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Interest Rate Term Structure Estimation with Exponential Splines: A Note

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ABSTRACT

Vasicek and Fong [11] developed exponential spline functions as models of the interest rate term structure and claim such models are superior to polynomial spline models. It is found empirically that i) exponential spline term structure estimates are no more stable than estimates from a polynomial spline model, ii) data transformations implicit in the exponential spline model frequently condition the data so that it is difficult to obtain approximations in which one can place confidence, and iii) the asymptotic properties of the exponential spline model frequently are unrealistic. Estimation with exponential splines is no more convenient than estimation with polynomial splines and gives substantially identical estimates of the interest rate term structure as well.

A RECENT PAPER BY Vasicek and Fong [11] outlined a method for smoothing interest rate term structure data by what the authors termed exponential spline fitting. The claims were made that the methodology had been applied successfully and that resulting term structure approximations "(have) desirable asymptotic properties for long maturities and exhibit(s) both a sufficient flexibility to fit a wide variety of shapes of the term structure, and a sufficient robustness to produce stable forward rate curves" [11, p. 340]. Since no evidence for the latter claim was presented, this paper offers some empirical applications of the technique.

The discussion to follow is divided into four sections that respectively treat I) the Vasicek and Fong model (hereafter, the VF model) and its relation to other spline smoothers, II) the implementation of the model, III) an examination of the circumstances in which it is likely to break down, and IV) a comparison of the performance of the model to that of a straightforward polynomial spline smoother.

I. The Vasicek and Fong Model

The basic measure of the interest rate term structure is the discount function $[D(t)]$ which relates the timing $[t]$ of a future unitary payment to its present value. Discount functions have been estimated ever more frequently with *ad hoc*

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smoothing techniques. The literature on these methods has grown substantially from the original papers by McCulloch [6, 7] and Schaefer [9] who first estimated the term structure with polynomial splines. Since then, contributions by several investigators such as Rose and Schworm [8], Jordan [2], and Shea [10] have noted that polynomial splines have drawbacks as models for the term structure. Polynomial spline models of the term structure commonly yield estimates of forward interest rate structures that are unstable, fluctuate widely, and often drift off to very large, even negative, values.

Vasicek and Fong assert [11, p. 345] that discount functions are principally exponential decays and since polynomials do not have the same curvature as do exponentials, a polynomial spline function will tend to weave around an exponential discount function. This will lead to poor approximations to the slope of the discount function and thus give the misleading estimates of forward rates typical of polynomial spline models. The solution which the authors propose is to estimate the discount function with exponential splines instead of polynomial splines.

An exponential spline is a piecewise exponential function in which the pieces are joined so that the function and its derivatives are continuous. Vasicek and Fong proposed that a transformation of the argument, t, of the discount function would simplify the estimation of an exponential spline approximation to the term structure. They defined the variable $x = 1 - e^{-\alpha t}$, where α is some constant. Payment timing, t, was thus defined as

$$t = -\frac{1}{\alpha}\log(1 - x), \qquad 0 \le x \le 1. \tag{1}$$

The discount function, $D(t)$, could therefore be defined as the transformation G, where

$$G(x) \equiv D\left(-\frac{1}{\alpha}\log(1 - x)\right) = D(t). \tag{2}$$

Whereas the discount function is nonlinear in t, since it is presumably an exponential decay, it is linear in the transformed argument, x. Thus, $G(x)$ could be estimated easily with polynomial splines in x. The advantages claimed for this approach were that i) it avoided the "use of complicated nonlinear estimation" and that ii) desired asymptotic properties could be enforced for the term structure [11, p. 346].

Vasicek and Fong completed their model of the discount function by defining $\{g_i(x)\colon 0 \le x \le 1, i = 1, 2, \cdots, m\}$ as a set of m polynomial functions that span all polynomials in the linear (G, x)-space. Thus, the functions G and D have representations as linear combinations of these g-basis functions so that

$$G(x) = \textstyle\sum_{i=1}^{m} \beta_i g_i(x). \tag{3}$$

The authors also assumed that $G(x)$ would have the boundary conditions that $G(0) = 1$ and $G(1) = 0$. The former condition guarantees that the present value of a current payment is equal to 1 and the latter guarantees that an infinitely remote payment is worthless. The asymptotic properties of the model are summarized in a first-order Taylor series expansion of $G(x)$ around 1,

$$G(x) = -G'(1)(1 - x) + o(1 - x), \tag{4}$$

which, by ignoring the error term, implies that

$$D(t) \cong -G'(1)e^{-\alpha t}. \tag{5}$$

If $G'(1) \neq 0$, the instantaneous continuously compounded forward rate for large t is defined

$$-\frac{D'(t)}{D(t)} = \alpha. \tag{6}$$

This asymptotic forward rate, that we shall call α, will be estimated along with the other parameters of the model.

II. Implementing the Model[1]

Even though Equation (3) is linear in the spline parameters (the β's), the spline g-basis functions are nonlinear in α. Estimation of the VF model therefore cannot avoid the use of nonlinear routines despite the contrary claims made by Vasicek and Fong. The first step in estimation of the VF model is to obtain starting values for the model's parameters. Since estimation of the β's is conditional upon an estimate of α, a starting value for the asymptotic forward rate must first be obtained. In experiments, it was found that any general measure for an interest rate term structure level would suffice for α's starting value. Sample observations of payment timings, t, are then subjected to the exponential transformation, $x = 1 - e^{-\hat{\alpha}t}$. Since data used in the examples to follow were prices for zero-coupon bonds, only the timing of each bond's payment of principal at maturity is transformed in this way.[2]

[1] There are other features of the VF model which are not central to the question of how well exponential splines smooth term structure data and thus are not implemented here. The matter of correcting the term structure for tax effects is avoided in this paper by using bid-ask averages for Nippon Telegraph and Telephone (NTT) long-term zero-coupon issues which, in secondary trade, are free from capital gains and interest taxation in Japan. Data come from publications of the Bond Underwriter's Association [3, 4].

A second auxiliary feature of the VF model postulates that residual errors are homoscedastic in yields and, consequently, are heteroscedastic in prices [11, p. 345]. The matter of how errors in prices and yields are respectively modelled is important, but cannot be settled without an investigation into the process that generates the errors. In this paper, it is assumed that errors in prices are random normally distributed. It was not observed in our work that estimated residuals displayed a systematic heteroscedastic character that required correction. However, care was taken that obviously stale or bad quotes were eliminated from the sample.

[2] The fundamental purpose of term structure estimation is to circumvent problems posed by coupon-bearing bond price data, but this study is unaffected by foregoing the use of such data. Although observed prices of discount bonds amount to direct observations of the interest rate term structure, the implementation of the Vasicek and Fong model is only slightly altered when price data for coupon-bearing bonds are used instead. The regression equation for zero-coupon bonds is $p = G(x) = \sum_{i=1}^{m} \beta_i g_i(x)$, where p is the price of the bond in units of face value. The appropriate regression equation for coupon-bearing issues is $p = \sum_{i=1}^{m} \beta_i [g_i(x_N) + c \sum_{j=1}^{N} g_i(x_j)]$, where c is the coupon in

A set of cubic polynomial normalized B-splines, $\{g_i(x)\}$, was then calculated subject to the conditions that the resulting spline approximation should be everywhere continuous and everywhere continuous in its first and second derivatives. This basis was chosen for its superior data conditioning properties, which are described by DeBoor [1], and for the convenience it affords in imposing the VF model's boundary conditions during estimation. Vasicek and Fong did not mention how their g-basis functions were constructed.

An initial estimate of the β-vector was then obtained by estimating Equation (3) via restricted linear least squares. Armed with initial estimates for α and the β-vector, the entire VF model was then estimated via an iterative nonlinear least squares estimation routine.[3] A number of programs were then developed to reliably simulate the estimated present value functions and their associated spot and forward yield curves.[4]

III. Model Performance

Splines are such flexible functional forms that it is relatively meaningless to compare alternative spline specifications on the basis of goodness-of-fit. For a like number of degrees of freedom, it is difficult to fault the VF model relative to any other spline function for its ability to smooth term structure data. One valid basis for judging model performance, however, is whether the investigator's prior expectations for the term structure approximations are reasonably met by the model. Such expectations may be justified by economic theory or may be totally *ad hoc*. In this regard, Vasicek and Fong have made two claims for the superiority of their model: i) the method of exponential splines is sufficiently "robust" to produce stable forward interest rate curves and ii) the model displays some desirable asymptotic properties. These claims will be examined below.

Since Vasicek and Fong believed that present value functions are exponential decays, they expected that estimated discount functions would be linear in x. They apparently were not willing to strictly impose this condition, or else they would have estimated an ordinary regression line in (G, x)-space instead of a polynomial spline. This allowed an estimated present value function to deviate from a strict exponential decay.

units of face value. The arguments $x_j = 1 - e^{-\alpha t_j}$ are calculated from t_j, $j = 1, \cdots, N$, where t_j is the timing of each of the respective N coupon payments. It is assumed that a principal of the size of the bond's face value is paid with the final coupon at time t_N. Estimated β's from a sample containing bonds with dissimilar coupons can be used, of course, to simulate the term structure and its equivalent zero-coupon bond spot and forward yield curves. As with other spline models, the VF method allows the investigator to transform coupon-bond data so that unbiased estimates of the zero-coupon bond interest rate structure can be obtained.

[3] The boundary value constraints specified in the VF model are imposed by constraining β_1 and β_m so that they equal 1 and -1, respectively. The nonlinear least squares program used here was ZXSSQ from the International Mathematical and Statistical Library. Linear estimation to get ordinary polynomial cubic spline estimates mentioned in this paper was carried out via algorithm LSE described by Lawson and Hanson [5, pp. 139–40].

[4] The subroutines BSPLVB, INTERV, and BVALUE from DeBoor [1] were used to evaluate B-splines and B-spline approximations.

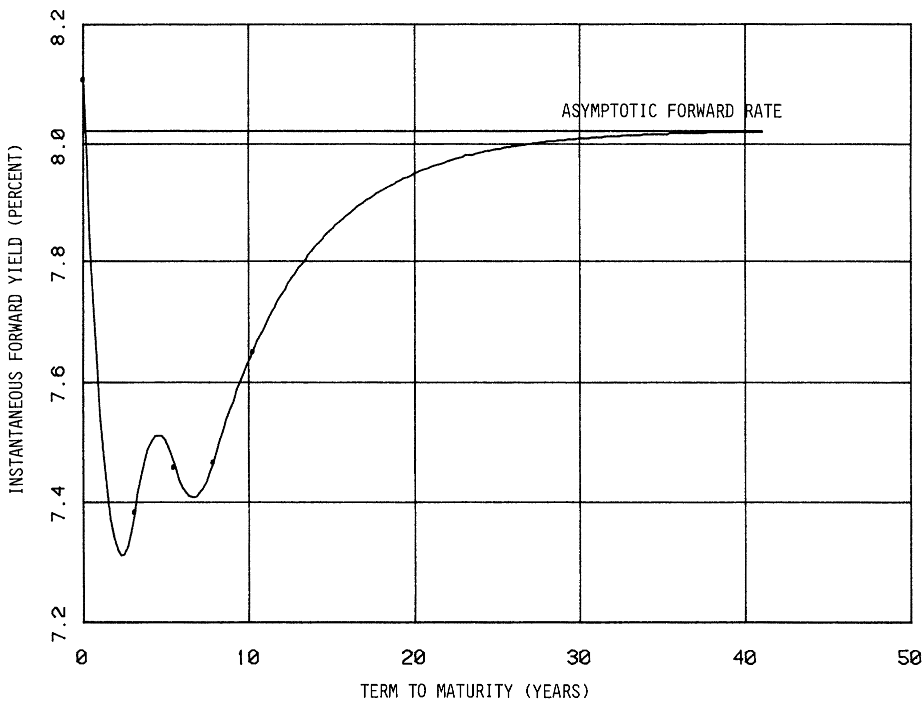

Figure 1. A continuously compounded instantaneous forward yield curve estimated for June 1970. The dark horizontal line denotes the yield curve's asymptote at long terms to maturity. Small ∘'s denote breakpoint locations. The curve plots the instantaneous rate of decay of the estimated discount function. The maximum observed term to maturity in this example was approximately 11 years.

It was found in numerous examples that estimated polynomial spline approximations for $G(x)$ were rarely linear or near linear. The asymptotic restrictions of the VF model were generally not helpful in defining an estimated discount function with the curvature of an exponential decay. In fact, departures from such decays were typically substantial and forward rate estimates were not as stable as Vasicek and Fong anticipated. Figure 1 illustrates an example of a forward yield curve. The curve exhibits wide changes over very short term-to-maturity intervals. Moreover, the asymptotic forward rate exhibits little influence over the shape or level of the forward rate curve at terms to maturity for which observable bond prices exist. In most cases, the asymptotics of the VF model apparently are not associated with sample behavior and, thus, do little to give an air of verisimilitude to the term structure estimates. The instability of VF model approximations is also exacerbated by the peculiar data-conditioning properties of the exponential transformation, $x = 1 - e^{-\alpha t}$. For small $\tilde{\alpha}$, observed x can become so bunched that substantial portions of the estimation interval, [0, 1], can become empty of data, leading to particularly unstable and unrealistic asymptotic forward rates. In these circumstances, the data also become so ill-conditioned that nonlinear estimation programs can only be coaxed to converge

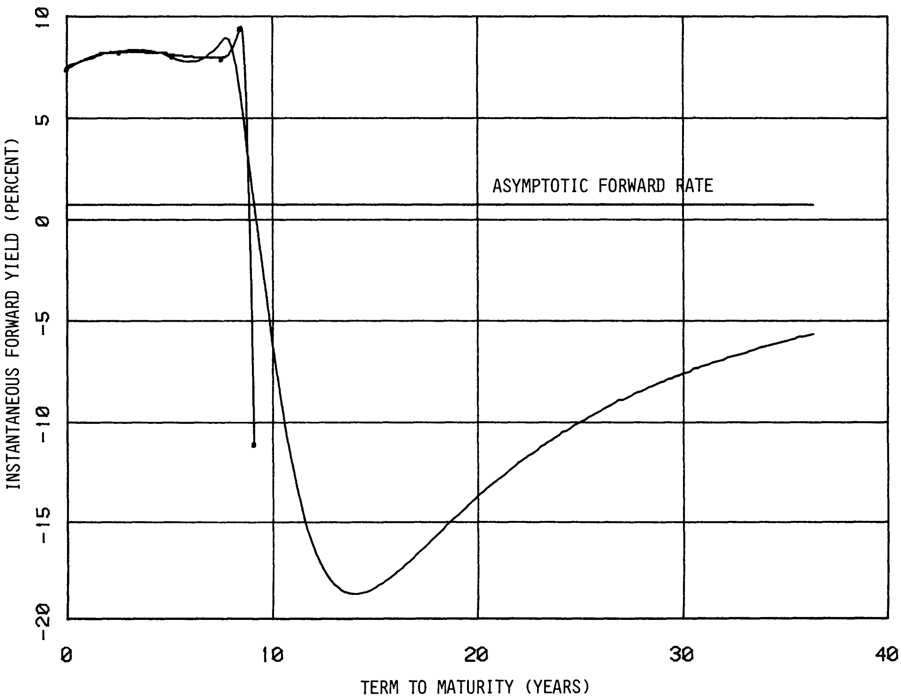

Figure 2. An estimated instantaneous forward yield curve for April 1975 using the VF model. Also illustrated, a forward yield curve derived from a cubic polynomial B-spline model. Small ∘'s denote breakpoint locations in the B-spline polynomial model; the B-spline model is estimated only as far as the maximum term to maturity observed in the sample.

to a solution with difficulty. Such an instance is illustrated in Figure 2.[5]

The VF model is no more likely to yield stable forward interest rate structures than are ordinary polynomial spline models. Indeed, term structure estimates arising from the VF model are often substantially the same estimates that would be obtained from an ordinary polynomial spline model. Figure 2 illustrates this with a cubic B-spline model of a forward yield curve in conjunction with one arising from the VF model. The VF model will give the same estimates as do polynomial splines, as long as the assumed exponential form of the discount function is modelled in only a piecewise fashion. By their piecewise nature, spline approximations can be viewed as a set of jointly determined but, nevertheless, local approximations to a function. While Vasicek and Fong correctly assert that polynomials have difficulty in modelling exponential functions, they neglect to point out that this state of affairs does not extend to *local* polynomial approximations to exponential functions; after all, local approximations to continuous

[5] The particular example pictured here was also extensively discussed in Shea [10] in the context of cubic polynomial spline approximations to the term structure. Estimated α in this example is 0.73 percent which guarantees that only payment timings more distant than 100 years ahead would have $1 - e^{-\alpha t} > 0.50$. More practically, we can expect that the [0, 1] estimation interval will be half empty of data for samples in which payment timings are on the order of 20 to 30 years in the future and forward interest rate levels are about 3 percent or less.

functions via polynomials are at the heart of the theory of Taylor series expansions. It is therefore not surprising that a piecewise polynomial function, each polynomial piece of which is estimated over a not too wide domain, might very well mimic a piecewise exponential. The difference between the two approximations will reflect the higher order components of the VF model's respective exponential pieces that are ignored by the *finite-order* polynomials which make up an ordinary polynomial spline. Except for that difference, it is reasonable to expect that these two very flexible piecewise approximation methods, exponential and polynomial splines, will look much the same.

IV. Conclusions

The examples included in this paper and many more available supporting examples suggest that modelling with exponential splines will give forward rates that are unstable and fluctuate much like forward rates obtained with polynomial splines. The VF model is just as capable of modelling the term structure as are ordinary polynomial splines, but it brings no practical advantages to the modelling task. Not only is it just as necessary to constrain and coax the VF model to obtain reasonable term structure approximations, but the exponential transformation of the data and the imposed asymptotics do not appear to add any more realism to the estimated term structure. There is the added computational burden of estimating a nonlinear rather than a linear model, and the exponential data transformation frequently makes for very ill-conditioned sample data. Since the resulting term structure estimates are likely to look very much like those from a polynomial spline model in any case, it is recommended that such ordinary spline techniques be used in preference to exponential splines.

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[Footnotes]

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