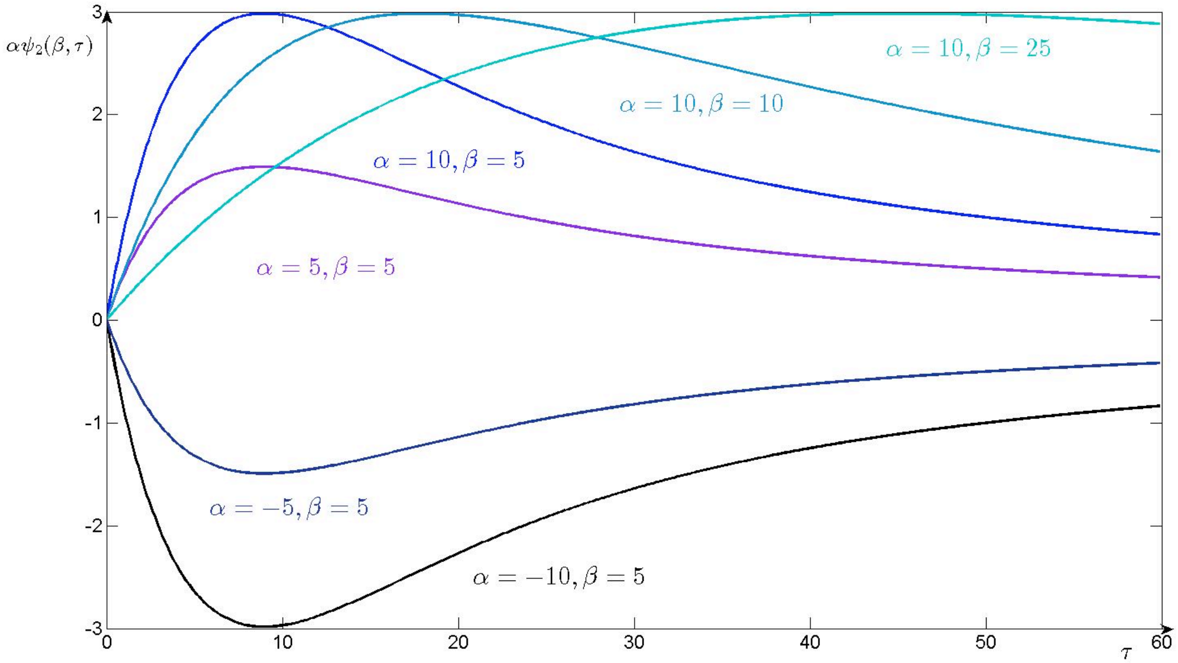
- $\alpha\psi_0(\beta, \tau) = \alpha$
- $\alpha\psi_1(\beta, \tau) = \alpha \left(\frac{1-\exp\left(-\frac{\tau}{\beta}\right)}{\frac{\tau}{\beta}} \right)$



The Svensson method

The different components of $\hat{R}(\tau) = \sum_{i=0}^{4} \alpha_i \psi_i(\beta_i, \tau)$ are:

- $$\alpha\psi_2(\beta,\tau) = \alpha\left(\frac{1-\exp\left(-\frac{\tau}{\beta}\right)}{\frac{\tau}{\beta}} - \exp\left(-\frac{\tau}{\beta}\right)\right)$$



Calibration of the Svensson model
Formulation of the optimization problem

We determine the parameters $\alpha \in \mathbb{R}^4$ and $\beta \in \mathbb{R}^2$ by minimization of the objective function

$$F(\alpha, \beta) = \sum_{j=1}^{J} \left(R(\tau_j) - \hat{R}(\tau_j) \right)^2 .$$

This can be written as the following minimization problem:

$$\min_{(\alpha,\beta)\in\mathbb{R}^4\times\mathbb{R}^2} \quad F(\alpha, \beta)$$

$$s.t. \quad \alpha_0, \beta_1, \beta_2 \geq \epsilon$$

$$\alpha_0 + \alpha_1 \geq \epsilon.$$

To ensure reasonable solutions one can introduce box-constraints:

$$(\alpha, \beta) \in G = \{(\alpha, \beta) \in \mathbb{R}^6 : lb \leq (\alpha, \beta) \leq ub\}.$$

For example:

$$lb = [\epsilon, -1, -10, -10, \epsilon, \epsilon]$$

$$ub = [1, 10, 10, 10, 50, 50].$$

Calibration of the Svensson model
Traditional method to solve the optimization problem

To solve the following nonlinear optimization problem:

$$\begin{aligned} \min_{(\alpha,\beta)\in\mathbb{R}^4\times\mathbb{R}^2} \quad & F(\alpha,\beta) \\ s.t. \quad & A\alpha \leq b \\ & lb \leq (\alpha,\beta) \leq ub \end{aligned}$$

one can use `fmincon` in Matlab which is local convergent.

- we need a good starting point.
- the solution might be a local minimum.

Therefore we will improve bad solutions by using a global method.

1. Which are “bad” solutions?
2. Which global method is appropriate?

Calibration of the Svensson model

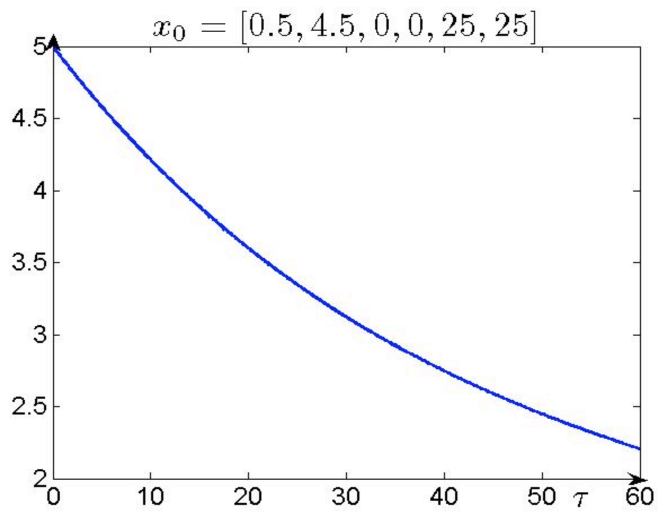
Traditional method to solve the optimization problem

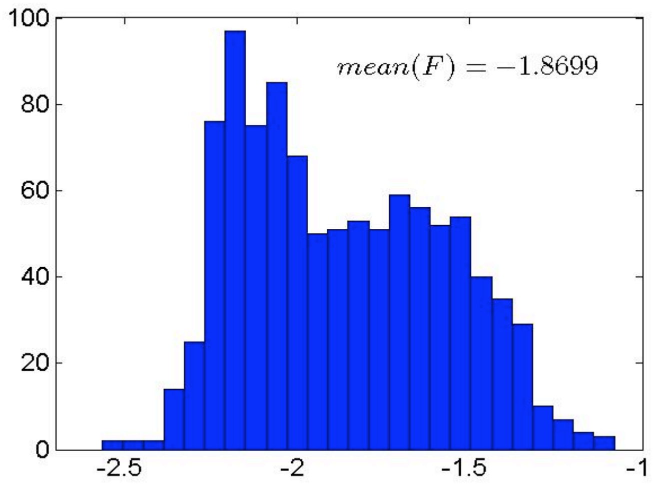
Some thoughts about the starting point:

1st idea:

$$x_0 = \frac{lb + ub}{2} = (0.5, 4.5, 0, 0, 25, 25)$$

this leads to the following yield curve and solutions of `fmincon` over 1000 scenarios with 7 given times to maturity:

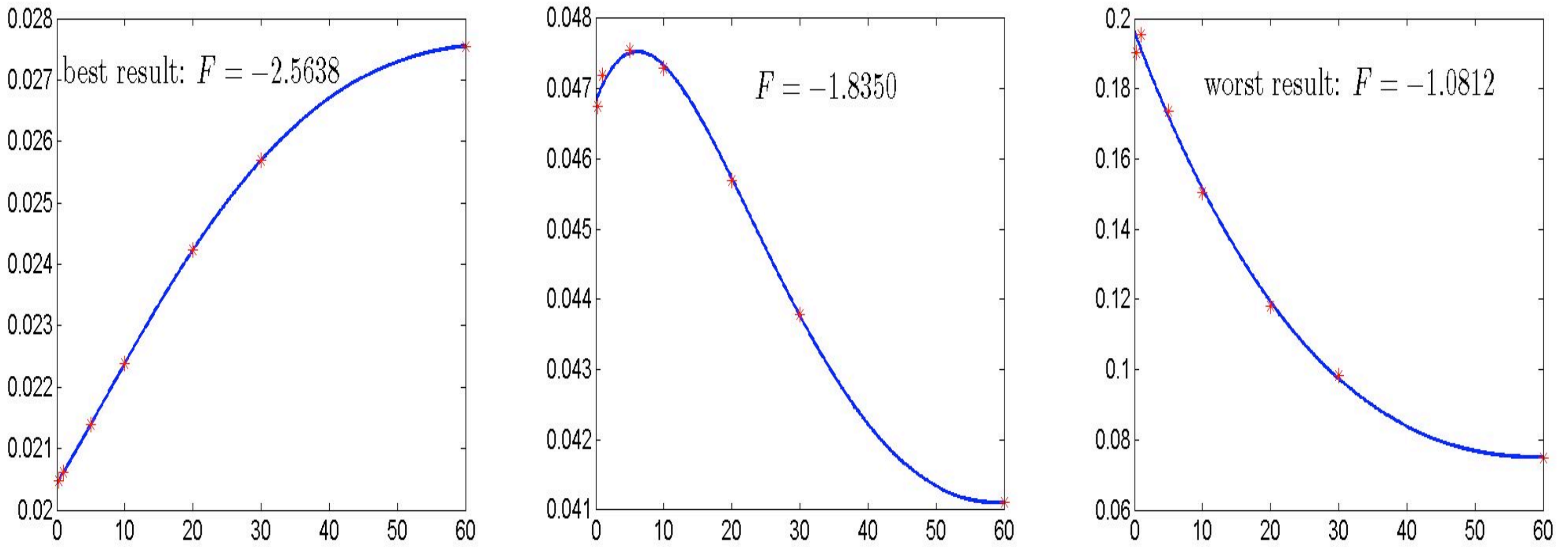




The objective value is $F = \log_{10}(\sqrt{f})$, where $f = \sum_{j=1}^{J} \left(R(\tau_j) - \hat{R}(\tau_j) \right)^2$.

Calibration of the Svensson model
Traditional method to solve the optimization problem

Some examples of the solutions with this starting point:



The 1st figure shows the best result, the 2nd shows one with an average objective value and the 3rd shows the worst result.

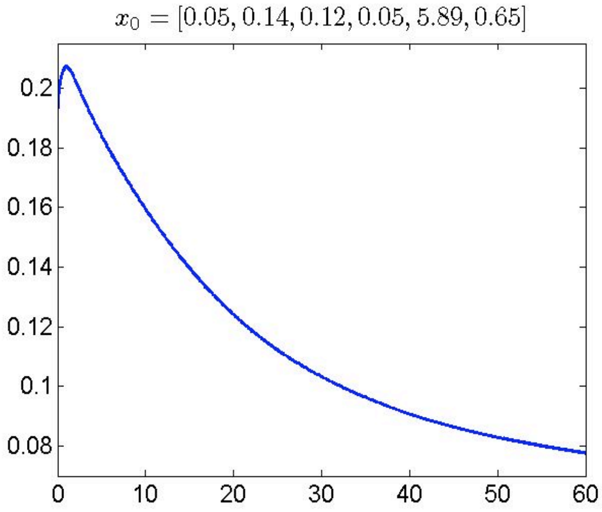
Calibration of the Svensson model
Traditional method to solve the optimization problem

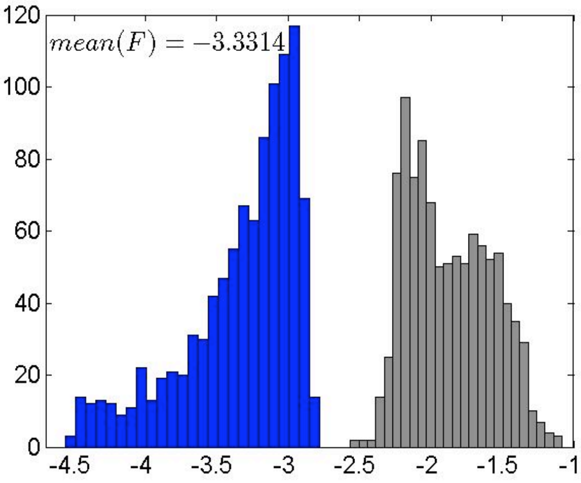
Some thoughts about the starting point:

2nd idea:

$$x_0 = (0.0512, 0.1422, 0.1217, 0.0538, 5.8940, 0.6510)$$

this leads to the following yield curve and solutions of `fmincon` over 1000 scenarios:



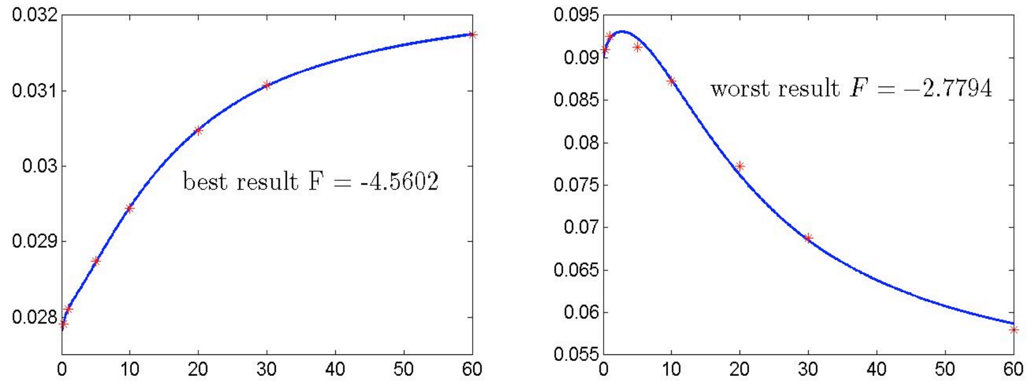


The computation time is reduced by 80%, the average objective value is improved by 1.46.

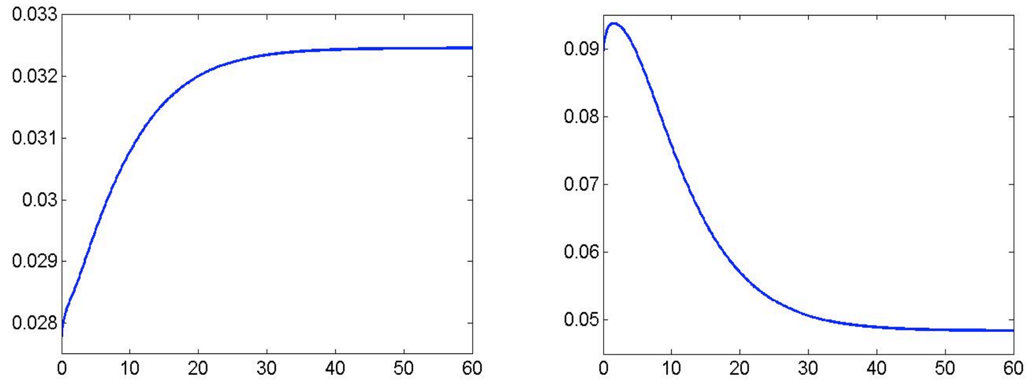
Calibration of the Svensson model

Results of the optimization with `fmincon`.

The best and worst solutions of the optimization with the 2nd starting point are:



The corresponding instantaneous forward rates are:



Calibration of the Svensson model
Improvement by global optimization

We use an adaptive, iterative method which is based on the HCP algorithm (Novak, Ritter 1995) and uses sparse grids.

Advantages and disadvantages of this method:

- converges to the global minimum of the objective function.
- the objective function must (only) be in $C^0(G)$.
- slow for difficult objective functions and high dimensional problems.

As this method is too slow for our application we have to reformulate the optimization problem.

Calibration of the Svensson model
Reformulation of the problem: splitting

The original optimization problem

$$\min_{(\alpha,\beta)\in\mathbb{R}^4\times\mathbb{R}^2} \quad F(\alpha,\beta)$$

$$s.t. \quad A\alpha \leq b$$

$$lb \leq (\alpha,\beta) \leq ub.$$

is equivalent to the nondifferentiable problem

$$\min_{\beta\in\mathbb{R}^2} \quad H(\beta)$$

$$s.t. \quad lb \leq \beta \leq ub.$$

Where H is the following optimization problem:

$$H(\beta) = \min_{\alpha\in\mathbb{R}^4} \quad F(\alpha,\beta)$$

$$s.t. \quad A\alpha \leq b$$

$$lb \leq \alpha \leq ub.$$

Calibration of the Svensson model
Reformulation of the problem: splitting

We solve

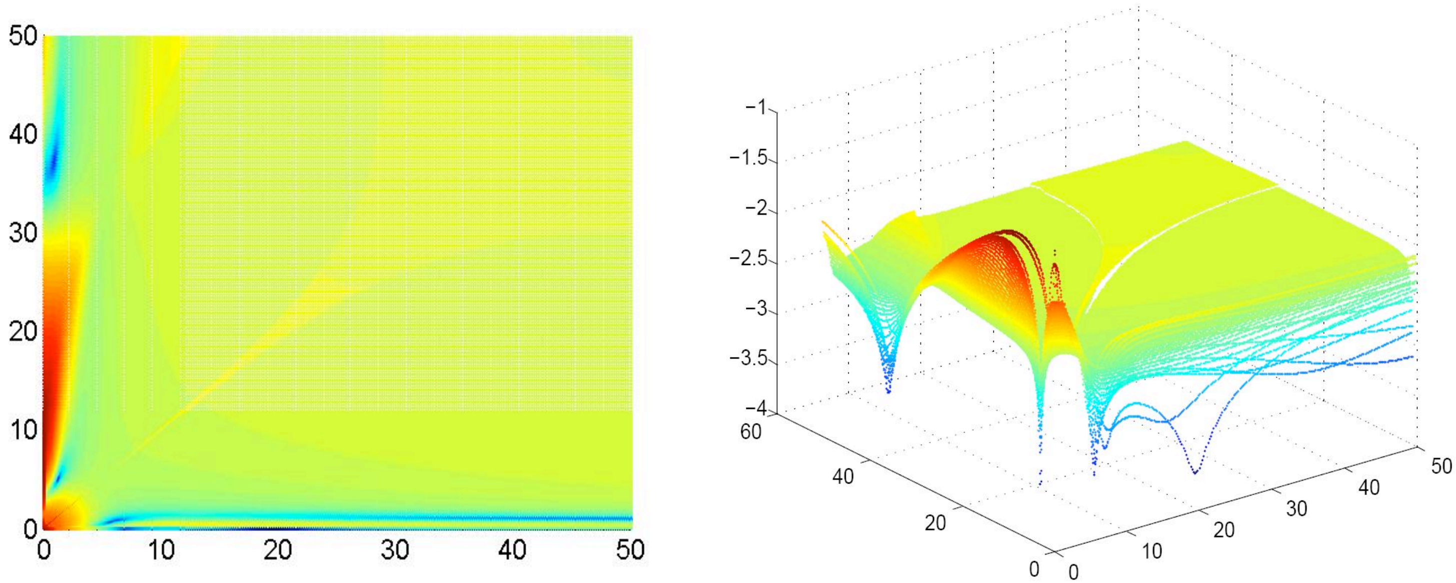
$$\min_{\beta \in \mathbb{R}^2} \quad H(\beta)$$
$$s.t. \quad lb \leq \beta \leq ub$$

where $lb = [\epsilon, \epsilon]$ and $ub = [50, 50]$ with the global optimization algorithm.

- global optimization on two parameters is much faster than in six parameters.
- the evaluation of the objective function takes longer (one easy minimization problem per evaluation).

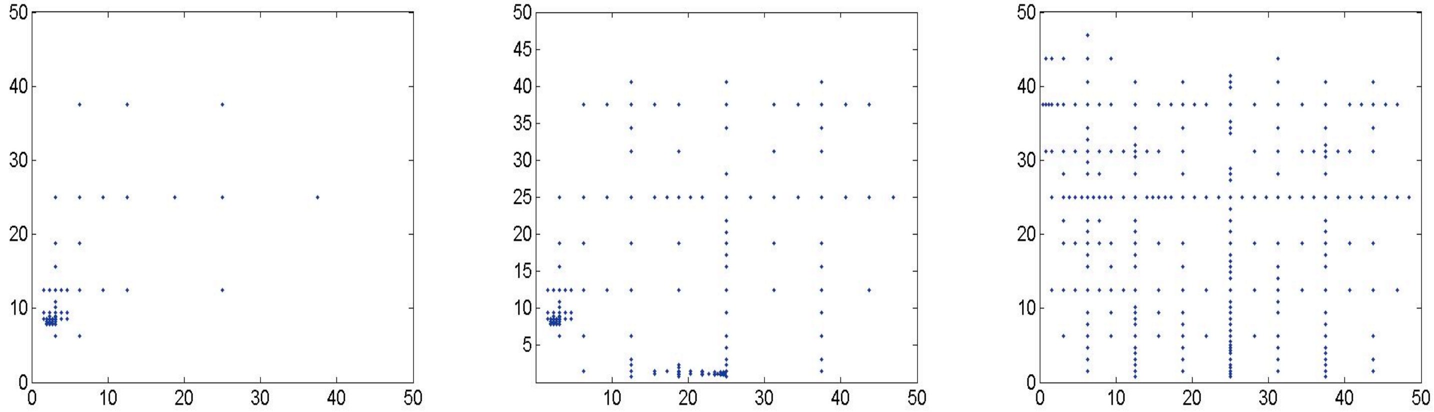
Calibration of the Svensson model
Reformulation of the problem: splitting

The objective values of the new objective function $H(\beta)$ are:



Calibration of the Svensson model
Reformulation of the problem:splitting

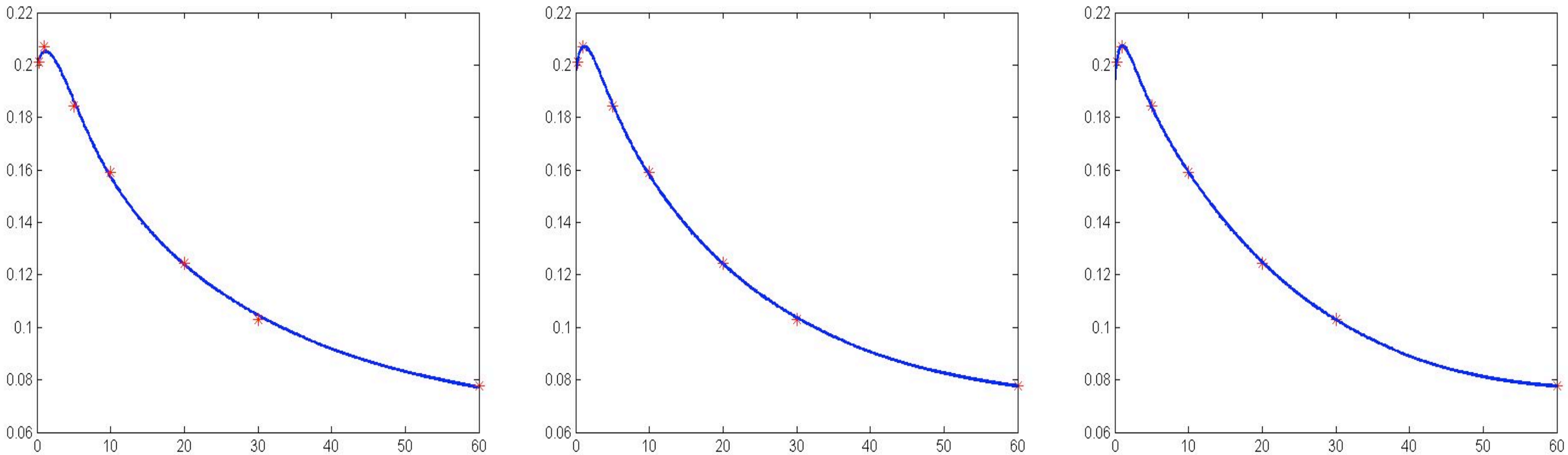
The global optimization of the 2 dimensional problem yields



adaptiveness	#points	time (sec.)	objective value	minimizer
0.5	100	5	-2.3824	[2.3533, 7.8210]
0.3	150	8	-2.7723	[24.8097, 1.1816]
0.1	250	14	-3.1283	[0.7911, 37.5025]

Calibration of the Svensson model
Reformulation of the problem:splitting

The corresponding yield curves for this problem are:



The objective values are -2.3824, -2.7723 and -3.1283.

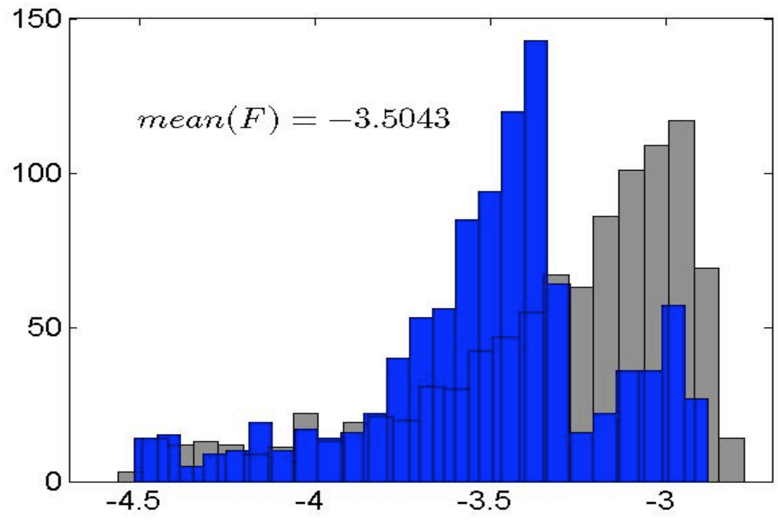
Calibration of the Svensson model
Variable-transformation: polar coordinates

Idea: introduce polar coordinates for the β's:

$$\min_{\gamma \in \mathbb{R}^2} \quad H(\gamma)$$
$$s.t. \quad lb \leq \gamma \leq ub.$$

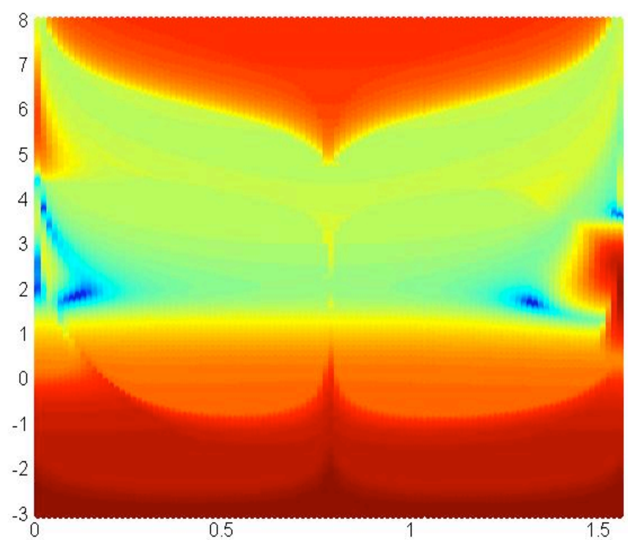
Where $\gamma_1 = \sqrt{\beta_1^2 + \beta_2^2}$ and $\gamma_2 = \arccos\left(\frac{\beta_1}{\rho}\right)$ and $lb = [\epsilon, -3], ub = [\frac{\pi}{2}, 8]$.

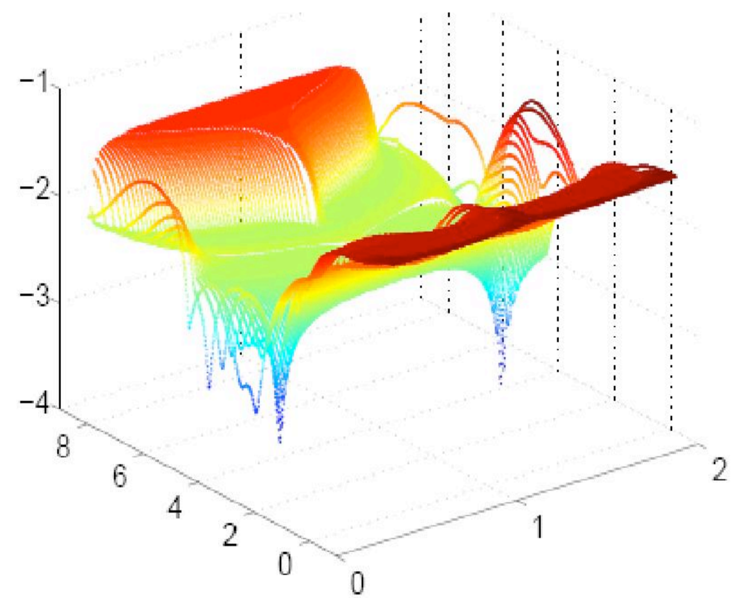
The minimization over all parameters yields an improvement of 0.1729:



Calibration of the Svensson model
Variable-transformation: polar coordinates

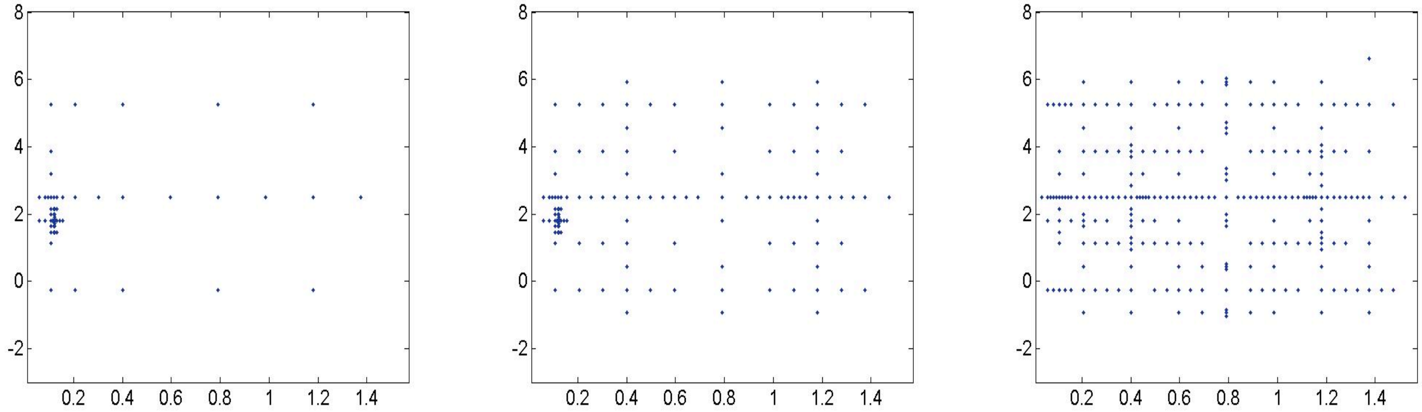
The objective values of $H(\gamma)$ are:





Calibration of the Svensson model
Results of the global optimization

As expected, the global algorithm performs better:



adaptiveness	#points	time (sec.)	objective value	minimizer
0.5	100	5	-2.3824	[2.3533, 7.8210]
0.3	150	8	-2.7723	[24.8097, 1.1816]
0.1	250	14	-3.1283	[0.7911, 37.5025]
0.5	100	4	-3.3526	[0.1213, 1.8125]
0.3	150	7	-3.3526	[0.1213, 1.8125]
0.1	250	14	-3.1122	[0.1075,1.8125]

Calibration of the Svensson model

New objective function

We take another look at the original objective function:

$$F(\alpha, \beta) = \sum_{j=1}^{J} \left(R(\tau_j) - \hat{R}(\tau_j) \right)^2$$

where $\alpha \in \mathbb{R}^4$ and $\beta \in \mathbb{R}^2$.

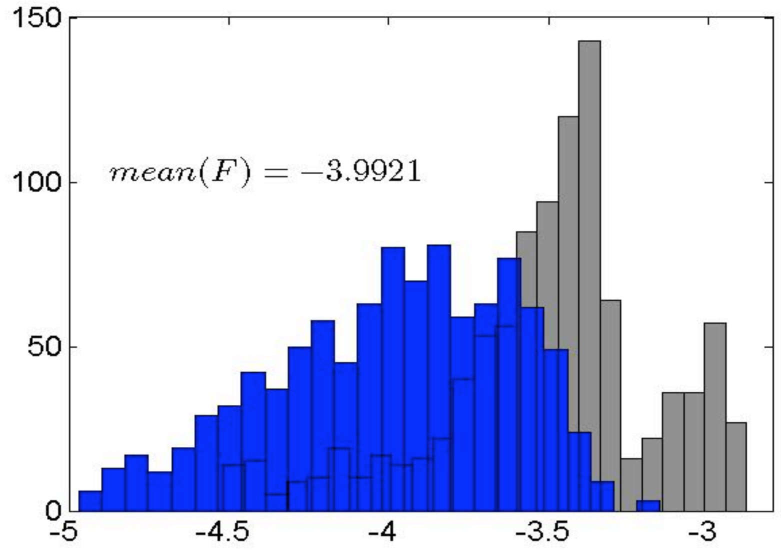
this is equivalent to

$$F^*(\alpha, \beta) = \sqrt{\sum_{j=1}^{J} \left(R(\tau_j) - \hat{R}(\tau_j) \right)^2}.$$

This objective function has the effect that `fmincon` makes more iterations, whereas the behavior of the global method is not influenced.

Calibration of the Svensson model
New objective function

The minimization of F^* over all parameters with `fmincon` yields an improvement of 0.4878:



The figures show the results of `fmincon` for polar coordinates.

Calibration of the Svensson model
Algorithm

The previous considerations lead to the following algorithm:

```
for each time step and each scenario do
```

1. `minimize` $F^*(\alpha, \gamma)$ `over all parameters with fmincon;`

2. `if the solution is below a certain threshold:`

   2.1 `minimize` $H(\gamma)$ `(`$\gamma \in \mathbb{R}^2$`)` `with the global algorithm;`
   x^* `is the minimizer of` H`;`

   2.2 `minimize` $F^*(\alpha, \gamma)$ `with fmincon;`
   `use` x^* `as the starting point;`

Calibration of the Svensson model
Summary of the results

We calibrated the svensson model for 1000 scenarios and one time step. The global algorithm was adaptive and used at most 100 points.

For cartesian coordinates:

	min	mean	max	time (sec.)
after step 1	-4.9443	-3.9417	-3.3351	90
after step 2.1.	-4.9443	-3.8862	-2.3824	721
after step 2.2.	-4.9443	-3.9696	-3.3584	8
total change	-	0.0279	0.0233	819

For polar coordinates:

	min	mean	max	time (sec.)
after step 1	-4.9691	-3.9921	-3.1508	92
after step 2.1.	-4.9691	-3.9451	-2.7187	715
after step 2.2.	-4.9691	-3.9939	-3.3351	10
total change	-	0.0018	0.1843	817