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Integrating market and credit risk: A simulation and optimisation perspective

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Abstract

We introduce a modelling paradigm which integrates credit risk and market risk in describing the random dynamical behaviour of the underlying fixed income assets. We then consider an asset and liability management (ALM) problem and develop a multistage stochastic programming model which focuses on *optimum risk decisions*. These models exploit the dynamical multiperiod structure of credit risk and provide insight into the corrective recourse decisions whereby issues such as *the timing risk of default* is appropriately taken into consideration. We also present an index tracking model in which risk is measured (and optimised) by the CVaR of the tracking portfolio in relation to the index. In-sample as well as out-of-sample (backtesting) experiments are undertaken to validate our approach. The main benefits of backtesting, that is, ex-post analysis are that (a) we gain insight into asset allocation decisions, and (b) we are able to demonstrate the feasibility and flexibility of the chosen framework.

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1. Introduction

Credit markets have experienced significant growth over the last few years in Europe and in the US. The BAA Credit Derivatives Report 2002 estimated the size of the global market (excluding asset swaps) over one trillion $ by the end of 2001. They also estimate it to grow to $4.8 trillion by the end of 2004, with London being the dominant center in global credit derivative markets.

As a consequence considerable amount of research has taken place on the valuation of financial instruments which are exposed to credit risk, see Schönbucher (2003) for a comprehensive treatment of credit derivative pricing models.

In addition to valuation models, several approaches for managing portfolios of credit risky exposures emerged (see, for example, the CreditMetrics model (Risk Metrics Group, 1997). By capturing marginal default risk and dependence, these models attempt to give an indication of the total portfolio risk.

Overall, practical credit risk management incorporates many of the features that rely heavily on security valuation models. An example is the timing of defaults that lead to a liquidation of positions if default seems unavoidable under a given set of scenarios. Hence, modern credit risk management tools need to go beyond the traditional approaches of risk control that limit (arbitrarily) sector, geographical or issuer exposures. Quantitative approaches to credit risk management that allow portfolio managers to quantify the overall risk in their positions, and in particular optimisation models (e.g. portfolio optimisation models under credit risk) are still at the early stages of their development. While clearly the tail of the overall credit loss distribution of a portfolio of obligors is critical, practical estimation of tail risk measures or downside risk measures is challenging.

According to Ramaswamy (2002) diversification of credit risk is much more difficult than that market risk, which is particularly true in case of the risk of overexposure to a particular issuer or industry (concentration risk). To avoid this, the traditional approach has been to include a large number of issuers in the portfolio. However, such a strategy is not based on any efficient quantitative framework and implies considerable transactions costs.

The literature on portfolio optimisation under credit risk is not extensive and only recently practitioners and academics have started to investigate a number of alternative approaches. Some of these methods build on the mean–variance analysis of Markowitz where the mean and variance refer to the actual loss distribution of the portfolio (e.g. Kealhofer, 1998; Ramaswamy, 2002, or Dynkin et al., 2001). For a general discussion of portfolio construction using alternative risk measures the readers are referred to Mitra et al. (2003). However, measuring the risk of downgrading and default by the standard deviation (of losses) does not allow any statement about the worst losses or the severity of losses at a certain percentile of the return/loss distribution. Such a model only reduces the standard deviation, hoping that the chance of catastrophic events is reduced, too. It is well known that standard deviation penalizes positive as well as negative deviations equally and is less suitable for asymmetric distributions. These approaches may have an immediate appeal to practitioners who are familiar with the mean–variance framework, however, optimisation methods based on downside risk and tail measures (especially CVaR) are much more appropriate given the nature of credit risk.

Andersson et al. (2001) consider a single period (anticipative) model that minimizes the conditional value at risk of a emerging market bond portfolio. For a discrete and finite sample distribution, the CVaR minimization model is formulated as a linear program (see Rockafellar and Uryasev (2000)). Linear programming algorithms are very efficient and hence, large scale real world models can be tackled. Furthermore Andersson et al. (2001) show that given a specific initial portfolio, optimisation leads to reductions in many risk measures, such as CVaR, VaR, expected loss, and standard deviation of the losses.

Jobst and Zenios (2001a) investigate the adequacy of different risk metrics in a credit risk optimisation context. Single period Mean-Absolute-Deviation (MAD) and CVaR models are investigated. In Jobst and Zenios (2001b) a single period tracking model is investigated and extensive backtesting results based on real-world data are reported.

The rest of this paper is organized in the following way. In Section 2 we describe how to integrate market risk and credit risk. We also discuss the simulator and illustrate some key properties of credit risky assets using the simulation results. In Section 3 we further expand on this theme and demonstrate how the asset allocation decisions can be influenced by the price behaviour of the 'credit risky' assets. We conclude this section by explaining how the simulator is used as an ex-ante scenario generator for the decision model. In Section 4 we introduce the salient features of the dynamic ALM decision model for "credit risky" assets. We set out two decision models: (a) the anticipative model and (b) non-anticipative model which incorporates recourse decisions as future realisations take place. Three case studies in a summary form are presented in Section 5. The purpose of these case studies is to highlight the importance of the decision models. To gain insight into the nature of these decisions ex-post simulation analysis are carried out and reported. Our conclusions are presented in Section 6.

2. Market risk and credit risk: Integration and simulation

The majority of research conducted on credit risk is devoted to either the pricing of a single credit derivative, or a portfolio of credit risky assets. The standard model for credit portfolios—the Gaussian copula default time approach—attempts to capture default dependence via correlated latent variables (asset values). While considerable research effort is devoted to dependency aspects (see, e.g. Schönbucher (2003)), the default risk of each single obligor is described by a constant, or at most time-dependent (but deterministic) description of the likelihood of default via intensity (hazard) functions.

While such a modelling framework may be adequate to capture correlated default events, the asset specific component is highly simplified. Although this may be justified for certain applications such as the risk analysis of synthetic CDOs, other problems in credit markets require a more holistic approach to the risks inherent in a portfolio of credits.

The effect of interest rate and spread risk on the overall risk of credit portfolios, or more generally, the structure of credit risk, has received relatively little attention, with the exception of Kiesel et al. (2001) and Jobst and Zenios (2001). In the last paper the impact of spread and interest rate uncertainty is highlighted, showing that these risks do not diversify away in a large portfolio context. This is particularly true for high quality credits. Whenever a thorough assessment of the overall portfolio risk is required, an integrated market and credit risk framework needs to be developed. The complexity of such a task generally prevents the use of analytical approximations and Monte Carlo methods and/or scenario analysis are employed to calculate portfolio performance/loss statistics. Because of the diverse application and complex nature of credit risk models, no single approach has yet been developed that can be employed across the wide spectrum.

We have developed a simulation model (see Jobst and Zenios (2001)) in which the risk of the future value of a credit risk sensitive instrument is decomposed into

(i) the risk that a firm's rating changes (including the risk of default),
(ii) the correlation between credit events,
(iii) the risk that changes occur on the average spread of exposures with the same final rating as the firm,[1]
(iv) the effect of interest rate uncertainty.

In the rest of this section we discuss these features further.

2.1. Credit events: Rating migrations and defaults

We consider n firms simultaneously. The migration process κ^j describes the evolution of the credit rating of firm j, where $\kappa^j \in S$ given the state space $S = \{1, 2, \ldots, K\}$, with K denoting the default state. The future credit rating $\kappa_{\overline{T}} \in S$ of each bond is simulated according to the actual migration process under $\mathscr{P}$ (the real measure). We assume that the probability of changes from rating l to rating m over one time period is a constant π_{lm} and let the migration matrix be denoted by $Q := [\pi_{lm}]_{l,m \in S}$. Table 1 shows a typical matrix regularly published by rating agencies, such as Standard and Poor's. Defaults take place when the process hits the absorbing state K, hence, $\pi_{Kl} = 0$, $l = 1, \ldots, K-1$, and $\pi_{KK} = 1$.

[1] We do not model the risk that the gap between the idiosyncratic spread and the average spread changes. A similar assumption is taken in Kiesel et al. (2001). Given a certain exposure in rating l, we apply the OAS methodology to match market prices and assume the same OAS forward in time if the bond stays in the same rating class. Given a rating change, we assume the bond is priced at the average spread of the new rating class (fair market value).

Table 1
Average transition matrix (1981–1999) expressed in %

	AAA	*AA*	*A*	*BBB*	*BB*	*B*	*CCC/C*	*D*
AAA	91.9	7.4	0.5	0.1	0.1	0.0	0.0	0.0
AA	0.7	91.9	6.7	0.6	0.1	0.1	0.0	0.0
A	0.1	2.3	91.7	5.2	0.5	0.2	0.0	0.0
BBB	0.0	0.3	5.2	88.6	4.7	0.8	0.1	0.3
BB	0.0	0.1	0.5	6.9	82.6	7.8	1.0	1.1
B	0.0	0.1	0.3	0.4	6.0	83.6	3.9	5.7
CCC/C	0.2	0.0	0.4	1.1	2.2	11.2	59.5	25.6
D	0.0	0.0	0.0	0.0	0.1	0.0	0.0	0.1

Source: Standard & Poor's.

We incorporate correlation between migrations and defaults following the latent variable approach employed in the CreditMetrics methodology of JP Morgan (1997). There, latent factors z^j that drive the transitions of a single exposure (firm j) are assumed to be multivariate normal ($Z \sim \Phi(0,\Sigma)$), where Σ denotes the correlation matrix). Conditional on the initial rating $l \in S$, let Z_{lm}, $m = 1,\ldots,K-1$ represent cut-off points (or barriers) calibrated to the firms migration probabilities. In practice, the assumption on the level of asset value (latent-variable) correlation is of paramount importance for many portfolio credit derivatives. The interpretation of the CreditMetrics model as a general asset value model frequently motivates the use of equity correlations as proxies for asset correlations. Unfortunately, these estimates are frequently driven by factors unrelated to credit risk (e.g. general market swings or liquidity). Instead, we calibrated the correlation parameters from empirically observed default and/or migration events using Standard and Poor's CreditPro ratings and default database. Recently, several techniques have been proposed and a comparative analysis of these techniques in Jobst and de Servigny (2005) reveals an average correlation of two firms in the same (in different) industry of 3–6% (15–20%).

Hence, rating transitions and default events can be simulated by sampling the multivariate latent random factors z^j and determining whether or not the random variable crosses one of the calibrated barriers. We denote simulated scenario sets as *credit* or *rating scenarios*.

2.2. The term structure of interest rates and credit spreads

For interest rate and credit spread modelling, we have adopted the Gaussian approach of Kijima and Muromachi (2000) that allows us to derive closed form bond prices consistent with the current term structure of interest rates and credit spreads. Kijima and Muromachi assume the general short rate and spread processes to follow the extended Vasicek (1977) model, i.e.,

$$\mathrm{d}x(t) = (\theta_x(t) - a_x x(t))\,\mathrm{d}t + \sigma_x\,\mathrm{d}W_x(t), \quad \mathcal{Q}\text{—a.s.}, \tag{1}$$

with constant a_x, mean reversion $\theta_x(t)$, and spot volatility function $\sigma_x(t)$ under the martingale (pricing) measure $\mathcal{Q}$. Here, this process is considered for the short interest

Table 2
Parameter estimates for the extend Vasicek interest rates and term structure models

	r	AAA	AA	A	BBB	BB	B
σ	0.009	0.004	0.008	0.012	0.02	0.025	0.03
a	0.022	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001

rate (i.e., $\Delta x = r$), and for credit spreads ($x = h_k$) for $k = 1,\ldots,K$, where 1 denotes the highest rating (AAA) and $K-1$ denotes the lowest rating CCC–C.

The advantage of the Gaussian model is that the solution $x(t)$ is known explicitly (see for example Kijima (2000) for details) as a Gauss–Markov process that is normal and hence may become negative.

For simulation purposes in a risk management context, however, we may consider the dynamic evolution of all processes under the real measure $\mathscr{P}$. We assume the following structure under $\mathscr{P}$,

$$\mathrm{d}x(t) = (\theta_x(t) - \hat{a}_x x(t))\,\mathrm{d}t + \sigma_x\,\mathrm{d}\widehat{W}_x(t), \quad \mathscr{P}\text{—a.s.,} \tag{2}$$

where for $x = \{r, h_1, \ldots h_{K-1}\}$.

$$a_x = \hat{a}_x + \lambda_x(\sigma_x)^2,$$
$$W_x = \widehat{W}_x + \int_0^t \gamma_x(s)\,\mathrm{d}s,$$

with

$$\gamma_x(t) = \lambda_x \sigma_x x(t).$$

2.2.1. Parameter estimation

Under the Gaussian dynamics, we can easily derive the parameters a_r, a_k, σ_r, and σ_k, which specify the stochastic process in the risk neutral world, from historical bond price volatilities and credit spread volatilities. In particular, the approach of Hull and White (1998) can be employed for the estimation of interest rates and credit spreads when they are assumed to be independent. The outcome of such a calibration is given in Table 2.[2]

These results are based on term structures for estimating the parameters rather than individual bond prices of tradable securities. The initial term structure, estimated from our bond index data. Given the time series of term structures, we estimate the parameters for the short term interest rate process and the credit spread processes for rating classes AAA, AA, and A and use these volatility estimates throughout the section.

[2] Due to data availability, parameters were only estimated for AAA, AA, A and BBB rating classes, while the parameters for lower rating classes are assumptions. Furthermore, in the following simulation and optimisation examples, we only work with risk-neutral processes.

2.3. The structure of credit risk: Some simulation insights

In this section, we assess the impact of the alternative risk factors on large portfolios. In particular, we consider single rating grade CVaRs at 5%, 1%, and 0.1% confidence levels in order to get insight into the importance of the alternative factors for different credit quality portfolios. All results are reported for June 30, 2001 and for a one year risk horizon. In the following, we consider portfolios of 200 equally weighted zero coupon bonds maturing on January 31, 2004. We report Value at Risk (VaR) and Conditional Value at risk (CVaR) numbers for 5%, 1%, and 0.1% confidence levels. In the base scenario, we suppose that the correlation coefficient between all the latent random variable is equal to 0.2, a number frequently employed in empirical studies, that is in line with the empirical estimates for obligors within the same industry.

We simulate the portfolios in three different ways (modes) and calculate the corresponding risk measures. In the first case, denoted by DT, we simulate only rating migration (and default) events and recovery rates, assuming future interest rates and credit spreads to be non-stochastic. In the calculation of the bond price at the risk horizon, we employ the forward interest rates and credit spreads implied from the current term-structure. Hence, the DT model is similar to standard mark-to-market CreditMetrics implementations. The $DT+S$ model extends the DT model by incorporating spread risk. Spreads are simulated from the stochastic processes, whereas interest rates are still assumed non-stochastic. The $DT+S+I$ model extends the previous model once more by adding stochastic interest rates, too. The resulting risk measures are scaled by the initial portfolio value and based on 1000 rating scenarios and 50 economic scenarios, hence a total of 50,000 scenarios.[3]

Table 3 reports the single rating grade CVaR statistics for a 1 year holding period.

Overall, the impact of spread risk and interest rate risk for high quality bonds is striking. At a 1% confidence level, the CVaRs for *Aaa* rated securities are six times higher, for *Aa* and *A* rated securities about three times higher when spread risk is incorporated ($DT+S$ versus DT). Furthermore, when market risk is added ($DT+S+I$), the risk for *Aaa* rated securities more than doubles that of $DT+S$. This increase in risk is less considerable as we decrease in credit quality. For lower ratings, spread and market risk is less influential, suggesting that transition and recovery risk is more significant.

Hence, in particular for high quality exposures, spread and market risk need to be taken into consideration as otherwise the portfolio risk is underestimated significantly.

[3] We check the sensitivity of the results by computing standard error bounds. We consider a mixed quality portfolio if 200 exposures (equally weighted) and run the simulation repeatedly (50 times) with different initializations. We consider rating migrations only (CM mode) and simulate 1000 scenarios. The mean 95%-VaR is 1.76%, with an upper limit of 1.79% an a lower bound of 1.75% (under a 95% confidence level). Increasing the number of simulations improves the error bound, e.g. for 10,000 rating scenarios we can get accurate results up to the second decimal place. However, we conclude that 1000 scenarios are efficient and numerically more tractable for most of the results discussed in the following sections.

Table 3
Conditional Value at Risk (CVaR) statistics. Portfolios consist of 200 exposures of equal face value. CVaRs are measured in percentage of the expected value. The correlation coefficient of the latent random variables triggering credit events is 0.2. DT indicates Default and Transition simulations (standard CreditMetrics type), $DT+S$ indicates the extension of the DT model to incorporate spread risk, $DT+S+I$ indicates the extension of the $DT+S$ model to incorporate interest rate risk

Rating	VaR-level (%)	DT	$DT+S$	$DT+S+I$
AAA	95	0.16	1.22	2.72
	99	0.28	1.33	2.93
	99.9	0.38	1.58	3.21
AA	95	0.68	2.21	3.21
	99	1.36	2.94	3.97
	99.9	1.98	3.55	5.04
A	95	0.72	3.66	5.47
	99	1.39	3.97	6.36
	99.9	2.06	4.21	6.46
BBB	95	2.17	5.34	6.35
	99	4.15	6.25	6.99
	99.9	5.67	7.48	8.07
BB	95	6.55	7.85	7.66
	99	11.67	12.31	12.22
	99.9	16.72	17.65	17.52
B	95	15.07	16.43	16.54
	99	23.66	24.41	24.48
	99.9	29.58	30.74	30.86
CCC–C	95	30.86	31.69	31.76
	99	40.42	40.93	40.96
	99.9	45.72	46.35	46.41

We calculate CVaRs under different assumptions about the correlation between the latent random variables, next. In the following, we consider a lower correlation of 0.1 and an increased correlation of 0.3. In particular, we compare the impact of this correlation on the VaRs of portfolios of different credit qualities. Table 4 reports the results for the $D+T+I$ approach.

Correlation appears to impact the overall portfolio risk of all portfolios. While the impact is moderate for high quality portfolios, low quality portfolios are significantly affected. Fig. 1 illustrates that impact when only default and transitions are modelled. As we can see, for high quality debt when considering all risk factors, increasing the asset correlation has no significant impact on the overall portfolio risk (factors close to 1). However when ignoring market and spread risk, asset correlation seems to be highly significant; hence ignoring all risk factors may lead to wrong conclusions about the structure of credit risk. On the other hand, for lower quality portfolios, an increase in asset correlation increases the overall portfolio risk across all simulation models. For a portfolio of B rated bonds, increasing asset correlations from ten to thirty percent approximately doubles the CVaR of the portfolio.

Table 4
Value at Risk (VaR) statistics under the $DT+S+I$ simulation model for different correlation coefficients. Portfolios consist of 200 exposures of equal face value. DT indicates Default and Transition simulations (standard CreditMetrics type), $DT+S$ indicates the extension of the DT model to incorporate spread risk, $DT+S+I$ indicates the extension of the $DT+S$ model to incorporate interest rate risk

Rating	VaR-level (%)	Correlation		
		0.1	0.2	0.3
AAA	95	2.53	2.54	2.54
	99	2.84	2.82	2.81
	99.9	2.98	3.06	3.18
AA	95	2.01	2.05	2.15
	99	3.57	3.57	3.62
	99.9	4.14	4.48	5.08
A	95	4.69	4.70	4.71
	99	6.29	6.31	6.33
	99.9	6.38	6.39	6.41
BBB	95	5.19	5.33	5.48
	99	6.68	6.71	6.79
	99.9	7.12	7.39	9.51
BB	95	4.64	5.20	5.84
	99	6.64	9.29	12.76
	99.9	9.65	15.83	22.96
B	95	9.18	11.47	13.98
	99	13.66	20.18	26.37
	99.9	19.64	29.03	38.31
CCC–C	95	18.06	24.82	30.50
	99	26.22	36.99	44.06
	99.9	33.42	45.00	50.80

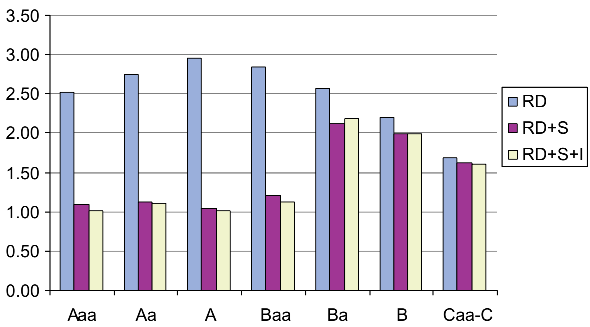

Fig. 1. The structure of credit risk: spread and interest rate uncertainty and sensitivity to changes in asset correlation.

3. Simulator, scenarios and optimum decisions

The price behaviour of credit risky assets clearly influences the asset allocation decisions. In this section we discuss how the simulator can be used as a scenario generator and then incorporated in the (optimizing) decision model.

Fig. 2 illustrates that the success of optimisation techniques in such a setting is largely due to the adequate specification of the risk factors (see Jobst and Zenios (2001b) for further details). The figure shows the performance of a portfolio (optimisation) tracking model (tracking the Merrill Lynch Eurodollar index) when the underlying scenario sets include defaults, migrations, spreads and interest rate simulations (left panel) and when we omit interest rate and spread scenarios. The good tracking performance diminishes when specifying scenario sets inadequately (the Eurodollar index is an investment grade index which requires the consideration of spreads and interest rate uncertainty).

In addition to sound simulation methods, adequate risk measures are necessary to develop successfully quantitative credit risk management tools. For example, managing large portfolios using standard industry solutions[4] are based on Value-at-Risk. However, as Frey and McNeil (2002) point out, the conceptual weaknesses of VaR (e.g. the lack of subadditivity in the framework of coherent risk measures) is exploited if one tries to maximise the expected return of a portfolio subject to some constraints on VaR. They also illustrate the inconsistency of VaR in credit portfolios and the dangers of mean–VaR portfolio optimisations.

Fig. 3 reports the mean-CVaR efficient frontier (solid line) and also the CVaR of the MAD-optimal portfolio (.99 CVaR(MAD*)). We can clearly see that the output of the corresponding MAD optimisation leads to highly inefficient portfolios in a CVaR perspective. Given the asymmetry in the presence of credit risk, such a MAD framework needs to be treated with care.

In addition to adequate simulation and optimisation paradigms, most approaches to credit risk optmisation are single period in nature, hence failing to incorporate dynamic aspects in both, the simulation and decision (optimisation) process. In a credit risk context, dynamic aspects such as the default timing is important and should be adequately addressed.[5] For example, the optimisation models must include recourse (corrective) decisions such as liquidation of positions if defaults seem unavoidable under certain scenarios.

Taking into consideration these arguments, we can say consequently that a scenario based optimisation model that incorporates alternative risk measures and incorporates the dynamical aspects (timing of risk) is a desirable model. We develop such multistage stochastic programming models that incorporate recourse decisions (such as buying and selling at future periods) and hence provide a step forward in developing successful quantitative credit risk management (optimisation) tools.

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[4] For example, the CreditMetrics model proposed by the Risk Metrics Group (1997), or the CreditRisk$^+$ model proposed by Credit Suisse Financial Products (1997).

[5] An example are CDO's where the timing of cashflows is mainly driven by the performance of the underlying collateral through time.

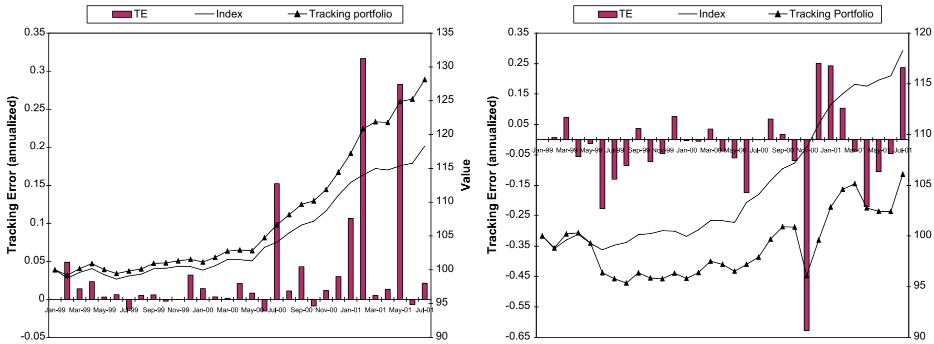


Fig. 2. Performance of the tracking model versus the index and corresponding tracking errors when the scenario generation does not include uncertainty in interest rates and credit spreads.

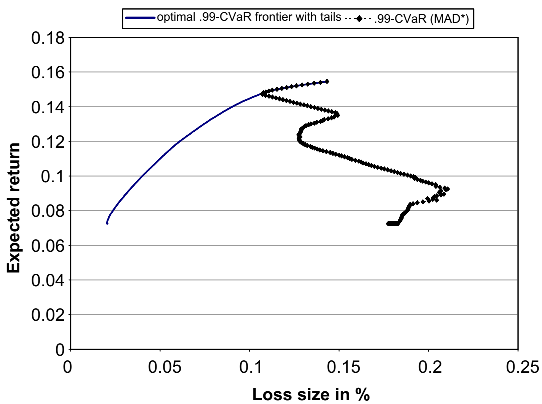


Fig. 3. Performance of the tracking model versus the index and corresponding tracking errors when the scenario generation does not include uncertainty in interest rates and credit spreads.

4. Dynamic asset and liability management modelling under credit risk

For real world applications it is meaningful to study multi-time period models which capture the dynamical aspects of both risk factors and ALM investment decisions. Multistage stochastic programming in particular is well applied to process ALM models, for instance see Mulvey and Vladimirou (1992), Ziemba and Mulvey (1998), Consigli and Dempster (1998) and Kouwenberg and Zenios (2006).

We develop credit risk optimisation models within a stochastic programming framework and present a generic multistage stochastic ALM model that maximises the expected value of terminal wealth under limited CVaR risk constraints. These constraints are imposed on the portfolio value at future points in time, and in a liability matching context.

4.1. Model structure and model development

We show how the portfolio composition can be optimised by maximizing the expected final value or return of the portfolio under given constraints that ensure coverage of the liabilities of a company at a maximum tolerated risk. One set of restrictions is due to a minimum required cashflow per period to cover liabilities, the second set limits the risk of portfolio wealth. Both risks are measured by the conditional value at risk (CVaR) to account for the downside risk and extreme losses. These CVaR based reformulation of liability restrictions are essential in the presence of credit risk due to the default events which imply a stop in coupon income.

The essential descriptors of our decision model comprises (a) Sets and indices, (b) data parameters and (c) decision variables.

Sets and indices	
i	index defining each bond ($i = 1, \ldots n$)
ω	scenario index $\omega \in \Omega$
$\mathcal{T}$	discrete time steps $\mathcal{T} = \{t_0, t_1, \ldots, t_{m_T}\}$, with $t_0 := 0$ and $t_{m_T} := T$
U	universe of n bonds: $U = \{1, \ldots, n\}$
$\mathcal{T}_{\mathrm{B}}$	set of timesteps with benchmark restrictions $\mathcal{T}_{\mathrm{B}} = \{T_1^{\mathrm{B}}, \ldots, T_{m_{\mathrm{B}}}^{\mathrm{B}}\} \subset \mathcal{T}$
$\mathcal{T}_{\mathrm{L}}$	set of timesteps with liability restrictions $\mathcal{T}_{\mathrm{L}} = \{T_1^{\mathrm{L}}, \ldots, T_{m_{\mathrm{L}}}^{\mathrm{L}}\} \subset \mathcal{T}$
$\mathcal{N}_t$	set of nodes at time t in scenario tree
D_o	bundle of scenarios passing through node $o \in \mathcal{N}_t$, $t = t_1, \ldots, T-1$
Data parameters	
π^{ω}	probability of scenario ω
r_i^{ω}	holding period return of bond i under scenario ω
β_i, (β_i^c)	weight of security i in the index (in class c)
I^{ω}, (I_c^{ω})	index (asset class index) return under scenario ω
b_i	initial face value holding of bond i
P_{i0}	initial price of bond i
P_{it}^{ω}	price of bond i at time t under scenario ω, $P_{it}^{\omega} = P_i^{\omega}(t, T_i)$
r_0^{ω}	riskless rate of return under scenario ω
tc_i^{s}, tc_i^{b}	transaction costs for selling, buying of security i
W_0	initial portfolio wealth
W^{ω} (W_t^{ω})	portfolio wealth under scenario ω (at time t)
$\mathcal{T}$	discrete time steps $\mathcal{T} = \{t_0, t_1, \ldots, t_{m_T}\}$, with $t_0 := 0$ and $t_{m_T} := T$
T_i	the maturity of bond i
r^{ca}	continuous cash rate
$\overline{F}_i(t, t_k)$	default adjusted cashflow (multiple coupons/recovery) at time t_k, $t \in (t_{k-1}, t_k]$
$\mathrm{Liab}^{\omega}(t)$	liability at time t under scenario l
$B(t)$	value of benchmark at time t with benchmark return $r^{\mathrm{B}}(t)$

$B^{\mathrm{CVaR}}(t)$	tolerated benchmark CVaR (right hand side) at time t
$L^{\mathrm{CVaR}}(t)$	tolerated liability CVaR (right hand side) at time t
Decision variables	
c_0	initial cash holding
x_i	weight of (or face value holding in) security i in the portfolio
v (v_t^{ω})	amount invested in cash (at time t under scenario ω)
x_i^{b}	face value purchased of security i
x_i^{s}	face value sold of security i
ζ	Value-at-Risk
ζ_t^{B} (ζ_t^{L})	Value at Risk at time t for benchmark (liability) risk
$y_{\mathrm{B}}^{\omega}(t)$ ($y_{\mathrm{L}}^{\omega}(t)$)	auxiliary variable in CVaR calculation for benchmark (liability) risk
α_{B} (α_{L})	confidence level in benchmark (liability) risk restrictions

4.1.1. Cashflows and recovery payments

Coupon payments $F_i(t)$ between liability dates and time periods are put into a cash account, and reinvested at a continuous interest rate r^{ca}, that is

$$\widehat{F}_i(t,t_k) := F_i(t)\cdot \mathrm{e}^{r^{\mathrm{ca}}(t,t_k)(t_k-t)}, \quad t\in(t_{k-1},t_k]. \tag{3}$$

In the scenario generation, we handle recovery payments ${}_i(\tau)$ in a similar way. Given default at time $\tau\in(t_{k-1},t_k]$, we assume to receive a cashflow

$$\widehat{\Upsilon}_i(\tau,t_k) := \Upsilon_i(\tau)\cdot \mathrm{e}^{r^{\mathrm{ca}}(\tau,t_k)(t_k-\tau)}, \tag{4}$$

where ${}_i(\tau)$ either specified as a constant amount, a random variable or a fraction of the pre-default value of the bond. Hence, in the scenario generation we use indicator functions to denote the cashflow at time $t\in(t_{k-1},t_k]$, that is

$$F_i^d(t,t_k) := \mathbf{1}_{\{\tau>t\}}\widehat{F}_i(t,t_k) + \mathbf{1}_{\{\tau\leqslant t_k\}}\widehat{\Upsilon}_i(t,t_k). \tag{5}$$

Multiple cashflows of bond i in the interval $(t_{k-1},t_k]$ are treated correspondingly, that is

$$\overline{F}_i(t_k) := \left(\sum_{j=1\ldots n_i,\, t_{ij}\in(t_{k-1},t_k]} \mathbf{1}_{\{\tau>t_{ij}\}}\widehat{F}_i(t_{ij},t_k)\right) + \mathbf{1}_{\{\tau\leqslant t_k\}}\widehat{\Upsilon}_i(t,t_k), \tag{6}$$

where t_{ij} denotes the time of the j's coupon payment of bond i, and n_i denotes the number of scheduled coupon payments (see Fig. 4).

Given default at time $\tau\in(t_{k-1},t_k]$ and the corresponding cash recovery payment, we assume that no further cashflows $F_i(t)$, for $t>\tau$ will be received. We assume that the recovery payment is put into the cash account, and furthermore $P_i(t,T_i) := 0$, $t\geqslant\tau$. Hence, the value of the cash account given portfolio cash inflows between t_{k-1} and t_k is given by

$$v(t_k) = v(t_{k-1})\mathrm{e}^{r^{\mathrm{ca}}(t_{k-1},t_k)(t_k-t_{k-1})} + \sum_{i=1}^{n} x_i\overline{F}_i(t_k) - \mathrm{Liab}(t_k), \tag{7}$$

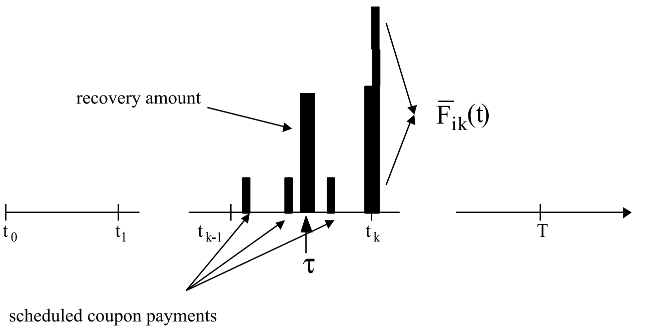


Fig. 4. Cashflow handling in discrete time-period setup.

where $\mathrm{Liab}(t_k)$ denotes the desired liability payments and x_i denote the portfolio holding in asset i, which is assumed to be constant throughout this planning horizon in a so called anticipative model, i.e., $x_i := x_i(t_0)$.

In extended multistage recourse models,[6] we also need to include the corresponding rebalancing decisions, which amounts to recourse actions, that is, (7) has to be replaced by

$$v(t_k) = v(t_{k-1})\mathrm{e}^{r^{\mathrm{ca}}(t_{k-1},t_k)(t_k-t_{k-1})} + \sum_{i=1}^{n} x_i(t_{k-1})\overline{F}_i(t_k) - \mathrm{Liab}(t_k) + \sum_{i=1}^{n} x_i^{\mathrm{s}}(t_k)P_i(t_k,T_i)(1-tc_i^{\mathrm{s}}) - \sum_{i=1}^{n} x_i^{\mathrm{b}}(t_k)P_i(t_k,T_i)(1+tc_i^{\mathrm{b}}), \quad (8)$$

where x_i^{s} denotes the units of bond i sold and x_i^{b} denote the units of bond i bought, and where we assume constant transaction costs tc_i^{s} and tc_i^{b} as a fraction of the bonds value.[7]

4.1.2. Portfolio wealth

Then, the future portfolio wealth, given price scenarios $P_i^{\omega}(t,T_i)$, $t \in (t_{k-1}, t_k]$, is given in the anticipative model by

$$W_{t_k}^{\omega} := \sum_{i \in U} x_i(t_0)P_i^{\omega}(t_k,T_i) + v_{t_k}^{\omega}, \quad (9)$$

where $v_{t_k}^{\omega}$ is given according to Eq. (7) as

$$v_{t_k}^{\omega} := v_{t_{k-1}}^{\omega}\mathrm{e}^{r^{ca,\omega}(t_{k-1},t_k)(t_k-t_{k-1})} + \sum_{i=1}^{n} x_i(t_0)\overline{F}_i^{\omega}(t_k) - \mathrm{Liab}_{t_k}^{\omega}.$$

[6] Both, anticipative and recourse models allow for a scenario based representation of uncertainty. The main difference lies in ability of the latter approach to allow for corrective actions (recourse decision).

[7] Alternative transaction cost assumptions could be easily introduced. If, for example, an fixed amount should be charged as soon as a transaction takes place, we can employ binary variables following the modelling principles presented in Jobst et al. (2001) at the cost of higher computational requirements.

In the corresponding multistage recourse model, we assume that the portfolio wealth is given as

$$W_{t_k}^{\omega} := \sum_{i\in U} x_i^{\omega}(t_{k-1})P_i^{\omega}(t_k, T_i) + v_{t_{k-1}}^{\omega}\mathrm{e}^{r_{t_{k-1}}^{ca,\omega}(t_k-t_{k-1})} + \sum_{i\in U} x_i^{\omega}(t_{k-1})\overline{F}_i^{\omega}(t_k), \tag{10}$$

which corresponds to the value just before portfolio rebalancing takes place at time t_k. After rebalancing, the portfolio value is given by

$$\widetilde{W}_{t_k}^{\omega} := \sum_{i\in U} x_i^{\omega}(t_k)P_i^{\omega}(t_k, T_i) + v_{t_k}^{\omega}. \tag{11}$$

It is easily seen that (10) and (11) are equivalent, if no transaction costs are considered.

4.1.3. Risk constraints

In both, the anticipative and the recourse models downside risk constraints are introduced. To restrict the downside risk of the future portfolio value at times $t \in \mathcal{T}_{\mathrm{B}}$, we consider CVaR constraints with respect to a specific benchmark $B(t) \in \mathbb{R}$. If we are given a benchmark return $r^{\mathrm{b}}(t)$ for the period $(0, t]$, the corresponding benchmark value is given by $B(t) = W_0(1 + r^{\mathrm{b}}(t))$. Hence, the portfolio loss under a given scenario is given by

$$\mathrm{Loss}_{\mathrm{B}}^{\omega}(t_k) := B(t_k) - W^{\omega}(t_k), \tag{12}$$

or in relative terms

$$\mathrm{RLoss}_{\mathrm{B}}^{\omega}(t_k) := \frac{B(t_k) - W^{\omega}(t_k)}{B(t_k)}, \tag{13}$$

where $W^{\omega}(t_k)$ is defined by (9) in the anticipative model and by (10) in the multistage recourse model. We introduce a set of auxiliary variables, y_{B}^{ω} ($y_{\mathrm{B}}^{\omega} \geqslant 0$), as

$$y_{\mathrm{B}}^{\omega}(t) \geqslant \mathrm{Loss}_{\mathrm{B}}^{\omega}(t) - \zeta_t^{\mathrm{B}} \tag{14}$$

and define the set of CVaR constraints as

$$\zeta_t^{\mathrm{B}} + \frac{\sum_{\omega\in\Omega} \pi^{\omega} y_{\mathrm{B}}^{\omega}(t)}{1-\alpha_{\mathrm{B}}} \leqslant B^{\mathrm{CVaR}}(t),$$

where $B^{\mathrm{CVaR}}(t)$ denotes the maximum level of risk (CVaR) tolerated, and α_{B} is the percentile. For further details, see Rockafellar and Uryasev (2000).

To restrict the risk in the liability stream we introduce $\mathrm{Loss}_{\mathrm{L}}^{\omega}(t)$ and variables $y_{\mathrm{L}}^{\omega}(t),\ t \in \mathcal{T}_{\mathrm{L}}$. The loss at time $t \in \mathcal{T}_{\mathrm{L}}$ is defined by

$$\mathrm{Loss}_{\mathrm{L}}^{\omega}(t) := 0 - v^{\omega}(t) = -v^{\omega}(t). \tag{15}$$

The following constraints are added

$$y_{\mathrm{L}}^{\omega}(t) \geqslant \mathrm{Loss}_{\mathrm{L}}^{\omega}(t) - \zeta_t^{\omega}, \tag{16}$$

$$y_{\mathrm{L}}^{\omega}(t) \geqslant 0, \tag{17}$$

$$\zeta_t^{\mathrm{L}} + \frac{\sum_{\omega\in\Omega} \pi^{\omega} y_{\mathrm{L}}^{\omega}(t)}{1-\alpha_{\mathrm{L}}} \leqslant L^{\mathrm{CVaR}}(t), \tag{18}$$

where $L^{\mathrm{CVaR}}(t)$ denotes the maximum allowed tolerance risk of not matching the liabilities. These restrictions are not necessary when only market risk is considered, in particular when treasury coupon bonds are used to finance the liabilities. In that case, the problem relaxes to exact cashflow matching. However, give the risk of default the restrictions ensure that liabilities will be covered up to some probabilistic tolerance (confidence level).

4.1.4. Non-anticipativity constraints

In the multistage recourse model, we need also a set of constraints that ensure that scenarios sharing a common history up to any point in time must also share common decisions up to that point in time. This requirement is known as non-anticipativity, and greatly complicates the processing of SP models as it ties together optimisation problems pertaining to separate scenarios. In order to ensure consistency of the solution, if two scenarios ω_a and ω_b, $a \neq b$, are indistinguishable up to a given time period t, then, the related decisions up to that period must be the same. In order to define these constraints we need the information embodied in the scenario structure. Let S_o be the bundle of scenarios passing through node o and let $\mathcal{N}_t$ define the set of all notes at time t, $t = t_1, \ldots, T$. Then, non-anticipativity constraints for a decision vector x_t, $t \in \mathcal{T}$, can be formalised as $x_{t,\omega_a} = x_{t,\omega_b}$, $a \neq b$, $\forall \omega_a, \omega_b \in S_o$, with $o \in \mathcal{N}_t$, $t = 1, \ldots, T-1$.

5. Bond index tracking experiments: Ex post decision analysis

In this section, we will illustrate the simulation and optimisation models developed above in the context of asset allocation decisions faced by a fixed-income fund manager. While Jobst and Zenios (2001) successfully apply single-period (antithetic) optimisation models at an individual security level, the asset allocation approach discussed here focuses on the (ex post) difference between anticipative and recourse models.

We amend the model to determine the optimal allocation in several asset classes in order to track the Merrill Lynch Eurodollar index. We represent these asset classes by synthetic bonds that are approximated from index data. In particular, we consider the four investment grade ratings and four different maturity buckets; thus for each rating, we aggregate bonds in four groups with 1–3 years, 3–5 years, 5–7 years and 7+ years of maturity. For example, on January 31, 1999 we obtain the following asset class details which correspond to the market weighted average of all bonds belonging to each asset class (Table 5). Instead of using the weighted average prices we use the exact prices implied from the current term structures and assume that these asset classes or synthetic bonds are priced in the market at the fair value. In the backtesting experiment we will, however, use the real, observed holding period return for each of these synthetic bonds to assess the model performance. Similar statistics can then be used for the index one month later, on February 28, 1999. Of course the details may differ significantly from the previous month. Hence at the end of the month the price for an asset class differs from the price at the beginning

Table 5
Asset class details (synthetic bonds) on January 31, 1999. The holding period return (HPR) corresponds to the return over the next one month

Rat.	Mat.	Mat. Bucket	CPN	Dur.	Yield	OAS	Hold.	Price	HPR
AAA	12/4/00	1	6.24	1.69	5.29	58.44	11.85	102.62	0.00
	11/16/02	2	6.05	3.34	5.21	52.45	19.16	104.08	−0.01
	2/24/05	3	6.56	4.91	5.30	53.04	6.13	109.28	−0.02
	2/7/08	4	5.99	6.90	5.36	40.38	6.48	107.34	−0.03
AA	12/14/00	1	6.47	1.71	5.45	74.17	13.01	102.61	0.00
	9/12/02	2	6.48	3.14	5.45	73.21	19.50	105.83	−0.01
	12/23/04	3	6.59	4.80	5.51	73.60	5.18	106.04	−0.02
	11/27/07	4	6.48	6.62	5.67	71.76	6.72	106.73	−0.03
A	11/22/00	1	6.77	1.65	5.66	94.13	3.11	103.18	0.00
	10/30/02	2	6.50	3.25	5.70	97.95	5.39	104.26	−0.01
	9/12/05	3	6.84	5.24	6.16	136.88	0.96	106.26	−0.02
	9/19/07	4	6.91	6.31	7.86	298.18	0.37	96.67	0.01
BBB	3/3/01	1	7.68	1.94	7.89	322.80	0.34	102.75	0.01
	1/11/03	2	6.82	3.36	7.45	275.97	1.24	98.25	−0.01
	3/1/06	3	8.44	5.09	10.16	533.19	0.15	94.96	0.01
	10/4/08	4	7.48	6.68	7.52	255.31	0.42	102.14	−0.02

of the next month. We adjust the next month holding after pricing the synthetic bonds such that the total market value in each asset class at the beginning of the new month is equal to the end of the month market value from the previous month.

Fig. 5 plots the quoted option adjusted spreads (OAS) for the shortest and longest maturity buckets in every rating class over a period of 2.5 years. We can observe a period of spread widening (especially for all long maturity bonds) between December 1999 and November/December 2000, and a long period of spread tightening afterwards until July/August 2001.

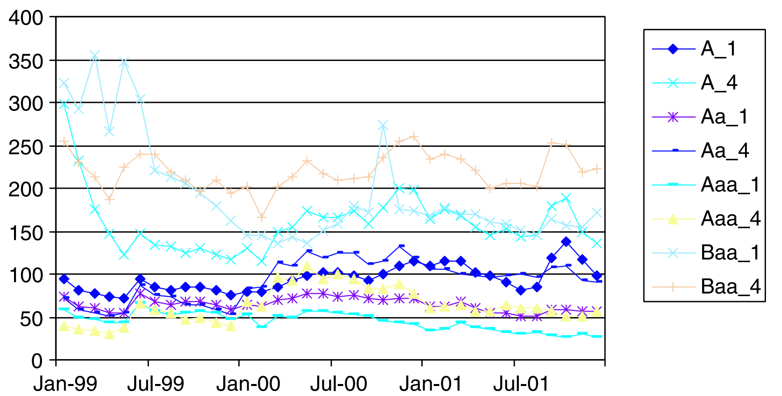


Fig. 5. OAS of eight asset classes from January 31, 1999 to December 31, 2001.

5.1. The CVaR-Index tracking model

Portfolio value or loss distributions of a given portfolio in the presence of credit risk exhibit fat tails and non-normality. When we are tracking a bond index, we are less interested in the absolute risk, but more concerned about the relative risk, that is, losses with respect to the performance of a bond index, which is a random variable itself. The following Fig. 6 plots the loss distribution (at a risk horizon of one year) for an investment grade portfolio when we overweight *BBB* rated bonds. We observe that the downside risk is significantly higher, compared to the upside potential (indicating a maximum loss of 10%). We therefore need to develop optimisation models that account for the tail events in the tracking context. We have chosen once more the CVaR methodology. Instead of defining the losses with respect to the initial portfolio value or expected value with respect to a pre-specified (deterministic) benchmark (as in the previous section) we define the losses in this section with respect to the random index, that is

$$\mathrm{ILoss}_t^{\omega} := I_t^{\omega} - W_t^{\omega}, \quad t \in \mathcal{T}, \ \omega \in \Omega, \tag{19}$$

or in relative terms as

$$\mathrm{RILoss}_t^{\omega} := \frac{I_t^{\omega} - W_t^{\omega}}{I_t^{\omega}}, \quad t \in \mathcal{T}, \ \omega \in \Omega, \tag{20}$$

where I_t^{ω} denotes the index value and W_t^{ω} denotes the value of the tracking portfolio (including reinvested cash payments) at time t under scenario ω. In the following case study, we only consider one set of constraints at the end of the horizon $T = 6$ m. Given this loss definition we derive the tracking models from the ALM models introduced in the previous section.

We can derive the anticipative CVaR tracking model from the ALM model of the previous section by setting Liab_{t_k} equal to zero and deleting the corresponding liability risk constraints (15)–(18) in the Anticipative ALM model. We also have to adjust constraint (14) and replace $y_{L_{t_L}}$ by y_{IT} where

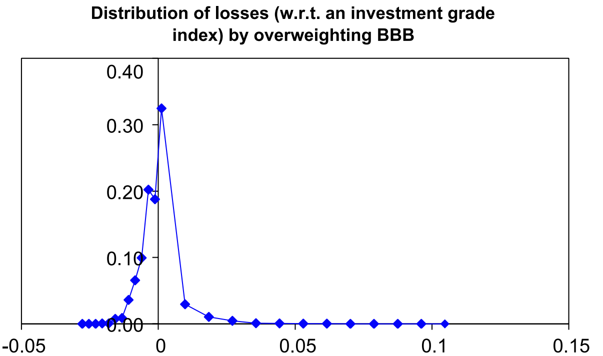


Fig. 6. Distribution of losses with respect to the index.

$$y_{IT} \geqslant \frac{I_T^{\omega} - W_T^{\omega}}{I_T^{\omega}} - \zeta_T^I.$$

These constraints model the CVaR of the portfolio with respect to the index at the end of the horizon T. Similar adjustment can be made to the MS-recourse model resulting in the multi-stage recourse index tracking (asset allocation) model. A detailed discussion of these models and investigations is given in Jobst (2002).

5.2. Empirical investigation: Ex post results

We investigate now the tracking performance of the Merrill Lynch Eurodollar index when we implement the optimal portfolio suggested by the anticipative and the two-stage models ($\mathscr{T} = \{0, 3\text{ m}, 6\text{ m}\}$). Of course, we only implement the first stage decision for the two-stage model. We also investigate the tracking of a government bond index with US Treasury securities and corporate bonds and conduct the experiments once again.

We present some interesting backtesting results next. In the following we consider transaction costs which are rating class dependent, that is 5 bp for *Aaa*, 10 bp for *Aa*, 20 bp for *A* and 40 bp for *Baa*.[8] We implement the CVaR constraint at a 95% level and allow for 100 bp CVaR ($I_T^{\text{CVaR}} = 1\%$), that is, the expected losses with respect to the index below the 95%-VaR have to be less than one percent. The anticipative model allows for only one decision at time $t_0 = 0$, whereas the two-stage model also allows for portfolio rebalancing at $t_1 = 3$ m. The backtesting is conducted by implementing the suggested portfolio and re-running the simulation/optimisation models after one month.

Scenario sets We generate all scenarios with a 20% correlation between the latent random variables triggering default. This is at the upper end of correlation estimates from actual default event data, but compares well with correlations frequently assumed in the market. We simulate 12,000 scenarios over a 6 month period. The event tree has 200 branches in the root node (10 economic and 20 credit scenarios), and 60 branches (6 economic and 10 credit scenarios) at time step 3 m at each parent node.[9]

5.2.1. Case study 1: Corporate bond index tracking

Anticipative model. We study the tracking of the Eurodollar index by investing in the asset classes according to the optimal solution of the anticipative model. Fig. 7 plots the portfolio value versus the index value over the backtesting period of January 1999 to December 2001. We also plot the implemented or optimal asset allocation by rating class, as suggested by the model (right panel).

Overall we observe a good tracking performance with a large holding in high quality, *Aaa* rated bonds. Surprisingly, apart from the first few months, hardly any capital is invested in *Aa* rated bonds.

[8] This transaction costs assumption stems from discussions with market participants.

[9] Solving the two-stage model with 12,000 scenarios on a Pentium 4 2 GHz processor, 512 MB RAM, using FortMP (2002), takes approximately 20 min to 1 h.

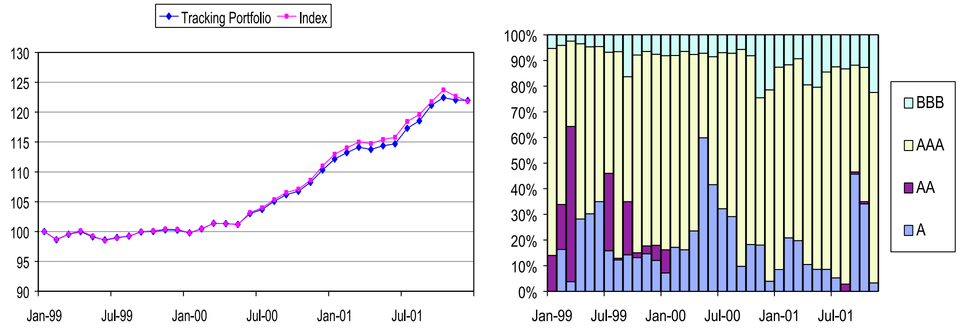


Fig. 7. Backtesting results anticipative model: portfolio value versus the index value (Eurodollar). Investment in corporate bonds is possible, only.

Two-stage recourse model. We now repeat the previous example, however we allow for recourse decisions at month 3 in the model. We plot the portfolio value evolution compared to the index, and the first stage investment decision in Fig. 8, and observe a significantly larger final portfolio value compared to the one stage (anticipative) model. Especially interesting is to take a closer look at the optimal portfolios. We observe overall a significant holding in *AAA* rated bonds. In particular, during the period of spread widening (December 1999 to November 2000), the model shifts the investment out of *BBB* rated bonds to *AAA* assets. This behaviour is intuitive and encouraging, given that in periods of spread widening, lower rated bonds were more affected. After that, we can observe a shift of a significant part of the portfolio value to *BBB* rated securities (going long credit), during the period of spread tightening. Overall, the model does what a portfolio manager should have done. Knowing that the model can correct initial decisions in subsequent periods, leads to portfolios that differ more significantly from the index structure, however as explained above, this does not always imply riskier portfolios. In both examples, we observe as good tracking performance and especially in the two-stage implementa-

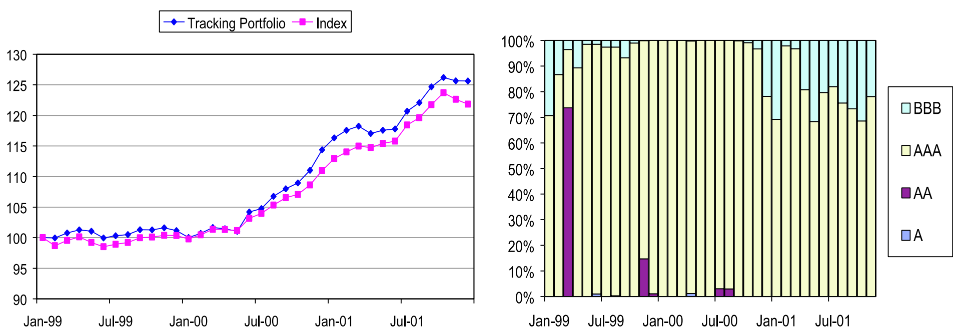


Fig. 8. Backtesting results two-stage model: portfolio value versus the index value (Eurodollar). Investment in corporate bonds is possible, only. Backtesting results: asset allocation according to the optimal first stage solution of the two-stage stochastic program.

tion we observe a good reaction to market developments. Overall this supports the choice of our simulation and optimisation paradigm.

5.2.2. Case study 2: Tracking a government bond index

Anticipative model. We consider the Merrill Lynch Government Bond index aggregated into six different maturity buckets, with 1–3, 3–5, 5–7, 7–9, 9–21, and 21–30 years of maturity. We add to this the 16 synthetic corporate bonds representing investment grade assets. Fig. 9 plots the portfolio performance and asset allocation throughout the backtesting period.

Overall, we observe once more a good tracking performance, with a very significant holding of 70–80% in high quality (mainly *AAA*) corporate bonds. This is somewhat surprising, however consistent with market practice. Also, over the short risk horizon and given the nature of the transition matrix employed, default events are extremely rare for the high rating class exposures. Therefore, all constraints can still be satisfied. We can also observe a complete shift to corporate products at the end of the backtesting period, which makes sense due to the extreme tightening in corporate spreads.

Two-stage recourse model. We re-run the previous experiment by applying the two-stage model, that is, allowing for a corrective recourse decision after 3 months. Fig. 10 reports the portfolio performance and asset allocation throughout the backtesting period, which shows a significant improvement in performance with respect to final portfolio wealth.

Particularly interesting is once more that the flexibility of recourse decisions leads to a portfolio that differs more significantly from the index structure, while satisfying all risk constraints. This flexibility leads to decisions that respond to market developments by rebalancing (or recourse) actions. We observe that during the spread widening period (beginning of 1999 to the end of 2000), the model suggests significant investment in government bonds, whereas in the spread tightening periods (before and after the widening), the model invests almost entirely in corporate products. Once more, the model reacts to the market developments in an intuitive way.

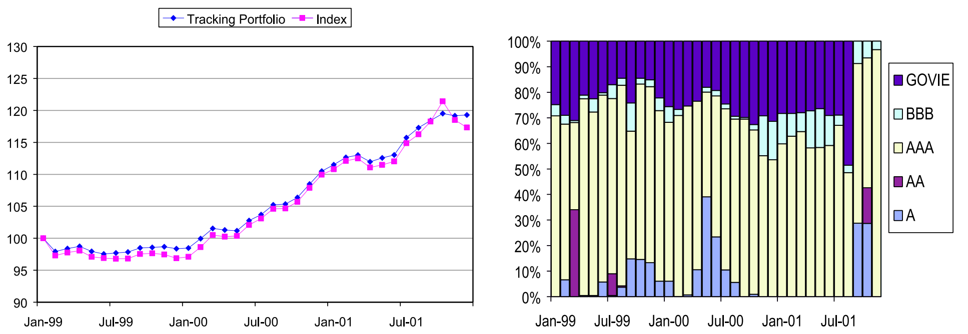
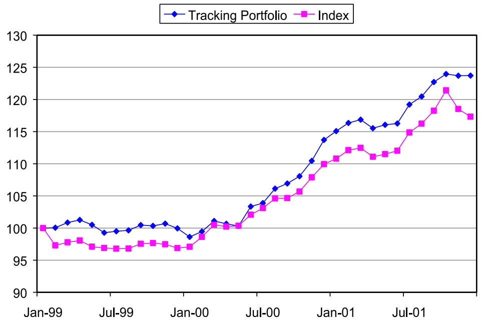


Fig. 9. Backtesting results anticipative model: portfolio value versus the index value (Government Index). Investment in treasury and corporate bonds is possible. Backtesting results: asset allocation according to the optimal solution of the anticipative model.



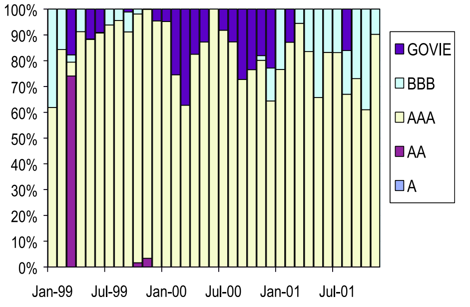


Fig. 10. Backtesting results two-stage model: portfolio value versus the index value (Government Index). Investment in treasury and corporate bonds is possible. Backtesting results: asset allocation according to the optimal first-stage solution of the two-stage model.

5.2.3. Case study 3: Credit portfolio management for funding fixed liabilities

We consider now the problem of funding a fixed liability using credit risky assets. Such problems appear in portfolio dedication and portfolio immunization for pension funds, and in numerous business settings. In order to avoid the risks associated with default these problems have traditionally being tackled using government bonds only. We show in this section that the two-stage stochastic programming models allow us to introduce credit risky assets in these important financial planning problems.

We setup an example whereby an initial endowment of 10,000 Euro must be invested over an 18 month horizon in order to achieve a target annualized return of 5% while paying a fixed liability of 2000 Euro at month 12. An asset universe of *AAA*, *AA*, *A* and *BBB* rated bonds of different maturities is used to construct an optimal portfolio. The optimality criterion was to minimize the CVaR against the target 5% return at the 99% confidence level.

As a benchmark we solve a single period CVaR optimization model without any constraints for funding the target liability. Fig. 11 illustrates the portfolio terminal wealth for the generated scenarios and the cash (reserve) account created through coupon payments, bonds maturing or cash investment from the beginning. Note that at month 12 the cash account has reached a level of 1200 Euro, which is significantly below the target. The portfolio expected return over the 18 month period is 15.2%. This figure can be used to benchmark the results of the following two experiments.

We then solve a second single period model that imposes CVaR constraints on the target liability. That is we aim at accumulating sufficient cash reserves to finance the liability. Fig. 12 illustrates the portfolio terminal wealth and the cash (reserve) account created through coupon payments and bond maturing. Note that at month 12 the cash account has reached the target level a level of 2000 Euro. However, this was possible through substantial investments in cash at the start of the planning period, and as a result the expected return over the 18 month horizon is reduced to 13%, indicating a significant cost in expected return in exchange for financing the liability.

Finally, we solve a two-stage model that imposes CVaR constraints on the target liability and allows us to rebalance the portfolio and sell bonds at period 12 in order

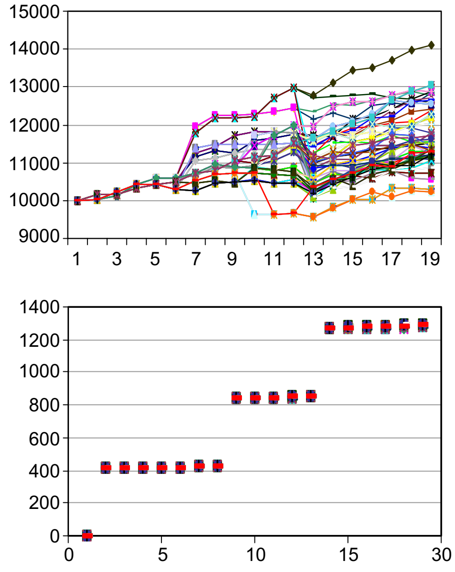


Fig. 11. Terminal wealth (top) and cash reserve (bottom) for a portfolio optimized using single period optimization without risk constraints in funding the target liability. The expected return is 15.2%.

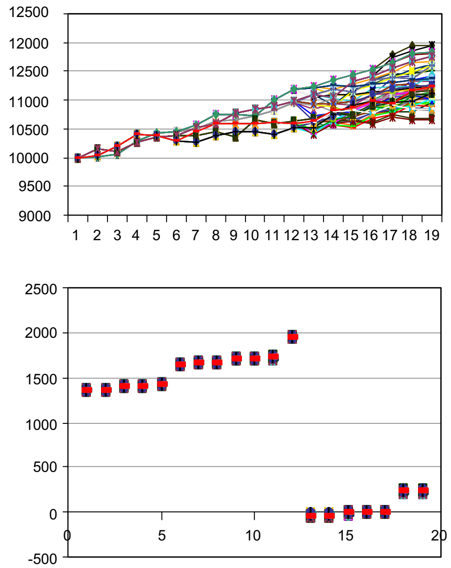


Fig. 12. Terminal wealth (top) and cash reserve (bottom) for a portfolio optimized using single period optimization with risk constraints in funding the target liability. Sufficient cash is at hand at month 12 to finance the liability, primarily through cash investment at the start of the planning period. The expected return is 13%.

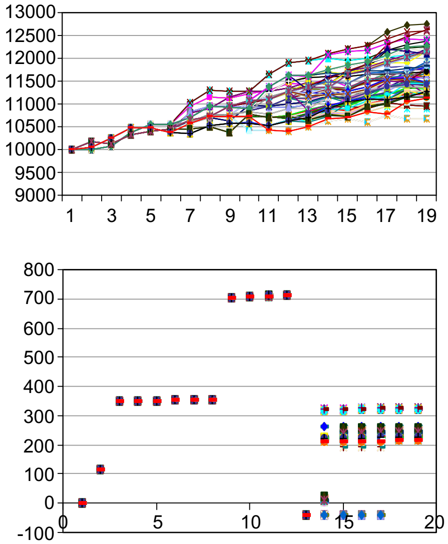


Fig. 13. Terminal wealth (top) and cash reserve (bottom) for a portfolio optimized using the two-stage model, with risk constraints in funding the target liability. Cash at month 12 is below the target liability, but portfolio rebalancing covers the shortfall. The expected return is 15.9%.

to satisfy the risk constraints. Fig. 13 illustrates the portfolio terminal wealth and the cash (reserve) account created through coupon payments and bond maturing. Note that at month 12 the cash account is only 700 (a substantial reduction from the previous experiment), however with proper rebalancing we can still meet the liabilities. Note that some cash remains after the portfolio is rebalanced, and the amount of cash depends on the scenarios—hence the multiple data points for each time period after month 12. The expected return over the 18 month horizon is reduced to 15.9% thus indicating that superior returns can be achieved while funding the liability when a stochastic programming model is used that allows us to model the complex dynamics of the problem.

6. Conclusion and further research

We have considered an important problem of financial planning involving fixed income assets which are subject to credit and market risk. In this paper we have introduced a number of innovations which we list below:

(i) We have combined models of credit risk and market risk which makes our generation of time varying prices of *fixed income* assets relatively more accurate compared to other approaches which do not jointly take into account these multiple sources of risk.

(ii) Within the decision making perspective we extend our earlier work and those of other researchers based on a simulation only paradigm of prices. We use a multistage decision making framework which takes into consideration (a) dynamical behaviour of the assets and (b) recourse decisions which respond appropriately to *the timing of default*.
(iii) Finally, our framework reveals one of the most powerful aspects of stochastic optimisation, that is, bringing together the optimum decision models with the descriptive approach of simulation models.
(iv) Overall we find the proposed models to perform well in the illustrative case study. The multi-stage models suggest more flexible and intuitive investment strategies.

Future research needs to address the interaction and incorporation of modern default risk models with the optimisation paradigm as well as further investigation of the stability of the optimisation/simulation results; hence, methods of limiting the inherent model risk need further development.

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References

Andersson, F., Mauser, H., Rosen, D., Uryasev, S., 2001. Credit risk optimization with conditional Value-at-Risk criterion. Mathematical Programming, Series B 89, 273–291.

Consigli, G., Dempster, M.A.H., 1998. The CALM stochastic programming model for dynamic asset and liability management. In: Ziemba, W.T., Mulvey, J.M. (Eds.), Worldwide Asset and Liability Modeling. Cambridge University Press, pp. 464–500.

Credit-Suisse-Financial-Products, 1997. Creditrisk^{+} a credit risk management framework. Technical Report, Credit-Suisse-Financial-Products.

Dynkin, L., Hyman, J., Konstantinovsky, V., 2001. Sufficient diversification in credit portfolios. Technical Report, Lehman Brothers, Fixed Income Research.

Ellison, E.F.D., Hajian, M., Levkovitz, R., Maros, I., Mitra, G., Sayers, D., 2002. FortMP manual, release 3. Technical Report, OptiRisk Systems and Brunel University, Available from: <http://www.optirisk-systems.com>.

Frey, R., McNeil, A.J., 2002. Var and expected shortfall in portfolios of dependent credit risks: Conceptual and practical insights. Technical Report, Swiss Banking Institute, University of Zürich, Department of Mathematics, ETH Zürich.

Hull, J., White, A., 1998. Using RiskMetrics to get a and σ in the Hull–White model. Training Material.

Jobst, N.J., 2002. On credit risk modelling, measurement and optimisation. PhD thesis, Brunel University, Department of Mathematical Sciences.

Jobst, N.J., de Servigny, A., 2005. An empirical investigation of equity default swaps II: Multivariate insights. Technical Report, Standard & Poor's.

Jobst, N.J., Horniman, M.D., Lucas, C.A., Mitra, G., 2001. Computational aspects of alternative portfolio selection models in the presence of discrete asset choice constraints. Quantitative Finance I, 1–13.

Jobst, N.J., Zenios, S.A., 2001a. The tail that wags the dog: Integrating credit risk in asset portfolios. The Journal of Risk Finance 3 (1), 31–43.

Jobst, N.J., Zenios, S.A., 2001b. Tracking bond indices in an integrated market and credit risk environment. Quantitative Finance 3 (2).

JP Morgan, 1997. CreditMetrics: The Benchmark for Understanding Credit Risk. Technical Document, New York.

Kealhofer, S., 1998. Portfolio management of default risk. Net Exposure, The Electronic Journal of Financial Risk 1 (2).

Kiesel, R., Perraudin, W., Taylor, A., 2001. The structure of credit risk: Spread volatility and ratings transitions. Technical Report, Bank of England, ISSN 1268-5562.

Kijima, M., 2000. A Gaussian Term Structure Model of Credit Spreads. Technical Report, Tokyo Metropolitan University.

Kijima, M., Muromachi, Y., 2000. Evaluation of Credit Risk of a Portfolio with Stochastic Interest Rate and Default Processes. Technical Report, Tokyo Metropolitan University.

Kouwenberg, R., Zenios, S.A., 2006. Stochastic programming for asset and liabiity management. In: Zenios, S.A., Ziemba, W.T., (Eds.), The Handbook of Asset and Liability Management, North-Holland, Amsterdam, The Netherlands, in print.

Mitra, G., Kyriakis, T., Lucas, C., Pirbhai, M., 2003. A review of portfolio planning: Models and systems. In: Satchell, S.E., Scowcroft, A.E. (Eds.), Advances in Portfolio Construction and Implementation, pp. 1–39.

Mulvey, J.M., Vladimirou, H., 1992. Stochastic network programming for financial planning problems. Management Science 38, 1642–1664.

Ramaswamy, Srichander, 2002. Managing credit risk in a corporate bond portfolio. Journal of Portfolio Management, 67–72.

RiskMetrics-Group. 1997. Creditmetrics—technical document. Technical Report, RiskMetrics-Group.

Rockafellar, R.T., Uryasev, S., 2000. Optimization of conditional Value-at-Risk. The Journal of Risk 2 (3), 21–41.

Schönbucher, P., 2003. Credit Derivatives Pricing Models. Wiley Finance.

Vasicek, O.A., 1977. An equilibrium characterization of the term structure. Journal of Financial Econometrics 5, 177–188.

Ziemba, W.T., Mulvey, J.M. (Eds.), 1998. Worldwide Asset and Liability Modeling. Cambridge University Press, Cambridge, England.