JOURNAL OF FINANCIAL AND QUANTITATIVE ANALYSIS VOL 19, NO 3, SEPTEMBER 1984

A New Approach to Estimation of the Term Structure of Interest Rates

Donald R. Chambers, Willard T. Carleton, and Donald W. Waldman*

I. Introduction

The term structure of interest rates is the pricing relationship that exists at any point in time between default-free discount securities arrayed by maturity. Both normative (e.g., the development of portfolio immunization strategies) and positive (e.g., tests of alternative specifications of how the structure evolves stochastically through time) uses of the term structure require that it be observable—either directly or through statistical procedures. Unfortunately, the absence of discount government securities with maturities greater than one year and the presence of pricing disturbances that include maturity-related heteroschedasticity have vastly complicated empirical estimation of the term structure. McCulloch [7], [8], Schaefer [10], and Carleton and Cooper [1], as well as others, have estimated term structures from samples of bond prices with varying degrees of success. In this paper, we propose and estimate an alternative specification in which each present-value coefficient in a bond's price function is expressed as an nth degree polynomial in time to payment, and for which a maturity-related heteroschedasticity specification is jointly estimated. Our alternative specification provides satisfactory results as estimated for 16 samples from 1976-1980. It also appears to provide a satisfactory resolution of problems associated with the disturbances of bond pricing functions as found by prior studies in this area.

II. Term Structure Estimation for Default-Free Financial Assets

For the purposes of this study, a default-free financial asset is defined as a bond in which the probability of occurrence of a state of nature in which the asset's actual cash flows deviate from its promised cash flows is zero.

* Pennsylvania State University, University of Arizona, and University of North Carolina, Chapel Hill, respectively. The authors would like to acknowledge the assistance of Nelson J. Lacey in preparing the figures.

The price of a default-free security can be expressed as a linear combination of a promised cash flow vector

$$P_i = \sum_{t=1}^{T} C_i(t)B(t) \,, \tag{1}$$

where t indexes time and

P_i = price of default-free asset i;
$C_i(t)$ = asset's promised cash flow at time t;
$B(t)$ = present value coefficient, defined below;
T = length of time to maturity

Haley and Schall [3] discuss in detail the assumptions under which the price of a financial asset may be expressed as a linear combination of promised cash flows and market prices, i.e., for which the "Value-Additivity Principle" is valid. These assumptions generally include the absence of transaction costs and information costs. The coefficient $B(t)$ is the price of a default-free security, which promises a payoff of \$1 at time t in every state of nature that may occur. $B(t)$ is referred to as the present-value function and may change through time. For each $B(t)$, an interest rate $R(t)$ may be defined as the rate at which $B(t)$ must be compounded to reach \$1 over the interval from the current time until time t. Each interest rate $R(t)$ is referred to as the risk-free interest rate corresponding to time t and forms a function of t referred to as the term structure of interest rates.

Because future cash flows of default-free bonds are known and prices are observable, if the number of bonds with linearly independent vectors of promised cash flows exceeded the number of cash-flow payment dates, then the present-value coefficients could be derived from equation (1). Additionally, the corresponding risk-free interest rates could be derived, and the exact price of any default-free bond could be calculated from the observed present-value function. However, because the set of assumptions that is sufficient for equation (1) to hold is somewhat restrictive, it may be useful to examine the empirical results. Carleton and Cooper [1] and others have regressed prices of default-free government bonds on their cash-flow payments and find that a unique vector of present-value coefficients that permits equation (1) to hold simultaneously for all bonds does not exist. In these studies, bond prices are allowed to deviate from the theoretical price formula of equation (1) by the inclusion of a pricing disturbance

$$P_i = \sum_{t=1}^{T} C_i(t)B(t) + \varepsilon_i \,. \tag{2}$$

One reason for the existence of a pricing disturbance is the fact that coupon-bearing default-free obligations with maturities of greater than one period contain information concerning more than one present-value coefficient. Consequently, there may be some imprecision in the price formation process. A second reason is that bond portfolios are not continuously rebalanced, so that at any moment each bond can deviate by some (presumably random) amount. A third reason is that the published quotations on bond prices employed in this and similar studies reflect averages of dealers' assessments as polled each day by the Federal Reserve. They do not constitute market-clearing closing prices at which all dealers will trade. The multiple-dealer nature of the market plus use of bid/ask spreads to induce trades thus implies some inherent imprecision to the concept of a single price—hence the inclusion of an error term. Instruments such as Treasury bills mature in one year or less and are traded on a discount basis, offering only one future cash flow. Therefore, from each Treasury bill one present-value coefficient can be determined easily. To estimate present-value coefficients corresponding to maturities of more than one year, regression analysis can be used. In order to justify the use of regression techniques, it is, of course, necessary to assume that the bond prices deviate from their theoretical value by the inclusion of a disturbance term.[1]

Since equation (2) is in the form of a classical linear regression model, observed bond prices could be regressed on a promised payoff matrix and the estimated coefficients used as point estimates of the present-value function. However, in practice, this procedure is complicated by the tendency of the data to create a singular payoff matrix. Two solutions to this problem have been attempted. The first solution, due to Carleton and Cooper [1], is the use of a carefully selected bond sample that has common payment dates to avoid the singularity problem. The second solution is to place constraints on the regression coefficients. We discuss these in turn.

A necessary condition for the regression of bond prices on cash flows is that the payoff matrix be nonsingular. Jordan [4] has discussed this requirement in detail and observes the following difficulties: (1) two payment dates in which identical interest payments are promised and in which no obligation matures create a singular payoff matrix resulting in an indeterminate solution; and (2) one payment date in which no obligation matures results in very large standard errors. To resolve these difficulties, the sample can be selected in order to include for each payment date at least one maturing obligation. In addition, the method provides estimates of the present-value function only at points corresponding to the payment dates. In order to obtain estimates of present-value coefficients at points other than the payment dates, it is necessary to interpolate between the two nearest payment dates. The properties of estimates obtained from interpolation are unclear. The reason is that each interpolated point estimate includes the effects of errors in estimation of the nearest two points and an unspecified error due to the imposition of a particular interpolation procedure.

Despite these difficulties, Carleton and Cooper [1] demonstrated the reasonableness of the procedure's results. The Carleton-Cooper estimation procedure is primarily useful in estimating the present-value function over maturities of less than ten years because the singularity difficulty is more easily avoided. The singularity difficulty is more acute with longer maturities due to the paucity of data.

[1] In this paper as in the prior studies, the assumption of error term additivity is made. Aside from its obvious convenience for estimation purposes, we cannot think of any rationale for an alternative specification, *a priori* or based on examination of residuals.

Another approach to the problem of a singular payoff matrix is to compress the parameter space by the imposition of functional restrictions on the present-value coefficients. Linear regression is then possible if the number of restrictions is greater than the degree to which the payoff matrix is short rank. An important decision is the selection of the function to be imposed.

Weierstrass' Approximation Theorem [9] indicates that there is a class of functions that may be used to approximate any continuous function over an interval with an arbitrarily small degree of error. An example of such a function is a simple polynomial. Therefore, it may be shown that the term structure, if assumed continuous, may be approximated with arbitrary accuracy by a polynomial. McCulloch [7], [8] has found that the use of a simple polynomial introduces substantive problems. Specifically, the polynomial appears to conform too strongly to shapes at the far end of the term structure while smoothing over shapes implied by data at the near end. The conclusion reached by McCulloch is that a simple polynomial is not able to fit adequately both ends of the term structure simultaneously.

Several methodologies have been suggested to avoid the problems with using a simple polynomial. McCulloch [7] used various spline functions to enable the estimated term structure to conform to shapes throughout the maturity ranges. Schaefer [10] used a Bernstein polynomial set (see [9]) to approximate the term structure. The important characteristic of these approximating functions is that they permit different parts of the term structure to be approximated without severely affecting or being affected by other portions of the term structure. The result of these approaches is to produce an estimate of the term structure that is reasonably well fitted over its entire range.

Estimating present-value coefficients may be accomplished either by unconstrained linear regression or by constraining the coefficients to conform to a given functional form. The best method for a particular study depends upon data availability and how the estimated coefficients are to be used. This study explores further the functional approach. We propose an exponential polynomial and allow the data to determine the length of the polynomial in a heuristic way. We generalize the specification of the pricing disturbance in equation (2) to allow for marked heteroschedasticity in the data and find that we can reject the simpler specification of homoschedasticity by using formal statistical tests.

III. The Model

In the following empirical analysis, it is assumed that the term structure of continuously compounded interest rates may be expressed as a simple polynomial

$$R_s(T) = \sum_{j=1}^{J} x_{s,j} t^{j-1} \tag{3}$$

$R_s(T)$ = spot rate of time T at time s;

$x_{s,j}$ = j^{th} polynomial coefficient at time s;

$t = T - s$, the time to maturity;

J = length of polynomial.

Under the assumptions of continuous compounding, the present-value function also can be expressed by using the polynomial and the exponential function. The resulting function is henceforth referred to as an "exponential/polynomial" function

$$B_s(t) = \exp\left(- \sum_{j=1}^{J} x_{s,j} t^j \right). \tag{4}$$

By using the functional constraint of equation (4), the present-value function is estimated with nonlinear least squares regression. The results of the regression are then analyzed to explore a more appropriate fitting criterion.

The exponential/polynomial coefficients are estimated by regressing the bond prices on the cash payoff matrix

$$P_{i,s} = \left[\sum_{t=1}^{T} C_i(s + t, s) \right] \exp\left(- \sum_{j=1}^{J} x_{s,j} t^j \right) + \varepsilon_{i,s} \tag{5}$$

where s indicates the time period under consideration and $C_i(s + t,s)$ is the cash flow promised for $s + t$. As discussed in the previous section, the disturbance term represents the temporary mispricing of the bond and is for the moment assumed to be independently and homoschedastically distributed. Accordingly, the criterion used to estimate the exponential/polynomial coefficients is the minimization of the sum of the squared residuals. (For details of the estimation procedure, see [2].)

There are two primary issues to be explored using the regressions: (1) the appropriate length of the polynomial used in the approximating function; and (2) the reasonableness of the procedure in estimating the term structure over its entire range. The first issue is analyzed by comparing the regression results obtained by using different polynomial lengths. The reasonableness of the term structure estimates may be analyzed both by examining the estimates themselves and by examining the residuals.

IV. Data Description

The set of obligations that initially qualifies for inclusion in the sample is the set of all U.S. government obligations whose nonpayment would represent government default. Unfortunately, most obligations (e.g., Treasury bills and Treasury notes) are concentrated at the short end of the maturity range rather than being spread evenly. To obtain a sample that is somewhat evenly spread over the maturity range and that has a manageable number of issues, the sample is restricted to those Treasury notes and Treasury bonds with maturities on the 15th day of either February, May, August, or November. Because few of these issues have maturities of greater than ten years, the sample is further restricted to include only those obligations with time to maturity of ten years or less.

In addition to the above restrictions, it is necessary to eliminate some issues due to attributes that might affect their prices. Issues with less than $1 billion in outstanding obligations are dropped because of their potential for mispricing due to infrequent trading or liquidity differentials. All callable obligations are eliminated due to the uncertainty regarding their future cash flows. Finally, all flower bonds are dropped due to their favorable tax status.

The observation period to be analyzed is quarterly on coupon payment dates from November 15, 1976 to August 15, 1980. These sixteen quarters not only have more data than earlier years but also correspond to a period of interest rate volatility and coupon differentials that should provide interesting tests for the model. A total of 68 Treasury notes and Treasury bonds appears in the sample at least once during the observation period. Table 1 describes the sample size for each of the sixteen quarters.

Current prices are measured as the mean between bid and asked prices as published in the *Wall Street Journal*. This approach is consistent with [1], [7], [8], and [10]. Jordan [4] has analyzed this approach empirically and is unable to produce systematic support for an alternative.

At each of the sixteen observation dates, approximately one-half of the sample makes an interest payment and accordingly is selling flat (no accrued interest). The rest of the sample has an accrued interest payment of approximately one-half of its biannual interest payment. Rather than calculating accrued interest to the nearest day, this study approximates accrued interest to the nearest quarter. In addition, the times to maturity of each cash flow are rounded to the nearest quarter. Both approximations subject the measurements to possible errors of up to three days.[2]

V. Analysis of Regression Results

Nonlinear least squares regressions of equation (5) are performed on all 16 of the samples which are described in Table 1. The mean squared errors (MSE) of these regressions are shown in Table 2. Five regressions are performed on each sample to analyze the effects of varying the polynomial length from one to five degrees. It appears that much of the variation in prices may be explained by using a third degree polynomial. The marginal explanatory power of including a fourth or fifth polynomial degree is slight. A fourth degree polynomial results in an increase in MSE in five cases, approximately the same MSE in three samples (8, 13, 14), and a noticeable decrease in MSE in eight cases. Fifth degree polynomial regressions produce higher MSE's in nearly half of the samples. Therefore, it is concluded by inspection of Table 2 that a third or fourth degree polynomial has adequate length to approximate the term structure over the sample period.

[2] A referee has pointed out, correctly, that our estimation method can handle exact dates. In fact, examination of the residuals at each sample observation date, stratified between securities selling flat and selling with accrued interest, indicates that our approximation has no material impact on the results.

TABLE 1

Sample Size Description

Sample Number	Date	Day	Number of Bonds	Number of Notes	Total
1	11-15-76	Mon.	3	33	36
2	2-15-77	Tues.	3	35	38
3	5-16-77	Mon.	3	34	37
4	8-15-77	Mon.	4	34	38
5	11-15-77	Tues.	4	36	40
6	2-15-78	Wed.	4	37	41
7	5-15-78	Mon.	4	36	40
8	8-15-78	Tues.	4	37	41
9	11-15-78	Wed.	4	38	42
10	2-15-79	Thurs.	4	38	42
11	5-15-79	Tues.	4	38	42
12	8-15-79	Wed.	4	37	41
13	11-15-79	Thurs.	4	37	41
14	2-15-80	Fri.	4	39	43
15	5-15-80	Thurs.	5	40	45
16	8-15-80	Fri.	4	42	46

TABLE 2

Least Squares Mean Squared Error (MSE)

Sample Number	Length of Approximating Polynomial				
	1	2	3	4	5
1	2.009	0.521	0.347	0.357	0.343
2	1.839	0.725	0.444	0.457	0.442
3	1.449	0.341	0.208	0.212	0.214
4	0.622	0.317	0.250	0.254	0.227
5	0.290	0.058	0.054	0.049	0.050
6	0.295	0.159	0.129	0.112	0.112
7	0.121	0.037	0.034	0.028	0.028
8	0.088	0.019	0.020	0.020	0.020
9	0.452	0.248	0.080	0.049	0.047
10	0.321	0.235	0.135	0.122	0.118
11	0.442	0.363	0.177	0.171	0.175
12	0.221	0.192	0.094	0.084	0.085
13	0.952	0.547	0.304	0.302	0.309
14	1.260	1.096	0.715	0.714	0.716
15	1.021	0.207	0.192	0.174	0.170
16	1.062	0.529	0.528	0.537	0.535

The second issue is the reasonableness of the function in estimating the term structure over the entire ten-year maturity range. For each sample period, the regression results are used to derive an estimated term structure and the implied forward rates. Both interest rate structures appear reasonable over most of the maturity range. Near the far range of the term structure, the implied forward rates are erratic; however, this is attributable to the paucity of data in that maturity range. Therefore, casual observation of the estimated interest rate structures reveals no difficulties within an eight- or nine-year maturity range.

Another approach in evaluating the usefulness of the estimation procedure is the analysis of the residuals for each time period. Three major problems appear evident upon inspection of the residuals: (1) residuals of obligations with short maturities tend to share the same sign; (2) in most regressions, the residuals demonstrate heteroschedasticity with respect to maturity; and (3) several residuals in each regression are abnormally large in absolute value.

The first difficulty is the tendency of the residuals of obligations with short maturity to share the same sign. To quantify this perception, the residuals in each time period are ranked from shortest time to maturity to longest time to maturity. In cases of equal times to maturity, the residuals are ranked from lowest coupon to highest. Table 3 shows the number of sign changes observed progressing from the first through the tenth residuals. In the case of independent, mean zero, normally distributed variables, the expected number of sign changes is 4.5. Although regression residuals are not independently distributed, a heuristic assumption is that this result should be approximated.

TABLE 3

Number of Sign Changes in First Ten Residuals: Least Squares

Sample Number	Length of Approximating Polynomial				
	1	2	3	4	5
1	0	0	1	1	0
2	0	0	1	1	0
3	0	0	2	4	3
4	0	0	4	0	1
5	0	0	2	5	3
6	0	0	1	5	1
7	0	0	1	3	3
8	0	2	2	2	2
9	0	0	1	3	1
10	0	0	2	4	4
11	0	0	5	0	3
12	0	0	2	4	3
13	0	0	4	4	4
14	0	0	3	1	3
15	0	3	2	3	4
16	0	0	0	2	3
mean	0.00	0 31	2 06	2.63	2 38

Table 3 reveals that first and second degree polynomials tend to give the first ten residuals the same sign. Even the third, fourth, and fifth degree polynomials produce far less than acceptable results. In fact, the inclusion of a fifth polynomial coefficient appears to worsen the problem.

In intuitive terms, the difficulty is in the tendency of the polynomial to ignore shapes in the term structure at the short end. This difficulty is well established in the literature on the estimation of term structure as reviewed in Section 2. It should be possible to approximate better the shapes at the near end by using a higher order polynomial, but, as Table 3 indicates, increasing the length of the polynomial does not help. Additional polynomial coefficients tend to alter the shape at the far end of the estimating function to reduce a few large residuals. The residuals at the near end are rather small in absolute value and, accordingly, do not seem to affect substantially the function's shape. It is possible that the apparent inability of the polynomial to conform simultaneously to shapes of the term structure at both ends is due to the second difficulty, heteroschedasticity.

The heteroschedasticity appears to be directly related to the times to maturity of the obligations. Plotting of the residual against maturity for each time period reveals severe heteroschedasticity in many periods and homoschedasticity in some periods. McCulloch [7], Jordan [4]. and Vasicek and Fong [15] have discussed this problem and examined solutions. The heteroschedasticity can be associated with the bid-asked price spread and adjustment can be made by dividing all observations by their spread. Because of the strong relationship between time to maturity and spread, the heteroschedasticity may be associated with either variable. Several explanations of the heteroschedasticity are plausible. Issues with longer time to maturity may have less frequent trading and, therefore, prices may be less accurate. Also, the larger price spreads may inhibit arbitrage of issues with longer times to maturity. The role of tax effects, which are not dealt with in our model, may include heteroschedasticity. On the other hand, there is no generally accepted way to incorporate tax effects into the term structure at this time [5]. Analysis of the residuals indicates that the severity of the heteroschedasticity changes substantially between sample periods. Therefore, each time period must be examined to determine a specific heteroschedasticity correction.

The third difficulty, abnormally large residuals, appears to be related to the difference between Treasury bonds and Treasury notes. The abnormal residuals consistently belong to Treasury bonds. Because the bonds tend to have lower coupons than the notes, one could suspect that an omitted capital gains tax effect is responsible. Inspection reveals no consistent residual signs, however, so tax effects are unlikely to be responsible. Also, the residuals of each Treasury bond did not exhibit stability through time, indicating as an alternative explanation that the difficulty may be attributable to relatively inactive trading. Because Treasury bonds are issued with longer maturities than Treasury notes, the bonds in the sample have been in existence much longer than the notes. Accordingly, it is reasonable to suspect that these bonds are not traded so frequently. With infrequent trading, the published prices of the bonds may deviate further than notes from theoretically correct prices. Due to the paucity of Treasury bonds in the sample and their potential effects on the functional estimate, the Treasury bonds are not included in later analysis. The resulting samples include only Treasury notes as summarized in Table 1.

VI. Maximum Likelihood Estimation

To correct the problems of the first regression, a second set of regressions is estimated in which Treasury bonds are excluded from the sample and a correction is made for the maturity-related heteroschedasticity. Correction for the maturity-related heteroschedasticity involves the specification of the variance of the disturbance terms. Kmenta [6] discusses the selection of heteroschedasticity specification and suggests the following

$$(6) \qquad V\left(\varepsilon_{i,s}\right) = \sigma_s^2\left(Z_{i,s}\right)^{d_s}$$

where

$V(\cdot)$ = variance operator;
σ_s^2 = variance parameter at time s;
$Z_{i,s}$ = variable of bond i at time s to which heteroschedasticity is related;
d_s = heteroschedasticity parameter at time s.

This specification of the variance is general in the sense that it permits varying degrees of heteroschedasticity that include the homoschedastic case ($d = 0$). Maximum likelihood estimation is proposed with $Z_{i,s}$ = time to maturity.

We now assume that the $\varepsilon_{i,s}$ are independent random variables that follow a normal distribution with zero mean and variance given by equation (6). For a sample of N bonds, the resulting log-likelihood function, apart from a constant, is

$$(7) \qquad \begin{aligned} L\left(P_{i,s}, C_i(s+t,s) \mid \underline{x}_s, \sigma_s^2, d_s, J\right) &= -\frac{1}{2}\sum_{i=1}^{N}\left(\log s_s^2 + d_s \log Z_{i,s}\right) \\ &-\frac{1}{2\sigma_s^2}\sum_{i=1}^{N} Z_{i,s}^{-d_s}\left[P_{i,s} - \sum_{t=1}^{T} C_i(s+t,s)\exp\left(-\sum_{j=1}^{J} t^j x_{s,j}\right)\right]^2 \end{aligned}$$

where $\underline{x}_s = (x_{s,1}, \ldots, x_{s,J})'$. Maximizing this function produces estimates of $\underline{x}_s$, σ_s^2, and d_s that are consistent and efficient.

Notice that the log-likelihood function is also a function of J, the length of the approximating polynomial. The decision to truncate a polynomial of degree $j + 1$ to degree j may be viewed as the imposition of a constraint that the coefficient $j + 1$ is zero. Minus twice the logarithm of the resulting likelihood ratio asymptotically approaches a chi-squared distribution with one degree of freedom (see, e.g., [14], pp. 396-397). Denoting the maximum likelihood parameter estimates by "^", this statistic is

$$(8) \qquad -2\frac{L\left(P_{i,s}, C_i(s+t,s) \mid \hat{\underline{x}}_s, \hat{\sigma}_s^2, \hat{d}_s, j\right)}{L\left(P_{i,s}, C_i(s+t,s) \mid \hat{\underline{x}}_s, \hat{\sigma}_s^2, \hat{d}_s, j+1\right)} \rightarrow \chi^2(1).$$

The statistic is reported in Table 4 for $j = 1$ to $j = 6$ for all 16 samples. Unfortunately, the distributional properties of this statistic are asymptotic and the sample in this study is small. In addition, the appropriate significance level increases with each test in a sequence on a particular sample. Therefore, a unique critical value cannot be established. The reported critical regions apply only to large samples and to the first test in a sequence. Accordingly, the sequence of statistics may not be regarded formally as a sequence of significance tests so that our discussion of significance is heuristic. With this caveat in mind, in each case where the statistic may be significant, the hypothesis that coefficient $j + 1$ is equal to zero would be rejected at the appropriate significance level. The results indicate that a seventh degree polynomial coefficient is significant in four samples, fifth and sixth polynomial coefficients are frequently significant, and the first four polynomial coefficients are almost always significant. Because of the substantial computational requirements of using higher order polynomials, the results suggest the use of a sixth degree polynomial.

TABLE 4

Chi-Squared Tests of Polynomial Coefficient Significance

Sample Number	Length of Values of: $N/N + 1$					
	1/2	2/3	3/4	4/5	5/6	6/7
1	90.72	44.90	13.94	8.22	8.36	0.32
2	83.20	63.50	19.20	11.64	2.90	0.00
3	90.68	63.60	11.82	0.04	4.88	3.52
4	93.08	30.70	9.40	2.12	0.44	7.60
5	105.20	10.50	19.16	8.70	0.34	5.28
6	68.06	26.52	33.10	7.34	0.02	0.80
7	53.24	5.86	9.10	2.38	18.72	1.40
8	85.84	0.28	0.68	3.60	10.34	0.42
9	17.26	43.84	18.76	1.46	2.38	0.16
10	4.54	26.82	24.74	0.66	3.88	2.46
11	9.68	57.88	8.00	4.56	7.44	0.36
12	5.34	33.72	30.32	3.14	0.30	10.54
13	32.84	33.34	13.20	0.02	0.50	0.90
14	17.60	34.54	5.88	5.96	12.52	1.12
15	92.34	0.00	2.22	0.00	2.72	0.08
16	82.10	34.32	10.08	8.74	6.84	0.22

Critical Regions: 3.84 rejects null at 95 percent confidence.
7.88 rejects null at 99.5 percent confidence.

Note: These critical regions apply only to the first test in a sequence and to large samples.

Another approach to analyzing the reasonableness of the results is the inspection of residuals as performed in the previous subsection. The first ten residuals (ranked by time to maturity) are analyzed to determine the number of sign changes. Table 5 reveals the results of this procedure using polynomials of from one to the seventh degree. The mean results of the third and higher degree polynomials in the maximum likelihood estimation are superior to all mean results of the least squares regressions (see Table 3). In the maximum likelihood results, the sixth degree polynomial performed best. Additionally, it is the only polynomial length that has a three or greater in every sample period. These results indicate that the approximating polynomial is conforming to term structure shapes at the near end.

Concentrating on the sixth degree polynomial as a reasonable approximating function, Table 6 displays the estimated value of the heteroschedasticity parameter d, its t-statistic, the estimated value of the first polynomial coefficient, and the maximized value of the log-likelihood function for each sample period.

TABLE 5

Number of Sign Changes in First Ten Residuals: Maximum Likelihood Estimates

Sample Number	Length of Approximating Polynomial						
	1	2	3	4	5	6	7
1	0	0	1	1	3	3	3
2	0	0	1	3	5	5	5
3	0	0	4	6	6	4	6
4	0	0	4	4	4	4	2
5	0	0	1	5	5	5	3
6	0	0	1	5	3	3	4
7	0	1	3	3	3	5	5
8	1	1	2	2	2	3	3
9	0	0	2	4	4	4	4
10	0	0	3	3	3	3	3
11	0	0	5	1	5	4	2
12	0	0	2	3	1	3	5
13	0	0	2	4	4	4	4
14	0	1	5	1	5	4	4
15	0	3	3	3	3	3	3
16	0	0	4	4	4	6	6
mean	0.06	0.38	2.69	3 25	3.75	3.94	3 88

Inspection of the heteroschedasticity estimates reveals the rather unstable degree of heteroschedasticity throughout the study period. However, at this point it is difficult to establish a connection between the movement of the heteroschedasticity parameter and any other variable. The t-statistic indicates that the heteroschedasticity parameter is insignificant in samples 9 and 10, significant at the .01 level in samples 11 and 12, and significant at the .005 level in all remaining samples. Accordingly, the null hypothesis that the least squares regression (which assumes homoschedasticity) is appropriate may be rejected. The first polynomial coefficient may be interpreted as an estimate of the instantaneous interest rate. The rate begins the study period near 5 percent, climbs to a high of about 12 percent, and then drops to about 9 percent. The final column indicates the total explanatory power of the model, with larger values indicating a higher degree of explanation.

A final assessment of the polynomial's fit of the present-value function is through visual inspection of the term structure and forward rate estimates. Under the assumption of continuous compounding of interest, the sixth degree exponential/polynomial present-value function estimate is used to derive an estimate of the continuous term structure of interest rates and the implied forward rates. These structures are shown in the 16 figures of the Appendix. Although many of the samples contain one or two obligations with maturities of ten years, the term structures are plotted out only to 9.5 years and the forward rates are plotted only to 9.0 years. The reason is that the term structure and forward rates are rather erratic at the far end. Rather than attach economic meaning to the erratic behavior of the estimates at the far end, it is hypothesized that this behavior is attributable to the paucity of observations within that maturity range. All implied forward rates are nonnegative.

TABLE 6

Maximum Likelihood Sixth Degree Polynomial Results

Sample Number	Heteroschedasticity Parameter (t-statistic)	First Polynomial Coefficient	Log-Likelihood Optimum
1	2.02 (8.63)	0.045	27.45
2	1.87 (9.89)	0.045	31.26
3	1.82 (8.13)	0.050	32.60
4	1.00 (5.71)	0.060	30.94
5	1.01 (5.87)	0.064	47.45
6	1.12 (6.55)	0.066	39.58
7	1.23 (6.00)	0.067	31.36
8	1.09 (3.42)	0.074	31.93
9	0.04 (0.13)	0.101	7.76
10	0.43 (1.44)	0.100	14.83
11	0.77 (2.48)	0.098	20.00
12	0.79 (2.42)	0.102	22.88
13	0.21 (1.22)	0.120	−1.27
14	1.17 (4.92)	0.121	−1.38
15	1.27 (5.55)	0.087	1.73
16	2.20 (11.34)	0.087	19.83

The figures also reveal the problem with using the yield curve as an estimate of the term structure. The yield of a bond is a complex average of the spot rates corresponding to each of the cash flows. Accordingly, the yield will be a biased estimate of the spot rate corresponding to the maturity. Specificially, yields will be biased downward when the term structure slope is upward and vice versa. The extent of the bias depends on both the size of the bond's coupon and the degree of slope in the term structure. Additionally, an appropriate criterion for fitting the yield curve is not clear.

VII. Summary and Conclusions

The purpose of this study is to extend the analysis of estimation of the term structure. The ability of the exponential/polynomial present-value function of equation (5) to approximate the theoretical present-value function is analyzed empirically. This study confirms previous results that indicate that the function does not provide an acceptable fit using least squares regression. In most samples, the near end of the estimated term structure appears substantially in error. Lengthening the polynomial tends to improve the fit only at the far end. Since the data appear to be subject to maturity-related heteroschedasticity, we have generalized the disturbance variance specification to allow for this possibility.

Our correction permits determination of sample-specific degrees of heteroschedasticity. The reason is that the intensity of the heteroschedasticity appears to change from period to period. Although previous studies have attempted heteroschedasticity corrections (see [4]), the same specification was imposed on each sample in such studies. Maximum likelihood estimation was proposed and implemented in this paper. The results indicate that the difficulties of providing a good fit using the exponential/polynomial function are substantially eliminated.

Recent literature regarding term structure estimation emphasizes the selection of more sophisticated estimating functions to provide a reasonable fit throughout the entire maturity range. This study emphasizes more careful modeling of the pricing disturbances. The prior approach involves somewhat complicated estimating functions and a potential loss of efficiency due to the lack of influence that each portion of the term structure is able to exert on other portions (with spline functions, for example.) Our maximum likelihood approach is somewhat difficult to implement and computationally burdensome. Nevertheless, it appears to provide a useful term structure estimation procedure.

References

[1] Carleton, W. T., and I. A. Cooper. "Estimation and Uses of the Term Structure of Interest Rates." *Journal of Finance*, Vol. 31 (September 1976), pp. 1067-1083.

[2] Chambers, D. R. *The Management of Default-Free Bond Portfolios*. Unpublished Ph.D. dissertation, University of North Carolina, Chapel Hill (1981).

[3] Haley, C. W., and L. D. Schall. *The Theory of Financial Decisions*. New York: McGraw Hill Book Company (1979).

[4] Jordan, J. V. *Studies in Direct Estimation of the Term Structure*. Unpublished Ph.D. dissertation, University of North Carolina, Chapel Hill, (1980).

[5] ________________. "Discussion." *Journal of Finance*, Vol. 38 (May 1982), pp. 354-356.

[6] Kmenta, J. *Elements of Econometrics*. New York: MacMillan Publishing Company (1971).

[7] McCulloch, J. H. "Measuring the Term Structure of Interest Rates." *Journal of Business*, Vol. 44 (January 1971), pp. 19-31.

[8] ________________. "An Estimate of the Liquidity Premium," *Journal of Political Economy*, Vol. 83 (February 1975), pp. 95-119.

[9] Phillips, G. M., and P. J. Taylor. *Theory and Applications of Numerical Analysis*. London: Academic Press (1973).

[10] Schaefer, S. M. "On Measuring the Term Structure of Interest Rates." Paper presented at the International Workshop on Recent Research in Capital Markets (September 1973).

[11] ________________. "Bond Prices and Taxation," Unpublished paper (November 1977).

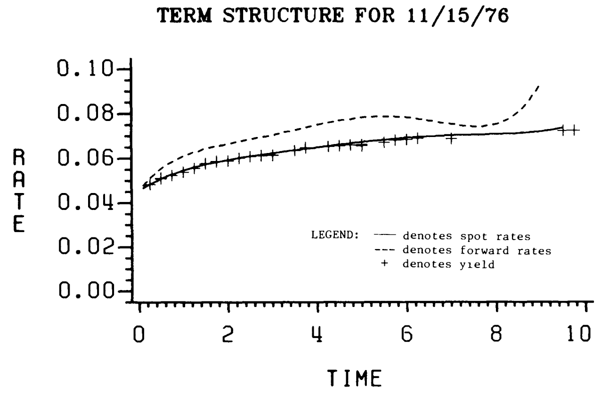
[12] ________________. "Measuring a Tax-Specific Term Structure of Interest Rates in the Market for British Government Securities." Unpublished paper (August 1979).

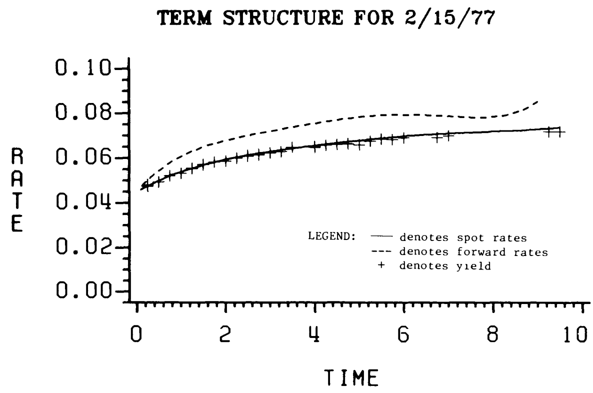
[13] ——————————. "Discussion: Conditional Predictions of Bond Prices and Returns," *Journal of Finance*, Vol. 35 (May 1980), pp. 417-419.

[14] Theil, H. *Principles of Econometrics*. New York: John Wiley & Sons, Inc. (1971).

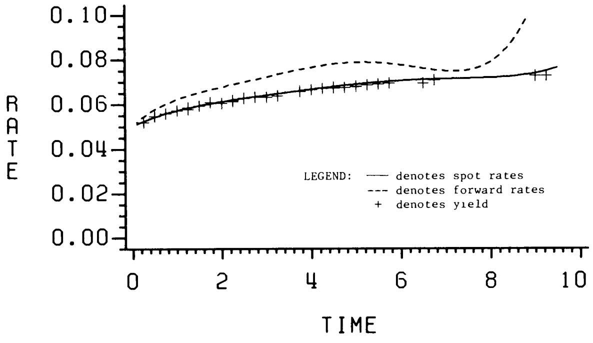
[15] Vasicek, O. A., and G. Fong. "Term Structure Estimation Using Exponential Splines." *Journal of Finance*, Vol. 38 (May 1982), pp. 339-348.

Appendix

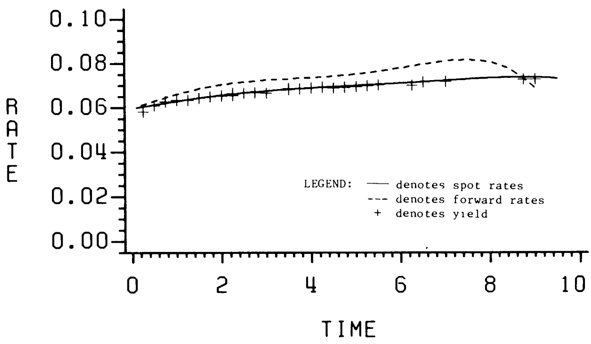




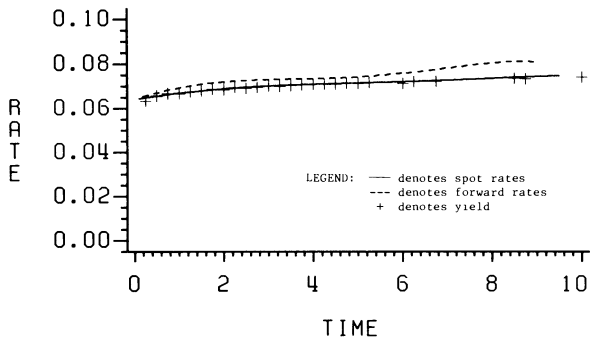
TERM STRUCTURE FOR 5/16/77



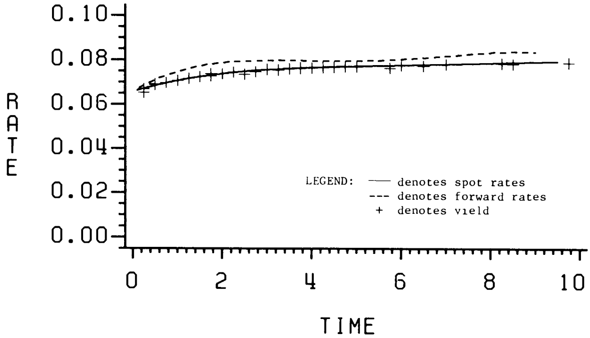
TERM STRUCTURE FOR 8/15/77



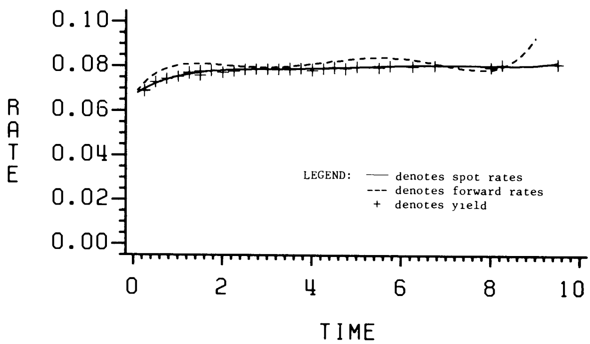
TERM STRUCTURE FOR 11/15/77



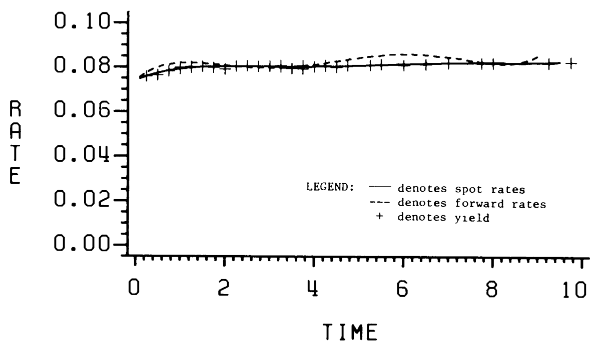
TERM STRUCTURE FOR 2/15/78



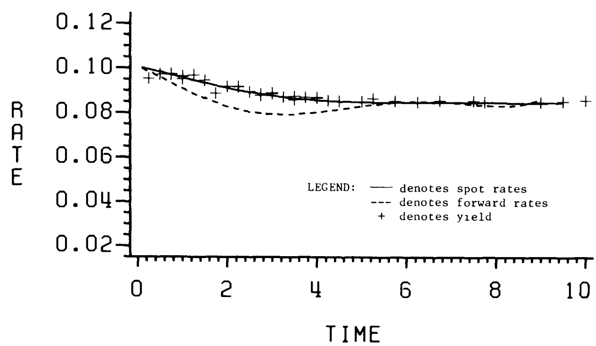
TERM STRUCTURE FOR 5/15/78



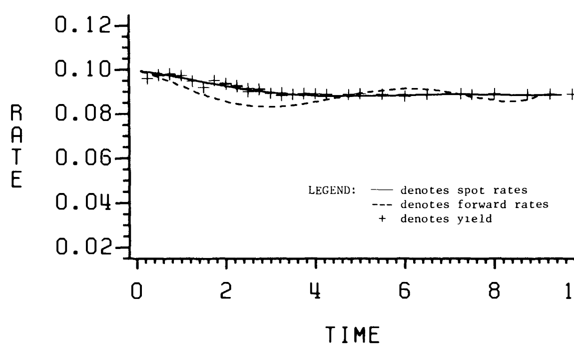
TERM STRUCTURE FOR 8/15/78



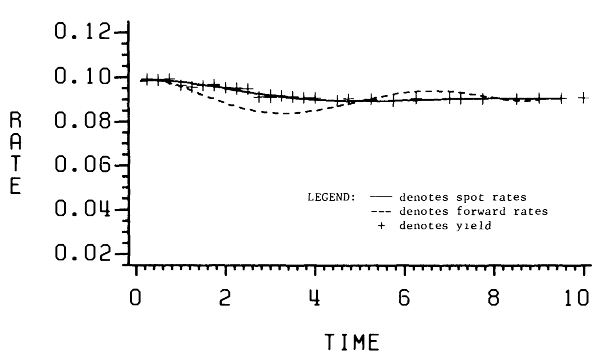
TERM STRUCTURE FOR 11/15/78



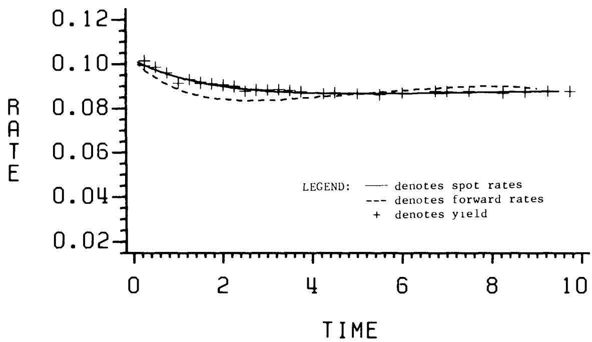
TERM STRUCTURE FOR 2/15/79



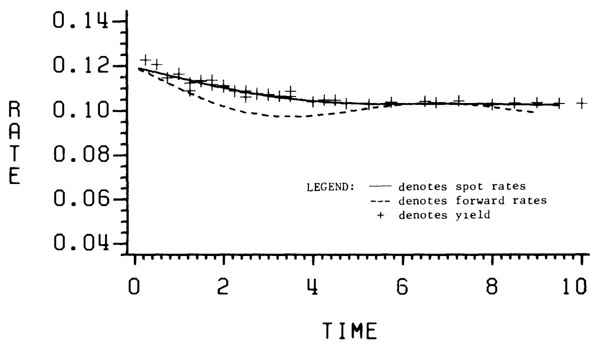
TERM STRUCTURE FOR 5/15/79



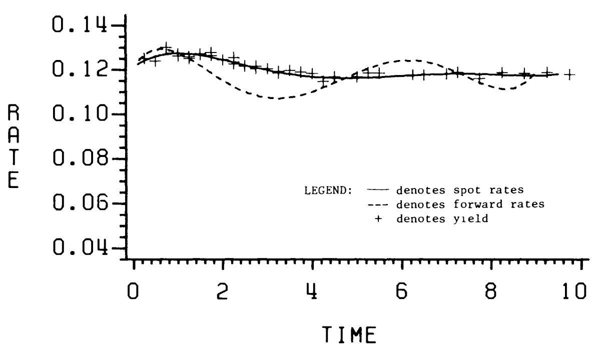
TERM STRUCTURE FOR 8/15/79



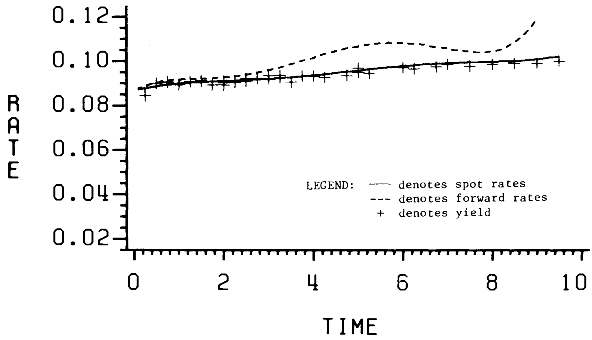
TERM STRUCTURE FOR 11/15/79



TERM STRUCTURE FOR 2/15/80



TERM STRUCTURE FOR 5/15/80



TERM STRUCTURE FOR 8/15/80