THE JOURNAL OF FINANCE • VOL. XXXI, NO. 4 • SEPTEMBER 1976

ESTIMATION AND USES OF THE TERM STRUCTURE OF INTEREST RATES

WILLARD T. CARLETON* AND IAN A. COOPER*

I. INTRODUCTION

EMPIRICAL STUDIES employing the term structure of interest rates have generally suffered from the defect that yield curves have been used as surrogate measures of the term structure. A number of scholars and professional bond analysts have, in the past decade, called attention to the problem (Malkiel [12], Buse [2], Weingartner [21], Homer and Leibowitz [8], and most recently Carr, Halpern and McCallum [4]) that only when the yield curve is flat can it be so used, legitimately. Only McCulloch [13] and Schaefer [18] have attempted to measure the term structure directly.

In this paper we summarize briefly the analytical issues; suggest some of the uses to which term structure estimates could be put; propose an estimation method and describe some preliminary results; and finally, present an empirical application of term structure estimates in a stock valuation model. Most research papers conclude with a promise of more research. To a greater than usual degree the present paper is a promissory note because of the implications of our approach (and McCulloch's [13] and Schaefer's [18] for that matter) for a wide range of topics in financial and monetary economics.

II. REVIEW OF THE PROBLEM

Let us first make some commonplace observations about bond valuation models, beginning with three alternative present value formulations:

$$P_{0,n}=\frac{C}{(1+i_n)}+\frac{C}{(1+i_n)^2}+\cdots+\frac{C+S_n}{(1+i_n)^n} \tag{1}$$

is the "yield" relationship,

* Both, University of North Carolina, Chapel Hill.

Earlier versions of this paper have been given at a number of university seminars and professional meetings, including one entitled "Direct Estimation of the Term Structure of Interest Rates" by the senior author and William R. Bryan at a joint session of the Econometric Society and American Finance Association, Toronto, December 1972. As a consequence, the number of helpful persons who have provided helpful suggestions is too long to enumerate. We would like to thank especially, however, Ed Kane for a couple of important suggestions, and Michael Caulfield and Bryan Stanhouse for assistance in data collection and programming activities. The Tuck School Associates Program and Institute for Quantitative Research in Finance have also provided financial support in generous measure, for which we are grateful.

where $P_{0,n}$ = price, at period 0, of an n-period bond
C = coupon rate
S_n = face value of bond
n = number of periods to maturity
i_n = yield to maturity

Alternatively,

$$P_{0,n}=\frac{C}{(1+R_1)}+\frac{C}{(1+R_2)^2}+\cdots+\frac{C+S_n}{(1+R_n)^n} \tag{2}$$

where R_t = spot rate of return for tth period payments, contracted for at period 0.

Finally,

$$P_{0,n}=\frac{C}{(1+r_1)}+\frac{C}{(1+r_1)(1+r_2)}+\cdots+\frac{C+S_n}{\prod_1^n(1+r_t)} \tag{3}$$

where r_t = one period forward rate for period t.

Among the factors to note about these algebraically equivalent valuation formulas:

a. What we observe for any bond is $P_{0,n}$, C, S and n. Interest rate measures such as i_n, $\{R_t\}$ and $\{r_t\}$ are derived or implied. Indeed, for coupon securities, market quotes are in terms of price, not interest rates. Moreover, while an i_t may be derivable as a matter of arithmetic from a single security, $\{R_t\}$ and $\{r_t\}$ of necessity are derived from a set of securities.

b. As many people have shown, the concept of yield to maturity, i_n, is an ambiguous concept. For a conventional bond (if not for some hybrid financial contracts or real asset purchases), the expected cash flow pattern implies a unique i_n as a solving—or internal—rate of return. Its economic meaning is moot, however, inasmuch as reinvestment of intermediate cash flows at the solving rate is implied. To borrow a concept from the capital budgeting literature, the price of an asset equals its present value only in the sense that its cash flows have been discounted at the market's return requirements (i_n, $\{R_t\}$ or $\{r_t\}$). It is true that associated with each bond is a *derived* i_n, but that does not give us license to say that i_n is a market-required rate, exogenous to the individual bond in question.

c. The Hicksian formulation of the expectations hypothesis relates the term structure of spot interest rates, $\{R_t\}$, to the forward rate structure, $\{r_t\}$, under well-known behavioral assumptions regarding market participants *and given the objective existence of* $\{R_t\}$. That is, it is well-known that forward rates contain economic meaning in an expectational sense[1] only under profit seeking behavior leading to the equalization of R's across all (equivalent) securities. Otherwise the r_t's are mere numerical artifacts. What is not so well appreciated is that the term

1. For the time being we ignore the possible existence of term, or liquidity, premia and the like.

structure, $\{R_t\}$, is not observed directly on coupon securities so that validity of the forward rate structure also depends upon the properties of $\{R_t\}$ estimates used.

d. The empirical assumption normally made is that "the" yield curve, for the class of bonds under investigation, averaged in some way over a period of time, is a useful estimate of the term structure of interest rates, $\{R_t\}$.

There are several difficulties with this procedure. In the first place, a set of yields $\{i_{jt}\}$, calculated from the price of each security j in maturity class t, is a set of internal rates of return which may bear only accidental relationship to the R_t's of the expectations hypothesis. As a matter of fact, as Buse [2] and others have noticed, the yield curve (for securities with common C) will be identical with the term structure only when all R_t's are equal. Second, the averaging process, both across time and across yields for bonds of identical maturity, destroys information. In particular, the fact that several different bond yields can be observed for a given maturity does not in itself indicate data noise or disequilibrium if internal rate of return is an artifact. In fact, the observation of several bond prices for that maturity might constitute important sample information. For example, one important source of yield and price difference, on bonds of equal maturity, is coupon difference. The range of coupon rates on all classes of bonds has increased in recent years. It is not clear what the implications of this have been for (i) the calculation of smoothed yield curve and (ii) derivation of forward rates, but we think that suspicion is the appropriate attitude.[2]

What we are left with is that yields are converted, *ad hoc*, into a yield curve and the yield curve is used as the basic data source for tests of the expectations hypothesis (and other studies requiring estimates of the term structure). There appears to be increasing professional awareness of the problem, and indeed, two substantial efforts have been made to measure the term structure directly (McCulloch [13] and Schaefer [18]). Before presenting our estimation procedure and contrasting it with these other efforts, however, it would be useful to suggest (nonexhaustively) the range of topics for which such estimates would be useful.

III. Applications of the Term Structure

a. *Expectations hypothesis-related models.* As alluded to in the last section, the use of more objective estimation procedures to generate the raw material, $\{R_t\}$, of these models would lend additional credibility to their results. This comment applies to the whole range of models currently being tested: from pure expectations versions with or without liquidity premia, to supply-affected models, to preferred habitat versions. As an editorial aside, some of the best-known recent work in this field seems to be moving in the direction of employing still cruder (and measurement error-ladden) data (e.g., Modigliani-Sutch [18]).

b. *Tax effects models.* It suffices to refer to the Robichek-Niebuhr study which allocates the effect of capital gains-ordinary income tax differentials on discount bonds to the maturity date of such securities. Our present measurement procedure, which identifies a single yield with each security, undoubtedly makes this a

2. It would be a detour to go over the effects of coupon differences in detail in this paper. The interested reader should consult Buse's excellent article.

necessary assumption. But there is no economic theory we know of (especially one consistent with the usual statements of the hypothesis) which requires the present value incidence of such tax effects to be allocated to each bond's maturity date. A more general model, in which at least a pre-tax term structure of rates independent of specific securities was employed, might permit the testing of alternative tax effects hypotheses.

c. *Credit-risk effects on non-U. S. government bonds and on stocks.* The most general form of a present value bond equation is:

$$P_{0,n} = \frac{\alpha_1 C}{1+r_1} + \frac{\alpha_2 C}{(1+r_1)(1+r_2)} + \cdots + \frac{\alpha_n (C+S_n)}{\prod_1^n (1+r_t)}$$

where α_t = certainly equivalent discount factor for period t, an adjustment for period t credit risk. It is clear that knowledge of $P_{0,n}$, C, S_n *and* $\{r_t\}$ for a sample of securities could in principle permit estimation and interpretation of $\{\alpha_t\}$. Work has been done in this area (Silvers [19], Cohan [5]), but under the disadvantage that the time discount rates employed were yields (i_t) rather than holding period (R_t) or forward (r_t) rates.

In the matter of common stock present value equations, the most general expression is:

$$P_{0,n} = \frac{D_1}{(1+k_1)} + \frac{D_2}{(1+k_2)^2} + \cdots + \frac{P_{n,n}}{(1+k_n)^n}$$

where $P_{0,n}$ = present value at period zero, for a valuation horizon of n periods
$P_{n,n}$ = present value at period n, of cash flows after period n
D_t = dividends expected for period t

Under usually harmless assumptions, *viz.*, that $D_t = D_{t-1}(1+g)$ where g is an expected constant growth rate, that $n \rightarrow \infty$, and that $k_t = k_{t'}$, $(t \neq t')$ we get the well known "cost of capital" version of k:

$$k = \frac{D_1}{P_0} + g$$

For present purposes we note that this set of assumptions places too great a burden on the internal rate of return k. In particular, as Robichek and Myers discussed [17], anomalous results are obtained when k is assumed to be the sole vehicle for expressing stockholder risk aversion. They suggest the alternative present value equation:

$$P_{0,n} = \frac{\alpha_1 D_1}{(1+r_1)} + \frac{\alpha_2 D_2}{(1+r_1)(1+r_2)} + \cdots + \frac{\alpha_n P_{n,n}}{\prod_1^n (1+r_t)}$$

where α_t = discount applied to expected dividends, D_t, to incorporate non-interest related risks.

Such a model has appeal,[3] but its empirical articulation presupposes knowledge of $\{r_t\}$, data very imperfectly derived from bond yields.

d. *"Fisher effects" in a multi-year context.* The usual specification is that $\bar{i}=\rho+E(\Delta P/P)$,

where $\bar{i}$ = average, nominal yield across a broad range of maturities of U. S. government bonds, measured at a point in time;

ρ = real required yield, an average for some indefinite number of years into the future;

$E(\Delta P/P)$ = rate of growth of the price level expected for some indefinite number of years into the future.

Normally, (Yohe-Karnosky [22]), some distributed lag model is employed to produce an estimate of $E(\Delta P/P)$. It seems clear that a richer variety of models could be employed if we did away with the convenient fiction of $\bar{i}$ and tried to explain the forward rates currently obtaining, $\{r_t\}$, each as sum of a real rate and the expected rate of inflation. But this presupposes the availability of reasonable estimates of $\{r_t\}$.

e. *Bond trading models.* The standard bond trading opportunity is described in terms of yield spreads between pairs of bonds: if a spread follows a stable frequency distribution over time, then the random occurrence of an unusually large spread (in a quality control chart sense) signals a quasi-arbitrage opportunity[4] for the trader. He takes a short position in the lower yield instrument and a long position in the higher yield one, anticipating a profit when the spread returns to its normal range. As Kramer [11] notes, however, because of the effects of coupon and maturity on yield comparisons "the search for arbitrage opportunities is usually conducted only among issues of at least intermediate maturity with 'similar' coupon rates, say within 3/4 of 1%, and similar maturities, say within two to three years in the intermediate maturities, and five to eight years in the longer maturities." (p. 32) Many financial institutions have developed statistical models to assist their bond traders in the identification and exploitation of quasi-arbitrage opportunities. To the extent that these are based on yield spreads, however, they do not escape the coupon and maturity problems. Estimates of the term structure internal to a set of securities with different coupons and maturities, on the other hand, would permit the direct comparison of every pair of bond prices relative to their equilibrium values. Quasi-arbitrage opportunities would thus be measured in terms of differences between pairs of residuals from a common underlying bond price model.

IV. A Procedure for Direct Estimate of the Term Structure of Interest Rates

With such an extensive set of preliminary remarks, we hope that our emperor has

3. Financial planning models (e.g., Carleton) which can incorporate a term structure or forward structure of capital costs would be the obvious beneficiaries of such a stock valuation model's parameter estimates.

4. We use the expression, quasi-arbitrage opportunity, because the trading profit has an expected value greater than zero but is not certain.

some clothes. As to criteria, it would seem reasonable that any method for estimating the term structure of interest rates should:

a. Recognize that the natural unit of measurement of the value of a bond is its price;

b. Recognize that even independent of tax effects, i_t does not have to be equal for each security of period t maturity;

c. Use all of the sample observations in a cross section of bonds, not some arbitrary average;

d. Maintain the key assumption of the expectations hypothesis that the rate of return for tth period payments is the same for all securities in the sample.

Next, we observe that another way of expressing a present value equation (2) is:

$$P_{0,j} = b_{0,j} + b_{1,j}x_{1,j} + b_{2,j}x_{2,j} + \cdots + b_{n,j}x_{n,j} \tag{2'}$$

where $P_{0,j}$ = present value of jth bond at period 0.

$b_{t,j}$ = present value per unit of expected cash flow for period t from bond j. That is, $b_{t,j} = (1 + R_{t,j})^{-t}$.

$x_{t,j}$ = cash flow, C_j, or $C_j + S$ (or zero, as the case may be, if bond j's maturity is less than n), for period t.

If (2′) is made into a statistical model, by addition of a disturbance term with appropriate characteristics, it is possible to admit that $P_{0,j}$ may differ from its expected value, at least randomly. The result is almost a regression equation. A few more assumptions are needed. First, $b_{t,j} = b_{t,j'} = b_t$ $(j \neq j')$ so that $R_{t,j} = R_{t,j'}$. Secondly, $b_0 = 0$. Third, $b_1 \leqslant 1$, so that r_1, or r_1, estimates will be greater than or equal to zero. Fourth, $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_n \geqslant 0$, so that $r_t \geqslant 0$ in the formula $1 + r_t = b_{t-1}/b_t$. The result is a constrained regression of the form:

$$P = Xb = e \tag{4}$$

where $Rb < d$ and $b > 0$. While it seems reasonable to estimate b in this structure using constrained least squares procedures,[5] before describing data and results a few additional comments should be made:

a. Economic theory is sufficiently robust with respect to interest rate phenomena that it is appropriate to treat our constraints in a deterministic—not probabilistic—fashion.

b. The rest of the usual regression model assumptions are likewise acceptable, notably the fixity of X under repeated samples.

c. It is useful to summarize the economic implications of equation (3). First, for any cross section sample of bonds, the expectations hypothesis is assumed to hold in expected present value terms:[6] there are no systematic arbitrage opportunities

5. Since ordinary least squares involves minimizing an expression of the form $(P - Xb)'(P - Xb)$, the addition of linear inequality and non-negativity constraints converts the method to one of quadratic programming. See Judge and Takayama [10].

6. Some statistical awkwardness arises over this point. If, in an unconstrained regression, the resulting present value coefficients are unbiased, the same property does not adhere to the derived term structure

available, since the same expected present value attaches to a dollar of period t future payment independent of the security from which it is to be received. Second, the presence of a random disturbance term reflects the assumption that while the expectations hypothesis holds expectationally, the market is not continuous so that, for example, quasi-arbitrage opportunities of the sort described in Section III may occur. What these two assumptions imply together is that residuals of individual bonds from a series of cross section regressions should also be random with zero means.

d. Regardless of whether one makes the usual assumptions about the disturbance vector, e, our model's parameter estimates do not possess all of the usual least squares properties (see Judge-Takayama, also Theil), if constrained estimation procedures are necessary, so that resulting estimates would have to be assessed on the basis of non-statistical criteria. This suggests that ordinary least squares first be employed.

V. Prior Estimation Efforts and our Empirical Model and Data

As mentioned previously there have been two other efforts, that we are aware of, to estimate the term structure of interest rates via some variant of (4). McCulloch [13] worked with samples of U. S. Government and corporate securities. He assumed continuous coupon and principal payments and compounding in order to generate a continuous discount function. This function was estimated with a set of approximating polynomials, piecewise quadratic. McCulloch was able through this specification to avoid quadratic programming estimation, but his derived forward rates display "knuckles," a direct consequence of the form of his approximating polynomials. Schaefer, by way of contrast, worked with British Government obligations. He assumed periodic payments, but continuous compounding, thus also requiring the use of approximating polynomials to estimate a continuous discount function. The form of his polynomials necessitated quadratic programming estimation, but his paper [18] does not disclose whether any of the constraints were binding.

By exploiting the structure of our sample data we have been able to avoid McCulloch's assumption of continuous payments, McCulloch's and Schaefer's use of approximating polynomials, and (at least in the data thus far examined) Schaefer's use of quadratic programming procedures. The first thing to note is that U. S. Government coupon securities (notes and bonds) with rare exception make

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or forward rate estimates, which are non-linear transformations of the coefficients. We do not find this too troublesome, however, because bond price (hence present value units) is the dimension in which market trades occur, and rate of return transformations put the results in a more convenient and recognizable framework. On the other hand, this does place the Malkiel proposition in a new light: It is true that the Hicksian behavioral assumption of a market acting on single period forward rate expectations over a long horizon produces results equivalent to the behavioral assumption of the market acting on one period expectations for the maturity structure of bond prices, as a matter of theory. The statistical model implications of these two approaches, once randomness and estimation problems enter the picture, are not equivalent. Neither McCulloch nor Schaefer called attention to this problem in their studies. Obviously, further research in this area is necessary.

semi-annual payments on only four days of each year: February 15, May 15, August 15 and November 15. For such securities there is thus a natural, discrete present value function whose coefficients, should in principle be amenable to direct least squares estimation. Second, Federal Home Loan Bank (FHLB) securities, which bear a U. S. Government guarantee, make payments on the 25th to 27th of the same months. Since they bear this guarantee the assumption arises that the observed lower prices on FHLB securities otherwise equivalent to Treasury issues are a consequence of liquidity and payment date differences only. This suggests a simple procedure for adjusting in one step for both differences, thus allowing present value coefficients to be estimated for pooled (thus enlarged) samples of Treasury and FHLB coupon obligations.

If we specify that

$$\tilde{b}_t = (1-\alpha)^t b_t$$

where $\tilde{b}_t$ = present value of period t payment on FHLB security;
b_t = present value of period t payment on U. S. Government note or bond;
α = constant;

this is equivalent to the requirement that

$$1 + r_t = (1-\alpha)(1+\tilde{r}_t)$$

where $\tilde{r}_t$ and r_t are forward rates applicable to FHLB and U. S. Government securities, respectively. The estimation results, described in the next section, involved determination of the α which best fits the data for each sample date.

Our final samples are of U. S. Government coupon issues (not including flower bonds or exchange refund notes) and FHLB securities with the standard payment dates just described,[7] maturing no later than sixteen quarters after price was was recorded with bid, ask and mean prices recorded for delivery dates of January 1, 1971, (38 observations); December 31, 1971 (41 observations); December 30, 1972 (39 observations); and January 1, 1974 (49 observations).

VI. Estimation Results

For each of the four sample dates and for bid, ask and mean prices, equations were fit of the form:

$$P_{0,j} = b_1 X_{1,j} + b_2 X_{2,j} + \cdots + b_{16} X_{16,j} + e_j \tag{5}$$

where $P_{0,j}$ = closing price of security j as reported in the New York Times, plus accrued interest in accordance with market conventions;
$X_{t,j}$ = payment at period t expected on security j;
e_j = disturbance term distributed as $N(0,\sigma^2)$.

7. This sample data, taken from the *New York Times*, is available upon request. Also, it should be noted that estimation required the inclusion of one nonstandard payment date security in two of the samples. This is explained in the next section.

TABLE 1

Payment Quarter	1/1/71 $\hat{b}_t$	Std. Error	$\hat{R}_t$	12/31/71 $\hat{b}_t$	Std. Error	$\hat{R}_t$	12/30/72 $\hat{b}_t$	Std. Error	$\hat{R}_t$	1/1/74 $\hat{b}_t$	Std. Error	$\hat{R}_t$
1	99.560	.137	.036	99.662	.117	.027	99.396	.151	.048	98.982	.123	.087
2	98.279	.137	.048	98.599	.134	.039	98.031	.175	.055	97.035	.143	.085
3	97.132	.159	.048	97.514	.135	.041	96.708	.175	.055	95.520	.124	.064
4	95.955	.138	.049	96.500	.135	.042	95.343	.305	.056	93.891	.143	.071
5	94.368	.123	.053	95.248	.117	.044	94.042	.174	.056	92.579	.111	.057
6	93.156	.159	.053	94.278	.166	.044	92.536	.214	.059	91.005	.124	.073
7	91.766	.278	.054	92.800	.135	.047	91.128	.176	.059	89.589	.124	.064
8	90.592	.277	.054	91.047	.747	.051	89.644	.214	.060	88.188	.143	.065
9	88.898	.276	.057	90.304	.135	.049	88.484	.152	.060	86.790	.124	.065
10	87.832	.274	.057	89.012	.169	.050	86.829	.175	.061	85.342	.124	.071
11	86.393	.160	.057	87.683	.166	.051	85.646	.215	.061	83.815	.143	.074
12	85.401	.915	.056	86.747	.167	.051	84.555	.215	.060	82.591	.248	.060
13	83.494	.195	.060	85.209	.136	.053	83.020	.176	.061	81.207	.175	.069
14	82.230	.276	.060	83.981	.167	.053	81.772	.215	.061	71.759	.250	.077
15	81.014	.197	.060	82.500	.166	.055	80.157	.301	.063	78.463	.175	.067
16	79.998	.199	.060	81.516	.168	.054	78.853	.303	.063	77.194	.177	.069
R^2	.99			.99			.99			.99		
Standard error of est.	2.84			2.40			3.12			2.55		
d.o.f.	22			25			23			33		

This was accomplished in two steps. First, for each sample date, successive trial values of α were employed in accordance with the procedure given in Section V to transform FHLB securities' payments to the form $(1-\alpha)^t X_{t,j}$. Then regressions were estimated for each of the resulting bid, ask and mean price versions of equation (5). Our criterion was to minimize standard error of estimate and make insignificant a FHLB dummy variable, while forcing a zero intercept. Uniformly the $\hat{b}_t$ coefficients declined monotonically with t in this procedure and the dummies were insignificant at the .001 level. The best values of α were:

.001	January 1, 1971
.001	December 31, 1971
.00075	December 30, 1972
.00075	January 1, 1974

The final equations were fit with these values of α and omitting the FHLB dummy variables. For economy of presentation and because the results were very similar only mean price equation results are presented and analyzed further in the paper. Table 1 gives the regression results, as well as spot rates, $\hat{R}_t$, expressed in annual rate of return units.

There are several points to be made with respect to these results, as well as the follow-up analyses and tests of model specification:

a. R^2 and standard errors of coefficients are not formally assessed in significance tests. This is because R^2 values tend to be high by nature of the data, and because the standard errors properly should be employed in a joint test of the signs of implicit forward rates. Inasmuch as the coefficients always were less than one, monotonically declining and greater than zero, this seems an unneccesary formalism. What is of interest is the small size of the regression standard errors of estimate, which are given in units of $1,000.00 bonds. That is, when present value functions are estimated, the standard error of estimate is in the vicinity of $2.50 to $3.00 for a $1,000.00 par security.

b. The only constraint imposed on the least squares estimation was a zero intercept. When an intercept was permitted the coefficients always were monotonically declining, but sometimes beginning at values greater than one. We have no explanation, economic or statistical, for this phenomenon and it may deserve further research. On the other hand it is the only way in which unconstrained least squares procedures by themselves failed to produce coefficient estimates both reasonable and well within the requirements of economic theory.[8]

c. Inspection of coefficient b_{12} in the January 1, 1971 regression and b_8 in the December 31, 1971 regression—as well as the corresponding forward rates in Figure 1—reveal some anomalies which are easily explained by a small compromise we had to make in our "no payments interpolation" requirement. In each sample no securities existed which matured on the standard dates indicated; in order to generate sample observations we interpolated linearly payments on the one non-standard FHLB security with maturity date between the standard dates. The obviously out of line behavior of the resulting coefficients calls attention to two things: first, that the concerns expressed by McCulloch [13] and Schaefer [18] as to the consequences of such interpolations appear well-founded; but, second, that the problem may not be serious if only one or two observations are involved. It is remarkable that there is a body of data which lends itself so readily to direct term structure estimation

d. A number of tests were made for specification error. Following a procedure suggested in Johnston [9, (p. 219)], Spearman rank correlation coefficients were calculated for absolute value of residual versus maturity, and absolute value of residual versus size of discount from par. Two of the former, but none of the latter were significant at the .05 level, indicating the possibility of heteroskedasticity. Arrays of the residuals displayed no obvious patterns of heteroskedasticity. Because of this and the small size of regression standard errors of estimate, no attempt to transform the variables was made. If any heteroskedasticity truly is

8. Following a helpful suggestion by Marshall Blume, we experimented with one other present value coefficient estimation procedure: Let $\hat{b}_1$ equal the mean ratio of payments to price for all securities maturing in one period. Let $\hat{b}_2$ equal the mean ratio of payments to price less $\hat{b}_1 C_1$ for all securities maturing in two periods; and so on. The term structures derived from this recursive, bootstrap procedure were similar to those derived from our least squares regression coefficients, with the property that the differences were larger for short maturities. The use of alternative term structure estimates will ultimately determine which (McCulloch, Schaefer, Carleton-Cooper, Blume, or other) method is most valid but it seems clear that the recursive bootstrap does not eliminate the zero intercept problem, except by assumption. The calculation of mean residual yields also by construction assures a zero intercept, and, in addition, throws away sample information, if our model's specification is correct.

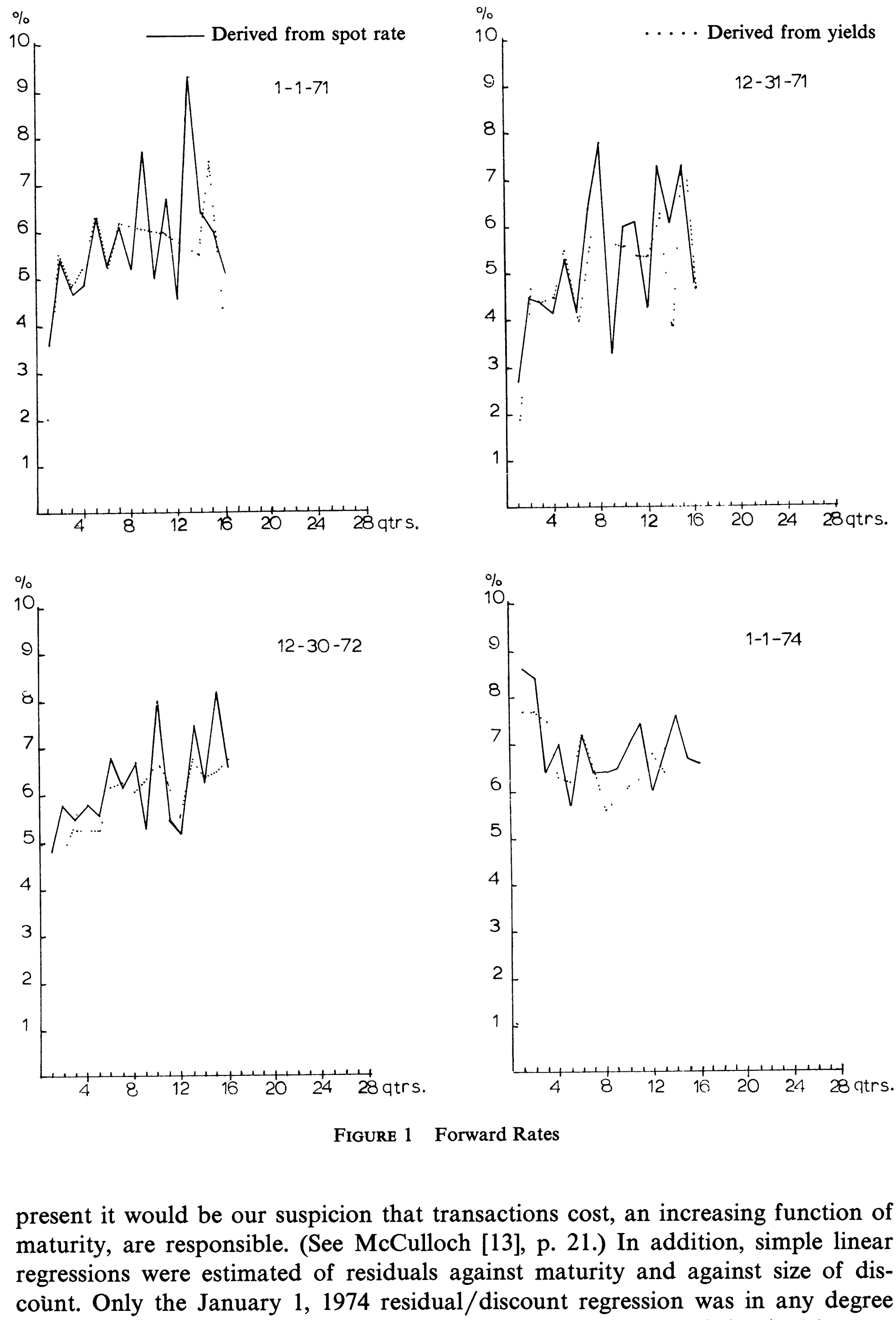

FIGURE 1 Forward Rates

present it would be our suspicion that transactions cost, an increasing function of maturity, are responsible. (See McCulloch [13], p. 21.) In addition, simple linear regressions were estimated of residuals against maturity and against size of discount. Only the January 1, 1974 residual/discount regression was in any degree significant, and it was so highly, at the .01 level. Given the high level of interest rates on this date, it is quite likely that ordinary income/capital gains tax differentials were significantly influencing relative prices. On the other hand, given the goodness of fit of the present value equation for this date, this tax effect, while significant statistically, is arithmetically small. As mentioned in Section III, this topic is appropriate for considerable research in its own right. Finally, in a search for bond-specific effects, mean residuals and standard deviations were calculated for the seven securities present in all four regressions, four Treasury and three FHLB obligations. One mean was as large as its standard deviation; the rest were considerably smaller. Given the small number of estimation dates, this finding is not very conclusive, but it supports our specification.

e. Our estimates of the term structure are displayed graphically with the yield structure[9] in Figure 2 and a similar comparison of forward rates derived from these, in Figure 1. In Figure 2, for payment dates beyond the sixteenth quarter, non-statistical estimates of the term structure, generated recursively, from equation 2, are included. Where, in this extension, there are gaps in the maturity structure, such rates are calculated on the assumption of a level term structure over the gap and date of next maturing instrument. Our term structures for both maturity segments follow the general shape and height of yield structures, especially after the first two quarters. On the other hand, in Figure 1 it can be seen that the two sets of forward rates differ substantially, even apart from the two anomalies discussed above. Without asserting that our forward rate estimates are the best ultimately possible, we would argue that our findings suggest caution in the use of yield-derived forward rates in, e.g. error learning models of the expectations hypothesis.

VII. An Application of our Term Structure Estimates: Stock Valuation

In Section III it was noted that an attractive alternative to the constant cost of equity stock valuation model is of the form:

$$P_{0,n}=\sum_{1}^{n}\alpha_t D_t\prod_{1}^{\tau}(1+r_\tau)^{-1}+\alpha_n P_{n,n}\prod_{1}^{n}(1+r_\tau)^{-1}$$

$$=\sum_{1}^{n}\alpha_t D_t(1+R_t)^{-t}+\alpha_n P_{n,n}(1+R_n)^{-n}$$

in which time value discounting with the term structure $\{R_t\}$, of the dividend stream is separated from credit risk, or dividend uncertainty, discounting. The $\{\alpha_t\}$ vector performs the latter function, reducing expected dividends, $\{D_t\}$, to their certainty equivalent values. A reasonable further specification is that dividend uncertainty increases with time, so that $1\geqslant\alpha_1\geqslant\alpha_2\geqslant\cdots\geqslant\alpha_n\geqslant 0$. The exact form of the α_t function is difficult to derive theoretically, but it seems reasonable to approximate it as:

$$\alpha_t=\alpha^t \qquad \alpha=\text{a constant}.$$

By way of justification, the certainty equivalent functions estimated by Silvers for

9. The yield observations are for single securities when only one matures on a particular date; otherwise, they are mean yields of all securities maturing on that date.

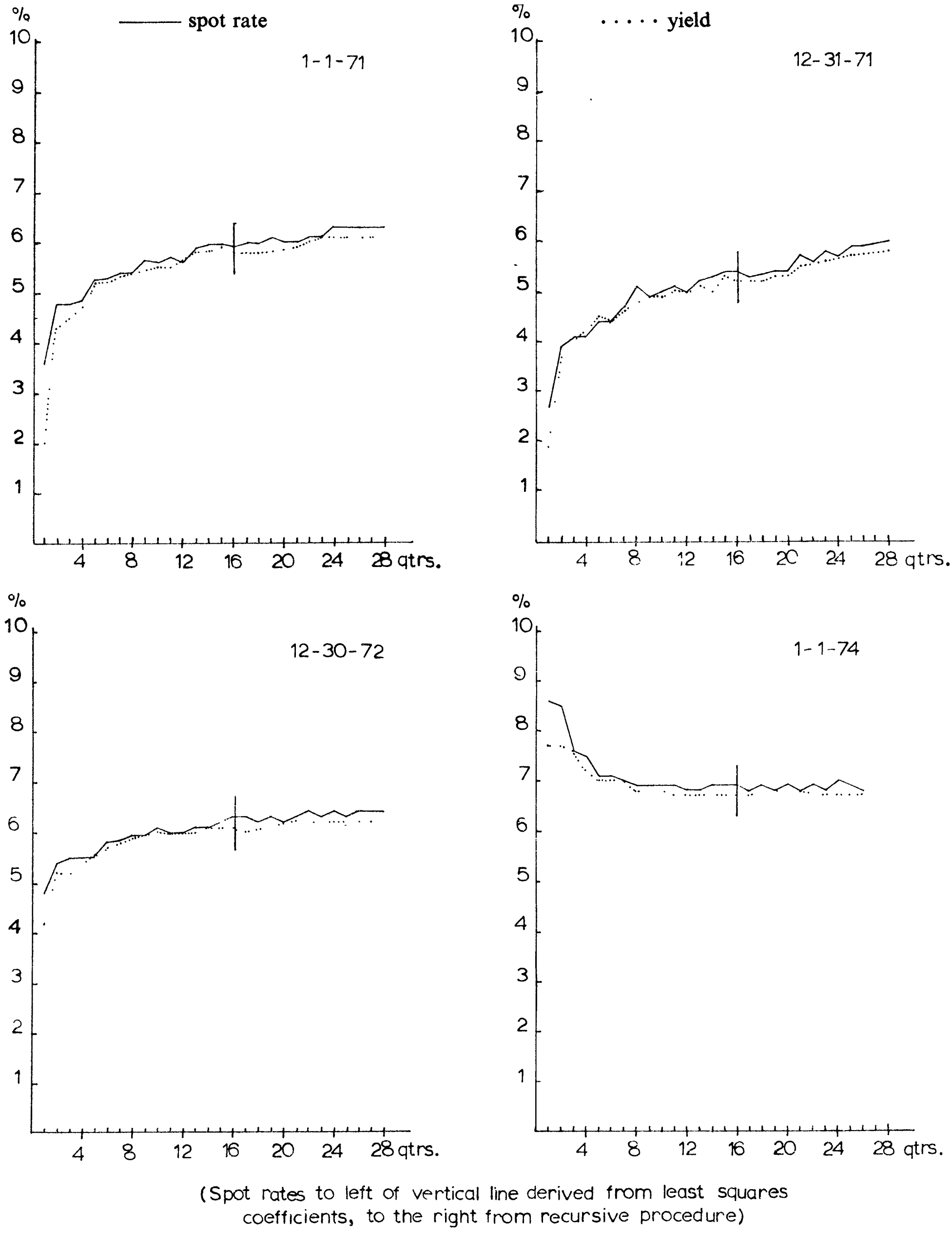

FIGURE 2 Spot Rate and Yield Structures (in annual rate of return units)

bonds were close to this form. With this assumption and the conventional assumption that $n \rightarrow \infty$ our stock valuation equation becomes:

$$P_0 = \sum_1^\infty \alpha^t D_t (1 + R_t)^{-t} \tag{6}$$

The electric utility industry was chosen for empirical estimation of equation 6, because of two features. First, firms in the industry have similar risk characteristics, capital structures and dividend policies so that at any given valuation date it can be maintained that the certainty equivalent coefficient vector is the same for all firms. Second, models of expectations formation regarding dividend growth and horizons for above non earnings opportunities have been developed and tested (Gordon [6], Higgins [7]), and we can employ some of their findings.

Data used were for the seventy electric utilities on the Compustat 1974 Quarterly Utility Tape for which the following information was available:

a. Quarterly earnings for the period 1964 through 1973.
b. Year end closing stock prices for the period 1970 through 1973.
c. Fiscal year ending with December.

From these records the following data was generated for each company for each of the four years 1970, 1971, 1972, 1973: year-end closing stock price, fourth quarter dividend payment, and exponentially smoothed annual growth rate of earnings[10]. For each of the four years the industry mean growth rate, $\bar{g}_y = (1/n) \sum_{i=1}^{n} g_{y,i}$ was also calculated.

The growth estimates derived in this way were found by Gordon to be fairly stable in their effect as estimators of dividend growth estimators for the period 1958–68. Over a similar period Higgins estimated the non-normal earnings horizon for the industry to be about four years from stock valuation dates. We assumed that Gordon's and Higgins' findings continued to be valid over the period we studied. Specifically, we assumed that at valuation dates dividends are expected at the firm's estimated growth rate for four years, then assumed the industry average rate. Thus:

$$D_{t,i} = D_{0,i}(1 + g_{y,i})^t \qquad t = 1, \ldots, 4$$

$$D_{t,i} = D_{0,i}(1 + g_{y,i})^t (i + \bar{g}_y)^{t-4} \qquad t > 4$$

where $D_{t,i}$ = per share dividends expected for year t, company i.

10. The precise form of the smoothing function was: Let $E_{y,q,i,}$ = per share earnings in year y, quarter q, company i and $E_{y,i}$ = per share earnings in year y, for company i. Then $g_{y,i}$ = per share earnings growth rate in year y, for company i is calculated recursively, applying the formula:

$$g_{y,i} = .85 g_{y-1,i} + .15[(E_{t,i}/E_{y-1,i}) - 1] y = 1964, \ldots, Y-1$$

$$g_{Y,i} = .85 g_{Y-1,i} + .15\left[\left(\sum_{q=1}^{3} E_{Y,q,i} / \sum_{q=1}^{3} E_{Y-1,q,i}\right) - 1\right]$$

$$g_{1963,i} = .05$$

This procedure uses only information which is available at the date of the observed stock price, i.e., at the end of the year Y.

Under the further assumption that the dividend payments of all firms would be made annually at mid-year appropriate present value coefficients, β_t, were generated from our term structure regression estimates. We further assumed that the term structure from year four onward would be horizontal at the four year rate: $R_t = R_4 = \bar{R}$ for $t = 4, \ldots, \infty$.

The result of the above assumptions was a final valuation model of the form:

$$P_{0,i} = \sum_1^3 \beta_t \alpha^t D_{t,i} + \beta_4 \sum_0^\infty \alpha^{t+4} D_{4,i} (1+\bar{g}_y)^t (1+\bar{R})^{-t}$$

$$= \sum_1^3 \beta_t \alpha^t D_{t,i} + \beta_4 \alpha^4 D_{t,i} (1+\bar{R}) \left[1+\bar{R} - \alpha(1+\bar{g}_y)\right]^{-1} \qquad (6')$$

A two step test of this model was made, using the regression equation

$$V_{0,i} = a + bP_{0,i} + e_i \qquad (7)$$

First, the equation was estimated with α, the certainty equivalent parameter, set equal to unity. This permitted identification and deletion of outliers, about four in each year, whose residuals were remarkably out of line with the rest of their respective samples. In a more thorough investigation this procedure would be suspect. For present demonstration purposes, we felt it was sufficient to insure sample homogeneity. Next, equation (7) was re-estimated iteratively for each year, employing values of α from .92 to 1.0. Table 2 below gives the regression results for each year for that value of α which produced an estimate for $\hat{b}$ equal to 1.0. In no year did the variation of α alter any statistics other than $\hat{b}$ by more than three percent.

TABLE 2

Year End	α	$\hat{a}$	$\hat{b}$	Std. Error of Regression	R^2	No. of Companies
1970	.951	4.14	1.0	4.10	.68	66
1971	.948	3.26	1.0	3.82	.71	66
1972	.944	4.59	1.0	6.11	.45	66
1973	.945	.57	1.0	2.45	.79	65

Our conclusions from this application exercise are: First, the high R^2 values suggest that the Gordon and Higgins models of expectations are applicable to 1970–73 data even though developed on pre-1968 data. Second, in all years the value of $\hat{a}$ was insignificant whatever value of α was used. The most striking result, however, is the constancy of the critical value of α. This suggests that attitudes toward risk/estimates of future risk of dividends did not change much over the period, even though stock prices fluctuated considerably. We may deduce that the changes in stock prices for this industry which occurred between sample dates was due rather to revisions in growth rate expectations and in shifts in the term structure of interest rates.

VIII. Conclusions

At the beginning of this paper it was argued that considerable value would attach to better estimates of the term structure than those derived from yield data. By exploiting the special structure of Treasury and FHLB securities we were able to produce estimates directly, employing ordinary least squares procedures with a minimal set of constraints, certainly relative to other efforts in this area. In general, the results appear reasonable, and at the very least are consistent with the expectations hypothesis itself. Tests for specification error yielded hints of heteroskedasticity and tax effects, not sufficient for us to reject our own model but certainly sufficient for us to reach the traditional conclusion that more research is needed. Finally, the application of our term structure estimates to a traditional valuation model for a much-analyzed industry was crude, perhaps, but demonstrated the kinds of problems to which better term structure estimates can be usefully employed.

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