The Hodge Theorem

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10/14/2016

ABSTRACT. In this talk we will sketch a proof of the Hodge Theorem using Pseudodifferential Operators and Microlocal Analysis.

0. The Hodge Theorem

Let M be a closed (compact, no boundary) oriented Riemmanian manifold of dimension n, or even a submanifold of $\mathbf{R}^m$. We will denote by

$$C^\infty(M; \Lambda^k T^*M)$$

the space of differential k-forms on M. The Riemmanian metric lets us introduce an inner product on $\Lambda^k T_p^*M$ and $C^\infty(M; \Lambda^k T^*M)$. Let $dx_1, \ldots, dx_n$ denote an orthonormal basis of $\Lambda^k T_p^*M$. The **hodge star** is a linear map

$$\star : \Lambda^k T_p^*M \to \Lambda^{n-k} T_p^*M$$

defined by the condition that

$$dx_I \wedge \star dx_I = dx_1 \wedge \cdots \wedge dx_n,$$

for a multi-index I. The right-hand side is of course the Riemannian volume form. In other words,

$$\star dx_I = \pm dx_J,$$

where J is the multi-index containing those indices not appearing in I. The sign depends on k, n. $\star$ is certainly an isomorphism. We defined an inner product on $\Lambda^k T_p^*M$ by specifying

$$\langle \alpha, \beta \rangle d\text{vol} = \alpha \wedge *\beta.$$

Integrating, we have an inner product on $C^\infty(M; \Lambda^k T^*M)$ given by

$$\langle \alpha, \beta \rangle = \int_M \langle \alpha, \beta \rangle \ d\text{vol}.$$

This is the L^2 inner product of forms. Of course one can speak of $L^2(M; \Lambda^k T^*M)$ in which case the above inner product is the L^2 inner product and the resulting space is complete. For reasons which will be made clear below, we will in fact be dealing with complex-valued forms. This changes nothing except that we must complexify the metric and the hodge-star becomes conjugate linear.

The exterior derivative $d : C^\infty(M; \Lambda^k T^*M) \to C^\infty(M; \Lambda^{k+1} T^*M)$ is a differential operator. This means that d is an operator involving only taking derivatives and applying a linear map

$$\Lambda^k T^*M \to \Lambda^{k+1} T^*M.$$

We can thus ask what its adjoint δ is with respect to the L^2 inner product, i.e. for a map δ for which

$$\langle d\alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle.$$

Using Stoke's theorem we see that $\delta = \pm \star d \star$. We define the Hodge Laplacian Δ as

$$\Delta = d\delta + \delta d.$$

Notice that Δ is formally self-adjoint. Why is it called Δ? Let's see how it acts on $C^\infty(M) = C^\infty(M; \Lambda^0 M)$. If $f \in C^\infty(M)$, then

$$\Delta f = - \star d \star df,$$

which we easily see is (the negative of) the ordinary Laplacian.

Theorem 0.1 (Hodge)**.** *We have the following L^2 orthogonal decomposition:*

$$\ker d = \ker \Delta \oplus \operatorname{im} d.$$

In particular, a Riemmanian metric induces an isomorphism

$$H^k_{dR}(M) \cong \ker \Delta \subseteq C^\infty(M; \Lambda^k M)$$

Let's see what we can prove from this. First of all, we have Poincaré duality: if $\alpha \in \ker \Delta$, then $\star\alpha \in \ker \Delta$. Thus $\star$ is an isomoprhism between $\ker \Delta$ inside $C^\infty(M; \Lambda^k M)$ and $\ker \Delta \subseteq C^\infty(M; \Lambda^{n-k} M)$. In particular, $\star$ gives an isomorphism $H^k_{\mathrm{dR}} \cong H^{n-k}_{\mathrm{dR}}(M)$. If $\dim M = n = 2m$, then we also see using $\star$ that the intersection form given by

$$\int_M \langle \alpha, \cdot \rangle$$

is non-degenerate.

We now "prove" the Hodge theorem.

Proof. Δ is Fredholm on C^∞. This means that $\ker \Delta$ is finite-dimensional, $\operatorname{im} \Delta$ is closed, and we have the orthogonal decomposition

$$C^\infty(M; \Lambda^k M) = \ker \Delta \oplus \operatorname{im} \Delta^* = \ker \Delta \oplus \operatorname{im} \Delta.$$

Integrating by parts using d and δ (i.e. invoking the adjoint property) and using that $\langle, \cdot, \cdot \rangle$ is a positive-defininite inner product, one checks that $\ker \Delta = \ker d \cap \ker \delta$, and that $\operatorname{im} d, \operatorname{im} \delta$ are orthogonal to each other and to $\ker \Delta$. Since $\operatorname{im} \Delta \subseteq \operatorname{im} d \oplus \operatorname{im} \delta$, we have the orthorgonal decomposition

$$C^\infty(M; \Lambda^k M) = \ker \Delta \oplus \operatorname{im} d \oplus \operatorname{im} \delta.$$

If $\omega \in \ker d$, we write $\omega = \alpha + \beta + \gamma$. Since $d\omega = 0$, $d\gamma = 0$, and so $\gamma = 0$. Thus

$$\ker d = \ker \Delta + \operatorname{im} d,$$

as desired. This proof also shows that $\ker \Delta$, and hence $H^k_{dR}(M)$ is finite-dimensional, something which is not at all obvious from the definition. $\square$

The rest of this talk will be to justify the Fredholm property using pseudodifferential operators.

1. Differential Operators

Suppose P is a differential operator on and open set $\Omega \subseteq \mathbf{R}^n$. For example we can consider a constant-coefficient operator like

$$\partial_{x_1} \partial_{x_1} + 2\partial_{x_2},$$

or a variable coefficient operator like

$$\sin(x)\partial_x + y^2 \partial_y.$$

P is effectively a polynomial with coefficients in $C^\infty(\Omega)$ and with the variables replaced by the differential operators $D_j = -i\partial_{x_j}$, i.e. $P = p(x, D)$ for some polynomial

$$p \in C^\infty(\Omega)[\xi_1, \dots, \xi_n].$$

We call p the **symbol** of P. p is also the "Fourier transform" of P. Explicitly, if $u \in C_c^\infty(\Omega)$, then

$$Pu(x) = \frac{1}{(2\pi)^n} \int \int e^{i(x-y)\xi} p(x, \xi) u(y) \; dy d\xi. \tag{1.1}$$

For example, the symbol of the identity is 1, the symbol of D_j is ξ_j, and the symbol of $\sin(x) D_x$ is $\sin(x)\xi$.

The **principal symbol** $\sigma_P(x, \xi)$ of P is just the highest-degree part of p. Notice that if P is a differential operator of order k, then $\sigma_P(x, \xi)$ is homogeneous of degree k, too. We also notice that

$$e^{-ix\lambda\xi} P(e^{iy\lambda\xi} u(y))(x) = \lambda^k \sigma_P(x, \xi) u(x) + O(\lambda^{k-1}). \tag{1.2}$$

For example, if $P = D_x^2 + D_x$ on $\mathbf{R}$, then the only terms with a λ^2 in from of them will come from hitting $e^{iy\lambda\xi}$ with two derivatives, which will bring down a factor of $-\xi^2$.

It is clear that $\sigma_P(x,\xi)$ is multiplicative and additive. Thus it gives a homomorphism between the filterered algebra Diff of differential operators of order k and the graded algebra of polynomials. Moreover,

$$\sigma : \mathrm{Diff}_k/\mathrm{Diff}_{k-1} \to \mathrm{Hom}_k \tag{1.3}$$

is an isomorphism of algebras (where the right-hand side is the set of homogeneous polynomials of degree k). A differential operator P is called **elliptic** if $\sigma_P(x,\xi)$ is invertible away from $\xi = 0$. Examples include the ordinary Laplace operator

$$\Delta = D_{x_1}^2 + \cdots + D_{x_n}^2$$

on $\mathbf{R}^n$, whose symbol is $|\xi|^2$.

On a manifold, there is no consistent (i.e. coordinate-free) way to define the complete symbol of a differential operator. Nonetheless, the principal symbol becomes a well-defined map on T^*M. Namely if $\xi_1,\ldots,\xi_n$ are the dual coordinates to $x_1,\ldots,x_n$, so that an element of T_x^*M is written $\omega = \xi_1 dx_1 + \cdots \xi_n dx_n$, we may define $\sigma_P(x,\xi)$ coordinate-wise. If $(x,\xi) \in T^*M$ and $d\varphi(x) = \xi$, then we have that

$$e^{-i\lambda\varphi}P(e^{i\lambda\varphi}u(y))(x) = \lambda^k\sigma_P(x,\xi)u(x) + O(\lambda^{k-1}).$$

The above discussion generalizes to differential operators between sections vector bundles, which are locally the same as matrices of differential operators. The symbol is no longer a real number, but is now a linear map between the vector bundles.

2. Pseudodifferential Operators

Equation **??** is highly suggestive. What if instead of a polynomial, $p(x,\xi)$ is a more general symbol? Then the integral need not be well-defined, but we can still make sense of it. If $p(x,\xi)$ is a reasonable function, we call the operator $P(x,D)$ defined by the integral a Pseudodifferential operator. The functions we will allow for p will be called symbols. For a real number m we define $S^m(\Omega)$ to be the set of all smooth functions $p : \Omega \times \mathbf{R}^n \to \mathbf{C}$ satisfying the estimates

$$|\partial_x^\alpha \partial_\xi^\beta p(x,\xi)| \le C\sqrt{1+|\xi|^2}^{m-|\beta|}$$

on compact sets in x. In other words, hitting with an x derivative does not do anything, but hitting with a ξ derivative gives better decay. All polynomials are clearly symbols. If $p \in S^k$, we write $p(x,D) \in \Psi^k$.

We will set $S^{-\infty} = \bigcap S^m$. $S^{-\infty}$ consists of all symbols of rapid decay. The integrals defining the associated differential operators actually converge, and so if $P \in \Psi^{-\infty}$ is such a pseudodifferential operators, then

$$Pu(x) = \int K(x,y)u(y)\ dy$$

for some smooth K. We call P a **smoothing operator**.

Symbols are asymptotically complete, in the following sense: If $m_j \to -\infty$ is a decreasing sequence, and $p_j \in S^{m_j}$ are symbols, then there is a symbol $p \in S^m$ such that

$$p \sim \sum p_j$$

in the sense that

$$p - (p_1 + \cdots + p_k) \in S^{m_{k+1}}$$

for all j. Moreover p is unique modulo $S^{-\infty}$. This is like an asymptotics series.

To every Pseudodifferential Operator P, we can associate to it its principal symbol $\sigma_P(x, \xi) \in S^m(\Omega)$ by the same formula as **??**.

The pseudofferential operators form an algebra filtered by Ψ^m the pseudodifferential operators of order m. This means that we have a well-defined addition and composition which behaves exactly as one would expect. The algebra

$$S = \bigcup S^m / S^{-\infty}$$

is a graded algebra. The map σ is again a map between the filetered algebra Ψ of all pseudodifferential operators and the graded algebra S which factors almost as in **??**. What we lose is that the symbol map can no longer distinguish between two factors differing by an element in S^{m-1}. Thus we have the isomorphism

$$\sigma : \Psi^k / \Psi^{m-1} \to S^m / S^{m-1}.$$

Using charts, we can define Pseudodifferential operators on a manifold. Again, the "total symbol" is not well-defined, but the principal symbol is.

Pseudodifferential operators are unfortunately non-local; that is $Pu(x)$ depends on the behaviour of u everywhere, not just around x. The main utilitiy of Pseudodiffential operators is that they have mapping proprties which are the same as one would expect for differential operators. Namely if $P \in \Psi^m$, then P is continuous between the following spaces (with Ω a compact manifold. Similar results hold for open subsets of $\mathbf{R}^d$):

$$C^\infty(\Omega) \to C^\infty(\Omega),$$

$$H^s(\Omega) \to H^{s-m}(\Omega),$$

$$\mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega).$$

Here $H^s(\Omega)$ denotes the L^2 Sobolev space of functions with "s derivatives in $L^2(\Omega)$" (if s is a positive integer, this is a valid definition. If s is not, then one uses the Fourier transform). $\mathcal{D}'(\Omega)$ denotes the distributions.

Differential operators can be thought of as built up of smooth functions and vector fields, just like the natural numbers are built up of 0 and 1. An analogy is that the Pseudodifferential operators are like the real numbers to the natural numbers of the differential operators.

3. Elliptic Operators

The above analogy motivates that we can do the following. If P is an elliptic differential operator of order k, then σ_P is invertible away from 0. Thus one should expect to find an "almost-inverse" Q to P in Ψ^{-k}. We show how to do this. If χ is a smooth cutoff around 0, then set $q_1 = \frac{(1-\chi)}{\sigma_P} \in S^{-k}$. Let's examine

$$R_1 = \mathrm{Id} - q_1(x, D)p(x, D).$$

The principal symbols of both summands are the same, modulo a rapidly decreasing function. It follows that $R_1 \in \Psi^{-1}$ by the properties of principal symbols. Set $r_1 = \sigma_{R_1}$. Set $q_2 = \frac{(1-\chi)r_1}{\sigma_P} \in S^{-k-1}$. Then consider

$$R_2 = \mathrm{Id} - (q_1(x, D) + q_2(x, D))(p(x, D)) = R_1 - q_2(x, D)p(x, D).$$

The principal symbols of both summands are the same, and so $R_2 \in \Psi^{-2}$. Continuing, for any k we may find a symbol $\tilde{q}_k = q_1 + \cdots + q_k$ such that

$$\mathrm{Id} - \tilde{q}_k(x, D)p(x, D) \in \Psi^{-k}.$$

By asymptotic completeness, we may set $q \sim \sum q_j$ so that $q - (q_1 + \cdots q_k) \in S^{-k-1}$ for any k. Thus

$$\mathrm{Id} - q(x, D)p(x, D) \in \Psi^{-k}$$

for all k, and hence

$$\mathrm{Id} - q(x, D)p(x, D) \in \Psi^{-\infty}$$

is smoothing. Likewise we may find a (perhaps different q) which works on the right. Standard group theory shows that both q are the same up to a smoothing operator, i.e.

$$q(x, D)p(x, D) = \mathrm{Id} - R_1, \quad p(x, D)q(x, D) = \mathrm{Id} - R_2,$$

where $R_1, R_2 \in \Psi^{-\infty}$ are smoothing operators.

It is easy to see that if M is a closed manifold, then smoothing operators are bounded between any two $H^s(M)$ spaces, and hence are compact between any two (for instance by Reillich-Kondrashov). Thus if $p(x, D)$ is elliptic, then $p(x, D)$ is invertible modulo compact operators on any $H^s(M)$, and is thus by Atkinson's theorem Fredholm on all of them (i.e. has finite-dimensional kernel, cokernel, and has closed range). Taking intersections and using Sobolev embedding, we see that

$$C^\infty(M) = \ker P \oplus \operatorname{im} P^*,$$

and $\ker P$ is finite-dimensional, and $\operatorname{im} P^*$ is closed. Here $\operatorname{im} P^*$ denotes the adjoint to P, and $\sigma_{P^*} = (\sigma_P)^*$.

Naturally, the above discussion also generalizes to vector bundles.

4. Δ is elliptic

The last thing we need to show is that the Hodge Laplacian Δ is elliptic. We show that

$$\sigma_\Delta(x,\xi)\omega = |\xi|^2\omega.$$

Since the symbol map is an algebra homomorpshim, and the symbol of δ is the adjoint to the symbol of d, we just need to find the symbol of d. We check

$$e^{-i\lambda\varphi}d(e^{i\lambda\varphi}\omega) = i\lambda d\varphi \wedge \omega \pm d\omega.$$

Thus $\sigma_d(x,\xi)\omega = i\xi \wedge \omega$. Since δ, d are adjoints, $\sigma_\delta(x,\xi)\omega = -i\xi \lrcorner \omega$, where $\lrcorner$ denotes interior multiplication via the dual vector to ξ. So

$$\sigma_\Delta(x,\xi)\omega = \xi \wedge (\xi \lrcorner \omega) + \xi \lrcorner (\xi \wedge \omega).$$

To evaluate this, we pick a convenient basis and test against basis elements. Suppose $\tau = \xi/|\xi|$, and extend τ to an orthonormal basis $\{\tau_1 = \tau, \tau_2, \dots, \tau_n\}$ of T_x^*M. Set

$$\tau_I = \tau_{i_1} \wedge \cdots \wedge \tau_{i_k} \in \Lambda^k T_x^* M.$$

Suppose first that $1 \in I$. Then $\xi \wedge \tau_I = 0$, but, denoting $\tilde{I} = (i_2, \dots, i_k)$, $\xi \lrcorner \tau_I = \tau_1(\xi)\tau_{\tilde{I}} = |\xi|\tau_{\tilde{I}}$. Then

$$\xi \wedge |\xi|\tau_{\tilde{I}} = |\xi|^2 \tau \wedge \tau_{\tilde{I}} = |\xi|^2\tau_I.$$

A similar thing happens if $1 \notin I$.

5. Elliptic Complexes

Suppose M is a manifold (without boundary), and $E_1, E_2, \dots, E_k$ are vector bundles over M. Suppose $d_i : C^\infty(M; E_i) \to C^\infty(M; E_{i+1})$ are differential operators which satisfy $d_{i+1}d_i = 0$. Such data is called a **differential complex**. We will call a differential complex **elliptic** if for all $\xi \neq 0$, the sequence

$$0 \to E_1 \to E_2 \to \ldots \to E_k \to 0,$$

with the maps at each arrow being $\sigma_{d_i}(x,\xi)$ is exact away from the zero section $\xi = 0$.

Examples of elliptic complexes are the de Rham complex of differential forms, any two trivial bundles linked by an elliptic operator. It is easy to see (just algebra) that a complex is elliptic iff either of the following two conditions holds: let δ_i denote the adjoint to d_i. Then a complex is elliptic iff the associated Laplace opertors $\Delta_i = \delta_i d_i + d_{i-1}\delta_{i-1}$ are elliptic. Denoting $E = \bigoplus E_i$, we also have a map $d : E \to E$ and its adjoint $\delta : E \to E$. Then the complex is elliptic iff $d + \delta$ is elliptic.

Define the index of a differential complex to the alternating sum of the differences of dimensions of the kernels and cokernels of the d_i, i.e. the index is

$$\sum_{i=0}^{k} (-1)^{i+1}(\dim\ker d_{i+1} - \dim\operatorname{cok} d_i).$$

If M is compact, then we have actually proved the following: if E is an elliptic complex, then the index of E is well-defined. The Hodge Theorem in particular says that the index of the de Rham complex is the Euler characteristic of M.