Modeling the term structure of interest rates: a review of the literature

R. GIBSON, F.-S. LHABITANT, D. TALAY*

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Abstract

The last two decades have seen the development of a profusion of theoretical models of the term structure of interest rates. This study provides a general overview and a comprehensive comparative study of the most popular ones among both academics and practitioners. It also discusses their respective advantages and disadvantages in terms of bond and/or interest rate contingent claims continuous time valuation or hedging, parameter estimation, and calibration. Finally, it proposes a unified approach for model risk assessment. Despite the relatively complex mathematics involved, financial intuition rather than mathematical rigour is emphasised throughout. The classification by means of general characteristics should enable the understanding of the different features of each model, facilitate the choice of a model in specific theoretical or empirical circumstances, and allows the testing of various models with nested as well as non nested specifications.

*H.E.C. University of Lausanne, INRIA Sophia-Antipolis and Thunderbird, the American Graduate School of International Management. We wish to aknowledge financial support from RiskLab (Zürich). This work is a part from the RiskLab project entitled "*Interest rate risk management and model risk*". We thank Jessica James (FNB Chicago) and Mireille Bossy (INRIA Sophia-Antipolis) for helpful suggestions. Preliminary version. All comments are welcome. Contact: francois@lhabitant.net

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1 Introduction

Understanding and modeling the term structure of interest rates represents one of the most challenging topics of financial research. There are in fact many benefits from a better understanding of the term structure of interest rates. Since the introduction of option trading on bonds and other interest rate dependant assets, much attention has been given to the development of models to price and hedge interest rate dependant assets or to manage the risk of interest rates contingent portfolios.

While the Black and Scholes (1973) model has rapidly established himself as "the" model for stock contingent claims, a large number of continuous time approaches are simultaneously used among academics and practitioners in the field of interest rates contingent claims. Despite the widespread use of Black (1976) model[1] to value interest rate derivatives such as bond options, caps, or swaps, interest rate derivatives have some major differences from those on stocks that need to be resolved. For instance, let us consider the case of a simple bond: unlike stocks, the bond price at maturity is fixed and known, and the Wiener process used to model stock prices is inappropriate. Furthermore, bond prices are dependant on interest rates, which exhibit a complex stochastic behavior and are not directly tradable, which means that the dynamic replication strategy is more complex. Similar differences can be found in the case of more complex interest rate derivatives such as bond options, caps, floors, or interest rate swaps. Thus, pricing, hedging or managing the risk of interest rate derivatives is a complex task, and each of the existing models has its own advantages and drawbacks.

The aim of this paper is to provide a comprehensive review of the modeling techniques of the term structure applicable to default-free bonds and other interest rate derivatives. We propose a typology, describe the most important models and methodologies in a common framework, explain their advantages and differences, report the most relevant analytical and empirical results and provide references for their derivations. Given the vast array of issues in the field as well as the large number of existing surveys, we attempted to provide an analysis from an overall perspective, focusing on what the central issues were rather than on specific details.

[1] The essence of Black (1976) model is that the underlying variable (bond price for a bond option, interest rates of the constituent caplets for a cap, swap rate for a swap) is lognormal at the maturity of the derivative. This allows the use of a slightly modified version of the Black and Scholes (1973) formula for stock options.

This paper is organized as follows: section 1 introduces the definitions and notations we will used throughout the paper; section 2 presents the basic theories of the term structure of interest rates; section 3 proposes a model taxonomy for interest rate models. Section 4 reviews simple factor interest rate models, with a strong emphasis on the pricing of contingent claims using partial differential equations or martingales theory. It also provides numerous examples of such single-factor models, with both constant and time varying parameters. Section 5 considers the extension to multifactor models. Section 6 reviews the extension to a multidimensional space, from the pioneering work of Ho and Lee (1986) to the most recent work on random fields and including the Heath, Jarrow and Morton (1992) family of models. Section 7 presents some of the empirical findings comparing the performance of alternatives over a brad range of applied interest rate models. Section 8 concludes.

1.1 Definitions

We will first recall some usual notations. All our models will be set up in a given complete probability space (Ω, F_t, P) and an augmented filtration $(F_t)_{t\geq 0}$ generated by a standard Brownian motion $W(t)$ in R (to keep things simple, unless explicitly mentioned, the uncertainty will be represented by a one-dimensional process).

To focus on pure questions of valuation, we will ignore taxes and transaction costs. We denote by $B(t,T)$ the price at time t of a **discount bond**, i.e. a zero-coupon bond which pays one currency unit at time T and nothing else at any other time. It follows immediately that $B(T,T) = 1$. At time t, the **yield to maturity** $R(t,T)$ of the discount bond $B(t,T)$ is the continuously compounded rate of return that causes the bond price to rise to one at time T

$$B(t,T)e^{(T-t)R(t,T)} = 1$$

or, solving for the yield

$$R(t,T) = -\frac{\ln B(t,T)}{T-t} \tag{1}$$

For a fixed time t, the shape of $R(t,T)$ as T increases determines the **term structure of interest rates**. In our framework, the **yield curve** is the same as the term structure of interest rates, as we only work with zero-coupon bonds. Finance traditionally views bonds as contingent claims and interest rates as underlying assets.

We denote by $r(t)$ the **instantaneous risk-free interest rate**, also called **short term rate**, i.e. the yield on the currently maturing bond. Thus,

$$r(t) = \lim_{T \to t} R(t,T)$$

A roll over position at the short term rate $r(t)$ is called **money market account**. The value of the money market account initialized at time 0 with one dollar investment is

$$\beta(t) = e^{\int_0^t r(s)ds}$$

We denote by $f(t, T_1, T_2)$ the **forward rate**, i.e. the rate that can be agreed upon at time t for a risk-free loan starting at time T_1 and finishing at time T_2.

$$f(t, T_1, T_2) = \frac{\ln B(t, T_1) - \ln B(t, T_2)}{T_2 - T_1}$$

Of particular interest is the **instantaneous forward rate**

$$f(t,T) \equiv f(t,T,T)$$

It is the rate that one contracts at time t for a loan starting at time T for an instantaneous period of time. We have

$$f(t,T) = -\left.\frac{\partial \ln B(t,\tau)}{\partial \tau}\right|_{\tau=T} = -\frac{1}{B(t,T)}\frac{\partial B(t,T)}{\partial T}$$

assuming that bond prices are differentiable. Equivalently, one can define the bond price in terms of forward rates as

$$B(t,T) = e^{-\int_t^T f(t,s)ds}$$

Note that we can write

$$r(t) = f(t,t) \tag{2}$$

1.2 Remarks

Let us recall that there exist a set of bond specific arbitrage restrictions:

- any bond price process has a non stochastic terminal value at the end of its life
$$B(T,T) = 1$$
- a bond price will never exceed its terminal value plus the outstanding coupon payments
- a zero-coupon price cannot exceed the price of any zero-coupon with a shorter maturity
- the value of a zero-coupon bond must be equal to a value of a replicating portfolio composed of zero-coupon bonds.
- interest rates should not be negative.

In the following, we will always assume that these arbitrage restrictions are fulfilled, unless explicitly mentioned.

2 Theories of the term structure of interest rates

How can we explain the shape of the term structure of interest rates ? In other terms, in which manner are the spot rates or discount factors determined ? What explains the shape of the term structure of interest rates is clealy related to the existence and the value of term premia. There exist three major theories to explain the relationships between the interest rates of various maturities: the expectation hypothesis, the liquidity preference, the preferred habitat theory.

2.1 The expectation hypothesis

According to the expectation hypothesis, the term structure is driven by the investor's expectations on future spot rates. The forward rate is an unbiased estimator of the future prevailing spot rate. The rate of return on a bond maturing at time T should be equal to the geometric average of the expected short-term rate from t to T, and the term structure is given by

$$R(t,T) = \frac{1}{T-t}\int_t^T E_t(r(s))ds$$

Note that in fact, there exist four continuous-time interpretations of the expectation hypothesis:

- the **naive expectation hypothesis** states that the expected return on any strategy for any holding period is the same. In particular, the investor should be indifferent between holding a long term bond and rolling over a short term one.

$$-\frac{\ln B(t,T)}{T-t} = E\left[\frac{1}{T-t}\int_t^T r(s)ds\right]$$

- the **local expectation hypothesis** states that for any bond on the market

$$E\left[\frac{dB(t,T)}{B(t,T)}\right] = r(t)dt$$

  or equivalently that

$$B(t,T) = E\left[e^{-\int_t^T r(s)ds}\,|r(t)\right]$$

- the **return to maturity expectations hypothesis** - which is also called the Lutz hypothesis - states that the expected return on holding any bond up to its maturity will have the same expected return as rolling over a set of short term bonds.

$$\frac{1}{B(t,T)} = E\left[\exp\int_t^T r(s)ds\,|r(t)\right]$$

- the **unbiased expectation hypothesis** - which is also called the Malkiel hypothesis - states that the forward rate is equal to the future expected spot rate

$$\frac{\partial B(t,T)/\partial T}{B(t,T)} = E\left[r(T)\right]$$

which is equivalent to

$$-\ln B(t,T) = \int_t^T E(r(s)ds)$$

Using Jensen inequality, one can show that these theories are mutually inconsistent, with the exception of the unbiased expectation hypothesis with the naive expectation hypothesis.

2.2 The liquidity preference theory

According to the liquidity preference theory, investors are risk-averse, tend to prefer short term maturities and will require a premium to engage in long term lending. Borrowers prefer long-term securities and agree to pay this premium. The term structure is then given by

$$R(t,T) = \frac{1}{T-t}\left[\int_t^T E_t(r(s))ds + \int_t^T L(s,T)ds\right]$$

where $L(t,T) > 0$ denotes the instantaneous term premium at time t for a bond maturing at time T.

An important consequence is that the expected return from a buy and hold strategy will be higher than the expected return fn a roll over strategy. The resulting term structure of interest rates should be upward sloping.

2.3 The preferred habitat theory

According to the preferred habitat theory, investors and borrowers have different specific time-horizons. The term structure is still given by

$$R(t,T) = \frac{1}{T-t}\left[\int_t^T E_t(r(s))ds + \int_t^T L(s,T)ds\right]$$

but depending on the offer and the demand, the risk premium $L(s,T)$ attached to bonds of various maturities can be positive, negative or equal to zero. Thus, the term structure of interest rates can have any shape.

3 Interest rate models taxonomy

In order to better understand interest rate models, it is helpful to identify some of their characteristic features and distinctions. Unfortunately, these are not mutually exclusive, and categories are frequently overlapping.

3.1 Continuous versus discrete models

Should we model the term structure dynamics in a discrete or a continuous framework ?

- as far as the time dimension is concerned, most interest rates models were specified in a continuous time framework. The power of continuous time stochastic calculus allows more elegant derivations and proofs, and provides an adequate framework to produce more precise theoretical solutions and more refined empirical hypothesis, unfortunately at the cost of a considerably higher degree of mathematical sophistication.

- on the space dimension, until recently, diffusion models were the rule. But should we use a diffusion model, or allow for discontinuity ? Recently, models based on jumps or point processes have appeared in order to model discontinuous real world phenomena such as the central bank interventions.

3.2 Bond prices, interest rate versus yield curve models

What should we model ?

- early models of the term structure attempted to model the bond price dynamics. Their results did not allow for a better understanding of the term structure, which is hidden in the bond prices.

- many interest rate models are simply models of the stochastic evolution of a given interest rate (often chosen to be the short term rate). This interest rate is often defined as Markovian: its future evolution only depends on its current value, not on the historical path it followed to arrive there. As we will see, this translates the valuation problem into a partial differential equation that can be solved analytically or numerically.

- an alternative is to specify the stochastic dynamics of the entire term structure of interest rates, either by using all yields or all forward rates. The approach is intuitively attractive, but the model complexity increases. This has prevented whole yield curve models from coming in more widespread use.

However, the three approaches are not independent, as there exist relationships that must hold - even without assuming that markets are free of arbitrage - between bond prices, short term rates and forward rates or yields.

3.3 Single versus multi-factor models

Factor models assume that the term structure of interest rates is driven by a set of variables or factors. Most empirical studies using a principal component analysis have decomposed the motion of the interest rate term structure into three independent and non-correlated factors (see Wilson (1994)):

- the first one is a shift of the term structure, i.e., a parallel movement of all the rates. It usually accounts for up to 80-90 percent of the total variance (the exact number depending on the market and on the period of observation).

- the second one is a twist, i.e. a situation in which long rates and short term rates move in opposite directions. It usually accounts for an additional 5-10 percent of the total variance.

- the third one is called a butterfly (the intermediate rate moves in the opposite direction of the short and long term rate). Its influence is generally small (1-2 percent of the total variance).

As the first component generally explains a large fraction of the yield curve movements, it may be tempting to reduce the problem to a one-factor model. It must be stressed at this point that this does not necessarily imply that the whole term structure is forced to move in parallel, but simply that one single source of uncertainty is sufficient to explain the movements of the terms structure (or the price of a particular interest rate contingent claim). For example, almost all arbitrage-based single factor models derive or assume the instantaneous spot rate to be the single state variable.

On the other hand, some securities are sensible to the shape of the term structure (or to other aspects such as the volatility term structure deformations, as we will see later) and not only to its level. They will require at least a two factor model. Generally, a second state variable such as the long rate or the rate of inflation is added.

3.4 Fitted versus non fitted

In fitted models, a term structure (of interest rates, of forward rates, of yield volatilities, etc.) is determined exogenously, generally using market data, and the stochastic differential equation of some state variables is specified such that this term structure is obtained at a particular date. In a sense, the models is build specifically to fit an arbitrary (exogenous) initial term structure.

In other models, we first specify the dynamics of the state variables. As a consequence of a particular specification, we will obtain endogeneously a given term structure. These models generally do not fit well the initially observed term structure[2].

3.5 Arbitrage free versus equilibrium models

A fundamental question from the theoretical point of view - but not necessarily in practice - is the distinction between arbitrage-free and equilibrium models. Arbitrage-free models start with assumptions about the stochastic behavior of one or many interest rates and about a specific market price of risk and derive the price of all contingent claims assuming that there are no arbitrage opportunities on the market. In other terms, there is no risk-free financial strategy with zero-setup cost that should give with certainty a positive return[3]. In contrast, equilibrium models start from a description of the economy, including the utility function of a representative investor and derive the term structure of interest rates, the risk premium and other assets prices endogenously, assuming that the market is at equilibrium.

[2] Note that Dybvig (1989) proposes a methodology to convert an endogenous model into an exogenous one.

[3] The formal definition of the no-arbitrage condition (see Harrisson and Kreps (1979)) would require a rigorous definition of a complete market, a self-financing strategy, an attainable contingent claim and an implicit price-system.

But the distinction is subtle, since equilibrium models should be arbitrage-free (otherwise, the economy would not be at equilibrium), and as some so-called "arbitrage-free" models were shown later on to allow for arbitrage opportunities. Furthermore, as pointed out by Duffie and Kan (1993), it is always possible to support any (regular) short term rate process in an equilibrium model based on a representative agent with an appropriate utility function and consumption stream constructed on the interest rate process.

4 Single factor models

Single factor models assume that all the information about the term structure at any point in time can be summarized by one single specific factor. Although any interest rate could be chosen for this single factor, it is usually specified as the short term interest rate $r(t)$. As a consequence, only the short term interest rate and the time to maturity will affect the price of any interest rate contingent claim. In particular, for of a zero-coupon bond maturing at time T ($T \geq t$), we have

$$B(t,T) \equiv B(t,T,r(t)) \tag{3}$$

Single factor models start by specifying the stochastic differential equation driving the spot rate stochastic process $\{r(t)\}_{t\geq 0}$ and deduce from there the price of various interest rate dependant assets[4]. The specification can be exogenous or exogenous (as the result of an equilibrium in the economy). As they differ in their specification of the dynamics of the short term rate, they also provide lousy empirical results.

In this section, we will first review the two basic methodologies for pricing interest rate contingent claims in a single factor framework, namely the partial differential equation and the martingale approach. By the theorem of Feynman-Kac, these two approaches are equivalent. The former creates an instantaneous risk-free portfolio to obtain by arbitrage a second order partial differential equation that any interest rate contingent claim must satisfy. The latter relies on the argument of Harrison and Pliska (1979) that in a complete market, in the absence of arbitrage, there exists an equiva-

[4]Note that if the assumption of one unique underlying factor implies that all rates move in the same direction over any short period interval, but not necessarily by the same amount, it does not imply that the term structure has always the same shape.

lent martingale measure under which asset prices can be computed as an expectation.

In a second step, we will illustrate the methodology by specific examples of single-factor models.

4.1 Interest rate derivatives pricing: the partial differential equation approach

Let us consider that the short term interest rate is the single factor driving the entire term structure. We assume that the dynamics of the short term rate is given by

$$dr(t) = \mu_r()dt + \sigma_r()dW(t) \tag{4}$$

where $W(t)$ is a Wiener process, $\mu_r() \equiv \mu_r(t, r(t))$ and $\sigma_r() \equiv \sigma_r(t, r(t))$ are given real valued functions whose form will totally determine the behavior of the short term rate[5]. When $\mu_r()$ and $\sigma_r()$ are at most function of the state variable $r(t)$ and do not depend on time, the model is called **time homogeneous**.

Let us denote by $V(t)$ the value at time t of an interest rate contingent claim with maturity T. In reality, V could be a discount or a coupon bond, a bond option, a cap, a floor, an interest rate swap, etc. As it derives from the single factor model assumption, only the short term rate $r(t)$ and the time to maturity $(T-t)$ will affect the price of our claim, and we can write

$$V(t) \equiv V(t, T, r(t)) \tag{5}$$

By Ito's lemma,

$$dV(t) = \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}(dr)^2$$

[5]Mathematically, these functions must fulfill some regularity conditions so that this stochastic differential equation has only a unique solution:

- μ, σ must be measurable functions from $R_+ \times R$ to R.
- $\exists\ k_1 > 0$ such that the Lipschitz condition holds, i.e. $\forall t \in [0,T]; x, y \in R$

$$|\mu(t,x) - \mu(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq k_1 |x-y|$$

- $\exists\ k_2 > 0$ such that the growth condition is specified, i.e. $\forall t \in [0,T]; x \in R$

$$|\mu(t,x)|^2 + |\sigma(t,x)|^2 \leq k_2 (1 + |x|)$$

Using (4) to compute dr and $(dr)^2$ yields the claim dynamics

$$dV(t) = \left[\frac{\partial V}{\partial t} + \mu_r()\frac{\partial V}{\partial r} + \frac{\sigma_r^2()}{2}\frac{\partial^2 V}{\partial r^2}\right] dt + \left[\frac{\partial V}{\partial r}\sigma_r()\right] dW_t$$

Dividing both sides by $V(t)$ yields the instantaneous return on the contingent claim:

$$\frac{dV(t)}{V(t)} = \frac{1}{V(t)}\left[\frac{\partial V}{\partial t} + \mu_r()\frac{\partial V}{\partial r} + \frac{\sigma_r^2()}{2}\frac{\partial^2 V}{\partial r^2}\right] dt + \frac{1}{V(t)}\left[\frac{\partial V}{\partial r}\sigma_r()\right] dW_t$$

that is,

$$\frac{dV(t)}{V(t)} = \mu_V()dt + \sigma_V()dW(t) \tag{6}$$

where $\mu_V()$ and $\sigma_V()$ are functions of t, T, and $r(t)$.

Now, let us consider two distinct interest rate contingent claims V_1 and V_2 with maturity T_1 and T_2 and let us form a portfolio P made of x_1 currency unit of the claim $V_1(t) \equiv V(t, T_1, r(t))$ and x_2 currency units of the claim $V_2(t) \equiv V(t, T_2, r(t))$. The portfolio value will be described by a process denoted $\{P(t); 0 \leq t \leq T \leq \min(T_1, T_2)\}$, and we have

$$P(t) = n_1 V_1(t) + n_2 V_2(t)$$

As V_1 and V_2 are interest rate contingent claims, their prices verify (6), and we will denote

$$\begin{cases} \dfrac{dV_1(t)}{V_1(t)} = \mu_{V_1}()dt + \sigma_{V_1}()dW(t) \\ \dfrac{dV_2(t)}{V_2(t)} = \mu_{V_2}()dt + \sigma_{V_2}()dW(t) \end{cases}$$

Thus, the variations of the portfolio value are given by

$$\begin{aligned} dP(t) &= x_1\frac{dV_1(t)}{V_1(t)} + x_2\frac{dV_2(t)}{V_2(t)} \\ &= \left(x_1\mu_{V_1}() + x_2\mu_{V_2}()\right) dt + \left(x_1\sigma_{V_1}() + x_2\sigma_{V_2}()\right) dW(t) \end{aligned}$$

We can easily select x_1 and x_2 to cancel out the instantaneous risk of the position, i.e. to reduce the volatility of $dP(t)$ to zero. In such a case, in order to avoid arbitrage opportunities, the return on the portfolio must be equal to the risk-free rate. This gives the following system of equations

$$\begin{cases} x_1\sigma_{B_1}() + x_2\sigma_{B_2}() = 0 \\ x_1(\mu_{B_1}() - r(t)) + x_2(\mu_{B_2}() - r(t)) = 0 \end{cases}$$

which has a non trivial solution if and only if

$$\frac{\mu_{B_1}() - r(t)}{\sigma_{B_1}()} = \frac{\mu_{B_2}() - r(t)}{\sigma_{B_2}()}$$

As this relationship must hold for any T_1 and T_2, we must have the risk premium per unit of risk constant for all maturities. We denote

$$\frac{\mu_B() - r(t)}{\sigma_B()} = \lambda(t, r(t)) \tag{7}$$

where $\lambda(t, r(t))$ is called the **market risk-premium and is independent of T**. This allows us to express the instantaneous return on the bond as

$$\mu_B() = r(t) + \underbrace{\lambda(t, r(t))\sigma_B()}_{\text{total risk premium}}$$

which is quite similar to the Arbitrage Pricing Theory and to classical theories of the interest rate structure[6].

Substituting $\mu_B()$ and $\sigma_B()$ in (7) by their definitions from equation (6) gives us a second order partial differential equation (called the Feynman-Kac equation) that must be satisfied by **any** interest rate contingent claim in a no-arbitrage one factor model:

$$\frac{\partial V}{\partial t} + (\mu_r() - \lambda(t, r(t))\sigma_r())\frac{\partial V}{\partial r} + \frac{\sigma_r^2()}{2}\frac{\partial^2 V}{\partial r^2} - r(t)V = 0 \tag{8}$$

with one boundary condition. The term $\mu_r() - \lambda_B()\sigma_r()$ is often called the **risk adjusted drift**.

Equation (8) will be the fundamental equation. Any interest-rate contingent claim price can be computed as the solution to such a partial differential equation subject to an appropriate boundary condition[7]. Of course, different one factor models will produce partial differential equations of identical structure, but with different $\mu_r()$ and $\sigma_r()$ as inputs, while different interest rate contingent claims will produce the same partial differential equation, but with different boundary conditions. For instance, if one considers V as

[6] As mentioned already, these theories distinguish themselves depending on the existence of a risk premium and its sign. For instance, the expectations hypothesis assumes that $\mu_B = r(t)$, whatever the maturity, which implies $\lambda = 0$ (no liquidity preference).

[7] Mathematically, this corresponds to a Green function.

- a zero-coupon bond $B(t, T)$ with maturity date T, we have

$$\frac{\partial B}{\partial t} + (\mu_r() - \lambda()\sigma_r())\frac{\partial B}{\partial r} + \frac{\sigma_r^2()}{2}\frac{\partial^2 B}{\partial r^2} - r(t)B = 0 \tag{9}$$

with the boundary condition

$$B(T, T) = 1$$

- a plain vanilla call option on $B(t, T)$ with maturity date $T_C < T$, we have

$$\frac{\partial V}{\partial t} + \frac{\sigma_r^2()}{2}\frac{\partial^2 V}{\partial r^2} + (\mu_r() - \lambda()\sigma_r())\frac{\partial V}{\partial r} - r(t)V = 0$$

with the boundary condition

$$V(T_C) = \max(B(t, T_C) - K, 0)$$

- a swap of a fixed rate r^* against a floating rate r with maturity date T, we have

$$\frac{\partial V}{\partial t} + \frac{\sigma_r^2()}{2}\frac{\partial^2 V}{\partial r^2} + (\mu_r() - \lambda()\sigma_r())\frac{\partial V}{\partial r} - r(t)V + (r - r^*) = 0$$

with the boundary condition

$$V(0) = 0$$

- a caplet at rate r^*, we have

$$\frac{\partial V}{\partial t} + \frac{\sigma_r^2()}{2}\frac{\partial^2 V}{\partial r^2} + (\mu_r() - \lambda()\sigma_r())\frac{\partial V}{\partial r} - r(t)V + \min(r, r^*) = 0$$

with the boundary condition

$$V(T) = \max(r(T) - r^*, 0)$$

- a floorlet at rate r^*, we have

$$\frac{\partial V}{\partial t} + \frac{\sigma_r^2()}{2}\frac{\partial^2 V}{\partial r^2} + (\mu_r() - \lambda()\sigma_r())\frac{\partial V}{\partial r} - r(t)V + \max(r, r^*) = 0$$

with the boundary condition

$$V(T) = \max(r^* - r(T))$$

In the particular case of a zero-coupon bond, solving this partial differential equation will give us the bond price $B(t,T)$, from which we get the discount function and the whole yield curve using (1).

Proposition 1 *The solution to equation (9) for $V(t,T) \equiv B(t,T)$ under the terminal condition $B(T,T) = 1$ is given by*

$$B(t,T) = E_P\left[e^{-\int_t^T r(s)ds - \frac{1}{2}\int_t^T \lambda^2(s,r(s))ds - \int_t^T \lambda(s,r(s))dW(s)} | F_t\right] \tag{10}$$

where F_t is the sigma-algebra generated by the past information of process $W(s)$ up to time t and P is the historical probability measure.

The relation (9) allows us in theory to compute the price of a zero coupon knowing the real valued functions $\mu_r()$, $\sigma_r()$, and $\lambda()$. Thus, specifying these functions will fully specify the model. The fact that these functions are exogenously specified only gives us a **partial equilibrium model**: the equilibrium is not unique, as different specifications will determine different equilibrium. Specifying $\mu_r(t,r(t))$, $\sigma_r(t,r(t))$ can be done examining long-term statistical properties of the short term interest rate. But specifying $\lambda(t,r(t))$ is a harder task, as it is not observable.

Note that if $\lambda(t,r(t)) = 0$, equation (7) takes the form of the local expectation hypothesis, a form of expectation hypothesis compatible with the no-arbitrage requirement, and the bond price simplifies into

$$B(t,T) = E_P\left[e^{-\int_t^T r(s)ds} | F_t\right] \tag{11}$$

But in general, $\lambda(t,r(t))$ differs from zero, and there are two more terms in the expectation operator, which correspond to non-anticipated variations of the short term rate.

4.2 Interest rate derivatives pricing: the martingale approach

The second approach to bond pricing under stochastic interest rates takes a more probabilistic view. It is based on the martingale framework proposed Harrison and Kreps (1979) and extended by Artzner and Delbaen (1989) and Heath, Jarrow, Morton (1992) for term structure modeling.

4.2.1 The methodology

An essential point is the choice of the numeraire, that is, the common unit on the basis of which asset prices are expressed. Any asset price can be selected as a numeraire, as long as it has a strictly positive value in any state of the world. If we choose asset $N(t)$ as a numeraire, we will denote by

$$V_i^*(t) = \frac{V(t)}{N(t)}$$

the **relative price** of asset V under the new numeraire N at time t, where $V(t)$ and $N(t)$ are the prices at time t expressed in the old numeraire.

Proposition 2 *Under some regularity conditions, a complete market is arbitrage free if there exists an equivalent martingale measure, i.e. a probability measure Q equivalent to P (the historical or actual probability), such that the relative price process of any security is a Q-martingale.*

$$E_Q\left[V_i^*(T)\,|F_t\right] = V_i^*(t) \tag{12}$$

where F_t denotes the filtration (i.e. all the information) known at time t.

The Girsanov theorem provides the necessary framework to transform a probability measure in another equivalent (i.e. sharing the same support) measure.

Proposition 3 *Any martingale $V_i^*(t)$ can be represented as*

$$dV_i^*(t) = \sigma(t)V_i^*(t)dW^*(t)$$

where $W^*(t)$ is the Wiener process $W(t)$ under the measure Q.

A particular choice of numeraire is the **money market account** $\beta(t)$, i.e. the price at time t of one currency unit continuously reinvested at the short-term rate since a specified initial time τ.

$$\beta(t) = e^{\int_\tau^t r(s)ds} \tag{13}$$

Under this new numeraire, the relative price of asset V is given by

$$V^*(t) = \frac{V(t)}{\beta(t)} \tag{14}$$

Using (12) when the asset V is a zero-coupon bond with maturity T and the numeraire is β with $\tau = t$ yields

$$E_Q\left[\frac{B(T,T)}{\beta(T)}|F_t\right] = \frac{B(t,T)}{\beta(t)} = B(t,T)$$

that simplifies into

$$B(t,T) = E_Q\left[e^{-\int_t^T r(s)ds}|F_t\right] \tag{15}$$

Thus, **the price of a zero-coupon bond is equal to the expectation under Q of the reciprocal of the money market account**[8].

This explains why some researchers focus on models in which $\int_t^T r(s)ds$ is normally distributed. If a random variable x is normally distributed $N(\mu, \sigma^2)$, we know that

$$E(e^{ax}) = e^{a\mu + \frac{b^2}{2}\sigma^2}$$

Thus, having a normal distribution for $\int_t^T r(s)ds$ will imply an analytic expression for the bond price.

In a similar way, one could show that for any interest rate contingent claim $V(t)$ maturing at time T, we will have

$$V(t) = E_Q\left[V(T)e^{-\int_t^T r(s)ds}|F_t\right]$$

In other terms, **under Q, discounted asset prices are martingales**. Therefore, using (13) and (14), one can show that the instantaneous return on V under the measure Q is the risk-free rate:

$$\frac{dV(t)}{V(t)} = r(t)dt + \sigma_V()dW^*(t)$$

There are two possible interpretations of this:

- the first one is that we are working in a risk-neutral world using **risk-neutral probabilities** (Cox and Ross (1976)). The latter are also referred to as the **equivalent martingale measure** (Harrison and Kreps (1979)), the **artificial probabilities** (Cox, Ross, and Rubinstein (1979)), or the **objective probabilities**.

- the second one is that the local expectations hypothesis (Cox, Ingersoll, and Ross (1981)) holds, in which case the artificial probabilities are the actual probabilities.

[8] This will be a fundamental equation when using lattices to price interest rate contingent claims.

4.2.2 From one world to another

How do we go from the "real" world to this "risk-neutral" world ? We start with the original risk-free interest rate dynamics

$$dr(t) = \mu_r()dt + \sigma_r()dW(t)$$

which defines a probability distribution P for $r(t)$. Starting from the P-Brownian motion $W(t)$ and the risk premium $\lambda(t)$, we can build a new stochastic process $W^*(t)$ such that

$$W^*(t) = W(t) - \int_0^t \lambda(s)\sigma_r()ds$$

Under some technical conditions, using Girsanov's theorem, we know that there exist a probability measure Q such that $W^*(t)$ is a Q-Brownian motion[9]. Investors agree on the unique probability measure Q, given by

$$dP = \rho(t, \lambda)dQ$$

where $\rho(t, \lambda)$ is the Radon-Nikodym derivative defined by

$$\rho(t, \lambda) = \exp\left(\int_0^t \lambda dW^*(t) - \frac{1}{2}\int_0^t \lambda^2 ds\right)$$

We can write the evolution of the short term interest rate under Q as

$$dr(t) = (\mu_r() - \lambda(t)\sigma_r())\,dt + \sigma_r()dW^*(t)$$

From there, it is easy to verify that under the new probability, the zero-coupon bond price is the expected value of its final value discounted at the instantaneous rate, that is, equation (15) holds.

Of course, both methods are equivalent. We obtain the same zero-coupon price under (11) and (15). Using Q rather than P, we have simply removed the adjustment factors due to the uncertainty in the economy (terms depending on $\lambda(t)$) from our pricing formula and sent them in our probability measure[10].

[9] To be mathematically rigorous, we should check for each model that the equivalent probability exists and that it is unique.

[10] Thus, the problem of estimating the risk-premium $\lambda(t)$ remains a matter of concern.

4.2.3 Model specification: P or Q ?

Some interest rate pricing models specify (4), the dynamics of the short term rate under the historical probability P. This may be the source of problems, as the equivalent probability measure Q may not be unique.

However, it is important to notice that even combined with a no-arbitrage restriction, (4) specified under Q is not sufficient to determine uniquely the price of a particular bond. As in the Black and Scholes framework, we have one source of randomness and one state process, but the short term rate $r(t)$ is not the price of a traded asset. Thus, the market is clearly incomplete, and the corresponding equivalent martingale measure is not necessarily unique. Fortunately, if we include one single bond in the exogenously specified market, then we are able to price all other bonds in term of this "benchmark bond", as bonds with different coupon rates and maturities must satisfy certain internal consistency conditions in order to avoid arbitrage possibilities.

The choice between P and Q also has some important consequences on the parameter estimation: a common features of these models is that there is a set of observable parameters $\underline{\theta}$ (reversion level or speed, volatility, etc.) that must be estimated from the "real world process", i.e. under P and not under Q. But $\underline{\theta}$ enters in a partial differential equation collectively with a set of market prices of risk $\underline{\lambda}$. Thus, estimating $\underline{\theta}$ alone is not sufficient, and we need market traded instruments (bonds, caps, options, etc.) to find the best $(\underline{\theta}, \underline{\lambda})$ combination that optimally fits cross-sectionally their prices. Once all the parameters have been estimated, the partial differential equation can be solved numerically or analytically.

4.3 Some specific properties of short term interest rate models

If one is ready to impose restrictions on the drift and diffusion parameters of the short term rate, very useful results can be obtained. Three particular restrictions will be presented here: the affine models, the Gaussian models, and the lognormal models.

4.3.1 The affine class of short term rate models

Most of the one factor models considered in the financial literature are in the class of what Duffie and Kan (1993) call "affine factor models". A model is said to be **affine** if the zero-coupon bond price takes the form

$$B(t,T) = \exp\left[a(t,T)r(t) + b(t,T)\right]$$

where $a(t,T)$ and $b(t,T)$ are deterministic functions in C^1. The term "affine" is justified by the observation that in such models, the term structure of interest rates is an affine function of the short rate:

$$R(t,T) = \frac{-a(t,T)}{T-t}r(t) + \frac{b(t,T)}{T-t}$$

Note that forward rates will also be an affine function of the short term rate.

Proposition 4 *If under Q, $\mu_r()$ and $\sigma_r^2()$ are affine in $r(t)$, then the model is affine.*

4.3.2 The Gaussian class of short term rate models

A short term interest rate model is said to belong to the **Gaussian class** if it can be written as the following linear differential equation

$$\begin{aligned} dr(t) &= \mu_r(t, r(t))dt + \sigma_r(t, r(t))dW(t) \\ &= \left(\mu_1(t)r(t) + \mu_2(t)\right)dt + \sigma_2(t)dW(t) \end{aligned}$$

This clearly shows that Gaussian models are a particular class of affine models.

Proposition 5 *In a Gaussian model, $r(t)$ is normally distributed, and*

$$r(t) = \phi(t)\left[r(0) + \int_0^t \phi^{-1}(u)\mu_2(u)du + \int_0^t \phi^{-1}(u)\sigma_2(u)dW(u)\right]$$

where $\phi(t)$ solves

$$\left\{ \begin{array}{l} d\phi(t) = \mu_1(t)\phi(t)dt \\ \phi(0) = 1 \end{array} \right.$$

Proposition 6 *Under Q, $-\int_t^T r(u)du$ is normally distributed with a mean m and a variance v that are easy to calculate. Bond prices are lognormally distributed and are given by*

$$B(t,T) = e^{m+\frac{v}{2}}$$

This will allow us to compute bond prices easily. Unfortunately, by definition, Gaussian interest rate models do not prevent the interest rate from becoming negative, which is economically unrealistic.

4.3.3 The log-normal class of short-term rate models

A short term interest rate model is said to be **lognormal** if and only if $\ln r(t)$ is Gaussian. The major advantage of lognormal models over Gaussian is that by definition, lognormal rate models cannot generate negative interest rates. Unfortunately, as we will see, they generally lack analytical tractability. For instance, if we take the price of a zero-coupon bond as given by (15), we have to know the distribution of $\int_t^T r(s)ds$. As $r(s)$ is lognormally distributed and the sum of lognormal random variables is not lognormal, the computational problem turns out to be quite hard.

However, some of them are very popular among practitioners as they are simple to calibrate, allow simultaneous fitting to both the yield curve and the volatility term structure, and provide good pricing of specific instruments such as caps and floors.

4.4 Some specific examples of one-factor time-invariant processes

In this section, we will review in detail some specific examples of one-factor time-invariant processes, namely Merton (1973), Vasicek (1977), and Cox, Ingersoll and Ross (1985) models. These models are the most famous and have some interesting computational features. Unless explicitly mentioned, we assume that the dynamics are specified under the historical probability P (i.e. in the "real" world).

4.4.1 Merton (1973)

Merton (1973) was the first to propose a general stochastic process as a model for the short rate. Under the historical probability P, the short term rate is

$$dr(t) = \mu_r dt + \sigma_r dW(t) \tag{16}$$

where μ_r and σ_r are constant and $W(t)$ is a standard Brownian motion. Furthermore, Merton assumes a constant risk premium λ.

Short term rate The explicit solution to (16) is

$$r(t) = r(s) + \mu_r t + \sigma_r \int_s^t dW(s)$$

for any $t \geqslant s$. Given the set of information at time s, the short term rate $r(t)$ is normally distributed

$$r(t)\,|F_s \sim N\left(r(s) + (t-s)\mu_r, (t-s)\sigma_r^2\right)$$

The unboundedness of the first and second moment of the distribution allows the rate to become negative or infinite. In a sense, the model lacks stability.

Discount bond price The stochastic differential equation to be solved is

$$\frac{\partial B}{\partial t} + \frac{\sigma_r^2}{2}\frac{\partial^2 B}{\partial r^2} + (\mu_r - \lambda\sigma_r)\frac{\partial B}{\partial r} - r(t)B = 0$$

with the boundary condition $B(T,T) = 1$. Its solution is

$$B(t,T) = e^{-(T-t)r(t) - \frac{(T-t)^2(\mu_r - \lambda\sigma_r)}{2} + \frac{(T-t)^3\sigma_r^2}{6}}$$

From there, it is easy to see that the bond price is an increasing function of the maturity date. In particular, one can show that an infinite maturity discount bond will have an infinite price, which is unrealistic.

Under P, the bond price dynamics is given by

$$\frac{dB(t,T)}{B(t,T)} = [r(t) - \lambda(T-t)\sigma_r]\,dt - [(T-t)\sigma_r]\,dW_t$$

which clearly shows the convergence toward a known value: as $T-t$ becomes smaller, the diffusion term vanishes.

Term structure The term structure is given by the sum of the short term rate and of a quadratic function of the time to maturity

$$\begin{aligned} R(t,T) &= -\frac{\ln B(t,T)}{T-t} \\ &= r(t) + \frac{(T-t)(\mu_r - \lambda\sigma_r)}{2} - \frac{(T-t)^2\sigma_r^2}{6} \end{aligned}$$

This implies that changes in the short rate will result in parallel shifts of the term structure. In addition, yields are a concave function of the term structure, increases in volatility will result in increase in the curvature of the term structure, and

$$\lim_{T\to\infty} R(t,T) = -\infty$$

Deriving $R(t,T)$ with respect to T gives the slope of the term structure

$$\frac{\partial R(t,T)}{\partial T} = \frac{(\mu_r - \lambda\sigma_r)}{2} - \frac{(T-t)\sigma_r^2}{3}$$

If $\mu_r > \lambda\sigma_r$, the term structure is humped with a maximum for the maturity $\frac{3(\mu_r - \lambda\sigma_r)}{2\sigma_r^2}$. If $\mu_r \leq \lambda\sigma_r$, the term structure is decreasing. In no case, the term structure can be increasing.

Given the set of information at time $s \leq t$, the yield to maturity $R(t,T)$ is normally distributed

$$R(t,T)\,|F_s \sim N\left(R(s,T) + \theta(t-s), \sigma_r^2(t-s)\right)$$

This implies that the yield volatility term structure is flat and independent of the maturity, whereas in practice, we generally observe a larger volatility on short term rates.

Option price The value at time t of a European call option $C(t)$ with maturity T_C, with exercise price K, on a zero-coupon bond with maturity $T_B \leq T_C$ is given by

$$\begin{aligned} C(t) &= E_Q\left[e^{-\int_t^{T_C} r(s)ds} Max\left[B(T_C,T_B) - K, 0\right]\right] \\ &= B(t,T_B)N(d_1) - KB(t,T_C)N(d_2) \end{aligned}$$

with

$$\begin{aligned} d_1 &= \frac{1}{v}\ln\left(\frac{B(t,T_B)}{KB(t,T_C)}\right) + \frac{1}{2}v \\ d_2 &= d_1 - v \\ v^2 &= \sigma_r^2(T_B - T_C)^2(T_C - t) \end{aligned}$$

4.4.2 Vasicek (1977)

Vasicek (1977) proposes to model the short term interest rate as an Ornstein-Uhlenbeck process:

$$dr(t) = \kappa(\theta - r(t))dt + \sigma dW(t)$$

where κ, θ, and σ are positive constants and $W(t)$ is a standard Wiener process. This defines an elastic random walk around a trend, with a mean reverting characteristic: when $r(t)$ goes over (respectively: under) θ, the expected variation of $r(t)$ becomes negative (respectively: positive) and $r(t)$ tends to come back to its average long term level θ at an adjustment speed κ. In addition, Vasicek postulates a constant risk premium λ.

Short term rate The explicit solution to this stochastic differential equation is

$$r(t) = \theta + (r(s) - \theta)e^{-\kappa(t-s)} + \sigma_r \int_s^t e^{-\kappa(t-s)} dW(u)$$

for any $t \geqslant s$. Given the set of information at time s, the short term rate $r(t)$ is normally distributed

$$r(t)\,|F_s \sim N\left(\theta + (r(s) - \theta)e^{-\kappa(t-s)}, \frac{\sigma_r^2}{2\kappa}(1 - e^{-2\kappa(t-s)})\right)$$

As a consequence, interest rates can become negative, which is incompatible with no arbitrage in the presence of cash in the economy.

For very large values of t, the expected value and variance of the short-term rate are θ and $\frac{\sigma_r^2}{2\kappa}$. The mean reversion process precludes these two values to explode, reducing the probability of unreasonably large or low interest rates.

Discount bond price The stochastic differential equation to be solved by the bond price is

$$\frac{\partial B}{\partial t} + \frac{\sigma_r^2}{2}\frac{\partial^2 B}{\partial r^2} + (\kappa(\theta - r(t)) - \lambda\sigma_r)\frac{\partial B}{\partial r} - rB = 0 \tag{17}$$

with the boundary condition $B(T, T) = 1$. Alternatively, the bond price can be obtained by computing the discounted expected terminal value of the bond with respect to Q.

$$B(t, T) = E_Q\left[e^{-\int_t^T r(s)ds}\,|F_t\right]$$

The solution is

$$B(t, T) = e^{a(t,T)r(t)+b(t,T)}$$

with

$$
\begin{aligned}
a(t,T) &= \frac{1}{\kappa}(e^{-(T-t)\kappa} - 1) \\
b(t,T) &= \frac{\sigma_r^2}{4\kappa^3}\left(1 - e^{-2(T-t)\kappa}\right) + \frac{1}{\kappa}\left(\theta - \frac{\lambda\sigma_r}{\kappa} - \frac{\sigma_r^2}{\kappa^2}\right)\left(1 - e^{-(T-t)\kappa}\right) \\
&\quad - \left(\theta - \frac{\lambda\sigma_r}{\kappa} - \frac{\sigma_r^2}{\kappa^2}\right)(T-t)
\end{aligned}
$$

Under the original measure P, the bond price dynamics is given by

$$
\frac{dB}{B} = \left[r(t) + \frac{\lambda\sigma_r}{\kappa}\left(e^{-(T-t)\kappa} - 1\right)\right]dt + \frac{\sigma_r}{\kappa}\left(e^{-(T-t)\kappa} - 1\right)dW(t)
$$

which implies that bond prices are lognormally distributed. Note that the volatility term increases with T, but is bounded with respect to the time to maturity.

Term structure The term structure is given by

$$
\begin{aligned}
R(t,T) &= -\frac{1}{T-t}\left\{\frac{1}{\kappa}\left(e^{-(T-t)\kappa} - 1\right)r(t) + \frac{\sigma_r^2}{4\kappa^3}\left(1 - e^{-2(T-t)\kappa}\right)\right. \\
&\quad + \frac{1}{\kappa}\left(\theta - \frac{\lambda\sigma_r}{\kappa} - \frac{\sigma_r^2}{\kappa^2}\right)\left(1 - e^{-(T-t)\kappa}\right) \\
&\quad \left. - \left(\theta - \frac{\lambda\sigma_r}{\kappa} - \frac{\sigma^2}{2\kappa^2}\right)(T-t)\right\}
\end{aligned}
$$

Noticing that the infinite maturity interest rate is constant and does not depend on $r(t)$

$$
R(t,\infty) \equiv \lim_{T\to\infty} R(t,T) = \theta - \frac{\lambda\sigma_r}{\kappa} - \frac{\sigma_r^2}{2\kappa^2}
$$

we have

$$
\begin{aligned}
R(t,T) &= R(t,\infty) + \frac{1 - e^{-(T-t)\kappa}}{(T-t)\kappa}\left(r(t) - R(t,\infty)\right) \\
&\quad + \frac{\sigma_r^2}{4(T-t)\kappa^3}\left(1 - e^{-(T-t)\kappa}\right)^2
\end{aligned}
$$

The term structure can be positively shaped ($r(t) < R(t,\infty) - \frac{\sigma^2}{4\kappa^2}$), negatively shaped ($r(t) > R(t,\infty) + \frac{\sigma^2}{2\kappa^2}$) or humped (other values of $r(t)$).

Given the set of information at time $s \leq t$, the yield to maturity $R(t,T)$ is normally distributed

$$R(t,T)\,|F_s \sim N\left(\mu_R(), \sigma_R^2()\right)$$

with

$$\begin{aligned}
\mu_R() &= (1 - e^{-\kappa(t-s)})\left(R(t,\infty) + \frac{1-e^{-\kappa T}}{\kappa T}(\theta - R(t,\infty)) + \frac{\sigma_r^2(1-e^{-\kappa T})^2}{4\kappa^3 T}\right) \\
&\quad + e^{-\kappa(t-s)}R(s,T) \\
\sigma_R() &= \left(\frac{1-e^{-\kappa T}}{\kappa T}\right)^2 (1 - e^{-2\kappa(t-s)})\frac{\sigma_r^2}{2\kappa}
\end{aligned}$$

The volatility term structure of the yields is a decreasing function of the time to maturity, with limiting value zero.

Options prices Jamshidian (1989) derives analytic solutions for the prices of European call and put options on discount bonds. The option pricing formula has similarities with the Black & Scholes formula, since discount bond prices are also lognormally distributed in the model. The price at time t of a European call $C(t)$ and put options $P(t)$, with strike K, maturing at time T_C, on a zero-coupon bond with maturity date T_B are

$$\begin{aligned}
C(t) &= B(t,T_B)N(d_1) - KB(t,T_C)N(d_2) \\
P(t) &= KB(t,T_C)N(-d_2) - B(t,T_B)N(-d_1)
\end{aligned} \tag{18}$$

with

$$\begin{aligned}
d_1 &= \frac{1}{v}\ln\left(\frac{B(t,T_B)}{KB(t,T_C)}\right) + \frac{1}{2}v \\
d_2 &= d_1 - v \\
v^2 &= \frac{1}{2}\frac{\sigma_r^2}{\kappa^3}\left(1 - e^{-2\kappa(T_C-t)}\right)\left(1 - e^{-\kappa(T_B-T_C)}\right)^2
\end{aligned}$$

Jamshidian (1989) also provides expressions for the price of call and put options on coupon-paying bonds. His approach is of general validity for any one-factor model as long as the option price is a monotonic function of the state variable. Let us examine the case of a call option. Suppose the bond provides n coupon payments after the option maturity (T). The coupon are denoted by C_1, C_2, ..., C_n and their corresponding payment dates by T_1, T_2, ..., T_n. Let r^* be the value of the short term interest rate at time T that causes the coupon bearing bond price to equal the strike price of the option (K), and let $B^*(T, T_i)$ be the value at time T of a zero-coupon bond paying one currency unit at time T_i (i = 1,...,n) when $r(T) = r^*$. At time T, the payoff from the option is

$$\max \left[0, \sum_{i=1}^{n} C_i B(T, T_i) - K \right]$$

Since all rates are an increasing function of $r(t)$, all bond prices are a decreasing function of $r(t)$. Thus, the option will be in the money at time T and should be exercised if and only if $r < r^*$. Furthermore, the zero-coupon bond $B(T, T_i)$ is worth more than $C_i B^*(T, T_i)$ if and only if $r < r^*$. The payoff from the option is therefore

$$\sum_{i=1}^{n} C_i \max \left[B(T, T_i) - B^*(T, T_i) \right]$$

that is, the sum of n options on the underlying zero-coupon-bonds.

Finally, let us note that Sundaresan (1989) provides an analytical solution in the case of fixed to float interest rate swaps.

4.4.3 Cox, Ingersoll, Ross (1985b)

Cox, Ingersoll and Ross (1985b) have developed an equilibrium model in which interest rates are determined by the supply and demand of individuals having a **logarithmic utility function**. The result is a single factor model in which the short term rate satisfies

$$dr(t) = \kappa(\theta - r(t))dt + \sigma\sqrt{r(t)}dW(t) \tag{19}$$

where κ, θ, and σ are positive constants, $W(t)$ is a standard Brownian motion and the risk premium at equilibrium is shown to be

$$\lambda(r, t) = \lambda\sqrt{r(t)}$$

The endogeneously derived short term rate process, also known as the **square-root process,** is similar to Vasicek (1977), but its variance is proportional to the short rate rather than constant. This means that as the short term interest rate increases, its standard deviation increases as well. Furthermore, if it hits the zero-boundary (which is only possible if $\sigma^2 > 2\kappa\theta$), it will never become negative[11].

Short term rate The unique positive solution to the short rate stochastic differential equation (19) is

$$r(t) = \theta + (r(s) - \theta)e^{-\kappa(t-s)} + \sigma_r e^{-\kappa(t-s)} \int_s^t e^{\kappa(u-s)} \sqrt{r(u)} dW(u)$$

for any $t \geqslant s$. Given the set of information at time s, Feller (1951) has shown that the short term rate $r(t)$ is distributed as a non central chi-squared

$$r(t) \,| F_s \sim \chi(2cr(t), 2q + 2, 2u)$$

with $2q + 2$ degrees of freedom and non central parameter $2u$, where

$$\begin{aligned} c &= \frac{2\kappa}{\sigma_r^2(1 - e^{-\kappa(t-s)})} \\ u &= cr(s)e^{-\kappa(t-s)} \\ v &= cr(s) \\ q &= \frac{2\kappa\theta}{\sigma_r^2} - 1 \end{aligned}$$

The distribution can be written explicitly[12] as

$$f(r(t)\,|r(s)) = ce^{-u-v} \left(\frac{u}{v}\right)^{\frac{q}{2}} I_q \left[2\sqrt{uv}\right] \tag{20}$$

[11]See Feller (1951) for a proof. Intuitively, the variance $\sigma_r^2 r(t)$ is very small if $r(t)$ is close to zero and is dominated by the drift component.

[12]This distribution is not a chi-squared. But if we define $x(t) \equiv 2cr(t)$, then $x(t)$ is a non central chi-squared with distribution

$$f(x(t)\,|x(s)) = \frac{1}{2} e^{-c(x(s)+2u)} \left(\frac{x(s)}{2u}\right)^{q/2} I_q \left[\sqrt{2ux(s)}\right]$$

where $2u$ is the degree of non-centrality and $2(q + 1)$ is the degree of freedom.

where $I_q[]$ is the modified Bessel function of the first type and of order q.

According to this distribution, the mean and variance of $r(t)$ given $r(s)$ are

$$E\left(r(t)\left|r(s)\right.\right) = \theta + (r(s) - \theta)e^{-\kappa(t-s)}$$

and

$$V\left(r(t)\left|r(s)\right.\right) = r(s)\frac{\sigma_r^2}{\kappa}(e^{-\kappa(t-s)} - e^{-2\kappa(t-s)}) + \theta\frac{\sigma_r^2}{2\kappa}(1 - e^{-\kappa(t-s)})^2$$

respectively. When $t \to \infty$, the limits are θ and $\theta\frac{\sigma_r^2}{2\kappa}$ respectively.

Note that the risk-neutral density f_{RN} corresponding to (20) is obtained by replacing κ by $(\kappa + \lambda)$. The resulting distribution has the same number of degree of freedom, but the degree of non-centrality changes.

Discount bond price The stochastic differential equation to be solved is

$$\frac{\partial B}{\partial t} + \frac{\sigma_r^2}{2}\frac{\partial^2 B}{\partial r^2}r(t) + \left(\kappa(\theta - r(t)) - \lambda\sigma_r\right)\frac{\partial B}{\partial r} - r(t)B = 0 \tag{21}$$

with the usual boundary condition $B(T,T) = 1$. Alternatively, the bond price can be obtained by computing the discounted expected terminal value of the bond with respect to Q,

$$\begin{aligned} B(t,T) &= E_Q\left[e^{-\int_t^T r(s)ds}\left|F_t\right.\right] \\ &= \int_0^\infty e^{-\int_t^T r(s)ds} f_{RN}\left(r(T)\left|r(t)\right.\right)dr(T) \end{aligned}$$

Unlike the normal case, the distribution of $-\int_t^T r(s)ds$ is not known, and we have to solve the Laplace transform to obtain it, which also leads to solving a partial differential equation.

The solution is

$$B(t,T) = a(t,T)\exp\left(-b(t,T)r_t\right)$$

with

$$a(t,T) = \left(\frac{2\gamma e^{(\widetilde{\kappa}+\gamma)\frac{T}{2}}}{(\widetilde{\kappa}+\gamma)(e^{\gamma T}-1)+2\gamma}\right)^{(\widetilde{\kappa}+\gamma)\frac{R(t,\infty)}{\sigma_r^2}}$$

$$
\begin{aligned}
b(t,T) &= \frac{2(e^{\gamma T}-1)}{(\widetilde{\kappa}+\gamma)(e^{\gamma T}-1)+2\gamma} \\
\gamma &= \sqrt{\widetilde{\kappa}^2+2\sigma_r^2} \\
\widetilde{\kappa} &= \kappa+\lambda\sigma_r
\end{aligned}
$$

Under the original measure P, the bond price dynamics is given by

$$
\frac{dB}{B} = [r(t)(1-\lambda\sigma_r b(t,T)]\,dt + \sqrt{r(t)}\sigma_r b(t,T)dW(t)
$$

Term structure The rate $R(t,T)$ linearly depends on $r(t)$ and $R(t,\infty)$.We have

$$
R(t,T) = \frac{B(T)}{T}r(t) - \frac{\kappa+\lambda\sigma_r+\gamma}{2\kappa\theta}\frac{\ln A(T)}{T}R(t,\infty)
$$

and

$$
R(t,\infty) = \frac{2\kappa\theta}{\kappa+\lambda\sigma_r+\gamma}
$$

Thus, the value of $r(t)$ determines the level of the term structure at time t, but not its shape. As in the case of Vasicek, upward-sloping, downward sloping and humped yield curves are admissible.

Options prices Cox, Ingersoll and Ross provide formulas for the price of European call and put options with strike K and maturity date T_C on zero-coupon bonds maturing at time T_B. They are slightly complicated, as they involve integrals of the non-central chi-square distribution. For instance, the call option formula is:

$$
\begin{aligned}
C(t) &= B(t,T_B)\chi\left(d_1, \frac{4\kappa\theta}{\sigma_r^2}, \frac{\phi^2 r(t)e^{\gamma T_C}}{d_1/r^*}\right) \\
&\quad -KB(t,T_C)\chi\left(d_2, \frac{4\kappa\theta}{\sigma_r^2}, \frac{\phi^2 r(t)e^{\gamma T_C}}{d_2/r^*}\right)
\end{aligned}
$$

where

$$
\begin{aligned}
d_1 &= 2r^*(\phi+\psi+b(t,T_B-T_C)) \\
d_2 &= 2r^*(\phi+\psi)
\end{aligned}
$$

$$\phi = \frac{2\gamma}{\sigma^2(e^{\gamma T_C} - 1)}$$
$$\psi = \frac{\widetilde{\kappa} + \gamma}{\sigma_r^2}$$

and

$$r^* = \frac{\ln\left(\frac{a(t,T_B - T_C)}{K}\right)}{b(t, T_B - T_C)}$$

is the rate at maturity such that the option is exactly at the money.

Nevertheless, the formula can be interpreted in the same way as the Black and Scholes framework: the first term is the discounted expected value of the bond conditional upon an in the money option at maturity. The second term is the discounted exercise price times the probability of ending up in the money.

European options on coupon-bearing bonds can be valued using the same approach as Jamshidian (1989) in the case of the Vasicek model. The Cox, Ingersoll and Ross (1985b) model has been used widely in the literature to develop pricing formulas for other contingent claims. Longstaff (1990) has shown how to value European options on yields, Dunn and McConnell (1981) mortgage-backed securities, Ramaswamy and Sundaresan (1986) futures and options on futures, Sundaresan (1989) interest rates swaps, Chesney, Eliott and Gibson (1993) American options on yields.

Transformation Define

$$h(t) = \frac{\sigma_r^2}{4\kappa}(e^{\kappa t} - 1), h^{-1}(u) = \frac{1}{\kappa}\log(1 + \frac{4\kappa u}{\sigma_r^2})$$

and

$$r(u) = e^{\kappa h^{-1}(u)} r(h^{-1}(u))$$

Then, the process

$$d\widehat{r}(u) = \frac{4\kappa\theta}{\sigma_r^2}du + 2\sqrt{\widehat{r}(u)}d\widehat{W}(t)$$

is of the class BESQ($\frac{4\kappa\theta}{\sigma_r^2}$), which is extensively studied by Revuz and Yor (1991)

4.5 One-factor time-varying (fitted) processes: Hull and White

Most of the time-invariant models that we reviewed in the previous section suffer from the shortcomings that the short term rate dynamics implies an endogenous term structure, which is not necessarily consistent with the observed one. As these models cannot be calibrated to effective yield curves, practitioners are very reluctant to apply them. Furthermore, these models cannot at the same time fit the initial term structure and a predefined future behavior for the short term rate volatility.

This is why Hull and White (1990) introduced a class of models which allows both and that is consistent with a whole class of existing models. The Hull and White (1993) most general specification is

$$dr(t) = (\theta(t) - \kappa(t)r(t))dt + \sigma(t)r^{\beta}(t)dW(t) \tag{22}$$

with an exogeneously specified risk premium

$$\lambda(r,t) = \lambda r^{\gamma}$$

and with $\lambda, \gamma \geqslant 0$. The functions $\theta(t)$, $\kappa(t)$ and $\sigma(t)$ are time-varying and can be use to calibrate exactly the model to current market prices (in fact, what is called non-stability of parameters in calibrating the time-invariant model is developed here at time-varying parameters)[13]. The price to be paid for this exact calibration is that bond and bond options prices are no longer analytically obtainable.

4.5.1 Should we consider all parameters as time varying ?

It may be tempting to set $\kappa(t)$ and $\sigma(t)$ as time-varying in order to exactly match the initial term structure. However, an important consequence is that the resulting volatility term structure will generally be non stationary and will often evolve in a quite unpredictable way (Carverhill (1995)). As a consequence, option prices computed with this volatility should be taken very cautiously. Very fluctuating values from the parameters can often point to a misspecified or a misestimated model. Hull and White (1995) themselves wrote:

[13]Note that specifications with $\beta \in]0; 0.5[$ should be taken cautiously, because the solution to (22) is not necessarily unique(see Arnold (1973, p. 124)). Empirically, Chan, Karolyi, Longstaff, and Sanders (1992a) suggest $\beta \approx 1.5$.

> "It is always dangerous to uses time-varying model parameters so that the initial volatility curve is fitted exactly. Using all the degrees of freedom in a model to fit the volatility exactly constitutes an **over-parametrization** of the model. It is our option that there should be no more than one time varying parameter used in Markov models of the term structure evolution, and this should be used to fit the initial term structure".

This explains why, in practice, the model (22) is often implemented with $\kappa(t)$ and $\sigma(t)$ constant and $\theta(t)$ as time-varying.

4.5.2 Example: the extended Vasicek model

In the Hull and White framework, the extended Vasicek (1977) model can be written as (22) with $\beta = 0$, or equivalently as

$$dr(t) = \kappa\left(\frac{\theta(t)}{\kappa} - r(t)\right)dt + \sigma_r dW(t)$$

where κ and σ are positive constants. In a sense, it is both a Ho and Lee (1986) model with mean reversion at rate κ and a Vasicek model with time-dependant reversion level (at time t, the short term rate reverts to $\frac{\theta(t)}{\kappa}$ at rate κ).

The parameter $\theta(t)$ can be estimated using the initial term structure

$$\theta(t) = \frac{\partial}{\partial t}F(0,t) + \kappa F(0,t) + \frac{\sigma_r^2}{2\kappa}(1 - e^{-2\kappa t})$$

Discount bond prices Discount bond prices are given by

$$B(t,T) = e^{a(t,T)r_t + b(t,T)}$$

with

$$\begin{aligned} a(t,T) &= \frac{1}{\kappa}(e^{-(T-t)\kappa} - 1) \\ b(t,T) &= \ln\left(\frac{B(0,T)}{B(0,t)}\right) - B(t,T)\frac{\partial \ln(B(0,t))}{\partial t} \\ &\quad - \frac{1}{4\kappa^3}\sigma^2(e^{-\kappa T} - e^{-\kappa t})^2(e^{2\kappa t} - 1) \end{aligned}$$

These equations specify the price of bonds at time t in terms of the short rate and the price of zero-coupon bonds today at time t.

Option prices The price at time t of a European call and put options with strike K maturing at time T on a zero-coupon bond with principal amount L_B and maturity date T_B are

$$\begin{aligned} C(t) &= L_B B(0,T_B) N(h) - K B(0,T) N(h-\sigma_P) \\ P(t) &= K B(0,T) N(-h+\sigma_P) - L_B B(0,T_B) N(-h) \end{aligned} \tag{23}$$

where

$$h = \frac{1}{\sigma_P} \ln\left(\frac{L_B B(0,T_B)}{B(0,T)K}\right) + \frac{\sigma_P}{2}$$

and

$$\sigma_P = \frac{\sigma_r}{\kappa}(1-e^{-\kappa(T_B-T)})\sqrt{\frac{1-e^{-2\kappa T}}{2\kappa}}$$

is the standard deviation of the logarithm of the bond price at time T. Equation (23) is the same as (18), but with $t = 0$. It gives the same results as using Black's model with a volatility of $\frac{\sigma_P}{\sqrt{T}}$.

European options on coupon-bearing bonds can be valued using the same approach as Jamshidian (1989) in the case of the Vasicek model.

Critiques In addition to the previous analytical expressions, Hull and White (1990) have developed a very elegant trinomial lattice methodology that can be used to calibrate the model to market data. But despite its positive features, the extended Vasicek model still suffers from important problems

- it still allows for negative interest rates (even if due to the presence of mean reversion, the probability of this occurrence is limited).
- caution is needed when calibrating the model to cap prices, especially for some combinations of term structure shapes and market cap volatilities, which give very strong or very low levels of mean reversion (see Rebonato (1996)).
- an exact day to day fitting to the term structure generally produces an implausible extreme unstable behavior, in particular for $\theta(t)$. This explains why the model is often implemented with a constant reversion speed.

4.5.3 Other extended models

In the Hull and White (1993) framework, the extended Cox, Ingersoll, Ross (1985b) model can be written as (22) with $\beta = 0.5$. In this case, however, closed form solutions for the zero-coupon bond price and option prices are not available. The solution involves numerical procedures to solve the partial differential equation of the bond price. Note that the extended Brennan and Schwartz (1977) and Courtadon (1982) model can be extended as (22) with $\beta = 1$.

Studying the family of models given by (22), Jamshidian (1993b) proves that if $\theta(t)/\sigma^2(t)$ is a constant, then the bond price is analytically tractable. Strickland (1993) and Carverhill (1994) also discuss the constraints on the volatility functions for such extended models.

4.6 Some specific examples of lognormal models

In most of the models we have seen, either the sort rate or the forward rate were modeled as Gaussian processes. This popularity is due to the analytical tractability of Gaussian processes, but it also implies that there is a positive probability of negative rates, which implies arbitrage opportunities in the presence of cash. This has led some authors to propose models with lognormal rates, thus avoiding negative rates.

4.6.1 Black, Derman, and Toy's (1987, 1990)

Black, Derman, and Toy (1987) have proposed a one factor binomial model whose continuous time equivalent is[14]

$$d\ln(r(t)) = (\theta(t) - a\ln(r(t)))\,dt + \sigma_r dW(t)$$

[14] Using Ito's lemma, it can be rewritten as

$$dr(t) = r(t)\left(\theta(t) - \kappa\ln(r(t)) + \frac{1}{2}\sigma_r^2\right)dt + \sigma_r r(t)dW(t)$$

Another possible form often seen in the litterature is:

$$r(t) = u(t)e^{\sigma_r(t)z(t)}$$

where κ is the speed of mean reversion. The model is similar to the Ho and Lee (1986) model in approach, but it incorporates the mean reverting behavior of interest rates. Furthermore, it assumes a log-normal process for the short rate, which precludes negative values. In 1990, they extended the model to allow for time dependant volatility:

$$d\ln(r(t)) = (\theta(t) - a\ln(r(t)))\,dt + \sigma_r(t)dW(t)$$

The model is very popular among practitioners for various reasons:

- it can be constructed to price exactly any set of discount bonds, as it uses the initially observed term structure to estimate the expected means and standard deviations of future short rates (in practice, it requires numerical fitting to both the interest rate level and the volatility term structure)

- (plain vanilla) swap rates, which are a linear combination of discount bonds, can be priced exactly for any volatility structure,

- caps or swaptions quotes (i.e., implied volatilities) can be used directly to calibrate the model.

- the current market information can be represented by a simple recombining binomial tree with equally likely up and down moves, which eases computation and understanding.[15]

Unfortunately, the model lack analytical properties, and its implications and implicit assumptions are unknown. Furthermore, it has been shown that the model gives infinite expected roll-over returns (see Sandmann and Sondermann (1993), Haugan and Weintraub (1993)).

An interesting reformulation of the model is obtained by applying Ito's lemma to the function $r(t, z(t))$ with

$$z(t) = \frac{\ln r(t) - \ln u(t)}{\sigma_r(t)}$$

where $u(t)$ is the **median** (which has no analytical expression for a log-normal distribution). We obtain

$$d\ln(r(t)) = \left(\frac{\partial \ln u(t)}{\partial t} + \frac{\partial \sigma_r(t)}{\partial t}\right)(\ln(u(t)) - \ln(r(t)))\,dt + \sigma_r(t)dW(t)$$

[15] Jamshidian (1991b) suggests a forward induction methology to build the short rate tree.

which is very useful in lattice constructions and Monte-Carlo methodologies. Furthermore, it is easy to see that

- as the reversion speed which determines the volatilities for various maturities is a function of the short rate volatility, the term structure of volatilities is completely determined by the future volatility of the short rate
- if σ_r is constant, the model does not display any mean reversion

4.6.2 Black and Karasinski (1991)

Black and Karasinski (1991) suggested a binomial tree approach with time steps of varying lengths. The continuous time version of their model is

$$d\ln(r(t)) = (\theta(t) - \kappa(t)\ln(r(t)))\,dt + \sigma_r(t)dW(t)$$

which is an extension of the Black, Derman, and Toy (1987) model with a time-varying speed reversion speed ($\kappa(t)$). They postulate that the model fits the yield curve, the volatility curve, and the cap-curve.

4.6.3 Sandmann and Sondermann (1993b)

Modeling lognormally distributed rates is the simplest solution to avoid the problem of negative rates. unfortunately, as we have seen, no closed-form solution have been found for these models. But they suffer from a more important drawback. As evidenced by Hogan and Weintraub (1993), rates explode with positive probability, implying infinite roll over return whatever the maturity, zero prices for bonds, and thus, arbitrage opportunities. As a consequence, such models cannot value one of the most widely used hedging instrument, namely the Eurodollar futures contract, which is worth minus infinity...

As evidenced by Sandmann and Sondermann (1993b), the explosion results from the choice of the instantaneous period as a compounding period. Specifying the instantaneous rate as log-normally distributed will result in exponential of exponential functions. Thus, research is now focusing on modeling **simple** interest rates $r^*(t)$ over a fixed finite period - rather than instantaneous rates - within the lognormal framework. We have

$$r(t) = \ln(1 + r^*(t))$$

Assuming lognormality for the simple rate

$$\frac{dr^*(t)}{r^*(t)} = \mu_{r^*}(t)dt + \sigma_{r^*}(t)dW(t)$$

implies that the continuously compounded rate follows a diffusion that is neither normal, nor lognormal, but a dynamic combination of both.

$$dr(t) = (1 - e^{-r(t)t})\left(\left(\mu_{r^*}(t) - (1 - e^{-r(t)})\frac{\sigma_{r^*}^2(t)}{2}\right)dt + \sigma_{r^*}(t)dW(t)\right)$$

When $r^*(t) \to \infty$, the dynamics converges to the normal diffusion

$$dr(t) = (\mu_{r^*}(t) - \frac{\sigma_{r^*}^2(t)}{2})dt + \sigma_{r^*}(t)dW(t)$$

while when $r^*(t) \to 0$, the dynamics of $r^*(t)$ and $r(t)$ coincide.

4.6.4 Miltersen, Sandman and Sondermann (1997)

Miltersen, Sandman and Sondermann (1997) developed a lognormal term structure model for simple annual forward rates. They define the **simple** forward rate $f^*(t, T_1, T_2)$ as the interest rate set at time t over a fixed period from T_1 to $T_2 \geqslant T_1$. We have

$$B(t, T_2) = B(t, T_1)\frac{1}{1 + \alpha f^*(t, T_1, T_2)}$$

with $\alpha = T_2 - T_1$. The limit case $\alpha = 0$ corresponds to continuously compounded forward rates. This simple forward rate is log-normally distributed

$$\frac{df^*(t, T_1, T_2)}{f^*(t, T_1, T_2)} = \mu_{f^*}(s, T_1, T_2)dt + \sigma_{f^*}(s, T_1, T_2)dW(t)$$

The initially observed term structure is used to calibrate the model. The model is very similar to Heath, Jarrow, Morton (1992), except that it is based on the simple forward rate rather than the continuously compounded forward rate. Note that their model can be stated in a Heath, Jarrow and Morton (1992) framework by using a specific form for the volatility function of the instantaneous forward rate. This form is state dependent.

Miltersen, Sandman and Sondermann (1997) have obtained closed form solutions for European bond options in such a framework. For a European call option with maturity date T_C and strike K, on a zero-coupon bond with maturity T_B, we have

$$C(t) = (1-K)B(t,T_B)N(d_1) - K\left[B(t,T_C) - B(t,T_B)\right]N(d_2)$$

with

$$d_{1/2} = \frac{1}{a(t,T_C,T_B)}\left[\ln\frac{B(t,T_B)(1-K)}{\left(B(t,T_C)-B(t,T_B)\right)K} \pm \frac{a^2(t,T_C,T_B)}{2}\right]$$

and

$$a^2(t,T_C,T_B) = \int_t^T \sigma_{f^*}^2(s,T_C,T_B)ds$$

They have also obtained closed form solutions for caps and floors, which are very similar to those obtained with Black (1976) formula which is widely used in the market.

4.7 Other models

In this section, we will briefly discuss some other one-factor interest rate models. These models are generally less popular than the ones we have examined already; either they do not provide analytical solutions, either they rely on too restrictive or unrealistic assumptions. However, we mention them briefly as they are often the starting point of a more elaborate and accepted model. We will first present models based on a volatility that is proportional to the level of the short term rate, or to a power of the level of the short term rate. Then, we will expose models that focus on the zero-coupon bond price dynamics rather than on the short term rate. We will end by presenting Black (1976) model, which is widely used among practitioners dealing with caps and floors on interest rates. However, despite the use of these models to price specific derivatives, one should always remember that their conceptual background remains fragile. In particular, they are not necessarily arbitrage-free.

4.7.1 Dothan (1978), Brennan and Schwartz (1977)

In Dothan (1978) model, the short term rate follows a geometric Brownian motion without drift:

$$dr(t) = \sigma_r r(t)dW(t)$$

Given the set of information at time $s \leqslant t$, the short term rate $r(t)$ is log-normally distributed

$$r(t)\,|F_s \sim Ln\left(r(s), e^{\sigma_r^2(t-s)} - 1\right)$$

and thus cannot become negative. The model is also called the geometric random walk, or the elastic random walk. The resulting term structure is a monotonically decreasing function of the time to maturity, an increasing concave function of $r(t)$, and a decreasing convex function of σ_r^2.

Courtadon (1982) has shown that by the Law of Iterated Logarithms, in Dothan (1978) model,

$$\lim_{t\to\infty} r(t) = 0$$

As a consequence, the model is inadequate to represent the long term behavior of interest rates. Other major restrictions for the use of Dothan (19878) model are that there is no known distribution for the integral of $r(t)$, nor for its Laplace transform. As a result, there is no simple solution for bond or option prices. Despite this, the model was used numerically by Dothan (1978) to value discount bonds and by Brennan and Schwartz (1977) to value savings, retractable and callable bonds.

4.7.2 Brennan and Schwartz (1980), Courtadon (1982)

Brennan and Schwartz (1980) have proposed to extend Dothan (1978) model by adding a mean reverting term:

$$dr(t) = \kappa(\theta - r(t))dt + \sigma_r r(t)dW(t)$$

But there is no known distribution for $r(t)$, and contingent claim prices must be computed using numerical procedures. Brennan and Schwartz (1980) use the model to price convertible bond and Courtadon (1982) to price discount bond prices.

4.7.3 Rendleman and Bartter (1980)

Rendleman and Bartter (1980) assumed that $r(t)$ follows a geometric Brownian motion with a constant drift and diffusion parameters.

$$dr(t) = \mu_r r(t)dt + \sigma_r r(t)dW(t)$$

The model was also studied by Marsh and Rosenfeld (1983).

4.7.4 Cox, Ingersoll and Ross(1980)

Cox, Ingersoll and Ross(1980) have proposed the following diffusion

$$dr(t) = \sigma_r r(t)^{3/2} dW(t)$$

to study variable-rate securities. The model was also used by Constantinides and Ingersoll (1984) to value bond in the presence of taxes.

4.7.5 Cox (1975), Cox and Ross (1976)

Cox (1975) and Cox and Ross (1976) have proposed using a constant-elasticity of variance diffusion to model the short term rates dynamics.

$$dr(t) = \mu r(t)dt + \sigma_r r^{\gamma}(t) dW_t$$

This model nests the Dothan (1978), Brennan and Schwartz (1980) and Cox, Ingersoll and Ross (1980) models. The application of this process is discussed in Marsh and Rosenfeld (1983), footnote 4.

4.7.6 Longstaff (1989) and the double square-root model

Longstaff (1989) modified the Cox, Ingersoll and Ross (1985b) model as follows:

$$dr(t) = \kappa(\theta - \sqrt{r(t)})dt + \sigma\sqrt{r(t)}dW(t)$$

This model is sometimes referred to as the double square-root model. Longstaff (1989) provides a closed form expression for the price of a zero-coupon bond. His empirical tests suggest that this model outperforms the Cox, Ingersoll and Ross (1985b) model in most situations.

4.7.7 Black and Scholes (1973) and Merton (1973)

Black and Scholes (1973) and Merton (1973) have developed the complete framework for option valuation on an asset whose price follows a geometric Brownian motion. While this asset is traditionally considered as being a stock, the pricing of bond option was already discussed by Merton (1973), who used the zero coupon bond price itself as a state variable to obtain a preference free closed form formulae for an European call option price on a zero-coupon bond. In this framework, the discount bond price follows a diffusion process

$$\frac{dB(t,T_B)}{B(t,T_B)} = \mu dt + \sigma dW(t)$$

The price C_t of a call option (with maturity T_c and strike K) on such a bond is given by:

$$C_t = B(t,T_B)N(d_1) - Ke^{-r(T_C-t)}N(d_2)$$

where

$$\begin{aligned} d_1 &= \frac{\ln(B(t,T_B)/K) + (r-\sigma^2/2)(T_c-t)}{\sigma\sqrt{T_c-t}} \\ d_2 &= d_1 - \sigma\sqrt{T_c-t} \end{aligned}$$

This is a direct extension of the original formula for stocks. It is used widely for short-dated options on long term bonds. despite its simplicity, at the conceptual level, it suffers from major drawbacks:

- The volatility parameter is known to decrease as we get closer to the bond maturity, as the redemption price is known with certainty.
- If the bond price is lognormally distributed, the instantaneous return will be normally distributed. In a one factor model, this implies that the short term rate is also normally distributed and can take negative values.
- How can one imagine that the short term rate is constant, while long term bond prices are stochastic ?
- The information included in the term structure is not used at all. Furthermore, the implied term structure is not necessarily compatible with the observed term structure, which may create arbitrage opportunities.

4.7.8 Ball and Torous (1983)

Ball and Torous (1983) incorporated the constraint of bond price approaching its face value at maturity by assuming that the bond price follows a Brownian bridge process rather than the original geometric Brownian motion of Black and Scholes (1973).

The model suffers from important drawbacks:

- it is incompatible with the initial term structure
- the instantaneous variance of the bond price is constant. Thus, the volatility of the corresponding yield to maturity increases without bounds as we get close to maturity.
- assuming the existence of an equivalent risk-neutral probability, Ball and Torous (1983) derive closed-form solutions for options on zero-coupon bonds. But the model is in fact not arbitrage free: as evidenced by Cheng (1989), the corresponding risk-neutral probability does not exist.

4.7.9 Black (1976)

The property of positive interest rates can be recovered by assuming that interest rates are lognormally distributed. A possible alternative is the use of the Black (1976) model to price options on forward contracts. If F is the forward price, the price of a call option on the forward price becomes

$$C_t = e^{-r(T_C-t)}\left[FN(d_1) - KN(d_2)\right]$$

where

$$d_1 = \frac{\ln(F/K) + (\sigma^2/2)(T_c - t)}{\sigma\sqrt{T_c - t}}$$

$$d_2 = d_1 - \sigma\sqrt{T_c - t}$$

This model is widely used to price caplets and floorlets, as they can be considered as options on the forward rate multiplied by a nominal value. In such a case, we replace F and K by the forward rate and the strike rate.

5 Extensions to a multi-dimensional space

So far, we have been mostly working with interest rate models where the short rate $r(t)$ was the only explanatory variable. These models were characterized by their analytical tractability and their ease of use. However, single factor models have been often criticized on various grounds:

- the long rate is a deterministic function of the short term rate and that the prices of bonds with different maturities are perfectly correlated (or equivalently: there is a perfect correlation between movements in rates of different maturities).

- models often fail to match observed prices, as we will see later on in the empirical review.
- from an economic point of view, it seems unreasonable to assume that the entire term structure is governed only by the short rate.
- it is difficult to obtain realistic volatility structures for the forward rate without going to very complicated specifications for the short rate.

For these considerations, some authors have suggested using more than one explanatory factor to model the interest rates uncertainty. By going from a single factor to a multi-factor, one should get an improved fit. The price to pay is generally a loss of tractability, partial differential equations of a higher dimensionality, and slower results. The choice of the correct factors is also important. Here again, we find models based on the no-arbitrage condition and equilibrium models

Most multi-factor models are in fact based on two factors. Cox, Ingersoll and Ross (1985b) and Richard (1978) used the spot rate and the rate of inflation, Longstaff and Schwartz (1991) the spot rate and its volatility, Duffie and Khan (1993) the yields on a fixed set of bonds, Brennan and Schwartz (1979) the long and the short rate, Schaefer and Schwartz (1987) the short rate and the spread, Fong and Vasicek (1991) the short rate and its volatility, Das and Foresi (1996) the short rate and its mean, etc. More recently, three factor models have been developed. A comprehensive analysis of specific forms of such three factor models can be found in Chen (1994).

Since the analysis of multi-factor models is rather lengthy, we will only provide hereafter the major results of some multi-factor models. We will successively examine a sample of arbitrage and fitted models. As in the case of their single-factor equivalents, multifactor arbitrage models create an instantaneous risk-free portfolio with respect to all the considered factors in order to obtain by arbitrage a partial differential equation that any interest rate contingent claim must satisfy. Fitted models will extend them by allowing for time-varying parameters..

5.1 Simple extensions of single-factor models: Langetieg (1980)

The simplest extension of a single-factor model is obtained by defining the short term rate as being the sum of two stochastic factors. More generally, one can always extend a single factor model to a multi-factor model by defining the short term rate as a function of a set of stochastic factors.

It should be noted that some single-factor model do not necessarily gain anything by being extended to a multi-factor case. For instance, let us extend Merton (1973) model by defining the short term rate as being the sum of n factors

$$r(t) = \sum_{i=1}^{n} x_i(t)$$

where each factor obeys

$$dx_i(t) = \theta_i dt + \sigma_i dW_i(t)$$

and where $W_i(t)$ are n independent Brownian motions, θ_i and σ_i being constant. Then, given $r(s)$, at time $t \geqslant s$, $r(t)$ is again normally distributed with mean $r(s) + (t-s)\theta$, with $\theta = \sum \theta_i$, and variance $\sigma^2(t-s)$, with $\sigma^2 = \sum \sigma_i^2$. Thus, the multifactor extension resumes in a single factor case.

However, in some cases, extensions of single factor models provide useful results. For instance, Langetieg (1980) extended the Vasicek (1977) model by considering the short term rate as the sum of n factors which obey

$$dx_i(t) = \kappa_i(\theta_i - x_i(t))dt + \sigma_i dW_i(t) \tag{24}$$

where $W_i(t)$ are n independent Brownian motions, κ_i, θ_i and σ_i being constant. In such a case, one can show that the discount bond price is given by

$$B(t,T) \equiv B(r(t),t,T) = \prod_{i=1}^{n} B(x_i(t),t,T) \tag{25}$$

and the value of a European call option $C(t)$ with maturity T_C and exercise price K on a zero-coupon bond with maturity T_B is given by

$$C(t) = B(t,T_B)N(d_1) - KB(t,T_C)N(d_2)$$

where

$$\begin{aligned} d_1 &= \frac{1}{v}\ln\frac{B(t,T_B)}{KB(t,T_C)} + \frac{1}{2}v \\ d_2 &= d_1 - v \\ v^2 &= \sum_{i=1}^{n}\frac{1}{2}\frac{\sigma_i^2}{\kappa_i^3}\left(\left(1-e^{-\kappa_i(T_B-T_C)}\right)^2 - \left(e^{-\kappa_i(T_B-t)} - e^{-\kappa_i(T_C-t)}\right)^2\right) \end{aligned}$$

The Cox, Ingersoll and Ross (1985b) model can be extended in a similar way. We obtain again (25) for the bond price, and the expression for the option price can be derived in a similar way.

For a thorough discussion of these extensions and specific examples, see Langetieg (1980), Hull and White (1990), Buser, Hendershott and Sanders (1990) for the Vasicek case and Cox, Ingersoll and Ross (1985b), Hull and White (1990) and Richard (1978) - presented hereafter - for the Cox, Ingersoll and Ross case. However, one has to remember that adding another factor may be easy, but its interpretation in terms of economic signification has to be coherent.

5.2 Duffie and Kan (1993) and the affine models

Duffie and Kan (1993) have introduced models in which the drifts and volatility coefficients of the state-variable processes are affine functions. Two approaches have been pursued in the term structure literature:

- the first one assumes that the short term rate is a linear combination of an unobserved state vector $\underline{Y}(t)$, which itself follows an affine diffusion model which remains to be specified.

  $$r(t) = \underline{\delta Y}(t)$$

  In this framework, one can decompose the term structure movements in factor such as "curvature", "twist", "slope", or "level". Examples of these are the square-root diffusion models used by Chen and Scott (1993), Pearson and Sun (1994), and Duffie and Singleton (1996).

- the second one posits a model for the short rate in terms of its own lag and other state variables, such as long term mean and volatility of $r(t)$. Examples of these are Chen (1996), Balduzzi, Das and Foresi (1995), and Backus, Foresi and Telmer (1996).

Yield-factor models are multi-factor model of the term structure in which the factors are the yields of zero-coupon bonds of n various fixed maturities. Each yield factor is defined by a Markov process, is **observable** on the current yield curve, and its increment can have an arbitrarily specified correlation with other yields. Discount bond prices are given as the solution to an ordinary differential Ricatti equation, and path-independent contingent claims can be priced using the traditional partial differential equation approach. These models have also proved an interesting result, that is, for the forward rate to be affine in the spot rate, the volatility of the short term interest rate must be restricted to the form

$$\sigma_r^2(t, r(t)) = a(t) + b(t)r(t)$$

Ritchken and Sankarasubramanian (1996) have extended this result by showing that the class of volatility structures is the same if the forward rate is a finite degree polynomial of the short rate.

5.3 Richard (1978), Cox, Ingersoll, Ross (1985b)

Richard (1978) proposed a model in which the term structure of interest rates is determined by two factors: the real short term rate $q(t)$ and the expected instantaneous inflation rate $\pi(t)$. Both factors are assumed to follow independent diffusion process

$$\begin{cases} dq(t) = \mu_q(t)dt + \sigma_q(t)dW_q(t) \\ d\pi(t) = \mu_\pi(t)dt + \sigma_\pi(t)dW_\pi(t) \end{cases}$$

where W_q and W_π are two independent Brownian motions. The model is also presented in the Cox, Ingersoll, Ross (1985b) paper.

Applying Ito's lemma to the price of a zero-coupon gives us a diffusion process for the bond price, and a partial differential equation to be solved for the price of any interest rate contingent claim:

$$\frac{\sigma_q^2}{2}\frac{\partial^2 B}{\partial q^2} + \frac{\sigma_\pi^2}{2}\frac{\partial^2 B}{\partial \pi^2} + (\mu_q - \lambda_q\sigma_q)\frac{\partial B}{\partial q} + (\mu_\pi - \lambda_\pi\sigma_\pi)\frac{\partial B}{\partial \pi} - rB + \frac{\partial B}{\partial t} = 0$$

But this partial differential equation now depends on $\pi(t)$, $q(t)$, and $r(t)$, plus the two risk premium λ_q and λ_π! Fortunately, it is possible (after some manipulations) to express $r(t)$ as a function of $\pi(t)$ and $q(t)$ and to rewrite our partial differential equation in a simplified form involving seven parameters: the short real rate dynamics (μ_q, σ_q^2), the short expected inflation rate dynamics (μ_π, σ_π^2), the inflation volatility, and two risk premia (λ_π, λ_q). Assuming a representative investor economy with a logarithmic utility function, using a square-root process for $dq(t)$ and $d\pi(t)$ and proportional risk-premia, Richard obtains a complicated, but analytical, solution for the zero-coupon bond price.

5.4 Brennan and Schwartz (1979, 1982)

Brennan and Schwartz (1979) have suggested a two factor model, in which the term structure of interest rates depends on both the short term rate $r(t)$ and the long term rate $l(t)$. The long term rate is defined as[16]

$$l(t) = \lim_{T\to\infty} R(t,T)$$

The short term and long term rates dynamics in the real world are given by a joint diffusion process:

$$\begin{cases} dr(t) = \mu_r()dt + \sigma_r()dW_r(t) \\ dl(t) = \mu_l()dt + \sigma_l()dW_l(t) \end{cases} \tag{26}$$

where $W_r(t)$ and $W_l(t)$ are two correlated standard Brownian motions, with

$$E(W_r(t), W_l(t)) = \rho t \qquad \forall t \in [0,T]$$

and $\mu_r()$, σ_r, $\mu_l()$, $\sigma_l()$ are functions of t, $r(t)$, and $l(t)$. The specification allows the model to reflect the assumption that the long term rate contains some information about the future value of the short rate.

The zero-coupon bond price is defined as a function of the time to maturity, the short term and the long term rate.

$$B(t,T) \equiv B(t,T,r(t),l(t))$$

[16] In practice, the long term rate can be approximated by the yield on a consol bond (infinite time-to-maturity bond that only pays coupons) which is quoted on some markets. However, hogan (1993) shows that this may yields some problems such as the explosion of the solution of the partial differential equation we will obtain and on the compatilibility between the dynamics specifications for both rates.

Applying Ito's lemma, we have

$$dB(t,T) = \mu_B()dt + \sigma_{B,r}()dW_r(t) + \sigma_{B,l}()dW_l(t)$$

with

$$\begin{aligned}
\mu_B() &= \frac{\partial P}{\partial t} + \mu_r \frac{\partial P}{\partial r} + \mu_l \frac{\partial P}{\partial l} + \frac{\sigma_r^2}{2}\frac{\partial^2 P}{\partial r^2} + \frac{\sigma_l^2}{2}\frac{\partial^2 P}{\partial l^2} + \rho\sigma_r\sigma_l \frac{\sigma_r^2}{2}\frac{\partial^2 P}{\partial r \partial l} \\
\sigma_{B,r}() &= -\sigma_r \frac{\partial P}{\partial r} \\
\sigma_{B,l}() &= -\sigma_l \frac{\partial P}{\partial l}
\end{aligned}$$

Applying the same arbitrage methodology we used for the one-factor model gives

$$\mu_B() - r(t)B(t,T) = \lambda_r(t,r,l)\sigma_{B,r}() + \lambda_l(t,r,l)\sigma_{B,l}() \tag{27}$$

which is the equivalent of (7) in the two factors framework. The functions $\lambda_1(t,r,l)$ and $\lambda_2(t,r,l)$ do not depend on the maturity date T; they are call the risk premium for the short rate and the risk premium for the long rate.

Substituting $\mu_B()$, $\sigma_{B,r}()$ and $\sigma_{B,l}()$ in (27) by their definitions gives us the partial differential equation with the boundary condition that must be verified by the bond price:

$$\frac{\partial B}{\partial t} + (\mu_r - \lambda_r\sigma_r)\frac{\partial B}{\partial r} + (\mu_l - \lambda_l\sigma_l)\frac{\partial B}{\partial l} + \frac{\sigma_r^2}{2}\frac{\partial^2 B}{\partial r^2} + \frac{\sigma_l^2}{2}\frac{\partial^2 B}{\partial l^2} + \rho\sigma_r\sigma_l\frac{\partial^2 B}{\partial r \partial l} - rB = 0 \tag{28}$$

with the usual boundary condition $B(T,T) = 1$, where μ_r, σ_r, μ_l, σ_l, λ_r, and λ_l are real valued functions of t, $r(t)$, and $l(t)$ that have to be specified. The general solution is of the form

$$B(t,T) = E_P\left[e^{-\int_t^T r(s)ds - \frac{1}{2}\int_t^T(\lambda_r^2+\lambda_l^2)ds + \int_t^T \lambda_r dW_r(s) + \int_t^T \lambda_l dW_l(s)} | F_t\right]$$

Fortunately, equation (28) is valid in particular for a consol bond paying a continuous time coupon, whose price is given by $L(t,\infty) = \frac{1}{l(t)}$. From there, we can compute the various derivatives of $L(t,\infty)$ and replace them in (28). We obtain

$$\mu_l - \lambda_l\sigma_l = \frac{\sigma_l^2}{l} + l(l-r)$$

that is, the partial differential equation (28) can be rewritten in a form that is independent of μ_l and λ_l:

$$\begin{aligned} 0 &= \frac{\partial B}{\partial t} + (\mu_r - \lambda_r \sigma_r)\frac{\partial B}{\partial r} + \left(\frac{\sigma_l^2}{l} + l(l-r)\right)\frac{\partial B}{\partial l} \\ &\quad + \frac{\sigma_r^2}{2}\frac{\partial^2 B}{\partial r^2} + \frac{\sigma_l^2}{2}\frac{\partial^2 B}{\partial l^2} + \rho\sigma_r\sigma_l\frac{\partial^2 B}{\partial r \partial l} - rB \end{aligned}$$

with the usual boundary condition $B(T,T) = 1$, where $\mu_r - \lambda_r\sigma_r$ and $\frac{\sigma_l^2}{l} + l(l-r)$ are the risk adjusted drifts. Thus, the bond price will depend on the stochastic processes parameters μ_r, σ_r, σ_l, ρ, and on one preference parameter λ_r. The equation must be solved numerically or by simulation.

In order to perform a quantitative estimation, Brennan and Schwartz choose specific forms of the drift and volatility functions in equations (26), namely,

$$\begin{cases} dr(t) = (a_1 + b_1(l(t) - r(t))dt + r(t)\sigma_1 dW_1(t) \\ dl(t) = l(t)(a_2 + b_2 r(t) + c_2 l(t))dt + l(t)\sigma_2 dW_2(t) \end{cases}$$

However, this specification approach has been questioned recently by Hogan (1993) and Duffie, Ma and Yong (1994), who proves that there are no real-valued solutions to the diffusion equations, thus allowing for the existence of arbitrage.

5.5 Schaefer and Schwartz (1984)

Schaefer and Schwartz (1984) have also suggested a two factor model of the term structure of interest rates, but they have expressed their model in terms of the long term rate $l(t)$ and the spread $s(t)$ between the long rate and the short rate. The choice of these variables was based on the empirical evidence of orthogonality between $dl(t)$ and $ds(t)$. It allowed them to obtain an (approximated) analytical formula for the bond price.

The specific form of the stochastic processes assumed is given by

$$\begin{cases} ds(t) = m(\mu - s(t))dt + \gamma dW_1(t) \\ dl(t) = \beta(s(t), l(t), t)dt + \sigma\sqrt{l(t)}dW_2(t) \end{cases}$$

i.e. a mean-reverting Ornstein-Uhlenbeck process for the spread (which allows for negative spread), and long rate with a variance of change that is proportional to the current long rate. As $\beta(s(t), l(t), t)$ will not enter in the valuation equation, it can be a function of the parameters t, $r(t)$ and $l(t)$. In addition, Schaefer and Schwartz assume that λ_s, the market price of risk of the spread, is a constant.

Under these assumptions, Schaefer and Schwartz show that the partial differential equation for the discount bond price is given by

$$\begin{aligned} 0 = \quad & \frac{1}{2}\gamma\frac{\partial^2 B}{\partial s^2} + \frac{1}{2}\sigma^2 l\frac{\partial^2 B}{\partial l^2} + \frac{\partial V}{\partial s}m(\mu - \frac{\lambda_s\gamma}{m} - s) \\ & + \frac{\partial B}{\partial l}(\sigma^2 - ls) - (l+s)B - \frac{\partial B}{\partial \tau} \end{aligned} \tag{29}$$

with $\tau = T - t$ and subject to the boundary condition

$$B(T, T) = 1$$

This equation has no known analytical solution, but Schaefer and Schwartz provide an approximate analytical solution. They show that (29) is closely related to

$$\begin{aligned} 0 \quad = \quad & \frac{1}{2}\gamma\frac{\partial^2 B}{\partial s^2} + \frac{1}{2}\sigma^2 l\frac{\partial^2 B}{\partial l^2} + \frac{\partial V}{\partial s}m(\mu - \frac{\lambda_s\gamma}{m} - s) \\ & + \frac{\partial B}{\partial l}(\sigma^2 - l\widehat{s}) - (l+s)B - \frac{\partial B}{\partial \tau} \end{aligned} \tag{30}$$

where $\widehat{s}$ is a constant. The solution to (30) subject to the terminal bond price boundary condition can be written as

$$B(t, T) = X(s, \tau)Y(l, \tau)$$

where $X(s, \tau)$ solves

$$\begin{cases} \frac{1}{2}\gamma^2\frac{\partial^2 X}{\partial s^2} + \frac{\partial X}{\partial s}m(\mu - \frac{\lambda_s\gamma}{m} - s) - sX - \frac{\partial X}{\partial \tau} = 0 \\ X(s, 0) = 1 \end{cases} \tag{31}$$

and $Y(l, \tau)$ solves

$$\begin{cases} \frac{1}{2}\sigma^2 l\frac{\partial^2 Y}{\partial l^2} + (\sigma^2 - l\widehat{s})\frac{\partial Y}{\partial l} - lY - \frac{\partial Y}{\partial \tau} = 0 \\ Y(s, 0) = 1 \end{cases} \tag{32}$$

Equation (31) is isomorphic to (17) derived in the Vasicek model. Its solution is

$$X(s, \tau) = \exp\left[\frac{1}{m}(1 - e^{-m\tau})(s_\infty - s) - \tau s_\infty - \frac{\gamma^2}{4m^3}(1 - e^{-m\tau})\right]$$

with

$$s_\infty = \mu - \frac{\lambda_s \gamma}{m} - \frac{1}{2}\frac{\gamma^2}{m^2}$$

Equation (32) is isomorphic to (21) derived in the Cox, Ingersoll and Ross (1985) model. Its solution is

$$Y(l,\tau) = A(\tau)\exp[-B(\tau)l]$$

with

$$\begin{aligned} A(\tau) &= \left[\frac{2\alpha \exp\left((\widehat{s}+\alpha)\frac{\tau}{2}\right)}{(\widehat{s}+\alpha)(\exp(\alpha\tau)-1)+2\alpha}\right]^2 \\ B(\tau) &= \frac{2(\exp(\alpha\tau)-1)}{(\widehat{s}+\alpha)(\exp(\alpha\tau)-1)+2\alpha} \\ \alpha &= \sqrt{\widehat{s}^2+2\sigma^2} \end{aligned}$$

Thus, the bond price is analytically determined by the product of $X(s,\tau)$ and $Y(l,\tau)$. This analytical bond price is an accurate approximation to the solution of (29).

5.6 Longstaff and Schwartz (1992)

Longstaff and Schwartz (1992) developed an equilibrium model of the economy and derived from there a two-factor term structure model. The two factors are (indirectly) the short term rate $r(t)$ and the variance of changes in the short term rate $v(t)$.

In their framework, the representative investor has a logarithmic utility and has the choice between investing or consuming the only good available in the economy, whose price $P(t)$ follows the following stochastic differential equation

$$\frac{dP(t)}{P(t)} = (\mu X(t) + \theta Y(t))dt + \sigma\sqrt{Y}dW_1(t)$$

where $X(t)$ and $Y(t)$ are two specific economic factors and $W_1(t)$ is a Brownian motion. The two factors are chosen in such a way that $X(t)$ is the expected return part that is unrelated to $dW_1(t)$ and $Y(t)$ is the factor correlated with $dP(t)$. The two factors dynamics are given by

$$\begin{cases} dX(t) = (a - bX(t))dt + c\sqrt{X(t)}dW_2(t) \\ dY(t) = (d - eY(t))dt + f\sqrt{Y(t)}dW_2(t) \end{cases}$$

where $W_2(t)$ and $W_3(t)$ are non correlated Brownian motions and a, b, c, d, e, $f > 0$.

Longstaff and Schwartz do not provide any intuitive interpretation for the two factors $X(t)$ and $Y(t)$. But they show that in this specific framework, $X(t)$ and $Y(t)$ can be related to observable quantities, as the equilibrium instantaneous rate $r(t)$ and the variance of its changes $v(t)$ are given by a weighted sum of these two factors

$$\begin{cases} r(t) = \alpha X(t) + \beta Y(t) \\ v(t) = \alpha^2 X(t) + \beta^2 Y(t) \end{cases}$$

with $\alpha = \mu c^2$ and $\beta = (\theta - \sigma^2) f^2$, so that $r(t)$ and $v(t)$ are non-negative for all feasible values of the state variables. This can be rewritten as

$$\begin{cases} X(t) = \dfrac{\beta r(t) - V(t)}{\alpha(\beta - \alpha)} \\ Y(t) = \dfrac{V(t) - \alpha r(t)}{\beta(\beta - \alpha)} \end{cases}$$

that is, we can easily go back and forth between $X(t), Y(t)$ and $r(t), V(t)$. The model can be seen as an affine two factor model, in which one is the short term rate and the second one is its volatility.

If the representative investor maximizes his expected utility of wealth and consumes $C(t)$, Longstaff and Schwartz (1992) compute his wealth dynamics and deduce from there the general partial differential equation obeyed by any interest rate contingent claim V:

$$\frac{1}{2} x \frac{\partial^2 V}{\partial x^2} + \frac{1}{2} y \frac{\partial^2 V}{\partial y^2} + (\gamma - \delta x) \frac{\partial V}{\partial x} + (\eta - (\xi + \lambda) y) \frac{\partial V}{\partial y} - rV = \frac{\partial V}{\partial \tau} \qquad (33)$$

subject to a boundary condition with $x = \frac{X}{c^2}$, $y = \frac{Y}{f^2}$, $\gamma = \frac{a}{c^2}$, $\delta = b$, $\eta = \frac{d}{f^2}$, $\xi = e$, and λ represents the market price of the risk of changes in the level of production uncertainty (which is governed by $Y(t)$). It is important here to note that the fact that the market price of risk is proportional to y is endogenously determined by the model rather than exogenously imposed, which ensures that the risk premium is consistent with the absence of arbitrage.

Solving (33) with $B(T,T) = 1$ as a boundary condition will give us the price of the discount bond as

$$B(t,T) = A^{2\gamma}(\tau) B^{2\eta}(\tau) \exp\left(\kappa\tau + C(\tau) r(t) + D(\tau) V(t)\right)$$

with $\tau = T - t$ and

$$\begin{aligned}
A(\tau) &= \frac{2\phi}{(\delta+\phi)(e^{\phi\tau}-1)+2\phi} \\
B(\tau) &= \frac{2\psi}{(v+\psi)(e^{\psi\tau}-1)+2\psi} \\
C(\tau) &= \frac{\alpha\phi(e^{\psi\tau}-1)B(\tau)-\beta\psi(e^{\phi\tau}-1)A(\tau)}{\phi\psi(\beta-\alpha)} \\
D(\tau) &= \frac{-\phi(e^{\psi\tau}-1)B(\tau)+\psi(e^{\phi\tau}-1)A(\tau)}{\phi\psi(\beta-\alpha)} \\
v &= \lambda+\xi \\
\phi &= \sqrt{2\alpha+\delta^2} \\
\psi &= \sqrt{2\beta+v^2} \\
\kappa &= \gamma(\delta+\phi)+\eta(v+\psi)
\end{aligned}$$

The discount bond price depends on six parameters α, β, γ, δ, η, and v. It is important to notice that the market price of risk enters in the equation only through v, the sum of λ and ξ. Thus, both parameters need not to be separately specified. This reduces the number of parameters to be specified, but also implies that there is an infinity of values of λ and ξ that give rise to an identical fit of a given term structure.

Longstaff and Schwartz (1992) also provide analytical expression for the case of an option on a zero-coupon bond. For instance, a European call price on a discount bond satisfies (33) with

$$C(T) = Max\,[0, B(T_c, T_B) - K]$$

as a boundary condition. If we denote by $\tau = T_C - t$ the time to maturity of the option, we have

$$\begin{aligned}
C(t) &\equiv C(r, V, \tau, K, T_B) \\
&= B(r, V, T_B)\Psi(\theta_1, \theta_2; 4\gamma, 4\eta, \omega_1, \omega_2) \\
&\quad -KB(r, V, \tau)\Psi(\theta_3, \theta_4; 4\gamma, 4\eta, \omega_3, \omega_4)
\end{aligned}$$

where

$$\theta_1 = \frac{4\zeta\phi^2}{\alpha(e^{\phi\tau}-1)^2 A(T_B)} \qquad \theta_2 = \frac{4\zeta\psi^2}{\beta(e^{\psi\tau}-1)^2 B(T_B)}$$
$$\theta_3 = \frac{4\zeta\phi^2}{\alpha(e^{\phi\tau}-1)^2 A(\tau)A(T_B-\tau)} \quad \theta_4 = \frac{4\zeta\psi^2}{\beta(e^{\psi\tau}-1)^2 B(\tau)B(T_B-\tau)}$$
$$\omega_1 = \frac{4\phi e^{\phi\tau}A(T_B)(\beta r - V)}{\alpha(\beta-\alpha)(e^{\phi\tau}-1)A(T_B-\tau)} \quad \omega_2 = \frac{4\psi e^{\psi\tau}B(T_B)(V-\alpha r)}{\beta(\beta-\alpha)(e^{\psi\tau}-1)B(T_B-\tau)}$$
$$\omega_3 = \frac{4\phi e^{\phi\tau}A(\tau)(\beta r - V)}{\alpha(\beta-\alpha)(e^{\phi\tau}-1)} \qquad \omega_2 = \frac{4\psi e^{\psi\tau}B(\tau)(V-\alpha r)}{\beta(\beta-\alpha)(e^{\psi\tau}-1)}$$

and

$$\zeta = \kappa T + 2\gamma \ln A(T_B-\tau) + 2\eta \ln B(T_B-\tau) - \ln K$$

The function $\Psi(\theta_1, \theta_2; 4\gamma, 4\eta, \omega_1, \omega_2)$ is the bivariate non-central chi-square distribution function given by

$$\Psi(\theta_1, \theta_2; 4\gamma, 4\eta, \omega_1, \omega_2) = \int_0^{\theta_1}\int_0^{\theta_2-\theta_2\frac{u}{\theta_1}} \chi^2(u; 4\gamma, \omega_1)\chi^2(v; 4\gamma, \omega_2)dvdu$$

where $\chi^2(.; p, q)$ is the non-central chi-square density[17] with p degrees of freedom and non-centrality parameter q.

Given that the Longstaff and Schwartz (1992) model is affine, it provide closed-form solutions for zero-coupon bonds and European options. However, the difficulty is now to estimate the numerous input parameters. Longstaff and Schwartz (1993) have outlined a parameter estimation method that uses the historical time series of interest rates and interest rates volatilities, but Clewlow and Strickland (1994) have shown that its implementation was rather difficult due to the financial time series generally available in practice.

5.7 Fong and Vasicek (1991, 1992a, 1992b)

In a series of papers, Fong and Vasicek (1991, 1992a, 1992b) have derived a two-factor model using the same factors as the Longstaff-Schwartz (1992) paper, i.e. the short term rate $r(t)$ and the variance of changes in the short term rate $v(t)$. In their framework, the short term rate evolves under the risk-neutral measure according to

$$dr(t) = \alpha(\overline{r} - r(t))dt + \sqrt{v(t)}dW_1(t)$$

[17] See Johnson and Kotz (1970), chapter 28, page 133

where $\overline{r}$ is the long term mean of the short term rate, and $v(t)$ is its instantaneous volatility. This process is very similar to Vasicek (1977), with an extra uncertainty resulting from the variance that is stochastic and that evolves under the risk-neutral measure according to

$$dv(t) = \gamma(\overline{v} - v(t))dt + \xi\sqrt{v(t)}dW_2(t)$$

where $\overline{v}$ is the long term mean of the volatility. The two variations $dW_1(t)$ and $dW_2(t)$ are correlated. Note that the specification does not preclude the short term rate from becoming negative.

In this framework, Fong and Vasicek derive the general partial differential equation obeyed by any interest rate contingent claim V

$$\frac{\partial V}{\partial t} + \alpha(\overline{r} - r)\frac{\partial V}{\partial r} + \gamma(\overline{v} - v)\frac{\partial V}{\partial v} + \frac{v}{2}\frac{\partial^2 V}{\partial r^2} + \frac{\xi^2 v}{2}\frac{\partial^2 V}{\partial v^2} + V\xi v\frac{\partial^2 V}{\partial v \partial r} - rV = 0 \quad (34)$$

subject to a boundary condition.

Solving (34) with $B(T, T) = 1$ as a boundary condition will give us the price of the discount bond as

$$B(t, T) = \exp(-rA(\tau) + vB(\tau) + C(\tau))$$

where $\tau = T - t$ and

$$A(\tau) = \frac{(1 - e^{-\alpha\tau})}{\alpha}$$

is the duration measure of the Vasicek (1977) paper. The functions $B(\tau)$ and $C(\tau)$ are slightly complicated and require the use of complex - as opposed to real - algebra. Selby and Strickland (1995) provide efficient series approximations.

5.8 Chen (1996)

Chen (1996) proposed a three factor model of the term structure. In his model, the short rate dynamics depends on the current short rate, the stochastic mean of the short rate, and the stochastic volatility of the short rate.

$$\begin{cases} dr(t) = \kappa(\theta(t) - r(t))dt + \sqrt{\sigma(t)}\sqrt{r(t)}dW_2(t) \\ d\theta = v(\overline{\theta} - \theta(t))dt + \xi\sqrt{\theta}dW_1(t) \\ d\sigma(t) = \mu(\overline{\sigma} - \sigma(t))dt + \eta\sqrt{\sigma}dW_3(t) \end{cases}$$

A discrete time version of the model can be implemented using a four dimensional lattice. Closed form solutions for discount bonds and some interest rate derivatives are obtained in very specific cases.

5.9 Hull and White (1994b)

In order to overcome the limitations of their one factor model, Hull and White (1994b) have also proposed a two factors model that is a new extension of their extended Vasicek model:

$$\begin{cases} dr(t) = (\theta(t) + u - r(t))\,dt + \sigma_1 dW_1(t) \\ du(t) = -bu(t)dt + \sigma_2 dW_2(t) \end{cases}$$

with $E(dW_1(t), dW_2(t)) = \rho dt$ and $u(0) = 0$. The model is similar to their one factor model (1990), but with a stochastic drift: the short term rate is mean reverting, u is a component of $r(t)$ mean-reversion level, and u itself is mean reverting to 0 at a rate b. Note that there is no loss of generality in the specification, as if u reverts to some level c, $u^* = u - ct$ reverts to 0; we can define u as the second factor and absorb the difference between u and u^* in $\theta(t)$. The parameter $\theta(t)$ is used to make the model consistent with the initial term structure.

The differential equation satisfied by the bond price is

$$\begin{aligned} &\frac{\partial B}{\partial t} + (\theta(t) + u - ar)\frac{\partial B}{\partial r} - bu\frac{\partial B}{\partial u} \\ &+\frac{1}{2}\sigma_1^2\frac{\partial^2 B}{\partial r^2} + \frac{1}{2}\sigma_2^2\frac{\partial^2 B}{\partial u^2} + \rho\sigma_1\sigma_2\frac{\partial^2 B}{\partial r \partial u} - rB = 0 \end{aligned}$$

As the model belongs to the affine class, discount bond prices have a very lengthy and complicated but analytical expression, that can be found in Hull and White (1994b), Annex B.

6 Extension to an infinite dimensional space

Even with a multi-factor model, the term structure of interest rates has a rather limited number of degrees of freedom. The observed term structure does not necessarily match the theoretical one, at least in non-fitted models. Furthermore, fitted finite factor models suffer from the difficulty that they need to be recalibrated constantly to remain consistent with the observed term structure. Thus, an alternative approach to single and multi-factor interest rate modeling is to specify the dynamics of the term structure of interest rates as a whole. Rather than using a finite number of state variables, we use **one state variable of infinite dimension**, namely, the term structure itself.

For instance, forward curve based models attempt to model the forward rate stochastic processes $\{F(t,T)\}_{0\leq t\leq T}$. The first contribution to this approach was made by Ho and Lee (1986) in discrete time, but the most significant one was made by Heath, Jarrow and Morton (1992). One should note that such forward interest rate models are an infinite dimensional stochastic system, as there is one equation for each fixed T. The advantage of the approach is that if at time 0, we set the theoretical forward rate $f(0,T)$ equal to the observed one $f^*(0,T)$, we have a perfect fitting of the whole current term structure and the problem of inverting the yield curve to calibrate is avoided.

6.1 Ho and Lee (1986)

The discrete multi-period binomial model of Ho and Lee (1986) is especially important since it was the first to model movements in the entire term structure. Ho and Lee take the initial term structure as exogenously given at a point in time (by a set of zero-coupon prices) and derive the subsequent feasible term structure movements so that they are compatible with no-arbitrage opportunity. Note that they create a binomial lattice of the term structure rather than a binomial process for the bond price.

6.1.1 The lattice model

Ho and Lee define a set of equidistant trading dates separated by a period of Δt (the step size is time dependant to ensure that the bond price converges to the face value at maturity). At time $t_0 = 0$, the term structure is set equal to the currently observed term structure. Then, they introduce two maturity dependant perturbation functions $h(\tau)$ and $h^*(\tau)$. At time $t + \Delta t$, there is a draw of an upstate and a down state (the draws are i.i.d. over time), and the new term structure is equal to the original one at time t multiplied by the perturbation function $h(\tau)$ with probability π or by the perturbation function $h^*(\tau)$ with probability $1-\pi$. To ensure that π and $(1-\pi)$ are correctly interpreted as risk-neutral probabilities, they set

$$\pi h(t) + (1-\pi)h^*(t) = 1$$

and

$$h(0) = h^*(0) = 1$$

This defines a binomial tree that they assume to be recombining (there is a path independence of price movements of all zero-coupon bonds). This allows them to compute explicitly the perturbation functions $h(\tau)$ and $h^*(\tau)$ in terms of the measure π and of a given parameter δ:

$$h(\tau) = \frac{1}{(\pi + (1-\pi)\delta^\tau)} \tag{35}$$

and

$$h^*(\tau) = \frac{\delta^\tau}{(\pi + (1-\pi)\delta^\tau)} \tag{36}$$

Which of the up-state or down-state corresponds to an upward shift in the yield curve depends on whether δ is larger or smaller than one. In term of discount bond prices, if the up-state occurs at time t,

$$B(t,T) = h(T-t)\frac{B(t-1,T)}{B(t-1,t)} \tag{37}$$

and

$$B(t,T) = h^*(T-t)\frac{B(t-1,T)}{B(t-1,t)} \tag{38}$$

if the down-state occurs at time t.

6.1.2 Forward rates

The one period forward rate, i.e. the rate quoted at time t for borrowing from T to $T+1$ is given by

$$f(t,T) = -\log\left(\frac{B(t,T+1)}{B(t,T)}\right)$$

which can be transformed in

$$f(t,T) = f(t-1,T) + \log\left(\frac{\pi + (1-\pi)\delta^{T+1-t}}{\pi + (1-\pi)\delta^{T-t}}\right) \tag{39}$$

using (37) and (35) for the upstate, and in

$$f(t,T) = f(t-1,T) + \log\left(\frac{\pi + (1-\pi)\delta^{T+1-t}}{\pi + (1-\pi)\delta^{T-t}}\right) - \log(\delta) \tag{40}$$

using (38) and (36) for the downstate.

Equations (39) and (40) can be rewritten as

$$f(t,T) = f(t-1,T) + \log\left(\frac{\pi + (1-\pi)\delta^{T+1-t}}{\pi + (1-\pi)\delta^{T-t}}\right) - (1-\pi)\log(\delta) + \epsilon_t \tag{41}$$

where

$$\epsilon_t = \begin{cases} (1-\pi)\log(\delta) & \text{if there is an up-state} \\ -\pi\log(\delta) & \text{if there is a down-state} \end{cases}$$

so that $E(\epsilon_t) = 0$. This implies that the new forward rate is the sum of the old forward rate, a constant depending on the time-to-maturity and the probabilities, plus an i.i.d. random noise term that is the same for all maturities.

6.1.3 Short term rate

Aggregating (41) over time, we obtain

$$\begin{aligned} f(t,T) &= f(0,T) + \sum_{j=1}^{t}\left(\log\left(\frac{\pi + (1-\pi)\delta^{T+1-j}}{\pi + (1-\pi)\delta^{T-j}}\right) - (1-\pi)\log(\delta) + \epsilon_j\right) \\ &= f(0,T) + \log\left(\frac{\pi + (1-\pi)\delta^{T}}{\pi + (1-\pi)\delta^{T-t}}\right) - t(1-\pi)\log(\delta) + \sum_{j=1}^{t}\epsilon_j \end{aligned}$$

and using (2),

$$r(t) = f(t,t) = f(0,t) + \log\left(\pi + (1-\pi)\delta^{t}\right) - t(1-\pi)\log(\delta) + \sum_{j=1}^{t}\epsilon_j$$

Differentiating with respect to time gives

$$\begin{aligned} r(t) &= r(t-1) + (f(0,t) - f(0,t-1)) \\ &\quad + \log\left(\frac{\pi + (1-\pi)\delta^{t}}{\pi + (1-\pi)\delta^{t-1}}\right) - (1-\pi)\log(\delta) + \epsilon_t \end{aligned}$$

that is, the interest rate at time t depends on the interest rate one period ago, the relevant slope of the yield curve, a constant depending on time, π and δ, plus some noise.

6.1.4 Estimation

The probability π of the up and down movements of the term structure and the parameter δ have to be estimated from the data (and it is not obvious how sensitive the model depends on them). Since the methods uses the whole term structure as an input, bond prices are already in the model and π and δ have to be estimated using other contingent claims (for instance, options on bonds).

6.1.5 Continuous time limit

As shown by Dybvig (1988) and Jamshidian (1991a), in the continuous time equivalent version of the Ho and Lee (1986) model, the short rate is driven under Q by the stochastic differential equation

$$dr(t) = \mu_r(t)dt + \sigma_r dW(t)$$

where $\mu_r(t)$ is deterministic and bounded and σ is constant. As shown by Heath, Jarrow, Morton (1992), discount bonds prices can be valued analytically:

$$B(t,T) = \frac{B(0,T)}{B(0,t)} e^{-\frac{\sigma_r^2}{2}Tt(T-t)-\sigma_r(T-t)W(t)}$$

Options on discount bond also have an analytical solution which is close to the Black and Scholes (1973) formula. For instance, for a call option with maturity T and strike price K, on a bond with maturity time T_B, we have

$$C(t) = B(t,T_B)N(h) - KB(t,T)N\left(h - \sigma_r(T_B - T)\sqrt{T-t}\right)$$

with

$$h = \frac{\ln\left(\dfrac{B(t,T_B)}{KB(t,T)}\right) + \frac{1}{2}\sigma_r(T_B - T)^2(T-t)}{\sigma_r(T_B - T)\sqrt{T-t}}$$

European options on coupon-bearing bonds can be valued using the same approach as Jamshidian (1989) in the case of the Vasicek model.

6.1.6 Critiques and extensions

By definition, the Ho and Lee (1986) model is a Markov analytically tractable model that fits perfectly the observed term structure. But

- it will always generate upward sloping term structures.
- it does not incorporate any mean-reversion feature. Regardless of how high or low interest rates are, the average direction in which they move over the next period of time is always the same. Thus, at extreme points, interest rates can become infinite or negative, and the zero coupon price can exceed the face value. This can lead to serious mispricing.[18]
- it implies that all spot and forward rates have the same instantaneous constant standard deviation σ_r.
- it is not necessarily arbitrage-free.

One should also note that there exist numerous extensions to the Ho and Lee (1986) model. For instance, Bliss and Ronn (1989) develop a trinomial model based on the Ho and Lee's binomial model; they incorporate state dependant shifts that are determined by observable state variables. Sandmann and Sondermann (1993a) propose a path independent binomial model in which the current term structure is reflected, but that does not generate negative interest rates. Their perturbation function is based on the local volatility of the spot rate and on the transition probability to an up or a down state. By choosing particular volatility and probability, they obtain specific continuous time limits of the short term rate process.

6.2 Heath, Jarrow, Morton (1992)

Heath, Jarrow and Morton (1992) have significantly extended the Ho and Lee (1986) model in three directions. First, they consider forward rates rather than bond prices as their basic building blocks. Second, they allow for continuous trading, which results in a valuation formula which is independent of the pseudo probabilities π that we have in the Ho and Lee model . Third, they extend it from a one factor model to allow for multiple factors.

Although their model is not explicitly derived in an equilibrium framework, the Heath, Jarrow and Morton (1992) model is a model that explains the whole term structure dynamics in an arbitrage-free framework in the

[18] If one sets up a boundary condition, this will cut off some undesirable paths in the tree, and the volatility resulting from the model will not correspond anymore to the input data volatiliy.

spirit of Harrison and Kreps (1979), and it is fully compatible with an equilibrium model. The same model was developed independently and simultaneously by Babbs (1991).

Heath, Jarrow and Morton (1992) set the forward rate for each fixed maturity T to evolve under the historical probability P as

$$df(t,T) = \mu_f(t,T)dt + \sigma_f(t,T)dW(t) \tag{42}$$

where $\mu_f(t,T)$ and $\sigma_f(t,T)$ are adapted processes for each T. This specification is very general as the drifts $\mu_f(t,T)$ and volatilities $\sigma_f(t,T)$ can in fact depend on the history of the Brownian motion $W(t)$ and on the forward rates themselves up to time t. The model can be written in an integral form as

$$f(t,T) = f(0,T) + \int_0^t \mu_f(s,T)ds + \int_0^t \sigma_f(s,T)dW(s)$$

As a boundary value at time 0, we use the observed forward curve $f^*(0,T)$, that is, for all T, we set

$$f(0,T) = f^*(0,T)$$

By construction, the model will perfectly fit the observed term structure.

One should note that a direct implication is that there is no such thing as "the" Heath, Jarrow, Morton model; rather, there exists a whole class of models, each being characterized by specific functional forms of the drifts and volatilities.

6.2.1 Restrictions on the possible parameters values

The major result of Heath, Jarrow, Morton is the following proposition which is essential for the existence of a unique equivalent martingale measure[19], that is, for computing the price of contingent claims by discounting their terminal expected values.

Proposition 7 *The following conditions are equivalent:*

1. *the market price of risk $\lambda(t)$ is independent of the maturity dates*

[19] Artzner and Delbaen (1989) also propose a martingale approach to model the term structure of interest rates, in which they price discount bonds in a first step and contingent claims in a second one. Heath, Jarrow and Morton (1992) have only one step in their methodology because they take the bond prices and forward prices processes as given.

2. *a unique martingale measure exists*

3. *the parameters $\mu_f(t,T)$ and $\sigma_f(t,T)$ cannot be freely specified: drifts of forward rates under the risk-neutral probability are entirely determined by their volatility and by the market price of risk*

The third part of the proposition is probably the major contribution of the Heath, Jarrow, Morton model, as it allows the model to be arbitrage-free[20], a major improvement over the Ho and Lee (1986) and other similar models.

Proposition 8 *Assume that the family of forward rates are given by (42). Then, in order to avoid arbitrage opportunities, there must exist an adapted process $\lambda(t)$ which is independent of the maturity T such that under P*

$$\mu_f(t,T) = \sigma_f(t,T)\left[\int_t^T \sigma_f(t,s)ds - \lambda(t)\right] \tag{43}$$

One can show that $\lambda(t)$ represents the instantaneous market price of risk and that it is independent of the maturity T.

This proposition is similar to the arbitrage condition used in the one factor models. But it implies that the choice of a particular model from the general specification of Heath, Jarrow and Morton can be reduced to the specification of the volatility coefficient. For a particular risk premium, the drift coefficient can easily be retrieved using (43). Furthermore, as the volatility is the same under the risk-neutral and the subjective probability, it can be estimated using historical data.

Accounting for this new no-arbitrage condition, equation (42) can be rewritten as

$$df(t,T) = \sigma_f(t,T)\left[\int_t^T \sigma_f(t,s)ds - \lambda(t)\right]dt + \sigma_f(t,T)dW(t) \tag{44}$$

[20] In addition to this no-arbitrage condition, additional restrictions are often postulated. A first necessary condition on the volatility function is that

$$\sigma_f(t,t) = 0$$

whatever t, as there is certain redemption at par of any discount bond. Another interesting restriction is developed in Miltersen (1994) to restrict the forward rates from becoming negative.

under P, the historical probability.

Under the corresponding risk-neutral measure Q, we can suppress the explicit dependance on the market price of risk, and we can rewrite equation (43) as

$$\mu_f(t,T) = \sigma_f(t,T)\int_t^T \sigma_f(t,s)ds$$

and equation (44) as

$$df(t,T) = \sigma_f(t,T)\left[\int_t^T \sigma_f(t,s)ds\right]dt + \sigma_f(t,T)dW^*(t)$$

or equivalently in an integral form as

$$f(t,T) = f(0,T) + \int_0^t \sigma_f(s,T)\left[\int_s^t \sigma_f(s,u)du\right]ds + \int_0^t \sigma_f(s,T)dW^*(s) \tag{45}$$

where $dW^*(t)$ is a standard Wiener process.

6.2.2 Short term rate

Since $r(t) = f(t,t)$, the dynamics of the short rate under the historical probability P can be obtained from (44) as

$$dr(t) = df(t,t) + \left.\frac{d}{dT}f(t,T)\right|_{T\to t} dt$$

that is,

$$dr(t) = \left[\left.\frac{\partial f(t,T)}{\partial T}\right|_{T=t} - \sigma_f(t,T)\lambda(t)\right]dt + \sigma_f(t,T)dW(t)$$

Later on, we will show that particular specifications of the functions $\sigma_f(s,t)$ will allow us to obtain specific well-known short term interest rate models. But this is just a consequence of the model, as the short term rate is just a specific forward rate.

Under the corresponding risk-neutral measure Q, the explicit dependence on the market price of risk can be suppressed, and we obtain

$$dr(t) = \left[\left.\frac{\partial f(t,T)}{\partial T}\right|_{T=t}\right]dt + \sigma_f(t,T)dW^*(t)$$

or in an integral form

$$r(t) = f(0,t) + \int_0^t \sigma_f(s,t) \int_s^t \sigma_f(s,u) du ds + \int_0^t \sigma_f(s,t) dW^*(s) \tag{46}$$

where $dW^*(t)$ is a standard Wiener process generated by the risk-neutral probability measure Q. As we could expect, the principal difficulty of estimating a Heath, Jarrow, Morton model will arise because of the non-Markovian term in equation (46), which depends on the history of the process from time 0 to time t.

6.2.3 Discount bond price

The bond prices are contained in the forward rate informations, as bond prices can be written down by integrating over the forward rate between t and T in terms of the risk-neutral process.

$$B(t,T) = e^{-\int_t^T f(t,s)ds}$$

Thus, an exogenous specification of the forward rates is equivalent to a specification of the bond prices for all maturities. Integrating the original equation for the forward rates $f(t,T)$, using Ito's lemma and a generalized version of the Fubini's theorem, Heath, Jarrow and Morton found that

$$B(t,T) = e^{-\left(\int_0^t \left(\int_0^T \sigma_f(s,u)du\right)dW(s) + \int_t^T f(0,u)du + \int_0^t \int_t^T \mu_f(s,u)duds\right)}$$

that is, the bond price dynamics is given by

$$\frac{dB(t,T)}{B(t,T)} = \mu_B(t,r(t))dt + \sigma_B(t,r(t))dW_t \tag{47}$$

with

$$\mu_B(t,r(t)) = \underbrace{f(t,t)}_{r(t)} - \int_t^T \mu_f(t,u)du + \frac{1}{2}\left|\int_t^T \sigma_f(t,u)du\right|^2 \tag{48}$$

$$\sigma_B(t,r(t)) = -\int_t^T \sigma_f(t,u)du \tag{49}$$

In general, $\mu_B(t,r(t))$ and $\sigma_B(t,r(t))$ could depend on the entire information set at time t. However, using the no-arbitrage condition (43), equation (48) can be rewritten as

$$\mu_B(t,r(t)) = r(t) + \lambda(t)\sigma_B(t,T)$$

or equivalently as

$$\lambda(t) = \frac{\mu_B(t, r(t)) - r(t)}{\sigma_B(t, T)}$$

Note that under the objective probability Q, all bond prices continuously discounted at the spot rate are martingales:

$$\frac{dB(t,T)}{B(t,T)} = r(t)dt + \sigma_B(t, r(t))dW_t^*$$

6.2.4 Additional possible restrictions to obtain a Markovian model

As we mentioned already, most models of forward rates evolution in the Heath, Jarrow and Morton family result in non Markovian models (i.e. path-dependant) of the short term interest rate evolution. But numerical methods for Markovian models are usually more efficient than those necessary for non-Markovian models. Thus, a common, but very restrictive condition set on interest rates models is that the short term rate has to be Markovian (as this allows the mapping on a recombining lattice). Hull and White (1993) and Carverhill (1994, 1995) have shown that the equivalent condition was that the volatility function had to be of the form

$$\sigma_f(t, T) = x(t)\left(y(T) - y(t)\right)$$

where $x(\tau)$ and $y(\tau)$ are appropriately well behaved functions[21].

Ritchken and Sankarasubramanian (1995) have extended Carverhill (1994, 1995) results showing that if the volatilities of forward rates were differentiable with respect to their maturity date, for any given initial term structure, a necessary and sufficient condition for the prices of all interest rate contingent claims to be completely determined by a two-state Markov process is that the volatility of forward rate is of the form

$$\sigma_f(t, T) = \sigma_r(t) e^{-\int_t^T \kappa(s)ds} \tag{50}$$

Examples of such volatility structures are

$$\sigma_f(t, T) = \sigma_r(t) e^{\kappa(T-t)}$$

[21] Jeffrey (1995) derives the conditions for a Heath, Jarrow, Morton model to result in a Markovian spot interest rate and have a term structure at time t function of time, maturity, and spot interest rate at time t.

or

$$\sigma_f(t,T) = \sigma_r(t) r^{\gamma}(t)$$

Under (50), Ritchken and Sankarasubramanian show that the price of a zero-coupon bond is of the form

$$B(t,T) = \frac{B(0,T)}{B(0,t)} e^{-\frac{1}{2}\beta^2(t,T)\phi(t)+\beta(t,T)[f(0,t)-r(t)]}$$

where

$$\begin{aligned} \beta(t,T) &= -\frac{\sigma_B(t,T)}{\sigma_f(t,T)} = \int_t^T e^{-\int_t^u \kappa(s)ds} du \\ \phi(t) &= \int_0^t \sigma_f^2(s,t)ds \end{aligned}$$

that is, the term structure is uniquely determined by the spot interest rate $r(t)$ and the integrated variance $\phi(t)$.

Bhar and Chiarella (1997) have also shown that if the volatility of the forward rates is defined as deterministic function of time of the form

$$\sigma_f(t,T) = p(T-t)e^{-a(T-t)}$$

where $p(\tau)$ is a n-degree polynomial function of u and a is a number (both to be estimated), then, the instantaneous spot rate $r(t)$ and bond price $B(t,T)$ are determined by a (n+2) dimensional Markovian stochastic differential equation. When the volatility of the forward rates are defined as the same function times a function of the short term rate, i.e.

$$\sigma_f(t,T) = p(T-t)e^{-a(T-t)}b(r(t))$$

the instantaneous spot rate $r(t)$ and bond price $B(t,T)$ are determined by a $\frac{1}{2}(n^2+7n+8)$ dimensional Markovian stochastic differential equation.

If, in addition to the Markovian property we require the volatility function to depend only on the time to maturity $(T-t)$, and not on the calendar time t, Carverhill (1995) has shown that the volatility function had to be of the form

$$\sigma_f(t,T) = \frac{k(1-e^{-a(T-t)})}{a}$$

Additional restrictions and conditions on the volatility function for the short rate to be Markovian can be found in Carverhill (1994). In particular, it is shown that when the volatility of the zero-coupon bond is a function that depends only on time, the prices of options on zero-coupon bonds can be recovered analytically, even if the resulting term structure evolution is path-dependant.

6.2.5 Practical implementation

For practical purposes, when one uses an Heath, Jarrow, Morton model, it is necessary to proceed as follows:

1. specify the volatilities $\sigma_f(t,T)$. In theory, any specification of $\sigma_f(t,T)$ can be used. In practice, however, one should check for the consequences of a particular specification. We do not need to specify the forward rates drifts, as they will be uniquely determined by

$$\mu_f(t,T) = \sigma_f(t,T) \int_t^T \sigma_f(t,u)du$$

1. Observe the effective forward rate structure $f^*(0,T)$ for $T \geq 0$.

2. Compute the forward rate according to

$$f(t,T) = f^*(0,T) + \int_0^t \mu_f(s,T)ds + \int_0^t \sigma_f(s,T)dW(s)$$

3. Compute bond prices according to

$$B(t,T) = \exp\left(-\int_t^T f(t,s)ds\right)$$

Example 9 *Consider the case of a constant volatility $\sigma_f(t,T) = \sigma$. The corresponding drift is $\mu_f(t,T) = \sigma^2(T-t)$. The forward rate is given by*

$$\begin{aligned} f(t,T) = f^*(0,T) + \int_0^t \sigma^2(T-s)ds + \int_0^t \sigma dW(s) \\ = f^*(0,T) + \sigma^2 t(T - \frac{t}{2}) + \sigma W(t) \end{aligned}$$

We have

$$\int_t^T f(t,T) = \int_t^T f^*(0,s)ds + \sigma^2 tT(T-t) + \sigma(T-t)W(t)$$

so that

$$\begin{aligned} B(t,T) &= \exp\left(-\int_t^T f(t,s)ds\right) \\ &= \frac{B^*(0,T)}{B^*(0,t)} e^{-\frac{1}{2}\sigma^2 tT(T-t)-\sigma(T-t)W(t)} \end{aligned}$$

We can compute the spot rate $r(t)$ as

$$r(t) = f(t,t) = f^{*(0,t)} + \sigma^2 \frac{t^2}{2} + \sigma W(t)$$

from which we can extract $\sigma W(t)$ and replace it in the bond price to obtain

$$B(t,T) = \frac{B^*(0,T)}{B^*(0,t)} e^{(T-t)f^*(0,t)-\frac{1}{2}\sigma^2 t(T-t)^2-(T-t)r(t)}$$

that is, the bond price as a function of $r(t)$.

6.2.6 From the Heath, Jarrow, Morton model to short term interest rate models and back

What is the link that exists between short term interest rate models and Heath, Jarrow, Morton models ? A short term rate is simply a particular forward rate. This implies that specifying a given model in the Heath, Jarrow, Morton framework will result in a particular behavior for the short term interest rate. **Particular specifications of the forward rates dynamics allows us to recover specific short term interest rate stochastic processes.** The link between both specifications can be evidenced by recalling equation (46), which specifies the short term rate dynamics under Q:

$$r(t) = f(0,t) + \int_0^t \sigma_f(s,t) \int_s^t \sigma_f(s,u)duds + \int_0^t \sigma_f(s,t)dW^*(s)$$

The converse is not true: any short term interest rate model is not necessarily compatible with an Heath, Jarrow, Morton model. The compatibility is verified only when condition (43) is also verified by the short term rate drift and volatility.

As an illustration, we provide some examples hereafter of original short term interest rate models and the "corresponding" compatible Heath, Jarrow, Morton model. However, in practice, it is impossible to do this **analytically** for most short-rate Markov models, and numerical techniques have to be used.

Ho and Lee (1986) The Ho and Lee (1986) model

$$dr(t) = \mu_r dt + \sigma_r dW(t)$$

corresponds to a constant volatility

$$\sigma_f(t, T) = \sigma_r$$

In Heath, Jarrow, Morton terms, the equivalent model is

$$\begin{aligned} df(t,T) &= \sigma_r^2 (T-t)dt + \sigma_r dW(t) \\ f(0,T) &= r(0) + \frac{1}{2}\sigma_r^2 T^2 + \int_0^T \mu_r dt \end{aligned}$$

that is, all forward rates are normally distributed and display exactly the same volatility.

Generalized Ho and Lee The generalized Ho and Lee model

$$dr(t) = \mu_r(t)dt + \sigma_r(t)dW(t)$$

corresponds to

$$\begin{aligned} df(t,T) &= \sigma_r^2(t)(T-t)dt + \sigma_r(t)dW(t) \\ f(0,T) &= r(0) + \frac{1}{2}\int_0^T \sigma_r^2(s)(T-s)ds + \int_0^T \mu_r(s)ds \end{aligned}$$

Forward rates (and instantaneous short rates) are normally distributed.

Vasicek (1977) The Vasicek (1977) model

$$dr(t) = \kappa(\theta - r(t))dt + \sigma dW(t)$$

corresponds to the specific forward volatility function

$$\sigma_f(t,T) = \sigma_r e^{-\kappa(T-t)}$$

Compared to the Ho and Lee (1986) model, this introduces a maturity dependance on the volatility surface. In Heath, Jarrow, Morton terms, the equivalent model is

$$f(0,T) = \theta + e^{-\kappa T}(r(0) - \theta) - \frac{\sigma^2}{2\kappa}(1 - e^{-\kappa T})^2$$

Generalized Vasicek (1977) The Hull and White model

$$dr(t) = \kappa(t)(\theta(t) - r(t))dt + \sigma(t)dW(t)$$

corresponds to a specific exponentially decaying volatility

$$\sigma_f(t,T) = \sigma_r(t)\gamma(t,T)$$

with $\gamma(t,T) = e^{-\int_t^T \kappa(u)du}$. In Heath, Jarrow, Morton terms, the equivalent model is

$$\begin{aligned} f(0,T) \quad &= \quad r(0)\gamma(0,T) + \int_0^T \kappa(t)\theta(t)\gamma(s,T)ds \\ &\quad - \int_0^T \sigma_r^2(s)\gamma(s,T)\left(\int_s^T \gamma((s,u)du\right)ds \end{aligned}$$

and the forward rates are normally distributed, which means that the bond prices are log-normally distributed. Both the short term rate and the forward rates can become negative.

Generalized Cox, Ingersoll and Ross (1985b) The generalized Cox, Ingersoll and Ross (1985b) model

$$dr(t) = \kappa(t)(\theta(t) - r(t)) + \sigma(t)r(t)dW(t)$$

corresponds to

$$\sigma_f = \sigma_r(t)\sqrt{r(t)}\frac{\partial X(t,T)}{\partial T}$$

where $X(t,T)$ solves the following Ricatti equation

$$\begin{cases} \frac{\partial X}{\partial t} = \frac{1}{2}\sigma_r^2(t)X^2(t,T) + \kappa(t)(t,T) - 1 \\ X(T,T) = 0 \end{cases}$$

This equation has no analytic solution, but is well studied numerically. In Heath, Jarrow, Morton terms, the equivalent model is

$$f(0,T) = r(0)X(0,T) + \int_0^T \kappa(s)\theta(s)X(s,T)ds$$

6.3 Multifactor generalization of the Heath, Jarrow, Morton (1992)

Multifactor generalizations of the Heath, Jarrow, Morton (1992) model have been developed in the literature as

$$df(t,T) = \mu_f(t,T)dt + \sum_{i=1}^{K} \sigma_{f,i}(t,T)dW_i(t)$$

where $\mu_f(t,T)$ is the drift of the forward rate with maturity T, $\sigma_{f,i}(t,T)$ are its volatility coefficients and $W_i(t)$ are independent standard Brownian motions[22]. This can be rewritten in an integral form as

$$f(t,T) = f(0,T) + \int_0^t \mu_f(s,T)ds + \sum_{i=1}^{K} \int_0^t \sigma_{f,i}(s,T)dW_i(s)$$

As in the single factor versions, since bond prices depend on forward rates, and we have

$$B(t,T) = B(0,T) + \int_0^t \mu_B(s,T)B(s,T)ds + \sum_{i=1}^{K} \int_0^t \sigma_{B,i}(s,T)B(s,T)dW_i(s)$$

[22] Note that in the most general specification, $\mu_f(t,T)$ and $\sigma_f(t,T)$ could depend on the path of the Brownian motions.

with $(i = 1, 2, ..., K)$

$$\begin{aligned}
\sigma_{B,i}(t,T) &= -\int_t^T \sigma_{f,i}(t,s)ds \\
\mu_B(t,T) &= r(t) - \int_t^T \mu_f(t,s)ds + \frac{1}{2}\sum_{i=1}^{K}\sigma_{B,i}^2(t,T)
\end{aligned}$$

Such a K-factor model is arbitrage free if there exist K market prices of risk $\lambda_i(t)$ such that

$$\lambda_i(t) = \frac{\mu_B(t,T) - r(t)}{\sum_{i=1}^{K}\sigma_{B,i}(t,T)}$$

for all finite T. Then, the forward rate drift is uniquely determined by the volatility structure and the market price of risk

$$\mu_f(t,T) = \sum_{i=1}^{K}\sigma_{f,i}(t,T)\left[\lambda_i(t) + \int_t^T \sigma_{f,i}(t,s)ds\right]$$

There exist a K-factor martingale representation theorem which is similar to the one-factor model representation theorem. In this context, Jamshidian (1991) also introduced a forward risk-adjusted measure in order to facilitate the calculation of closed form solutions for European style interest rate contingent claims.

Particular cases of such multifactor models have a deterministic volatility structure $\sigma_f(t,T)$. They are called **Gaussian models**, as forward rates become normally distributed and bond prices are normally distributed [23] and have analytical expressions in some circumstances.

For instance, Heath, Jarrow and Morton (1992) propose a two factor model which is basically a combination of a Ho and Lee (1988) and Vasicek (1977), in which

$$\begin{aligned}
\sigma_{f,1}(t,T) &= \sigma_1 \\
\sigma_{f,2}(t,T) &= \sigma_2 e^{-\kappa(T-t)}
\end{aligned}$$

[23] Jamshidian (1991) uses a more restricted definition of Gaussian models, as he imposes specific constraints on the volatility structure so that the resulting short term rate is Gaussian.

Heitmann and Trautmann (1995) propose the combination of two Vasicek (1977) processes

$$\begin{aligned}\sigma_{f,1}(t,T) &= \sigma_1 e^{-\kappa_1(T-t)}\\ \sigma_{f,2}(t,T) &= \sigma_2 e^{-\kappa_2(T-t)}\end{aligned}$$

6.4 Goldstein (1997), Kennedy (1997)

Goldstein (1997) and Kennedy (1997) recently introduced a new model of the term structure based on random fields. This is an infinite factor model in which the forward rate follows the following diffusion

$$df(t,T) = \mu_f(t,T)dt + \sigma_f(t,T)dW_T(t)$$

The innovation is the following: corresponding to each maturity date T, there is one unique Brownian motion $W_T(t)$, with the characteristics that $W_T(t)$ does not correspond to any linear combination of a finite number of other Brownian motions $W_{T'}(t)$, $T' \neq T$. The random field is characterized by the correlation

$$dW_{T_1}(t)dW_{T_2}(t) = Corr(t, T_1, T_2)dt$$

As in the Heath, Jarrow, Morton (1992) case, the drift coefficient $\mu_f(t,T)$ is completely determined by the volatility structure $\sigma_f(t,T)$ and the correlation structure $Corr(t, T_1, T_2)$.

Unfortunately, no partial differential equation exists (yet !) for interest rate contingent claims.

7 Empirical assessment of alternative models

The empirical issues relative to the performance of alternatives term structure models are large enough to call for a separate survey. In the following, we will briefly review some of the issues that have been addressed in the empirical work to date and their major findings. Without discussing implementation details, models of the term structure can be tested in three different ways:

- **directly, using a cross-section** across bonds of different maturities. This technique has the advantage of explicitly identifying the market price of risk, but makes it difficult to test for a whole class of models in a nested fashion.

- **directly, using a time series approach**, since we can infer the parameters from actual interest rates time series. Apart from the fact the method does not provide the market price of risk, the series of observation must be large enough and is subject to a discretisation bias.

- **indirectly**, through based on alternative models results in pricing or hedging interest rate contingent claims, or through an examination of how well do models fit the term structure of interest rates or the term structure of volatilities.

Despite a bewildering large set of models, relatively little work has been done to examine how these models compare in terms of their ability to capture the behavior of the short term rate, of the term structure, or the pricing of interest rate contingent claims. Because of the lack of a common framework, most studies have been focusing on specific models rather than on a comparison across models.

Many authors have shown that one-factor models do not fit well the yield curve: see for instance Chen and Scott (1993) or Pearson and Sun (1990) using maximum likelihood, Heston (1989) or Gibbons and Ramaswamy (1993) using the generalized method of moments, or Litterman and Scheinkman (1991) using factor analysis. De Munnick and Schotman (1992) has found similar quality of fit for the Vasicek and Cox, Ingersoll and Ross model on the Dutch bond market. However, these results are not glaring and often depend on the set of data and the econometric methodology used. For instance, Brown and Dybvig (1986) tested the Cox, Ingersoll and Ross (1985b) model with cross-sectional data. They concluded that the long term mean and the volatility parameter were unstable over time, which may indicate a misspecification of the model. This conclusion was rejected by Gibbons and Ramaswamy (1993) on short term T-Bill data using generalized method of moments estimators of the unconditional short rate distribution, but was confirmed by Pearson and Sun (1994) using maximum likelihood estimates.

Chan, Karolyi, Longstaff and Sanders (1992) develop a general framework to estimate and compare a set of nested single factor models for the short term interest rate on the U.S. market. They obtain surprising results, such as an elasticity of 1.5, which would imply that the series is non-stationary. According to their study, Vasicek (1977) and Cox, Ingersoll and Ross (1985b) perform poorly compared to less famous models such as Dothan (1978) or Cox, Ingersoll and Ross (1980). Furthermore, it is not important to have mean reversion, but it is critical to model the volatility correctly: models that better describe the dynamics of the short term rate allow the volatility to depend on the rate level. But these conclusions may be due to inefficient estimation methods: they use generalized method of moments, assuming that the distributions are ergodic, but they are not in this case (see Eom (1994)). Furthermore, Nowman (1997) obtains opposite results regarding the mean-reversion and the volatility importance on the U.K. market. On the Swedish and Danish markets, Dahlquist (1994) finds that mean-reversion is important, that there seems to be positive relation between interest rate levels and volatility, and that there is evidence of a structural change in the Danish interest rate process.

Using non parametric techniques over the 1965-1985 period, Stanton (1997) finds non linearities in the drift coefficient of single-factor models: the mean reversion is low for low rates, but increases as the short term rate level increases. Aït Sahalia (1996) estimates the diffusion coefficient non-parametrically by comparing the marginal density implied by each model with that implied by the data, given a linear specification for the drift. His conclusions are that the fit is bad and the tests reject "every parametric model of the spot rate previously proposed in the literature", that is, all linear drift short term interest rate models.

Stambaugh (1988), Longstaff and Schwartz (1992), Litterman, Scheinkman and Weiss (1991) observe that adding factors improves the fit, and suggest using two factors. But Pearson and Sun (1994) do not find that two-factor model are sufficient. The shift, twist and change in curvature would tend to imply a three factor model. However, the implementation of such models for bond options is extremely complex.

In the Heath, Jarrow, Morton (1992) framework, Carverhill (1995) finds that the specific one factor Markovian models provide good results in pricing and hedging; more complex models capture more of the properties of the term structure evolution, but they are harder to calibrate and understand and too slow in implementation. Amin and Morton (1994) tested contingent claim pricing implications of six alternative models with absolute, square root, proportional, linear absolute, exponential, and linear proportional volatility, respectively. Their result are that the implied volatility functions are unstable, and that two factor models provide a better pricing, but increase the instability. Heath, Jarrow, Morton (1992) propose a two factor model that is a combination between the continuous time version of the Ho and Lee (1986) model and the Vasicek (1977) model. But Heitmann and Trautmann (1995) recommend using a two-factor model from the Heath, Jarrow, Morton family, and more specifically a two-factor Vasicek (1977) model, which provides better fitting.

8 Conclusions

In this paper, we have reviewed a number of specifications of diffusion based term structure of interest rates models. Rather than being exhaustive, we have presented an overview of the most popular models by means of some general characteristics. Our primary goal was to expose the models one **could** use rather than specifying the models one **should** use. Given the profusion of models, one may wonder if there are any empirical or numerical aspects which can motivate the choice of one family of models for a given application. Whereas for equity contingent claims, the lognormality of prices (as in the Black and Scholes model) is the standard starting point, there is no equivalent in the interest rate world. Each of these models have its own advantages as well as disadvantages. Practitioners use a large variety of models based on wide range of assumptions, objectives, and/or constraints.

Of course, one may use different models at the same time to value different assets. But when the hedging or the risk management of the interest rate sensitive global portfolio comes into consideration, or when pricing arbitrage relationships have to be studied, can we freely mix these models, even if they are based on contradictory assumptions[24] ? This question is essential for derivatives traders or bank regulators.

Which model should one use ? Unfortunately, there is no simple answer to this question. An ideal interest rate model should be theoretically consistent, flexible, simple, well-specified (in that required inputs can be observed or estimated) and realistic; it should provide a good fit[25] of the data (i.e. be

[24]As an example, cap prices and percentage volatilities are quoted in the market on the basis of the Black model (i.e. with lognormal rates). These volatilities are often plugged into a different model. What are the consequences ? For instance, if the model assumes normal rates, there very little difference for at the money strikes, but larger and larger ones for different moneyness.

[25]Note that an exact fitting is not necessarily desirable, as there are possible sources of "errors" in quotations of the market price of bonds (liquidity, tax effects, market imperfections, bid-ask spread, etc.), so that the discount rates that would exactly match the observed term structure resulting from a set of quotations would include these bond specific effects.

relevant with both interest rate volatility and the term structure's shape), be relevant with theories of the term structure of interest rates, and be consistent with a bond market equilibrium situation, or at least consistent with the absence of arbitrage paradigm; finally, hedging and pricing of derivatives should be tractable, efficient, and implemented analytically or using an efficient numerical algorithm. When all conditions are not met, the trade-offs start. For instance, single factor time-invariant models do not fit well the term structure, do not explain some humped yield curves, do not allow for particular volatility structures and cannot match at the same time caps and swaptions prices[26]. But they provide analytical solutions for bonds and bond options prices.

In fact, the answer to the model choice will certainly depend on the specific use of the model. The main questions are: how many factors do we need ? which factors ? is the model incremental complexity justified in light of their pricing and risk management effectiveness ? what is the main goal of the model ? An increased model generality is not always an advantage, as there is a trade-off between the incremental effort necessary for parameter estimation and model accuracy, while a model that has closed form expressions for the price of bonds will allow derivatives pricing, but will not necessarily fit the initial term structure.

Currently, a lot of empirical research is still required to evaluate and assess all these different models performance. A comparison framework is needed. The Heath, Jarrow and Morton model displays a great financial and intellectual appeal as well as undeniable elegance. It starts with a family of forward rate processes and initializes it to an arbitrary but fixed initial forward curve. As such, it offers several advantages:

- first, the current term structure is matched by construction, without requiring an arbitrary time-varying parameter.
- second, it assumes nothing regarding investors preferences, and it yields a pricing function that is uniquely determined by the specification of the variance structure of interest rates changes, as the volatility parameter

[26] A cap is a portfolio of independent options, while a swaption is an option on portfolio of rates and therefore depends on the imperfect correlation between them. As single factor models assume perfect instantaneous correlations between all rates, they can account for all the cap prices, but not for any swaption simultaneously. As shown by Rebonator and Cooper (1996), two-factor models do not do much better: they can match all cap prices, plus one swaption.

in the Black and Scholes (1973) equation. Drifts estimates are not needed, which simplifies the estimation procedure.

- third, all other interest rates models based on diffusion processes are nested into its general specification, including non-Markovian ones.

For these reasons, we believe that currently the Heath, Jarrow and Morton (1992) class of term structure models is the most broadly defined yet unified framework for model comparison and model risk assessment.

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