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THE TAX-ADJUSTED YIELD CURVE

J. HUSTON MCCULLOCH*

IT HAS BEEN DEMONSTRATED, by Robichek and Niebuhr [8], that tax-induced bias can substantially alter the shape of the yield curve if it is constructed from quotations on bonds selling below par. The apparent before tax yield curve can be upward sloping at the same time that the tax-adjusted yield curve is downward sloping. Thus, the inclusion of tax effects can actually reverse qualitative conclusions concerning the direction in which investors expect interest rates to move under the expectations hypothesis.

Furthermore, simple before tax term structure estimation does not satisfactorily explain the market prices of low coupon bonds selling at a discount, because of the long-term capital gains tax advantage on these securities. Treating these bonds as outliers is unsatisfactory, since they constitute the bulk of observations for some maturities.[1]

The present paper modifies our technique for regression fitting the term structure of interest rates, described in an earlier paper [4], to eliminate this tax-induced bias and reconcile observations on high and low coupon bonds.

I. SECURITY PRICES AND THE DISCOUNT FUNCTION

As in our earlier paper, it is most convenient to begin with the discount function $\delta(m)$. In the present paper, this curve gives the present value of \$1.00 *after tax* repayable after m years.

If we could ignore tax effects, the price of a bond with par value 100, coupon rate c, and terminal maturity date m would be given by

$$p = 100\delta(m) + c\int_0^m \delta(\mu)d\mu.^2 \qquad (1)$$

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1. Fisher [1] partially adjusts for this effect by including coupon terms in his yield-curve regression. However, directly fitting the yield curve is not based on the summation principle of equation (1). Weingartner [9] compensates for coupons through a series of approximations in a way that gives a yield curve similar to our regression-produced yield curve in [4]. However, he does not adjust for tax effects. Williams [10, chapters 10 and 20] gives an algorithm for fitting security prices exactly that uses the summation principle, although he too makes no adjustment for taxes. McCallum [3] and Pye [6] treat the effect of capital gains taxation, though not in a term-structure context.

2. As in [4], we assume for the sake of simplicity that coupons arrive in a continuous stream, instead of in semi-annual installments. We therefore interpret p as the quoted "and interest" price, rather than as the "flat" price at which the security actually changes hands. This convention simplifies the analysis and reduces computation time, but introduces a slight inaccuracy, especially in the maturities where bills interface with short-term notes and bonds. (This problem was called to my attention by Allen Lerman.) This difficulty can be easily corrected by replacing the integrals in the

However, when we take taxes into account, we must use three different equations, one for bonds selling below par, one for bonds selling above par, and one for bills.

Let t be the marginal income tax rate, expressed as a fraction of unity, on ordinary income and let t_g be the marginal tax rate on long-term capital gains income. For most practical purposes, we may assume that t_g is one half t.[3]

For a bond selling below par, the coupon payments are taxed at the rate t as they fall due, while the capital gain difference between the face value of 100 and the purchase price is taxable at t_g when the bond is cashed in at maturity. Therefore, assuming that it is held to maturity, its price should be

$$p = c(1 - t)\int_0^m \delta(\mu)d\mu + [100 - t_g(100 - p)]\delta(m).^4 \tag{2}$$

Under current U.S. tax law, if a bond is purchased above par, the owner has the option of amortizing the premium. This option, which is usually exercised, means he may deduct a linearly prorated share of the premium from his ordinary income each year, instead of taking a long-term capital loss on redemption. If the bond is callable after m^c years, the premium may only be amortized to the call date or the maturity date, whichever gives a smaller deduction. In the case of securities like U.S. Treasury bonds, which are callable at par, the maturity date m will always give a smaller deduction. However, it is customary to assume that bonds callable at par and selling above par will be redeemed on the call date. If so, the remaining, unamortized, premium may then be written off against ordinary income. Taking into account the savings from deducting the premium from other taxable income, the price of the bond should be

$$\begin{aligned} p = {} & [c(1 - t) + t(p - 100)/m]\int_0^{m^c} \delta(\mu)d\mu \\ & + [100 + t(p - 100)(m - m^c)/m]\delta(m^c). \end{aligned} \tag{3}$$

Equation (3) can be made valid for non-callable bonds selling above par by setting m^c equal to m.[5]

The discount on a Treasury bill is taxable at ordinary income rates even if it is held more than six months. The tax is payable in the year in which it matures. Therefore a bill purchased at p will repay 100 − t(100 −

text with appropriate sums adding the present value of each semi-annual coupon to that of the principal.

3. Prior to 1970, the maximum rate on long-term capital gains was .25. However, since 1972 the alternative tax rate for long-term capital gains has been .35, so that the alternative method is seldom advantageous to any taxpayers but those in the .70 bracket.

4. If the bond has less than six months to go to maturity, formula (2) still gives the value of the bond to a potential purchaser, but only if t_g is set equal to the short-term capital gains tax rate t.

5. Strictly speaking, (3) is only valid for bonds originally issued at par. However, this is approximately true for all U.S. Treasury bonds.

A tax-induced bias similar to that reported by Robichek and Niebuhr also appears when only quotations on bonds selling above par are used. However, the bias is not nearly as large in magnitude as when bonds are selling below par.

p) after tax at maturity. Its price should then be related to the discount function by

$$p = [100(1 - t) + tp]\delta(m).^{6} \tag{4}$$

II. Regression Fitting the Discount Function

In order to estimate $\delta(m)$ from observations on the prices of n securities, we express it as

$$\delta(m) = 1 + \sum_{j=1}^{k} a_j f_j(m) \tag{5}$$

where the $f_j(m)$ are postulated functions satisfying

$$f_j(0) = 0 \tag{6}$$

and the a_j are unknown parameters to be estimated by a linear regression. Note that condition (6) forces $\delta(0)$ equal to unity. The functional form of (5) is desirable because it makes (2), (3) and (4) amenable to linear regression.

The crucial issues of the choice of k and the selection of the $f_j(m)$ will be addressed in a later section. In the examples below, these functions were chosen so that $\delta(m)$ would follow a "cubic spline," a type of curve that provides great flexibility. However, in the derivations below we will maintain a high level of generality so that the reader may substitute other functional forms if he desires.

Using (5), equations (2), (3), and (4) can all be rewritten for the i-th security in the form

$$b_i p_i - d_i = \sum_{j=1}^{k} a_j [e_{ij} p_i + g_{ij}]. \tag{7}$$

For a bond selling below par, we have

$$b_i = 1 - t_g \tag{8a}$$
$$d_i = 100(1 - t_g) + c_i(1 - t)m_i \tag{8b}$$
$$e_{ij} = t_g f_j(m_i) \tag{8c}$$
$$g_{ij} = 100(1 - t_g)f_j(m_i) + c(1 - t)\int_0^{m_i} f_j(\mu)d\mu. \tag{8d}$$

For a bond selling above par, we have

$$b_i = 1 - t \tag{9a}$$
$$d_i = (100 + c_i m_i^c)(1 - t) \tag{9b}$$
$$e_{ij} = [t(m_i - m_i^c)f_j(m_i^c) + t\int_0^{m_i^c} f_j(\mu)d\mu]/m_i \tag{9c}$$
$$g_{ij} = 100[1 - t(m_i - m_i^c)/m_i]f_j(m_i^c) + [c_i(1 - t) - 100t/m_i]\int_0^{m_i^c} f_j(\mu)d\mu. \tag{9d}$$

6. The price of a bill may be determined from its quoted banker's discount yield r by the familiar formula

$$p = 100 - r \cdot DTM/360,$$

where DTM is the number of days to maturity.

For a bill, we have

$$b_i = 1 - t \tag{10a}$$
$$d_i = 100(1 - t) \tag{10b}$$
$$e_{ij} = t\, f_j(m_i) \tag{10c}$$
$$g_{ij} = 100(1 - t)f_j(m_i). \tag{10d}$$

Since b_i, d_i, e_{ij}, and g_{ij} are computed from c_i, m_i, m_i^c, t, t_g, and our postulated functions $f_j(m)$, we may regard them as known constants for the purposes of our regression.

Equation (7) will in general not hold exactly when n, the number of observations, is greater than k, the number of parameters estimated. For one thing, the value of the security may be different to different investors. All we know is that it lies above $p_i^b - q$ and below $p_i^a + q$, where p_i^b is the bid price, p_i^a is the asked price, and q is the broker's fee. (The value of q is zero for dealer traded U.S. Governments.) Even if we replace p_i by $\bar{p}_i = (p_i^b + p_i^a)/2$ in (7), we would still find an unexplained residual. As in our earlier paper those errors can be caused by transactions costs, callability, convertibility in the case of some corporates, announced exchange privileges in the case of some Governments, ineligibility for commercial bank purchase (an important factor prior to the 1951 Treasury-Federal Reserve Accord), default risk, the ability to be surrendered at par in payment of estate taxes (true of so-called flower bonds), and the rigidity imposed by specifying any particular functional form for the $f_j(m)$. However, one major source of error in our earlier model, namely that caused by the apparent premium on bonds selling below par and therefore subject to the capital gains tax advantage, should now be eliminated. There still will be a substantial error due to the fact that not all investors pay the same marginal tax rates, as we are implicitly assuming. Nevertheless, this error is considerably less than that caused by our previous assumption that each investor pays no taxes at all. Since transactions costs, as reflected by the bid-asked spread and any brokerage fee, are easily quantifiable and are one of the most important single sources of error, we will assume that the standard error of Equation (7)'s discrepancy is proportional to half the spread plus any brokerage fee: $v_i = (p_i^a - p_i^b)/2 + q$. This quantity is the difference between $\bar{p}$ and the upper or lower limit ($p_i^a + q$ or $p_i^b - q$) discussed above. If we divide equation (7) through by this v_i, the errors will therefore all have the same variance σ^2, provided the "p_i" we use is $\bar{p}_i$. Setting

$$y_i = (b_i\bar{p}_i - d_i)/v_i \tag{11}$$

and

$$x_{ij} = (e_{ij}\bar{p}_i + g_{ij})/v_i, \tag{12}$$

equation (7) written in matrix form with an error term added then becomes

$$y = Xa + \epsilon, \tag{13a}$$

where

$$Cov(\epsilon) = \sigma^2 I.^7 \tag{13b}$$

Our first thought would be to estimate a by means of the familiar ordinary least squares formula

$$\hat{a} = (X'X)^{-1}X'y. \tag{14}$$

Unfortunately, $\bar{p}_i$, which is responsible for the unexplained variance in the dependent variable y_i, is also used to calculate the supposedly independent variables x_{ij}. This implies that $\hat{a}$ is an inconsistent estimator of a.

In order to obtain a consistent estimate of a, we may employ the method of instrumental variables. We define

$$z_{ij} = (100e_{ij} + g_{ij})/v_i. \tag{15}$$

This instrumental variable contains no stochastic element, since $\bar{p}_i$ in (12) has been replaced by the par value of 100. The consistent instrumental variable estimate of a is

$$\tilde{a} = (Z'X)^{-1}Z'y.^8 \tag{16}$$

The variance-covariance matrix of $\tilde{a}$ may be estimated consistently by

$$C = \tilde{\sigma}^2(Z'X)^{-1}Z'Z(X'Z)^{-1} \tag{17}$$

where

$$\tilde{\sigma}^2 = \frac{1}{n-k}(y - X\tilde{a})'(y - X\tilde{a}). \tag{18}$$

Note that when the tax rate is zero, $z_{ij} = x_{ij}$, so that the ordinary least squares and instrumental variables estimates of a coincide.

Using data on U.S. Government bills, notes and bonds for the close of June 1965 and the close of July 1973, a was estimated using both ordinary least squares and instrumental variables.[9] The estimates differed perceptibly, but in no case by more than a small fraction of the estimated standard errors of the $\tilde{a}_j$, as given by the square roots of the main diagonal elements of the matrix C defined in (17). In every instance the discrepancy was well less than one-tenth the estimated standard error. Such close agreement suggests both that the inconsistency in ordinary least squares is not serious, and that our instrumental variables Z are relatively efficient. The reason the inconsistency is so small is probably that the fit of our regression equation is very tight, compared to that in

7. When dealing with bonds which are not really homogenous in all respects other than coupon and maturity, some thought might be given to relaxing the assumption of spherical disturbances made in (13b). For instance, $cov(u_i, u_j)$ could be assumed equal to $.5\sigma^2$ if securities i and j had the same issuer, and 0 otherwise. The a_j could then be estimated by Aitken's formula, or by an instrumental variables version of Aitken's formula.

8. See, e.g., Goldberger [2], pp. 284-286.

9. The ordinary income tax rates used were .30 and .50 for 1965, using Salomon Brothers and Hutzler quotations, and the s-minimizing value of .21 for 1973, using Federal Reserve Bank of New York composite quotations.

other applications that involve potential inconsistency. The economic content of our equation is simply that the present value of a future dollar is the same, no matter how it is packaged. We would expect discrepancies to be very small, compared to those in the estimation of the consumption function, for example. Although there would appear to be no practical objection to the employment of ordinary least squares to estimate (13), at least not for "small" values of n (94 in this case), we have used the theoretically preferable instrumental variables approach in the examples given in this paper and in our program for the Treasury.

Figure 1 shows the estimated discount function computed from (5)

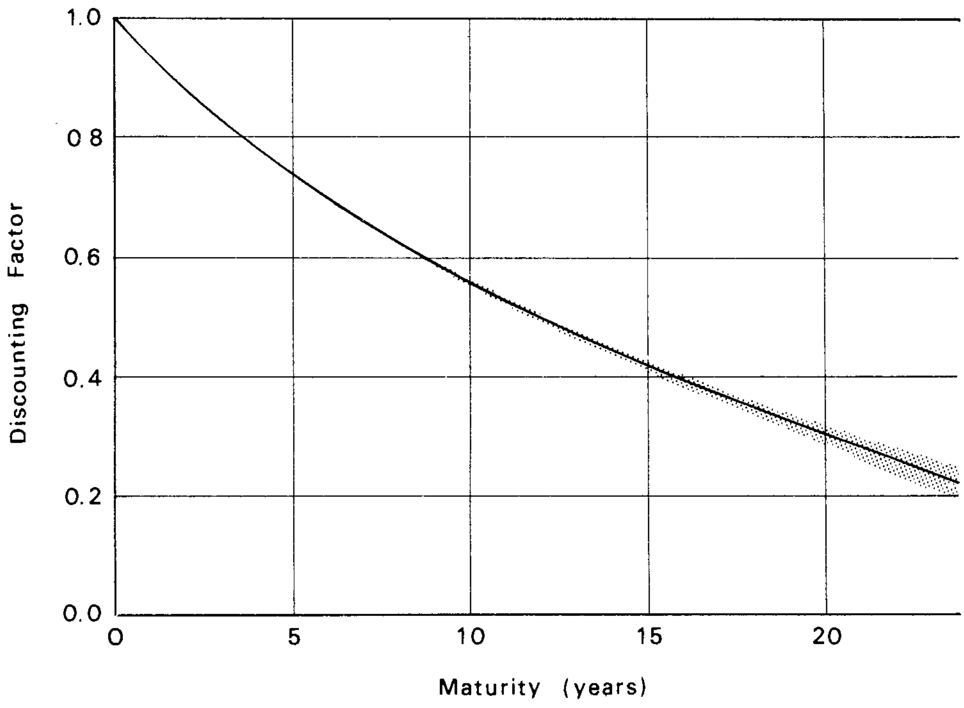

FIGURE 1

The discount function $\delta(m)$, by which after-tax payments are discounted. Based on quotations dated July 31, 1973 for 94 U.S. Government bills, notes, and bonds, using an ordinary tax rate of 0.19 and a long term capital gains rate of 0.095. Ten parameters were estimated. The half-tone band extends one standard error above and below the point estimate. The s of this regression was 2.82.

using estimates of a based on dealer quotations for 94 U.S. Government bills, notes and bonds. The quotations used were those of Salomon Brothers and Hutzler, Inc., dated July 31, 1973, for delivery on August 2, 1973. The tax rates used were $t = .19$ and $t_g = .095$. The curve was plotted with a band extending one standard error above and below the best estimate. However, these errors are so small, relative to the resolution of the diagram, that the symbol employed to represent the band does not appear for short maturities.

III. Predicted Security Values

Given estimates of the parameters a, the value of a given security can be estimated by solving (7) for p_i:

$$\tilde{p}_i = \frac{d_i + \sum_{j=1}^{k} \tilde{a}_j g_{ij}}{b_i - \sum_{j=1}^{k} \tilde{a}_j e_{ij}}. \tag{19}$$

This value is the bid-asked mean price the security would command if it were valued on the same basis as the other securities in the regression.

Table 1 shows predicted prices based on the same regression that produced the discount function of Figure 1. The table gives p_i^b, p_i^a, $\bar{p}_i$ and $\tilde{p}_i$, and the weighted error $(\bar{p}_i - \tilde{p}_i)/v_i$. The regression used only the 94 securities marked with a "T" in the last column.

The Federal National Mortgage Association 6½ of 6/10/77 was excluded since it is a security of a former U.S. Government Agency, rather than an actual Treasury issue, and therefore may not be valued on the same basis as the other securities. Using the discounting factors appropriate for Treasury securities, our formulas calculate that it would be worth \$.96 per \$100 of par value more than its actual bid-asked mean price, or 1.92 times its half-spread v_i. Since a weighted error of 1.92 is small compared to the average of 2.82 for securities included in the regression, we may conclude that the market is not discriminating greatly, if at all, against this particular issue as compared with actual Treasury issues.

IV. Estate Tax Bonds

Most of the older outstanding marketable Treasury bonds are acceptable at par in payment of estate taxes if owned by the decedent at the time of his death. These estate bonds, or flower bonds as they are called, all were issued before 1963. They therefore happen to have coupons lower than current interest rates, and so are worth less than par. If the marginal holder of any of these bonds is planning to utilize the estate tax privilege directly or indirectly by selling it to someone who will exercise this option, the bonds may command a premium. Except in special cases when there are bonds without the privilege with the same coupon and similar maturities, the size of this premium is difficult to measure. Other bonds with a similar terminal maturity but a higher coupon rate have a smaller capital gains tax advantage since they sell closer to par. Furthermore, they reflect shorter term interest rates, since the coupons comprise a larger part of their value. However, our curve fitting technique is specifically designed to adjust for tax effects while allowing different discount rates for the coupons and principal. It may therefore be used to measure the value of this premium.

TABLE 1
PREDICTED VALUES
N = 94 K = 10 SE = 2.82
BASED ON ORDINARY INCOME TAX RATE OF .19, LTCG RATE OF .095

Coupon Rate	Maturity Date	Call Date	Estimated Redem. Date	DTM	Bid Price	Asked Price	Mean Price	Predicted Price, SE	Absolute Error	Weighted Error		Included
.000				7	99.835	99.849	99.842	99.841 ± .013	.001	.09		T
3.125	8 15 73		8 15 73		99.875	100.000	99.937	99.995 ± .014	−.057	−.92	Note	T
4.000	8 15 73		9 15 73		99.750	99.875	99.812	99.848 ± .014	−.036	−.57		T
.000				14	99.669	99.699	99.684	99.638 ± .015	.001	.05		T
.000				21	99.522	99.539	99.530	99.52 ± .016	.005	.55		T
.000				26	99.401	99.422	99.411	99.413 ± .016	−.002	−.16		T
.000				28	99.354	99.378	99.366	99.368 ± .015	−.002	−.17		T
.000				35	99.193	99.222	99.208	99.210 ± .016	−.003	−.19		T
.000				42	99.032	99.067	99.049	99.053 ± .018	−.003	−.19		T
.000				49	98.870	98.911	98.891	98.894 ± .021	−.004	−.17		T
.000				54	98.755	98.800	98.778	98.781 ± .022	−.003	−.14		T
.000				56	98.709	98.756	98.732	98.735 ± .023	−.003	−.13		T
1.500	10 1 73		10 1 73		98.000	99.000	98.500	98.892 ± .023	−.392	−.78	Note	T
.000				63	98.548	98.600	98.574	98.575 ± .023	−.002	−.06		T
.000				70	98.386	98.444	98.415	98.415 ± .021	.000	.01		T
.000				77	98.257	98.278	98.268	98.254 ± .018	.014	1.28		T
.000				82	98.109	98.155	98.132	98.139 ± .017	−.006	−.28		T
.000				84	98.087	98.110	98.098	98.092 ± .017	.006	.51		T
.000				91	97.897	97.922	97.910	97.931 ± .019	−.021	−1.67		T
.000				98	97.741	97.795	97.768	97.769 ± .022	−.001	−.04		T
4.125	11 15 73		11 15 73		98.625	98.750	98.687	98.786 ± .026	−.098	−1.57	Estate	T
.000				105	97.579	97.638	97.608	97.607 ± .026	.001	.05		T
.000				110	97.464	97.525	97.494	97.491 ± .028	.003	.10		T
.000				113	97.395	97.458	97.426	97.422 ± .029	.004	.12		T
.000				119	97.256	97.323	97.289	97.284 ± .031	.006	.17		T
.000				126	97.095	97.165	97.130	97.123 ± .031	.007	.21		T
.000				133	96.934	97.008	96.971	96.961 ± .030	.009	.25		T
.000				138	96.818	96.895	96.857	96.846 ± .029	.011	.28		T

TABLE 1 (Continued)

PREDICTED VALUES

N = 94 K = 10 SE = 2.82

BASED ON ORDINARY INCOME TAX RATE OF .19, LTCG RATE OF .095

Coupon Rate	Maturity Date	Call Date	Estimated Redem. Date	DTM	Bid Price	Asked Price	Mean Price	Predicted Price, SE	Absolute Error	Weighted Error		Included
.000				140	96.772	96.850	96.811	96.800 ± .029	.011	.29		T
.000				147	96.611	96.693	96.652	96.638 ± .028	.014	.33		T
.000				154	96.449	96.535	96.492	96.476 ± .028	.017	.39		T
.000				161	96.310	96.400	96.355	96.313 ± .029	.043	.95		T
.000				166	96.196	96.280	96.242	96.195 ± .031	.046	1.01		T
.000				168	96.150	96.197	96.173	96.149 ± .031	.025	1.06		T
.000				175	95.965	96.014	95.990	95.984 ± .034	.006	.25		T
.000				182	95.753	95.804	95.779	95.817 ± .037	−.039	−1.53		T
.000				194	95.473	95.581	95.527	95.530 ± .042	−.003	−.05		T
7.750	2 15 74		2 15 74		99.312	99.437	99.375	99.581 ± .038	−.206	−3.30	Note	T
4.125	2 15 74		2 15 74		97.625	97.750	97.687	97.882 ± .038	−.194	−3.11	Estate	T
.000				222	94.820	94.943	94.882	94.850 ± .047	.031	.51		T
1.500	4 1 74		4 1 74		94.000	96.000	95.000	95.783 ± .044	−.783	−.78	Note	T
.000				250	94.236	94.375	94.306	94.166 ± .049	.140	2.02		T
.000				278	93.513	93.668	93.591	93.484 ± .051	.106	1.38		T
7.250	5 15 74		5 15 74		98.625	98.750	98.687	98.868 ± .049	−.181	−2.89	Note	T
4.250	5 15 74		5 15 74		96.625	96.750	96.687	96.827 ± .049	−.140	−2.24	Estate	T
.000				306	92.979	93.064	93.021	92.815 ± .059	.206	4.86		T
.000				334	92.225	92.318	92.272	92.167 ± .069	.105	2.26		T
5.625	8 15 74		8 15 74		96.875	97.125	97.000	97.042 ± .077	−.042	−.34	Note	T
6.000	9 30 74		9 30 74		96.937	97.062	97.000	97.135 ± .084	−.135	−2.16	Note	T
1.500	10 1 74		10 1 74		92.000	94.000	93.000	92.668 ± .083	.332	.33	Note	T
5.750	11 15 74		11 15 74		96.375	96.625	96.500	96.667 ± .087	−.167	−1.33	Note	T
3.875	11 15 74		11 15 74		94.437	94.687	94.562	94.624 ± .086	−.061	−.49	Estate	T
5.875	12 31 74		12 31 74		96.375	96.625	96.500	96.628 ± .087	−.128	−1.03	Note	T
5.875	2 15 75		2 15 75		96.312	96.562	96.437	96.487 ± .088	−.049	−.39	Note	T
5.750	2 15 75		2 15 75		96.125	96.375	96.250	96.326 ± .088	−.076	−.60	Note	T
1.500	4 1 75		4 1 75		90.000	92.000	91.000	90.280 ± .090	.720	.72	Note	T

TABLE 1 (Continued)
PREDICTED VALUES
N = 94 K = 10 SE = 2.82
BASED ON ORDINARY INCOME TAX RATE OF .19, LTCG RATE OF .095

Coupon Rate	Maturity Date	Call Date	Estimated Redem. Date	DTM	Bid Price	Asked Price	Mean Price	Predicted Price, SE	Absolute Error	Weighted Error		Included
6.000	5 15 75		5 15 75		96.312	96.562	96.437	96.430 ± .099	.008	.06	Note	T
5.875	5 15 75		5 15 75		96.125	96.375	96.250	96.244 ± .099	.006	.05	Note	T
5.785	8 15 75		8 15 75		95.687	95.937	95.812	95.853 ± .115	−.040	−.32	Note	T
1.500	10 1 75		10 1 75		88.000	90.000	89.000	88.105 ± .118	.895	.89	Note	T
7.000	11 15 75		11 15 75		97.937	98.187	98.062	97.818 ± .124	.244	1.95	Note	T
6.250	2 15 76		2 15 76		96.250	96.500	96.375	96.169 ± .121	.206	1.65	Note	T
5.875	2 15 76		2 15 76		95.062	95.313	95.187	95.400 ± .121	−.213	−1.70	Note	T
1.500	4 1 76		4 1 76		86.000	88.000	87.000	85.845 ± .113	1.155	1.15	Note	T
6.500	5 15 76		5 15 76		96.312	96.562	96.437	96.451 ± .114	−.014	−.11	Note	T
5.750	5 15 76		5 15 76		94.437	94.687	94.562	94.777 ± .114	−.215	−1.72	Note	T
7.500	8 15 76		8 15 76		98.687	98.937	98.812	98.618 ± .110	.194	1.55	Note	T
6.500	8 15 76		8 15 76		96.125	96.375	96.250	96.207 ± .109	.043	.34	Note	T
1.500	10 1 76		10 1 76		84.500	86.500	85.500	83.574 ± .103	1.926	1.93	Note	T
6.250	11 15 76		11 15 76		95.125	95.375	95.250	95.316 ± .111	−.066	−.53	Note	T
8.000	2 15 77		2 15 77		100.062	100.312	100.187	99.852 ± .135	.336	2.69	Note	F
1.500	4 1 77		4 1 77		83.000	85.000	84.000	81.390 ± .120	2.610	2.61	Note	T
6.500	6 10 77		6 10 77		94.000	95.000	94.500	95.461 ± .134	−.961	−1.92	FNMA	F
7.750	8 15 77		8 15 77		98.937	99.187	99.062	99.190 ± .143	−.127	−1.02	Note	T
1.500	10 1 77		10 1 77		81.500	83.500	82.500	79.345 ± .138	3.155	3.16	Note	T
6.250	2 15 78		2 15 78		94.125	94.375	94.250	94.139 ± .153	.111	.88	Note	T
1.500	4 1 78		4 1 78		80.000	82.000	81.000	77.431 ± .146	3.569	3.57	Note	T
6.000	11 15 78		11 15 78		92.250	92.500	92.375	92.682 ± .152	−.307	−2.46	Note	T
6.250	8 15 79		8 15 79		93.812	94.062	93.937	93.294 ± .143	.644	5.15	Note	T
6.625	11 15 79		11 15 79		94.937	95.187	95.062	94.872 ± .144	.191	1.52	Note	T
4.000	2 15 80		2 15 80		81.750	82.250	82.000	82.742 ± .140	−.742	−2.97	Estate	T
6.875	5 15 80		5 15 80		95.125	95.375	95.250	95.997 ± .154	−.747	−5.98	Note	T
3.500	11 15 80		11 15 80		77.625	78.125	77.875	79.096 ± .167	−1.221	−4.88	Estate	T
7.000	8 15 81		8 15 81		99.000	99.500	99.250	96.774 ± .223	2.476	9.90		T

TABLE 1 (Continued)
PREDICTED VALUES
N = 94 K = 10 SE = 2.82
BASED ON ORDINARY INCOME TAX RATE OF .19, LTCG RATE OF .095

Coupon Rate	Maturity Date	Call Date	Estimated Redem. Date	DTM	Bid Price	Asked Price	Mean Price	Predicted Price, SE	Absolute Error	Weighted Error		Included
6.375	2 15 82		2 15 82		93.625	94.125	93.875	93.316 ± .251	.559	2.24		T
3.250	6 15 83	78	6 15 83		70.375	71.375	70.875	73.446 ± .287	−2.571	−5.14	Estate	T
6.375	8 15 84		8 15 84		92.375	92.875	92.625	92.910 ± .331	−.285	−1.14		T
4.250	5 15 85	75	5 15 85		75.875	76.875	76.375	77.660 ± .316	−1.285	−2.57	Estate	T
3.250	5 15 85		5 15 85		70.375	71.375	70.875	70.546 ± .306	.329	.66	Estate	T
6.125	11 15 86		11 15 85		90.750	91.250	91.000	90.560 ± .346	.440	1.76		T
3.500	2 15 90		2 15 90		69.875	70.875	70.375	66.394 ± .380	3.981	7.96	Estate	T
4.250	8 15 92	87	8 15 92		71.250	72.250	71.750	70.532 ± .447	1.218	2.44	Estate	T
4.000	2 15 93	88	2 15 93		71.000	72.000	71.500	67.629 ± .444	3.871	7.74	Estate	T
6.750	2 15 93		2 15 93		91.125	91.625	91.375	93.997 ± .498	−2.622	−10.49		T
4.125	5 15 94	89	5 15 94		70.125	71.125	70.625	67.486 ± .436	3.139	6.28	Estate	T
3.000	2 15 95		2 15 95		69.750	70.750	70.250	55.358 ± .400	14.892	29.78	Estate	F
7.000	5 15 98	93	5 15 98		92.875	93.375	93.125	93.113 ± .671	.012	.05		T
3.500	11 15 98		11 15 98		69.875	70.875	70.375	55.239 ± .741	15.136	30.27	Estate	F

The two longest maturity, most deeply discounted flower bonds, namely the 3's of 1995 and 3½'s of 1998, definitely carry a premium and therefore were excluded from the regression. As may be seen from Table 1, the coupon and principal of these two bonds are worth only about \$55, and yet they are selling for about \$70 each. The value of the estate privilege is therefore about \$15. The three next lowest priced estate bonds, the 3½'s of 1990, and 4's of 1993/88, and the 3's of 1995, appear to carry premiums of \$3.14 to \$3.98. Since this size of discrepancy is not large, relative to the half spread, when compared to that observed for a few of the other bonds included in the regression, these three were not excluded. However, this discrepancy probably does represent a true premium due to the estate feature.

V. The Bond Yield Curve

A frequently asked question that we may answer with our estimated discount function is, "What coupon rate would a bond with terminal maturity m have to have in order to sell at par?" We call y(m), the function that answers this question, the "*bond yield curve.*" Bonds selling at par (but not necessarily bonds selling at other than par, even in the absence of taxation) should lie on this curve.

This function must obey the following equation:

$$(1 - t)y(m)\int_0^m \delta(\mu)d\mu + 100\delta(m) = 100. \tag{20}$$

Solving for y(m), we get

$$y(m) = \frac{100[1 - \delta(m)]}{(1 - t)\int_0^m \delta(\mu)d\mu}. \tag{21}$$

This expression may be estimated, using the estimated discount function and integrating (5) with respect to m. Figure 2 shows the bond yield curve corresponding to the discount curve of Figure 1. When a zero tax rate was tried in place of the 19 per cent rate used for these diagrams, the curve had the same qualitative shape. However, it was generally lower, hitting a minimum of 7.16 per cent at 15 years instead of 7.33 per cent, and turned up more at the long end.

VI. The Point Payment Yield Curve and Forward Interest Rates

Although the bond yield curve y(m) is what financiers think of as "the" yield curve, economists who are concerned with the relation between forward interest rates and subsequent spot rates usually prefer to work in terms of a *point payment yield curve* $\eta(m)$ that gives the yield to maturity on a hypothetical pure discount bond that pays no coupons prior to final maturity. Early theorists, such as Irving Fisher and Hicks, simply assumed yields on such bonds could be observed. Later workers, including Meiselman and Kessel, perforce used observations from the

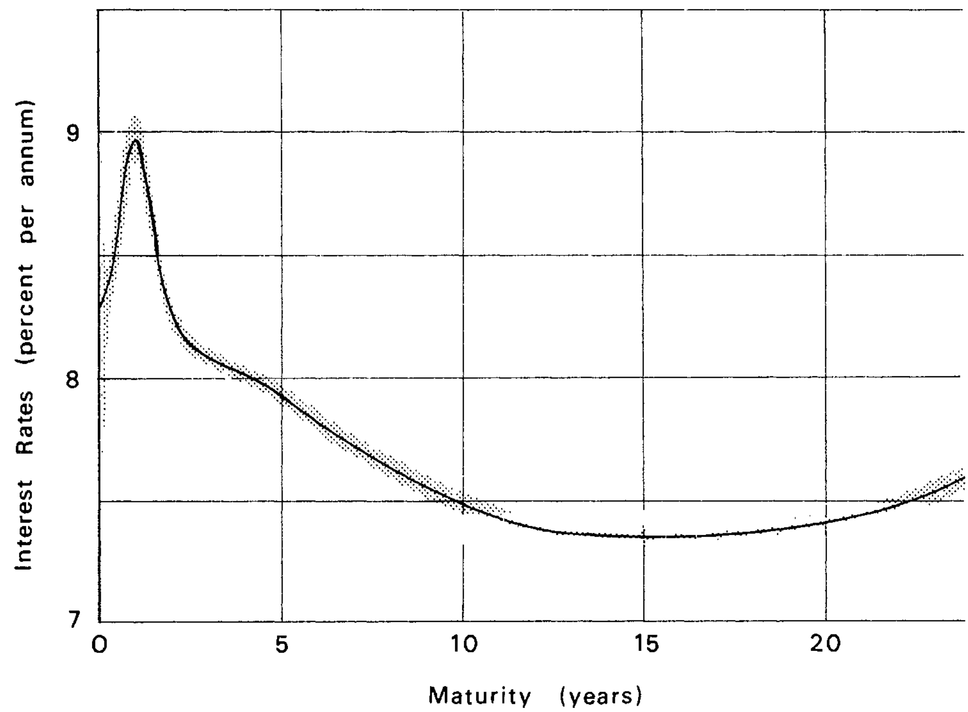

FIGURE 2
Bond yield curve $y(m)$, which gives the coupon rate a bond of maturity m would have to have in order to sell at par. This curve is based on the discount function of Figure 1.

bond yield curve except in the short maturities where bills were available, while commenting that it would be preferable to have pure discount yields. On an after tax basis, this point yield would be given by $-100 \ln \delta(m)/m$, using continuous compounding. However, we are not always certain just which tax rate is the one to use. In order to make intertemporal comparisons of forward rates corresponding to this point yield curve less sensitive to possible error in the estimated tax rate, it is desirable to convert the point yield to a before tax basis by dividing through by $1 - t$:

$$\eta(m) = \frac{-100}{m(1-t)} \ln\delta(m). \tag{22}$$

This curve gives the before tax yield such a hypothetical bond would have if the interest on it, as reflected by its discount, were taxed at the ordinary income tax rate as the bond appreciated in value. This does not correspond to actual tax practice, but then this curve does not correspond to actual securities either. The point payment yield curve may be estimated by substituting the estimated discount function into (22).

Corresponding to the point payment yield curve is the *instantaneous forward rate curve*:

$$\rho(m) = \frac{-100\delta'(m)}{(1-t)\delta(m)}. \tag{23}$$

It can be estimated using the estimated discount function and differentiating (5) with respect to m. This curve gives the implicit before tax forward rate on a hypothetical forward loan to begin in m years and end an instant later, assuming the interest would be taxed at ordinary income rates.

Figure 3 shows the estimated $\tilde{\eta}(m)$ and $\tilde{\rho}(m)$ corresponding to Figures 1 and 2. The vertical scale is more compressed than in Figure 2, in order to

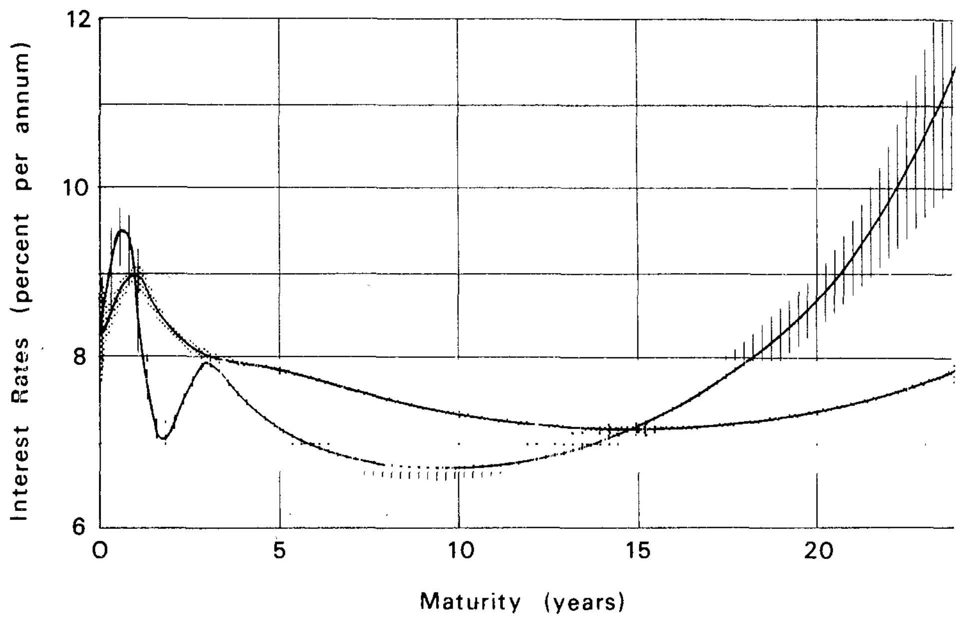

FIGURE 3

Point payment yield curve $\eta(m)$ (with stippled band) and instantaneous forward rate curve $\rho(m)$ (with hatched band). The bands extend one standard error above and below the point estimates.

fit all the forward rates in. The basic shape of $\tilde{\eta}(m)$ can be seen to be very similar to that of $\tilde{y}(m)$. This is to be expected, since the bond yield curve is a smoothed version of the point payment yield curve. The high forward rates at the long end are implied by the upturn in the yield curve in these maturities. However, the half-tone band extending one standard error above and below the best estimate curve indicates that these high forward rates do not have a high degree of accuracy. Figure 4 shows the first two years of Figure 3 in greater detail.

As m approaches 0, y(m) and $\eta(m)$ both approach $\rho(m)$. Therefore, although (21) and (22) cannot be evaluated at m = 0, the common value of y(0) and $\eta(0)$ may be calculated from (23). This value may be interpreted as a hypothetical "call money" rate.

The *mean forward rate* on a point payment loan to begin at time m_1 and have duration m_2 (that is, to end at time $m_3 = m_1 + m_2$) is given by averaging together values of $\rho(m)$:

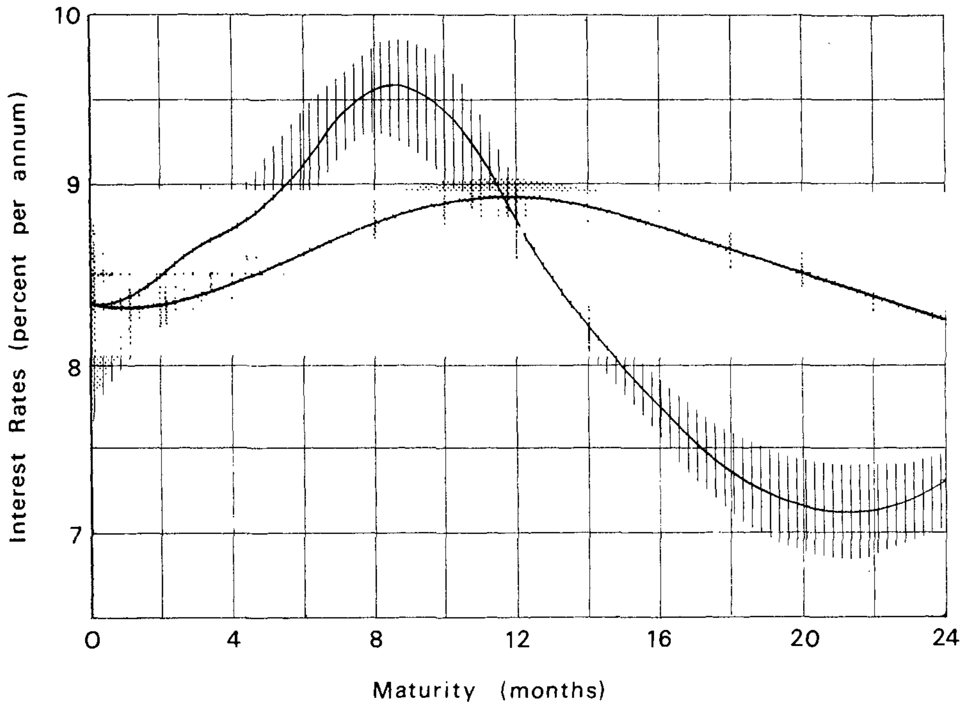

FIGURE 4
Detail of the first two years of Figure 3.

$$r(m_1, m_3) = \frac{1}{m_2}\int_{m_1}^{m_3} \rho(m)dm \tag{24}$$

$$= \frac{100}{(1 - t)m_2} \ln \frac{\delta(m_1)}{\delta(m_3)}. \tag{25}$$

Expression (25) may be estimated using (5).

Mean forward rates, with m_2 equal to one "period" (usually a week, month, quarter or year), are what economists usually think of as "forward interest rates." However, a financier might find a *"forward bond yield,"* $b(m_1, m_3)$ a more useful forward rate concept. This is the coupon rate a bond issued m_1 years from now and maturing after m_2 years (or $m_3 = m_1 + m_2$ years from now) would have to have in order for the present discounted value of its after-tax payments to just equal the present discounted value of its par value on its issue date. This means it must satisfy the following equation:

$$100\delta(m_1) = b(m_1, m_3)(1 - t)\int_{m_1}^{m_3}\delta(m)dm + 100\delta(m_3). \tag{26}$$

Solving, we get

$$b(m_1, m_3) = \frac{100[\delta(m_1) - \delta(m_3)]}{(1 - t)\int_{m_1}^{m_3}\delta(m)dm}. \tag{27}$$

Under the expectations hypothesis, this forward bond yield is the coupon rate that the market would expect a bond with maturity m_2 years to have to have in order to sell at par if issued m_1 years from now. If $m_1 = 0$, it becomes identical to the spot bond yield $y(m_2)$.

VII. The Variance-Minimizing Tax Rate

One unsatisfactory aspect of fitting the term structure without taking taxes into account is that it fails to account for the prices of deep discount bonds. We may measure how closely we have fit the observed prices by calculating the following expression:

$$s = \sqrt{\frac{1}{n-k}\sum_{i=1}^{n}\frac{\bar{p}_i - \tilde{p}_i}{v_i}}. \tag{28}$$

This s is the root mean square of the weighted errors shown in Table 1.[10]

Of course, not all investors are in the same marginal income tax bracket. Some are tax-exempt institutions, some are large corporations paying 48 per cent, and some are individuals paying a variety of personal income tax rates. Furthermore, investors in different brackets will find different securities attractive. Individuals in the highest brackets will probably hold tax-exempt state and municipal bonds. Those in moderately high brackets will prefer deep discounts. Those in low brackets will be attracted by the high yield on taxables near par.

Nevertheless, if we want one tax rate that best explains the whole structure of prices for the securities observed, it would make most sense to use the rate that minimizes s. This s-minimizing tax rate t^* may to some extent be regarded as an effective tax rate, the approximate rate at which the Treasury recaptures its interest payments when it floats new debt.[11]

For some points in time, s is not very sensitive to t. At the close of June 1950, for instance, s fell from 20.02 at $t = 0$ to 19.99 at $t^* = .13$. At that time, all bonds were selling above par, so that the capital gains tax advantage did not come into play. Instead, the major source of variance was the ineligibility of some bonds for commercial bank purchase, a factor not taken into account by our formulas. At such times, an estimate of t^* cannot be regarded as very accurate.

However, when there are many bonds selling at a deep discount with a small bid-asked spread, s can be very sensitive to t. The close of June 1963, June 1964, June 1965, and March 1966 were such times. At each of

10. When $t = 0$, s coincides exactly with $\bar{\sigma}$ from equation (18). In our experience, s has always been larger than $\bar{\sigma}$ for $t > 0$. For $t > .50$, the difference can become appreciable. As t increases from 0, s and $\bar{\sigma}$ usually fall off smoothly, pass a minimum, and then rise. By .60, they are usually considerably larger than their common value at 0.

11. This interpretation is only valid to the extent that the tax rate is not serving as a proxy for other related factors, such as estate tax features, or (as was pointed out by Fischer Black) call protection. However, bonds with a conspicuous flower premium may be excluded from the regression, while the call protection factor is probably small compared with other sources of error.

these dates, the variance of the residuals (i.e., s^2) fell by at least 73 per cent. Values of t^* were found ranging from .22 to .30. For June 1965, s fell from 11.51 at $t = 0$ to a respectable value of 3.31 at $t^* = .27$, a decline of 92 per cent in s^2. These t^* values probably give a fairly accurate indication of the effective marginal tax rate that governs U.S. Treasury security prices, and the average tax rebate to the Treasury on its coupon payments.[12]

Using quotations for July 31, 1973, s falls from 3.31 at $t = 0$ to 2.82 at $t^* = .19$. Since bid-asked spreads on long-term deep discount bonds are considerably wider than they were in 1965, t is not now as critical a parameter as formerly.

VIII. Measurement Errors

The parameters a may only be estimated with some uncertainty, because the fit of our regression is not exact. Therefore the various quantities derived from these parameters are also uncertain. It is worth our while to have an index of this uncertainty, lest we jump to conclusions.

Given C, the estimated variance-covariance matrix of $\tilde{a}$, the variance of a differentiable function $\psi(\tilde{a})$ may be approximated by the quadratic form

$$\text{var}\ \psi(\tilde{a}) \approx w'Cw, \tag{29a}$$

where

$$w_j = \partial\psi(\tilde{a})/\partial\tilde{a}_j \tag{29b}$$

[2, pp. 122-125]. Standard errors may then be approximated by taking square roots.

The particular formulas for the variance of our estimates of the discount function and the various forward rate and yield curve concepts may be derived from their definitions and (29). They are also available from the author on request.

IX. The Form of the Functions $f_j(m)$

A broad family of approximating functions, known as "splines," are useful in many curve-fitting applications [7, Vol. II, 123-167; 5]. An r-degree spline is piecewise an r-degree polynomial, with $r - 1$ continuous derivatives. Its r-th derivative is therefore a step-function.

If the "knot" points, where the discontinuities in the r-th derivative occur, are equally spread, the spline will be able to fit equally complex shapes for all values of the abscissa. If they are unequally spaced, the spline will be able to fit more complex shapes where the knots are closest

12. These values are in the same range as Pye's C(0, 1) estimates which equalize the net return on securities trading at par with that on securities whose coupon rates are 1 to 2 per cent below their yields [6, 578]. However, Pye obtains higher estimates of the effective marginal tax rate using other comparisons such as taxables versus tax exempts and moderate discounts versus 1½ per cent exchange notes. His C(0, 1) estimates are most nearly comparable to our estimates, because our regression gives little weight to the exchange notes and we have not included tax exempts.

together. Since the high concentration of bill and note maturities at the short end allows us to distinguish considerably more detailed shape there in the discount function, and indirectly in the yield and forward curves, a spline seems ideal to our purpose, provided we space the knots so that an equal number of issues fall between adjacent knots.

In our earlier paper, a quadratic spline was employed. In the present paper, we have used a cubic spline, because it gives a smoother shape to the forward curve. Although we still regard a quadratic spline as satisfactory, it tends to produce an awkward scalloped shape in the forward curve.

An ordinary polynomial is unsatisfactory when used with quotations that are bunched at short maturities and sparse at the long end. When a 10-th degree polynomial was used with 1965 data, it over-smoothed at the short end and failed to pick up a distinct upward slope in the first few months of the yield curve. On the other hand, it under-smoothed at the long end. It conformed too closely to the two longest bonds, in such a way that the resulting discount curve implied forward rates ranging from −9 up to 57 per cent per annum! A 10-parameter third degree spline, on the other hand, clearly showed the upward slope at the short end of the yield curve, and gave credible forward rates, all between 3.5 and 4.5 per cent. In all fairness to the polynomial, it should be noted that it does a better job of fitting a pure exponential decay discount function, implied by artificial data reflecting a perfectly flat yield curve, than does the cubic spline. It is only with real-world data that the polynomial does worse.

Formulas for the $f_j(m)$ which generate a cubic spline are given in Appendix A.

As in our earlier paper, we set k equal to the nearest integer to the square root of n. This formula has the desirable properties that both k and the ratio n/k will become large as n becomes large, giving at once greater resolution and greater accuracy as more information is added. For a cubic spline, we must have k at least equal to 3, but ordinarily we will have well over 9 observations, so this will be no problem.

X. Conclusion

Taking the effects of differential taxation of ordinary and long-term capital gains income into account can reduce the unexplained variance when we regression curve-fit the term structure of interest rates by as much as 92 per cent. The resulting tax-adjusted bond yield curve should be free of the tax-induced bias discussed by Robichek and Niebuhr, and should give an accurate estimate of the coupon rate necessary to float new debt at par. The forward interest rates associated with the point payment yield curve will also be free from these distortions, which otherwise could actually reverse inferences about the direction in which the market expects interest rates to move. By searching over tax rates, we have found that the effective tax rate that best explains the prices of U.S. Treasury securities lies somewhere in the range .22 to .30.

APPENDIX

To define a k-parameter cubic spline, we need to fix $k - 1$ knot points d_j, instead of k as with the quadratic spline. Assume that the redemption date m_i of the n securities are arranged in ascending order. Set

$$d_j = m_h + \theta(m_{h+1} - m_h) \tag{A.1a}$$

where

$$h = \text{greatest integer in } \frac{(j-1)n}{k-2} \tag{A.1b}$$

and

$$\theta = \frac{(j-1)n}{k-2} - h. \tag{A.1c}$$

This specification places an equal number of security maturities between adjacent knots. It sets $d_1 = 0$ and $d_{k-1} = m_n$.

In order to generate the family of cubic splines relative to these knots, we define for $m < d_{j-1}$

$$f_j(m) = 0. \tag{A.2}$$

For $d_{j-1} \leqslant m < d_j$, we define

$$f_j(m) = \frac{(m - d_{j-1})^3}{6(d_j - d_{j-1})}. \tag{A.3}$$

When $d_j \leqslant m < d_{j+1}$, we define

$$f_j(m) = \frac{c^2}{6} + \frac{ce}{2} + \frac{e^2}{2} - \frac{e^3}{6(d_{j+1} - d_j)} \tag{A.4a}$$

where

$$c = d_j - d_{j-1} \tag{A.4b}$$

and

$$e = m - d_j. \tag{A.4c}$$

For $d_{j+1} \leqslant m$, define

$$f_j(m) = (d_{j+1} - d_{j-1})\left[\frac{2d_{j+1} - d_j - d_{j-1}}{6} + \frac{m - d_{j+1}}{2}\right]. \tag{A.5}$$

(Set $d_{j-1} = d_j = 0$ when $j = 1$.)

The above formulas apply when $j < k$. When $j = k$, we define

$$f_k(m) = m, \tag{A.6}$$

regardless of m.

These formulas may be integrated and differentiated with respect to m in order to evaluate the bond yield curve and forward curve formulas.[13]

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