Calibration of the Svensson model to simulated yield curves.

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Outline.

1. **Application.**
2. Simulated yield curves.
3. Traditional calibration.
4. Enhanced calibration.

Application.
MCEV – Risk Capital – ALM.

Three main applications in insurance companies.

- MCEV – market consistent economic value.
- Risk Capital – risk measure based on change of MCEV.
- ALM – asset liability management.

Solution approach at major players.

- MCEV – risk neutral (= market consistent) valuation, scenario based.
- Risk Capital – shocks (full MCEV distribution is not available).
- ALM – real world simulation, scenario based.

Stochastic modelling.
Computationally very demanding approach.

Main instruments.

- Interest rate based: bonds, floaters, mortgages.
- Equity based: equity and real estate indices.
- Derivatives: (CMS) swaps, forwards, options.

$\Longrightarrow$ **zero curve is required in each scenario node.**

Main facts.

- 1000 scenarios.
- 40 time steps (annual) or 480 time steps (monthly).
- 5 main economies.

$\Longrightarrow$ **# zero curves $\geq$ 200000.**

Outline.

1. Application.

2. **Simulated yield curves.**

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Simulated yield curves.
Logarithmic Black-Karasinski 2-factor model.

Consequences of the BK2 yield curve model.

- No analytical solution available for bond prices.
- Approximation of yield curve by discretization (60 bonds).
- Spline interpolation for intermediate maturities.

$\Longrightarrow$ **huge disk space required.**

Two ways of addressing the disk space problem.

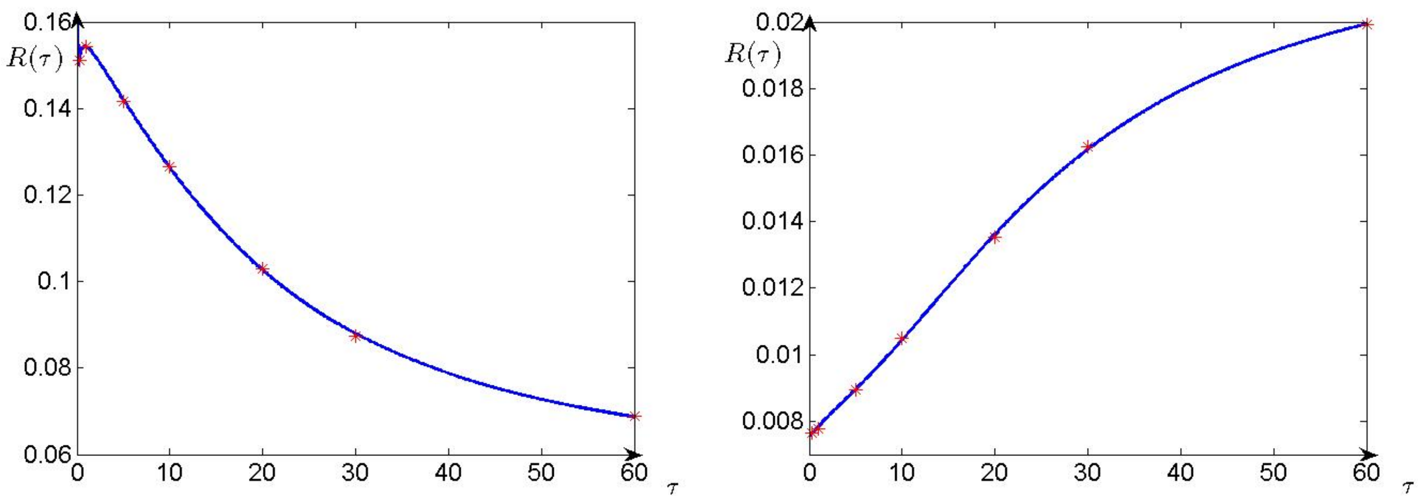
- Move to different yield curve model (not possible in Barrie&Hibbert).
- Parametric fitting of yield curve.

$\Longrightarrow$ **need for appropriate approximation.**

Simulated yield curves.

Large variety of shapes.

Both normal and inverse yield curves may occur.



Notation.

- T time of maturity, t current time, $\tau = T - t$ time to maturity.
- $P(t,T)$ price of a zero coupon bond at time t with maturity T.
- $R(t,T) = -\frac{1}{\tau}\ln(P(t,T))$ is the zero rate.
- $f(t,T) = -\frac{\partial}{\partial T}\ln(P(t,T))$ instantaneous forward rate.

Simulated yield curves.

Large variety of shapes.

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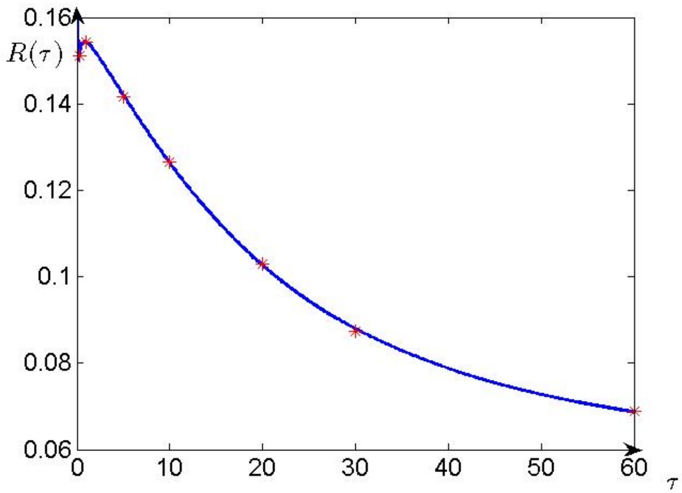
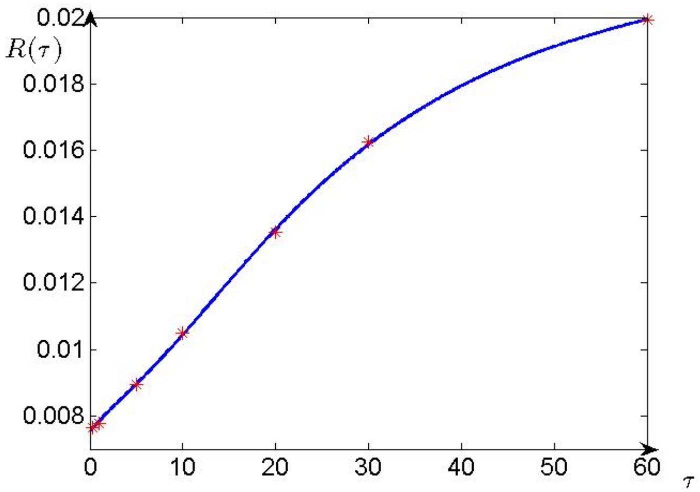




Requirements on the approximation.

- Low dimensional approximation (i.e. at most 6 parameters).
- Flexible approximation to fit to all possible shapes.
- Fast and reliable calibration required.
- Preferably interpretable parameters.

Parametric model.
Approximation of the instantaneous forward rate.

Modelling the most basic curve.

Parametric model which assumes for the instantaneous forward rate

$$\hat{f}_{\alpha,\beta}(\tau) = \sum_{i=1}^{K} \alpha_i \varphi_i(\beta_i, \tau).$$

Integration yields the model for the zero rates

$$\hat{R}_{\alpha,\beta}(\tau) = \frac{1}{\tau} \sum_{i=1}^{K} \alpha_i \int_0^{\tau} \varphi_i(\beta_i, s) ds = \sum_{i=1}^{K} \alpha_i \psi_i(\beta_i, \tau).$$

- The *global* parameter vector β determines the shape of the ansatz functions.
- For fixed β (i.e. fixed family of ansatz functions) the calibration of α is straightforward.
- Both parameters α and β are calibrated such that the fit to the given curve is optimal.

Parametric model.
The Nelson and Siegel method (1987).

Ansatz functions of Nelson and Siegel.

Parametric model which assumes for the instantaneous forward rate

$$\hat{f}_{\alpha,\beta}(\tau) = \alpha_0 + \alpha_1 \exp\left(-\frac{\tau}{\beta_1}\right) + \alpha_2 \left(\frac{\tau}{\beta_1} \exp\left(-\frac{\tau}{\beta_1}\right)\right).$$

Integration yields the model for the zero rates

$$\hat{R}_{\alpha,\beta}(\tau) = \alpha_0 + \alpha_1 \left(\frac{1 - \exp\left(-\frac{\tau}{\beta_1}\right)}{\frac{\tau}{\beta_1}}\right) + \alpha_2 \left(\frac{1 - \exp\left(-\frac{\tau}{\beta_1}\right)}{\frac{\tau}{\beta_1}} - \exp\left(-\frac{\tau}{\beta_1}\right)\right).$$

Interpretation.

- long rate: $\alpha_0 > 0$.
- short rate: $\alpha_0 + \alpha_1 > 0$
- hump: α_2 determines height and direction, $\beta_1 > 0$ the position.

Parametric model.
The Svensson method (1994).

Enhanced ansatz functions of Svensson.

Enhanced parametric model which assumes for the instantaneous forward rate

$$\hat{f}_{\alpha,\beta}(\tau) = \alpha_0 + \alpha_1 \exp\left(-\frac{\tau}{\beta_1}\right) + \alpha_2 \left(\frac{\tau}{\beta_1} \exp\left(-\frac{\tau}{\beta_1}\right)\right) + \alpha_3 \left(\frac{\tau}{\beta_2} \exp\left(-\frac{\tau}{\beta_2}\right)\right).$$

Integration yields the model for the zero rates

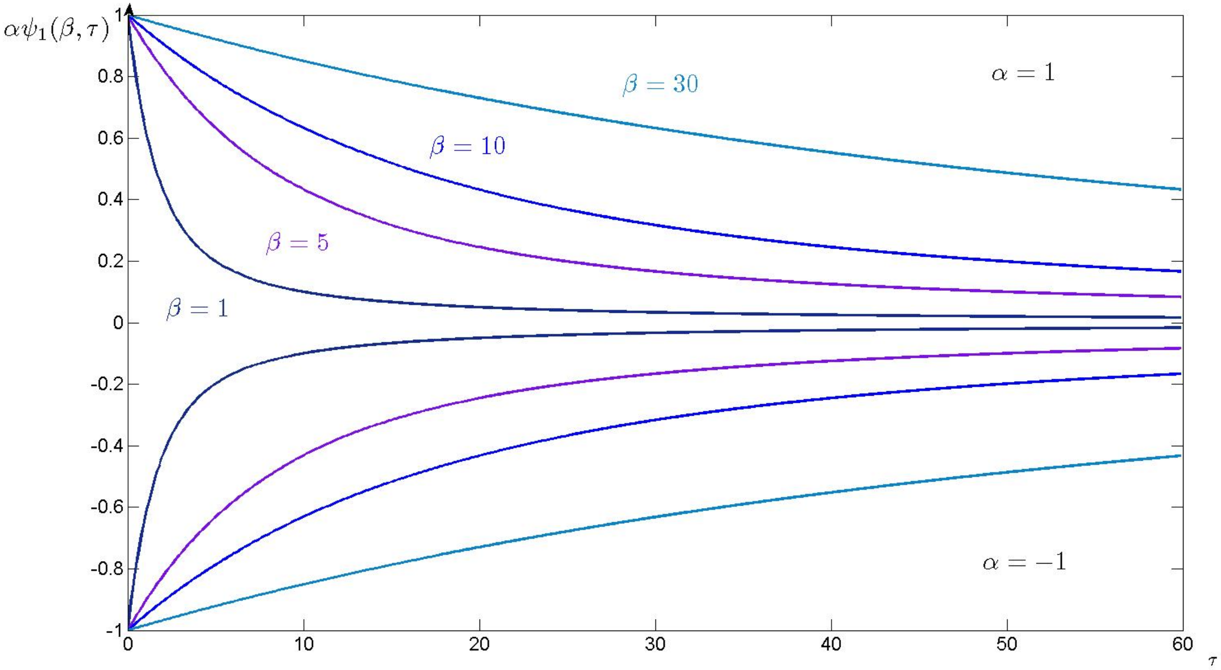
$$\hat{R}_{\alpha,\beta}(\tau) = \alpha_0 + \alpha_1 \left(\frac{1-\exp\left(-\frac{\tau}{\beta_1}\right)}{\frac{\tau}{\beta_1}}\right) + \alpha_2 \left(\frac{1-\exp\left(-\frac{\tau}{\beta_1}\right)}{\frac{\tau}{\beta_1}} - \exp\left(-\frac{\tau}{\beta_1}\right)\right) + \alpha_3 \left(\frac{1-\exp\left(-\frac{\tau}{\beta_2}\right)}{\frac{\tau}{\beta_2}} - \exp\left(-\frac{\tau}{\beta_2}\right)\right).$$

Remarks.

- Svensson enlarged the Nelson-Siegel model by one additional function.
- Additional function allows for larger variety in shapes.

Parametric model.
The Svensson method (1994).

Visualization of the Svensson functions (first term).

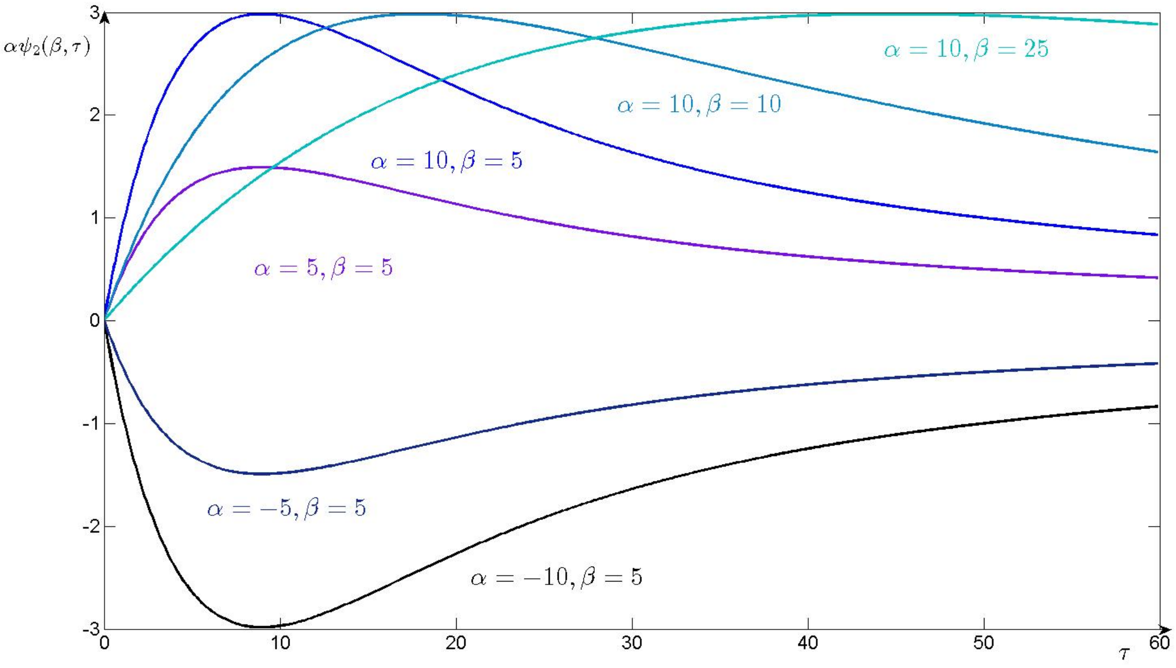


$$\alpha\psi_1(\beta,\tau) = \alpha\left(\frac{1-\exp\left(-\frac{\tau}{\beta}\right)}{\frac{\tau}{\beta}}\right)$$

Parametric model.

The Svensson method (1994).

Visualization of the Svensson functions (second / third term).



$$\alpha\psi_2(\beta,\tau) = \alpha\left(\frac{1-\exp\left(-\frac{\tau}{\beta}\right)}{\frac{\tau}{\beta}} - \exp\left(-\frac{\tau}{\beta}\right)\right)$$

Outline.

1. Application.

2. Simulated yield curves.

3. **Traditional calibration.**

4. Enhanced calibration.

Traditional calibration.

Calibration via non-convex least squares.

Least squares approach.
The parameters α and β are calibrated to the zero rates coming from the simulated yield curve:

$$\min_{\alpha,\beta} F(\alpha,\beta)$$
$$\alpha_0 \geq 0, \alpha_0 + \alpha_1 \geq 0,$$
$$\beta_1 \geq \varepsilon, \beta_2 \geq \varepsilon.$$

where the objective function F is defined as

$$F(\alpha,\beta) = \sum_{j=1}^{J} \left(R(\tau_j) - \hat{R}_{\alpha,\beta}(\tau_j) \right)^2.$$

Remarks.

- The objective function is not convex.
- The objective function is convex (linear) in α only.
- Non-convexity is introduced via β.

Traditional calibration.

Drawbacks of non-convex least squares.

Theoretical observations.

- As the objective is non-convex, local minima (may) exist.
- Non-convex least squares problems require special solvers to ensure convergence to a local minimum.

Numerical observations.

- Most instances possess local minima.
- Solution quality is heavily depending on the starting point for the solver.
- When looking at the *bad instances*, we can manually choose a better starting point and then reach a sufficient quality.
- Computation time also depends on the starting point.

$\Longrightarrow$ **manual process cannot (yet) be automized.**

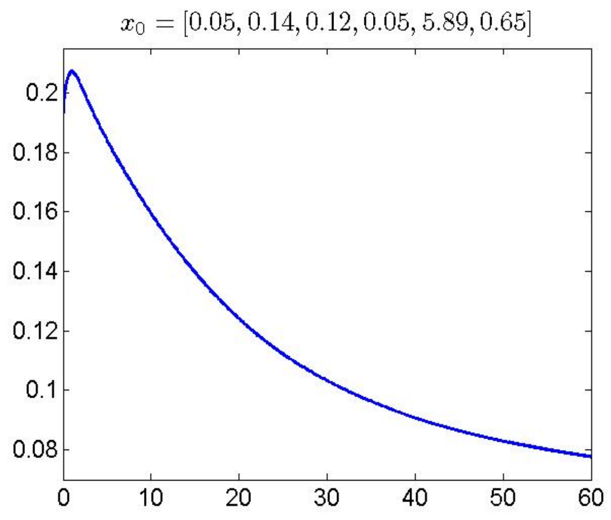
Traditional calibration.

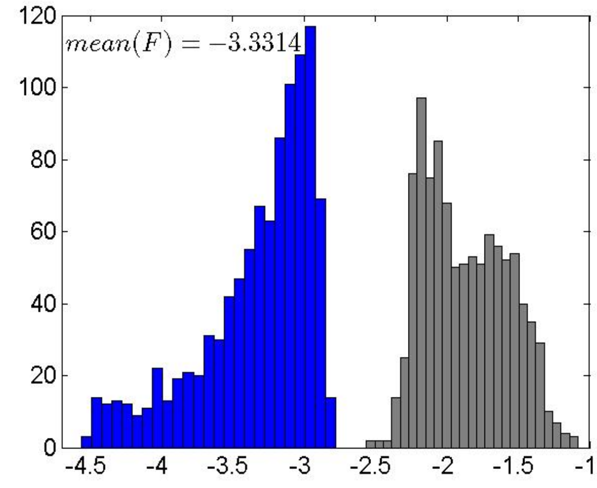
Improvement of non-convex least squares.

An interesting aspect on the starting point.

- Investigations on the shape of the interest curves in risk neutral scenarios show that the large majority of yield curves is inverse.

$\Longrightarrow$ **use different starting point (inverse one) instead of today's curve.**



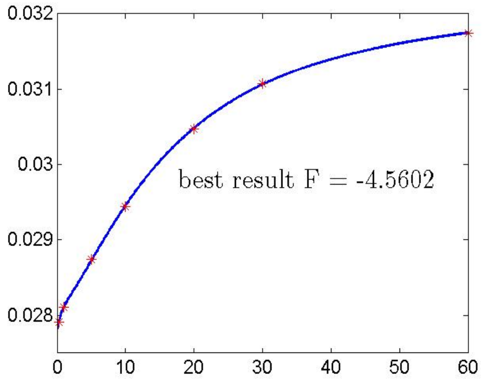


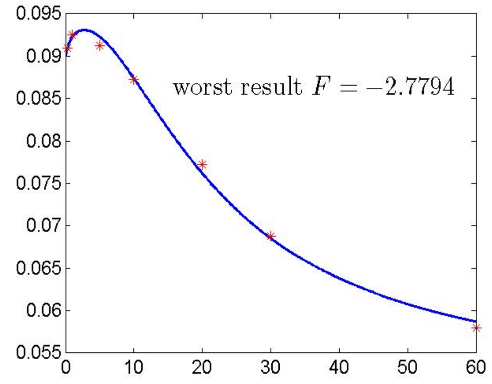
Impact on the performance is dramatic.

The computation time for 1000 curves is reduced by 80% (to 30s), the objective is improved by 1.46 (valid digits on average).

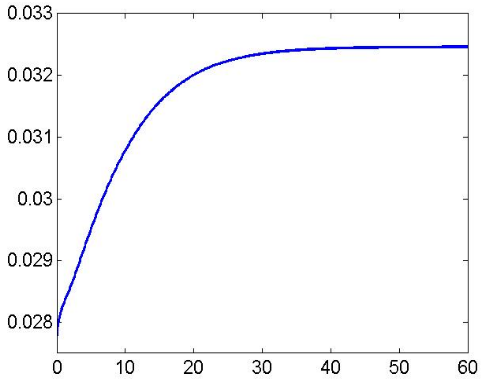
Calibration of the Svensson model.
Illustration of the quality of results.

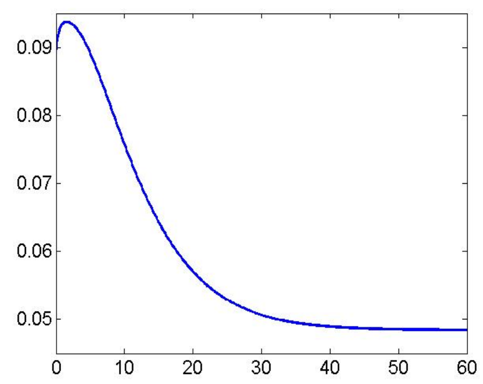
Best and worst solution with the new starting point:





Corresponding instantaneous forward rates:





Outline.

1. Application.
2. Simulated yield curves.
3. Traditional calibration.
4. **Enhanced calibration.**

Enhanced calibration.
Further improvement is possible.

Intermediate conclusion.

- Calibration works within reasonable time (20min for each currency).
- Calibration sometimes yields results with insufficient quality.
  - for ALM quality is always sufficient.
  - for MCEV and Risk Capital high quality is crucial.

$\Longrightarrow$ **improvement on bad curves required.**

Approaches for further improvement.

1. Split of parameters in local (α) and global (β) component.
2. Improvement of condition of objective function.
3. Reparametrization of global component.
4. Global solver for calibration of global component.

Enhanced calibration.

Split of parameters in local and global component.

Original problem.

$$\min_{\alpha,\beta} F(\alpha,\beta)$$
$$\alpha_0 \geq 0, \alpha_0 + \alpha_1 \geq 0,$$
$$\beta_1 \geq \varepsilon, \beta_2 \geq \varepsilon.$$

Problem after split of variables.

$$\min_{\beta \geq \varepsilon} \min_{\substack{\alpha_0 \geq 0 \\ \alpha_0 + \alpha_1 \geq 0}} F(\alpha,\beta)$$

or, equivalently,

$$\min_{\beta \geq \varepsilon} H(\beta)$$

with

$$H(\beta) = \min_{\substack{\alpha_0 \geq 0 \\ \alpha_0 + \alpha_1 \geq 0}} F(\alpha,\beta)$$

Enhanced calibration.

Split of parameters in local and global component.

Solution of splitted problem.

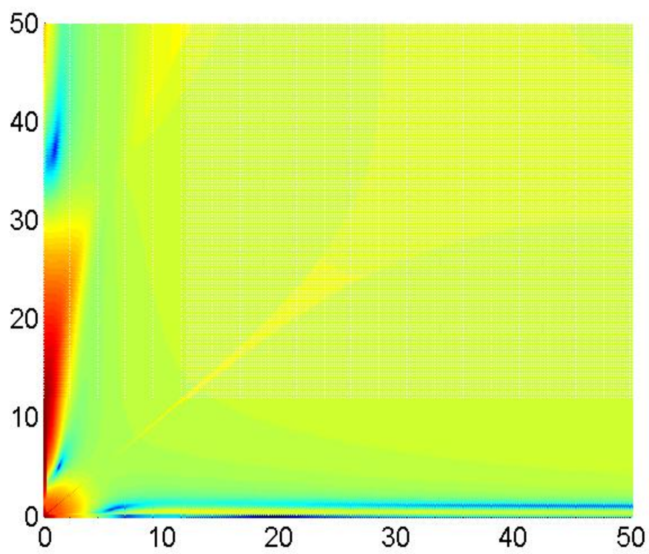
- The outer optimization problem in β is a global (i.e. non-convex) problem in two variables.
- Dimension for the global optimization has been reduced from six to two.
- The inner problem is a standard least squares problem under linear constraints.
- The inner problem is very easy to solve, efficient codes are available.
- The objective function of the outer problem is non-smooth.
- For the outer problem, a specialized global optimizer is needed.

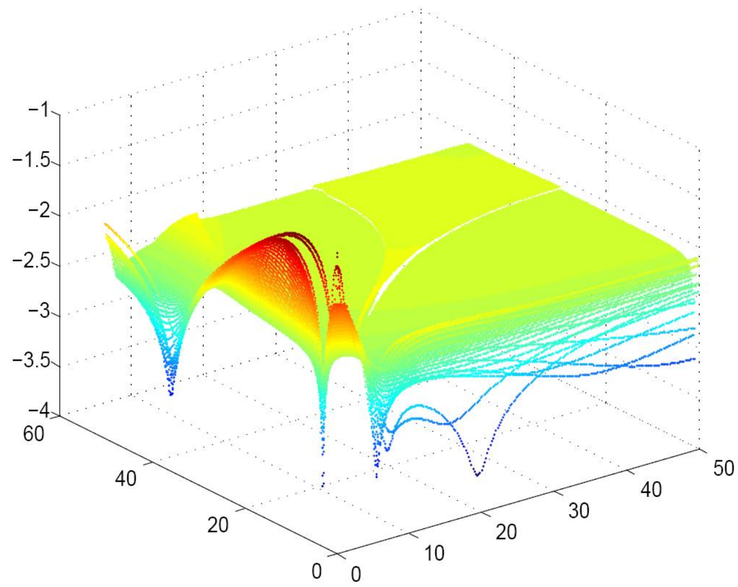
Global optimization routine.

- HCP algorithm (Novak, Ritter 1995): sparse grid + adaptiv search.
- Further improvement by using different type of sparse grid.

Enhanced calibration.
The global component.

Visualization of the non-convex function H.

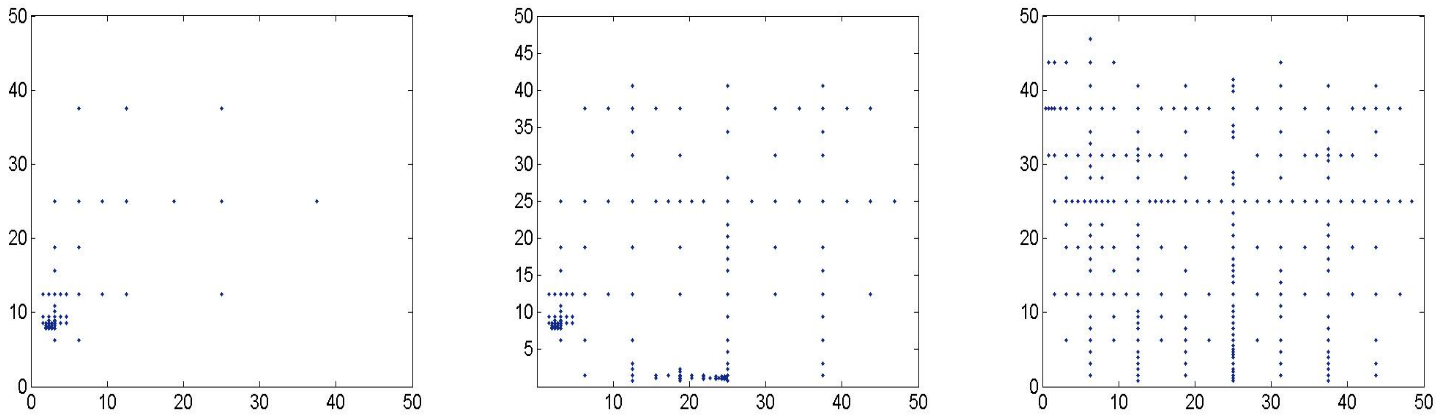




Enhanced calibration.

Optimizing the global component.

Illustration of the global optimizer.

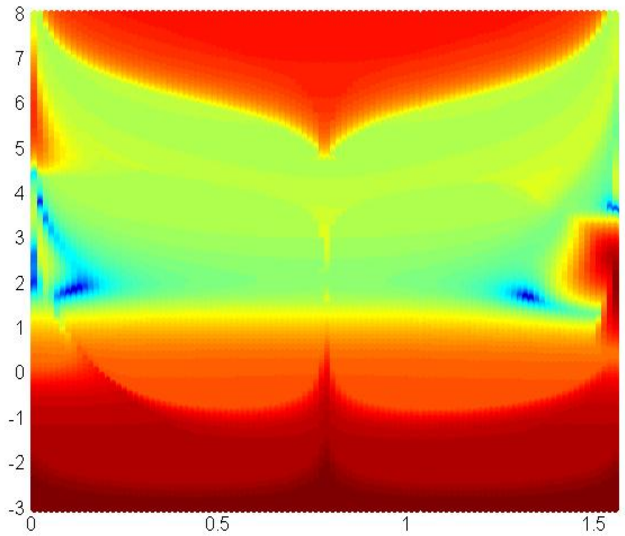


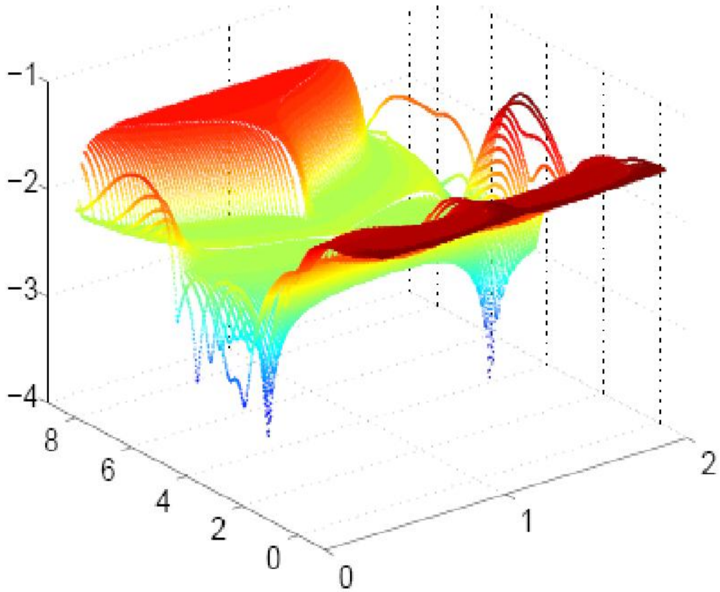
- Adaptivity can be controlled by the user.
- The global optimization routine does not return the global optimum.
- The global optimization routine returns a starting point which is sufficiently close.
- Global optimization applied to the worst 150 results yields an average improvement of 0.36.

Enhanced calibration.

Improving the global optimization.

Coordinate transformation to polar coordinates.





- Instead of cartesian coordinates β (logarithmic) polar coordinates γ are used.
- $\beta_1 = \exp(\gamma_2) \cdot \cos(\gamma_1)$, $\beta_2 = \exp(\gamma_2) \cdot \sin(\gamma_1)$.
- Reparametrization yields an average improvement of 0.17.

Enhanced calibration.

Improving condition of objective function.

Observations.

- We observed that the least square problem in α runs into numerical difficulties.
- Investigations have shown that this is due to very different scaling of variables and objective function.
- Taking the square root of the objective function does not change the minimum but improves efficiency.

Further remark.

- Taking the square root may seem to destroy convexity at first glance.
- It is easy to see that we stay within the class of SOCPs (second-order cone programs) and retain convexity.

Final algorithm.

Calibration of the Svensson model.

Calibration algorithm.

`for each time step and each scenario do:`

1. `local optimization of non-convex problem in six dimensions (with improved starting point, well-conditioned objective and in polar coordinates).`
2. `if the solution quality is below a certain threshold:`
   - 2.1 `global optimization of outer non-convex problem in two dimensions.`
   - 2.2 `local optimization of non-convex problem in six dimensions with new starting point from global optimization.`

Final algorithm.

Summary of numerical performance.

Numerical observations.

- As step 1 has been strongly improved, only few calibrations are below the threshold (3.5 valid digits) $\Longrightarrow$ no improvement on average solution quality.
- Global optimization of bad instances often leads to further improvement.
- Strong improvement on bad solutions is still not always possible.

	min	mean	max	time (sec.)
after step 1	-4.9691	-3.9921	-3.1508	92
after step 2	-4.9691	-3.9939	-3.3351	10
total change	-	0.0018	0.1843	

- Total runtime of optimization is 1.5 min for 1000 scenarios (i.e. 60 min per currency).

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