



**UNIVERSITÀ  
DEGLI STUDI  
DI TRIESTE**

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DIPARTIMENTO DI MATEMATICA, INFORMATICA E GEOSCIENZE

Corso di Laurea Magistrale in Matematica

**Large Deviations Principle for the  
Korteweg-de Vries Equation for long timescales**

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**Anno Accademico 2023/2024**



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## Abstract

In this thesis we study the formation of rogue waves from a probabilistic perspective. We consider the periodic Korteweg-De Vries equation, which is used to model the evolution of long water waves in a rectangular canal, with small random initial data of average size  $\varepsilon$ .

We first establish the local and global well-posedness of the KdV equation in  $L^2(\mathbb{T})$  as well as the local well-posedness in  $\mathcal{FL}^{0,1}(\mathbb{T})$ . We exploit this approach in order to obtain an approximation of the dynamics of the KdV equation up to timescales of the form  $\varepsilon^{-2}$ . We use this approximation to obtain a large deviations principle (LDP) for the KdV equation up to said timescales, *i.e.* a sharp asymptotic development of the fluctuations around the null solution of the KdV equation.

We then exploit normal form transformations in order to derive a nonlinear approximation to the solution of the KdV equation for longer timescales. In particular, we give a formal construction of a normal form transformation which puts the whole KdV hierarchy in normal form up to an arbitrary order. A rigorous analysis of this normal form up to order 4 then allows us to extend the upper bound of the LDP to timescales of the form  $\varepsilon^{-3}$ .

## Sommario

In questa tesi analizziamo la formazione di onde anomale da un punto di vista probabilistico. Consideriamo l'equazione di Korteweg-De Vries periodica, che modellizza l'evoluzione di onde d'acqua in un canale rettangolare, con un dato iniziale aleatorio piccolo, di ampiezza media  $\varepsilon$ .

Dapprima ci concentriamo sulla buona positura locale e globale in  $L^2(\mathbb{T})$ , e sulla buona positura locale in  $\mathcal{FL}^{0,1}(\mathbb{T})$ . Sfruttiamo questo approccio per ottenere un'approssimazione della dinamica dell'equazione KdV fino a tempi di ordine  $\varepsilon^{-2}$ . Utilizziamo poi questa approssimazione per ottenere un principio delle grandi deviazioni (LDP) per l'equazione KdV fino a tali tempistiche, ovvero un preciso sviluppo asintotico delle fluttuazioni attorno alla soluzione nulla dell'equazione KdV.

Infine utilizziamo la forma normale per ricavare un'approssimazione nonlineare della soluzione dell'equazione KdV per tempi più lunghi. In particolare forniamo una costruzione formale di una trasformazione che mette in forma normale tutti gli integrali primi di KdV fino ad un ordine arbitrario. Un'analisi rigorosa della forma normale fino all'ordine 4 ci permetterà di estendere il limite superiore del LDP fino a tempi di ordine  $\varepsilon^{-3}$ .

# Chapter 1

## Introduction

### 1.1 Background and motivation

Oceanographers define rogue waves as deep-water waves whose height exceeds twice the characteristic wave height, which is four times the standard deviation of the sea surface height. They appear suddenly in deep sea, with a vertical size of the order of 20 – 30m, and they are a serious threat for naval infrastructures. There is a large debate regarding the mechanism of formation of rogue waves, as well as many *deterministic* studies.

Researchers began to approach this problem with techniques from probability and statistics, in particular after the works by Dematteis et al. [9, 10]. In these two works they consider two dispersive equations on the 1-dimensional torus: the model proposed in [10] comes from Dysthe's modified NLS equation, while in [9] they use the cubic NLS equation. In particular, they consider a random initial datum

$$u_0(x) = \sum_{k \in \mathbb{Z}} c_k \theta_k e^{ikx} \quad (1.1.1)$$

for  $x \in \mathbb{T}$ . The coefficients  $(c_k)_k$  in [10], which are fast-decaying, are chosen according to observations in the North Sea (Joint North Sea Wave Project)<sup>1</sup>. The  $\theta_k$ 's are independent identically distributed (IID) standard complex Gaussian random variables over the same probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ .

This way the solution  $u(t, x)$  is a random variable, and the problem is to estimate the probability

$$\mathbb{P}(\|u(t)\| > \lambda) \quad (1.1.2)$$

where  $\lambda > 0$  and the norm depends on the functional space to which  $u$  belongs. The usual choice is the sup norm

$$\|u(t)\|_{L^\infty} = \sup_{x \in \mathbb{T}} |u(t, x)|,$$

given that we are interested in the height of the wave.

Dematteis et al. conjectured the existence of a Large Deviations Principle (LDP), *i.e.* a sharp asymptotic development for the probability (1.1.2) as  $\lambda \rightarrow \infty$ <sup>2</sup>, backed by a series of numerical results. On the one hand, they numerically estimate (1.1.2) by a Monte Carlo approach. On the other hand, they use Large Deviations techniques to predict the typical profile of rogue waves, which is shown to agree with their Monte Carlo simulations. Let us elaborate on these ideas in the next few paragraphs.

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<sup>1</sup>We will see that the choice of these coefficients directly impacts the statistics of rogue wave formation, see Theorem 1.2.1.

<sup>2</sup>We will give more details in Section 1.6.

With the former approach Dematteis et al. numerically solve the Dysthe equation, up to a fixed time  $t$ , for a large family of initial data created by sampling the  $\theta_k$  according to their distribution. Out of the many solutions they obtain, they only keep those that reach a minimal amplitude  $\lambda > 0$  and disregard the rest. It turns out that, once the peak of the remaining solutions is properly translated to  $x = 0$ , they all have a similar profile, see Figure 1.1. Moreover, one can then give an empirical estimate of (1.1.2) as the ratio between the number of initial data which for which (1.1.2) is satisfied and the total amount of initial data.

They compared these results with an approximation of this profile obtained using Large Deviations Techniques. More precisely, they write (1.1.2) as a Gaussian integral over the non-explicit region  $\{\|u(t)\|_{L^\infty} > \lambda\}$ . Inspired by a Laplace-type approximation, whose idea is suggested in Theorem A.4.1, they numerically maximize the density function over the integration region, which leads to a specific profile. This profile, depicted as the black curve in Figure 1.1, is shown to empirically agree with the typical profile obtained using Monte Carlo simulations (dark blue curve).

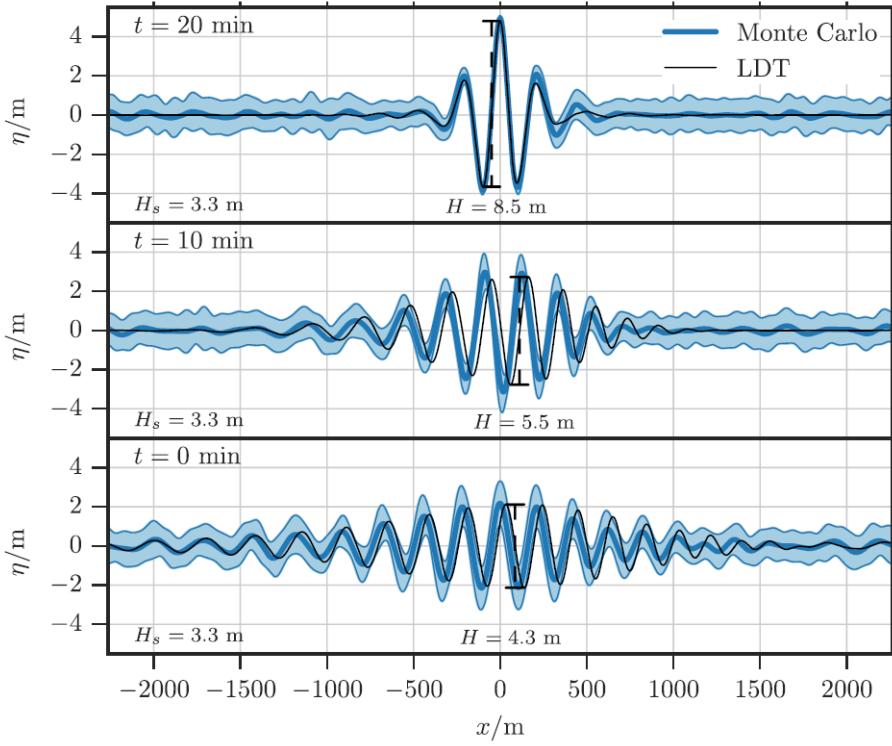


Figure 1.1: Comparison between Monte Carlo sampling and the Large Deviation Technique provided in [10]. The dark blue curve represents the mean of Monte Carlo realizations such that  $\sup_x |u(t, x)| \geq 4.8$  m, with  $t = 0, 10, 20$  min, while the black one represents the solution to the optimization problem. As we can see, they are very close to one another especially around the rogue wave ( $t = 20$  min).

These results are only numerical: a rigorous mathematical theory is missing. In [15] Garrido et al. rigorously derived a LDP for the solution to the cubic NLS equation in a weakly nonlinear setting. As explained in Section 1.6 the optimization problem proposed in [10] is very hard to solve in practice. The strategy proposed by Garrido et al. consists in approximating the solution with another random variable whose statistical properties are easier to study. One of the aims of this thesis is to test the robustness of these techniques in the context of a different dispersive PDE: the KdV equation.

From a physical perspective, this model is particularly interesting as it describes the evolution of long water waves in a rectangular canal [22]. From a technical perspective, this equation is more challenging than the NLS equation treated in [15] on account of the derivatives in the nonlinear term, which give rise to a loss of derivatives unless properly handled. Moreover, the KdV equation is a good approximation to the Dysthe equation, the model used in the simulations in [10], for which no well-posedness theory<sup>3</sup> is available on  $\mathbb{T}$ .

Indeed, starting from the Dysthe equation, up to a time scaling, a Galilean transformation  $x \mapsto x + ct$  and a complex rotation of an angle  $ax + bt$  we can arrive to an equation of the form

$$v_t + v_{xxx} = N(v),$$

where the linear part is exactly the same of the KdV equation and the nonlinear terms are cubic and involve at most one derivative of the function  $v$ , as in the case of the KdV equation.

That being said, the KdV equation has one important advantage: it is a completely integrable Hamiltonian system. This allows us to exploit strong techniques from normal form theory to study its dynamics.

## 1.2 Statement of results

We consider the periodic KdV equation

$$u_t + u_{xxx} + uu_x = 0, \quad x \in \mathbb{T} \tag{1.2.1}$$

with a real-valued, mean-zero<sup>4</sup> random initial datum

$$u_0(x) = \varepsilon \sum_{k \neq 0} c_k \eta_k e^{ikx} \tag{1.2.2}$$

where  $\varepsilon > 0$ ,  $c_k = ae^{-b|k|}$  or  $c_k = ae^{-bk^2}$  with  $a, b > 0$ , and  $(\eta_k)_{k \geq 1}$  are IID standard complex Gaussian random variables with  $\overline{\eta_k} = \eta_{-k}$ . We remark that the Cauchy problem associated to the KdV equation is unconditionally globally well-posed<sup>5</sup> in  $L^2(\mathbb{T})$ , and locally in  $\mathcal{FL}^{0,1}(\mathbb{T})$ . Moreover, we will show that, for the timescales that we are interested in, the solution lives in  $\mathcal{FL}^{0,1}(\mathbb{T})$ .

The most important result of this thesis is the following Large Deviations Principle for the solution of the KdV equation:

**Theorem 1.2.1** (LDP for the KdV equation). *Consider the KdV equation with random initial value (1.2.2). Let  $u_\varepsilon$  be the corresponding solution. Assume that  $t \lesssim \varepsilon^{-2\alpha+\gamma}$  with  $\alpha \in (0, 1)$  and  $\gamma > 0$  as small as desired. Then for any  $\lambda > 0$  we have that*

$$\lim_{\varepsilon \rightarrow 0^+} \varepsilon^{2-2\alpha} \log \mathbb{P} \left( \sup_{x \in \mathbb{T}} |u_\varepsilon(t, x)| \geq \lambda \varepsilon^\alpha \right) = - \frac{\lambda^2}{4 \sum_{k=1}^{+\infty} c_k^2}. \tag{1.2.3}$$

**Remark 1.2.2.** The choice of the functional space  $\mathcal{FL}^{0,1}(\mathbb{T})$  is due to the fact that  $\mathcal{FL}^{0,1}(\mathbb{T}) \subseteq L^\infty(\mathbb{T})$ , therefore we can take the sup-norm of the solution, which captures the height of the wave. A priori we don't know if the solution (which belongs almost surely to  $L^2(\mathbb{T})$ ) remains in  $\mathcal{FL}^{0,1}(\mathbb{T})$  for all times, but with a bootstrap argument<sup>6</sup> we are able to prove that the solution belongs to  $\mathcal{FL}^{0,1}(\mathbb{T})$  almost surely for  $t \lesssim \varepsilon^{-2\alpha+\gamma}$ .

<sup>3</sup>To the best of our knowledge, only some Strichartz and multilinear-estimates are known, see [17].

<sup>4</sup>By Lemma 2.1.2 it will be clear that we can make this choice without loss of generality.

<sup>5</sup>See Definition 2.1.1.

<sup>6</sup>See Proposition 1.4.1.

**Remark 1.2.3.** The right-hand side in (1.2.3) doesn't depend on the time  $t$ . This might seem very surprising, but the reason is the following: in order to prove (1.2.3) we will derive a Large Deviations Principle for the solution  $u_{\text{app},\varepsilon}(t,x)$  to the linearized equation

$$u_t + u_{xxx} = 0, \quad x \in \mathbb{T}$$

with  $u(0,x) = u_0(x)$ , whose statistic doesn't depend on time, and we will approximate the solution  $u_\varepsilon(t,x)$  with  $u_{\text{app},\varepsilon}(t,x)$  uniformly in  $t$ , for  $t \lesssim \varepsilon^{-2\alpha+\gamma}$ .

The most surprising thing is that the *same* LDP holds for longer timescales, for example  $t \lesssim \varepsilon^{-3\alpha+\gamma}$ . This is due to the complete integrability of the KdV equation, as we shall see soon. We will prove the following

**Theorem 1.2.4.** *Consider the KdV equation with random initial value (1.2.2). Let  $u_\varepsilon$  be the corresponding solution. Assume that  $t \lesssim \varepsilon^{-3\alpha+\gamma}$  with  $\alpha \in (0,1)$  and  $\gamma > 0$  as small as desired. Then for any  $\lambda > 0$  we have that*

$$\limsup_{\varepsilon \rightarrow 0^+} \varepsilon^{2-2\alpha} \log \mathbb{P} \left( \sup_{x \in \mathbb{T}} |u_\varepsilon(t,x)| \geq \lambda \varepsilon^\alpha \right) \leq -\frac{\lambda^2}{4 \sum_{k=1}^{+\infty} c_k^2}. \quad (1.2.4)$$

**Remark 1.2.5.** We conjecture that the same holds for the lower bound, but we need different techniques, see Section 4.4 for more details. Moreover, using normal form techniques, we conjecture that we should be able to prove (1.2.4) for  $t \lesssim \varepsilon^{-n\alpha+\gamma}$  for all  $n \in \mathbb{N}$ .

In the next few pages we discuss the main ideas of the techniques employed in the proof of these results.

### 1.3 The Korteweg-de Vries equation

The Korteweg-de Vries (KdV) equation is a nonlinear partial differential equation, introduced by Boussinesq [7] in 1877 and then re-discovered by Korteweg and De Vries in 1895 [21]. It is used to model the evolution of long water waves in a rectangular canal. The general form of the KdV equation is

$$u_t + au_{xxx} + buu_x = 0 \quad a, b \in \mathbb{R}. \quad (1.3.1)$$

The choice of the coefficients  $a$  and  $b$  is not so important because we can consider them equal to one after rescaling and, if necessary, a time inversion  $t \mapsto -t$ . The KdV equation is a nonlinear *dispersive* PDE.

**Definition 1.3.1** (Dispersive PDE). A nonlinear *dispersive* PDE is a partial differential equation of the form

$$u_t + \sum_{k=0}^n a_k \partial_x^k u = \mathcal{N}(u), \quad (1.3.2)$$

with  $n \in \mathbb{N}$  and  $a_k \in \mathbb{C}$ , where  $\mathcal{N}(u)$  is the non-linearity and the *dispersive relation*

$$h(\xi) = \sum_{k=0}^n a_k i^{k-1} \xi^k$$

is a real-valued polynomial in  $\xi \in \mathbb{R}$ . This implies that, when we consider the linearized equation in the Fourier setting, we find

$$\hat{u}(t, \xi) = e^{-ith(\xi)} \hat{u}(0, \xi) \quad \text{or} \quad u_k(t) = e^{-ith(k)} u_k(0)$$

depending on the choice  $x \in \mathbb{R}$  or  $x \in \mathbb{T}$ . In both cases, for a fixed time  $t$ , the solution to the linearized equation has exactly the same  $H^s$ -regularity of the initial datum, with the same norm. Informally, "dispersion" refers to the fact that different frequencies in these equations will tend to propagate at different velocities, thus dispersing the solution over time. This dispersion may encounter some obstacle, in particular on compact domains such as the torus.

From a modelling perspective, we are interested in the periodic KdV equation, so we'll consider  $x \in \mathbb{T}$ . The period in the simulations in [10] is roughly 5km, which we normalize after rescaling.

The KdV equation is also an *integrable system*, meaning that it has infinitely many conserved quantities in involution<sup>7</sup>. These first integrals are functionals of  $u$  of the form

$$F^{(n)} = \int_{\mathbb{T}} P_n(u, \partial_x u, \dots, \partial_x^n u) dx,$$

with  $n \geq 0$ , where  $P$  is a polynomial in  $u$  and a finite number of its spacial derivatives. Such quantities are, a priori, dependent on time  $t$  for a general function  $u$ , but they are constant when evaluated on a solution to (1.3.1), as proved in Appendix A.2.

Our first approach to the KdV equation is classical: we consider a deterministic initial datum  $u_0 \in L^2(\mathbb{T})$  and we prove the local and global well-posedness, *i.e.* the existence of a unique solution in  $L^2(\mathbb{T})$ , which depends *continuously* on the initial data. This argument can be found in [12], which follows from ideas in [2]. It is convient to perform the computations in Fourier space: we first exploit that the mean is a conserved quantity to reduce our analysis to the case of mean-zero initial data. Then, after introducing the *interaction variables*  $\nu_k = u_k e^{-ik^3 t}$ , where  $(u_k)_k$  are the Fourier coefficients of  $u$ , we rewrite (1.2.1) as follows:

$$\nu_k(t) = \nu_k(0) - \frac{ik}{2} \int_0^t \sum_{k_1+k_2=k} e^{-3ikk_1k_2s} \nu_{k_1}(s) \nu_{k_2}(s) ds. \quad (1.3.3)$$

Note that, due to the fact that we have a derivative in the non-linearity, we cannot apply the Banach fixed point theorem to (1.3.3) in the space  $C_t^0 \ell_k^2([-\delta, \delta] \times (\mathbb{Z} \setminus \{0\}))$ , for some  $\delta > 0$ . One way to overcome this difficulty is by integrating by parts the complex exponential in order to absorb the factor of  $k$ . Naturally, we may not integrate by parts when the exponent  $3kk_1k_2$  is zero: such tuples  $(k_1, k_2)$  are called *resonant*. Luckily there are no resonant terms the first time we integrate by parts, thanks to the fact that the mean is zero, *i.e.*  $\nu_0 = 0$ . The reason why we have to integrate by parts twice is that, after integrating by parts once, in the integral remains the convolution of  $\ell^2$ -sequences which loses summability.<sup>8</sup> This will be clear in Lemma 2.1.8, in particular in the equation (2.1.14).

This procedure is analogous to Normal Form Transformations typical of Hamiltonian systems, as we shall see in Chapter 3. Once we have performed integration by parts two times, we separate high and low frequencies and then we apply the Banach fixed point theorem to obtain local well-posedness. Separating frequencies is indeed necessary: we already discussed why (1.3.3) is not enough to prove well-posedness. Unfortunately, the formula obtained after integrating by parts twice is also not entirely satisfactory, as one has to deal with the boundary terms. As a result, it is not enough to shrink the time interval  $[-\delta, \delta]$  to obtain a contraction, one must also bound these boundary terms. The

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<sup>7</sup>See Appendix A.2 for the precise definition.

<sup>8</sup>When proving the local well-posedness in  $\mathcal{FL}^{0,1}(\mathbb{T})$  we decided to exploit the same strategy as in  $L^2(\mathbb{T})$ , but in that case one integration by parts is enough. This follows by the fact that  $\ell^1$  is an algebra, Remark A.1.2.

precise way of combining these ingredients in order to prove well-posedness may be found in Theorem 2.1.11.

Note that the global well-posedness will be a consequence of the conservation of the  $L^2$ -norm for the solutions of the KdV equation.

Since we are interested in the height of the wave, we shall work in some functional space  $X \subseteq L^\infty(\mathbb{T})$ . Different spaces  $X$  are possible, such as  $H^s(\mathbb{T})$  with  $s > \frac{1}{2}$  or the Fourier-Lebesgue space  $\mathcal{FL}^{0,1}(\mathbb{T})$ , which is the choice we made in this thesis. The local well-posedness theory in  $\mathcal{FL}^{0,1}$  admits the same arguments as that in  $L^2$ . In particular, since  $\mathcal{FL}^{0,1} \subseteq L^2$ , arguments based on integration by parts are still justified. Moreover, we will show that the solution remains in  $\mathcal{FL}^{0,1}$  for long times via a bootstrap argument, which we explain next.

## 1.4 Estimates on the solution and bootstrap arguments

A key step in the proof of Theorem 1.2.1 is obtaining a good approximation  $u_{\text{app}}$  to the solution  $u$  to the KdV equation (1.2.1). Assuming that  $\nu_k(t) = \mathcal{O}(\varepsilon)$  (which is true for  $t = 0$ ), it is clear that the more we integrate by parts (1.3.3), the smaller the remaining terms will be. In particular, this suggests that a first approximation  $u_{\text{app}}$  might consist of the top order terms after integrating by parts (1.3.3) once. This is precisely  $e^{-t\partial_x^3}u(0)$ , the solution to the linearized KdV equation around zero, also known as the *Airy equation*,

$$u_t + u_{xxx} = 0.$$

In particular, the integration-by-parts process yields not only the local well-posedness theory, but also estimates on the solution. We can prove that

$$\left\| u(t) - e^{-t\partial_x^3}u(0) \right\|_{\mathcal{FL}^{0,1}} \leq \|u(t)\|_{\mathcal{FL}^{0,1}}^2 + \|u(0)\|_{\mathcal{FL}^{0,1}}^2 + t \sup_{s \in [0,t]} \|u(s)\|_{\mathcal{FL}^{0,1}}^3, \quad (1.4.1)$$

If we knew that  $u(t) = \mathcal{O}(\varepsilon)$ , it would be clear from (1.4.1) that

$$\left\| u(t) - e^{-t\partial_x^3}u(0) \right\|_{\mathcal{FL}^{0,1}} \lesssim \varepsilon^2 + t\varepsilon^3 \ll \varepsilon$$

as long as  $t \ll \varepsilon^{-2}$ . Thus heuristically the linear flow is a good approximation of the nonlinear dynamics up to times  $t \ll \varepsilon^{-2}$ . However, this intuitive computation heavily relies on the assumption that  $u(t) = \mathcal{O}(\varepsilon)$  for all such times, which is only known at time  $t = 0$ . One way to provide a rigorous proof of this fact from the a priori bound (1.4.1) is the bootstrap principle. This principle, also known as continuity method, is a continuous analogue of mathematical induction. An abstract way of stating it is the following:

**Proposition 1.4.1** (Bootstrap principle - Proposition 1.21 in [31]). *Let  $[0, T]$  be a time interval. Let  $H_t$  be an hypothesis at time  $t \in [0, T]$ , and let  $C_t$  be a conclusion. Assume that:*

- (1)  $H_0$  is true;
- (2) If  $H_t$  is true, then  $C_t$  is true;
- (3) If  $C_t$  is true, then there exists  $\delta > 0$  such that  $H_{t'}$  is true for  $t' \in (t - \delta, t + \delta)$ ;
- (4) If  $C_{t_n}$  is true for  $t_n \rightarrow t$ , then  $C_t$  is true.

Then  $C_t$  is true  $\forall t \in [0, T]$ .

*Proof.* With these hypothesis it is trivial to prove that  $C_t$  holds in a non-empty, open and closed subspace  $I \subseteq [0, T]$ . But  $[0, T]$  is connected, therefore  $I = [0, T]$ .  $\square$

The reader can find an easy, but concrete, example of this argument used to prove the Gronwall's inequality, in Theorem A.1.8.

After the second integration by parts, we will find a better (nonlinear) approximation  $u_{\text{app}}(t)$  such that  $\|u_{\text{app}}(t)\|_{\mathcal{F}L^{0,1}} = \|u(0)\|_{\mathcal{F}L^{0,1}}$  and

$$\begin{aligned} \|u(t) - u_{\text{app}}(t)\|_{\mathcal{F}L^{0,1}} &\lesssim \|u(t)\|_{\mathcal{F}L^{0,1}}^2 + \|u(t)\|_{\mathcal{F}L^{0,1}}^3 + \|u(0)\|_{\mathcal{F}L^{0,1}}^2 + \|u(0)\|_{\mathcal{F}L^{0,1}}^3 \\ &+ t \sup_{s \in [0,t]} \left( \|u(s)\|_{\mathcal{F}L^{0,1}}^4 + \|u(s)\|_{\mathcal{F}L^{0,1}}^9 \right). \end{aligned} \quad (1.4.2)$$

The precise statement with the proof can be found in Lemma 2.3.3. Thus, again using the bootstrap principle, we will prove that

$$\|u(t) - u_{\text{app}}(t)\|_{\mathcal{F}L^{0,1}} \ll \varepsilon$$

as long as  $t \ll \varepsilon^{-3}$ . Hence  $u_{\text{app}}(t)$  approximates the solution  $u(t)$  to the KdV equation for longer timescales.

The goal then becomes to improve the estimates (1.4.1) and (1.4.2) finding another function  $u_{\text{app}}(t)$  such that a bound similar to (1.4.2) holds with an higher exponent in the term multiplied by  $t$ . Theoretically one can perform other integrations by parts to obtain a more precise approximation for the dynamics, but this becomes quite difficult in practice. Indeed, an explicit expression for the resonant terms requires to find all the integer solutions to a cubic polynomial with more and more variables, which is a very hard problem in number theory. This justifies the research of a more systematic approach: *normal form transformations*.

## 1.5 Normal form and conserved quantities

The Normal Form is a more robust approach to our problem because it exploits the Hamiltonian structure of the KdV equation. This allows us to exploit the first integrals of the KdV equation, *i.e.* the quantities which are conserved along the flow. Let us carefully explain all the ingredients at our disposal.

It is a well known fact that it is possible to write the KdV equation as a Hamiltonian system. Actually<sup>9</sup> the KdV equation admits a Bi-Hamiltonian structure, meaning that we can choose between two different Hamiltonian functions, with the corresponding Poisson structures. Let

$$H^{(1)}(u) = \frac{1}{2} \int_{\mathbb{T}} \left( u_x^2 - \frac{1}{3} u^3 \right) dx, \quad H^{(2)}(u) = \frac{1}{2} \int_{\mathbb{T}} u^2 dx$$

and

$$J_1 = \partial_x, \quad J_2 = - \left( \partial_{xxx} + \frac{2}{3} u \partial_x + \frac{1}{3} u_x \right).$$

We can write the KdV equation in physical space as

$$\partial_t u = J_1 \frac{dH^{(1)}}{du} = J_2 \frac{dH^{(2)}}{du} \quad (1.5.1)$$

where  $d/du$  denotes the  $L^2$ -differential of the functional  $H^{(j)}$  and  $J_1, J_2$  are skew-adjoint with respect to the  $L^2$ -scalar product. The fact that  $J_1$  and  $J_2$  are skew-adjoint is crucial,

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<sup>9</sup>This was firstly observed by Lax [25, 24] and Magri [27].

because it allows us to introduce a Poisson structure. Given two functionals  $F(u)$  and  $G(u)$ , we can define

$$\{F, G\}_j = \int_{\mathbb{T}} \frac{dF}{du} J_j \frac{dG}{du} dx \quad j = 1, 2.$$

With this Bi-Hamiltonian structure we are able to explicitly construct all the conserved quantities of (1.3.1), following the works by Lax and Magri. Moreover these quantities Poisson-commute (equivalently, we say that they are in involution), meaning that  $\{F^{(n)}, F^{(m)}\} = 0$  for all  $n, m$ . The family  $(F^{(n)})_{n=-1}^{+\infty}$  is known as the *KdV hierarchy*. All these conserved quantities share a very particular form:

$$F^{(-1)} = \int_{\mathbb{T}} u dx, \quad F^{(n)} = \frac{1}{2} \int_{\mathbb{T}} [(\partial^n u)^2 + P_{\geq 3}^{(n)}(u, \partial u, \dots, \partial^{n-1} u)] dx$$

for  $n \geq 0$ , where  $P_{\geq 3}^{(n)}$  is a polynomial in  $u$  and its derivatives up to order  $n - 1$ , whose monomials are of degree at least 3.

Once we found all the conserved quantities, we choose the first Hamiltonian structure and we begin studying the KdV equation as a Hamiltonian system in the Fourier setting. We have

$$\mathcal{H}(\mathbf{u}, \bar{\mathbf{u}}) = \mathcal{H}_2 + \mathcal{H}_3 = \pi \sum_{k \neq 0} k^2 u_k \bar{u}_k - \frac{\pi}{3} \sum_{k_1+k_2+k_3=0} u_{k_1} u_{k_2} u_{k_3}$$

with the Poisson bracket

$$\{F, G\} = \sum_{k \neq 0} \frac{ik}{2\pi} \frac{\partial F}{\partial u_k} \frac{\partial G}{\partial \bar{u}_k}.$$

We highlight that the Poisson bracket loses derivatives on account of the factor  $k$ .

One can then write the KdV equation as

$$\partial_t u_k = \{u_k, \mathcal{H}\} \quad \forall k \in \mathbb{Z}. \quad (1.5.2)$$

We consider only the zero-mean case<sup>10</sup>.

A standard method to study the dynamics of a Hamiltonian system is to find new variables  $\mathbf{v}$  for which the Poisson structure is preserved. These are called *canonical* variables and the transformation  $\mathbf{u} = \Phi(\mathbf{v})$  is called *canonical*. We look for a canonical transformation  $\Phi$ , depending on  $n \geq 2$ , such that

$$\mathcal{H} \circ \Phi(\mathbf{v}) = \mathcal{H}_2 + \hat{\mathcal{H}}_4 + \hat{\mathcal{H}}_6 + \dots + \hat{\mathcal{H}}_n + R_{n+1}, \quad (1.5.3)$$

with

$$\begin{cases} \hat{\mathcal{H}}_n = 0 & \text{if } n \text{ is odd,} \\ \hat{\mathcal{H}}_n = \hat{\mathcal{H}}_n(\mathbf{I}) & \text{if } n \text{ is even,} \end{cases}$$

where  $\mathbf{I} = (|v_k|^2)_{k \neq 0}$  and  $\hat{\mathcal{H}}_{2j}(\mathbf{I})$  is a homogeneous formal polynomial<sup>11</sup> in the  $(|v_k|^2)_k$  with real coefficients. Finally the remainder  $R_{n+1}$  has a zero of order  $n + 1$  at the origin. This is known as the *normal form* of  $\mathcal{H}$  up to order  $n$ .

Constructing such a  $\Phi$  is one of the most important result of this thesis, see Section 3.4 for the full details. We highlight that the difference with respect to the results in [20] is the fact that our construction follows that of the standard Birkhoff normal form, which consists of removing non-resonant terms at each order  $n$  using auxiliary Hamiltonians of degree  $n$ . In order to construct a transformation  $\Phi$  that puts  $\mathcal{H}$  in normal form up to a given order, we will exploit the first integrals, their structure and the fact that they commute with  $\mathcal{H}$ .

<sup>10</sup>As we know how to generalize the result using Lemma 2.1.2

<sup>11</sup>See Definition 3.1.6.

In fact,  $\Phi$  will put the whole KdV hierarchy in normal form up to the same order. One drawback of this approach is that we work on a higher regularity than [20].

From a mathematical viewpoint, the most challenging part is proving that  $\Phi$  is an invertible transformation in a neighborhood of the origin of some functional space  $X$  and that it is close to the identity. We prove it rigorously for  $n = 3, 4$  in Section 3.2 and Section 3.3. For  $n \geq 5$ , it is still an open question, but we conjecture that our argument should hold. The key difference is that for orders 3 and 4 we can explicitly find the transformations and notice that we don't lose derivatives, despite the Poisson bracket. For example, to put the KdV Hamiltonian in normal form up to order 3, we use the transformation given by the time 1 flow associated to the auxiliary Hamiltonian.

$$G_3 = i \frac{\pi}{3} \sum_{k_1+k_2+k_3=0} \frac{u_{k_1} u_{k_2} u_{k_3}}{k_1^3 + k_2^3 + k_3^3}.$$

Using the fact that  $k_1 + k_2 + k_3 = 0$  we obtain that the denominator can be factored out as  $k_1^3 + k_2^3 + k_3^3 = 3k_1 k_2 k_3$ . This compensates the loss of derivatives given by the Poisson bracket. For orders  $n \geq 5$  we will use transformations given by the time 1 flow associated to auxiliary Hamiltonian functions of the form

$$G_n^{(l)} = -i \sum_{\substack{k_1+\dots+k_n=0 \\ k_1^{2l+1}+\dots+k_n^{2l+1} \neq 0}} \frac{\tilde{f}_n^{(l)}(k_1, \dots, k_n)}{k_1^{2l+1} + \dots + k_n^{2l+1}} u_{k_1} \dots u_{k_n}.$$

where  $l \in \mathbb{N}$  indicates the first integral  $F^{(l)}$  of the KdV hierarchy that we use to generate the transformation and  $\tilde{f}_n^{(l)}(k_1, \dots, k_n)$  are the coefficients of the homogeneous formal polynomial of degree  $n$  in  $F^{(l)}$  after we have put the whole KdV hierarchy in normal form up to order  $n - 1$ . This time we cannot factor out the denominator to control the loss of derivatives, and thus different ideas are required, some of which we outline in Chapter 3.

The upshot of these transformations is a simplified analysis of the dynamics of  $v$  thanks to the particular form of the *new* Hamiltonian  $\mathcal{H} \circ \Phi$ . The key observation is that the part of the Hamiltonian which is in normal form, contributes to the dynamics only with a complex rotation (for each Fourier mode). In particular, this implies that  $|v_k(t)| \approx |v_k(0)|$  over long timescales  $t \ll \varepsilon^{-n+1}$ . This key fact allows us to write the problem of estimating the probability (1.1.2) in terms of the random variables in the initial datum, whose statistics we know well. The detailed study of the dynamics can be found in Section 3.5.

The larger the integer  $n$  for which we can justify (1.5.3), the longer the timescale under which we will be able to approximate the dynamics of KdV. In particular, the results in this thesis are a first step towards showing that (1.5.3) should hold for any  $n \in \mathbb{N}$ . We thus conjecture that Theorem 1.2.4 holds for  $t \ll \varepsilon^{-n+1}$  for any  $n$ .

As previously explained, the key obstacle to turn this conjecture into a theorem is a rigorous justification that  $\Phi$  is an invertible transformation in a neighborhood of the origin of some functional space  $X$  contained in  $L_x^\infty$ .

## 1.6 Large Deviations

A Large Deviations Principle is a sharp description of the asymptotic behaviour of a sequence of probability distributions. In particular we are interested in the height of the wave, which we measure via the sup-norm of the solution. Since we are working with the zero-mean, real, KdV equation, we consider a random initial datum of the form

$$u_\varepsilon(0, x) = \varepsilon \sum_{k \neq 0} c_k \eta_k^\omega e^{ikx} \tag{1.6.1}$$

where  $0 < \varepsilon \ll 1$ ,  $(c_k)_k \in \ell^1(\mathbb{Z}^*, \mathbb{R}_+)$  with  $c_k = c_{-k}$  and  $(\eta_k)_{k \geq 1}$  are independent identically distributed (I.I.D.) complex Gaussian random variables with  $\eta_{-k} = \bar{\eta}_k$  and

$$\mathbb{E}\eta_k = 0, \quad \mathbb{E}\eta_k^2 = 0, \quad \mathbb{E}\eta_k \bar{\eta}_k = 1.$$

The coefficients  $(c_k)_k$  are chosen of the form

$$c_k = a e^{-b|k|} \quad \text{or} \quad c_k = a e^{-bk^2}$$

with  $a, b > 0$ , as in [10], which implies that

$$u_\varepsilon(0) \in H^\infty(\mathbb{T}) = \bigcap_{s \geq 0} H^s(\mathbb{T})$$

almost surely. The existence of a unique solution  $u_\varepsilon(t, x) \in L^\infty(\mathbb{T})$  follows from the well-posedness theory presented in Chapter 2. Our goal is to study the fluctuations of  $\|u_\varepsilon\|_{L^\infty}$  around the null solution:

$$\mathbb{P} \left( \sup_{x \in \mathbb{T}} |u_\varepsilon(t, x)| > \lambda_\varepsilon \right) \tag{1.6.2}$$

for some  $\lambda_\varepsilon > 0$ , possibly depending on  $\varepsilon$ , and for  $t \in [0, T_\varepsilon]$ .

Dematteis et al. propose a way to approximate (1.6.2) using Laplace's method [10], a common technique to estimate Gaussian integrals, see Theorem A.4.1 for an account of the main ideas. More precisely, they start from an initial datum with a finite number non-zero Fourier modes<sup>12</sup>,

$$u(0, x) = \sum_{|k| \leq N} \theta_k e^{ikx}$$

and define the set

$$\mathcal{A}(t, \lambda) = \left\{ \theta = (\theta_k)_k \in \mathbb{C}^{2N+1} \mid \sup_{x \in \mathbb{T}} |u(t, x)| > \lambda \right\}.$$

Then for a fixed  $t > 0$ , they claim that

$$\log \mathbb{P}(\mathcal{A}(t, \lambda)) \approx I(\theta^*(\lambda))$$

provided that the minimization problem

$$\theta^*(\lambda) = \arg \min_{\theta \in \mathcal{A}(t, \lambda)} I(\theta)$$

has a *unique* solution, where

$$I(\theta) = \max_{y \in \mathbb{C}^{2N+1}} [\langle y, \theta \rangle - S(y)], \quad S(y) = \log \mathbb{E} e^{\langle y, \theta \rangle}.$$

The main problem of this approach is that in practice we are not able to solve the minimization problem. Indeed the set  $\mathcal{A}(t, \lambda)$  is not convex, and it depends on the dynamics of a non-linear equation whose solution is not explicit.

For this reason we choose to follow the approach in [15]. The key idea is to approximate the dynamics with a function  $u_{\text{app}, \varepsilon}$  whose statistical properties are easier to study. We previously discussed how to obtain such an approximation in Section 1.4 and Section 1.5.

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<sup>12</sup>The  $\theta_k$ 's are again complex Gaussian random variables for any  $k \in \mathbb{Z}$  such that  $|k| \leq N$ , with mean zero and variance  $c_k^2$ . The difference with our problem is that they consider also complex-valued solutions and they don't restrict to the zero-mean case.

A LDP for the solution of the KdV equation will then follow from a LDP for the approximated solution  $u_{\text{app},\varepsilon}(t,x)$ . However, such approximations are precise enough only for certain timescales, hence in order to reach longer timescales we will require better and better approximations. In order to prove Theorem 1.2.1, *i.e.* for timescales  $t \ll \varepsilon^{-2\alpha}$ , a linear approximation is enough. Indeed, we will prove that for the solution  $u_\varepsilon(t,x)$  to the Cauchy problem

$$\begin{cases} u_t + u_{xxx} = 0, & x \in \mathbb{T}, \\ u_\varepsilon(0,x) = \varepsilon \sum_{k \neq 0} c_k \eta_k^\omega e^{ikx} \end{cases}$$

the following LDP holds for all times:

$$\lim_{\varepsilon \rightarrow 0^+} \varepsilon^2 \log \mathbb{P}(\|u_\varepsilon(t)\|_{\mathcal{F}L^{0,1}} > \lambda) = \lim_{\varepsilon \rightarrow 0^+} \varepsilon^2 \log \mathbb{P}(\|u_\varepsilon(t)\|_{L^\infty} > \lambda) = -\frac{\lambda^2}{4 \sum_{k=1}^{+\infty} c_k^2}. \quad (1.6.3)$$

We highlight that we need this LDP in both norms,  $L^\infty$  and  $\mathcal{F}L^{0,1}$ , to prove Theorem 1.2.1.

In order to reach timescales  $t \geq \varepsilon^{-2\alpha}$ , one needs a better approximation. In particular, we need to identify the main nonlinear terms which contribute to the dynamics of the KdV equation up to the desired timescale  $t$ . These nonlinear terms are precisely the *resonant* terms introduced in Section 1.3. Equivalently, these terms are exactly the ones which come from the polynomials  $\hat{\mathcal{H}}_j$  in (1.5.3). Their contribution may be explicitly integrated in order to come up with the following better approximation:

$$u_{\text{app},\varepsilon}(t,x) = \varepsilon \sum_{k \neq 0} c_k e^{i \int_0^t F_k(\mathbf{J}) ds} \eta_k e^{ikx} \quad (1.6.4)$$

where  $F_k$  is a polynomial with real coefficients in  $\mathbf{J} = (J_k)_k = \left( |\Phi^{-1}(\mathbf{u})_k|^2 \right)_k$ .

As explained in Remark 1.2.5, this approximation allows us to prove the upper bound in Theorem 1.2.1 up to timescales  $t \ll \varepsilon^{-3\alpha}$ , by using the following key observation

$$\sup_{x \in \mathbb{T}} |u_{\text{app},\varepsilon}(t,x)| \leq \varepsilon \sum_{k \neq 0} c_k |\eta_k|$$

together with (1.6.3).

We conjecture that our techniques will yield the upper bound of Remark 1.2.5 for all times  $t \ll \varepsilon^{-n}$  for any  $n \in \mathbb{N}$ . The only remaining ingredients are some properties on the transformation  $\Phi$  when  $n \geq 5$ . In particular one needs to find some functional space  $X$ , continuously embedded in  $\mathcal{F}L^{0,1}(\mathbb{T})$ , where the KdV equation is locally well-posed and where we have the invertibility of  $\Phi$ , as well as the fact that  $\Phi$ , together with its inverse, is close to the identity. This latter property will allow us to approximate the solution  $u$  of the KdV equation using the dynamics in the new variables  $v = \Phi^{-1}(u)$ .

We highlight that in order to extend Theorem 1.2.1 to timescales  $t \ll \varepsilon^{-n}$  for any  $n \in \mathbb{N}$ , one also needs sharp lower bounds. The main obstacle towards proving a lower bound, is that we need statistical information about  $\|u_{\text{app},\varepsilon}(t)\|_{L_x^\infty}$ , as opposed to  $\|u_{\text{app},\varepsilon}(t)\|_{\mathcal{F}L^{0,1}}$  which was sufficient for an upper bound. While in the case of the linear approximation it is easy to see that  $u_{\text{app},\varepsilon}(t,x)$  is Gaussian for each fixed  $x \in \mathbb{T}$ , c.f. Section 4.1, this is not necessarily the case for the nonlinear approximations (1.6.4). New ideas are thus needed in order to overcome these difficulties.

## 1.7 Notation

- We will write  $A \lesssim B$  if there exists a positive constant  $C$  such that  $A \leq CB$ . If the constant depends on some other quantity  $p$ , we will write  $A \lesssim_p B$ . Sometimes we will

write explicitly a constant  $C$ : even if we use the same letter, this constant may change from line to line.

- We will use different spaces of sequences  $\ell^p(\mathbb{Z})$ , with  $1 \leq p < \infty$ , where

$$\ell^p(\mathbb{Z}) = \left\{ x = (x_k)_k \subseteq \mathbb{C} \mid \sum_{k \in \mathbb{Z}} |x_k|^p < \infty \right\}$$

which are Banach spaces with the usual norm

$$\|x\|_{\ell^p} = \left( \sum_{k \in \mathbb{Z}} |x_k|^p \right)^{\frac{1}{p}}$$

and the usual generalization to  $p = \infty$ . In particular we are interested in the subspaces

$$\dot{\ell}^p(\mathbb{Z}) = \{x \in \ell^p(\mathbb{Z}) \mid x_0 = 0, \quad \overline{x_k} = x_{-k}\}.$$

- To simplify the notation, we will often write  $u_x$  instead of  $\partial_x u$ , meaning the partial derivative of  $u$  with respect to  $x$ .
- We will consider the torus  $\mathbb{T} \cong \frac{\mathbb{R}}{2\pi\mathbb{Z}}$ , and functions  $f : \mathbb{T} \rightarrow \mathbb{C}$ , identifying them with the corresponding  $2\pi$ -periodic functions  $f : \mathbb{R} \rightarrow \mathbb{C}$ .
- With  $\mathcal{F}L^{s,p}(\mathbb{T})$  we will denote the Fourier-Lebesgue space, namely the Banach space of measurable functions  $f : \mathbb{T} \rightarrow \mathbb{C}$  whose Fourier coefficients  $(f_k)_k$  satisfy

$$\|\langle k \rangle^s f_k\|_{\ell_k^p} < \infty$$

where  $\langle k \rangle = \sqrt{1 + k^2}$  is the usual Japanese bracket. In the case  $p = 2$  they coincide with the usual Sobolev spaces  $H^s(\mathbb{T})$ , which we will indicate with a small  $h$  if we mean the space of sequences, with the same convention of above for what concerns  $\dot{h}^s$ . Notice that

$$\mathcal{F}L^{r,p}(\mathbb{T}) \subseteq \mathcal{F}L^{s,p}(\mathbb{T}) \quad \text{if } s \leq r.$$

We will often use the subspace

$$\mathcal{F}L_0^{s,p}(\mathbb{T}, \mathbb{R}) = \{f \in \mathcal{F}L^{s,p}(\mathbb{T}, \mathbb{R}) \mid f_0 = 0\}.$$

- The scalar product in  $L^2(\mathbb{T}, \mathbb{R})$  is

$$\langle f, g \rangle_{L^2} = \int_{\mathbb{T}} f(x)g(x) dx,$$

while the scalar product in  $\ell^2$  is

$$\langle \mathbf{u}, \mathbf{v} \rangle_{\ell^2} = 2\pi \sum_{k \in \mathbb{Z}} u_k \overline{v_k}.$$

This choice is due to Parseval's inequality.

- We will write  $X \sim_d Y$  when two random variables have the same distribution.
- With the notation  $\sum^*$  we indicate the sum over all the indexes for which the denominator doesn't vanish.

- If it is not specified, with the symbol  $\mathbb{N}$  we denote the set of positive natural numbers

$$\mathbb{N} = \{1, 2, 3, \dots\}.$$

- With  $u \in C_t^0 E_x(I \times \mathbb{T})$ , where  $E$  is some Banach space of functions defined on the torus  $\mathbb{T}$ , we mean that the map

$$\begin{aligned} P_t : I &\longrightarrow E_x \\ t &\longmapsto u(t) \end{aligned}$$

which associates to each time  $t$  the function  $u(t) \in E_x$  is continuous (when on  $I$  we put the Euclidean topology and on  $E_x$  the topology induced by the norm).



## Chapter 2

# General theory and well-posedness of KdV equation

In this chapter we investigate the local and global well-posedness of the *real* Korteweg–De Vries equation on the 1-dimensional torus, from a *deterministic* point of view. The KdV equation is a non-linear partial differential equation of the form

$$u_t + u_{xxx} + uu_x = 0, \quad x \in \mathbb{T}. \quad (2.0.1)$$

It appears in many different formulations which can be obtained from this one with some transformation. For historical reasons it is often written as

$$u_t + u_{xxx} \pm 6uu_x = 0,$$

where the sign doesn't matter because we can change it substituting  $u = -v$ , and if we consider

$$u_t + u_{xxx} + 6uu_x = 0$$

it is enough to substitute  $6u = v$  to obtain

$$v_t + v_{xxx} + vv_x = 0.$$

It is a well known fact, as we shall investigate in Appendix A.2, that this equation is an *integrable system*. In particular it has infinitely many first integrals in involution. For instance, if

$$u(t, x) : I \times \mathbb{T} \rightarrow \mathbb{R}$$

is a smooth solution of (2.0.1), then

$$\int_{\mathbb{T}} u(t, x) dx = \int_{\mathbb{T}} u(0, x) dx, \quad (2.0.2)$$

indeed

$$\frac{d}{dt} \int_{\mathbb{T}} u(t, x) dx = \int_{\mathbb{T}} u_t dx = - \int_{\mathbb{T}} (u_{xxx} + uu_x) dx = - \int_{\mathbb{T}} \partial_x \left( u_{xx} + \frac{u^2}{2} \right) dx = 0.$$

Similarly one can prove that

$$\int_{\mathbb{T}} u^2(t, x) dx = \int_{\mathbb{T}} u^2(0, x) dx \quad (2.0.3)$$

using integration by parts. In particular all the smooth solutions of the KdV equation have constant mean and  $L^2$ -norm.

Before starting with the KdV equation, we spend a few words about the linear part of it, called the *Airy equation*, which reads

$$u_t + u_{xxx} = 0. \quad (2.0.4)$$

We introduce the Fourier series representation

$$u(t, x) = \sum_{k \in \mathbb{Z}} u_k(t) e^{ikx}, \quad (2.0.5)$$

where

$$u_k = \frac{1}{2\pi} \int_0^{2\pi} u(t, x) e^{-ikx} dx.$$

The Airy equation in the Fourier setting becomes

$$\dot{u}_k = ik^3 u_k, \quad \forall k \in \mathbb{Z}. \quad (2.0.6)$$

If we define the solutions of (2.0.4) as the functions whose Fourier coefficients satisfy (2.0.6), then we have that for any  $s \geq 0$  and for any  $u(0) \in H^s(\mathbb{T})$ , there exists a unique global solution  $u \in C_t^0 H_x^s(\mathbb{R} \times \mathbb{T})$  given by

$$u_k(t) = e^{ik^3 t} u_k(0).$$

We define the bounded linear operator

$$e^{-t\partial_x^3} : H^s(\mathbb{T}) \longrightarrow H^s(\mathbb{T}) \quad (2.0.7)$$

as the map in the physical space  $H^s(\mathbb{T})$  whose corresponding operator in the Fourier setting is the Fourier multiplier which multiplies the  $k^{\text{th}}$  Fourier coefficient of  $u$  by  $e^{ik^3 t}$ . Notice that this correspondence is given by the fact that the Fourier operator  $\mathcal{F} : H^s(\mathbb{T}) \longrightarrow h^s(\mathbb{T})$  which associates to each function its Fourier coefficient is an isometric isomorphism. Therefore the following diagram commutes

$$\begin{array}{ccc} H^s(\mathbb{T}) & \xrightarrow{e^{-t\partial_x^3}} & H^s(\mathbb{T}) \\ \downarrow \mathcal{F} & & \downarrow \mathcal{F} \\ h^s(\mathbb{Z}) & \xrightarrow{e^{ik^3 t}} & h^s(\mathbb{Z}). \end{array}$$

Notice also that the solution  $u(t, x) = e^{-t\partial_x^3} u(0)$  has exactly the same regularity of  $u(0)$  in the space variable, with the same  $H^s$ -norm. This is a typical phenomenon for dispersive equations, whose solutions typically don't gain regularity with respect to the initial datum (as opposed to, say, the heat equation).

## 2.1 Local and global well-posedness of KdV equation in $L^2(\mathbb{T})$

We focus now on the local well-posedness of the KdV equation with initial data in  $L^2(\mathbb{T}, \mathbb{R})$ . Consider the Cauchy problem

$$\begin{cases} u_t + u_{xxx} + uu_x = 0, & x \in \mathbb{T} \\ u(0, x) = g(x) \in L^2(\mathbb{T}). \end{cases} \quad (2.1.1)$$

**Definition 2.1.1.** (Well-posedness, [31, Definition 3.4]). We say that the problem (2.1.1) is *locally well-posed* in  $L_x^2(\mathbb{T})$  if for any  $g^* \in L^2(\mathbb{T})$  there exists  $\delta > 0$ , an open ball  $B \subseteq L^2(\mathbb{T})$  containing  $g^*$  and a subset  $X \subseteq C_t^0 L_x^2([-\delta, \delta] \times \mathbb{T}, \mathbb{R})$  such that for any  $g \in B$  there exists a unique solution  $u \in X$  to the integral equation<sup>1</sup>

$$u(t, x) = e^{-t\partial_x^3} g - \int_0^t e^{-(t-s)\partial_x^3} u(s, x) u_x(s, x) ds, \quad \forall x \in \mathbb{T} \quad (2.1.2)$$

where  $e^{-t\partial_x^3}$  was defined in (2.0.7). Furthermore the data-to-solution map  $g \mapsto u$  from  $B$  (with the  $L^2$  topology) to  $X$  (with the  $C_t^0 L_x^2$  topology) is continuous. If  $X = C_t^0 L_x^2([-\delta, \delta] \times \mathbb{T}, \mathbb{R})$ , we say that the local well-posedness is *unconditional*. If we can take  $\delta$  as large as we want, we say that the problem is *globally well-posed*.

We will prove that the KdV equation is unconditionally well-posed in  $L^2(\mathbb{T})$ . The first result presented allows us to consider only mean-zero initial data.

**Lemma 2.1.2.** Suppose that (2.1.1) is locally well-posed for mean-zero initial data in  $L^2(\mathbb{T})$ . Consider some initial datum  $g \in L^2(\mathbb{T})$  such that

$$C = \frac{1}{2\pi} \int_{\mathbb{T}} g(x) dx \neq 0,$$

which does not depend on time by (2.0.2). Then there exists  $v(t, x)$  mean-zero solution of (2.0.1) such that  $v(t, x - Ct) + C$  solves (2.1.1). In particular (2.1.1) is locally well-posed in the whole space  $L^2(\mathbb{T})$ . Moreover, the time of existence for the solution to (2.1.1) is exactly the same of the solution to the mean-zero problem (2.1.3).

*Proof.* Let  $v(t, x)$  be the unique solution of

$$\begin{cases} v_t + v_{xxx} + vv_x = 0, & x \in \mathbb{T} \\ v(0, x) = g(x) - C \in L^2(\mathbb{T}), \end{cases} \quad (2.1.3)$$

which is a problem with mean-zero initial datum. Then  $w(t, x) = v(t, x) + C$  satisfies

$$\begin{cases} w_t + w_{xxx} + ww_x - Cw_x = 0, & x \in \mathbb{T} \\ w(0, x) = g(x) \in L^2(\mathbb{T}). \end{cases}$$

Therefore  $u(t, x) = w(t, x - Ct) = v(t, x - Ct) + C$  satisfies

$$\begin{cases} u_t + u_{xxx} + uu_x = 0, & x \in \mathbb{T} \\ u(0, x) = g(x) \in L^2(\mathbb{T}). \end{cases} \quad (2.1.4)$$

Viceversa, if  $u(t, x)$  satisfies (2.1.4), then  $v(t, x) = u(t, x + Ct) - C$  satisfies (2.1.3). This yields existence and uniqueness for the solution of (2.1.1), once we have proved them for the particular case of mean-zero initial data. The continuity with respect to the initial data follows from the fact that

$$\begin{aligned} \|u_n - u\|_{L_t^\infty L_x^2} &= \|v_n(t, x - C_n t) + C_n - v(t, x - Ct) - C\|_{L_t^\infty L_x^2} \\ &\leq \sqrt{2\pi} |C_n - C| + \|v_n(t, x - C_n t) - v(t, x - Ct)\|_{L_t^\infty L_x^2} \\ &\quad + \|v(t, x - C_n t) - v(t, x - Ct)\|_{L_t^\infty L_x^2} \\ &\leq \sqrt{2\pi} |C_n - C| + \|v_n - v\|_{L_t^\infty L_x^2} + \|v(t, x) - v(t, x + (C_n - C)t)\|_{L_t^\infty L_x^2}. \end{aligned}$$

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<sup>1</sup>Known as Duhamel's formula.

If we assume that  $g_n \rightarrow g$  in  $L^2(\mathbb{T})$ , then  $|C_n - C| \rightarrow 0$  and by the local well-posedness in the mean-zero case we have that  $\|v_n - v\|_{L_t^\infty L_x^2} \rightarrow 0$ . Moreover, by Theorem A.1.5,

$$\begin{aligned} \|v(t, x) - v(t, x + (C_n - C)t)\|_{L_t^\infty L_x^2}^2 &= 2\pi \sup_{t \in [-\delta, \delta]} \sum_{k \neq 0} |e^{itk(C_n - C)} - 1|^2 |v_k(t)|^2 \\ &\leq 8\pi \sum_{|k| > N} |v_k(t)|^2 + 2\pi |tN(C_n - C)| \sum_{\substack{|k| \leq N \\ k \neq 0}} |v_k(t)|^2. \end{aligned}$$

But  $\{(v_k(t))_k \mid t \in [-\delta, \delta]\}$  is compact in  $\ell^2$  by the continuity of the solution in time, therefore it has equibounded tails<sup>2</sup>. Given  $\varepsilon > 0$ , we can choose  $N$  big enough, such that

$$\sup_{t \in [-\delta, \delta]} \sum_{|k| > N} |v_k(t)|^2 \leq \varepsilon.$$

Then

$$\sup_{t \in [-\delta, \delta]} 2\pi |tN(C_n - C)| \sum_{\substack{|k| \leq N \\ k \neq 0}} |v_k(t)|^2 \leq \delta N |C_n - C| \|v\|_{L_t^\infty L_x^2}^2 \leq \varepsilon$$

for  $n$  big enough since  $|C_n - C| \rightarrow 0$ , and the proof is complete by the arbitrariness of  $\varepsilon$ .  $\square$

**Remark 2.1.3.** With the conservation of the  $L^2$ -norm we will prove in Corollary 2.1.15 that the solutions in the mean-zero case are actually global. This argument extends trivially to the general case.

We will prove the local well-posedness with a fixed point argument. This argument is presented in [12, Section 3.4]. To this aim we will use the Fourier series representation (2.0.5) where  $u_k = \bar{u}_{-k}$  since  $u$  is real valued<sup>3</sup> and  $u_0 = 0$  since it has mean-zero.

**Lemma 2.1.4.** *If we let*

$$u_k(t) = \nu_k(t) e^{ik^3 t} \quad \forall k \in \mathbb{Z}, \tag{2.1.5}$$

and  $u$  is a solution of the KdV equation, then  $\nu_k(t)$  satisfies

$$\nu_k(t) = \nu_k(0) - \frac{ik}{2} \int_0^t \sum_{k_1+k_2=k} e^{-3ikk_1k_2s} \nu_{k_1}(s) \nu_{k_2}(s) ds. \tag{2.1.6}$$

These new variables are called interaction variables.

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<sup>2</sup>See Proposition A.1.7 in the Appendix.

<sup>3</sup>We will only treat the real KdV equation, *i.e.* we will find the real-valued solution to the KdV equation with a real-valued initial value. The equation when we consider complex-valued solutions is different, and actually it is not even locally well-posed in  $L^2$ , as shown in [6]. However the real KdV equation is a good approximation to model water waves, as explained in [22, Chapter 7]

*Proof.* The equation (2.0.1) in Fourier variables becomes

$$\begin{aligned}
\partial_t u_k &= ik^3 u_k - i \sum_{k_1+k_2=k} k_1 u_{k_1} u_{k_2} \\
&= ik^3 u_k - i \sum_{\substack{k_1+k_2=k \\ k_1 < k_2}} k_1 u_{k_1} u_{k_2} - i \sum_{\substack{k_1+k_2=k \\ k_1=k_2}} k_1 u_{k_1} u_{k_2} - i \sum_{\substack{k_1+k_2=k \\ k_1 > k_2}} k_1 u_{k_1} u_{k_2} \\
&= ik^3 u_k - i \sum_{\substack{k_1+k_2=k \\ k_1 < k_2}} k_1 u_{k_1} u_{k_2} - \frac{ik}{2} \sum_{\substack{k_1+k_2=k \\ k_1=k_2}} u_{k_1} u_{k_2} - i \sum_{\substack{k_1+k_2=k \\ k_1 < k_2}} k_2 u_{k_1} u_{k_2} \\
&= ik^3 u_k - \frac{ik}{2} \sum_{\substack{k_1+k_2=k \\ k_1 \neq k_2}} u_{k_1} u_{k_2} - \frac{ik}{2} \sum_{\substack{k_1+k_2=k \\ k_1=k_2}} u_{k_1} u_{k_2} \\
&= ik^3 u_k - \frac{ik}{2} \sum_{k_1+k_2=k} u_{k_1} u_{k_2}.
\end{aligned} \tag{2.1.7}$$

An alternative way to obtain this equation would be the following: note that KdV can be written as

$$u_t + u_{xxx} + \frac{1}{2} \partial_x u^2 = 0,$$

which in the Fourier setting becomes (2.1.7). Using (2.1.5) we find

$$(\partial_t \nu_k) e^{ik^3 t} + ik^3 \nu_k e^{ik^3 t} = ik^3 \nu_k e^{ik^3 t} - \frac{ik}{2} \sum_{k_1+k_2=k} \nu_{k_1} \nu_{k_2} e^{i(k_1^3 + k_2^3)t}$$

from which

$$\partial_t \nu_k = -\frac{ik}{2} \sum_{k_1+k_2=k} \nu_{k_1} \nu_{k_2} e^{i(k_1^3 + k_2^3 - k^3)t} = -\frac{ik}{2} \sum_{k_1+k_2=k} e^{-3ikk_1k_2t} \nu_{k_1} \nu_{k_2}$$

using the identity

$$(k_1 + k_2)^3 - k_1^3 - k_2^3 = 3(k_1 + k_2)k_1 k_2.$$

□

**Definition 2.1.5.** We say that  $u \in C_t^0 L_x^2([-\delta, \delta] \times \mathbb{T}, \mathbb{R})$  is a solution to (2.1.1) if

$$\boldsymbol{\nu}(t) = (\nu_k(t) = u_k(t) e^{-ik^3 t})_{k \neq 0} \in C_t^0 \dot{\ell}_k^2([-\delta, \delta] \times (\mathbb{Z} \setminus \{0\}))$$

satisfies (2.1.6) for each  $k$  and  $t$ , and  $\nu_k(0) = u_k(0) = g_k$ .

**Remark 2.1.6.** The previous definition of solution in the interaction variables is equivalent to (2.1.2). Indeed, if we apply both sides  $e^{t\partial_x^3}$  in (2.1.2) we obtain

$$e^{t\partial_x^3} u(t, x) = g(x) - \frac{1}{2} \int_0^t e^{s\partial_x^3} \partial_x(u^2(s, x)) ds$$

which in the Fourier setting is exactly (2.1.6).

We want to integrate by parts two times in (2.1.6) and then we will define a contraction on a suitable closed ball of  $C_t^0 \dot{\ell}_k^2([-\delta, \delta] \times (\mathbb{Z} \setminus \{0\}))$ , where  $\delta > 0$  will be chosen later.

**Lemma 2.1.7.** *If  $\nu \in C_t^0 \dot{\ell}_k^2([-\delta, \delta] \times (\mathbb{Z} \setminus \{0\}))$  satisfies (2.1.6), then*

$$\nu_k(t) = \nu_k(0) - B_2(\nu, \nu)_k(t) + B_2(\nu, \nu)_k(0) + 2 \int_0^t B_2(\nu, \nu_t)_k(s) ds \quad (2.1.8)$$

where

$$\begin{cases} B_2(\mathbf{f}, \mathbf{g})_k(t) = -\frac{1}{6} \sum_{k_1+k_2=k}^* \frac{e^{-3kk_1k_2t} f_{k_1} g_{k_2}}{k_1 k_2} & \text{if } k \neq 0 \\ B_2(\mathbf{f}, \mathbf{g})_0(t) = 0. \end{cases}$$

With  $\sum^*$  we indicate the sum over all the indexes for which the denominator doesn't vanish.

*Proof.* Suppose that  $k \neq 0$  (otherwise the equation is trivial). In (2.1.6) we can exchange the order of summation and integration thanks to Fubini's theorem. Indeed

$$\begin{aligned} \int_0^t \sum_{k_1+k_2=k} |e^{-3ikk_1k_2s} \nu_{k_1}(s) \nu_{k_2}(s)| ds &= \int_0^t \sum_{k_1+k_2=k} |\nu_{k_1} \nu_{k_2}| ds \\ &\leq \int_0^t \|\nu(s)\|_{\ell^2}^2 ds \leq \delta \|\nu\|_{L_t^\infty \ell_k^2}^2 < \infty \end{aligned}$$

using the Cauchy-Schwarz inequality. Then we can use integration by parts for absolutely continuous functions<sup>4</sup>, and we get

$$\begin{aligned} \nu_k(t) - \nu_k(0) &= -\frac{ik}{2} \sum_{k_1+k_2=k}^* \left( \left[ \frac{e^{-3ikk_1k_2s}}{-3ikk_1k_2} \nu_{k_1} \nu_{k_2} \right]_0^t - \int_0^t \frac{e^{-3ikk_1k_2s}}{-3ikk_1k_2} \partial_s(\nu_{k_1} \nu_{k_2}) ds \right) \\ &= \frac{1}{6} \sum_{k_1+k_2=k}^* \frac{e^{-3ikk_1k_2t}}{k_1 k_2} \nu_{k_1}(t) \nu_{k_2}(t) - \frac{1}{6} \sum_{k_1+k_2=k}^* \frac{\nu_{k_1}(0) \nu_{k_2}(0)}{k_1 k_2} \\ &\quad - \frac{1}{3} \int_0^t \sum_{k_1+k_2=k}^* \frac{e^{-3ikk_1k_2s}}{k_1 k_2} \nu_{k_1} \partial_s \nu_{k_2} ds \\ &= -B_2(\nu, \nu)_k(t) + B_2(\nu, \nu)_k(0) + 2 \int_0^t B_2(\nu, \nu_t)_k(s) ds. \end{aligned}$$

We have exchanged again the sum with the integral thanks to the fact that

$$\sup_{s \in [0, t]} |\partial_s \nu_{k_2}| \lesssim |k_2|$$

from (2.1.6), where the implicit constant depends only on  $\|\nu\|_{L_t^\infty \ell_k^2}$ .  $\square$

Now we integrate by parts another time, but we must be more careful because for some choice of indexes the exponential in (2.1.8) is 1, since its exponent is zero. These are called *resonant terms* and we have to treat them separately.

**Lemma 2.1.8.** *If  $\nu \in C_t^0 \dot{\ell}_k^2([-\delta, \delta] \times (\mathbb{Z} \setminus \{0\}))$  satisfies (2.1.6), then  $\forall k \neq 0$*

$$\nu_k(t) = \nu_k(0) + C(\nu)_k(t) - C(\nu)_k(0) + \int_0^t (\rho_k + D(\nu)_k)(s) ds \quad (2.1.9)$$

---

<sup>4</sup>See Theorem A.1.6 in the appendix.

where

$$\rho_k = -\frac{i}{6k}\nu_k|\nu_k|^2, \quad (2.1.10)$$

$$D(\boldsymbol{\nu})_k = -\frac{i}{72} \sum_{\substack{k_1+k_2+k_3+k_4=k \\ k_3+k_4 \neq 0 \\ k_1, k_2, k_3, k_4 \neq 0}}^* \frac{e^{is\psi(k_1, k_2, k_3, k_4)}(2k_3 + 2k_4 + k_1)\nu_{k_1}\nu_{k_2}\nu_{k_3}\nu_{k_4}}{k_1(k_1+k_2)(k_1+k_3+k_4)(k_2+k_3+k_4)} \quad (2.1.11)$$

with

$$\psi(k_1, k_2, k_3, k_4) = -3[(k_1+k_2)(k_1+k_3+k_4)(k_2+k_3+k_4) + k_3k_4(k_3+k_4)], \quad (2.1.12)$$

and

$$C(\boldsymbol{\nu})_k = \sum_{k_1+k_2=k}^* \frac{e^{-3ikk_1k_2s}\nu_{k_1}\nu_{k_2}}{6k_1k_2} - \sum_{\substack{k_1+k_2+k_3=k \\ k_1, k_2, k_3 \neq 0}}^* \frac{e^{-3is(k_1+k_2)(k_1+k_3)(k_2+k_3)}\nu_{k_1}\nu_{k_2}\nu_{k_3}}{18k_1(k_1+k_2)(k_1+k_3)(k_2+k_3)}. \quad (2.1.13)$$

*Proof.* Remember that

$$\begin{aligned} 2 \int_0^t B_2(\boldsymbol{\nu}, \boldsymbol{\nu}_t)_k(s) ds &= -\frac{1}{3} \int_0^t \sum_{k_1+k_2=k}^* e^{-3ikk_1k_2s} \frac{\nu_{k_1}}{k_1} \frac{\partial_s \nu_{k_2}}{k_2} ds \\ &= \frac{i}{6} \int_0^t \sum_{\substack{k_1+k_2=k \\ k_1, k_2 \neq 0}} e^{-3ikk_1k_2s} \frac{\nu_{k_1}}{k_1} \left( \sum_{\substack{\lambda+\mu=k_2 \\ \lambda, \mu \neq 0}} e^{-3ik_2\lambda\mu s} \nu_\lambda \nu_\mu \right) ds \end{aligned}$$

where we used (2.1.6). Notice that  $\lambda + \mu = k_2 \neq 0$  since  $\nu_0 = 0$ . Renaming the variables  $k_2 = \lambda$  and  $k_3 = \mu$  we obtain

$$\begin{aligned} 2 \int_0^t B_2(\boldsymbol{\nu}, \boldsymbol{\nu}_t)_k(s) ds &= \frac{i}{6} \int_0^t \sum_{\substack{k_1+k_2+k_3=k \\ k_2+k_3 \neq 0 \\ k_1, k_2, k_3 \neq 0}} \frac{\nu_{k_1}\nu_{k_2}\nu_{k_3}}{k_1} e^{-3is[kk_1(k_2+k_3)+k_2k_3(k_2+k_3)]} ds \\ &= \frac{i}{6} \int_0^t \sum_{\substack{k_1+k_2+k_3=k \\ k_2+k_3 \neq 0 \\ k_1, k_2, k_3 \neq 0}} \frac{e^{-3is(k_1+k_2)(k_1+k_3)(k_2+k_3)}}{k_1} \nu_{k_1}\nu_{k_2}\nu_{k_3} ds \end{aligned} \quad (2.1.14)$$

using the fact that

$$\begin{aligned} kk_1(k_2+k_3) + k_2k_3(k_2+k_3) &= (k_2+k_3)[kk_1+k_2k_3] \\ &= (k_2+k_3)[(k_1+k_2+k_3)k_1+k_2k_3] \\ &= (k_2+k_3)(k_1^2 + k_1k_2 + k_1k_3 + k_2k_3) \\ &= (k_1+k_2)(k_1+k_3)(k_2+k_3). \end{aligned}$$

We shall study for which indexes the exponent in (2.1.14) vanish. For  $k \in \mathbb{Z} \setminus \{0\}$ , the set of resonant indexes is

$$R_k = \{(k_1, k_2, k_3) \in \mathbb{Z}^3 \mid k_1, k_2, k_3 \neq 0, k_2 + k_3 \neq 0, k_1 + k_2 + k_3 = k, (k_1 + k_2)(k_1 + k_3) = 0\}. \quad (2.1.15)$$

We can split it into three parts:

$$\begin{aligned} R_{1k} &= \{k_1 + k_2 = 0\} \cap \{k_1 + k_3 = 0\} = \{(-k, k, k)\}, \\ R_{2k} &= \{k_1 + k_2 = 0\} \cap \{k_1 + k_3 \neq 0\} = \{(j, -j, k) \mid j \in \mathbb{Z} \setminus \{0, \pm k\}\}, \\ R_{3k} &= \{k_1 + k_2 \neq 0\} \cap \{k_1 + k_3 = 0\} = \{(j, k, -j) \mid j \in \mathbb{Z} \setminus \{0, \pm k\}\}. \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{(k_1, k_2, k_3) \in R_k} \frac{e^{-3is(k_1+k_2)(k_1+k_3)(k_2+k_3)}}{k_1} \nu_{k_1} \nu_{k_2} \nu_{k_3} &= \sum_{j=1}^3 \sum_{R_{jk}} \frac{\nu_{k_1} \nu_{k_2} \nu_{k_3}}{k_1} \\ &= \frac{\nu_{-k} \nu_k \nu_k}{-k} + \nu_k \sum_{j \in \mathbb{Z} \setminus \{0, \pm k\}} \frac{\nu_j \nu_{-j}}{j} + \nu_k \sum_{j \in \mathbb{Z} \setminus \{0, \pm k\}} \frac{\nu_j \nu_{-j}}{j} = -\frac{1}{k} \nu_k |\nu_k|^2 \end{aligned}$$

where the two sums vanish by symmetry and for the first term we used the fact that  $\nu_{-k} = \overline{\nu_k}$ . Hence

$$2 \int_0^t B_2(\boldsymbol{\nu}, \boldsymbol{\nu}_t)_k(s) ds = \int_0^t \rho_k ds \quad (2.1.16)$$

$$+ \frac{i}{6} \int_0^t \sum_{\substack{k_1+k_2+k_3=k \\ (k_1+k_2)(k_1+k_3)(k_2+k_3) \neq 0 \\ k_1, k_2, k_3 \neq 0}} \frac{e^{-3is(k_1+k_2)(k_1+k_3)(k_2+k_3)}}{k_1} \nu_{k_1} \nu_{k_2} \nu_{k_3} ds \quad (2.1.17)$$

and now in (2.1.17) we can exchange the order of summation and integration by Fubini's theorem and Hölder's inequality, since  $\boldsymbol{\nu} \in L_t^\infty \dot{\ell}_k^2$ . Then integrating by parts we get

$$\begin{aligned} 2 \int_0^t B_2(\boldsymbol{\nu}, \boldsymbol{\nu}_t)_k(s) ds &= \int_0^t \rho_k ds - \frac{1}{18} \sum_{\substack{k_1+k_2+k_3=k \\ k_2, k_3 \neq 0}}^* \frac{e^{-3it(k_1+k_2)(k_1+k_3)(k_2+k_3)} (\nu_{k_1} \nu_{k_2} \nu_{k_3})(t)}{k_1(k_1+k_2)(k_1+k_3)(k_2+k_3)} \\ &\quad + \frac{1}{18} \sum_{\substack{k_1+k_2+k_3=k \\ k_2, k_3 \neq 0}}^* \frac{(\nu_{k_1} \nu_{k_2} \nu_{k_3})(0)}{k_1(k_1+k_2)(k_1+k_3)(k_2+k_3)} \\ &\quad + \frac{1}{18} \int_0^t \sum_{\substack{k_1+k_2+k_3=k \\ k_2, k_3 \neq 0}}^* \frac{e^{-3is(k_1+k_2)(k_1+k_3)(k_2+k_3)} \partial_s (\nu_{k_1} \nu_{k_2} \nu_{k_3})}{k_1(k_1+k_2)(k_1+k_3)(k_2+k_3)} ds \end{aligned}$$

from which

$$\nu_k(t) = \nu_k(0) + C(\boldsymbol{\nu})_k(t) - C(\boldsymbol{\nu})_k(0) + \int_0^t \rho_k ds \quad (2.1.18)$$

$$+ \int_0^t \sum_{\substack{k_1+k_2+k_3=k \\ k_2, k_3 \neq 0}}^* \frac{e^{-3is(k_1+k_2)(k_1+k_3)(k_2+k_3)} \partial_s (\nu_{k_1} \nu_{k_2} \nu_{k_3})}{18k_1(k_1+k_2)(k_1+k_3)(k_2+k_3)} ds. \quad (2.1.19)$$

We have only to compute (2.1.19). Notice that the sum inside the integral is symmetric with respect to  $k_2$  and  $k_3$ , hence we can write it as

$$\begin{aligned} I_k(t) &= \frac{1}{18} \int_0^t \sum_{\substack{k_1+k_2+k_3=k \\ k_2, k_3 \neq 0}}^* \frac{e^{-3is(k_1+k_2)(k_1+k_3)(k_2+k_3)}}{k_1(k_1+k_2)(k_1+k_3)(k_2+k_3)} \partial_s (\nu_{k_1}) \nu_{k_2} \nu_{k_3} ds \\ &\quad + \frac{1}{9} \int_0^t \sum_{\substack{k_1+k_2+k_3=k \\ k_2, k_3 \neq 0}}^* \frac{e^{-3is(k_1+k_2)(k_1+k_3)(k_2+k_3)}}{k_1(k_1+k_2)(k_1+k_3)(k_2+k_3)} \nu_{k_1} \nu_{k_2} \partial_s (\nu_{k_3}) ds \end{aligned}$$

and using (2.1.6) it becomes

$$\begin{aligned} I_k(t) = & -i \int_0^t \sum_{\substack{k_1+k_2+k_3=k \\ k_1, k_2, k_3 \neq 0}}^* \frac{e^{-3is(k_1+k_2)(k_1+k_3)(k_2+k_3)} \nu_{k_2} \nu_{k_3}}{36(k_1+k_2)(k_1+k_3)(k_2+k_3)} \left( \sum_{\substack{\lambda+\mu=k_1 \\ \lambda, \mu \neq 0}} e^{-3ik_1 \lambda \mu s} \nu_\lambda \nu_\mu \right) ds \\ & -i \int_0^t \sum_{\substack{k_1+k_2+k_3=k \\ k_1, k_2, k_3 \neq 0}}^* \frac{e^{-3is(k_1+k_2)(k_1+k_3)(k_2+k_3)} k_3 \nu_{k_1} \nu_{k_2}}{18k_1(k_1+k_2)(k_1+k_3)(k_2+k_3)} \left( \sum_{\substack{\lambda+\mu=k_3 \\ \lambda, \mu \neq 0}} e^{-3ik_3 \lambda \mu s} \nu_\lambda \nu_\mu \right) ds. \end{aligned}$$

Notice that  $\lambda + \mu = k_3 \neq 0$ . By symmetry in the first integral we can exchange the role of  $k_1$  and  $k_3$ , substitute  $k_3 = \lambda + \mu$  and then rename  $k_3 = \lambda$  and  $k_4 = \mu$ . In the second integral we just substitute  $k_3 = \lambda + \mu$  and then rename  $k_3 = \lambda$  and  $k_4 = \mu$ . We obtain

$$\begin{aligned} I_k(t) = & -\frac{i}{36} \int_0^t \sum_{\substack{k_1+k_2+k_3+k_4=k \\ k_1, k_2, k_3, k_4 \neq 0 \\ k_3+k_4 \neq 0}}^* \frac{e^{is\psi(k_1, k_2, k_3, k_4)} \nu_{k_1} \nu_{k_2} \nu_{k_3} \nu_{k_4}}{(k_1+k_2)(k_1+k_3+k_4)(k_2+k_3+k_4)} ds \\ & -\frac{i}{18} \int_0^t \sum_{\substack{k_1+k_2+k_3+k_4=k \\ k_1, k_2, k_3, k_4 \neq 0 \\ k_3+k_4 \neq 0}}^* \frac{e^{is\psi(k_1, k_2, k_3, k_4)} (k_3+k_4) \nu_{k_1} \nu_{k_2} \nu_{k_3} \nu_{k_4}}{k_1(k_1+k_2)(k_1+k_3+k_4)(k_2+k_3+k_4)} ds \\ = & -\frac{i}{36} \int_0^t \sum_{\substack{k_1+k_2+k_3+k_4=k \\ k_1, k_2, k_3, k_4 \neq 0 \\ k_3+k_4 \neq 0}}^* \frac{e^{is\psi(k_1, k_2, k_3, k_4)} (k_1+2k_3+2k_4) \nu_{k_1} \nu_{k_2} \nu_{k_3} \nu_{k_4}}{k_1(k_1+k_2)(k_1+k_3+k_4)(k_2+k_3+k_4)} ds, \end{aligned}$$

where

$$\psi(k_1, k_2, k_3, k_4) = -3[(k_1+k_2)(k_1+k_3+k_4)(k_2+k_3+k_4) + k_3 k_4 (k_3+k_4)].$$

□

We found two different formulas for  $\nu_k$ , (2.1.6) and (2.1.9). The idea is to define the contraction using both of them, separating high and low frequencies. Namely we set

$$\Gamma_N(\boldsymbol{\nu})_k(t) = \begin{cases} g_k - \frac{ik}{2} \int_0^t \sum_{k_1+k_2=k} e^{-3ikk_1 k_2 s} \nu_{k_1}(s) \nu_{k_2}(s) ds & \text{if } |k| \leq N, \\ g_k + C(\boldsymbol{\nu})_k(t) - C(\boldsymbol{\nu})_k(0) + \int_0^t (\rho_k + D(\boldsymbol{\nu})_k)(s) ds & \text{if } |k| > N \end{cases} \quad (2.1.20)$$

where  $N \in \mathbb{N}$  will be chosen later and  $\rho_k, C(\boldsymbol{\nu})_k$  and  $D(\boldsymbol{\nu})_k$  where defined in (2.1.10), (2.1.13) and (2.1.11). To prove that this is a contraction on a suitable closed ball of  $C_t^0 \dot{\ell}_k^2([-\delta, \delta] \times (\mathbb{Z} \setminus \{0\}))$  we need the following estimates:

**Proposition 2.1.9.** *If  $\nu \in \ell^2(\mathbb{Z})$ , we have the following bounds*

$$\|E(\nu)\|_{\ell^2_{|k| \leq N}} \lesssim N^{\frac{3}{2}} \|\nu\|_{\ell^2}^2 \quad (2.1.21)$$

$$\|\rho\|_{\ell^2_{|k| > N}} \lesssim N^{-1} \|\nu\|_{\ell^2}^3 \quad (2.1.22)$$

$$\|C(\nu)\|_{\ell^2_{|k| > N}} \lesssim N^{-\frac{1}{4}} (\|\nu\|_{\ell^2}^2 + \|\nu\|_{\ell^2}^3) \quad (2.1.23)$$

$$\|D(\nu)\|_{\ell^2_{|k| > N}} \lesssim \|\nu\|_{\ell^2}^4, \quad (2.1.24)$$

where

$$E(\nu)_k = k \sum_{k_1+k_2=k} e^{-3ikk_1k_2s} \nu_{k_1} \nu_{k_2} \quad (2.1.25)$$

and  $\rho$ ,  $C(\nu)$  and  $D(\nu)$  where defined respectively in (2.1.10), (2.1.13) and (2.1.11). Moreover each operator is locally Lipschitz-continuous from  $\ell^2$  to  $\ell^2$ .

*Proof.* **Step 1:** To prove (2.1.21) notice that

$$\|E(\nu)\|_{\ell^2_{|k| \leq N}} \leq \|\nu * \nu\|_{\ell^\infty} \|k\|_{\ell^2_{|k| \leq N}} \lesssim N^{\frac{3}{2}} \|\nu\|_{\ell^2}^2$$

using Young's convolution inequality<sup>5</sup> and the formula

$$\sum_{k=1}^N k^2 = \frac{N(N+1)(2N+1)}{6}.$$

**Step 2:** Now consider  $|k| > N$ . The bound (2.1.22) follows from the fact that

$$\|\rho\|_{\ell^2_{|k| > N}} \lesssim \left\| \frac{\nu_k}{k} |\nu_k|^2 \right\|_{\ell^2_{|k| > N}} \leq N^{-1} \|\nu\|_{\ell^\infty}^2 \|\nu\|_{\ell^2} \lesssim N^{-1} \|\nu\|_{\ell^2}^3$$

from the continuous embedding  $\ell^2 \hookrightarrow \ell^\infty$ .

**Step 3:** Now let us focus on (2.1.23). Note that there must exists  $j \in \{1, 2\}$  such that  $|k_j| \geq |k|/2$  (otherwise  $|k| \leq |k_1| + |k_2| < |k|$ ). Without loss of generality, suppose that  $|k_1| \geq |k|/2$ . Then

$$\left| \sum_{k_1+k_2=k} \frac{e^{-3ikk_1k_2s} \nu_{k_1} \nu_{k_2}}{k_1 k_2} \right| \leq \sum_{k_1+k_2=k} \frac{|\nu_{k_1}| |\nu_{k_2}|}{|k_1| |k_2|} \lesssim N^{-1} \sum_{k_1+k_2=k} \frac{|\nu_{k_1}| |\nu_{k_2}|}{|k_2|}.$$

For the other term we need the following lemma.

**Lemma 2.1.10.** *If  $k_1, k_2, k_3 \in \mathbb{Z} \setminus \{0\}$ ,  $k_1 + k_2 + k_3 = k$  and  $(k_1 + k_2)(k_1 + k_3)(k_2 + k_3) \neq 0$ , then*

$$|(k_1 + k_2)(k_1 + k_3)(k_2 + k_3)| \gtrsim \max\{|k_1|, |k_2|, |k_3|\}.$$

*Proof.* Let  $j$  be the index such that

$$|k_j| = \max\{|k_1|, |k_2|, |k_3|\}.$$

We consider three cases:

- If there is an other index  $r \neq j$  such that  $k_j k_r \geq 0$ , then clearly

$$|(k_1 + k_2)(k_1 + k_3)(k_2 + k_3)| \geq \max\{|k_1|, |k_2|, |k_3|\}.$$

---

<sup>5</sup>See Theorem A.1.1 in Appendix A.1.

- If for both of the other indexes we have  $k_j k_r < 0$ ,  $k_j k_l < 0$ , but for at least one of them we have  $|k_r| \leq \frac{|k_j|}{2}$ , then

$$|(k_1 + k_2)(k_1 + k_3)(k_2 + k_3)| \gtrsim \max\{|k_1|, |k_2|, |k_3|\}.$$

- If we have  $k_j k_r < 0$ ,  $k_j k_l < 0$  and at the same time  $|k_r| > \frac{|k_j|}{2}$ ,  $|k_l| > \frac{|k_j|}{2}$ , then clearly it follows that

$$|(k_1 + k_2)(k_1 + k_3)(k_2 + k_3)| \gtrsim \max\{|k_1|, |k_2|, |k_3|\}.$$

□

Back to the proposition, we use the fact that

$$|k| \leq |k_1| + |k_2| + |k_3| \lesssim \max\{|k_1|, |k_2|, |k_3|\}$$

to obtain that

$$\max\{|k_1|, |k_2|, |k_3|\} = \max\{|k_1|, |k_2|, |k_3|\}^{\frac{1}{4}} \max\{|k_1|, |k_2|, |k_3|\}^{\frac{3}{4}} \gtrsim |k_2|^{\frac{3}{4}} |k|^{\frac{1}{4}}.$$

We found that

$$|C(\boldsymbol{\nu})_k| \lesssim N^{-1} \sum_{k_1+k_2=k} \frac{|\nu_{k_1}| |\nu_{k_2}|}{|k_2|} + N^{-\frac{1}{4}} \sum_{k_1+k_2+k_3=k}^* \frac{|\nu_{k_1}| |\nu_{k_2}| |\nu_{k_3}|}{|k_1| |k_2|^{\frac{3}{4}}}$$

and taking the  $\ell^2$  norm we have

$$\begin{aligned} \|C(\boldsymbol{\nu})\|_{\ell_{|k|>N}^2} &\lesssim N^{-1} \left\| |\nu_k| * \frac{|\nu_k|}{|k|} \right\|_{\ell^2} + N^{-\frac{1}{4}} \left\| |\nu_k| * \frac{|\nu_k|}{|k|} * \frac{|\nu_k|}{|k|^{\frac{3}{4}}} \right\|_{\ell^2} \\ &\leq N^{-1} \|\boldsymbol{\nu}\|_{\ell^2} \left\| \frac{|\nu_k|}{|k|} \right\|_{\ell^1} + N^{-\frac{1}{4}} \|\boldsymbol{\nu}\|_{\ell^2} \left\| \frac{|\nu_k|}{|k|} \right\|_{\ell^1} \left\| \frac{|\nu_k|}{|k|^{\frac{3}{4}}} \right\|_{\ell^1} \\ &\lesssim N^{-\frac{1}{4}} (\|\boldsymbol{\nu}\|_{\ell^2}^2 + \|\boldsymbol{\nu}\|_{\ell^2}^3) \end{aligned}$$

from the Young and the Cauchy-Schwarz inequalities. Notice that the choice of the exponent  $3/4$  was made only to obtain  $(k^{-3/4})_k \in \ell^2$ . Any other choice in  $(1/2, 1)$  is good for that purpose.

**Step 4:** It remains to prove (2.1.24). First notice that

$$\begin{aligned} |D(\boldsymbol{\nu})_k| &\lesssim \sum_{k_1+k_2+k_3+k_4=k}^* \frac{|-k_1+2k_1+2k_3+2k_4| |\nu_{k_1}| |\nu_{k_2}| |\nu_{k_3}| |\nu_{k_4}|}{|k_1| |k_1+k_2| |k_1+k_3+k_4| |k_2+k_3+k_4|} \\ &\lesssim \sum_{k_1+k_2+k_3+k_4=k}^* \frac{|\nu_{k_1}| |\nu_{k_2}| |\nu_{k_3}| |\nu_{k_4}|}{|k_1+k_2| |k_1+k_3+k_4| |k_2+k_3+k_4|} \end{aligned} \tag{2.1.26}$$

$$+ \sum_{k_1+k_2+k_3+k_4=k}^* \frac{|\nu_{k_1}| |\nu_{k_2}| |\nu_{k_3}| |\nu_{k_4}|}{|k_1| |k_1+k_2| |k_2+k_3+k_4|}. \tag{2.1.27}$$

We will estimate only (2.1.26) (the same argument will work for (2.1.27)) and then use the Minkowski inequality. By duality we have that

$$\begin{aligned} & \left\| \sum_{k_1+k_2+k_3+k_4=k}^* \frac{|\nu_{k_1}| |\nu_{k_2}| |\nu_{k_3}| |\nu_{k_4}|}{|k_1+k_2| |k_1+k_3+k_4| |k_2+k_3+k_4|} \right\|_{\ell^2} \\ &= \sup_{\|\boldsymbol{h}\|_{\ell^2}=1} \sum_{k \neq 0} \sum_{k_1+k_2+k_3+k_4=k}^* \frac{|\nu_{k_1}| |\nu_{k_2}| |\nu_{k_3}| |\nu_{k_4}| |h_{k_1+k_2+k_3+k_4}|}{|k_1+k_2| |k_1+k_3+k_4| |k_2+k_3+k_4|} \\ &\leq \sup_{\|\boldsymbol{h}\|_{\ell^2}=1} \left\{ \left[ \sum_{k_1,k_2,k_3,k_4}^* \frac{|\nu_{k_1}|^2 |\nu_{k_4}|^2}{|k_1+k_2|^2 |k_1+k_3+k_4|^2} \right]^{\frac{1}{2}} \cdot \right. \\ &\quad \left. \left[ \sum_{k_1,k_2,k_3,k_4}^* \frac{|\nu_{k_2}|^2 |\nu_{k_3}|^2 |h_{k_1+k_2+k_3+k_4}|^2}{|k_2+k_3+k_4|^2} \right]^{\frac{1}{2}} \right\}. \end{aligned} \quad (2.1.28)$$

$$\cdot \left[ \sum_{k_1,k_2,k_3,k_4}^* \frac{|\nu_{k_2}|^2 |\nu_{k_3}|^2 |h_{k_1+k_2+k_3+k_4}|^2}{|k_2+k_3+k_4|^2} \right]^{\frac{1}{2}} \} \quad (2.1.29)$$

using the Cauchy-Schwarz inequality. Now in (2.1.28) we sum firstly in  $k_2$  and  $k_3$ , while in (2.1.29) we sum firstly in  $k_1$  and  $k_4$ , obtaining that

$$\begin{aligned} & \left\| \sum_{k_1+k_2+k_3+k_4=k}^* \frac{|\nu_{k_1}| |\nu_{k_2}| |\nu_{k_3}| |\nu_{k_4}|}{|k_1+k_2| |k_1+k_3+k_4| |k_2+k_3+k_4|} \right\|_{\ell^2} \\ &\lesssim \left[ \sum_{k_1,k_4} |\nu_{k_1}|^2 |\nu_{k_4}|^2 \right]^{\frac{1}{2}} \left[ \sum_{k_2,k_3} |\nu_{k_2}|^2 |\nu_{k_3}|^2 \right]^{\frac{1}{2}} = \sum_{k_1,k_2} |\nu_{k_1}|^2 |\nu_{k_2}|^2 = \|\boldsymbol{\nu}\|_{\ell^2}^4. \end{aligned}$$

**Step 5:** Finally we prove the Lipschitz-continuity of these operators. We have to evaluate them on the difference  $\boldsymbol{\nu}^{(1)} - \boldsymbol{\nu}^{(2)}$  and use the same estimates. For the first one we have that

$$\begin{aligned} & \|E(\boldsymbol{\nu}^{(1)}) - E(\boldsymbol{\nu}^{(2)})\|_{\ell_{|k| \leq N}^2} = \left\| k \sum_{k_1+k_2=k} e^{-3ikk_1k_2s} \nu_{k_1}^{(1)} \nu_{k_2}^{(1)} - k \sum_{k_1+k_2=k} e^{-3ikk_1k_2s} \nu_{k_1}^{(2)} \nu_{k_2}^{(2)} \right\|_{\ell_{|k| \leq N}^2} \\ &= \left\| k \sum_{k_1+k_2=k} e^{-3ikk_1k_2s} (\nu_{k_1}^{(1)} \nu_{k_2}^{(1)} - \nu_{k_1}^{(2)} \nu_{k_2}^{(2)}) \right\|_{\ell_{|k| \leq N}^2} \\ &\leq \left\| k \sum_{k_1+k_2=k} |\nu_{k_1}^{(1)} \nu_{k_2}^{(1)} - \nu_{k_1}^{(1)} \nu_{k_2}^{(2)} + \nu_{k_1}^{(1)} \nu_{k_2}^{(2)} - \nu_{k_1}^{(2)} \nu_{k_2}^{(2)}| \right\|_{\ell_{|k| \leq N}^2} \\ &\leq \left\| k \sum_{k_1+k_2=k} |\nu_{k_1}^{(1)}| |\nu_{k_2}^{(1)} - \nu_{k_2}^{(2)}| \right\|_{\ell_{|k| \leq N}^2} + \left\| k \sum_{k_1+k_2=k} |\nu_{k_2}^{(2)}| |\nu_{k_1}^{(1)} - \nu_{k_1}^{(2)}| \right\|_{\ell_{|k| \leq N}^2} \\ &\lesssim N^{\frac{3}{2}} \left( \|\boldsymbol{\nu}^{(1)}\|_{\ell^2} + \|\boldsymbol{\nu}^{(2)}\|_{\ell^2} \right) \|\boldsymbol{\nu}^{(1)} - \boldsymbol{\nu}^{(2)}\|_{\ell^2}. \end{aligned}$$

For the second one we have that

$$\begin{aligned}
 \left\| \rho^{(1)} - \rho^{(2)} \right\|_{\ell^2_{|k|>N}} &\lesssim \left\| \frac{1}{k} \nu_k^{(1)} |\nu_k^{(1)}|^2 - \frac{1}{k} \nu_k^{(2)} |\nu_k^{(2)}|^2 \right\|_{\ell^2_{|k|>N}} \\
 &\leq N^{-1} \left\| \nu_k^{(1)} |\nu_k^{(1)}|^2 - \nu_k^{(1)} |\nu_k^{(2)}|^2 + \nu_k^{(1)} |\nu_k^{(2)}|^2 - \nu_k^{(2)} |\nu_k^{(2)}|^2 \right\|_{\ell^2} \\
 &\leq N^{-1} \left( \left\| \boldsymbol{\nu}^{(1)} \right\|_{\ell^2} \left\| |\boldsymbol{\nu}^{(1)}|^2 - |\boldsymbol{\nu}^{(2)}|^2 \right\|_{\ell^\infty} + \left\| \boldsymbol{\nu}^{(2)} \right\|_{\ell^\infty}^2 \left\| \boldsymbol{\nu}^{(1)} - \boldsymbol{\nu}^{(2)} \right\|_{\ell^2} \right) \\
 &\leq N^{-1} \left( \left\| \boldsymbol{\nu}^{(1)} \right\|_{\ell^2} \left\| |\boldsymbol{\nu}^{(1)}| + |\boldsymbol{\nu}^{(2)}| \right\|_{\ell^\infty} \left\| |\boldsymbol{\nu}^{(1)}| - |\boldsymbol{\nu}^{(2)}| \right\|_{\ell^\infty} + \left\| \boldsymbol{\nu}^{(2)} \right\|_{\ell^2}^2 \left\| \boldsymbol{\nu}^{(1)} - \boldsymbol{\nu}^{(2)} \right\|_{\ell^2} \right) \\
 &\lesssim N^{-1} \left[ \left\| \boldsymbol{\nu}^{(1)} \right\|_{\ell^2}^2 + \left\| \boldsymbol{\nu}^{(2)} \right\|_{\ell^2}^2 \right] \left\| \boldsymbol{\nu}^{(1)} - \boldsymbol{\nu}^{(2)} \right\|_{\ell^2}.
 \end{aligned}$$

For the third one notice that

$$\begin{aligned}
 \nu_{k_1}^{(1)} \nu_{k_2}^{(1)} \nu_{k_3}^{(1)} - \nu_{k_1}^{(2)} \nu_{k_2}^{(2)} \nu_{k_3}^{(2)} &= \nu_{k_1}^{(1)} \nu_{k_2}^{(1)} (\nu_{k_3}^{(1)} - \nu_{k_3}^{(2)}) + \nu_{k_3}^{(2)} (\nu_{k_1}^{(1)} \nu_{k_2}^{(1)} - \nu_{k_1}^{(2)} \nu_{k_2}^{(2)}) \\
 &= \nu_{k_1}^{(1)} \nu_{k_2}^{(1)} (\nu_{k_3}^{(1)} - \nu_{k_3}^{(2)}) + \nu_{k_1}^{(1)} \nu_{k_3}^{(2)} (\nu_{k_2}^{(1)} - \nu_{k_2}^{(2)}) + \nu_{k_3}^{(2)} \nu_{k_2}^{(2)} (\nu_{k_1}^{(1)} - \nu_{k_1}^{(2)}).
 \end{aligned}$$

Hence we have

$$\begin{aligned}
 \left\| C(\boldsymbol{\nu}^{(1)}) - C(\boldsymbol{\nu}^{(2)}) \right\|_{\ell^2_{|k|>N}} &\lesssim N^{-1} \left\| \left| \nu_k^{(1)} \right| * \frac{|\nu_k^{(1)} - \nu_k^{(2)}|}{|k|} \right\|_{\ell^2} + N^{-1} \left\| \left| \nu_k^{(2)} \right| * \frac{|\nu_k^{(1)} - \nu_k^{(2)}|}{|k|} \right\|_{\ell^2} \\
 &\quad + N^{-\frac{1}{4}} \left\| \left| \nu_k^{(1)} - \nu_k^{(2)} \right| * \frac{|\nu_k^{(1)}|}{|k|} * \frac{|\nu_k^{(1)}|}{|k|^{\frac{3}{4}}} \right\|_{\ell^2} \\
 &\quad + N^{-\frac{1}{4}} \left\| \left| \nu_k^{(2)} \right| * \frac{|\nu_k^{(1)}|}{|k|} * \frac{|\nu_k^{(1)} - \nu_k^{(2)}|}{|k|^{\frac{3}{4}}} \right\|_{\ell^2} \\
 &\quad + N^{-\frac{1}{4}} \left\| \left| \nu_k^{(2)} \right| * \frac{|\nu_k^{(1)} - \nu_k^{(2)}|}{|k|} * \frac{|\nu_k^{(2)}|}{|k|^{\frac{3}{4}}} \right\|_{\ell^2} \\
 &\lesssim N^{-1} \left( \left\| \boldsymbol{\nu}^{(1)} \right\|_{\ell^2} + \left\| \boldsymbol{\nu}^{(2)} \right\|_{\ell^2} \right) \left\| \boldsymbol{\nu}^{(1)} - \boldsymbol{\nu}^{(2)} \right\|_{\ell^2} \\
 &\quad + N^{-\frac{1}{4}} \left( \left\| \boldsymbol{\nu}^{(1)} \right\|_{\ell^2}^2 + \left\| \boldsymbol{\nu}^{(2)} \right\|_{\ell^2}^2 + \left\| \boldsymbol{\nu}^{(1)} \right\|_{\ell^2} \left\| \boldsymbol{\nu}^{(2)} \right\|_{\ell^2} \right) \left\| \boldsymbol{\nu}^{(1)} - \boldsymbol{\nu}^{(2)} \right\|_{\ell^2} \\
 &\lesssim N^{-\frac{1}{4}} \left( \left\| \boldsymbol{\nu}^{(1)} \right\|_{\ell^2}^2 + \left\| \boldsymbol{\nu}^{(2)} \right\|_{\ell^2}^2 + \left\| \boldsymbol{\nu}^{(1)} \right\|_{\ell^2}^2 + \left\| \boldsymbol{\nu}^{(2)} \right\|_{\ell^2}^2 \right) \left\| \boldsymbol{\nu}^{(1)} - \boldsymbol{\nu}^{(2)} \right\|_{\ell^2}.
 \end{aligned}$$

Finally for the last one we have that

$$\begin{aligned}
 \nu_{k_1}^{(1)} \nu_{k_2}^{(1)} \nu_{k_3}^{(1)} \nu_{k_4}^{(1)} - \nu_{k_1}^{(2)} \nu_{k_2}^{(2)} \nu_{k_3}^{(2)} \nu_{k_4}^{(2)} &= \nu_{k_1}^{(1)} \nu_{k_2}^{(1)} \nu_{k_3}^{(1)} (\nu_{k_4}^{(1)} - \nu_{k_4}^{(2)}) + \nu_{k_4}^{(2)} (\nu_{k_1}^{(1)} \nu_{k_2}^{(1)} \nu_{k_3}^{(1)} - \nu_{k_1}^{(2)} \nu_{k_2}^{(2)} \nu_{k_3}^{(2)}) \\
 &= \nu_{k_1}^{(1)} \nu_{k_2}^{(1)} \nu_{k_3}^{(1)} (\nu_{k_4}^{(1)} - \nu_{k_4}^{(2)}) + \nu_{k_1}^{(1)} \nu_{k_2}^{(1)} (\nu_{k_3}^{(1)} - \nu_{k_3}^{(2)}) \nu_{k_4}^{(2)} \\
 &\quad + \nu_{k_1}^{(1)} \nu_{k_3}^{(2)} (\nu_{k_2}^{(1)} - \nu_{k_2}^{(2)}) \nu_{k_4}^{(2)} + \nu_{k_3}^{(2)} \nu_{k_2}^{(2)} (\nu_{k_1}^{(1)} - \nu_{k_1}^{(2)}) \nu_{k_4}^{(2)},
 \end{aligned}$$

from which it follows that

$$\left\| D(\boldsymbol{\nu}^{(1)}) - D(\boldsymbol{\nu}^{(2)}) \right\|_{\ell^2_{|k|>N}} \lesssim \left( \left\| \boldsymbol{\nu}^{(1)} \right\|_{\ell^2}^3 + \left\| \boldsymbol{\nu}^{(2)} \right\|_{\ell^2}^3 \right) \left\| \boldsymbol{\nu}^{(1)} - \boldsymbol{\nu}^{(2)} \right\|_{\ell^2}.$$

□

We are now ready to apply the Banach fixed point theorem to find a unique fixed point of  $\Gamma_N$  in a suitable closed subspace of  $C_t^0 \dot{\ell}_k^2$ .

**Theorem 2.1.11** ( $\Gamma_N$  is a contraction). *Let  $g \in L^2(\mathbb{T})$  and consider the Cauchy problem (2.1.1). Assume  $g \neq 0$  and consider the closed ball*

$$X_\delta = \left\{ \boldsymbol{\nu} \in C_t^0 \dot{\ell}_k^2([-\delta, \delta] \times (\mathbb{Z} \setminus \{0\})) \mid \|\boldsymbol{\nu}\|_{L_t^\infty \dot{\ell}_k^2} \leq \sqrt{\frac{2}{\pi}} \|g\|_{L^2} \right\},$$

with  $\delta > 0$ . Then there exists  $N > 0$  depending on  $\|g\|_{L^2}$  and  $\delta > 0$  depending on  $N$  and  $\|g\|_{L^2}$  such that the operator  $\Gamma_N$  defined in (2.1.20) is a contraction on  $X_\delta$ . Notice that by Parseval's identity, if  $\boldsymbol{\nu}$  is the unique fixed point of  $\Gamma_N$ , then

$$\|g\|_{L^2} = \sqrt{2\pi} \|\boldsymbol{\nu}(0)\|_{\ell^2}.$$

If instead  $g = 0$ , then there exists  $\varepsilon_0 > 0$  such that for any  $0 < \varepsilon \leq \varepsilon_0$ , there exists  $\delta = \delta(\varepsilon) > 0$  such that  $\Gamma_1$  is a contraction on

$$X_{\delta, \varepsilon} = \left\{ \boldsymbol{\nu} \in C_t^0 \dot{\ell}_k^2([-\delta, \delta] \times (\mathbb{Z} \setminus \{0\})) \mid \|\boldsymbol{\nu}\|_{L_t^\infty \dot{\ell}_k^2} \leq \varepsilon \right\}.$$

*Proof.* Let us start with  $g \neq 0$ . Let  $\boldsymbol{\nu} \in X_\delta$ . From (2.1.20) and Proposition 2.1.9 it follows that

$$\begin{aligned} & \|\Gamma_N(\boldsymbol{\nu})\|_{L_t^\infty \dot{\ell}_k^2} \\ & \leq \frac{1}{\sqrt{2\pi}} \|g\|_{L^2} + C \left[ N^{-\frac{1}{4}} (\|\boldsymbol{\nu}\|_{L_t^\infty \dot{\ell}_k^2}^2 + \|\boldsymbol{\nu}\|_{L_t^\infty \dot{\ell}_k^2}^3) + \delta \left( N^{-1} \|\boldsymbol{\nu}\|_{L_t^\infty \dot{\ell}_k^2}^3 + \|\boldsymbol{\nu}\|_{L_t^\infty \dot{\ell}_k^2}^4 + N^{\frac{3}{2}} \|\boldsymbol{\nu}\|_{L_t^\infty \dot{\ell}_k^2}^2 \right) \right] \\ & \leq \frac{1}{\sqrt{2\pi}} \|g\|_{L^2} + C \left[ N^{-\frac{1}{4}} (\|g\|_{L^2}^2 + \|g\|_{L^2}^3) + \delta \left( N^{-1} \|g\|_{L^2}^3 + \|g\|_{L^2}^4 + N^{\frac{3}{2}} \|g\|_{L^2}^2 \right) \right]. \end{aligned} \tag{2.1.30}$$

Moreover, using the Lipschitz continuity of the operators in Proposition 2.1.9, we have that

$$\begin{aligned} \|\Gamma_N(\boldsymbol{\nu})(t_1) - \Gamma_N(\boldsymbol{\nu})(t_2)\|_{\ell^2} & \leq C \left[ N^{-\frac{1}{4}} (\|g\|_{L^2} + \|g\|_{L^2}^2) \|\boldsymbol{\nu}(t_1) - \boldsymbol{\nu}(t_2)\|_{\ell^2} \right. \\ & \quad \left. + |t_1 - t_2| \left( N^{-1} \|g\|_{L^2}^3 + \|g\|_{L^2}^4 + N^{\frac{3}{2}} \|g\|_{L^2}^2 \right) \right]. \end{aligned} \tag{2.1.31}$$

Finally

$$\begin{aligned} \|\Gamma_N(\boldsymbol{\nu}^{(1)}) - \Gamma_N(\boldsymbol{\nu}^{(2)})\|_{L_t^\infty \dot{\ell}_k^2} & \leq C \left[ N^{-\frac{1}{4}} (\|g\|_{L^2} + \|g\|_{L^2}^2) \right. \\ & \quad \left. + \delta \left( N^{-1} \|g\|_{L^2}^2 + \|g\|_{L^2}^3 + N^{\frac{3}{2}} \|g\|_{L^2} \right) \right] \|\boldsymbol{\nu}^{(1)} - \boldsymbol{\nu}^{(2)}\|_{L_t^\infty \dot{\ell}_k^2}. \end{aligned} \tag{2.1.32}$$

From (2.1.31) we have that

$$\Gamma_N : X_\delta \longrightarrow C_t^0 \dot{\ell}_k^2([-\delta, \delta] \times (\mathbb{Z} \setminus \{0\})).$$

Now fix  $N$  big enough such that

$$N \gtrsim (\|g\|_{L^2} + \|g\|_{L^2}^2)^4.$$

In particular we require that

$$CN^{-\frac{1}{4}}(\|g\|_{L^2} + \|g\|_{L^2}^2) \leq \frac{1}{3}.$$

Then choose  $\delta$  (depending also on  $N$ ) such that

$$C\delta \left( N^{-1} \|g\|_{L^2}^2 + \|g\|_{L^2}^3 + N^{\frac{3}{2}} \|g\|_{L^2} \right) \leq \frac{1}{3}.$$

From (2.1.30) we have that

$$\Gamma_N : X_\delta \longrightarrow X_\delta$$

and from (2.1.32) we have that it is a contraction. Assume now  $g = 0$ . With the same computations of (2.1.30), (2.1.31) and (2.1.32), we obtain

$$\|\Gamma_1(\nu)\|_{L_t^\infty \ell_k^2} \leq C [\varepsilon^2 + \varepsilon^3 + \delta(\varepsilon^2 + \varepsilon^3 + \varepsilon^4)], \quad (2.1.33)$$

$$\|\Gamma_1(\nu)(t_1) - \Gamma_1(\nu)(t_2)\|_{\ell^2} \leq C [(\varepsilon + \varepsilon^2) \|\nu(t_1) - \nu(t_2)\|_{\ell^2} + |t_1 - t_2|(\varepsilon^2 + \varepsilon^3 + \varepsilon^4)], \quad (2.1.34)$$

$$\left\| \Gamma_1(\nu^{(1)}) - \Gamma_1(\nu^{(2)}) \right\|_{L_t^\infty \ell_k^2} \leq C [(\varepsilon + \varepsilon^2) + \delta(\varepsilon + \varepsilon^2 + \varepsilon^3)] \left\| \nu^{(1)} - \nu^{(2)} \right\|_{L_t^\infty \ell_k^2}. \quad (2.1.35)$$

and we can conclude as in the previous case. Notice that we can choose  $\delta \sim \varepsilon^{-1}$ .  $\square$

**Remark 2.1.12.** Notice that if  $\|\nu(0)\|_{\ell^2} \ll 1$ , then  $N_{\min} = 1$  and

$$\delta_{\max} \gtrsim \frac{1}{\|\nu(0)\|_{\ell^2} + \|\nu(0)\|_{\ell^2}^2 + \|\nu(0)\|_{\ell^2}^3} \gtrsim \|\nu(0)\|_{\ell^2}^{-1}.$$

This means that if the initial value has a small enough  $\ell^2$ -norm, then the solution is defined at least for times of order  $\|\nu(0)\|_{\ell^2}^{-1}$ .

**Corollary 2.1.13** (Local well-posedness of KdV in  $L^2(\mathbb{T})$ ). *For any initial datum  $g \in L^2(\mathbb{T})$  there exists  $\delta > 0$  and a unique solution  $u$  of (2.1.1) in the space  $C_t^0 L_x^2([-\delta, \delta] \times \mathbb{T}, \mathbb{R})$ . Moreover the solution depends continuously from the initial data.*

*Proof.* By Theorem 2.1.11,  $\Gamma_N$  is a contraction on the closed ball

$$X_\delta \subseteq C_t^0 \dot{\ell}_k^2([-\delta, \delta] \times (\mathbb{Z} \setminus \{0\})).$$

By the Banach fixed point theorem we can conclude that there is a unique  $\nu \in X_\delta$  such that  $\Gamma_N(\nu) = \nu$ . The theorem also proves the continuous dependence from initial data. Indeed if  $\nu^{(1)}$  and  $\nu^{(2)}$  are solutions of the equation  $\Gamma_N(\nu) = \nu$  (for a certain big  $N$ ) corresponding to initial values  $\nu^{(1)}(0)$  and  $\nu^{(2)}(0)$ , then with a computation similar to (2.1.32) we obtain that

$$\left\| \nu^{(1)} - \nu^{(2)} \right\|_{L_t^\infty \ell_k^2} \leq \left\| \nu^{(1)}(0) - \nu^{(2)}(0) \right\|_{\ell^2} + L \left\| \nu^{(1)} - \nu^{(2)} \right\|_{L_t^\infty \ell_k^2}$$

where  $L < 1$  is the Lipschitz constant of the contraction relative to the initial datum with the biggest norm. Therefore

$$\left\| \nu^{(1)} - \nu^{(2)} \right\|_{L_t^\infty \ell_k^2} \leq \frac{1}{1-L} \left\| \nu^{(1)}(0) - \nu^{(2)}(0) \right\|_{\ell^2}. \quad (2.1.36)$$

Remembering that the Fourier series defines an isomorphism between  $L^2(\mathbb{T})$  and  $\ell^2(\mathbb{Z})$ , we can construct a unique  $u \in C_t^0 L_x^2([-\delta, \delta] \times \mathbb{T})$  such that  $u_k(t) = \nu_k(t)e^{ik^3 t}$ . Then  $u$  is the solution we are looking for. To prove this it is enough to show that the fixed point  $\nu$  satisfies (2.1.6) for all  $k \in \mathbb{Z}$ . Notice that all the solutions  $\nu$  of KdV equation in the interaction variables are fixed points of  $\Gamma_N$  thanks to the integration by parts process justified in the previous lemmas (a priori the opposite is not true, because we are not allowed to differentiate the high frequencies in  $\nu = \Gamma_N(\nu)$ ). In particular, smooth solutions (which exists by Remark 2.1.14) satisfy (2.1.6) for any  $k \in \mathbb{Z}$ . If we approximate  $g \in L^2(\mathbb{T})$  with a sequence of smooth functions  $(g_n)_n$ , then we have corresponding smooth solutions  $(\nu_n)_n$  which converge to  $\nu$  in the  $L_t^\infty \ell_k^2$ -norm by continuous dependence from initial data. Taking the limit in (2.1.6) we find that it is satisfied also by  $\nu$ .

The continuous dependence from the initial data for the solution  $u(t, x)$  in the original variables follows by (2.1.36) similarly to Lemma 2.1.2. Finally we prove uniqueness. Let  $\nu^{(1)}, \nu^{(2)} \in C_t^0 \dot{\ell}_k^2([-\delta, \delta] \times (\mathbb{Z} \setminus \{0\}))$  be two solutions with the same initial value  $\nu(0) \in \ell^2(\mathbb{Z} \setminus \{0\})$ . Let

$$t_0 = \sup\{t \in [0, \delta] \mid \nu^{(1)}(s) = \nu^{(2)}(s), \forall s \in [0, t]\}.$$

If  $t_0 = \delta$  there is nothing to prove. Assume by contradiction that  $t_0 < \delta$ . Then by continuity there exists  $\delta_1$  such that

$$\|\nu^{(1)}(t)\|_{\ell^2}, \|\nu^{(2)}(t)\|_{\ell^2} \leq 2 \|\nu^{(1)}(t_0)\|_{\ell^2}, \quad t \in [t_0, t_0 + \delta_1].$$

But then  $\nu^{(1)}, \nu^{(2)}$  are fixed points of  $\Gamma_N$  in  $[t_0, t_0 + \delta_1]$ , which is a contradiction.  $\square$

**Remark 2.1.14.** Bona and Smith proved, using an energy method, that the KdV equation is globally well-posed in  $H^s(\mathbb{T})$  for  $s \geq 2$ , [5]. If we consider a smooth initial datum, we have that the corresponding solution is smooth with respect to the  $x$  variable, *i.e.*

$$u \in C_t^0 C_x^\infty(\mathbb{R} \times \mathbb{T}, \mathbb{R}).$$

But since  $u_t = -u_{xxx} - uu_x$ , we have also as many time derivatives as we want.

**Corollary 2.1.15** (Global well-posedness of KdV in  $L^2(\mathbb{T})$ ). *The solutions defined in Corollary 2.1.13 are globally defined. In particular for any initial datum  $g \in L^2(\mathbb{T})$  there exists a unique solution  $u \in C_t^0 L_x^2(\mathbb{R} \times \mathbb{T})$  of (2.1.1). Moreover the solution depends continuously on the initial data.*

*Proof.* Remember that the  $L^2$  norm of a smooth solution to (2.0.1) is constant in time by (2.0.3). This property remains true for solutions of KdV in  $L^2(\mathbb{T})$  by approximation. Indeed, if  $u \in C_t^0 L_x^2([-\delta, \delta] \times \mathbb{T})$  is the solution corresponding to an initial datum  $g \in L^2(\mathbb{T})$ , we can approximate  $g$  in  $L^2$  with a sequence of smooth functions  $(g_n)_n$ . By continuous dependence from initial data, the sequence of corresponding solutions  $(u_n)_n$  converge to  $u$  in  $C_t^0 L_x^2([-\delta, \delta] \times \mathbb{T})$ . Therefore if we take the limit in

$$\int_{\mathbb{T}} u_n^2(t, x) dx = \int_{\mathbb{T}} u_n^2(0, x) dx$$

we get that

$$\int_{\mathbb{T}} u^2(t, x) dx = \int_{\mathbb{T}} u^2(0, x) dx.$$

Now let  $\delta > 0$  be as in Theorem 2.1.11, which depends on  $\|g\|_{L^2}$ . By local uniqueness we can construct a unique maximal solution  $u \in C_t^0 L_x^2([-T_{\max}, T_{\max}] \times \mathbb{T})$  gluing together local solutions as usual. Suppose by contradiction that  $T_{\max} < +\infty$ . Then we can solve (2.1.1) with initial value  $u(T_{\max} - \frac{\delta}{2}, x)$ , which has the same  $L^2$ -norm of  $g$ . This way we get a solution of KdV in  $[T_{\max} - \frac{3}{2}\delta, T_{\max} + \frac{\delta}{2}]$ , which we can glue with  $u$  by uniqueness. But this is a contradiction because  $u$  was the maximal solution.  $\square$

## 2.2 Local well-posedness of KdV equation in $\mathcal{F}L^{0,1}(\mathbb{T})$

In Section 2.1 we proved the global well-posedness of the KdV equation in  $L^2(\mathbb{T})$ . For the next sections, we are interested in the maximum height of the wave at a certain time  $t$ , *i.e.* the  $L^\infty$ -norm of the solution. To this end, we shall prove local well-posedness in a functional space which embeds in  $L^\infty(\mathbb{T})$ . We can make different choices, for example Sobolev spaces  $H^s(\mathbb{T})$  with  $s > \frac{1}{2}$ . In this section we work on the Fourier-Lebesgue space  $\mathcal{F}L^{0,1}(\mathbb{T})$ , the space of periodic measurable functions whose Fourier coefficients belong to  $\ell^1(\mathbb{Z})$ . The first thing to notice is that  $\mathcal{F}L^{0,1}(\mathbb{T}) \subseteq L^2(\mathbb{T})$  with continuous embedding, which follows immediately from the continuous embedding  $\ell^1(\mathbb{Z}) \subseteq \ell^2(\mathbb{Z})$  and from the fact that  $\mathcal{F} : L^2(\mathbb{T}) \rightarrow \ell^2(\mathbb{Z})$  and  $\mathcal{F} : \mathcal{F}L^{0,1}(\mathbb{T}) \rightarrow \ell^1(\mathbb{Z})$  are isometric isomorphisms.

$$\begin{array}{ccc} \mathcal{F}L^{0,1}(\mathbb{T}) & \xhookrightarrow{\quad} & L^2(\mathbb{T}) \\ \downarrow \mathcal{F} & & \downarrow \mathcal{F} \\ \ell^1(\mathbb{Z}) & \xhookrightarrow{\quad} & \ell^2(\mathbb{Z}). \end{array}$$

For these reasons the integration by parts process is justified, so it is enough to prove that  $\Gamma_N$  is a contraction even on a certain ball of

$$C_t^0 \dot{\ell}_k^1([-\delta, \delta] \times (\mathbb{Z} \setminus \{0\})),$$

with  $\delta$  small enough, that we will choose later. Firstly we adapt Proposition 2.1.9 to  $\ell^1$ .

**Proposition 2.2.1.** *Let  $\nu \in \dot{\ell}^1(\mathbb{Z})$ . We have the following bounds*

$$\|E(\nu)\|_{\ell_{|k| \leq N}^1} \lesssim N \|\nu\|_{\ell^1}^2 \tag{2.2.1}$$

$$\|\rho\|_{\ell_{|k| > N}^1} \lesssim N^{-1} \|\nu\|_{\ell^1}^3 \tag{2.2.2}$$

$$\|C(\nu)\|_{\ell_{|k| > N}^1} \lesssim N^{-1} (\|\nu\|_{\ell^1}^2 + \|\nu\|_{\ell^1}^3) \tag{2.2.3}$$

$$\|D(\nu)\|_{\ell_{|k| > N}^1} \lesssim \|\nu\|_{\ell^1}^4. \tag{2.2.4}$$

where the operators  $\rho, C(\nu), D(\nu)$  and  $E(\nu)$  where defined respectively in (2.1.10), (2.1.13), (2.1.11) and (2.1.25). Moreover each operator is locally Lipschitz-continuous from  $\dot{\ell}^1$  to  $\dot{\ell}^1$ .

*Proof.* The proof is almost the same of Proposition 2.1.9. The key difference is that we can apply Remark A.1.2 to obtain better estimates.  $\square$

This allows us to state the following

**Theorem 2.2.2** (Local well-posedness of KdV in  $\mathcal{F}L^{0,1}(\mathbb{T})$ ). *For any initial datum  $g \in \mathcal{F}L^{0,1}(\mathbb{T})$  there exists  $\delta > 0$  and a unique solution  $u$  of (2.1.1) in the space  $C_t^0 \mathcal{F}L_x^{0,1}([-\delta, \delta] \times \mathbb{T}, \mathbb{R})$ . Moreover the solution depends continuously from the initial data.*

*Proof.* The proof is essentially the same as in  $L^2(\mathbb{T})$ .  $\square$

**Remark 2.2.3.** Notice that if  $\|\nu(0)\|_{\ell^1} \ll 1$ , then  $N_{\min} = 1$  and

$$\delta_{\max} \gtrsim \frac{1}{\|\nu(0)\|_{\ell^1} + \|\nu(0)\|_{\ell^1}^2 + \|\nu(0)\|_{\ell^1}^3} \gtrsim \|\nu(0)\|_{\ell^1}^{-1}.$$

This means that if the initial value has a small enough  $\mathcal{FL}^{0,1}$ -norm, then the solution belongs to  $\mathcal{FL}^{0,1}$  at least for  $t \sim \|u(0)\|_{\mathcal{FL}^{0,1}}^{-1}$ .

## 2.3 Long time estimates for the solution with initial datum in $\mathcal{FL}^{0,1}(\mathbb{T})$

In Section 2.2 we proved that, if we consider  $u(0) = g \in \mathcal{FL}_0^{0,1}(\mathbb{T})$ , then there exists  $\delta > 0$  and a unique solution  $u \in C_t^0 \mathcal{FL}_x^{0,1}([-\delta, \delta] \times \mathbb{T})$  of the KdV equation

$$u_t + u_{xxx} + uu_x = 0, \quad x \in \mathbb{T}.$$

In the interaction variables  $u_k(t) = \nu_k(t)e^{ik^3t}$ , we have that  $\nu \in C_t^0 \dot{\ell}_k^1([-\delta, \delta] \times (\mathbb{Z} \setminus \{0\}))$ . Since our final aim is to derive a Large Deviations Principle for the solution of the KdV equation, it is very useful to find some functions which approximate  $u(t, x)$  for certain timescales, and whose statistical properties are easier to study. In this section we give two estimates of this type.

**Lemma 2.3.1.** *Let  $u \in C_t^0 \mathcal{FL}_x^{0,1}([-\delta, \delta] \times \mathbb{T})$  be the solution of (2.0.1) with  $u(0) \in \mathcal{FL}_0^{0,1}(\mathbb{T})$ . Then for all  $t \in [-\delta, \delta]$ , in the interaction variables we have the estimate*

$$\|\nu(t) - \nu(0)\|_{\ell^1} \leq \|\nu(t)\|_{\ell^1}^2 + \|\nu(0)\|_{\ell^1}^2 + t \sup_{s \in [0, t]} \|\nu(s)\|_{\ell^1}^3. \quad (2.3.1)$$

which in the original variables becomes

$$\left\| u(t) - e^{-t\partial_x^3} u(0) \right\|_{\mathcal{FL}^{0,1}} \leq \|u(t)\|_{\mathcal{FL}^{0,1}}^2 + \|u(0)\|_{\mathcal{FL}^{0,1}}^2 + Ct \sup_{s \in [0, t]} \|u(s)\|_{\mathcal{FL}^{0,1}}^3, \quad (2.3.2)$$

where  $e^{-t\partial_x^3}$  was defined in (2.0.7). Remember that  $e^{-t\partial_x^3} u(0)$  is nothing but the solution of the linearized KdV equation, also known as the Airy equation.

*Proof.* We showed in Lemma 2.1.4 that

$$\begin{aligned} \nu_k(t) - \nu_k(0) &= -\frac{ik}{2} \int_0^t \sum_{k_1+k_2=k} e^{-3ikk_1k_2s} \nu_{k_1}(s) \nu_{k_2}(s) ds \\ &= \frac{1}{6} \sum_{k_1+k_2=k} e^{-3ikk_1k_2t} \frac{\nu_{k_1}(t)}{k_1} \frac{\nu_{k_2}(t)}{k_2} - \frac{1}{6} \sum_{k_1+k_2=k} \frac{\nu_{k_1}(0)}{k_1} \frac{\nu_{k_2}(0)}{k_2} \\ &\quad - \frac{1}{3} \int_0^t \sum_{k_1+k_2=k} e^{-3ikk_1k_2s} \frac{\partial_s \nu_{k_1}}{k_1} \frac{\nu_{k_2}}{k_2} ds. \end{aligned}$$

Taking the  $\ell^1$  norm both sides we get

$$\begin{aligned} \|\nu(t) - \nu(0)\|_{\ell^1} &\leq \frac{1}{6} \left\| \frac{|\nu_k(t)|}{|k|} * \frac{|\nu_k(t)|}{|k|} \right\|_{\ell^1} + \frac{1}{6} \left\| \frac{|\nu_k(0)|}{|k|} * \frac{|\nu_k(0)|}{|k|} \right\|_{\ell^1} \\ &\quad + \frac{1}{3} \int_0^t \left\| \frac{|\partial_s \nu_k(s)|}{|k|} * \frac{|\nu_k(s)|}{|k|} \right\|_{\ell^1} ds. \end{aligned}$$

But

$$\frac{\partial_t \nu_k(t)}{k} = -\frac{i}{2} \sum_{k_1+k_2=k} e^{-3ikk_1k_2t} \nu_{k_1} \nu_{k_2},$$

therefore

$$\left\| \frac{|\partial_s \nu_k(s)|}{|k|} \right\|_{\ell^1} \leq \frac{1}{2} \|\boldsymbol{\nu}(s)\|_{\ell^1}^2$$

and we get

$$\|\boldsymbol{\nu}(t) - \boldsymbol{\nu}(0)\|_{\ell^1} \leq \frac{1}{6} \|\boldsymbol{\nu}(t)\|_{\ell^1}^2 + \frac{1}{6} \|\boldsymbol{\nu}(0)\|_{\ell^1}^2 + \frac{1}{6} \int_0^t \|\boldsymbol{\nu}(s)\|_{\ell^1}^3 ds$$

using Young's convolution inequality and the trivial bound

$$\left\| \frac{|\nu_k|}{|k|} \right\|_{\ell^1} \leq \|\nu_k\|_{\ell^1}.$$

Finally we have the estimate

$$\|\boldsymbol{\nu}(t) - \boldsymbol{\nu}(0)\|_{\ell^1} \leq \|\boldsymbol{\nu}(t)\|_{\ell^1}^2 + \|\boldsymbol{\nu}(0)\|_{\ell^1}^2 + t \sup_{s \in [0,t]} \|\boldsymbol{\nu}(s)\|_{\ell^1}^3.$$

□

For the second estimate, we use a trick usual in Normal Form arguments, as we shall investigate and improve upon in the next sections. For (2.3.2) we integrated by parts once in (2.1.6). Remember that, the second time we integrated by parts, we couldn't remove the integral of the resonant terms of degree 3, given by  $\rho_k$ . This is not a problem, because by the particular form of  $\rho_k$  we can get rid of it with a rotation. Let's start from (2.1.9) written in a differential form, *i.e.*

$$\partial_t \nu_k = \partial_t C(\boldsymbol{\nu})_k - \frac{i}{6k} \nu_k |\nu_k|^2 + D(\boldsymbol{\nu})_k \quad (2.3.3)$$

and introduce the new variables

$$w_k = \nu_k - C(\boldsymbol{\nu})_k. \quad (2.3.4)$$

Notice that

$$\|\boldsymbol{w}(t)\|_{\ell^1} \leq \|\boldsymbol{\nu}(t)\|_{\ell^1} + \|C(\boldsymbol{\nu})(t)\|_{\ell^1} \leq \|\boldsymbol{\nu}(t)\|_{\ell^1} + C(\|\boldsymbol{\nu}(t)\|_{\ell^1}^2 + \|\boldsymbol{\nu}(t)\|_{\ell^1}^3). \quad (2.3.5)$$

This implies that

$$\|\boldsymbol{w}(t)\|_{\ell^1} \lesssim \|\boldsymbol{\nu}(t)\|_{\ell^1} \quad \text{when} \quad \|\boldsymbol{\nu}(t)\|_{\ell^1} \ll 1.$$

In these new variables, the equation (2.3.3) becomes

$$\partial_t w_k = -\frac{i}{6k} w_k |w_k|^2 + R(\boldsymbol{\nu}, \boldsymbol{w})_k \quad (2.3.6)$$

**Lemma 2.3.2.** *We have that*

$$\|R(\boldsymbol{\nu}, \boldsymbol{w})\|_{\ell^1} \lesssim \|\boldsymbol{\nu}\|_{\ell^1}^4 + \|\boldsymbol{\nu}\|_{\ell^1}^9.$$

*Proof.* A straightforward computation yields

$$\begin{aligned} R(\boldsymbol{\nu}, \mathbf{w})_k &= D(\boldsymbol{\nu})_k + \frac{i}{6k}(w_k|w_k|^2 - v_k|v_k|^2) \\ &= D(\boldsymbol{\nu})_k + \frac{i}{6k}(w_k|w_k|^2 - w_k|\nu_k|^2 + w_k|\nu_k|^2 - v_k|v_k|^2) \\ &= D(\boldsymbol{\nu})_k - \frac{i}{6k}|\nu_k|^2 C(\boldsymbol{\nu})_k + \frac{i}{6k}w_k(|w_k|^2 - |\nu_k|^2), \end{aligned}$$

thus

$$\begin{aligned} \|R(\boldsymbol{\nu}, \mathbf{w})\|_{\ell^1} &\lesssim \|D(\boldsymbol{\nu})\|_{\ell^1} + \|\|\nu_k|^2 C(\boldsymbol{\nu})_k\|_{\ell^1} + \|w_k(|w_k|^2 - |\nu_k|^2)\|_{\ell^1} \\ &\lesssim \|\boldsymbol{\nu}\|_{\ell^1}^4 + \|\|\boldsymbol{\nu}\|^2\|_{\ell^2} \|C(\boldsymbol{\nu})\|_{\ell^2} + \|\mathbf{w}\|_{\ell^2} \|w_k|^2 - |\nu_k|^2\|_{\ell^2} \\ &\lesssim \|\boldsymbol{\nu}\|_{\ell^1}^4 + \|\boldsymbol{\nu}\|_{\ell^1}^5 + (\|\boldsymbol{\nu}\|_{\ell^1} + \|\boldsymbol{\nu}\|_{\ell^1}^2 + \|\boldsymbol{\nu}\|_{\ell^1}^3) \|w_k| - |\nu_k|\|_{\ell^1} \|w_k| + |\nu_k|\|_{\ell^1} \\ &\lesssim \|\boldsymbol{\nu}\|_{\ell^1}^4 + \|\boldsymbol{\nu}\|_{\ell^1}^5 + (\|\boldsymbol{\nu}\|_{\ell^1} + \|\boldsymbol{\nu}\|_{\ell^1}^2 + \|\boldsymbol{\nu}\|_{\ell^1}^3)^2 \|\nu_k| + |C(\boldsymbol{\nu})_k| - |\nu_k|\|_{\ell^1} \\ &\lesssim \|\boldsymbol{\nu}\|_{\ell^1}^4 + \|\boldsymbol{\nu}\|_{\ell^1}^9 \end{aligned}$$

using Cauchy-Schwarz inequality, the continuous embedding  $\ell^1 \subseteq \ell^2$ , Proposition 2.2.1 and the bound (2.3.5).  $\square$

Now we introduce other variables

$$z_k(t) = e^{-\frac{i}{6k} \int_0^t |w_k(s)|^2 ds} w_k(t). \quad (2.3.7)$$

Clearly  $|z_k| = |w_k|$ , hence we have

$$w_k(t) = e^{-\frac{i}{6k} \int_0^t |z_k(s)|^2 ds} z_k(t).$$

It follows that

$$\partial_t z_k = e^{-\frac{i}{6k} \int_0^t |w_k(s)|^2 ds} \left( \frac{i}{6k} w_k|w_k|^2 + \partial_t w_k \right) = e^{-\frac{i}{6k} \int_0^t |z_k(s)|^2 ds} R(\boldsymbol{\nu}, \mathbf{z})_k. \quad (2.3.8)$$

The integral form of this is

$$z_k(t) - z_k(0) = \int_0^t e^{-\frac{i}{6k} \int_0^{t'} |w_k(s)|^2 ds} R(\boldsymbol{\nu}, \mathbf{z})_k(t') dt'. \quad (2.3.9)$$

**Lemma 2.3.3.** Let  $u \in C_t^0 \mathcal{F}L_x^{0,1}([-\delta, \delta] \times \mathbb{T})$  be the solution of (2.0.1) with  $u(0) \in \mathcal{F}L_0^{0,1}(\mathbb{T})$ . Then for all  $t \in ]-\delta, \delta[$ , in the interaction variables we have the estimate

$$\begin{aligned} \|\nu_k(t) - e^{-\frac{i}{6k} \int_0^t |\nu_k - C(\boldsymbol{\nu})_k|^2 ds} \nu_k(0)\|_{\ell^1} &\lesssim \|\boldsymbol{\nu}(t)\|_{\ell^1}^2 + \|\boldsymbol{\nu}(t)\|_{\ell^1}^3 + \|\boldsymbol{\nu}(0)\|_{\ell^1}^2 + \|\boldsymbol{\nu}(0)\|_{\ell^1}^3 \\ &\quad + t \sup_{s \in [0, t]} \left( \|\boldsymbol{\nu}(s)\|_{\ell^1}^4 + \|\boldsymbol{\nu}(s)\|_{\ell^1}^9 \right). \end{aligned} \quad (2.3.10)$$

*Proof.* If we substitute (2.3.4) and (2.3.7) in (2.3.9), we obtain

$$e^{-\frac{i}{6k} \int_0^t |\nu_k - C(\boldsymbol{\nu})_k|^2 ds} (\nu_k - C(\boldsymbol{\nu})_k)(t) - (\nu_k - C(\boldsymbol{\nu})_k)(0) = \int_0^t e^{-\frac{i}{6k} \int_0^{t'} |\nu_k - C(\boldsymbol{\nu})_k|^2 dt''} R(\boldsymbol{\nu}, \mathbf{z}[\boldsymbol{\nu}])_k(t') dt'$$

from which

$$\begin{aligned} \nu_k(t) &= e^{-\frac{i}{6k} \int_0^t |\nu_k - C(\boldsymbol{\nu})_k|^2 ds} \nu_k(0) - e^{-\frac{i}{6k} \int_0^t |\nu_k - C(\boldsymbol{\nu})_k|^2 ds} C(\boldsymbol{\nu})_k(0) + C(\boldsymbol{\nu})_k(t) \\ &\quad + e^{-\frac{i}{6k} \int_0^t |\nu_k - C(\boldsymbol{\nu})_k|^2 ds} \int_0^t e^{-\frac{i}{6k} \int_0^{t'} |\nu_k - C(\boldsymbol{\nu})_k|^2 dt''} R(\boldsymbol{\nu}, \mathbf{z}[\boldsymbol{\nu}])_k(t') dt' \end{aligned} \quad (2.3.11)$$

which yields the thesis.  $\square$

# Chapter 3

## Normal form

### 3.1 Hamiltonian formalism

In this section we study the KdV equation has a Hamiltonian system. This allows us to use a new technique, called *Normal Form transformation*, to approximate the solution of the KdV equation for longer time-scales, with a systematic treatment. As in Chapter 2 we will consider only the mean-zero case, as we can obtain the solution in the general case by Lemma 2.1.2. It is well known that KdV is an *integrable sistem*, meaning that it has infinitely many conserved quantities *in involution*. In Appendix A.2 the reader can find the precise definition of integrable system, the derivation of all these quantities and the proof that they are in involution. One first integral is

$$F^{(1)}(u) = H(u) = \frac{1}{2} \int_{\mathbb{T}} \left[ u_x(t, x)^2 - \frac{1}{3} u(t, x)^3 \right] dx. \quad (3.1.1)$$

Indeed, if  $u(t, x)$  is a smooth solution to (2.0.1), then

$$\begin{aligned} \frac{d}{dt} H(u) &= \frac{1}{2} \int_{\mathbb{T}} (2u_x u_{xt} - u^2 u_t) dx = \frac{1}{2} \int_{\mathbb{T}} (-2u_{xx} u_t + u^2 u_{xxx} + u^3 u_x) dx \\ &= \frac{1}{2} \int_{\mathbb{T}} (2u_{xx} u_{xxx} + 2uu_x u_{xx} - 2uu_x u_{xx} + u^3 u_x) dx \\ &= \frac{1}{2} \int_{\mathbb{T}} \frac{\partial}{\partial x} \left( u_{xx}^2 + \frac{1}{4} u^4 \right) dx = 0. \end{aligned}$$

For the Hamiltonian  $H(u)$  to be well-defined, it suffices to consider functions  $u$  in the space

$$H_0^s(\mathbb{T}, \mathbb{R}) = \{u \in H^s(\mathbb{T}) \mid u_0 = 0\} \quad \text{with } s \geq 1,$$

or in  $\mathcal{FL}_0^{s,1}(\mathbb{T}, \mathbb{R})$ , with  $s \geq 1$ , as we shall see soon. Remember that, since  $u$  is real-valued, we have that  $u_{-k} = \overline{u_k}$ . To write the KdV equation as a Hamiltonian system we recall the definition of  $L^2$ -Fréchet differential.

**Definition 3.1.1.** Let  $H : H^s(\mathbb{T}) \rightarrow \mathbb{R}$  be a functional. We say that

$$\frac{dH}{du} : H^s(\mathbb{T}) \rightarrow H^{s_0}(\mathbb{T})$$

is the  $L^2$ -differential of  $H$  if

$$\lim_{\|f\|_{H^s} \rightarrow 0} \frac{|H(u + f) - H(u) - \langle \frac{dH}{du}, f \rangle_{L^2}|}{\|f\|_{H^s}} \rightarrow 0,$$

for some  $s_0 \in \mathbb{R}$  possibly different from  $s$ . Sometimes we will write as usual  $\frac{dH}{du}$  as an operator, meaning that

$$\frac{dH}{du}[f] = \int_{\mathbb{T}} \frac{dH}{du}(x) f(x) dx.$$

In the same way one can define the  $\ell^2$ -differential.

The KdV equation can be written as a Hamiltonian system as

$$\partial_t u = \frac{\partial}{\partial x} \frac{dH}{du}, \quad (3.1.2)$$

where  $\frac{dH}{du}$  denotes the  $L^2$ -Fréchet differential of  $H$ . Indeed

$$\begin{aligned} H(u + f) - H(u) &= \frac{1}{2} \int_{\mathbb{T}} \left[ (u_x + f_x)^2 - u_x^2 - \frac{1}{3}(u + f)^3 + \frac{1}{3}u^3 \right] dx \\ &= \frac{1}{2} \int_{\mathbb{T}} \left[ 2u_x f_x + f_x^2 - u^2 f - u f^2 - \frac{1}{3}f^3 \right] dx \\ &= \frac{1}{2} \int_{\mathbb{T}} \left[ -2u_{xx} f + f_x^2 - u^2 f - u f^2 - \frac{1}{3}f^3 \right] dx, \end{aligned}$$

hence  $\frac{dH}{du} = -u_{xx} - \frac{1}{2}u^2$ , which combined with (3.1.2) yields

$$u_t = -u_{xxx} - uu_x.$$

Indeed

$$\begin{aligned} \limsup_{\|f\|_{H^s} \rightarrow 0} \frac{|H(u + f) - H(u) - \langle \frac{dH}{du}, f \rangle_{L^2}|}{\|f\|_{H^s}} &= \limsup_{\|f\|_{H^n} \rightarrow 0} \frac{\left| \frac{1}{2} \int_{\mathbb{T}} [f_x^2 - (u - \frac{1}{3}f)f^2] dx \right|}{\|f\|_{H^s}} \\ &\leq \limsup_{\|f\|_{H^s} \rightarrow 0} \frac{\frac{1}{2} \|f_x\|_{L^2}^2 + \frac{1}{2} \|u - \frac{1}{3}f\|_{L^2} \|f^2\|_{L^2}}{\|f\|_{H^s}} \\ &\lesssim_{\|u\|_{L^2}} \lim_{\|f\|_{H^s} \rightarrow 0} (\|f_x\|_{L^2} + \|f\|_{L^2}) = 0. \end{aligned}$$

We want to analyze the same problem but in the Fourier setting. To avoid confusion, from now on we will indicate with  $u$  a function and with  $\mathbf{u}$  the vector  $(u_k)_k$  of the Fourier coefficients of  $u$ . Moreover, we will indicate with  $H$  the Hamiltonian defined on a space of functions, and with  $\mathcal{H}$  the Hamiltonian defined on a space of sequences.

**Lemma 3.1.2.** *The Hamiltonian in the Fourier setting can be written as*

$$\mathcal{H}(\mathbf{u}) = \mathcal{H}_2(\mathbf{u}) + \mathcal{H}_3(\mathbf{u}) = \pi \sum_{k \in \mathbb{Z}} k^2 |u_k|^2 - \frac{\pi}{3} \sum_{k_1+k_2+k_3=0} u_{k_1} u_{k_2} u_{k_3}. \quad (3.1.3)$$

This functional is well defined if

$$u \in \mathcal{F}L_0^{1,2}(\mathbb{T}) = H_0^1(\mathbb{T}).$$

*Proof.* First we notice that, by Plancherel's Theorem,

$$\frac{1}{2} \int_{\mathbb{T}} u_x(t, x)^2 dx = \pi \sum_{k \in \mathbb{Z}} k^2 |u_k|^2.$$

Moreover

$$\frac{1}{6} \int_{\mathbb{T}} u(t, x)^3 dx = \frac{\pi}{3} (u^3)_0 = \frac{\pi}{3} \sum_{k_1+k_2+k_3=0} u_{k_1} u_{k_2} u_{k_3}.$$

Therefore

$$\mathcal{H}(\mathbf{u}) = \pi \sum_{k \in \mathbb{Z}} k^2 |u_k|^2 - \frac{\pi}{3} \sum_{k_1+k_2+k_3=0} u_{k_1} u_{k_2} u_{k_3}.$$

To prove that it is well-defined in  $H^1(\mathbb{T})$ , it is enough to notice that, by the Cauchy-Schwarz inequality,

$$H^1(\mathbb{T}) \subseteq \mathcal{F}L^{0,1}(\mathbb{T})$$

Indeed

$$\|u\|_{\mathcal{F}L^{0,1}} = \sum_{k \in \mathbb{Z}} |u_k| = \sum_{k \in \mathbb{Z}} \langle k \rangle |u_k| \cdot \frac{1}{\langle k \rangle} \leq \left( \sum_{k \in \mathbb{Z}} \langle k \rangle^2 |u_k|^2 \right)^{1/2} \left( \sum_{k \in \mathbb{Z}} \frac{1}{\langle k \rangle^2} \right)^{1/2} \lesssim \|u\|_{H^1}.$$

Then, if  $u \in H^1$ , we have

$$\sum_{k_3 \in \mathbb{Z}} \overline{u_{k_3}} \sum_{k_1+k_2=k_3} u_{k_1} u_{k_2} < \infty$$

since  $(u_{k_3})_{k_3} \in \ell^2$  and  $\sum_{k_1+k_2=k_3} u_{k_1} u_{k_2} \in \ell^1$ , given that it is the convolution of  $\ell^1$  sequences. Therefore we can apply the Cauchy-Schwarz inequality.

□

**Remark 3.1.3.** Clearly  $\mathcal{F}L^{1,1}(\mathbb{T}) \subseteq H^1(\mathbb{T})$ , because

$$u \in \mathcal{F}L^{1,1}(\mathbb{T}) \iff (\langle k \rangle u_k)_k \in \ell^1 \subseteq \ell^2.$$

This means that the Hamiltonian is well defined in  $\mathcal{F}L_0^{1,1}(\mathbb{T})$ .

Now we write (3.1.2) in the Fourier setting.

**Proposition 3.1.4.** *The KdV equation in the Hamiltonian formalism (3.1.2) can be written in the Fourier setting as*

$$\frac{d}{dt} \mathbf{u} = J \frac{d\mathcal{H}}{d\mathbf{u}} \tag{3.1.4}$$

where  $\frac{d\mathcal{H}}{d\mathbf{u}}$  is the  $\ell^2$ -differential of  $\mathcal{H}$  with respect to  $\mathbf{u}$  and

$$J : \dot{h}^s(\mathbb{Z}) \longrightarrow \dot{h}^{s-1}(\mathbb{Z}) \tag{3.1.5}$$

is the operator such that  $J(\mathbf{v})_k = ikv_k$ . If we write (3.1.4) for the  $k^{\text{th}}$  component we have

$$\partial_t u_k = ik^3 u_k - \frac{ik}{2} \sum_{k_1+k_2=k} u_{k_1} u_{k_2}, \tag{3.1.6}$$

which is exactly the same as (2.1.7).

*Proof.* Notice that (3.1.4) is nothing but (3.1.2) in the Fourier setting, since  $J$  is the operator which sends the Fourier coefficients of  $f$  into the Fourier coefficients of  $\partial_x f$ . A straightforward computation yields

$$\begin{aligned} \mathcal{H}(\mathbf{u} + \mathbf{f}) - \mathcal{H}(\mathbf{u}) &= \pi \sum_{k \in \mathbb{Z}} k^2 (|u_k + f_k|^2 - |u_k|^2) \\ &\quad - \frac{\pi}{3} \sum_{k_1+k_2=k_3} [(u_{k_1} + f_{k_1})(u_{k_2} + f_{k_2})(\overline{u_{k_3}} + \overline{f_{k_3}}) - u_{k_1} u_{k_2} \overline{u_{k_3}}] \\ &= \pi \sum_{k \in \mathbb{Z}} k^2 (u_k \overline{f_k} + f_k \overline{u_k} + |f_k|^2) \\ &\quad - \frac{\pi}{3} \sum_{k_1+k_2=k_3} [u_{k_1} u_{k_2} \overline{f_{k_3}} + u_{k_1} f_{k_2} \overline{u_{k_3}} + u_{k_1} f_{k_2} \overline{f_{k_3}} \\ &\quad + f_{k_1} u_{k_2} \overline{u_{k_3}} + f_{k_1} u_{k_2} \overline{f_{k_3}} + f_{k_1} f_{k_2} \overline{u_{k_3}} + f_{k_1} f_{k_2} \overline{f_{k_3}}]. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{d\mathcal{H}}{d\mathbf{u}}[\mathbf{f}] &= \pi \sum_{k \in \mathbb{Z}} k^2 (u_k \overline{f_k} + f_k \overline{u_k}) - \frac{\pi}{3} \sum_{k_1+k_2=k_3} [u_{k_1} u_{k_2} \overline{f_{k_3}} + u_{k_1} f_{k_2} \overline{u_{k_3}} + f_{k_1} u_{k_2} \overline{u_{k_3}}] \\ &= 2\pi \sum_{k \in \mathbb{Z}} k^2 u_k \overline{f_k} - \pi \sum_{k_1+k_2=k_3} u_{k_1} u_{k_2} \overline{f_{k_3}} \end{aligned}$$

using the fact that  $u_{-k} = \overline{u_k}$ . It follows that

$$\frac{d\mathcal{H}}{d\mathbf{u}}[\mathbf{f}] = 2\pi \sum_{k \in \mathbb{Z}} \left( k^2 u_k - \frac{1}{2} \sum_{k_1+k_2=k} u_{k_1} u_{k_2} \right) \overline{f_k}$$

and so

$$\frac{d\mathcal{H}}{d\mathbf{u}} = k^2 u_k - \frac{1}{2} \sum_{k_1+k_2=k} u_{k_1} u_{k_2}.$$

□

We shall now understand which is the symplectic structure behind (3.1.6). We consider the coordinates  $(u_k, \overline{u_k})_k$  and remember that  $\overline{u_k} = u_{-k}$ . The Hamiltonian is

$$\mathcal{H}(\mathbf{u}, \overline{\mathbf{u}}) = \mathcal{H}_2 + \mathcal{H}_3 = \pi \sum_{k \neq 0} k^2 u_k \overline{u_k} - \frac{\pi}{3} \sum_{k_1+k_2+k_3=0} u_{k_1} u_{k_2} u_{k_3}.$$

Let us consider the Poisson bracket

$$\{F, G\} = \sum_{k \neq 0} \frac{ik}{2\pi} \frac{\partial F}{\partial u_k} \frac{\partial G}{\partial \overline{u_k}}. \quad (3.1.7)$$

which may also be written as

$$\{F, G\} = \left\langle \frac{dF}{d\mathbf{u}}, J \frac{dG}{d\mathbf{u}} \right\rangle_{\ell^2}$$

where  $J$  was defined in (3.1.5).

**Remark 3.1.5.** Notice that

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial u_k} &= 2\pi k^2 \overline{u_k} - \frac{\pi}{3} \left( \sum_{k+k_2=k_3} u_{k_2} \overline{u_{k_3}} + \sum_{k_1+k=k_3} u_{k_1} \overline{u_{k_3}} + \sum_{k_1+k_2=-k} u_{k_1} u_{k_2} \right) \\ &= 2\pi k^2 \overline{u_k} - \pi \sum_{k_1+k_2=k} \overline{u_{k_1} u_{k_2}} \end{aligned} \quad (3.1.8)$$

and

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial \bar{u}_k} &= 2\pi k^2 u_k - \frac{\pi}{3} \left( \sum_{-k+k_2=k_3} u_{k_2} \overline{u_{k_3}} + \sum_{k_1-k=k_3} u_{k_1} \overline{u_{k_3}} + \sum_{k_1+k_2=k} u_{k_1} u_{k_2} \right) \\ &= 2\pi k^2 u_k - \pi \sum_{k_1+k_2=k} u_{k_1} u_{k_2}, \end{aligned} \quad (3.1.9)$$

therefore we can write

$$\partial_t u_k = \{u_k, \mathcal{H}\} = \sum_{j \neq 0} \frac{ik}{2\pi} \frac{\partial u_k}{\partial u_j} \frac{\partial \mathcal{H}}{\partial \bar{u}_j} = ik^3 u_k - \frac{ik}{2} \sum_{k_1+k_2=k} u_{k_1} u_{k_2}. \quad (3.1.10)$$

We will work in a special family of functions  $F(\mathbf{u}, \bar{\mathbf{u}})$ , which is an algebra with respect to the Poisson bracket defined in (3.1.7).

**Definition 3.1.6.** We will call *homogeneous formal polynomial* a function of the form

$$F = \sum_{k_1+\dots+k_n=N}^* f(k_1, \dots, k_n) u_{k_1} \dots u_{k_n},$$

where  $n \in \mathbb{N}$  is the *degree* of the polynomial,  $N \in \mathbb{Z}$ ,  $f(k_1, \dots, k_n) \in \mathbb{C}(k_1, \dots, k_n)$ , i.e. it is the fraction of two polynomials in  $(k_1, \dots, k_n)$ , and the sum is taken over a subset of indeces<sup>1</sup> for which the denominator doesn't vanish<sup>2</sup>. Notice that, since the numerator has finite degree,  $F$  is well defined if  $u \in \mathcal{F}L_0^{s,1}(\mathbb{T})$  or  $u \in H_0^s(\mathbb{T})$ , with  $s$  sufficiently big. For the time being, we will assume that  $u$  has all the derivatives we need to work with the Poisson algebra of formal polynomials.

**Remark 3.1.7.** A formal polynomial of degree 1 is a single monomial  $f(k)u_k$ , which exists without requiring any regularity on  $u$ . If a formal polynomial  $F$  has degree  $n \geq 2$ , then by the Cauchy-Schwarz inequality and Theorem A.1.4 we can use  $\ell^2$ -based spaces such as  $H^s(\mathbb{T})$  instead of  $\ell^1$ -based spaces, with the same number of derivatives. As required in Theorem A.1.4, we need  $s > \frac{1}{2}$ . Therefore, when dealing with formal polynomials, we will always assume such regularity for  $u$ . It is clear that, when a homogeneous formal polynomial of degree  $n$  is well defined in some space  $X = \mathcal{F}L_0^{s,p}(\mathbb{T})$  with  $s > \frac{1}{2}$  and  $p \in \{1, 2\}$ , then

$$|F| \lesssim \|u\|_X^n.$$

**Lemma 3.1.8** (Algebra of formal polynomials). *If  $F$  and  $G$  are two homogeneous formal polynomials of the form*

$$F = \sum_{k_1+\dots+k_n=N} f(k_1, \dots, k_n) u_{k_1} \dots u_{k_n}, \quad G = \sum_{k'_1+\dots+k'_m=M} g(k'_1, \dots, k'_m) u_{k'_1} \dots u_{k'_m},$$

---

<sup>1</sup>This precision will be clear by the proof of Lemma 3.1.8.

<sup>2</sup>From now on we avoid putting the asterisk above the sum.

with  $n, m \in \mathbb{N}$  and  $N, M \in \mathbb{Z}$ , then

$$\{F, G\} = H = \sum_{k_1 + \dots + k_{n+m-2} = N+M} h(k_1, \dots, k_{n+m-2}) u_{k_1} \dots u_{k_{n+m-2}}$$

is again a homogeneous formal polynomial. Moreover, if  $p \in \{1, 2\}$  and  $F$  is well defined for  $u \in \mathcal{FL}_0^{s,p}(\mathbb{T})$  and  $G$  is well defined for  $u \in \mathcal{FL}_0^{r,p}(\mathbb{T})$ , then  $\{F, G\}$  is well defined at least in  $\mathcal{FL}_0^{2r+2s+1,p}(\mathbb{T})$ .

*Proof.* A straightforward computation yields

$$\begin{aligned} \{F, G\} &= \left\{ \sum_{k_1 + \dots + k_n = N} f(k_1, \dots, k_n) u_{k_1} \dots u_{k_n}, \sum_{k'_1 + \dots + k'_m = M} g(k'_1, \dots, k'_m) u_{k'_1} \dots u_{k'_m} \right\} \\ &= \sum_{\substack{k_1 + \dots + k_n = N \\ k'_1 + \dots + k'_m = M}} f(k_1, \dots, k_n) g(k'_1, \dots, k'_m) \left\{ u_{k_1} \dots u_{k_n}, u_{k'_1} \dots u_{k'_m} \right\} \\ &= \sum_{\substack{i=1, \dots, n \\ j=1, \dots, m}} \sum_{\substack{k_1 + \dots + k_n - k_i = N-k \\ k'_1 + \dots + k'_m - k'_j = M+k}} \frac{ik}{2\pi} f(k_1, \dots, k_{i-1}, k, k_{i+1}, \dots, k_n) \\ &\quad \cdot g(k'_1, \dots, k'_{j-1}, -k, k'_{j+1}, \dots, k'_m) \frac{u_{k_1} \dots u_{k_n} u_{k'_1} \dots u_{k'_m}}{u_{k_i} u_{k'_j}} \\ &= \sum_{k_1 + \dots + k_{n+m-2} = N+M} h(k_1, \dots, k_{n+m-2}) u_{k_1} \dots u_{k_{n+m-2}}. \end{aligned}$$

where

$$\begin{aligned} h(k_1, \dots, k_{n+m-2}) &= \frac{ik}{2\pi} [f(k, k_1, \dots, k_{n-1}) + \dots + f(k_1, \dots, k_{n-1}, k)] \\ &\quad + [g(-k, k'_1, \dots, k'_{m-1}) + \dots + g(k'_1, \dots, k'_{m-1}, -k)] \end{aligned}$$

after substituting the value of  $k$  and renaming the indexes. Notice that  $k \neq 0$ , hence  $h(k_1, \dots, k_{n+m-2})$  is set equal to zero for some choice of indexes even if the numerator doesn't vanish. When we substitute  $k = k'_1 + \dots + k'_{m-1} - M = -(k_1 + \dots + k_{n-1} - N)$  inside  $f$  and  $g$ , we obtain that the degree of the numerators of the fractions which appear in  $h(k_1, \dots, k_{n+m-2})$  increases at most by  $r+s+1$ .  $\square$

We will see that in our specific case we are able to control this loss of derivatives, which a priori seems too bad.

**Lemma 3.1.9.** *The operation  $\{\cdot, \cdot\}$  defined in (3.1.7) is a Poisson bracket, i.e. it is bilinear, symmetric, and it satisfies the Jacobi identity (3.1.11).*

*Proof.* The bilinearity is obvious. It is antisymmetric because

$$\{F, G\} = \sum_{k \neq 0} \frac{ik}{2\pi} \frac{\partial F}{\partial u_k} \frac{\partial G}{\partial u_k} = - \sum_{k \neq 0} \frac{ik}{2\pi} \frac{\partial F}{\partial u_{-k}} \frac{\partial G}{\partial u_{-k}} = - \sum_{k \neq 0} \frac{ik}{2\pi} \frac{\partial F}{\partial u_k} \frac{\partial G}{\partial u_k} = -\{G, F\}.$$

It remains to prove the Jacobi identity

$$\{\{F, G\}, H\} + \{\{H, F\}, G\} + \{\{G, H\}, F\} = 0. \quad (3.1.11)$$

Note that

$$\begin{aligned} \{\{F, G\}, H\} &= \sum_{k_1 \neq 0} \frac{ik_1}{2\pi} \frac{\partial}{\partial u_{k_1}} \left( \sum_{k_2 \neq 0} \frac{ik_2}{2\pi} \frac{\partial F}{\partial u_{k_2}} \frac{\partial G}{\partial \bar{u}_{k_2}} \right) \frac{\partial H}{\partial \bar{u}_{k_1}} \\ &= -\frac{1}{4\pi^2} \sum_{k_1, k_2 \neq 0} k_1 k_2 \left[ \frac{\partial^2 F}{\partial u_{k_1} \partial u_{k_2}} \frac{\partial G}{\partial \bar{u}_{k_2}} \frac{\partial H}{\partial \bar{u}_{k_1}} + \frac{\partial F}{\partial u_{k_2}} \frac{\partial^2 G}{\partial u_{k_1} \partial \bar{u}_{k_2}} \frac{\partial H}{\partial \bar{u}_{k_1}} \right]. \end{aligned}$$

Therefore

$$\begin{aligned} \{\{F, G\}, H\} + \{\{H, F\}, G\} + \{\{G, H\}, F\} &= \\ = -\frac{1}{4\pi^2} \sum_{k_1, k_2 \neq 0} k_1 k_2 &\left[ \frac{\partial^2 F}{\partial u_{k_1} \partial u_{k_2}} \frac{\partial G}{\partial \bar{u}_{k_2}} \frac{\partial H}{\partial \bar{u}_{k_1}} + \frac{\partial F}{\partial u_{k_2}} \frac{\partial^2 G}{\partial u_{k_1} \partial \bar{u}_{k_2}} \frac{\partial H}{\partial \bar{u}_{k_1}} + \frac{\partial^2 H}{\partial u_{k_1} \partial u_{k_2}} \frac{\partial F}{\partial \bar{u}_{k_2}} \frac{\partial G}{\partial \bar{u}_{k_1}} \right. \\ &\left. + \frac{\partial H}{\partial u_{k_2}} \frac{\partial^2 F}{\partial u_{k_1} \partial \bar{u}_{k_2}} \frac{\partial G}{\partial \bar{u}_{k_1}} + \frac{\partial^2 G}{\partial u_{k_1} \partial u_{k_2}} \frac{\partial H}{\partial \bar{u}_{k_2}} \frac{\partial F}{\partial \bar{u}_{k_1}} + \frac{\partial G}{\partial u_{k_2}} \frac{\partial^2 H}{\partial u_{k_1} \partial \bar{u}_{k_2}} \frac{\partial F}{\partial \bar{u}_{k_1}} \right] = 0 \end{aligned}$$

from the symmetry of the sums and the relation  $u_{-k} = \bar{u}_k$ .  $\square$

Formal polynomials share some properties with the usual polynomials in a finite number of variables. In particular we have the following identity principle, which allows us to say that a formal polynomial vanishes if and only if all its coefficient vanish. We prove it under slightly more general hypotheses.

**Lemma 3.1.10** (Identity principle for formal polynomials). *Let*

$$P_n(\mathbf{u}, \bar{\mathbf{u}}) = \sum_{j=0}^n \sum_{|\alpha^{(j)}|+|\beta^{(j)}|=j} P_j^{(\alpha^{(j)}, \beta^{(j)})} \mathbf{u}^{\alpha^{(j)}} \bar{\mathbf{u}}^{\beta^{(j)}},$$

where  $\alpha^{(j)}, \beta^{(j)} : \mathbb{N} \rightarrow \mathbb{N}_0$  are multi-indexes with finitely many non-zero components. Suppose that the sequence  $\mathbf{u}$  belongs to a space  $X$  and that  $P_n : X \times X \rightarrow \mathbb{C}$  is well defined. Then

$$P_n = 0 \iff P_j^{(\alpha^{(j)}, \beta^{(j)})} = 0 \quad \forall j, \alpha^{(j)}, \beta^{(j)}.$$

*Proof.* Suppose by contradiction that there exists  $j_0, \alpha^{j_0}, \beta^{(j_0)}$  such that  $P_{j_0}^{(\alpha^{j_0}, \beta^{j_0})} \neq 0$ . Then we can restrict ourselves to the subspace of  $X$  where we put equal to zero all the components of  $\mathbf{u}$  and  $\bar{\mathbf{u}}$  for which  $\alpha^{(j_0)}$  and  $\beta^{(j_0)}$  vanish. This way we obtain

$$Q_n(\mathbf{u}, \bar{\mathbf{u}}) = 0$$

where  $Q$  is a standard polynomial in a finite number of variables, and the coefficients of  $Q_n$  which don't vanish are the same to those of  $P_n$ . By the identity principle for polynomials we can conclude that all the coefficients of  $Q_n$  vanish, included  $P_{j_0}^{(\alpha^{j_0}, \beta^{j_0})}$ . This yields a contradiction and concludes the proof of the theorem.  $\square$

In order to identify the effective dynamics of the Fourier coefficients of the solution to the KdV equation, our strategy in Chapter 2 was to manipulate the integral equations by performing complex rotations and integrating by parts twice to exploit oscillations. In this chapter we present an equivalent, albeit more systematic, approach which achieves the same goal. One advantage of these transformations is that they are carried out at the level of the Hamiltonian, which avoids the many integration by parts and complex rotations which we had to perform in Chapter 2. We will require such transformations to preserve the Poisson bracket, a property which will greatly simplify our computations. Those which do so are known as *canonical transformations*.

**Definition 3.1.11** (Canonical transformation). Given a Banach space of functions  $X$  and an algebra of Hamiltonian functions with respect to the Poisson bracket  $\{\cdot, \cdot\}$ , we say that a transformation defined on a open subset of  $X$

$$\Phi : A \subseteq X \longrightarrow \Phi(A)$$

is *canonical* if it is a diffeomorphism, *i.e.* it is invertible and, together with its inverse, it is Fréchet-differentiable, and if it preserves the Poisson structure, *i.e.*

$$\{F \circ \Phi, G \circ \Phi\} = \{F, G\} \circ \Phi$$

for all the Hamiltonians  $F, G$ .

Our main goal is to transform a given Hamiltonian into its simplest form possible: its *normal form*. This is why such mappings are often known as *normal form transformations*. This new Hamiltonian yields easily the dynamics in the new coordinates  $\mathbf{v}$ . Finally we will use the inverse transformation to obtain the dynamics in the original variables  $\mathbf{u}$ . In the next few pages we present some elementary results found in [4] and [16], which we record here for completeness. The properties presented in Lemma 3.1.12, Lemma 3.1.13, Remark 3.1.14 and Proposition 3.1.15 are formal, in the sense that we assume  $u \in \mathcal{FL}_0^{s,p}$  with  $p \in \{1, 2\}$  and  $s$  sufficiently big, such that all the quantities involved are well defined. In the next sections we will apply these results rigorously, finding each time the exact space where all the procedure is precisely justified. One way to construct a normal form transformation is as the flow of an auxiliary Hamiltonian function, such that the Hamiltonian in the new coordinates has a very special form. In particular we have to solve the Cauchy problem

$$\begin{cases} \frac{d}{d\xi} u_k = \{u_k, G\} = (J \frac{dG}{du})_k \\ u_k(0) = v_k \end{cases}$$

whose solution is  $\mathbf{u}(\xi) = \Phi_G^\xi(\mathbf{v})$ .

This change of coordinates preserves the Poisson structure. This, and other classical results, are presented in [1], [11] and [13].

**Lemma 3.1.12** (Taylor series for the new Hamiltonian). *For any Hamiltonian function  $H$  we have*

$$\frac{d}{d\xi}(H \circ \Phi_G^\xi) = \{H, G\} \circ \Phi_G^\xi.$$

*Inductively we have*

$$\frac{d^n}{d\xi^n}(H \circ \Phi_G^\xi) = \{\{\{H, G\}, G\} \dots\} \circ \Phi_G^\xi. \quad (3.1.12)$$

*Set*

$$H_0 = H, \quad H_n = \{H_{n-1}, G\} \quad \text{for } n \geq 1, \quad (3.1.13)$$

*then the formal Taylor series of  $H \circ \Phi_G^\xi$  at  $\xi = 0$  is*

$$\sum_{n=0}^{\infty} \frac{H_n}{n!} \xi^n.$$

*Proof.*

$$\frac{d}{d\xi}(H \circ \Phi_G^\xi) = \left( \frac{dH}{du} \circ \Phi_G^\xi \right) \left[ \frac{d}{d\xi} \Phi_G^\xi \right] = \left\langle \frac{dH}{du} \circ \Phi_G^\xi, J \frac{dG}{du} \circ \Phi_G^\xi \right\rangle_{\ell^2} = \{H, G\} \circ \Phi_G^\xi.$$

□

**Lemma 3.1.13** (Hamiltonian flows preserve the Poisson structure). *For any  $\xi$  such that  $\Phi_G^\xi$  is well defined,  $\Phi_G^\xi$  formally preserves the Poisson bracket,*

$$\{F_1, F_2\} \circ \Phi_G^\xi = \{F_1 \circ \Phi_G^\xi, F_2 \circ \Phi_G^\xi\}.$$

*Proof.* The thesis is clearly true for  $\xi = 0$ , since  $\Phi_G^0$  is the identity. Moreover

$$\begin{aligned} & \frac{d}{d\xi} \left( \{F_1, F_2\} \circ \Phi_G^\xi - \{F_1 \circ \Phi_G^\xi, F_2 \circ \Phi_G^\xi\} \right) \\ &= (\{\{F_1, F_2\}, G\} - \{\{F_1, G\}, F_2\} - \{F_1, \{F_2, G\}\}) \circ \Phi_G^\xi = 0 \end{aligned}$$

by the Jacobi identity.  $\square$

**Remark 3.1.14.** If  $\Phi_G^\xi$  is well defined up to  $\xi = 1$ , then by the Taylor formula up to order  $N$  with the Lagrange's remainder evaluated at  $\xi = 1$  we have

$$\begin{aligned} H \circ \Phi_G^1 &= \sum_{n=0}^N \frac{H_n}{n!} + R_{N+1} \\ &= H + \{H, G\} + \frac{1}{2}\{\{H, G\}, G\} + \dots + \frac{1}{N!}\{\{\{H, G\}, G\} \dots\} \\ &\quad + \frac{1}{N!} \int_0^1 (1-\xi)^N H_{N+1} \circ \Phi_G^\xi d\xi. \end{aligned}$$

This formula will be useful to estimate the remainder after a normal form transformation.

The following proposition explains how we can study the dynamics of a Hamiltonian system after a canonical transformation, *i.e.* in the new coordinates, and then to come back to the original coordinates.

**Proposition 3.1.15** (Dynamics in the new coordinates). *If  $u = \Phi(v)$  is a canonical transformation and  $u(t)$  is a solution of the Hamiltonian system with Hamiltonian  $H(u)$ , then  $v(t)$  is a solution of the Hamiltonian system with Hamiltonian  $H \circ \Phi(v)$  and viceversa.*

*Proof.* We have that  $\dot{u}_k = \{u_k, \mathcal{H}\}$  and in general  $\frac{d}{dt} f(u) = \{f, H\}$ . If we apply this last formula to  $f = \Phi_k^{-1} = P_k \circ \Phi^{-1}$  where  $P_k$  is the projection to the  $k^{\text{th}}$  component, we obtain

$$\begin{aligned} \dot{v}_k &= \frac{d}{dt} (P_k \circ \Phi^{-1})(u) = \{P_k \circ \Phi^{-1}, H\}(u) = \{P_k \circ \Phi^{-1}, H \circ \Phi \circ \Phi^{-1}\}(u) \\ &= \{P_k, H \circ \Phi\} \circ \Phi^{-1}(u) = \{v_k, H \circ \Phi\} \end{aligned}$$

since the transformation is canonical. The reverse implication is analogous.  $\square$

## 3.2 First normal form transformation

We are ready to apply the techniques presented in Section 3.1 to the KdV equation. Recall that we are using the Hamiltonian  $\mathcal{H}$  defined in (3.1.3), which is the sum of two homogeneous formal polynomials  $\mathcal{H}_2$  and  $\mathcal{H}_3$ , according to Definition 3.1.6. We will consider a canonical

transformation  $\Phi_{G_3}$  generated by some homogeneous formal polynomial  $G_3$  of degree 3 which we will choose later. By Lemma 3.1.12, we have that

$$\mathcal{H} \circ \Phi_{G_3} = \mathcal{H}_2 \circ \Phi_{G_3} + \mathcal{H}_3 \circ \Phi_{G_3} \quad (3.2.1)$$

$$= \mathcal{H}_2 + \{\mathcal{H}_2, G_3\} + \mathcal{H}_3 \quad (3.2.2)$$

$$+ \mathcal{H}_2 \circ \Phi_{G_3} - \mathcal{H}_2 - \{\mathcal{H}_2, G_3\} \quad (3.2.3)$$

$$+ \mathcal{H}_3 \circ \Phi_{G_3} - \mathcal{H}_3 \quad (3.2.4)$$

$$= \mathcal{H}_2 + \hat{\mathcal{H}}_3 + R_4. \quad (3.2.5)$$

We will choose  $G_3$  such that

$$\{\mathcal{H}_2, G_3\} + \mathcal{H}_3 = \hat{\mathcal{H}}_3 \quad (3.2.6)$$

is in a very special form. In particular we want to remove from  $\mathcal{H}_3$  all the monomials that we can. The equation (3.2.6) is called *homological equation*. Moreover (3.2.3) and (3.2.5) will be of order  $\geq 4$ . This transformation of coordinates is called *normal form transformation*. Now we solve the homological equation.

**Theorem 3.2.1** (First normal form transformation). *If we choose*

$$G_3 = i \frac{\pi}{3} \sum_{k_1+k_2+k_3=0} \frac{u_{k_1} u_{k_2} u_{k_3}}{k_1^3 + k_2^3 + k_3^3} = i \frac{\pi}{9} \sum_{k_1+k_2+k_3=0} \frac{u_{k_1} u_{k_2} u_{k_3}}{k_1 k_2 k_3} \quad (3.2.7)$$

then we have

$$\{\mathcal{H}_2, G_3\} + \mathcal{H}_3 = \hat{\mathcal{H}}_3 = 0.$$

*Proof.* In this proof we will explain how to build such  $G_3$ . The first thing we have to notice is that, if we choose  $G_3$  as a polynomial of degree 3, then the degree of  $\{\mathcal{H}_2, G_3\}$  is 3, which is exactly the degree of  $\mathcal{H}_3$ . First we try to understand what would happen if we apply  $\{\mathcal{H}_2, \cdot\}$  to a generic monomial

$$u_{k_1}^{\sigma_1} u_{k_2}^{\sigma_2} u_{k_3}^{\sigma_3}$$

where  $\sigma_j \in \{\pm 1\}$  indicates the complex conjugation. We have that

$$\begin{aligned} \{\mathcal{H}_2, u_{k_1}^{\sigma_1} u_{k_2}^{\sigma_2} u_{k_3}^{\sigma_3}\} &= \sum_{k \neq 0} \frac{ik}{2\pi} \frac{\partial \mathcal{H}_2}{\partial u_k} \frac{\partial}{\partial \bar{u}_k} (u_{k_1}^{\sigma_1} u_{k_2}^{\sigma_2} u_{k_3}^{\sigma_3}) \\ &= \sum_{k \neq 0} ik^3 \bar{u}_k \cdot \left[ \delta(k_1 + \sigma_1 k = 0) u_{k_2}^{\sigma_2} u_{k_3}^{\sigma_3} + \delta(k_2 + \sigma_2 k = 0) u_{k_1}^{\sigma_1} u_{k_3}^{\sigma_3} + \delta(k_3 + \sigma_3 k = 0) u_{k_1}^{\sigma_1} u_{k_2}^{\sigma_2} \right]. \end{aligned}$$

We can write

$$G_3 = \sum_{\substack{k_1, k_2, k_3 \neq 0 \\ \sigma_1, \sigma_2, \sigma_3 = \pm 1}} g_3(k_1, k_2, k_3, \sigma_1, \sigma_2, \sigma_3) u_{k_1}^{\sigma_1} u_{k_2}^{\sigma_2} u_{k_3}^{\sigma_3}$$

and so, if we impose  $\{\mathcal{H}_2, G_3\} = -\mathcal{H}_3$ , we obtain

$$\begin{aligned} \{\mathcal{H}_2, G_3\} &= -i \sum_{\substack{k_1, k_2, k_3 \neq 0 \\ \sigma_1, \sigma_2, \sigma_3 = \pm 1}} g_3(k_1, k_2, k_3, \sigma_1, \sigma_2, \sigma_3) \\ &\quad \cdot \left[ \sigma_1 k_1^3 u_{\sigma_1 k_1} u_{k_2}^{\sigma_2} u_{k_3}^{\sigma_3} + \sigma_2 k_2^3 u_{\sigma_2 k_2} u_{k_1}^{\sigma_1} u_{k_3}^{\sigma_3} + \sigma_3 k_3^3 u_{\sigma_3 k_3} u_{k_1}^{\sigma_1} u_{k_2}^{\sigma_2} \right] \\ &= \frac{\pi}{3} \sum_{k_1+k_2+k_3=0} u_{k_1} u_{k_2} u_{k_3}. \end{aligned}$$

If we set  $g_3(k_1, k_2, k_3, \sigma_1, \sigma_2, \sigma_3) = 0$  whenever  $k_1 + k_2 + k_3 \neq 0$  or  $\sigma_1 = -1, \sigma_2 = -1, \sigma_3 = -1$ , we obtain

$$-i \sum_{k_1+k_2+k_3=0} g_3(k_1, k_2, k_3, \sigma_1, \sigma_2, \sigma_3)(k_1^3 + k_2^3 + k_3^3) u_{k_1} u_{k_2} u_{k_3} = \frac{\pi}{3} \sum_{k_1+k_2+k_3=0} u_{k_1} u_{k_2} u_{k_3}$$

and so by Lemma 3.1.10 we have to choose

$$G_3 = i \frac{\pi}{3} \sum_{k_1+k_2+k_3=0} \frac{u_{k_1} u_{k_2} u_{k_3}}{k_1^3 + k_2^3 + k_3^3}.$$

□

**Remark 3.2.2.** Notice that

$$\Omega_1 = k_1^3 + k_2^3 + k_3^3 = -3k_1^2 k_2 - 3k_1 k_2^2 = 3k_1 k_2 k_3 \neq 0,$$

which is the relation that allowed us to integrate by parts in Lemma 2.1.7.

Once we have found the auxiliary Hamiltonian  $G_3$  defined in (3.2.7), we have to show that its time 1 flow defines a diffeomorphism in an appropriate Banach space of functions. Then, by Lemma 3.1.13,  $\Phi_{G_3}$  preserves the canonical structure of the equation, and by Proposition 3.1.15 we can study the dynamics in the new coordinates, for which the Hamiltonian is *simpler*.

**Theorem 3.2.3** ( $\Phi_{G_3}$  is a local diffeomorphism). *Let  $X = H_0^s(\mathbb{T})$  or  $X = \mathcal{F}L_0^{s,1}(\mathbb{T})$  with  $s \geq 1$ . Let  $v \in X$ . Consider the Cauchy problem in the Fourier setting*

$$\begin{cases} \frac{d}{d\xi} u_k = \{u_k, G_3(\mathbf{u})\}(\xi), \\ u_k(0) = v_k. \end{cases} \quad (3.2.8)$$

Where  $G_3$  was defined in (3.2.7). There exists  $\varepsilon_0 > 0$  such that for all  $0 < \varepsilon \leq \varepsilon_0$  and for all  $v \in B_X(\varepsilon)$  the problem (3.2.8) has a unique solution  $u \in C_\xi^0 X([0, \Xi] \times \mathbb{T})$ , with  $\Xi > 1$  and

$$\sup_{\xi \in [0, \Xi]} \|u(\xi)\|_X \leq 2 \|v\|_X. \quad (3.2.9)$$

Moreover, if we set  $\Phi_{G_3}(v) = u = u(1)$ , we have that  $\Phi_{G_3} : B_X(\varepsilon) \rightarrow \Phi_{G_3}(B_X(\varepsilon))$  is a diffeomorphism, i.e. it is  $C^1$  and invertible, and its inverse  $\Phi_{G_3}^{-1}$  is  $C^1$ . Finally both  $\Phi_{G_3}$  and  $\Phi_{G_3}^{-1}$  are close to the identity, in the sense that

$$\|(I - \Phi_{G_3})(v)\|_X \leq C_{G_3} \|v\|_X^2, \quad (3.2.10)$$

$$\|(I - \Phi_{G_3}^{-1})(u)\|_X \leq C_{G_3} \|u\|_X^2. \quad (3.2.11)$$

*Proof.* Let us investigate the change of coordinates induced by  $G_3$ . We have to solve the Cauchy problem

$$\begin{cases} \frac{d}{d\xi} u_k = \{u_k, G_3(\mathbf{u})\}(\xi), \\ u_k(0) = v_k. \end{cases}$$

But

$$\begin{aligned} \{u_k, G_3(\mathbf{u})\} &= \frac{ik}{2\pi} \frac{\partial G_3}{\partial u_k} \\ &= -\frac{k}{18} \sum_{k_1+k_2+k_3=0} \left[ \frac{\delta(k = -k_1)u_{k_2}u_{k_3} + \delta(k = -k_2)u_{k_1}u_{k_3} + \delta(k = -k_3)u_{k_1}u_{k_2}}{k_1k_2k_3} \right] \\ &= \frac{1}{6} \sum_{k_1+k_2=k} \frac{u_{k_1}u_{k_2}}{k_1k_2} \end{aligned}$$

and so

$$u_k(\xi) = v_k + \frac{1}{6} \int_0^\xi \sum_{k_1+k_2=k} \frac{u_{k_1}u_{k_2}}{k_1k_2} d\xi'.$$

By Theorem A.1.9 there exists  $\varepsilon_0 > 0$  such that for any  $0 < \varepsilon \leq \varepsilon_0$  and  $\|v\|_X \leq \varepsilon$ , setting

$$u = \Phi_{G_3}(u) = u(1)$$

we have that  $\Phi_{G_3}$  is well defined (the solution exists up to times  $\xi > 1$ ) and it is a diffeomorphism in a neighborhood of the origin. Moreover we have the estimates (3.2.9), (3.2.10) and (3.2.11).

The hypothesis of the theorem are satisfied because we have

$$F(u)_k = \sum_{k_1+k_2=k} \frac{u_{k_1}u_{k_2}}{k_1k_2}, \quad (3.2.12)$$

$$dF(u)[h]_k = 2 \sum_{k_1+k_2=k} \frac{u_{k_1}h_{k_2}}{k_1k_2}, \quad (3.2.13)$$

$$d^2F(u)[h, w]_k = 2 \sum_{k_1+k_2=k} \frac{h_{k_1}w_{k_2}}{k_1k_2}, \quad (3.2.14)$$

and we can apply Corollary A.1.3 or Theorem A.1.4 depending on the choice of  $X$ .  $\square$

Now we want to understand what happens when we consider the Poisson bracket  $\{F_r, G_3\}$ , where  $F_r$  is a homogeneous formal polynomial of degree  $r$ . This is important in order to comprehend in which space the new Hamiltonian functions appearing after the transformation  $\Phi_{G_3}$  are defined. While in Lemma 3.1.8 we showed that the Poisson bracket typically loses derivatives, our next result shows that this is not the case when computing  $\{F_r, G_3\}$ .

**Theorem 3.2.4.** *Consider a homogeneous formal polynomial of degree  $r \geq 2$ ,*

$$F_r = \sum_{k_1+\dots+k_r=0}^* f_r(k_1, \dots, k_r) u_{k_1} \dots u_{k_r}$$

*with  $f_r \in \mathbb{C}(k_1, \dots, k_r)$  which is well defined in some space  $X = \mathcal{FL}_0^{s,p}(\mathbb{T})$  for  $p \in \{1, 2\}$  and  $s > \frac{1}{2}$ . Then the Poisson bracket  $\{F_r, G_3\}$ , where  $G_3$  was defined in (3.2.7), is a homogeneous formal polynomial of degree  $r+1$  which is well defined on the same space  $X$ .*

*Proof.* Since  $f_r$  is the ratio of two polynomials, we have that

$$|f_r(k_1, \dots, k_r)| \leq \sum_{|\alpha| \leq n} c_\alpha |k_1|^{\alpha_1} \dots |k_r|^{\alpha_r} \quad (3.2.15)$$

for some  $n \in \mathbb{N}$ , where  $\alpha = (\alpha_1, \dots, \alpha_r)$  is a vector with non-negative integer components and  $c_\alpha$  is a non-negative constant. Since  $|\alpha| = \alpha_1 + \dots + \alpha_r \leq n$ , (3.2.15) is a finite sum. Therefore, if we take

$$s = \max_{\alpha: c_\alpha \neq 0} \|\alpha\|_{\ell^\infty},$$

we have that  $F_r$  is well defined on  $X = \mathcal{FL}_0^{s,p}(\mathbb{T})$  for  $p \in \{1, 2\}$ , using Corollary A.1.3 or Theorem A.1.4. A straightforward computation yields

$$\begin{aligned} \{F_r, G_3\} &= i \frac{\pi}{9} \sum_{\substack{k_1+\dots+k_r=0 \\ k'_1+k'_2+k'_3=0}} \frac{f_r(k_1, \dots, k_r)}{k'_1 k'_2 k'_3} \{u_{k_1} \dots u_{k_r}, u_{k'_1} u_{k'_2} u_{k'_3}\} \\ &= i \frac{\pi}{9} \sum_{\substack{i=1, \dots, r \\ j=1, 2, 3}} \sum_{\substack{k_1+\dots+k_r-k_i=-k \\ k'_1+k'_2+k'_3-k'_j=k}} \frac{i}{2\pi} \frac{f_r(k_1, \dots, k_{i-1}, k, k_{i+1}, \dots, k_r)}{\frac{k'_1 k'_2 k'_3}{-k'_j}} \frac{u_{k_1} \dots u_{k_r} u_{k'_1} u_{k'_2} u_{k'_3}}{u_{k_i} u_{k'_j}}. \end{aligned}$$

Notice that we obtained the sum of  $3r$  formal polynomials of degree  $r+3-2$ . Moreover, if in  $f_r(k_1, \dots, k_{i-1}, k, k_{i+1}, \dots, k_r)$ , we substitute  $k'_1 + k'_2 + k'_3 - k'_j = k$ , we obtain that

$$|f_r(k_1, \dots, k_{i-1}, k, k_{i+1}, \dots, k_r)| \leq \sum_{|\beta| \leq n} c_\beta |\mathbf{k}|^\beta$$

where  $\beta = (\beta_1, \dots, \beta_{r+1})$  is again a multi-index with non-negative components and  $\mathbf{k}$  is the ordered vector whose components are taken from

$$\{k_1, \dots, k_r, k'_1, k'_2, k'_3\} \setminus \{k_i, k'_j\}.$$

But clearly  $\max_{\beta: c_\beta \neq 0} \|\beta\|_{\ell^\infty} \leq s$ . This means that  $\{F_r, G_3\}$  is well defined on the same space of  $F_r$ .  $\square$

**Remark 3.2.5.** Recall what we have obtained after the first normal form transformation. The original Hamiltonian was  $\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_3$ . Then, if we truncate at any order  $n \in \mathbb{N}$  the Taylor series, by Lemma 3.1.12 we have

$$\begin{aligned} \mathcal{H} \circ \Phi_{G_3}(\mathbf{v}) &= \mathcal{H}_2 + \{\mathcal{H}_2, G_3\} + \frac{1}{2} \{\{\mathcal{H}_2, G_3\}, G_3\} + \dots + \frac{1}{(n-2)!} \{\{\{\mathcal{H}_2, G_3\}, G_3\}, \dots, G_3\}^{(n-2)} \\ &\quad + \mathcal{H}_3 + \{\mathcal{H}_3, G_3\} + \frac{1}{2} \{\{\mathcal{H}_3, G_3\}, G_3\} + \dots + \frac{1}{(n-3)!} \{\{\{\mathcal{H}_3, G_3\}, G_3\}, \dots, G_3\}^{(n-3)} \\ &\quad + \int_0^1 \left[ \frac{(1-\xi)^{n-2}}{(n-2)!} \{\{\{\mathcal{H}_2, G_3\}, G_3\}, \dots, G_3\}^{(n-1)} \circ \Phi_{G_3}^\xi \right. \\ &\quad \left. + \frac{(1-\xi)^{n-3}}{(n-3)!} \{\{\{\mathcal{H}_3, G_3\}, G_3\}, \dots, G_3\}^{(n-2)} \circ \Phi_{G_3}^\xi \right] d\xi \\ &= \mathcal{H}_2 + \mathcal{H}_4 + \mathcal{H}_5 + \dots + \mathcal{H}_n + R_{n+1}, \end{aligned} \tag{3.2.16}$$

where

$$\mathcal{H}_j = \left( \frac{1}{(j-3)!} - \frac{1}{(j-2)!} \right) \{\{\{\mathcal{H}_3, G_3\}, G_3\}, \dots\}^{(j-3)} \quad j = 4, \dots, n$$

and, by Theorem 3.2.4 and (3.2.9),

$$|R_{n+1}| \leq C_{\mathcal{H}_3, G_3, n} \sup_{\xi \in [0, 1]} \|u(\xi)\|_X^{n+1} \lesssim \|v\|_X^{n+1}.$$

where  $X = \mathcal{FL}_0^{s,p}(\mathbb{T})$  with  $p \in \{1, 2\}$  and  $s \geq 1$ , which is required by lemma 3.1.2. Recall that our convention is to call  $u_k$  the old variables and  $v_k$  the new variables, but we will write the new Hamiltonian again in the  $u_k$ 's since we are doing several transformations.

### 3.3 Second normal form transformation

Recall that we started from the KdV Hamiltonian  $\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_3$  defined in (3.1.3) and we performed the canonical transformation  $\Phi_{G_3}$  generated by the auxiliary Hamiltonian  $G_3$  defined in (3.2.7). In Theorem 3.2.3 we proved that  $\Phi_{G_3}$  is a diffeomorphism in a neighborhood of the origin of  $H_0^s(\mathbb{T})$  or  $\mathcal{FL}_0^{s,1}(\mathbb{T})$ , with  $s \geq 1$ . Moreover, together with its inverse  $\Phi_{G_3}^{-1}$ , it is close to the identity, as given by (3.2.10) and (3.2.11). By Remark 3.2.5 we have that all the new Hamiltonian functions up to a fixed order  $n \in \mathbb{N}$  are well defined in  $\mathcal{FL}_0^{s,p}(\mathbb{T})$  for  $p \in \{1, 2\}$  and  $s \geq 1$ . To iterate the procedure, we have to compute explicitly  $\mathcal{H}_4$  with the formula given in Remark 3.2.5. Remember that

$$\mathcal{H}_4 = \frac{1}{2}\{\{\mathcal{H}_2, G_3\}, G_3\} + \{\mathcal{H}_3, G_3\} = \frac{1}{2}\{\mathcal{H}_3, G_3\}.$$

**Lemma 3.3.1.** *If  $u \in \mathcal{FL}_0^{s,p}(\mathbb{T})$  for  $p \in \{1, 2\}$  and  $s \geq 1$ , then the term  $\mathcal{H}_4$  in (3.2.16) admits the following explicit expression:*

$$\mathcal{H}_4 = -\frac{\pi}{12} \sum_{\substack{k_1+k_2+k_3+k_4=0 \\ k_2+k_3 \neq 0}} \frac{u_{k_1} u_{k_2} u_{k_3} u_{k_4}}{k_2 k_3}. \quad (3.3.1)$$

Remember that we use  $\mathbf{v}$  to denote the new variables and  $\mathbf{u}$  to denote the original ones. In this section  $\mathcal{H} \circ \Phi_{G_3}$  will be the "old" Hamiltonian, therefore we shall use the variables  $\mathbf{u}$ .

*Proof.*

$$\begin{aligned} \mathcal{H}_4 &= \frac{1}{2} \left\{ -\frac{\pi}{3} \sum_{k_1+k_2+k_3=0} u_{k_1} u_{k_2} u_{k_3}, i \frac{\pi}{9} \sum_{k'_1+k'_2+k'_3=0} \frac{u_{k'_1} u_{k'_2} u_{k'_3}}{k_1 k_2 k_3} \right\} \\ &= -\frac{i\pi^2}{54} \sum_{\substack{k_1+k_2+k_3=0 \\ k'_1+k'_2+k'_3=0}} \frac{1}{k'_1 k'_2 k'_3} \{u_{k_1} u_{k_2} u_{k_3}, u_{k'_1} u_{k'_2} u_{k'_3}\}. \end{aligned}$$

But

$$\begin{aligned} \{u_{k_1} u_{k_2} u_{k_3}, u_{k'_1} u_{k'_2} u_{k'_3}\} &= \sum_{k \neq 0} \frac{ik}{2\pi} \frac{\partial(u_{k_1} u_{k_2} u_{k_3})}{\partial u_k} \frac{\partial(u_{k'_1} u_{k'_2} u_{k'_3})}{\partial u_k} \\ &= \sum_{k \neq 0} \frac{ik}{2\pi} [\delta(k_1 = k) u_{k_2} u_{k_3} + \delta(k_2 = k) u_{k_1} u_{k_3} + \delta(k_3 = k) u_{k_1} u_{k_2}] \\ &\quad \cdot [\delta(k'_1 = -k) u_{k'_2} u_{k'_3} + \delta(k'_2 = -k) u_{k'_1} u_{k'_3} + \delta(k'_3 = -k) u_{k'_1} u_{k'_2}] \\ &= \frac{i}{2\pi} \left[ k_1 \delta(k_1 = -k'_1) u_{k_2} u_{k_3} u_{k'_2} u_{k'_3} + k_1 \delta(k_1 = -k'_2) u_{k_2} u_{k_3} u_{k'_1} u_{k'_3} \right. \\ &\quad + k_1 \delta(k_1 = -k'_3) u_{k_2} u_{k_3} u_{k'_1} u_{k'_2} + k_2 \delta(k_2 = -k'_1) u_{k_1} u_{k_3} u_{k'_2} u_{k'_3} \\ &\quad + k_2 \delta(k_2 = -k'_2) u_{k_1} u_{k_3} u_{k'_1} u_{k'_3} + k_2 \delta(k_2 = -k'_3) u_{k_1} u_{k_3} u_{k'_1} u_{k'_2} \\ &\quad + k_3 \delta(k_3 = -k'_1) u_{k_1} u_{k_2} u_{k'_2} u_{k'_3} + k_3 \delta(k_3 = -k'_2) u_{k_1} u_{k_2} u_{k'_1} u_{k'_3} \\ &\quad \left. + k_3 \delta(k_3 = -k'_3) u_{k_1} u_{k_2} u_{k'_1} u_{k'_2} \right]. \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{H}_4 = & \frac{\pi}{108} \left[ \sum_{\substack{k_1+k_2+k_3=0 \\ -k_1+k'_2+k'_3=0}} \frac{k_1 u_{k_2} u_{k_3} u_{k'_2} u_{k'_3}}{-k_1 k'_2 k'_3} + \sum_{\substack{k_1+k_2+k_3=0 \\ k'_1-k_1+k'_3=0}} \frac{k_1 u_{k_2} u_{k_3} u_{k'_1} u_{k'_3}}{-k'_1 k_1 k'_3} + \sum_{\substack{k_1+k_2+k_3=0 \\ k'_1+k'_2-k_1=0}} \frac{k_1 u_{k_2} u_{k_3} u_{k'_1} u_{k'_2}}{-k'_1 k'_2 k_1} \right. \\ & + \sum_{\substack{k_1+k_2+k_3=0 \\ -k_2+k'_2+k'_3=0}} \frac{k_2 u_{k_1} u_{k_3} u_{k'_2} u_{k'_3}}{-k_2 k'_2 k'_3} + \sum_{\substack{k_1+k_2+k_3=0 \\ k'_1-k_2+k'_3=0}} \frac{k_2 u_{k_1} u_{k_3} u_{k'_1} u_{k'_3}}{-k'_1 k_2 k'_3} + \sum_{\substack{k_1+k_2+k_3=0 \\ k'_1+k'_2-k_2=0}} \frac{k_2 u_{k_1} u_{k_3} u_{k'_1} u_{k'_2}}{-k'_1 k'_2 k_2} \\ & \left. + \sum_{\substack{k_1+k_2+k_3=0 \\ -k_3+k'_2+k'_3=0}} \frac{k_3 u_{k_1} u_{k_2} u_{k'_2} u_{k'_3}}{-k_3 k'_2 k'_3} + \sum_{\substack{k_1+k_2+k_3=0 \\ k'_1-k_3+k'_3=0}} \frac{k_3 u_{k_1} u_{k_3} u_{k'_1} u_{k'_2}}{-k'_1 k_3 k'_3} + \sum_{\substack{k_1+k_2+k_3=0 \\ k'_1+k'_2-k_3=0}} \frac{k_3 u_{k_1} u_{k_2} u_{k'_1} u_{k'_2}}{-k'_1 k'_2 k_3} \right]. \end{aligned}$$

These nine sums are all equal, hence

$$\mathcal{H}_4 = -\frac{\pi}{12} \sum_{\substack{k_1+k_2+k_3+k_4=0 \\ k_2+k_3 \neq 0}} \frac{u_{k_1} u_{k_2} u_{k_3} u_{k_4}}{k_2 k_3}.$$

□

We carry out another normal form transformation in order to cancel as many terms as possible from  $\mathcal{H}_4$ . Given that  $\mathcal{H}_4$  is a polynomial of degree 4, we need a Hamiltonian generated by a polynomial of degree 4, which we denote by  $G_4$ . By Lemma 3.1.12 the new Hamiltonian will be

$$\mathcal{H} \circ \Phi_{G_3} \circ \Phi_{G_4} = \mathcal{H}_2 + \{\mathcal{H}_2, G_4\} + \mathcal{H}_4 + R_5$$

where  $R_5$  has a zero of order  $\geq 5$  at the origin. We will see that in general the homological equation cannot be

$$\{\mathcal{H}_2, G_4\} + \mathcal{H}_4 = 0$$

as it was in the previous case. The general requirement that we can do is

$$\{\mathcal{H}_2, G_4\} + \mathcal{H}_4 = \hat{\mathcal{H}}_4 \tag{3.3.2}$$

where  $\hat{\mathcal{H}}_4 \in \text{Ker}(\{\mathcal{H}_2, \cdot\})$ . It will be clear that the monomials that we can't remove are exactly the ones which belong to  $\text{Ker}(\{\mathcal{H}_2, \cdot\})$ . Hence, to build a general strategy, we shall understand which monomials belong to that space.

**Theorem 3.3.2.** *If we choose*

$$G_4 = -i \frac{\pi}{36} \sum_{k_1+k_2+k_3+k_4=0}^* \frac{u_{k_1} u_{k_2} u_{k_3} u_{k_4}}{k_2 k_3 (k_1 + k_2) (k_1 + k_3) (k_2 + k_3)} \tag{3.3.3}$$

then we have

$$\{\mathcal{H}_2, G_4\} + \mathcal{H}_4 = \hat{\mathcal{H}}_4$$

with

$$\hat{\mathcal{H}}_4 = -\frac{\pi}{12} \sum_{k \neq 0} \frac{|u_k|^4}{k^2}. \tag{3.3.4}$$

*Proof.* Following the same strategy of Theorem 3.2.1, we compute

$$\begin{aligned} \{\mathcal{H}_2, u_{k_1} u_{k_2} u_{k_3} u_{k_4}\} &= \sum_{k \neq 0} i k^3 \bar{u}_k \frac{\partial}{\partial u_k} (u_{k_1} u_{k_2} u_{k_3} u_{k_4}) = \sum_{k \neq 0} i k^3 \bar{u}_k [\delta(k_1 = -k) u_{k_2} u_{k_3} u_{k_4} \\ &\quad + \delta(k_2 = -k) u_{k_1} u_{k_3} u_{k_4} + \delta(k_3 = -k) u_{k_1} u_{k_2} u_{k_4} + \delta(k_4 = -k) u_{k_1} u_{k_2} u_{k_3}] \\ &= -i(k_1^3 + k_2^3 + k_3^3 + k_4^3) u_{k_1} u_{k_2} u_{k_3} u_{k_4}. \end{aligned}$$

If we now impose that  $k_1 + k_2 + k_3 + k_4 = 0$ , then we have that

$$k_1^3 + k_2^3 + k_3^3 + k_4^3 = -3(k_1 + k_2)(k_1 + k_3)(k_2 + k_3).$$

Remember that in  $\mathcal{H}_4$  we have that  $k_2 + k_3 \neq 0$ , therefore this polynomial vanishes only on the set  $R_{-k_4}$  that we defined in (2.1.15). If we set

$$G_4 = \sum_{\substack{k_1+k_2+k_3+k_4=0 \\ k_2+k_3 \neq 0}} g(k_1, k_2, k_3, k_4) u_{k_1} u_{k_2} u_{k_3} u_{k_4},$$

then

$$\begin{aligned} &\{\mathcal{H}_2, G_4\} + \mathcal{H}_4 \\ &= \sum_{\substack{k_1+k_2+k_3+k_4=0 \\ k_2+k_3 \neq 0}} \left[ g(k_1, k_2, k_3, k_4) 3i(k_1 + k_2)(k_1 + k_3)(k_2 + k_3) - \frac{\pi}{12k_2k_3} \right] u_{k_1} u_{k_2} u_{k_3} u_{k_4}. \end{aligned}$$

We can impose

$$g(k_1, k_2, k_3, k_4) = \frac{\pi}{36ik_2k_3(k_1 + k_2)(k_1 + k_3)(k_2 + k_3)}$$

whenever  $(k_1, k_2, k_3) \notin R_{-k_4}$ , and so

$$G_4 = -i \frac{\pi}{36} \sum_{k_1+k_2+k_3=k_4}^* \frac{u_{k_1} u_{k_2} u_{k_3} u_{k_4}}{k_2 k_3 (k_1 + k_2)(k_1 + k_3)(k_2 + k_3)}.$$

In this way we obtain

$$\begin{aligned} \{\mathcal{H}_2, G_4\} + \mathcal{H}_4 &= \hat{\mathcal{H}}_4 = -\frac{\pi}{12} \sum_{k_4 \neq 0} \sum_{(k_1, k_2, k_3) \in R_{-k_4}} \frac{u_{k_1} u_{k_2} u_{k_3} \bar{u}_{k_4}}{k_2 k_3} \\ &= -\frac{\pi}{12} \left( \sum_{k \neq 0} \frac{u_{-k} u_k u_k \bar{u}_k}{k^2} - \sum_{\substack{k \neq 0 \\ j \notin \{0, \pm k\}}} \frac{u_j u_{-j} u_k \bar{u}_k}{jk} - \sum_{\substack{k \neq 0 \\ j \notin \{0, \pm k\}}} \frac{u_j u_k u_{-j} \bar{u}_k}{jk} \right) \\ &= -\frac{\pi}{12} \sum_{k \neq 0} \frac{|u_k|^4}{k^2}. \end{aligned}$$

□

**Remark 3.3.3.** We can check that  $\hat{\mathcal{H}}_4 \in \text{Ker}\{\mathcal{H}_2, \cdot\}$ . Indeed

$$\{\mathcal{H}_2, \hat{\mathcal{H}}_4\} = -\frac{i\pi}{24} \sum_{k \neq 0} k \frac{\partial}{\partial u_k} \left( \sum_{j \neq 0} j^2 u_j \bar{u}_j \right) \frac{\partial}{\partial \bar{u}_k} \left( \sum_{l \neq 0} \frac{1}{l^2} u_l^2 \bar{u}_l^2 \right) = -\frac{i\pi}{3} \sum_{k \neq 0} k |u_k|^2 = 0.$$

**Remark 3.3.4.** Notice that, if we study the dynamics relative to  $\hat{\mathcal{H}}_4$ , we have

$$\dot{u}_k = \{u_k, \hat{\mathcal{H}}_4\} = -\frac{\pi}{12} \cdot \frac{ik}{2\pi} \sum_{j \neq 0} \frac{\partial}{\partial \bar{u}_k} \frac{|u_j|^2}{j^2} = -\frac{i}{6k} |u_k|^2 u_k = \rho_k,$$

where  $\rho_k$  is the same defined in (2.1.10).

**Theorem 3.3.5** ( $\Phi_{G_4}$  is a local diffeomorphism). *Let  $X = H_0^s(\mathbb{T})$  or  $X = \mathcal{FL}_0^{s,1}(\mathbb{T})$  with  $s \geq 1$ . Let  $v \in X$ . Consider the Cauchy problem in the Fourier setting*

$$\begin{cases} \frac{d}{d\xi} u_k = \{u_k, G_4(\mathbf{u})\}(\xi), \\ u_k(0) = v_k. \end{cases} \quad (3.3.5)$$

Where  $G_4$  was defined in (3.3.3). There exists  $\varepsilon_0 > 0$  such that for all  $0 < \varepsilon \leq \varepsilon_0$  and for all  $v \in B_X(\varepsilon)$  the problem (3.3.5) has a unique solution  $u \in C_\xi^0([0, \Xi] \times \mathbb{T})$ , with  $\Xi > 1$  and

$$\sup_{\xi \in [0, \Xi]} \|u(\xi)\|_X \leq 2 \|v\|_X. \quad (3.3.6)$$

Moreover, if we set  $\Phi_{G_4}(v) = u = u(1)$ , we have that  $\Phi_{G_4} : B_X(\varepsilon) \rightarrow \Phi_{G_4}(B_X(\varepsilon))$  is a diffeomorphism, i.e. it is  $C^1$  and invertible, and its inverse  $\Phi_{G_4}^{-1}$  is  $C^1$ . Finally both  $\Phi_{G_4}$  and  $\Phi_{G_4}^{-1}$  are close to the identity, in the sense that

$$\|(I - \Phi_{G_4})(v)\|_X \leq C_{G_4} \|v\|_X^2, \quad (3.3.7)$$

$$\|(I - \Phi_{G_4}^{-1})(u)\|_X \leq C_{G_4} \|u\|_X^2. \quad (3.3.8)$$

*Proof.* A straightforward computation yields

$$\begin{aligned} \{u_k, G_4\} &= -i \frac{\pi}{36} \frac{ik}{2\pi} \sum_{k_1+k_2+k_3+k_4=0}^* \frac{\frac{\partial}{\partial \bar{u}_k} (u_{k_1} u_{k_2} u_{k_3} u_{k_4})}{k_2 k_3 (k_1 + k_2)(k_1 + k_3)(k_2 + k_3)} \\ &= \frac{k}{72} \sum_{k_1+k_2+k_3=k}^* \left( \frac{-2u_{k_1} u_{k_2} u_{k_3}}{kk_2(k_1 - k)(k_1 + k_3)(-k + k_3)} + \frac{2u_{k_1} u_{k_2} u_{k_3}}{k_1 k_2(-k + k_1)(-k + k_2)(k_1 + k_2)} \right) \\ &= \frac{k}{36} \sum_{k_1+k_2+k_3=k}^* \left( \frac{-u_{k_1} u_{k_2} u_{k_3}}{kk_2(k_1 + k_2)(k_1 + k_3)(k_2 + k_3)} + \frac{u_{k_1} u_{k_2} u_{k_3}}{k_1 k_2(k_1 + k_3)(k_2 + k_3)(k_1 + k_2)} \right) \\ &= \frac{1}{36} \sum_{\substack{k_1+k_2+k_3=k \\ k_2+k_3 \neq 0}}^* \frac{u_{k_1} u_{k_2} u_{k_3}}{k_1 k_2(k_1 + k_2)(k_1 + k_3)}. \end{aligned}$$

Then we can proceed as in Theorem 3.2.3.  $\square$

**Theorem 3.3.6.** *Consider a homogeneous formal polynomial of degree  $r \geq 2$ ,*

$$F_r = \sum_{k_1+\dots+k_r=0}^* f_r(k_1, \dots, k_r) u_{k_1} \dots u_{k_r}$$

with  $f_r \in \mathbb{C}(k_1, \dots, k_r)$  which is well defined in some space  $X = \mathcal{FL}_0^{s,p}(\mathbb{T})$  for  $p \in \{1, 2\}$  and  $s > \frac{1}{2}$ . Then the Poisson bracket  $\{F_r, G_4\}$ , where  $G_4$  was defined in (3.3.3), is a homogeneous formal polynomial of degree  $r+1$  which is well defined on the same space  $X$ .

*Proof.* A straightforward computation yields

$$\begin{aligned}
\{F_r, G_4\} &= -i \frac{\pi}{36} \sum_{\substack{k_1+\dots+k_r=0 \\ k'_1+k'_2+k'_3+k'_4=0}}^* \frac{f_r(k_1, \dots, k_r)}{k'_2 k'_3 (k'_1 + k'_2)(k'_1 + k'_3)(k'_2 + k'_3)} \{u_{k_1} \dots u_{k_r}, u_{k'_1} u_{k'_2} u_{k'_3} u_{k'_4}\} \\
&= -i \frac{\pi}{36} \sum_{i=1, \dots, r} \sum_{\substack{k_1+\dots+k_r-k_i=-k \\ j=1, 2, 3, 4 \\ k'_1+k'_2+k'_3+k'_4=k'}} \frac{i k}{2\pi} \frac{f_r(k_1, \dots, k_{i-1}, k, k_{i+1}, \dots, k_r)}{k'_2 k'_3 (k'_1 + k'_2)(k'_1 + k'_3)(k'_2 + k'_3)} \frac{u_{k_1} \dots u_{k_r} u_{k'_1} u_{k'_2} u_{k'_3} u_{k'_4}}{u_{k_i} u_{k'_j}} \\
&= \frac{1}{36} \sum_{i=1, \dots, r} \sum_{\substack{k_1+\dots+k_r-k_i=-k \\ k'_1+k'_2+k'_3=k}} \frac{f_r(k_1, \dots, k_{i-1}, k, k_{i+1}, \dots, k_r)}{k'_1 k'_2 (k'_1 + k'_2)(k'_1 + k'_3)} \frac{u_{k_1} \dots u_{k_r} u_{k'_1} u_{k'_2} u_{k'_3}}{u_{k_i}}.
\end{aligned}$$

Then we can proceed as in Theorem 3.2.4.  $\square$

**Remark 3.3.7.** For the same reasons as in Remark 3.2.5, by Theorem 3.2.4 and Theorem 3.3.6, if we consider  $v \in X$  where  $X = \mathcal{FL}_0^{s,p}(\mathbb{T})$  with  $s \geq 1$  and  $p \in \{1, 2\}$ , we can write the new Hamiltonian as

$$\begin{aligned}
\mathcal{H} \circ \Phi_{G_3} \circ \Phi_{G_4}(\mathbf{v}) &= \mathcal{H}_2 \\
&\quad + \mathcal{H}_4 + \{\mathcal{H}_2, G_4\} \\
&\quad + \mathcal{H}_5 \\
&\quad + \mathcal{H}_6 + \{\mathcal{H}_4, G_4\} + \{\{\mathcal{H}_2, G_4\}, G_4\} \\
&\quad + \mathcal{H}_7 + \{\mathcal{H}_5, G_4\} \\
&\quad \vdots \\
&\quad + \mathcal{H}_n + \{\mathcal{H}_{n-2}, G_4\} + \{\{\mathcal{H}_{n-4}, G_4\}, G_4\} + \dots \\
&\quad + \tilde{R}_{n+1}
\end{aligned}$$

where

$$|\tilde{R}_{n+1}| \lesssim \|v\|_X^{n+1},$$

the  $\mathcal{H}_j$ 's are defined in Remark 3.2.5,  $G_3$  and  $G_4$  were defined respectively in (3.2.7) and (3.3.3). Using the same letters with slight abuse of notation, and renaming  $\mathbf{u}$  the new variables, we have

$$\mathcal{H} \circ \Phi_{G_3} \circ \Phi_{G_4}(\mathbf{u}) = \mathcal{H}_2 + \hat{\mathcal{H}}_4 + \mathcal{H}_5 + \dots + \mathcal{H}_n + R_{n+1}. \quad (3.3.9)$$

We started from the KdV Hamiltonian  $\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_3$  defined in (3.1.3) and, after two transformations  $\Phi_{G_3} \circ \Phi_{G_4}$  generated by two auxiliary Hamiltonians  $G_3$  and  $G_4$  defined respectively in (3.2.7) and (3.3.3) we arrived to the new Hamiltonian (3.3.9). The procedure was simple: using Lemma 3.1.12 we looked for transformations which removed as many monomials as possible from order 3 and 4. The only monomials which we couldn't remove are the ones in  $\text{Ker}\{\mathcal{H}_2, \cdot\}$ . At order 3 we were able to remove everything, while at order 4 the term  $\hat{\mathcal{H}}_4$  remained, as defined in (3.3.4). This function is said to be in *normal form*, according to Definition 3.3.8 which we give below. Our aim now is to iterate this procedure in order to put the KdV Hamiltonian in normal form up to a given order  $n \geq 5$ . This will be more difficult: indeed, as we shall explain soon, even the construction of new transformations is difficult starting from order 5. Moreover, proving that these transformations are diffeomorphisms close to the identity in a neighborhood of the origin of some space  $X$  is much trickier than for  $\Phi_{G_3}$  and  $\Phi_{G_4}$ , as we saw in Theorem 3.2.3 and Theorem 3.3.5.

**Definition 3.3.8.** A function  $F = F(\mathbf{u})$  is said to be in *normal form* if  $F = F(\mathbf{I})$ , where  $\mathbf{I} = (I_k)_k = (|u_k|^2)_k$ . We will often write  $\hat{F}$  meaning that  $F$  is in normal form.

### 3.4 Normal form up to order $n \geq 5$ - Formal theory

In order to come up with a general algorithm which puts the KdV Hamiltonian in normal form up to order  $n$  for any  $n \geq 5$ , we have to generalize this procedure. We will see that starting from order 5 the problem changes. This is due to the fact that the *resonant relations* become very difficult to study from this order. Remember that, when we were dealing with order 3, resonant terms where the ones for which the three indexes  $(k_1, k_2, k_3)$  satisfied

$$\begin{cases} k_1 + k_2 + k_3 = 0 \\ k_1^3 + k_2^3 + k_3^3 = 0 \\ k_1, k_2, k_3 \neq 0. \end{cases} \quad (3.4.1)$$

It was easy to see that this system has no solutions. Then, when dealing with order 4, the system was

$$\begin{cases} k_1 + k_2 + k_3 + k_4 = 0 \\ k_1^3 + k_2^3 + k_3^3 + k_4^3 = 0 \\ k_1, k_2, k_3, k_4 \neq 0 \\ k_2 + k_3 \neq 0, \end{cases} \quad (3.4.2)$$

and again it was easy to solve this system. The key observation is that, when we substitute the first equation into the second, we obtain a polynomial which we are able to factor out. This will not be true for higher orders. The first thing to do is to understand which are the resonant relations for higher orders.

**Definition 3.4.1.** Given a Hamiltonian function

$$F_n = \sum_{k_1 + \dots + k_n = 0} f_n(k_1, \dots, k_n) u_{k_1} \dots u_{k_n}$$

we say that a monomial  $f_n(k_1, \dots, k_n) u_{k_1} \dots u_{k_n}$  is *resonant* if

$$f_n(k_1, \dots, k_n) u_{k_1} \dots u_{k_n} \in \text{Ker}\{\mathcal{H}_2, \cdot\}.$$

We call *resonant relation* the system

$$\begin{cases} k_1 + \dots + k_n = 0 \\ f_n(k_1, \dots, k_n) \neq 0 \\ f_n(k_1, \dots, k_n) u_{k_1} \dots u_{k_n} \in \text{Ker}\{\mathcal{H}_2, \cdot\}. \end{cases}$$

These are exactly the monomials which we are not able to remove using the same strategy of Section 3.2 and Section 3.3. In Theorem 3.4.2 we see that at each order the resonant relations share a common structure.

**Theorem 3.4.2.** All the resonant relations of the KdV equation are of the form

$$\begin{cases} k_1 + k_2 + \dots + k_j = 0 \\ k_1^3 + k_2^3 + \dots + k_j^3 = 0, \end{cases} \quad (3.4.3)$$

with some other inequalities in addition.

*Proof.* The first thing to notice is that the KdV Hamiltonian can be written as the sum of two homogeneous formal polynomials as in Definition 3.1.6, with  $N = 0$ . Indeed

$$\mathcal{H}(\mathbf{u}) = \mathcal{H}_2 + \mathcal{H}_3 = \pi \sum_{k_1+k_2=0} k_1^2 u_{k_1} u_{k_2} - \frac{\pi}{3} \sum_{k_1+k_2+k_3=0} u_{k_1} u_{k_2} u_{k_3}.$$

Also  $G_3$  and  $G_4$  are formal polynomials with  $N = 0$ . By Lemma 3.1.8, we know that the Poisson bracket preserves the structure of a formal polynomial, and that all the Hamiltonian functions involved in (3.3.9) are of the form

$$\mathcal{H}_j = \sum_{k_1+\dots+k_j=0} h_j(k_1, \dots, k_j) u_{k_1} \dots u_{k_j} \quad j = 5, 6, \dots, n.$$

Assume that we want to use  $\mathcal{H}_2$  to remove non-resonant monomials from  $\mathcal{H}_j$ . Then we have to solve the homological equation

$$\{\mathcal{H}_2, G_j\} + \mathcal{H}_j = \hat{\mathcal{H}}_j$$

with  $\hat{\mathcal{H}}_j \in \text{Ker}(\{\mathcal{H}_2, \cdot\})$ . For this reason, we look for

$$G_j = \sum_{k_1+\dots+k_j=0} g_j(k_1, \dots, k_j) u_{k_1} \dots u_{k_j},$$

so that

$$\begin{aligned} \{\mathcal{H}_2, G_j\} + \mathcal{H}_j &= i \sum_{k \neq 0} k^3 \bar{u}_k \frac{\partial}{\partial \bar{u}_k} \left( \sum_{k_1+\dots+k_j=0} g_j(k_1, \dots, k_j) u_{k_1} \dots u_{k_j} \right) \\ &\quad + \sum_{k_1+\dots+k_j=0} h_j(k_1, \dots, k_j) u_{k_1} \dots u_{k_j} \\ &= \sum_{k_1+\dots+k_j=0} [h_j(k_1, \dots, k_j) - i(k_1^3 + \dots + k_j^3) g_j(k_1, \dots, k_j)] u_{k_1} \dots u_{k_j} \end{aligned}$$

we can impose

$$g_j(k_1, \dots, k_j) = -i \frac{h_j(k_1, \dots, k_j)}{k_1^3 + \dots + k_j^3}$$

whenever

$$k_1^3 + \dots + k_j^3 \neq 0$$

and we obtain

$$\{\mathcal{H}_2, G_j\} + \mathcal{H}_j = \hat{\mathcal{H}}_j = \sum_{\substack{k_1+\dots+k_j=0 \\ k_1^3+\dots+k_j^3=0}} h_j(k_1, \dots, k_j) u_{k_1} \dots u_{k_j} \in \text{Ker}(\{\mathcal{H}_2, \cdot\}).$$

Notice that  $G_j$  is again a homogeneous formal polynomial with  $N = 0$ , so the transformation  $\Phi_{G_j}$  preserves the structure of all the Hamiltonians.  $\square$

**Remark 3.4.3.** The same resonant relations would be obtained by continuing the integration by parts procedure discussed in Chapter 2.

**Example 3.4.4.** Consider  $j = 5$ . We have

$$\begin{cases} k_1 + k_2 + k_3 + k_4 + k_5 = 0 \\ k_1^3 + k_2^3 + k_3^3 + k_4^3 + k_5^3 = 0 \end{cases}$$

If we substitute the first equation into the second we obtain

$$k_1^3 + k_2^3 + k_3^3 + k_4^3 - (k_1 + k_2 + k_3 + k_4)^3 = 0.$$

To understand the structure of the resonant terms, we should find the integer solutions of this equation. This is a tricky number theory problem, for which the solution is not known up to the knowledge of the author. For  $n \geq 5$ , using only (3.4.3), we are not able to prove that the resonant terms are in normal form, according to Definition 3.3.8. It would be better if we had several equations of this type, as explained in Theorem 3.4.5.

**Theorem 3.4.5.** Consider the system of equations

$$\begin{cases} \alpha_1 k_1 + \alpha_2 k_2 + \dots + \alpha_n k_n = 0 \\ \alpha_1 k_1^3 + \alpha_2 k_2^3 + \dots + \alpha_n k_n^3 = 0 \\ \vdots \\ \alpha_1 k_1^{2n-1} + \alpha_2 k_2^{2n-1} + \dots + \alpha_n k_n^{2n-1} = 0, \end{cases} \quad (3.4.4)$$

where  $\alpha_j \in \mathbb{N}$ . We require that  $k_j \neq 0$  for all  $j$ . Then:

- if  $S = \alpha_1 + \alpha_2 + \dots + \alpha_n$  is odd, there is no integer solution to (3.4.4).
- If  $S$  is even, all the solutions  $(k_1, k_2, \dots, k_n)$  are weightedly paired, meaning that if we build the vector

$$(k'_1, \dots, k'_S) = (k_{1,1}, \dots, k_{1,\alpha_1}, k_{2,1}, \dots, k_{2,\alpha_2}, \dots, k_{n,1}, \dots, k_{n,\alpha_n}) \in \mathbb{Z}^S,$$

where  $k_{j,1} = k_{j,2} = \dots = k_{j,\alpha_j} = k_j$ , there exists a permutation  $\sigma \in \text{Sym}(S)$  such that

$$k'_{\sigma(2i-1)} + k'_{\sigma(2i)} = 0 \quad \text{for } i = 1, \dots, \frac{S}{2}.$$

*Proof.* The first thing to notice is that we can write (3.4.4) as

$$\begin{bmatrix} k_1 & k_2 & \dots & k_n \\ k_1^3 & k_2^3 & \dots & k_n^3 \\ \vdots & \vdots & \ddots & \vdots \\ k_1^{2n-1} & k_2^{2n-1} & \dots & k_n^{2n-1} \end{bmatrix} \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

from which we have that

$$\begin{aligned}
 \det \begin{bmatrix} k_1 & k_2 & \dots & k_n \\ k_1^3 & k_2^3 & \dots & k_n^3 \\ \vdots & \vdots & \ddots & \vdots \\ k_1^{2n-1} & k_2^{2n-1} & \dots & k_n^{2n-1} \end{bmatrix} &= k_1 k_2 \dots k_n \cdot \det \begin{bmatrix} 1 & 1 & \dots & 1 \\ k_1^2 & k_2^2 & \dots & k_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ k_1^{2n-2} & k_2^{2n-2} & \dots & k_n^{2n-2} \end{bmatrix} \\
 &= k_1 k_2 \dots k_n \prod_{1 \leq i < j \leq n} (k_i^2 - k_j^2) \\
 &= k_1 k_2 \dots k_n \prod_{1 \leq i < j \leq n} (k_i - k_j)(k_i + k_j) = 0.
 \end{aligned} \tag{3.4.5}$$

This implies that there exist  $i, j$  such that  $k_i + k_j = 0$  or  $k_i - k_j = 0$ . In the former case we say that  $(i, j)$  is a *good pair*, in the latter one we say that it is a *bad pair*. We will prove this theorem by induction, in particular we will prove that if the theorem is true for  $n - 2$  and  $n - 1$ , then it is true for  $n$ . We need two base cases.

**Base case 1:**

$$\begin{cases} \alpha_1 k_1 + \alpha_2 k_2 = 0 \\ \alpha_1 k_1^3 + \alpha_2 k_2^3 = 0, \end{cases} \tag{3.4.6}$$

with  $\alpha_1, \alpha_2 > 0$ . This implies  $k_1^2 - k_2^2 = 0$  but we cannot have  $k_1 = k_2$  otherwise  $\alpha_1 + \alpha_2 = 0$  which is impossible. Therefore  $k_1 = -k_2 \neq 0$ . Now we distinguish on the two cases presented in the statement. If  $S$  is odd, clearly we have no solutions. If  $S$  is even, we have a solution only if  $\alpha_1 = \alpha_2$ , which means that the solutions are weightedly paired.

**Base case 2:**

$$\begin{cases} \alpha_1 k_1 + \alpha_2 k_2 + \alpha_3 k_3 = 0 \\ \alpha_1 k_1^3 + \alpha_2 k_2^3 + \alpha_3 k_3^3 = 0 \\ \alpha_1 k_1^5 + \alpha_2 k_2^5 + \alpha_3 k_3^5 = 0, \end{cases} \tag{3.4.7}$$

with  $\alpha_1, \alpha_2, \alpha_3 > 0$ . Here there are three possibilities:

- If  $k_i - k_j = 0$ , up to a permutation of indeces we can assume  $k_2 = k_3$  and we arrive to the base case 1

$$\begin{cases} \alpha'_1 k_1 + \alpha'_2 k_2 = 0 \\ \alpha'_1 k_1^3 + \alpha'_2 k_2^3 = 0, \end{cases}$$

with  $\alpha'_2 = \alpha_2 + \alpha_3$  and  $\alpha'_1 = \alpha_1$ . But we know that this has a solution only if  $\alpha'_1 = \alpha'_2$  and the solution is  $k_1 = -k_2$ . Then the solution of (3.4.7) is weightedly paired. Notice that in this case, if  $S$  were odd, there would be no solution since

$$S = \alpha_1 + \alpha_2 + \alpha_3 = \alpha'_1 + \alpha'_2 = S'$$

and the only case with solutions is when  $S = \alpha'_1 + \alpha'_2 = 2\alpha'_1$  which is even.

- If  $k_i + k_j = 0$  with  $\alpha_i - \alpha_j \neq 0$ , up to a permutation of indeces we arrive to the base case 1

$$\begin{cases} \alpha'_1 k_1 + \alpha'_2 k_2 = 0 \\ \alpha'_1 k_1^3 + \alpha'_2 k_2^3 = 0, \end{cases}$$

with  $\alpha'_2 = \alpha_2 - \alpha_3 > 0$  and  $\alpha'_1 = \alpha_1$ . But we know that this has a solution only if  $\alpha'_1 = \alpha'_2$  and the solution is  $k_1 = -k_2$ . Then the solution of (3.4.7) is weightedly paired. Notice that, setting  $S' = \alpha'_1 + \alpha'_2$ , we have  $S = S' + 2\alpha_3$  and so  $S - S' \equiv 0 \pmod{2}$ .

- If  $k_i + k_j = 0$  with  $\alpha_i - \alpha_j = 0$ , up to a permutation of indeces we arrive to  $k_1 = 0$  which is not possible.

Now we consider the general case (3.4.4). Again we have three possibilities.

- If  $k_i - k_j = 0$ , up to a permutation of indeces we can assume  $k_{n-1} = k_n$  and we arrive to

$$\begin{cases} \alpha'_1 k_1 + \alpha'_2 k_2 + \dots + \alpha'_{n-1} k_{n-1} &= 0 \\ \alpha'_1 k_1^3 + \alpha'_2 k_2^3 + \dots + \alpha'_{n-1} k_{n-1}^3 &= 0 \\ \vdots \\ \alpha'_1 k_1^{2n-3} + \alpha'_2 k_2^{2n-3} + \dots + \alpha'_{n-1} k_{n-1}^{2n-3} &= 0, \end{cases}$$

with  $\alpha'_{n-1} = \alpha_{n-1} + \alpha_n$ . By the inductive hypothesis, since  $S' = S$ , if  $S$  is even this system has weightedly paired solutions, therefore the same holds for (3.4.4), whereas if  $S$  is odd we have no solutions to (3.4.4).

- If  $k_i + k_j = 0$  with  $\alpha_i - \alpha_j \neq 0$ , up to a permutation of indeces we arrive to

$$\begin{cases} \alpha'_1 k_1 + \alpha'_2 k_2 + \dots + \alpha'_{n-1} k_{n-1} &= 0 \\ \alpha'_1 k_1^3 + \alpha'_2 k_2^3 + \dots + \alpha'_{n-1} k_{n-1}^3 &= 0 \\ \vdots \\ \alpha'_1 k_1^{2n-3} + \alpha'_2 k_2^{2n-3} + \dots + \alpha'_{n-1} k_{n-1}^{2n-3} &= 0, \end{cases}$$

with  $\alpha'_{n-1} = \alpha_{n-1} + \alpha_n$ . Notice that the  $k_j$  we have removed is paired yet. Again  $S - S' \equiv 0 \pmod{2}$ , hence by the inductive hypothesis this system has weightedly paired solutions if  $S$  is even (and the same holds for (3.4.4)), and no solutions if  $S$  is odd.

- If  $k_i + k_j = 0$  with  $\alpha_i - \alpha_j = 0$ , up to a permutation of indeces we arrive to

$$\begin{cases} \alpha'_1 k_1 + \alpha'_2 k_2 + \dots + \alpha'_{n-2} k_{n-2} &= 0 \\ \alpha'_1 k_1^3 + \alpha'_2 k_2^3 + \dots + \alpha'_{n-2} k_{n-2}^3 &= 0 \\ \vdots \\ \alpha'_1 k_1^{2n-5} + \alpha'_2 k_2^{2n-5} + \dots + \alpha'_{n-2} k_{n-2}^{2n-5} &= 0. \end{cases}$$

Notice that  $k_i$  and  $k_j$  that we have removed are paired yet. Again  $S - S' \equiv 0 \pmod{2}$ , hence by the inductive hypothesis this system has weightedly paired solutions if  $S$  is even (and the same holds for (3.4.4)), and no solutions if  $S$  is odd.

□

**Corollary 3.4.6.** *Consider the system of equations*

$$\begin{cases} k_1 + k_2 + \dots + k_n &= 0 \\ k_1^3 + k_2^3 + \dots + k_n^3 &= 0 \\ \vdots \\ k_1^{2n-1} + k_2^{2n-1} + \dots + k_n^{2n-1} &= 0, \end{cases}$$

*with the additional assumption that  $k_j \neq 0$  for any  $j$ . Then, if  $n$  is odd there is no solution, whereas if  $n = 2m$  is even all the solutions  $(k_1, k_2, \dots, k_{2m})$  are "paired", meaning that there exists a permutation  $\sigma \in \text{Sym}(2m)$  such that  $k_{\sigma(2i-1)} + k_{\sigma(2i)} = 0$  for  $i = 1, \dots, m$ .*

**Remark 3.4.7.** The same result holds if we have a system of the form

$$\begin{cases} k_1 + k_2 + \dots + k_n &= 0 \\ k_1^{2m+1} + k_2^{2m+1} + \dots + k_n^{2m+1} &= 0 \\ k_1^{4m+1} + k_2^{4m+1} + \dots + k_n^{4m+1} &= 0 \\ &\vdots \\ k_1^{2m(n-1)+1} + k_2^{2m(n-1)+1} + \dots + k_n^{2m(n-1)} &= 0, \end{cases}$$

with  $m \geq 1$ . Indeed the proof is the same, with the only difference that the determinant of the associated matrix is

$$k_1 k_2 \dots k_n \prod_{1 \leq i < j \leq n} (k_i^{2m} - k_j^{2m}) = k_1 k_2 \dots k_n \prod_{1 \leq i < j \leq n} (k_i^2 - k_j^2) \sum_{r=0}^{m-1} k_i^{2r} k_j^{2(m-r-1)},$$

which has the same real solutions.

It will be clear soon why we would be satisfied if we were able to remove all the terms but the one presented in Corollary 3.4.6. Indeed the dynamics would be very easy to approximate for all the times. To obtain these new relations, we have to use the KdV hierarchy, *i.e.* all the conserved quantities of the KdV equation. The reader can find the derivation of these quantities and some comments on their structure in the Appendix A.2. We can write them as

$$F^{(j)} = F_2^{(j)} + F_3^{(j)} + \dots + F_{j+2}^{(j)}, \quad j \in \mathbb{N} \quad (3.4.8)$$

where the  $j$ 's numerate the first integrals and the subscript stands for the degree of the homogeneous formal polynomial. Moreover  $F^{(1)} = \mathcal{H}$ . The reader can find the first 10 conserved quantities in [28]. By Theorem A.2.8 we have that

$$F_2^{(j)} = \pi \sum_{k \neq 0} k^{2j} |u_k|^2. \quad (3.4.9)$$

and

$$F_l^{(j)} = \sum_{k_1 + \dots + k_l = 0} f_l^{(j)}(k_1, \dots, k_l) u_{k_1} \dots u_{k_l}, \quad l = 3, \dots, j+2. \quad (3.4.10)$$

Furthermore, since  $F^{(j)}$  is the integral of a polynomial in  $u$  and its derivatives with real coefficients,  $f_l^{(j)}(k_1, \dots, k_l) \in \mathbb{Q}(i)[k_1, \dots, k_l]$ . The very important property we shall use to apply Corollary 3.4.6 is that these quantities are in involution, *i.e.*

$$\{F^{(i)}, F^{(j)}\} = 0 \quad \forall i, j \in \mathbb{N}. \quad (3.4.11)$$

Moreover, since we are dealing with canonical transformations, (3.4.11) holds also after a transformation  $\Phi$ . Indeed

$$0 = \{F^{(i)}, F^{(j)}\} \circ \Phi = \{F^{(i)} \circ \Phi, F^{(j)} \circ \Phi\}. \quad (3.4.12)$$

The property (3.4.11) allows us to say something more on (3.4.10).

**Proposition 3.4.8.** *If we consider a first integral of the KdV equation  $F^{(j)}$  defined in (3.4.8), (3.4.9) and (3.4.10), then actually  $f_l^{(j)}(k_1, \dots, k_l) \in \mathbb{Q}[k_1, \dots, k_l]$ .*

*Proof.* We proceed by induction on the degree  $l$  of the homogeneous formal polynomial  $F_l^{(j)}$ , for a fixed  $j \geq 2$ . Clearly the statement is true for  $l = 2$ , by (3.4.9). Consider now  $2 < l \leq j + 2$ . By (3.4.11) with  $i = 1$ , evaluated at order  $l$  using Lemma 3.1.10, we obtain

$$\{\mathcal{H}_2, F_l^{(j)}\} + \{\mathcal{H}_3, F_{l-1}^{(j)}\} = 0$$

where  $F^{(1)} = \mathcal{H} = \mathcal{H}_2 + \mathcal{H}_3$  was defined in (3.1.3). But by the inductive hypothesis, the definition of the Poisson bracket (3.1.7) and Lemma 3.1.8, the Poisson bracket  $\{\mathcal{H}_3, F_{l-1}^{(j)}\}$  is a formal polynomial with purely imaginary coefficients (which are polynomials in the  $k_n$ 's). Since the coefficients of  $\mathcal{H}_2$  are real, the same must hold for the coefficients of  $F_l^{(j)}$ .  $\square$

Now we generalize the result presented in Theorem 3.4.2, observing that all the kernels  $\text{Ker}\{F_2^{(j)}, \cdot\}$ , for  $j \geq 1$ , have the same particular form.

**Lemma 3.4.9.** *Let  $G$  be the homogeneous formal polynomial:*

$$G = \sum_{k_1+...+k_n=0} g(k_1, \dots, k_n) u_{k_1} \dots u_{k_n},$$

with  $g \in \mathbb{C}(k_1, \dots, k_n)$ , then

$$\{F_2^{(j)}, G\} = -i \sum_{k_1+...+k_n=0} g(k_1, \dots, k_n) (k_1^{2j+1} + \dots + k_n^{2j+1}) u_{k_1} \dots u_{k_n}.$$

*Proof.*

$$\begin{aligned} \{F_2^{(j)}, G\} &= \sum_{k \neq 0} \frac{ik}{2\pi} 2\pi k^{2j} \overline{u_k} \sum_{k_1+...+k_n=0} g(k_1, \dots, k_n) \frac{\partial}{\partial \overline{u_k}} (u_{k_1} \dots u_{k_n}) \\ &= -i \sum_{k_1+...+k_n=0} g(k_1, \dots, k_n) (k_1^{2j+1} + \dots + k_n^{2j+1}) u_{k_1} \dots u_{k_n}. \end{aligned}$$

$\square$

**Corollary 3.4.10.** *If*

$$G = \sum_{k_1+...+k_n=0} g(k_1, \dots, k_n) u_{k_1} \dots u_{k_n} \in \text{Ker}\{F_2^{(j)}, \cdot\},$$

then

$$G = \sum_{\substack{k_1+...+k_n=0 \\ k_1^{2j+1} + \dots + k_n^{2j+1} = 0}} g(k_1, \dots, k_n) u_{k_1} \dots u_{k_n}.$$

Another thing that is worth noticing is that the transformation  $\Phi_{G_3}$  (resp.  $\Phi_{G_4}$ ) put the whole KdV hierarchy in normal form up to order 3 (resp. 4). We will prove this fact using the idea of [20, Theorem G.2].

**Lemma 3.4.11.** *Given a function of the KdV hierarchy*

$$F^{(j)} = F_2^{(j)} + F_3^{(j)} + F_4^{(j)} + \dots, \quad j \in \mathbb{N},$$

*then, if we apply the first transformation  $\Phi_{G_3}$  defined in Theorem 3.2.3, we obtain*

$$F^{(j)} \circ \Phi_{G_3} = F_2^{(j)} + \tilde{F}_4^{(j)} + \tilde{F}_5^{(j)} + \dots$$

*i.e.  $\tilde{F}_3^{(j)} = 0$ , where  $F^{(j)} \circ \Phi_{G_3}$  can be written using Lemma 3.1.12 as we did for  $\mathcal{H}$  in (3.2.16).*

*Proof.* By (3.4.12) we have that

$$0 = \{\mathcal{H} \circ \Phi_{G_3}, F^{(j)} \circ \Phi_{G_3}\} = \{\mathcal{H}, F^{(j)}\} \circ \Phi_{G_3}. \quad (3.4.13)$$

Recall that the Hamiltonian  $\mathcal{H}$  defined in (3.1.3) after the transformation  $\Phi_{G_3}$  became (3.2.16) with  $\hat{\mathcal{H}}_3 = 0$ . By Lemma 3.1.10 we have that the term of degree 3 in (3.4.13) vanishes, thus we obtain

$$\{\mathcal{H}_2, \tilde{F}_3^{(j)}\} = 0$$

which, combined with Corollary 3.4.10 yields

$$\tilde{F}_3^{(j)} = \sum_{\substack{k_1+k_2+k_3=0 \\ k_1^3+k_2^3+k_3^3=0}} f_3^{(j)}(k_1, k_2, k_3) u_{k_1} u_{k_2} u_{k_3} = 0.$$

Indeed

$$\begin{cases} k_1 + k_2 + k_3 = 0 \\ k_1^3 + k_2^3 + k_3^3 = 0 \end{cases} \implies k_j = 0 \quad \text{for some } j.$$

□

**Example 3.4.12.** To convince ourselves of Lemma 3.4.11, let us perform the transformation for

$$\begin{aligned} F^{(2)} &= \int_{\mathbb{T}} \left( u_{xx}^2 + \frac{5}{6} u^2 u_{xx} + \frac{5}{36} u^4 \right) dx = F_2^{(2)} + F_3^{(2)} + F_4^{(2)} \\ &= 2\pi \sum_{k \neq 0} k^4 |u_k|^2 - \frac{5}{3}\pi \sum_{k_1+k_2+k_3=0} k_3^2 u_{k_1} u_{k_2} u_{k_3} + \frac{5}{18}\pi \sum_{k_1+k_2+k_3+k_4=0} u_{k_1} u_{k_2} u_{k_3} u_{k_4}. \end{aligned}$$

Applying the transformation  $\Phi_{G_3}$ , by Lemma 3.1.12 we obtain

$$F^{(2)} \circ \Phi_{G_3} = F_2^{(2)} + \{F_2^{(2)}, G_3\} + F_3^{(2)} + \frac{1}{2} \{\{F_2^{(2)}, G_3\}, G_3\} + \{F_3^{(2)}, G_3\} + F_4^{(2)} + \dots$$

Now we prove that

$$\{F_2^{(2)}, G_3\} + F_3 = 0.$$

A straightforward computation yields

$$\begin{aligned}
\{F_2^{(2)}, G_3\} &= \left\{ 2\pi \sum_{k_1 \neq 0} k_1^4 |u_{k_1}|^2, i\frac{\pi}{9} \sum_{\substack{k'_1 + k'_2 + k'_3 = 0 \\ k'_1 + k'_2 + k'_3 = 0}} \frac{u_{k'_1} u_{k'_2} u_{k'_3}}{k'_1 k'_2 k'_3} \right\} \\
&= \frac{2}{9} i\pi^2 \sum_{\substack{k_1 \neq 0 \\ k'_1 + k'_2 + k'_3 = 0}} \frac{k_1^4}{k'_1 k'_2 k'_3} \{|u_{k_1}|^2, u_{k'_1} u_{k'_2} u_{k'_3}\} \\
&= \frac{2}{3} \pi \sum_{k_1 + k_2 + k_3 = 0} \frac{k_1^4}{k_2 k_3} u_{k_1} u_{k_2} u_{k_3} \\
&= \frac{2}{3} \pi \sum_{k_1 + k_2 + k_3 = 0} \left( 8k_3^2 + 6k_2 k_3 + 2 \frac{k_3^3}{k_2} \right) u_{k_1} u_{k_2} u_{k_3}
\end{aligned}$$

by the symmetries of the sums. But

$$\begin{aligned}
\sum_{k_1 + k_2 + k_3 = 0} k_2 k_3 u_{k_1} u_{k_2} u_{k_3} &= -\frac{1}{2\pi} \int_{\mathbb{T}} u u_x^2 dx = \frac{1}{4\pi} \int_{\mathbb{T}} u^2 u_{xx} dx \\
&= -\frac{1}{2} \sum_{k_1 + k_2 + k_3 = 0} k_3^2 u_{k_1} u_{k_2} u_{k_3}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k_1 + k_2 + k_3 = 0} \frac{k_3^3}{k_2} u_{k_1} u_{k_2} u_{k_3} &= -\frac{1}{2\pi} \int_{\mathbb{T}} u u_{xxx} \partial_x^{-1} u dx = \frac{1}{2\pi} \int_{\mathbb{T}} u_{xx} (u^2 + u_x \partial_x^{-1} u) dx \\
&= \frac{1}{2\pi} \int_{\mathbb{T}} \left( u_{xx} u^2 - \frac{1}{2} u_x^2 u \right) dx = \frac{5}{8\pi} \int_{\mathbb{T}} u_{xx} u^2 dx \\
&= -\frac{5}{4} \sum_{k_1 + k_2 + k_3 = 0} k_3^2 u_{k_1} u_{k_2} u_{k_3}.
\end{aligned}$$

Therefore

$$\{F_2^{(2)}, G_3\} = \frac{5}{3} \pi \sum_{k_1 + k_2 + k_3 = 0} k_3^2 u_{k_1} u_{k_2} u_{k_3} = -F_3^{(2)}.$$

Passing to integrals is useful, but not necessary. One can also notice that

$$\begin{aligned}
\sum_{k_1 + k_2 + k_3 = 0} k_2 k_3 u_{k_1} u_{k_2} u_{k_3} &= \frac{1}{2} \sum_{k_1 + k_2 + k_3 = 0} [(k_2 + k_3)^2 - (k_2^2 + k_3^2)] u_{k_1} u_{k_2} u_{k_3} \\
&= \frac{1}{2} \sum_{k_1 + k_2 + k_3 = 0} (k_1^2 - k_2^2 - k_3^2) u_{k_1} u_{k_2} u_{k_3} \\
&= -\frac{1}{2} \sum_{k_1 + k_2 + k_3 = 0} k_3^2 u_{k_1} u_{k_2} u_{k_3}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k_1+k_2+k_3=0} \frac{k_3^3}{k_2} u_{k_1} u_{k_2} u_{k_3} &= \frac{1}{2} \sum_{k_1+k_2+k_3=0} \frac{k_1^3 + k_3^3}{k_2} u_{k_1} u_{k_2} u_{k_3} \\
&= \frac{1}{2} \sum_{k_1+k_2+k_3=0} \frac{(k_1+k_3)(k_1^2 - k_1 k_3 + k_3^2)}{k_2} u_{k_1} u_{k_2} u_{k_3} \\
&= -\frac{1}{2} \sum_{k_1+k_2+k_3=0} (2k_3^2 - k_1 k_3) u_{k_1} u_{k_2} u_{k_3} \\
&= -\frac{5}{4} \sum_{k_1+k_2+k_3=0} k_3^2 u_{k_1} u_{k_2} u_{k_3}.
\end{aligned}$$

To prove that the same is true for order 4, we need an auxiliary result, which says that all the functions in normal form Poisson-commute, *i.e.* they are in involution.

**Lemma 3.4.13.** *If  $F$  and  $G$  are formal polynomials in normal form. *i.e.* they are only functions of  $\mathbf{I} = (|u_k|^2)_k$ , then they are in involution, *i.e.**

$$\{F, G\} = 0.$$

*Proof.* We can write

$$\begin{aligned}
F &= F_2 + F_4 + \dots + F_{2n}, \\
G &= G_2 + G_4 + \dots + G_{2m},
\end{aligned}$$

with

$$F_{2j} = \sum_{|\alpha|=j} F_{2j}^{(\alpha)} \mathbf{u}^\alpha \bar{\mathbf{u}}^\alpha, \quad G_{2l} = \sum_{|\alpha|=l} G_{2l}^{(\alpha)} \mathbf{u}^\alpha \bar{\mathbf{u}}^\alpha,$$

where  $\alpha : \mathbb{N} \rightarrow \mathbb{N}_0$  is a multi-index with finitely many non-zero components, which are positive integers whose sum is  $|\alpha|$ , and

$$\mathbf{u}^\alpha = \prod_{k \in \mathbb{Z} \setminus \{0\}} u_k^{\alpha_k}.$$

But

$$\begin{aligned}
\{F_{2j}, G_{2l}\} &= \sum_{k=1}^{+\infty} \frac{ik}{2\pi} \sum_{\substack{|\alpha|=j \\ \alpha_k>0}} \alpha_k F_{2j}^{(\alpha)} \frac{\mathbf{u}^\alpha \bar{\mathbf{u}}^\alpha}{u_k} \sum_{\substack{|\beta|=l \\ \beta_k>0}} \beta_k G_{2l}^{(\beta)} \frac{\mathbf{u}^\beta \bar{\mathbf{u}}^\beta}{\bar{u}_k} \\
&\quad + \sum_{k=-\infty}^{-1} \frac{ik}{2\pi} \sum_{\substack{|\alpha|=j \\ \alpha_{-k}>0}} \alpha_{-k} F_{2j}^{(\alpha)} \frac{\mathbf{u}^\alpha \bar{\mathbf{u}}^\alpha}{\bar{u}_k} \sum_{\substack{|\beta|=l \\ \beta_{-k}>0}} \beta_{-k} G_{2l}^{(\beta)} \frac{\mathbf{u}^\beta \bar{\mathbf{u}}^\beta}{u_k} = 0
\end{aligned}$$

by symmetry. We can conclude by the bilinearity of the Poisson bracket.  $\square$

**Lemma 3.4.14.** *Given a function of the KdV hierarchy (3.4.8)*

$$F^{(j)} = F_2^{(j)} + F_3^{(j)} + F_4^{(j)} + \dots,$$

*if we apply the first two transformations  $\Phi_{G_3} \circ \Phi_{G_4}$ , defined in Theorem 3.2.3 and Theorem 3.3.5 we obtain*

$$F^{(j)} \circ \Phi_{G_3} \circ \Phi_{G_4} = F_2^{(j)} + \hat{F}_4^{(j)} + \tilde{F}_5^{(j)} + \dots$$

*where  $\hat{F}_4^{(j)}$  is in normal form.*

*Proof.* As in Lemma 3.4.11, by (3.4.12) we have

$$0 = \{\mathcal{H} \circ \Phi_{G_3} \circ \Phi_{G_4}, F^{(j)} \circ \Phi_{G_3} \circ \Phi_{G_4}\}. \quad (3.4.14)$$

By Remark 3.3.7

$$\mathcal{H} \circ \Phi_{G_3} \circ \Phi_{G_4} = \mathcal{H}_2 + \hat{\mathcal{H}}_4 + \mathcal{H}_5 + \dots,$$

with  $\hat{\mathcal{H}}_4$  defined in (3.3.4) and, a priori, by Lemma 3.4.11

$$F^{(j)} \circ \Phi_{G_3} \circ \Phi_{G_4} = F_2^{(j)} + \tilde{F}_4^{(j)} + \tilde{F}_5^{(j)} \dots,$$

By Lemma 3.1.10 we can isolate the term of order 4 in (3.4.14) and we obtain

$$\{\mathcal{H}_2, \tilde{F}_4^{(j)}\} + \{F_2^{(j)}, \hat{\mathcal{H}}_4\} = \{\mathcal{H}_2, \tilde{F}_4^{(j)}\} = 0,$$

where in the last equality we applied Lemma 3.4.13. This, combined with Theorem 3.4.2 yields

$$\tilde{F}_4^{(j)} = \sum_{\substack{k_1+k_2+k_3+k_4=0 \\ k_1^3+k_2^3+k_3^3+k_4^3=0}} f_4^{(j)}(k_1, k_2, k_3, k_4) u_{k_1} u_{k_2} u_{k_3} u_{k_4}$$

which is in normal form. Indeed

$$\begin{cases} k_1 + k_2 + k_3 + k_4 = 0 \\ k_1^3 + k_2^3 + k_3^3 + k_4^3 = 0 \end{cases} \implies \begin{cases} (k_1 + k_2)(k_1 + k_3)(k_2 + k_3) = 0 \\ k_1 + k_2 + k_3 + k_4 = 0 \end{cases}$$

which implies that the four  $k_j$ 's are paired. □

We are now ready to prove the most important theorem of this section. We will build canonical transformations that puts the whole KdV hierarchy in normal form up to a fixed order  $n \geq 2$ . We will see that at each order we need more than one transformation. The reason is that, to apply Corollary 3.4.6, we need several relations on the indeces  $k_j$ 's. Each of these relations is obtained with a transformation. Remember that each part of the new Hamiltonian  $\mathcal{H} \circ \Phi$  is of the form

$$\mathcal{H}_j = \sum_{k_1+\dots+k_j=0} h_j(k_1, \dots, k_j) u_{k_1} \dots u_{k_j}$$

by Lemma 3.1.8. Moreover each index is always different from zero. We say that a tuple  $(k_1, \dots, k_j)$  is *admissible* if

$$\begin{cases} k_1 + \dots + k_j = 0 \\ k_1, \dots, k_j \neq 0. \end{cases}$$

**Theorem 3.4.15.** *Given  $n \geq 2$ , there exists a number  $N(n)$  of canonical transformations*

$$\Phi_1, \Phi_2, \dots, \Phi_N$$

*generated by some auxiliary Hamiltonian fluxes, such that for any  $j \in \mathbb{N}$ ,*

$$F^{(j)} \circ \Phi = F^{(j)} \circ \Phi_N \circ \dots \circ \Phi_1 = F_2^{(j)} + \hat{F}_4^{(j)} + \hat{F}_6^{(j)} + \dots + \hat{F}_n^{(j)} + R_{\geq n+1}^{(j)} \quad (3.4.15)$$

*where*

$$\begin{cases} \hat{F}_n^{(j)} = 0 & \text{if } n \text{ is odd,} \\ \hat{F}_n^{(j)} = \hat{F}_n^{(j)}(\mathbf{I}) & \text{if } n \text{ is even,} \end{cases}$$

*where  $\mathbf{I} = (|v_k|^2)_k$ . This means that these transformations put in normal form the whole KdV hierarchy up to order  $n$ .*

*Proof.* We already know that the theorem is true for  $n = 2, 3, 4$ , with  $N(2) = 0$ ,  $N(3) = 1$  and  $N(4) = 2$ . Now we use induction, separating the odd case from the even one. Suppose that  $n = 2m + 1$  is odd. By the inductive hypothesis we start from

$$F^{(j)} \circ \Phi = F_2^{(j)} + \hat{F}_4^{(j)} + \dots + \hat{F}_{2m}^{(j)} + \tilde{F}_{2m+1}^{(j)} + \dots \quad (3.4.16)$$

where

$$\tilde{F}_{2m+1}^{(j)} = \sum_{k_1+\dots+k_{2m+1}=0} \tilde{f}_{2m+1}^{(j)}(k_1, \dots, k_{2m+1}) u_{k_1} \dots u_{k_{2m+1}}.$$

Remember that the  $k_j$ 's are different from zero and  $k_1 + \dots + k_{2m+1} = 0$ . For any admissible tuple  $(k_1, \dots, k_{2m+1})$ , by Theorem 3.4.5 and Remark 3.4.7 we can find  $l_1, l_2, \dots, l_{2m}$  in arithmetic progression such that

$$k_1^{2l+1} + \dots + k_{2m+1}^{2l+1} \neq 0 \quad (3.4.17)$$

for some  $l \in \{l_1, \dots, l_{2m}\}$ . If this is true for several  $l$ 's, we choose the smallest one, in order to split all the tuples. Then, for this  $l$ , we consider the canonical transformation generated by

$$G_{2m+1}^{(l)} = -i \sum_{\substack{k_1+\dots+k_{2m+1}=0 \\ k_1^{2l+1}+\dots+k_{2m+1}^{2l+1} \neq 0}} \frac{\tilde{f}_{2m+1}^{(l)}(k_1, \dots, k_{2m+1})}{k_1^{2l+1} + \dots + k_{2m+1}^{2l+1}} u_{k_1} \dots u_{k_{2m+1}}. \quad (3.4.18)$$

This transformation removes some monomials from  $\tilde{F}_{2m+1}^{(l)}$ , indeed

$$\tilde{F}_{2m+1}^{(l)} = \tilde{F}_{2m+1}^{(l)} + \{F_2^{(l)}, G_{2m+1}^{(l)}\} = \tilde{F}_{2m+1}^{(l)} - \sum_{\substack{k_1+\dots+k_{2m+1}=0 \\ k_1^{2l+1}+\dots+k_{2m+1}^{2l+1} \neq 0}} \tilde{f}_{2m+1}^{(l)}(k_1, \dots, k_{2m+1}) u_{k_1} \dots u_{k_{2m+1}}.$$

Moreover it removes some monomials from  $\tilde{F}_{2m+1}^{(i)}$ ,  $\forall i \neq l$ , without adding other terms. Indeed, by the fact that  $\{F^{(i)}, F^{(l)}\} = 0$  and (3.4.16), isolating the terms of degree  $2m + 1$  we obtain that

$$\{F_2^{(i)}, \tilde{F}_{2m+1}^{(l)}\} + \{\tilde{F}_{2m+1}^{(i)}, F_2^{(l)}\} = 0.$$

By Lemma 3.1.10,

$$(k_1^{2i+1} + \dots + k_{2m+1}^{2i+1}) \tilde{f}_{2m+1}^{(l)}(k_1, \dots, k_{2m+1}) = (k_1^{2l+1} + \dots + k_{2m+1}^{2l+1}) \tilde{f}_{2m+1}^{(i)}(k_1, \dots, k_{2m+1})$$

and so, using (3.4.17),

$$\tilde{f}_{2m+1}^{(i)}(k_1, \dots, k_{2m+1}) = \frac{\tilde{f}_{2m+1}^{(l)}(k_1, \dots, k_{2m+1})}{k_1^{2l+1} + \dots + k_{2m+1}^{2l+1}} (k_1^{2i+1} + \dots + k_{2m+1}^{2i+1}). \quad (3.4.19)$$

Therefore

$$\begin{aligned} \tilde{F}_{2m+1}^{(i)} &= \tilde{F}_{2m+1}^{(i)} + \{F_2^{(i)}, G_{2m+1}^{(l)}\} \\ &= \tilde{F}_{2m+1}^{(i)} - \sum_{\substack{k_1+\dots+k_{2m+1}=0 \\ k_1^{2l+1}+\dots+k_{2m+1}^{2l+1}\neq 0}} \frac{\tilde{f}_{2m+1}^{(l)}(k_1, \dots, k_{2m+1})}{k_1^{2l+1} + \dots + k_{2m+1}^{2l+1}} (k_1^{2i+1} + \dots + k_{2m+1}^{2i+1}) u_{k_1} \dots u_{k_{2m+1}} \\ &= \tilde{F}_{2m+1}^{(i)} - \sum_{\substack{k_1+\dots+k_{2m+1}=0 \\ k_1^{2l+1}+\dots+k_{2m+1}^{2l+1}\neq 0}} \tilde{f}_{2m+1}^{(i)}(k_1, \dots, k_{2m+1}) u_{k_1} \dots u_{k_{2m+1}}. \end{aligned}$$

After  $2m$  transformations, the whole KdV hierarchy is in normal form up to order  $n = 2m+1$ . In the even case  $n = 2m+2$ , the procedure is the same, the only difference being that we cannot remove all the admissible tuples using Corollary 3.4.6, but only the ones which are not *paired*. This is enough, because it means that we put  $\hat{F}_{2m+2}^{(j)}$  in normal form. Another slight difference is that, using  $\{F^{(i)}, F^{(j)}\} = 0$ , isolating the terms of degree  $2m+2$  we obtain

$$\{F_2^{(i)}, \tilde{F}_{2m+2}^{(j)}\} + \{\hat{F}_4^{(i)}, \hat{F}_{2m}^{(j)}\} + \dots + \{\hat{F}_{2m}^{(i)}, \hat{F}_4^{(j)}\} + \{\tilde{F}_{2m+2}^{(i)}, F_2^{(j)}\} = 0$$

but the middle terms vanish by Lemma 3.4.13 and the inductive hypothesis. This concludes the proof of the theorem. Notice that, if  $n \geq 5$ , with

$$N = 2 + 4 + 5 + \dots + (n-1) = \frac{n(n-1)}{2} - 4$$

transformations we put in normal form the whole KdV hierarchy up to order  $n$ .  $\square$

**Remark 3.4.16.** Let  $\hat{\mathcal{H}}_{2r} = \hat{F}_{2r}^{(1)}$  be the polynomial of order  $2r \leq n$  in the Hamiltonian (3.4.15) for  $j = 1$ . We can write it as

$$\hat{\mathcal{H}}_{2r} = \sum_{|\alpha|=r} \hat{h}_{2r}^{(\alpha)} \mathbf{u}^\alpha \bar{\mathbf{u}}^\alpha$$

where  $\alpha : \mathbb{N} \rightarrow \mathbb{N}_0$  is, as usual, a multi-index with finitely many non-zero components, whose sum is  $r$ . Then  $\hat{h}_{2r}^{(\alpha)} \in \mathbb{R}$ , indeed the starting Hamiltonian  $\mathcal{H}$  has real coefficients and all the polynomials  $G$  which generate transformations of coordinates, which the reader can find in (3.2.7), (3.3.3) and (3.4.18), have purely imaginary coefficients by Proposition 3.4.8. By the definition of the Poisson bracket (3.1.7), Lemma 3.1.8 and Lemma 3.1.12, after a transformation the coefficients of the new Hamiltonian are all real.

## 3.5 Dynamics and estimates on the solution

After we have written the KdV Hamiltonian in normal form up to order  $n$  in Theorem 3.4.15, thanks to a canonical transformation  $\Phi$  which is the composition of all the transformations we need, we can study the dynamics in the new coordinates  $\mathbf{v} = (v_k)_{k \neq 0}$ . Finally we will return to our original coordinates  $\mathbf{u}$ .

**Theorem 3.5.1.** Suppose that we have put  $\mathcal{H}$  in normal form up to order  $n$ . We have

$$\mathcal{H} \circ \Phi = \mathcal{H}_2 + \hat{\mathcal{H}}_4 + \hat{\mathcal{H}}_6 + \dots + \hat{\mathcal{H}}_n + R_{n+1}.$$

Then, if we consider the flow of this new Hamiltonian, given by the solution to

$$\begin{cases} \partial_t v_k = \{v_k, \mathcal{H} \circ \Phi\} \\ v_k(t=0) = v_k(0), \end{cases}$$

for all  $k \neq 0$  we have

$$v_k(t) = v_k(0) + \int_0^t [iF_k(\mathbf{I})v_k + \tilde{R}_{n,k}](s) ds, \quad (3.5.1)$$

where  $F$  is a formal polynomial in the variables  $\mathbf{I} = (|v_j|^2)_{j \neq 0}$  with real coefficients and  $\tilde{R}_{n,k}$  has a zero of order  $n$  at the origin.

*Proof.* Consider first the case  $k > 0$ . It is enough to notice that

$$\{v_k, \hat{\mathcal{H}}_{2r}\} = \frac{ik}{2\pi} \frac{\partial}{\partial \bar{v}_k} \sum_{|\alpha|=r} \hat{h}_{2r}^{(\alpha)} \mathbf{v}^\alpha \bar{\mathbf{v}}^\alpha = \frac{ik}{2\pi} \sum_{\substack{|\alpha|=r \\ \alpha_k \neq 0}} \alpha_k \hat{h}_{2r}^{(\alpha)} \frac{\mathbf{v}^\alpha \bar{\mathbf{v}}^\alpha}{\bar{v}_k} = iv_k F_k^{(2r-2)}$$

where again  $F_k^{(2r-2)}(\mathbf{I})$  is a homogeneous polynomial of degree  $r-1$  in  $\mathbf{I} = (|v_k|^2)_k$ , with real coefficients. Therefore

$$\partial_t v_k = iF_k(\mathbf{I})v_k + \frac{ik}{2\pi} \frac{\partial R_{n+1}}{\partial \bar{v}_k} = iF_k(\mathbf{I})v_k + \tilde{R}_{n,k} \quad (3.5.2)$$

where

$$F_k(\mathbf{I}) = \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor} F_k^{(2r-2)}(\mathbf{I})$$

is a polynomial in  $(I_k)_k$  with real coefficients. If we consider  $k < 0$  we have

$$\{v_k, \hat{\mathcal{H}}_{2r}\} = \frac{ik}{2\pi} \frac{\partial}{\partial \bar{v}_k} \sum_{|\alpha|=r} \hat{h}_{2r}^{(\alpha)} \mathbf{v}^\alpha \bar{\mathbf{v}}^\alpha = \frac{ik}{2\pi} \sum_{\substack{|\alpha|=r \\ \alpha_{-k} \neq 0}} \alpha_{-k} \hat{h}_{2r}^{(\alpha)} \frac{\mathbf{v}^\alpha \bar{\mathbf{v}}^\alpha}{\bar{v}_{-k}} = iv_k F_k^{(2r-2)}$$

and the situation is the same.  $\square$

**Corollary 3.5.2.** If for all  $k \neq 0$  we consider

$$z_k(t) = e^{-i \int_0^t F_k(\mathbf{I}) ds} v_k(t),$$

where  $v_k(t)$  satisfies (3.5.1) for all  $k \neq 0$ , then we have

$$z_k(t) = z_k(0) + \int_0^t e^{-i \int_0^{t'} F_k(\mathbf{I}) dt''} \tilde{R}_{n,k}[\mathbf{v}(z)] dt'.$$

*Proof.* First notice that  $I_k = |v_k|^2 = |z_k|^2$ . We have

$$v_k(t) = e^{i \int_0^t F_k(\mathbf{I}) ds} z_k(t).$$

If we substitute this expression in (3.5.2), we obtain

$$iF_k(\mathbf{I})e^{i \int_0^t F_k(\mathbf{I}) ds} z_k + e^{i \int_0^t F_k(\mathbf{I}) ds} \dot{z}_k = iF_k(\mathbf{I})e^{i \int_0^t F_k(\mathbf{I}) ds} z_k + \tilde{R}_{n,k}[\mathbf{v}(\mathbf{z})]$$

which yields

$$\partial_t z_k = e^{-i \int_0^t F_k(\mathbf{I}) ds} \tilde{R}_{n,k}[\mathbf{v}(\mathbf{z})].$$

□

**Remark 3.5.3.** If we return to coordinates  $\mathbf{v} = (v_k)_k$  we find

$$v_k(t) = e^{i \int_0^t F_k(\mathbf{I}) ds} v_k(0) + e^{i \int_0^t F_k(\mathbf{I}) ds} \int_0^t e^{-i \int_0^{t'} F_k(\mathbf{I}) dt''} \tilde{R}_{n,k}(\mathbf{v}) dt'.$$

Remember that  $u = \Phi(v)$ . If we assume that there exists a Banach space  $X$ , continuously embedded in  $\mathcal{FL}_0^{0,1}(\mathbb{T})$ , where the KdV equation is locally well-posed, such that  $\Phi$  is invertible in a small neighborhood of the origin, *i.e.* there exists  $\varepsilon_0 > 0$  such that for all  $0 < \varepsilon \leq \varepsilon_0$ ,

$$\begin{aligned} \Phi : B_X(\varepsilon) &\longrightarrow B_X(2\varepsilon) \\ \Phi^{-1} : B_X(\varepsilon) &\longrightarrow B_X(2\varepsilon) \end{aligned}$$

and both  $\Phi$  and  $\Phi^{-1}$  are closed to the identity in a Banach space  $Y \subseteq L^\infty$  such that  $X \hookrightarrow Y$ , meaning that

$$\|(I - \Phi)(v)\|_Y \lesssim \|v\|_X^2 \tag{3.5.3}$$

$$\|(I - \Phi^{-1})(u)\|_Y \lesssim \|u\|_X^2 \tag{3.5.4}$$

then we have the following estimate for the solution  $u$  to the KdV equation. Let us call  $J_k = |(\Phi^{-1}(\mathbf{u}))_k|^2$  and  $\mathbf{J} = (J_k)_k$ , then

$$\begin{aligned} u_k(t) &= e^{i \int_0^t F_k(\mathbf{J}) ds} u_k(0) + (I - \Phi^{-1})(\mathbf{u})_k(t) - e^{i \int_0^t F_k(\mathbf{J}) ds} (I - \Phi^{-1})(\mathbf{u})_k(0) + \\ &+ e^{i \int_0^t F_k(\mathbf{J}) ds} \int_0^t e^{-i \int_0^{t'} F_k(\mathbf{J}) dt''} (\tilde{R}_{n,k} \circ \Phi^{-1})(\mathbf{u}) dt'. \end{aligned}$$

which gives the estimate

$$\left\| u_k(t) - e^{i \int_0^t F_k(\mathbf{J}) ds} u_k(0) \right\|_Y \leq C \|u(t)\|_X^2 + C \|u(0)\|_X^2 + Ct \sup_{s \in [0, t]} \|u(s)\|_X^n. \tag{3.5.5}$$

This estimate holds true only for times  $t$  for which  $v \in B_X(\varepsilon)$ , or equivalently  $u \in \Phi(B_X(\varepsilon))$ .

Notice that this (3.5.5) holds at least for  $n \in \{3, 4\}$  with  $X = Y = \mathcal{FL}^{s,p}(\mathbb{T})$ ,  $s \geq 1$  and  $p \in \{1, 2\}$  by Theorem 3.2.3 and Theorem 3.3.5.



# Chapter 4

# Probability

## 4.1 Preliminaries

Consider a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ , where  $\mathcal{A}$  is a  $\sigma$ -algebra of subsets of  $\Omega$  and  $\mathbb{P}$  is a probability measure. Each subset  $E \in \mathcal{A}$  is called *event*. We will denote an element of  $\Omega$  with  $\omega$ . If  $X : \Omega \rightarrow \mathbb{C}$  is a complex random variable such that  $X \in L^2(\Omega, \mathbb{P})$ , we will call

$$\mathbb{E}[X] = \int_{\Omega} X d\mathbb{P}$$

the *mean* of  $X$  and

$$\mathbb{E}[|X - \mathbb{E}[X]|^2] = \int_{\Omega} |X - \mathbb{E}[X]|^2 d\mathbb{P} = \mathbb{E}[|X|^2] - \mathbb{E}[X]\mathbb{E}[\bar{X}]$$

its *variance*.

We will focus on *absolutely continuous* random variables.

**Definition 4.1.1.** A real random variable  $X : \Omega \rightarrow \mathbb{R}$  is said to be *absolutely continuous* if there exists a non-negative function  $f \in L^1(\mathbb{R})$  known as *density*, such that for any Borel set  $A \subset \mathbb{R}$ ,

$$\mathbb{P}(X \in A) = \int_A f(x) dx.$$

A common example of absolutely continuous random variable is the *Gaussian (or normal)* random variable.

**Definition 4.1.2.** A *real Gaussian (or normal) random variable* is an absolutely continuous random variable  $X : \Omega \rightarrow \mathbb{R}$  with density

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where one can easily check that  $\mu$  is the mean and  $\sigma^2$  is the variance of  $X$ .

For any  $k \in \mathbb{N}$  consider the complex Gaussian random variables

$$\eta_k^\omega = \operatorname{Re}(\eta_k) + i\operatorname{Im}(\eta_k),$$

all independent and identically distributed<sup>1</sup>. In particular, for  $k \in \mathbb{N}$ ,  $\operatorname{Re}(\eta_k)$  and  $\operatorname{Im}(\eta_k)$  are independent real Gaussian random variables with zero mean and variance  $\sigma^2 = \frac{1}{2}$ . By

---

<sup>1</sup>The existence of a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$  where it is possible to construct a sequence of independent Gaussian random variables is guaranteed by the Kolmogorov's existence theorem, [3, Theorem 2.2].

the linearity of the mean  $\mathbb{E}$  and the independence of  $Re(\eta_k)$  and  $Im(\eta_k)$ , it follows that

$$\mathbb{E}\eta_k = 0, \quad \mathbb{E}|\eta_k|^2 = 1, \quad \mathbb{E}\eta_k^2 = 0. \quad (4.1.1)$$

**Lemma 4.1.3.** *It is possible to write*

$$\eta_k = R_k e^{i\phi_k}$$

where  $R_k = |\eta_k|$  follows a Rayleigh distribution, in particular it has density function

$$f(x) = \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} = 2xe^{-x^2}.$$

Moreover the angle  $\phi_k$  has a uniform distribution  $Unif[0, 2\pi]$  independent from  $R_k$ .

*Proof.*  $R_k = \sqrt{X^2 + Y^2}$ , where  $X, Y$  are independent, mean-zero, real, Gaussian random variables. If  $A \subseteq \mathbb{R}_+$  is a Borel set, then

$$\begin{aligned} \mathbb{P}(\sqrt{X^2 + Y^2} \in A) &= \int_{\{(x,y) \in \mathbb{R}^2 \mid \sqrt{x^2+y^2} \in A\}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} dx dy \\ &= \frac{1}{2\pi\sigma^2} \int_0^{2\pi} d\theta \int_A r e^{-\frac{r^2}{2\sigma^2}} dr \\ &= \int_A \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} dr. \end{aligned}$$

Moreover

$$\mathbb{E}|\eta_k| = \int_0^{+\infty} 2x^2 e^{-x^2} dx = \left[ -xe^{-x^2} \right]_0^{+\infty} + \int_0^{+\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

Now let us study the distribution of the angle variable  $\phi_k$ . If  $\Theta \subseteq [0, 2\pi]$  is a Borel set, then

$$\begin{aligned} \mathbb{P}(\phi_k \in \Theta) &= \frac{1}{\pi} \int_{\{(x,y) \in \mathbb{R}^2 \mid x+iy=re^{i\theta}, \theta \in \Theta\}} e^{-(x^2+y^2)} dx dy \\ &= \frac{1}{\pi} \int_{\Theta} d\theta \int_0^{+\infty} r e^{-r^2} dr = \frac{1}{2\pi} \int_{\Theta} d\theta, \end{aligned}$$

which means that the distribution of  $\phi_k$  is uniform. For the independence notice that

$$\begin{aligned} \mathbb{P}(R_k \in A, \phi_k \in \Theta) &= \frac{1}{\pi} \int_{\{(x,y) \in \mathbb{R}^2 \mid x+iy=re^{i\theta}, r \in A, \theta \in \Theta\}} e^{-(x^2+y^2)} dx dy \\ &= \int_{\Theta} \frac{1}{2\pi} d\theta \int_A 2r e^{-r^2} dr = \mathbb{P}(R_k \in A) \mathbb{P}(\phi_k \in \Theta). \end{aligned}$$

□

More information about the Rayleigh distribution can be found in [30]. Consider now the Cauchy problem associated to the KdV equation with a *small* probabilistic initial datum

$$\begin{cases} u_t + u_{xxx} + uu_x = 0 \\ u_{\varepsilon}(0, x)^{\omega} = \varepsilon \sum_{k \in \mathbb{Z}} c_k \eta_k^{\omega} e^{ikx} = \varepsilon \tilde{u}_0, \end{cases} \quad (4.1.2)$$

with  $\varepsilon > 0$ ,  $(c_k)_k \in \ell^1(\mathbb{Z}, \mathbb{R}_+)$  rapidly decreasing, for example  $c_k = ae^{-b|k|}$  or  $c_k = ae^{-bk^2}$  with  $a, b > 0$ . We are interested in a mean-zero, real-valued initial datum, so we take  $c_0 = 0$ ,  $c_k = c_{-k}$  and  $\eta_{-k} = \bar{\eta}_k$ . Then we have that

$$\begin{aligned} u_\varepsilon(0, x) &= \varepsilon \sum_{k=1}^{+\infty} c_k (\eta_k^\omega e^{ikx} + \bar{\eta}_k^\omega e^{-ikx}) \\ &= \varepsilon \sum_{k=1}^{+\infty} c_k [Re(\eta_k) \cos(kx) + iRe(\eta_k) \sin(kx) + iIm(\eta_k) \cos(kx) - Im(\eta_k) \sin(kx) + \\ &\quad + Re(\eta_k) \cos(kx) - iRe(\eta_k) \sin(kx) - iIm(\eta_k) \cos(kx) - Im(\eta_k) \sin(kx)] \\ &= 2\varepsilon \sum_{k=1}^{+\infty} c_k [Re(\eta_k) \cos(kx) - Im(\eta_k) \sin(kx)] \end{aligned}$$

where

$$Re(\eta_k), Im(\eta_k) \sim \mathcal{N}_{\mathbb{R}} \left( 0, \frac{1}{2} \right)$$

and they are independent for all  $k \in \mathbb{N}$ . By the properties of the Gaussian<sup>2</sup> we can bring inside a factor  $\sqrt{2}$  and we obtain

$$u_\varepsilon(0, x) = \sqrt{2}\varepsilon \sum_{k=1}^{+\infty} c_k [\theta_{k,1} \cos(kx) + \theta_{k,2} \sin(kx)]$$

where

$$\theta_{k,j} \sim \mathcal{N}_{\mathbb{R}}(0, 1)$$

and they are independent for any  $k \in \mathbb{N}$  and  $j = 1, 2$ . Finally, again by Corollary A.3.2 and Lemma A.3.3, we can notice that

$$\theta_{k,1} \cos(kx) + \theta_{k,2} \sin(kx) \sim \mathcal{N}_{\mathbb{R}}(0, 1),$$

hence, for each fixed  $x \in \mathbb{T}$ , we can write

$$u_\varepsilon(0, x) \sim_d \sqrt{2}\varepsilon \sum_{k=1}^{+\infty} c_k \theta_k, \quad \text{with } \theta_k \sim \mathcal{N}_{\mathbb{R}}(0, 1). \quad (4.1.3)$$

**Lemma 4.1.4.** *Consider the random initial datum of the Cauchy problem (4.1.2), with  $\varepsilon > 0$ ,  $x \in \mathbb{T}$  and  $(c_k)_k \in \ell^1(\mathbb{Z}, \mathbb{R}_+)$  satisfying  $c_k = c_{-k}$  and  $c_0 = 0$ . Suppose that  $(\eta_k)_{k \geq 1}$  are independent identically distributed complex Gaussian random variables satisfying (4.1.1), and that  $\eta_{-k} = \bar{\eta}_k$ . Then we have that  $u_\varepsilon(0) \in L^2(\Omega, \mathbb{P})$ . Moreover*

$$\mathbb{E}|u_\varepsilon(0, x)|^2 = 2\varepsilon^2 \sum_{k=1}^{+\infty} c_k^2, \quad \mathbb{E}u_\varepsilon(0, x) = 0, \quad \mathbb{E}\|u_\varepsilon(0)\|_{\mathcal{F}L^{0,1}} \approx \varepsilon.$$

The last equality implies that  $u_\varepsilon(0) \in \mathcal{F}L^{0,1}(\mathbb{T})$  almost surely. Moreover  $u_\varepsilon(0)$  has itself a real Gaussian distribution.

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<sup>2</sup>See Corollary A.3.2 and Lemma A.3.3 in the appendix.

*Proof.* For the first equality notice that

$$\mathbb{E}|u_\varepsilon(0, x)|^2 = \varepsilon^2 \int_{\Omega} \sum_{k_1, k_2 \neq 0} c_{k_1} \overline{c_{k_2}} \eta_{k_1}^\omega \overline{\eta_{k_2}^\omega} e^{ix(k_1 - k_2)} d\mathbb{P}$$

and we can apply Fubini's theorem because

$$\begin{aligned} \varepsilon^2 \sum_{k_1, k_2 \neq 0} \int_{\Omega} \left| c_{k_1} \overline{c_{k_2}} \eta_{k_1}^\omega \overline{\eta_{k_2}^\omega} e^{ix(k_1 - k_2)} \right| d\mathbb{P} &= \varepsilon^2 \sum_{k_1, k_2 \neq 0} c_{k_1} c_{k_2} \mathbb{E}[|\eta_{k_1}| |\overline{\eta_{k_2}}|] \\ &\leq \frac{\varepsilon^2}{2} \sum_{k_1, k_2 \neq 0} c_{k_1} c_{k_2} \mathbb{E}[|\eta_{k_1}|^2 + |\eta_{k_2}|^2] \\ &= 4\varepsilon^2 \left( \sum_{k=1}^{+\infty} c_k \right)^2 < \infty \end{aligned}$$

using the inequality

$$2|\eta_{k_1} \overline{\eta_{k_2}}| \leq |\eta_{k_1}|^2 + |\eta_{k_2}|^2.$$

Therefore

$$\mathbb{E}|u_\varepsilon(0, x)|^2 = \varepsilon^2 \sum_{k_1, k_2 \neq 0} c_{k_1} c_{k_2} e^{ix(k_1 - k_2)} \mathbb{E}[\eta_{k_1} \overline{\eta_{k_2}}] = 2\varepsilon^2 \sum_{k=1}^{+\infty} c_k^2$$

using the independence of  $\eta_{k_1}$  and  $\eta_{k_2}$  for  $k_1 \neq k_2$  and (4.1.1).

For the second one we have that

$$\mathbb{E}u_\varepsilon(0, x) = \int_{\Omega} u_\varepsilon(0, x)^\omega d\mathbb{P} = \int_{\Omega} \varepsilon \sum_{k \neq 0} c_k \eta_k^\omega e^{ikx} d\mathbb{P} = \varepsilon \sum_{k \neq 0} c_k e^{ikx} \int_{\Omega} \eta_k^\omega d\mathbb{P} = 0$$

where we can apply Fubini's theorem since  $u_\varepsilon(0) \in L^2(\Omega) \subseteq L^1(\Omega)$ . Finally

$$\mathbb{E}\|u_\varepsilon(0)\|_{\mathcal{F}L^{0,1}} = \varepsilon \int_{\Omega} \sum_{k \neq 0} |c_k \eta_k^\omega| d\mathbb{P} = \varepsilon \sum_{k \neq 0} c_k \mathbb{E}|\eta_k| = \varepsilon \sqrt{\pi} \sum_{k=1}^{+\infty} c_k.$$

To show that

$$\tilde{u}_0 = \sum_{k \in \mathbb{Z}} c_k \eta_k e^{ikx}$$

is Gaussian we recall some elementary results in probability. Set

$$\tilde{u}_0^{(n)}(x) = \sum_{|k| \leq n} c_k \eta_k e^{ikx}.$$

By Corollary A.3.2 and Lemma A.3.3 we have that for every  $x \in \mathbb{T}$ ,  $\tilde{u}_0^{(n)}$  is a real Gaussian, being the linear combination of *independent* Gaussian random variables. Moreover

$$\tilde{u}_0^{(n)} \rightarrow \tilde{u}_0 \quad \text{in } L^2(\Omega, \mathbb{P}),$$

indeed

$$\mathbb{E}|\tilde{u}_0^{(n)} - \tilde{u}_0|^2 = \sum_{|k| > n} c_k^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This implies also the convergence in distribution<sup>3</sup>. Finally, since the characteristic function<sup>4</sup> of  $\tilde{u}_0^{(n)}$  is

$$\phi_n(t) = \exp\left(-t^2 \sum_{k=1}^n c_k^2\right),$$

which converges pointwise to

$$\phi(t) = \exp\left(-t^2 \sum_{k=1}^{+\infty} c_k^2\right),$$

we find that  $\tilde{u}_0$  is Gaussian as well by Theorem A.3.7.  $\square$

Finally we show that, if we consider rapidly decreasing coefficients  $(c_k)_k$ , our initial datum  $u_\varepsilon(0, x)$  has almost surely high regularity.

**Lemma 4.1.5** (Regularity of the initial datum). *Consider the initial datum  $u_\varepsilon^\omega(0)$  of the Cauchy problem (4.1.2), with the same hypotheses of Lemma 4.1.4. Let  $c_k = ae^{-bk^2}$  or  $c_k = ae^{-b|k|}$  with  $a, b > 0$ . Then*

$$u_\varepsilon^\omega(0) \in \bigcap_{s \geq 0} H^s(\mathbb{T}) \quad \mathbb{P} - \text{almost surely in } \Omega.$$

*Proof.* Consider a sequence of positive constants  $(M_k)_{k \in \mathbb{N}}$  to be fixed later. Using the independence of the  $\eta_k$ 's and the continuity of the probability measure we have that

$$\mathbb{P}(|\eta_k| \leq M_k, \forall k \in \mathbb{N}) = \lim_{n \rightarrow +\infty} \mathbb{P}\left(\bigcap_{k=1}^n \{|\eta_k| \leq M_k\}\right) = \lim_{n \rightarrow +\infty} \prod_{k=1}^n (1 - e^{-M_k^2}).$$

This is the limit of a monotone sequence, therefore it exists. Provided that it is positive, we have that

$$\log \left[ \lim_{n \rightarrow +\infty} \prod_{k=1}^n (1 - e^{-M_k^2}) \right] = \lim_{n \rightarrow +\infty} \log \prod_{k=1}^n (1 - e^{-M_k^2}) = \sum_{k=1}^{+\infty} \log (1 - e^{-M_k^2})$$

and if  $e^{-M_k^2} \leq \frac{1}{2}$  for every  $k \in \mathbb{N}$ , then

$$\log \left[ \lim_{n \rightarrow +\infty} \prod_{k=1}^n (1 - e^{-M_k^2}) \right] = - \sum_{k=1}^{+\infty} \sum_{j=1}^{+\infty} \frac{(e^{-M_k^2})^j}{j} = - \sum_{j=1}^{+\infty} \frac{1}{j} \sum_{k=1}^{+\infty} e^{-jM_k^2}.$$

Now choose  $M_k = M\sqrt{k}$ , so that we obtain

$$\log \left[ \lim_{n \rightarrow +\infty} \prod_{k=1}^n (1 - e^{-M_k^2}) \right] = - \sum_{j=1}^{+\infty} \frac{e^{-jM^2}}{j(1 - e^{-jM^2})} \gtrsim - \sum_{j=1}^{+\infty} \frac{e^{-jM^2}}{j} \geq \log(1 - e^{-M^2}) \rightarrow 0$$

as  $M \rightarrow +\infty$ . We have proved that

$$\lim_{M \rightarrow +\infty} \mathbb{P}(|\eta_k| \leq M\sqrt{k}, \forall k \in \mathbb{N}) = 1,$$

---

<sup>3</sup>It is a classical result of probability. See for example [32], Chapter 17.

<sup>4</sup>See Definition A.3.5

therefore, if we choose the  $c_k$ 's as in the hypothesis, for every  $s \geq 0$  we have that

$$\mathbb{P}(\|u_\varepsilon(0)\|_{H^s} < \infty) = \mathbb{P}\left(\varepsilon^2 \sum_{k \in \mathbb{Z}} c_k^2 (1 + k^2)^s |\eta_k|^2 < \infty\right) \geq \mathbb{P}\left(|\eta_k| \leq M\sqrt{k}, \forall k \in \mathbb{N}\right) \rightarrow 1.$$

To conclude the proof it is enough to notice that

$$\begin{aligned} \mathbb{P}\left(u_\varepsilon(0) \in \bigcap_{s \geq 0} H^s(\mathbb{T})\right) &= \mathbb{P}\left(u_\varepsilon(0) \in \bigcap_{n \in \mathbb{N}} H^n(\mathbb{T})\right) = \mathbb{P}\left(\lim_{n \rightarrow \infty} \bigcap_{k \leq n} \{u_\varepsilon(0) \in H^k(\mathbb{T})\}\right) \\ &= \lim_{n \rightarrow \infty} \mathbb{P}(u_\varepsilon(0) \in H^n(\mathbb{T})) = 1 \end{aligned}$$

by the continuity of the measure and the fact that  $H^r(\mathbb{T}) \subseteq H^s(\mathbb{T})$  if  $s \leq r$ .  $\square$

## 4.2 Large deviations principle for the Airy equation

We consider now the linear part of the KdV equation, known also as the *Airy equation*, with initial datum as in Section 4.1. Namely we consider the problem

$$\begin{cases} u_t + u_{xxx} = 0, & x \in \mathbb{T}, \\ u(0, x)^\omega = \varepsilon \sum_{k \neq 0} c_k \eta_k^\omega e^{ikx} = \varepsilon \tilde{u}_0(x), \end{cases} \quad (4.2.1)$$

where  $c_k = ae^{-b|k|}$  or  $c_k = ae^{-bk^2}$  with  $a, b > 0$ , and  $(\eta_k)_{k=1}^\infty$  are IID complex Gaussian random variables with mean zero and variance 1, with  $\bar{\eta}_k = \eta_{-k}$ . From the equation we obtain, in the Fourier setting,

$$u_\varepsilon(t, x)^\omega \sim \varepsilon \sum_{k \in \mathbb{Z}} c_k \eta_k^\omega e^{i(kx + k^3 t)}, \quad (4.2.2)$$

where the series converge almost surely by Lemma 4.1.4. Notice that

$$\|u_\varepsilon(t)\|_{\mathcal{F}L^{0,1}} = 2\varepsilon \sum_{k=1}^{+\infty} c_k |\eta_k^\omega|.$$

Of course  $\|u_\varepsilon(t)\|_{\mathcal{F}L^{0,1}}$  is a random variable, but it is not a Gaussian, because it takes only positive values.

**Exercise 4.2.1.** If we have only two Fourier modes, corresponding to opposite integers, *i.e.*

$$u_\varepsilon(0, x)^\omega = \varepsilon c_k (\eta_k^\omega e^{ikx} + \eta_{-k}^\omega e^{-ikx}),$$

then for every  $\lambda > 0$ ,

$$\mathbb{P}(\|u(t)\|_{\mathcal{F}L^{0,1}} > \lambda) = e^{-\frac{\lambda^2 \varepsilon - 2}{4c_k^2}}.$$

*Solution.* We have that

$$\|u_\varepsilon(t)\|_{\mathcal{F}L^{0,1}} = 2\varepsilon c_k |\eta_k^\omega|,$$

therefore

$$\mathbb{P}(\|u_\varepsilon(t)\|_{\mathcal{F}L^{0,1}} > \lambda) = \mathbb{P}(2\varepsilon c_k |\eta_k^\omega| > \lambda) = \mathbb{P}\left(|\eta_k^\omega| > \frac{\lambda}{2\varepsilon c_k}\right) = \int_{\frac{\lambda}{2\varepsilon c_k}}^{+\infty} 2xe^{-x^2} dx = e^{-\frac{\lambda^2}{4\varepsilon^2 c_k^2}}.$$

**Lemma 4.2.2** (LDP for the Airy equation -  $\mathcal{F}L^{0,1}$ -norm). *For any  $\lambda > 0$  we have that*

$$\mathbb{P}(\|u_\varepsilon(t)\|_{\mathcal{F}L^{0,1}} > \lambda) \approx e^{-c\lambda^2\varepsilon^{-2}},$$

in the sense that

$$\lim_{\varepsilon \rightarrow 0^+} \varepsilon^2 \log \mathbb{P}(\|u_\varepsilon(t)\|_{\mathcal{F}L^{0,1}} > \lambda) = -c\lambda^2, \quad (4.2.3)$$

where

$$c = \frac{1}{4 \sum_{k \in \mathbb{Z}} c_k^2}.$$

*Proof.* We give a proof only in the case of four Fourier modes, corresponding to two couples of opposite integers. For the complete proof we refer to [15]. The first thing to observe is that if a random variable  $X$  follows a Rayleigh distribution with parameter  $\sigma^2$ , then  $kX$  follows a Rayleigh distribution with parameter  $k^2\sigma^2$ . This follows from the fact that, if  $A \subseteq \mathbb{R}_+$  is a Borel set, then

$$\mathbb{P}(kX \in A) = \mathbb{P}(X \in k^{-1}A) = \int_{k^{-1}A} \frac{x}{\sigma} e^{-\frac{x^2}{2\sigma^2}} dx = \int_A \frac{x}{k\sigma} e^{-\frac{x^2}{2k^2\sigma^2}} dx.$$

Let

$$u_\varepsilon(0, x)^\omega = \varepsilon \left[ c_1(\eta_{k_1}^\omega e^{ik_1 x} + \eta_{-k_1}^\omega e^{-ik_1 x}) + c_2(\eta_{k_2}^\omega e^{ik_2 x} + \eta_{-k_2}^\omega e^{-ik_2 x}) \right],$$

from which

$$\|u_\varepsilon(t)\|_{\mathcal{F}L^{0,1}} = 2\varepsilon(c_1|\eta_{k_1}^\omega| + c_2|\eta_{k_2}^\omega|).$$

Therefore

$$\mathbb{P}(\|u_\varepsilon(t)\|_{\mathcal{F}L^{0,1}} > \lambda) = \mathbb{P}(c_1|\eta_{k_1}^\omega| + c_2|\eta_{k_2}^\omega| > \gamma) = \int_\gamma^{+\infty} (f_1 * f_2)(x) dx$$

where  $\gamma = \frac{\lambda}{2\varepsilon}$  and

$$f_j(x) = \frac{2x}{c_j} e^{-\frac{x^2}{c_j^2}} \quad j = 1, 2$$

are the density functions of  $c_1|\eta_{k_1}^\omega|$  and  $c_2|\eta_{k_2}^\omega|$ .<sup>5</sup> If we substitute those we obtain

$$\begin{aligned} \mathbb{P}(\|u(t)\|_{\mathcal{F}L^{0,1}} > \lambda) &= \frac{4}{c_1^2 c_2^2} \int_\gamma^{+\infty} \int_0^x y e^{-\frac{y^2}{c_1^2}} (x-y) e^{-\frac{(x-y)^2}{c_2^2}} dy dx \\ &= \frac{4}{c_1^2 c_2^2} \left[ \int_\gamma^{+\infty} \int_0^\gamma y e^{-\frac{y^2}{c_1^2}} (x-y) e^{-\frac{(x-y)^2}{c_2^2}} dy dx + \int_\gamma^{+\infty} + \int_\gamma^x y e^{-\frac{y^2}{c_1^2}} (x-y) e^{-\frac{(x-y)^2}{c_2^2}} dy dx \right] \\ &= \frac{4}{c_1^2 c_2^2} \left[ \int_0^\gamma y e^{-\frac{y^2}{c_1^2}} \int_\gamma^{+\infty} (x-y) e^{-\frac{(x-y)^2}{c_2^2}} dx dy + \int_\gamma^{+\infty} y e^{-\frac{y^2}{c_1^2}} \int_y^{+\infty} (x-y) e^{-\frac{(x-y)^2}{c_2^2}} dx dy \right] \\ &= \frac{4}{c_1^2 c_2^2} \left[ \int_0^\gamma y e^{-\frac{y^2}{c_1^2}} \int_{-y+\gamma}^{+\infty} x e^{-\frac{x^2}{c_2^2}} dx dy + \int_\gamma^{+\infty} y e^{-\frac{y^2}{c_1^2}} \int_0^{+\infty} x e^{-\frac{x^2}{c_2^2}} dx dy \right] \\ &= \frac{2}{c_1^2} \int_0^\gamma y e^{-\frac{y^2}{c_1^2} - \frac{(-y+\gamma)^2}{c_2^2}} dy + e^{-\frac{\gamma^2}{c_1^2}} = \left( \frac{2}{c_1^2} \int_0^\gamma y e^{-\left( \frac{\sqrt{c_1^2+c_2^2}}{c_1 c_2} y - \frac{c_1 \gamma}{c_2 \sqrt{c_1^2+c_2^2}} \right)^2} dy \right) e^{-\frac{\gamma^2}{c_1^2+c_2^2}} + e^{-\frac{\gamma^2}{c_1^2}} \\ &= \frac{2c_2^2}{c_1^2 + c_2^2} \int_{-\frac{c_1 \gamma}{c_2 \sqrt{c_1^2+c_2^2}}}^{\frac{c_2 \gamma}{c_1 \sqrt{c_1^2+c_2^2}}} \left( z + \frac{c_1 \gamma}{c_2 \sqrt{c_1^2+c_2^2}} \right) e^{-z^2} dz + e^{-\frac{\gamma^2}{c_1^2}}. \end{aligned}$$

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<sup>5</sup>See Theorem A.3.1 in the appendix.

Set

$$I(w) := \int_{-\frac{c_1 w}{c_2 \sqrt{c_1^2 + c_2^2}}}^{\frac{c_2 w}{c_1 \sqrt{c_1^2 + c_2^2}}} \left( z + \frac{c_1 w}{c_2 \sqrt{c_1^2 + c_2^2}} \right) e^{-z^2} dz$$

and notice that, as  $w \rightarrow +\infty$ ,

$$I(w) \lesssim w \int_{-\frac{c_1 w}{c_2 \sqrt{c_1^2 + c_2^2}}}^{\frac{c_2 w}{c_1 \sqrt{c_1^2 + c_2^2}}} e^{-z^2} dz \lesssim w$$

and

$$\begin{aligned} I(w) &\gtrsim \int_0^{\frac{c_2 w}{c_1 \sqrt{c_1^2 + c_2^2}}} \left( z + \frac{c_1 w}{c_2 \sqrt{c_1^2 + c_2^2}} \right) e^{-z^2} dz \gtrsim \int_0^{\frac{c_2 w}{c_1 \sqrt{c_1^2 + c_2^2}}} e^{-z^2} dz \\ &= \int_0^{+\infty} e^{-z^2} dz - \int_{\frac{c_2 w}{c_1 \sqrt{c_1^2 + c_2^2}}}^{+\infty} e^{-z^2} dz \geq \frac{1}{2} \int_0^{+\infty} e^{-z^2} dz. \end{aligned}$$

We have proved that

$$0 < c \leq I(w) \leq Cw \quad \text{as } w \rightarrow +\infty.$$

But

$$\mathbb{P}(\|u(t)\|_{\mathcal{F}L^{0,1}} > \lambda) = \frac{2c_2^2 I(\frac{\lambda}{2\varepsilon})}{c_1^2 + c_2^2} e^{-\frac{\lambda^2}{4\varepsilon^2(c_1^2 + c_2^2)}} + e^{-\frac{\lambda^2}{4\varepsilon^2 c_1^2}},$$

therefore

$$\begin{aligned} &\lim_{\varepsilon \rightarrow 0^+} \varepsilon^2 \log \mathbb{P}(\|u(t)\|_{\mathcal{F}L^{0,1}} > \lambda) \\ &= \lim_{\varepsilon \rightarrow 0^+} \left[ -\frac{\lambda^2}{4(c_1^2 + c_2^2)} + \varepsilon^2 \log \left( \frac{2c_2^2 I(\frac{\lambda}{2\varepsilon})}{c_1^2 + c_2^2} + e^{-\frac{\lambda^2}{4\varepsilon^2 c_1^2} + \frac{\lambda^2}{4\varepsilon^2(c_1^2 + c_2^2)}} \right) \right] \\ &= \lim_{\varepsilon \rightarrow 0^+} \left[ -\frac{\lambda^2}{4(c_1^2 + c_2^2)} + \varepsilon^2 \log \left( \frac{2c_2^2 I(\frac{\lambda}{2\varepsilon})}{c_1^2 + c_2^2} + e^{-\frac{c_2^2 \lambda^2}{4\varepsilon^2 c_1^2(c_1^2 + c_2^2)}} \right) \right] = -\frac{\lambda^2}{4(c_1^2 + c_2^2)}. \end{aligned}$$

Indeed

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \varepsilon^2 \log \left( c + e^{-\frac{c_2^2 \lambda^2}{4\varepsilon^2 c_1^2(c_1^2 + c_2^2)}} \right) &\leq \lim_{\varepsilon \rightarrow 0^+} \varepsilon^2 \log \left( \frac{2c_2^2 I(\frac{\lambda}{2\varepsilon})}{c_1^2 + c_2^2} + e^{-\frac{c_2^2 \lambda^2}{4\varepsilon^2 c_1^2(c_1^2 + c_2^2)}} \right) \\ &\leq \lim_{\varepsilon \rightarrow 0^+} \varepsilon^2 \log \left( C\varepsilon^{-1} + e^{-\frac{c_2^2 \lambda^2}{4\varepsilon^2 c_1^2(c_1^2 + c_2^2)}} \right) \end{aligned}$$

which, by the squeeze theorem, yields

$$\lim_{\varepsilon \rightarrow 0^+} \varepsilon^2 \log \left( \frac{2c_2^2 I(\frac{\lambda}{2\varepsilon})}{c_1^2 + c_2^2} + e^{-\frac{c_2^2 \lambda^2}{4\varepsilon^2 c_1^2(c_1^2 + c_2^2)}} \right) = 0.$$

□

We have derived a large deviations principle for the  $\mathcal{F}L^{0,1}$ -norm of the solution of the Airy equation, which is the linear part of the KdV equation. In Section 4.3 we will derive a similar LDP for the  $L^\infty$ -norm of the solution of the nonlinear KdV equation, and initially we will extend the linear result by approximation. For the upper bound we need Lemma 4.2.2 and the following lemma.

**Lemma 4.2.3.** *We have the continuous embedding  $\mathcal{F}L^{0,1}(T) \subseteq L^\infty(\mathbb{T})$ .*

*Proof.* If a function  $f : \mathbb{T} \rightarrow \mathbb{C}$  has Fourier coefficients  $(f_k)_k \in \ell^1(\mathbb{Z})$ , then Fourier inversion holds and

$$f(x) = \sum_{k \in \mathbb{Z}} f_k e^{ikx} \quad \text{almost everywhere.}$$

Therefore

$$|f(x)| \leq \sum_{k \in \mathbb{Z}} |f_k| = \|f\|_{\mathcal{F}L^{0,1}} \quad \text{almost everywhere,}$$

which yields

$$\|f\|_{L^\infty} \leq \|f\|_{\mathcal{F}L^{0,1}}.$$

□

For the lower bound we need a result similar to (4.2.3), but with the  $L_x^\infty$  norm. Namely we want to prove the following theorem.

**Theorem 4.2.4** (LDP for the Airy equation -  $L^\infty$ -norm). *Consider the linear problem*

$$\begin{cases} u_t + u_{xxx} = 0, & x \in \mathbb{T} \\ u_\varepsilon(0, x) = \varepsilon \sum_{k \neq 0} c_k \eta_k^\omega e^{ikx} \end{cases} \quad (4.2.4)$$

and denote by  $u_\varepsilon(t, x)$  its solution. Then, for any  $\lambda > 0$ ,

$$\lim_{\varepsilon \rightarrow 0^+} \varepsilon^2 \log \mathbb{P} \left( \sup_{x \in \mathbb{T}} |u_\varepsilon(t, x)| \geq \lambda \right) = -\frac{\lambda^2}{4 \sum_{k=1}^{+\infty} c_k^2}. \quad (4.2.5)$$

*Proof.* We can write the linear solution (almost surely) as

$$u_\varepsilon(t, x) = \varepsilon \sum_{k \neq 0} c_k \eta_k^\omega e^{ikx + ik^3 t}.$$

The upper bound follows from the Lemma 4.2.3. Indeed

$$\begin{aligned} \mathbb{P} \left( \sup_{x \in \mathbb{T}} |u_\varepsilon(t, x)| \geq \lambda \right) &\leq \mathbb{P} (\|u_\varepsilon(t)\|_{\mathcal{F}L^{0,1}} \geq \lambda) = \mathbb{P} \left( 2\varepsilon \sum_{k=1}^{+\infty} c_k |\eta_k| \geq \lambda \right) \\ &= \mathbb{P} \left( \varepsilon \sum_{k=1}^{+\infty} c_k |\eta_k| \geq \frac{\lambda}{2} \right) \end{aligned}$$

and, by Lemma 4.2.2,

$$\limsup_{\varepsilon \rightarrow 0^+} \varepsilon^2 \log \mathbb{P} \left( \sup_{x \in \mathbb{T}} |u_\varepsilon(t, x)| \geq \lambda \right) \leq -\frac{\lambda^2}{4 \sum_{k=1}^{+\infty} c_k^2}.$$

For the lower bound, from (4.1.3) we have

$$\begin{aligned} \mathbb{P} \left( \sup_{x \in \mathbb{T}} |u_\varepsilon(t, x)| \geq \lambda \right) &\geq \mathbb{P} (|u_\varepsilon(t, 0)| \geq \lambda) \\ &= \mathbb{P} \left( \sqrt{2}\varepsilon \left| \sum_{k=1}^{+\infty} c_k [\theta_{k,1} \cos(k^3 t) + \theta_{k,2} \sin(k^3 t)] \right| \geq \lambda \right) \\ &= \mathbb{P}(X \geq \gamma) \end{aligned}$$

where  $X \sim_d \mathcal{N}_R(0, 1)$  and

$$\gamma = \frac{\lambda}{\varepsilon \sqrt{2 \sum_{k=1}^{+\infty} c_k^2}}.$$

But if we set  $x = \gamma + \frac{y}{\gamma}$  and take  $\gamma$  sufficiently big, we obtain

$$\mathbb{P}(X \geq \gamma) = \frac{1}{\sqrt{2\pi}} \int_{\gamma}^{+\infty} e^{-\frac{x^2}{2}} dx = \frac{e^{-\frac{\gamma^2}{2}}}{\sqrt{2\pi}\gamma} \int_0^{+\infty} \exp\left(-\frac{y^2}{2\gamma^2} - y\right) dy \gtrsim \frac{e^{-\frac{\gamma^2}{2}}}{\gamma},$$

since

$$\lim_{\gamma \rightarrow +\infty} \int_0^{+\infty} \exp\left(-\frac{y^2}{2\gamma^2} - y\right) dy = \int_0^{+\infty} e^{-y} dy = 1$$

by monotone convergence theorem. Then

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0^+} \varepsilon^2 \log \mathbb{P}\left(\sup_{x \in \mathbb{T}} |u_\varepsilon(t, x)| \geq \lambda\right) &\geq \liminf_{\varepsilon \rightarrow 0^+} \varepsilon^2 \log \left[ C\varepsilon \exp\left(-\frac{\lambda}{4\varepsilon^2 \sum_{k=1}^{+\infty} c_k^2}\right) \right] \\ &= -\frac{\lambda}{4 \sum_{k=1}^{+\infty} c_k^2}. \end{aligned}$$

□

### 4.3 Large deviations principle for KdV equation - $t \ll \varepsilon^{-2}$

In this section we will derive a large deviations principle for the real KdV equation with a random initial datum. In particular we consider the problem

$$\begin{cases} u_t + u_{xxx} + uu_x = 0, & x \in \mathbb{T} \\ u_\varepsilon(0, x) = \varepsilon \sum_{k \neq 0} c_k \eta_k^\omega e^{ikx} = \varepsilon \tilde{u}_0(x), \end{cases} \quad (4.3.1)$$

where  $\varepsilon > 0$ ,  $c_k = ae^{-b|k|}$  or  $c_k = ae^{-bk^2}$  with  $a, b > 0$ , and the  $(\eta_k)_{k \geq 1}$  are IID complex standard Gaussian random variables, with  $\eta_{-k} = \bar{\eta}_k$ . We want to prove that, for a fixed  $\alpha \in (0, 1)$  and for times  $t \ll \varepsilon^{-2\alpha}$ , the solution  $u_\varepsilon(t, x)$  of (4.3.1) satisfies

$$\lim_{\varepsilon \rightarrow 0^+} \varepsilon^{2-2\alpha} \log \mathbb{P}\left(\sup_{x \in \mathbb{T}} |u_\varepsilon(t, x)^\omega| \geq \lambda \varepsilon^\alpha\right) = -\frac{\lambda^2}{4 \sum_{k=1}^{+\infty} c_k^2}. \quad (4.3.2)$$

The idea behind the large deviations principle (4.3.2) is to approximate the solution  $u_\varepsilon(t, x)$  with another random variable  $u_{\text{app}, \varepsilon}(t, x)$  with a more precise statistical information, and then to estimate the error. For  $t \ll \varepsilon^{-2\alpha}$  a linear approximation will be enough. First we want to observe that  $u_\varepsilon(t, x) \in \mathcal{FL}^{0,1}$  almost surely. This, at least for a small time, follows from the local well-posedness of KdV in  $\mathcal{FL}^{0,1}(\mathbb{T})$  and from the fact that, by Lemma 4.1.4,  $u_\varepsilon(0) \in \mathcal{FL}^{0,1}$ . Moreover, by Lemma 4.2.3, we have that  $u_\varepsilon(t, x) \in L^\infty(\mathbb{T})$  almost surely. With a bootstrap argument we will show that actually  $u_\varepsilon(t, x) \in \mathcal{FL}^{0,1}(\mathbb{T})$  almost surely for the times that we need, provided that  $u_\varepsilon(0)$  is sufficiently small. We want to investigate the probability that the solution surpasses a certain amplitude  $\lambda \varepsilon^\alpha > 0$  when we let  $\varepsilon$  tending to zero. In particular we want to understand in which range  $\alpha$  lives.

**Lemma 4.3.1.** *Let  $u_\varepsilon(0) \in \mathcal{F}L_0^{0,1}(\mathbb{T})$  and let  $u_\varepsilon(t, x)$  be the solution of (4.3.1) with initial datum  $u_\varepsilon(0)$ . Fix  $\alpha \in (0, 1)$  and  $\gamma > 0$ . Let*

$$\delta > \max \left\{ \frac{\gamma\alpha}{\gamma + \alpha}, \alpha - \frac{\gamma^2}{2(\gamma + \alpha)} \right\}. \quad (4.3.3)$$

*Then there exists  $\varepsilon_0 > 0$  such that, if  $0 < \varepsilon \leq \varepsilon_0$ , for  $\|u_\varepsilon(0)\|_{\mathcal{F}L^{0,1}} \leq \varepsilon^\delta$  and  $t \lesssim \varepsilon^{-2\alpha+\gamma}$  we have that  $u_\varepsilon(t) \in \mathcal{F}L^{0,1}(\mathbb{T})$  and*

$$\sup_{s \in [0, t]} \|\nu_\varepsilon(s) - \nu_\varepsilon(0)\|_{\ell^1} \leq \varepsilon^{\delta + \frac{\gamma\alpha}{\gamma + \alpha}},$$

*where  $\nu_k(t)$  are the interaction variables defined in (2.1.5). In the original variables this means that*

$$\sup_{s \in [0, t]} \|u_\varepsilon(s) - e^{-s\partial_x^3} u_\varepsilon(0)\|_{\mathcal{F}L^{0,1}} \leq \varepsilon^{\delta + \frac{\gamma\alpha}{\gamma + \alpha}} \quad (4.3.4)$$

*where the operator  $e^{-t\partial_x^3}$  was defined in (2.0.7).*

*Proof.* By Lemma 2.3.1

$$\|\nu(t) - \nu(0)\|_{\ell^1} \leq \|\nu(t)\|_{\ell^1}^2 + \|\nu(0)\|_{\ell^1}^2 + t \sup_{s \in [0, t]} \|\nu(s)\|_{\ell^1}^3.$$

Let

$$I = \left\{ t \in [0, T] \mid \sup_{s \in [0, t]} \|\nu(s) - \nu(0)\|_{\ell^1} \leq \varepsilon^{\delta + \frac{\gamma\alpha}{\gamma + \alpha}} \right\},$$

with  $T$  to be determined. Notice that, once we prove that  $I = [0, T]$ , then  $u(t) \in \mathcal{F}L^{0,1}(\mathbb{T})$  at least for  $t \leq T + c\varepsilon^{-\delta}$ . Indeed

$$\|u(t)\|_{\mathcal{F}L^{0,1}} = \|\nu(t)\|_{\ell^1} \leq \|\nu(0)\|_{\ell^1} + \|\nu(t) - \nu(0)\|_{\ell^1} \lesssim \varepsilon^\delta$$

for  $t \leq T$  and we can prolong the solution using Remark 2.2.3. We want to use a bootstrap argument to prove that  $I = [0, T]$ . Clearly  $I$  is closed and non-empty, since  $0 \in I$  and the solution depends continuously on  $t$ . If  $t \in I$ , then Lemma 2.3.1 yields

$$\begin{aligned} & \|\nu(t) - \nu(0)\|_{\ell^1} \\ & \leq (\|\nu(t) - \nu(0)\|_{\ell^1} + \|\nu(0)\|_{\ell^1})^2 + \|\nu(0)\|_{\ell^1}^2 + t \sup_{s \in [0, t]} (\|\nu(s) - \nu(0)\|_{\ell^1} + \|\nu(0)\|_{\ell^1})^3 \\ & \leq 2 \|\nu(t) - \nu(0)\|_{\ell^1}^2 + 3 \|\nu(0)\|_{\ell^1}^2 + 8t \|\nu(0)\|_{\ell^1}^3 + 8t \sup_{s \in [0, t]} \|\nu(s) - \nu(0)\|_{\ell^1}^3 \\ & \leq 2 \|\nu(t) - \nu(0)\|_{\ell^1}^2 + 3\varepsilon^{2\delta} + 8t\varepsilon^{3\delta} + 8t\varepsilon^{3(\delta + \frac{\gamma\alpha}{\gamma + \alpha})}. \end{aligned}$$

Now we choose  $T \sim \varepsilon^{-2\alpha+\gamma}$ , so that

$$\begin{aligned} \|\nu(t) - \nu(0)\|_{\ell^1} & \leq 2 \|\nu(t) - \nu(0)\|_{\ell^1}^2 + 3\varepsilon^{2\delta} + C\varepsilon^{-2\alpha+\gamma+3\delta} + C\varepsilon^{-2\alpha+\gamma+3(\delta + \frac{\gamma\alpha}{\gamma + \alpha})} \\ & \leq 2 \|\nu(t) - \nu(0)\|_{\ell^1}^2 + \frac{1}{2} \varepsilon^{\delta + \frac{\gamma\alpha}{\gamma + \alpha}}, \end{aligned} \quad (4.3.5)$$

if  $\varepsilon$  is small enough, since

$$\begin{cases} \delta + \frac{\gamma\alpha}{\gamma + \alpha} < 2\delta \\ \delta + \frac{\gamma\alpha}{\gamma + \alpha} < -2\alpha + \gamma + 3\delta \\ \delta + \frac{\gamma\alpha}{\gamma + \alpha} < -2\alpha + \gamma + 3(\delta + \frac{\gamma\alpha}{\gamma + \alpha}) \end{cases}$$

provided that (4.3.3) holds. If we call  $x = \|\boldsymbol{\nu}(t) - \boldsymbol{\nu}(0)\|_{\ell^1}$ , we have that

$$2x^2 - x + \frac{1}{2}\varepsilon^{\delta+\frac{\gamma\alpha}{\gamma+\alpha}} \geq 0.$$

It follows that

$$x \leq \frac{1 - \sqrt{1 - 4\varepsilon^{\delta+\frac{\gamma\alpha}{\gamma+\alpha}}}}{4} \quad \text{or} \quad x \geq \frac{1 + \sqrt{1 - 4\varepsilon^{\delta+\frac{\gamma\alpha}{\gamma+\alpha}}}}{4},$$

provided that

$$1 - 4\varepsilon^{\delta+\frac{\gamma\alpha}{\gamma+\alpha}} \geq 0,$$

which is true if  $\varepsilon$  is small enough. By continuity we can conclude that

$$\|\boldsymbol{\nu}(t) - \boldsymbol{\nu}(0)\|_{\ell^1} \leq \frac{1 - \sqrt{1 - 4\varepsilon^{\delta+\frac{\gamma\alpha}{\gamma+\alpha}}}}{4} < \varepsilon^{\delta+\frac{\gamma\alpha}{\gamma+\alpha}}.$$

This strict inequality concludes the proof.  $\square$

**Theorem 4.3.2.** Consider the Cauchy problem with random initial value (4.3.1). Let  $u_\varepsilon(t, x)^\omega$  be the corresponding solution. Assume that  $t \lesssim \varepsilon^{-2\alpha+\gamma}$  with  $\alpha \in (0, 1)$  and  $\gamma > 0$ . Then for any  $\lambda > 0$  we have that

$$\lim_{\varepsilon \rightarrow 0^+} \varepsilon^{2-2\alpha} \log \mathbb{P} \left( \sup_{x \in \mathbb{T}} |u_\varepsilon(t, x)^\omega| \geq \lambda \varepsilon^\alpha \right) = -\frac{\lambda^2}{4 \sum_{k=1}^{+\infty} c_k^2}.$$

*Proof.* Let

$$\mathcal{A}_\varepsilon = \{\omega \in \Omega \mid \|u_\varepsilon(t)^\omega\|_{L_x^\infty} \geq \lambda \varepsilon^\alpha\}, \quad (4.3.6)$$

$$\mathcal{B}_\varepsilon = \{\omega \in \Omega \mid \|u_\varepsilon(t)^\omega - u_{\text{app}, \varepsilon}(t)^\omega\|_{\mathcal{F}L^{0,1}} \geq \varepsilon^{\delta+\frac{\gamma\alpha}{\gamma+\alpha}}\}, \quad (4.3.7)$$

$$\mathcal{C}_\varepsilon = \{\omega \in \Omega \mid \|u_{\text{app}, \varepsilon}(t)^\omega\|_{L_x^\infty} - \|u_\varepsilon(t)^\omega - u_{\text{app}, \varepsilon}(t)^\omega\|_{L_x^\infty} \geq \lambda \varepsilon^\alpha\}, \quad (4.3.8)$$

Clearly

$$\mathbb{P}(\mathcal{A}_\varepsilon) = \mathbb{P}(\mathcal{A}_\varepsilon \cap \mathcal{B}_\varepsilon) + \mathbb{P}(\mathcal{A}_\varepsilon \cap \mathcal{B}_\varepsilon^c),$$

but, by Lemma 4.2.3 and the triangular inequality we have that

$$\begin{aligned} \mathbb{P}(\mathcal{A}_\varepsilon \cap \mathcal{B}_\varepsilon^c) &= \mathbb{P} \left( \{\|u_\varepsilon(t)^\omega\|_{L_x^\infty} \geq \lambda \varepsilon^\alpha\} \cap \{\|u_\varepsilon(t)^\omega - u_{\text{app}, \varepsilon}(t)^\omega\|_{\mathcal{F}L^{0,1}} \leq \varepsilon^{\delta+\frac{\gamma\alpha}{\gamma+\alpha}}\} \right) \\ &\leq \mathbb{P} \left( \{\|u_\varepsilon(t)^\omega\|_{\mathcal{F}L^{0,1}} \geq \lambda \varepsilon^\alpha\} \cap \{\|u_\varepsilon(t)^\omega - u_{\text{app}, \varepsilon}(t)^\omega\|_{\mathcal{F}L^{0,1}} \leq \varepsilon^{\delta+\frac{\gamma\alpha}{\gamma+\alpha}}\} \right) \\ &\leq \mathbb{P} \left( \{\|u_\varepsilon(t)^\omega\|_{\mathcal{F}L^{0,1}} \geq \lambda \varepsilon^\alpha\} \cap \{\|u_\varepsilon(t)^\omega\|_{\mathcal{F}L^{0,1}} - \|u_{\text{app}, \varepsilon}(t)^\omega\|_{\mathcal{F}L^{0,1}} \leq \varepsilon^{\delta+\frac{\gamma\alpha}{\gamma+\alpha}}\} \right) \\ &\leq \mathbb{P} \left( \|u_{\text{app}, \varepsilon}(t)^\omega\|_{\mathcal{F}L^{0,1}} \geq \lambda \varepsilon^\alpha - \varepsilon^{\delta+\frac{\gamma\alpha}{\gamma+\alpha}} \right) \end{aligned}$$

and

$$\mathbb{P}(\mathcal{A}_\varepsilon \cap \mathcal{B}_\varepsilon) \leq \mathbb{P}(\mathcal{B}_\varepsilon) = \mathbb{P} \left( \|u_\varepsilon(t)^\omega - u_{\text{app}, \varepsilon}(t)^\omega\|_{\mathcal{F}L^{0,1}} \geq \varepsilon^{\delta+\frac{\gamma\alpha}{\gamma+\alpha}} \right).$$

Now it is the moment to choose  $u_{\text{app}, \varepsilon}(t, x)^\omega$ . For  $t \ll \varepsilon^{-2\alpha}$  it is enough to consider the solution to the linearized Cauchy problem, i.e. the solution to

$$\begin{cases} u_t + u_{xxx} = 0, & x \in \mathbb{T} \\ u(0, x) = \varepsilon \sum_{k \neq 0} c_k \eta_k^\omega e^{ikx} \end{cases} \quad (4.3.9)$$

which is

$$u_{\text{app},\varepsilon}(t,x)^\omega = e^{-t\partial_x^3} u_\varepsilon(0)^\omega = \varepsilon \sum_{k \neq 0} c_k \eta_k^\omega e^{i(kx+k^3t)}.$$

Remember that

$$\|u_{\text{app},\varepsilon}(t)\|_{\mathcal{F}L^{0,1}} = \|u_\varepsilon(0)\|_{\mathcal{F}L^{0,1}} = \varepsilon \|\tilde{u}_0\|_{\mathcal{F}L^{0,1}}.$$

By Lemma 4.3.1, if we consider  $\delta$  satisfying (4.3.3), we have that

$$\begin{aligned} \mathbb{P}\left(\|u_\varepsilon(t)^\omega\|_{L_x^\infty} \geq \lambda\varepsilon^\alpha\right) &\leq \mathbb{P}\left(\|u_{\text{app},\varepsilon}(t)\|_{\mathcal{F}L^{0,1}} \geq \lambda\varepsilon^\alpha - \varepsilon^{\delta+\frac{\gamma\alpha}{\gamma+\alpha}}\right) \\ &\quad + \mathbb{P}\left(\|u_\varepsilon(t) - u_{\text{app},\varepsilon}(t)\|_{\mathcal{F}L^{0,1}} > \varepsilon^{\delta+\frac{\gamma\alpha}{\gamma+\alpha}}\right) \\ &\leq \mathbb{P}\left(\|u_{\text{app},\varepsilon}(t)\|_{\mathcal{F}L^{0,1}} \geq \lambda\varepsilon^\alpha - \varepsilon^{\delta+\frac{\gamma\alpha}{\gamma+\alpha}}\right) + \mathbb{P}\left(\|u_{\text{app},\varepsilon}(t)\|_{\mathcal{F}L^{0,1}} \geq \varepsilon^\delta\right). \end{aligned}$$

We obtain that

$$\begin{aligned} &\limsup_{\varepsilon \rightarrow 0^+} \varepsilon^{2-2\alpha} \log \mathbb{P}\left(\|u_\varepsilon(t)^\omega\|_{L_x^\infty} \geq \lambda\varepsilon^\alpha\right) \\ &\leq \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{2-2\alpha} \log \left[ \mathbb{P}\left(\|u_{\text{app},\varepsilon}(t)\|_{\mathcal{F}L^{0,1}} \geq \lambda\varepsilon^\alpha - \varepsilon^{\delta+\frac{\gamma\alpha}{\gamma+\alpha}}\right) + \mathbb{P}\left(\|u_{\text{app},\varepsilon}(t)\|_{\mathcal{F}L^{0,1}} \geq \varepsilon^\delta\right) \right] \\ &= \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{2-2\alpha} \log \left[ \mathbb{P}\left(\|\tilde{u}_0\|_{\mathcal{F}L^{0,1}} \geq \lambda\varepsilon^{\alpha-1} - \varepsilon^{\delta+\frac{\gamma\alpha}{\gamma+\alpha}-1}\right) + \mathbb{P}\left(\|\tilde{u}_0\|_{\mathcal{F}L^{0,1}} \geq \varepsilon^{\delta-1}\right) \right]. \end{aligned}$$

Now it is the moment to choose  $\delta$ . But we are able to choose it such that

$$\begin{cases} \delta > \max\left\{\frac{\gamma\alpha}{\gamma+\alpha}, \alpha - \frac{\gamma^2}{2(\gamma+\alpha)}\right\} \\ \delta + \frac{\gamma\alpha}{\gamma+\alpha} - 1 > \alpha - 1 \\ \delta - 1 < \alpha - 1. \end{cases}$$

Therefore the second term in the logarithm is negligible with respect to the first one. Furthermore

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0^+} \varepsilon^{2-2\alpha} \log \mathbb{P}\left(\|u_\varepsilon(t)^\omega\|_{L_x^\infty} \geq \lambda\varepsilon^\alpha\right) &\leq \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{2-2\alpha} \log \mathbb{P}\left(\|\tilde{u}_0\|_{\mathcal{F}L^{0,1}} \geq \lambda\varepsilon^{\alpha-1} - \varepsilon^{\delta+\frac{\gamma\alpha}{\gamma+\alpha}-1}\right) \\ &= \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{2-2\alpha} \log \mathbb{P}\left(\|\tilde{u}_0\|_{\mathcal{F}L^{0,1}} \geq \lambda\varepsilon^{\alpha-1}\right) = -\frac{\lambda^2}{4 \sum_{k=1}^{+\infty} c_k^2}. \end{aligned}$$

by Lemma 4.2.2. Now let us prove the lower bound. We have that

$$\begin{aligned} \mathbb{P}\left(\|u_\varepsilon(t)^\omega\|_{L_x^\infty} \geq \lambda\varepsilon^\alpha\right) &\geq \mathbb{P}\left(\|u_{\text{app},\varepsilon}(t)\|_{L_x^\infty} - \|u_\varepsilon(t) - u_{\text{app},\varepsilon}(t)\|_{L_x^\infty} \geq \lambda\varepsilon^\alpha\right) \\ &= \mathbb{P}(\mathcal{C}_\varepsilon) \geq \mathbb{P}(\mathcal{C}_\varepsilon \cap \mathcal{B}_\varepsilon^c) \geq \mathbb{P}\left(\{\|u_{\text{app},\varepsilon}(t)\|_{L_x^\infty} \geq \lambda\varepsilon^\alpha + \varepsilon^{\delta+\frac{\gamma\alpha}{\gamma+\alpha}}\} \cap \mathcal{B}_\varepsilon^c\right) \\ &= \mathbb{P}\left(\|u_{\text{app},\varepsilon}(t)\|_{L_x^\infty} \geq \lambda\varepsilon^\alpha + \varepsilon^{\delta+\frac{\gamma\alpha}{\gamma+\alpha}}\right) - \mathbb{P}\left(\{\|u_{\text{app},\varepsilon}(t)\|_{L_x^\infty} \geq \lambda\varepsilon^\alpha + \varepsilon^{\delta+\frac{\gamma\alpha}{\gamma+\alpha}}\} \cap \mathcal{B}_\varepsilon\right) \\ &\geq \mathbb{P}\left(\|u_{\text{app},\varepsilon}(t)\|_{L_x^\infty} \geq \lambda\varepsilon^\alpha + \varepsilon^{\delta+\frac{\gamma\alpha}{\gamma+\alpha}}\right) - \mathbb{P}(\mathcal{B}_\varepsilon) \\ &\geq \mathbb{P}\left(\|u_{\text{app},\varepsilon}(t)\|_{L_x^\infty} \geq \lambda\varepsilon^\alpha + \varepsilon^{\delta+\frac{\gamma\alpha}{\gamma+\alpha}}\right) - \mathbb{P}\left(\|u_{\text{app},\varepsilon}(t)\|_{\mathcal{F}L^{0,1}} \geq \varepsilon^\delta\right) \end{aligned}$$

and we find

$$\begin{aligned} &\liminf_{\varepsilon \rightarrow 0^+} \varepsilon^{2-2\alpha} \log \mathbb{P}\left(\|u_\varepsilon(t)^\omega\|_{L_x^\infty} \geq \lambda\varepsilon^\alpha\right) \\ &\geq \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{2-2\alpha} \log \left[ \mathbb{P}\left(\|u_{\text{app},\varepsilon}(t)\|_{L_x^\infty} \geq \lambda\varepsilon^\alpha + \varepsilon^{\delta+\frac{\gamma\alpha}{\gamma+\alpha}}\right) - \mathbb{P}\left(\|u_{\text{app},\varepsilon}(t)\|_{\mathcal{F}L^{0,1}} \geq \varepsilon^\delta\right) \right] \\ &= -\frac{\lambda^2}{4 \sum_{k=1}^{+\infty} c_k^2} \end{aligned}$$

by Theorem 4.2.4 and Lemma 4.2.2 as before.  $\square$

## 4.4 Large deviations principle for KdV equation - $t \ll \varepsilon^{-3}$

In this section we will improve the result of Section 4.3, by making a different choice for  $u_{\text{app},\varepsilon}(t, x)$ . Again we will show that  $u_{\text{app},\varepsilon}$  shares the same large deviation principle as the linear approximation, and then we will use an argument similar to Theorem 4.3.2. Actually we will prove it only for the  $\mathcal{F}L^{0,1}$ -norm, therefore we will be able to prove only the upper bound of the LDP for KdV, for these longer timescales. Statistical information about  $\|u_{\text{app},\varepsilon}(t)\|_{L_x^\infty}$  are trickier to prove, but we expect that the result should be the same. We start from (2.3.10) in the original variables, which reads

$$\begin{aligned} \|u(t) - u_{\text{app}}(t)\|_{\mathcal{F}L^{0,1}} &\lesssim \|u(t)\|_{\mathcal{F}L^{0,1}}^2 + \|u(t)\|_{\mathcal{F}L^{0,1}}^3 + \|u(0)\|_{\mathcal{F}L^{0,1}}^2 + \|u(0)\|_{\mathcal{F}L^{0,1}}^3 \\ &\quad + t \sup_{s \in [0,t]} (\|u(s)\|_{\mathcal{F}L^{0,1}}^4 + \|u(s)\|_{\mathcal{F}L^{0,1}}^9). \end{aligned} \quad (4.4.1)$$

**Lemma 4.4.1.** *Let  $u_\varepsilon(0) \in \mathcal{F}L_0^{0,1}(\mathbb{T})$  and let  $u_\varepsilon(t, x)$  be the solution of (4.3.1) with initial datum  $u_\varepsilon(0)$ . Fix  $\alpha \in (0, 1)$  and  $\gamma > 0$ . Let*

$$\delta > \max \left\{ \frac{\gamma\alpha}{\gamma + \alpha}, \alpha - \frac{\gamma^2}{3(\gamma + \alpha)} \right\}. \quad (4.4.2)$$

*Then there exists  $\varepsilon_0 > 0$  such that, if  $0 < \varepsilon \leq \varepsilon_0$ , for  $\|u_\varepsilon(0)\|_{\mathcal{F}L^{0,1}} \leq \varepsilon^\delta$  and  $t \lesssim \varepsilon^{-3\alpha+\gamma}$  we have that  $u_\varepsilon(t) \in \mathcal{F}L^{0,1}(\mathbb{T})$  and*

$$\sup_{s \in [0,t]} \|u_\varepsilon(s) - u_{\text{app},\varepsilon}(s)\|_{\mathcal{F}L^{0,1}} \leq \varepsilon^{\delta + \frac{\gamma\alpha}{\gamma + \alpha}}. \quad (4.4.3)$$

*Proof.* Starting from (4.4.1) the proof is essentially the same of Lemma 4.3.1.  $\square$

**Remark 4.4.2.** Since

$$\|u_{\text{app},\varepsilon}(t)\|_{\mathcal{F}L^{0,1}} = \|u_\varepsilon(0)\|_{\mathcal{F}L^{0,1}},$$

for any  $\lambda > 0$  and  $\forall t \in \mathbb{R}$  we have that

$$\lim_{\varepsilon \rightarrow 0^+} \varepsilon^2 \log \mathbb{P} (\|u_{\text{app},\varepsilon}(t)\|_{\mathcal{F}L^{0,1}} \geq \lambda) = -\frac{\lambda^2}{4 \sum_{k=1}^{+\infty} c_k^2}.$$

**Theorem 4.4.3** (LDP for the KdV equation -  $t \ll \varepsilon^{-3\alpha}$  - upper bound). *Consider the Cauchy problem with random initial value (4.3.1). Let  $u_\varepsilon(t, x)^\omega$  be the corresponding solution. Assume that  $t \lesssim \varepsilon^{-3\alpha+\gamma}$  with  $\alpha \in (0, 1)$  and  $\gamma > 0$ . Then for any  $\lambda > 0$  we have that*

$$\limsup_{\varepsilon \rightarrow 0^+} \varepsilon^{2-2\alpha} \log \mathbb{P} \left( \sup_{x \in \mathbb{T}} |u_\varepsilon(t, x)^\omega| \geq \lambda \varepsilon^\alpha \right) \leq -\frac{\lambda^2}{4 \sum_{k=1}^{+\infty} c_k^2}. \quad (4.4.4)$$

*Proof.* It is the same of the proof of the upper bound in Theorem 4.3.2.  $\square$

We conjecture that an estimate similar to (4.4.4) holds also for the lower bound. More precisely we expect the following result.

**Conjecture 4.4.4.** Consider the Cauchy problem with random initial value (4.3.1). Let  $u_\varepsilon(t, x)^\omega$  be the corresponding solution. Assume that  $t \lesssim \varepsilon^{-3\alpha+\gamma}$  with  $\alpha \in (0, 1)$  and  $\gamma > 0$ . Then for any  $\lambda > 0$  we have that

$$\lim_{\varepsilon \rightarrow 0^+} \varepsilon^{2-2\alpha} \log \mathbb{P} \left( \sup_{x \in \mathbb{T}} |u_\varepsilon(t, x)^\omega| \geq \lambda \varepsilon^\alpha \right) = -\frac{\lambda^2}{4 \sum_{k=1}^{+\infty} c_k^2}. \quad (4.4.5)$$

The problem when proving the lower bound is that we need statistical information on the  $L^\infty$ -norm of  $u_{\text{app}}(t, x)$  which we can write in the interaction variables as

$$\nu_{\text{app}, k}(t) = e^{-\frac{i}{6k} \int_0^t |\nu_k - C(\nu)_k|^2(s) ds} \nu_k(0), \quad \forall k \neq 0. \quad (4.4.6)$$

Unlike the linear approximation, this one is not Gaussian for a fixed  $x \in \mathbb{T}$ . We could try to use a slightly different approximation such as

$$\tilde{\nu}_{\text{app}, k}(t) = e^{-\frac{it}{6k} |\nu_k(0)|^2} \nu_k(0), \quad \forall k \neq 0.$$

whose statistical properties are easier. Unfortunately the function  $\tilde{\nu}_{\text{app}, k}(t)$  is not a good approximation of the nonlinear dynamics over timescales  $t \sim \varepsilon^{-3\alpha+\gamma}$ , in the sense that there are choices of initial data for which

$$|\nu_k(t) - \tilde{\nu}_{\text{app}, k}(t)| \not\lesssim \varepsilon$$

for such times. For this reason, obtaining lower bounds requires more precise statistical information about the random variables (4.4.6). We leave this as an interesting open question.

## 4.5 Almost global large deviations principle for KdV equation

The better large deviations principle we can give is based on the estimate (3.5.5), that we reached with the normal form. We conjecture the following result:

**Conjecture 4.5.1.** Consider the Cauchy problem with random initial datum (4.3.1). Let  $u_\varepsilon^\omega(t, x)$  be the corresponding solution. Let  $n \geq 2$  and assume that  $t \lesssim \varepsilon^{-n\alpha+\gamma}$  with  $\alpha \in (0, 1)$  and  $\gamma > 0$ . Then for any  $\lambda > 0$  we have that

$$\limsup_{\varepsilon \rightarrow 0^+} \varepsilon^{2-2\alpha} \log \mathbb{P} \left( \sup_{x \in \mathbb{T}} |u_\varepsilon(t, x)^\omega| \geq \lambda \varepsilon^\alpha \right) \leq -\frac{\lambda^2}{4 \sum_{k=1}^{+\infty} c_k^2}. \quad (4.5.1)$$

For the upper bound the proof would be the same as Theorem 4.4.3, once we have proved that (3.5.5) holds. The missing ingredient is to prove rigorously the hypotheses of Remark 3.5.3. For the lower bound, the problem is the same as in Section 4.4.

**Remark 4.5.2.** For  $n \in \{3, 4\}$  we are able to prove the upper bound using (3.5.5), with the same strategy of Theorem 4.4.3. This follows by Theorem 3.2.3 and Theorem 3.3.5.



# Appendix A

## Auxiliary material

### A.1 Classical results

We often deal with convolution of sequences. The following classical results are very useful to give estimates. A deeper discussion can be found for example in [8] or [18].

**Theorem A.1.1** (Young's convolution inequality). *Let  $1 \leq p, q \leq \infty$  such that*

$$\frac{1}{p} + \frac{1}{q} \geq 1$$

and

$$\mathbf{x} = (x_n)_n \in \ell^p(\mathbb{Z}), \quad \mathbf{y} = (y_n)_n \in \ell^q(\mathbb{Z}).$$

Consider  $r$  such that

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$$

and the sequence

$$(\mathbf{x} * \mathbf{y})_n = \sum_{k_1+k_2=n} x_{k_1} y_{k_2}.$$

which is called the convolution of  $\mathbf{x}$  and  $\mathbf{y}$ . Then  $\mathbf{x} * \mathbf{y} \in \ell^r(\mathbb{Z})$  and

$$\|\mathbf{x} * \mathbf{y}\|_{\ell^r} \leq \|\mathbf{x}\|_{\ell^p} \|\mathbf{y}\|_{\ell^q}.$$

*Proof.* See [18, Theorem 20.18] and [8, Theorem 4.15].  $\square$

**Remark A.1.2.** This makes  $\ell^1(\mathbb{Z})$  an algebra. Indeed, if  $\mathbf{x}, \mathbf{y} \in \ell^1(\mathbb{Z})$ , then  $\mathbf{x} * \mathbf{y} \in \ell^1(\mathbb{Z})$ .

**Corollary A.1.3.** Suppose  $u, v \in \mathcal{F}L^{s,1}(\mathbb{T})$ , with  $s \geq 0$ , and denote by  $\mathbf{u} = (u_k)_k$ ,  $\mathbf{v} = (v_k)_k$ . Let  $f$  be the function whose Fourier coefficients compose the vector  $\mathbf{u} * \mathbf{v}$ . Then  $f \in \mathcal{F}L^{s,1}(\mathbb{T})$ .

*Proof.*

$$\begin{aligned} \|f\|_{\mathcal{F}L^{s,1}} &= \sum_{k_1 \in \mathbb{Z}} \langle k_1 \rangle^s \left| \sum_{k_2 \in \mathbb{Z}} u_{k_2} v_{k_1-k_2} \right| \leq \sum_{\substack{k, k_1, k_2 \\ k_1+k_2=k}} \langle k_1 + k_2 \rangle^s |u_{k_1}| |v_{k_2}| \\ &\lesssim_s \sum_{k_1, k_2 \in \mathbb{Z}} (\langle k_2 \rangle^s |u_{k_2}| |v_{k_1-k_2}| + \langle k_1 - k_2 \rangle^s |u_{k_2}| |v_{k_1-k_2}|) < \infty \end{aligned}$$

since  $(\langle k \rangle^s |u_k|)_k$  and  $(\langle k \rangle^s |v_k|)_k$  belong to  $\ell^1$ . Moreover

$$\|f\|_{\mathcal{F}L^{s,1}} \lesssim_s \|u\|_{\mathcal{F}L^{s,1}} \|v\|_{\mathcal{F}L^{s,1}}.$$

□

Corollary A.1.3 says that  $\mathcal{F}L^{s,1}(\mathbb{T})$  is an algebra for  $s \geq 0$ . For  $H^s(\mathbb{T})$  this is true only for  $s > \frac{1}{2}$ . Remember that Theorem A.1.1 guarantees only that if  $\mathbf{x}, \mathbf{y} \in \ell^2(\mathbb{Z})$ , then  $\mathbf{x} * \mathbf{y} \in \ell^\infty(\mathbb{Z})$ .

**Theorem A.1.4.** *If  $\mathbf{u}, \mathbf{v} \in h^s(\mathbb{Z})$  with  $s > \frac{1}{2}$  then  $\mathbf{u} * \mathbf{v} \in h^s(\mathbb{Z})$ .*

*Proof.* First notice that, if  $s > \frac{1}{2}$ , then

$$\|\mathbf{u}\|_{\ell^1} = \sum_{k \in \mathbb{Z}} |u_k| \leq \left( \sum_{k \in \mathbb{Z}} \frac{1}{\langle k \rangle^{2s}} \right)^{\frac{1}{2}} \left( \sum_{k \in \mathbb{Z}} \langle k \rangle^{2s} |u_k| \right)^{\frac{1}{2}} \lesssim \|\mathbf{u}\|_{h^s}$$

by the Cauchy-Schwarz inequality. Then have that

$$\begin{aligned} \|\mathbf{u} * \mathbf{v}\|_{h^s}^2 &= \sum_{k \in \mathbb{Z}} \langle k \rangle^{2s} \left| \sum_{k_1+k_2=k} u_{k_1} v_{k_2} \right|^2 \\ &\lesssim_s \sum_{k \in \mathbb{Z}} \left| \sum_{k_1+k_2=k} \langle k_1 \rangle^s u_{k_1} v_{k_2} \right|^2 + \sum_{k \in \mathbb{Z}} \left| \sum_{k_1+k_2=k} \langle k_2 \rangle^s u_{k_1} v_{k_2} \right|^2 \\ &\lesssim \|\mathbf{u}\|_{h^s}^2 \|\mathbf{v}\|_{h^s}^2 \end{aligned}$$

by Theorem A.1.1. Indeed each of those two terms is the  $\ell^2$ -norm of the convolution between an  $\ell^1$ -sequence and an  $\ell^2$ -sequence. □

The following classical result in Fourier analysis will be useful in order to compare norms in physical and Fourier spaces. A proof may be found in [29, Chapter 11].

**Theorem A.1.5** (Parseval). *Suppose*

$$f(x) \sim \sum_{k \in \mathbb{Z}} f_k e^{ikx}$$

with  $f \in L^2(\mathbb{T})$  and

$$f_k = \frac{1}{2\pi} \int_{\mathbb{T}} f(x) e^{-ikx} dx.$$

Then

$$\frac{1}{2\pi} \int_{\mathbb{T}} |f(x)|^2 dx = \sum_{k \in \mathbb{Z}} |f_k|^2.$$

We will often integrate absolutely continuous functions. The following result guarantees that their derivative exists almost everywhere and we may thus integrate by parts.

**Theorem A.1.6** (Integration by parts for absolutely continuous functions). *Let  $f, g \in L^1(a, b)$  and*

$$F(x) = \alpha + \int_a^x f(t)dt, \quad G(x) = \beta + \int_a^x g(t)dt.$$

*We say that  $F$  and  $G$  are absolutely continuous functions. It is possible to show that  $F$  and  $G$  are almost everywhere differentiable and*

$$F' = f, \quad G' = g \quad a.e.$$

*Moreover*

$$\int_a^b F(x)g(x)dx + \int_a^b f(x)G(x)dx = F(b)G(b) - F(a)G(a).$$

*Proof.* See [19], chapter 18. □

The following proposition concerns compact subsets of  $\ell^2(\mathbb{Z})$ . In particular it says that all the sequences in a compact subset of  $\ell^2$  have *equibounded tails*. We use it in Lemma 2.1.2, since we have a family of sequences depending continuously on the time  $t$ , which belongs to a compact subset of  $\mathbb{R}$ .

**Proposition A.1.7.** *If  $K \subseteq \ell^2(\mathbb{Z})$  is a compact set, then*

$$\lim_{N \rightarrow +\infty} \sup_{\mathbf{x} \in K} \sum_{n \geq N} |x_n|^2 = 0.$$

*Proof.* Suppose by contradiction that there exists  $\varepsilon > 0$  such that for any  $N \in \mathbb{N}$  there exists  $\mathbf{x}^{(N)} \in K$  such that  $\|\mathbf{x}^{(N)}\|_{\ell_{|n| \geq N}^2} > \varepsilon$ . Set  $\mathbf{y}^{(1)} = \mathbf{x}^{(1)}$ , i.e.  $N_1 = 1$ . Since  $\mathbf{x}^{(1)} \in \ell^2$ , there exists  $N_2 \in \mathbb{N}$  such that  $\|\mathbf{x}^{(N_1)}\|_{\ell_{|n| \geq N_2}^2} < \frac{\varepsilon}{2}$ . Then set  $\mathbf{y}^{(2)} = \mathbf{x}^{(N_2)}$  and so on. This way we construct a sequence  $(\mathbf{y}^{(j)})_j \in K$  such that

$$\|\mathbf{y}^{(j)}\|_{\ell_{|n| \geq N_j}^2} > \varepsilon \quad \text{and} \quad \|\mathbf{y}^{(j)}\|_{\ell_{|n| \geq N_{j+1}}^2} < \frac{\varepsilon}{2}.$$

But then, for any  $j > l$ ,

$$\|\mathbf{y}^{(j)} - \mathbf{y}^{(l)}\|_{\ell^2} \geq \|\mathbf{y}^{(j)} - \mathbf{y}^{(l)}\|_{\ell_{|n| \geq N_j}^2} \geq \left| \|\mathbf{y}^{(j)}\|_{\ell_{|n| \geq N_j}^2} - \|\mathbf{y}^{(l)}\|_{\ell_{|n| \geq N_j}^2} \right| \geq \frac{\varepsilon}{2}$$

which is a contradiction with the fact that  $K$  is compact. □

The following theorem is an easy example of application of a bootstrap argument, or continuity method.

**Theorem A.1.8** (Gronwall's inequality, Lemma 4.7 in [26]). *Let  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  be two non-negative functions. Suppose  $g$  is integrable and  $f$  is continuous. If there exists  $A \geq 0$  such that*

$$f(t) \leq A + \int_0^t f(s)g(s) ds \quad \forall t \in [0, T],$$

*then*

$$f(t) \leq A \exp \left( \int_0^t g(s) ds \right) \quad \forall t \in [0, T].$$

*Proof.* We prove this classical result using a bootstrap argument. Let  $\varepsilon > 0$  and consider

$$B_\varepsilon = \left\{ t \in [0, T] \mid f(s) \leq (1 + \varepsilon) A \exp \left( (1 + \varepsilon) \int_0^s g(r) dr \right) \quad \forall s \in [0, t] \right\}.$$

Clearly  $0 \in B_\varepsilon$  and  $B_\varepsilon$  is closed by the continuity of  $f$ . Moreover, if  $t \in B_\varepsilon$ , then

$$\begin{aligned} f(t) &\leq A + \int_0^t f(s)g(s) ds \leq A + A \int_0^t (1 + \varepsilon)g(s) \exp \left( (1 + \varepsilon) \int_0^s g(r) dr \right) ds \\ &= A \exp \left( (1 + \varepsilon) \int_0^t g(s) ds \right). \end{aligned}$$

This bound is strictly smaller than the one required for  $t \in B_\varepsilon$ . Hence, by continuity, there exists  $\delta > 0$  such that  $(t, t + \delta) \in B_\varepsilon$ . This proves that  $B_\varepsilon$  is open. Given that  $B_\varepsilon$  is closed, open and non-empty in a connected set, we have that  $B_\varepsilon = [0, T]$ . Finally, taking the limit  $\varepsilon \rightarrow 0$ , we find the thesis.  $\square$

By definition, for the local well-posedness of a differential equation, we require that the data-to-solution map is only continuous. Actually, under some hypothesis on the functions involved in the equation, when we apply a fixed point argument we automatically obtain that this map is more regular.

**Theorem A.1.9** ( $C^1$  dependence from initial data). *Consider a Banach space  $X$  and a function  $F : X \rightarrow X$ ,  $F \in C^2$ . Assume also that*

$$\|F(u)\|_X \lesssim \|u\|_X^n, \tag{A.1.1}$$

$$\|dF(u)\|_{\mathcal{L}(X)} \lesssim \|u\|_X^{n-1}, \tag{A.1.2}$$

$$\|d^2F(u)\|_{\mathcal{L}_2(X)} \lesssim \|u\|_X^{n-2}, \tag{A.1.3}$$

for some integer  $n \geq 2$ . Consider the Cauchy problem

$$\begin{cases} \dot{u} = F(u) \\ u(0) = v \end{cases} \tag{A.1.4}$$

which we can write as

$$u(t) = v + \int_0^t F(u(\tau)) d\tau. \tag{A.1.5}$$

Then (A.1.4) has a unique local solution  $u \in C_t^0([-\delta, \delta], X)$  for some  $\delta > 0$ .

Moreover we have that

$$\|u\|_{L_t^\infty X} \leq 2 \|v\|_X \tag{A.1.6}$$

$$\|u - v\|_{L_t^\infty X} \lesssim \|v\|_X^n \tag{A.1.7}$$

$$\|u(t) - v\|_X \lesssim \|u(t)\|_X^n. \tag{A.1.8}$$

Finally, for any fixed time  $t \in [-\delta, \delta]$ , the data-to-solution map

$$\begin{aligned} \Phi : X &\longrightarrow X \\ v &\longmapsto \Phi(v) = u(t) \end{aligned} \tag{A.1.9}$$

is  $C^1$  and the differential at the origin is the identity. Hence  $\Phi$  is a local diffeomorphism in a neighborhood of the origin.

*Proof.* Set

$$\Gamma(u)(t) = v + \int_0^t F(u(\tau)) d\tau.$$

By (A.1.1) we have that

$$\|\Gamma(u)\|_{L_t^\infty X} \leq \|v\|_X + C\delta \|u(\tau)\|_{L_t^\infty X}^n. \quad (\text{A.1.10})$$

Moreover, by the Taylor formula with integral remainder,

$$\begin{aligned} \Gamma(u)(t) - \Gamma(U)(t) &= \int_0^t [F(u(\tau)) - F(U(\tau))] d\tau \\ &= \int_0^t \int_0^1 dF((1-\lambda)u + \lambda U)[u - U](\tau) d\lambda d\tau. \end{aligned}$$

This, together with (A.1.2), implies that

$$\|\Gamma(u) - \Gamma(U)\|_{L_t^\infty X} \lesssim \delta \left( \|u\|_{L_t^\infty X}^{n-1} + \|U\|_{L_t^\infty X}^{n-1} \right) \|u - U\|_{L_t^\infty X}. \quad (\text{A.1.11})$$

Then using the Banach fixed point theorem we find a unique fixed point of (A.1.5) in

$$B = \{u \in C_t^0([- \delta, \delta], X) \mid \|u\|_{L_t^\infty X} \leq 2\|v\|_X\}.$$

Moreover  $\delta \sim \|v\|_X^{-n+1}$  if  $\|v\|_X \neq 0$ . The uniqueness follows in the usual way (as for the KdV equation, Corollary 2.1.13). This proves that the data-to-solution map (A.1.9) is well defined. Furthermore we have Lipschitz-continuity with respect to the initial data. Indeed

$$u(t) - U(t) = v - V + \int_0^t \int_0^1 dF((1-\lambda)u + \lambda U)[u - U](\tau) d\lambda d\tau$$

from which

$$\|u - U\|_{L_t^\infty X} \leq \|v - V\|_X + C\delta(\max\{\|v\|_X, \|V\|_X\})^{n-1} \|u - U\|_{L_t^\infty X}$$

and we can choose  $\delta$  such that

$$C\delta(\max\{\|v\|_X, \|V\|_X\})^{n-1} \leq \frac{1}{2}.$$

The estimates (A.1.6) and (A.1.7) follow immediately. If we consider the same Cauchy problem with initial datum  $u(t)$  we find also (A.1.8). Now we prove that  $\Phi \in C^1$ , where  $\Phi$  is the map (A.1.9). Using Taylor expansion we have that

$$\begin{aligned} u(t) - U(t) &= v - V + \int_0^t dF(u)[u - U](\tau) d\tau \\ &\quad + \int_0^t \int_0^1 (1-\lambda)d^2F((1-\lambda)u + \lambda U)[u - U]^2(\tau) d\lambda d\tau. \end{aligned}$$

This implies, by Duhamel's formula, that

$$\begin{aligned} u(t) - U(t) &= e^{\int_0^t dF(u) d\tau}[v - V] \\ &\quad + \int_0^t e^{\int_\tau^t dF(u) d\tau'} \int_0^1 (1-\lambda)d^2F((1-\lambda)u + \lambda U)[u - U]^2(\tau) d\lambda d\tau, \end{aligned}$$

which yields

$$\|\Phi(v + h) - \Phi(v) - e^{\int_0^t dF(u) d\tau}[h]\|_{L_t^\infty X} \leq C_1 \delta e^{C_2 \delta \|v\|_X^{n-1}} \|v\|_X^{n-2} \|h\|_X^2 = o(\|h\|_X)$$

if  $h \rightarrow 0$ , using all the bounds that we know and the Lipschitz continuity with respect to the initial data. As we can see the differential at the origin is the identity, hence  $\Phi$  is a local diffeomorphism by the implicit function theorem.  $\square$

## A.2 KdV Hierarchy

The KdV equation is an integrable system, meaning that it has infinitely many independent conserved quantities in involution

$$(F_n)_{n=-1}^{\infty} \quad \text{such that} \quad \{F_n, F_m\} = 0 \quad \forall n, m, \quad (\text{A.2.1})$$

with the Poisson bracket defined in (A.2.6) or (A.2.8). We will say also that these first integrals *Poisson-commute*, or simply *commute*. We already know three of these conservation laws:

$$F_{-1} = \int_{\mathbb{T}} u \, dx = 2\pi u_0, \quad (\text{A.2.2})$$

$$F_0 = \int_{\mathbb{T}} u^2 \, dx = 2\pi \sum_{k \in \mathbb{Z}} |u_k|^2, \quad (\text{A.2.3})$$

$$F_1 = \frac{1}{2} \int_{\mathbb{T}} \left( u_x^2 - \frac{u^3}{3} \right) \, dx = \pi \sum_{k \in \mathbb{Z}} k^2 |u_k|^2 - \frac{\pi}{3} \sum_{k_1+k_2+k_3=0} u_{k_1} u_{k_2} u_{k_3}. \quad (\text{A.2.4})$$

where  $F_1 = \mathcal{H}$  is the usual KdV Hamiltonian. We want to find a recursive formula which allows us to compute all the conservation laws and to understand their structure. To this aim, we introduce a *Bi-Hamiltonian structure* on the KdV equation. Recall that the first one is based on the Hamiltonian

$$H^{(1)}(u) = \frac{1}{2} \int_{\mathbb{T}} \left( u_x^2 - \frac{1}{3} u^3 \right) \, dx.$$

With this Hamiltonian function the KdV equation can be written as

$$\partial_t u = \partial_x \frac{dH^{(1)}}{du} = J_1 \frac{dH^{(1)}}{du}, \quad (\text{A.2.5})$$

where  $\frac{dH^{(1)}}{du}$  is the  $L^2$ -differential of  $H^{(1)}$  and  $J_1 = \partial_x$  is an antisymmetric operator with respect to the  $L^2$ -scalar product, which we can use to introduce the Poisson bracket proposed by Gardner in [14],

$$\{F, G\}_1 = \int_{\mathbb{T}} \frac{dF}{du} J_1 \frac{dG}{du} \, dx. \quad (\text{A.2.6})$$

Notice that, if we imagine

$$u = \sum_{k \in \mathbb{Z}} u_k e^{ikx}$$

as a function of its Fourier coefficients, then by the chain rule

$$\frac{\partial F}{\partial u_k} = \frac{dF}{du} \left[ \frac{\partial u}{\partial u_k} \right] = \int_{\mathbb{T}} \frac{dF}{du} e^{ikx} \, dx = 2\pi \left( \frac{dF}{du} \right)_{-k}$$

and so we have

$$\{F, G\}_1 = 2\pi \sum_{k_1+k_2=0} \left( \frac{dF}{du} \right)_{k_1} i k_2 \left( \frac{dG}{du} \right)_{k_2} = \sum_{k \neq 0} \frac{ik}{2\pi} \frac{\partial F}{\partial u_k} \frac{\partial G}{\partial \bar{u}_k}$$

which is the same as (3.1.7). The second Hamiltonian structure is based on the Hamiltonian

$$H^{(2)}(u) = \frac{1}{2} \int_{\mathbb{T}} u^2 \, dx.$$

We can write the KdV equation as

$$\partial_t u = - \left( \partial_{xxx} + \frac{2}{3} u \partial_x + \frac{1}{3} u_x \right) \frac{dH^{(2)}}{du}, \quad (\text{A.2.7})$$

where

$$J_2 = - \left( \partial_{xxx} + \frac{2}{3} u \partial_x + \frac{1}{3} u_x \right)$$

is antisymmetric with respect to the  $L^2$ -scalar product. Indeed

$$\begin{aligned} \langle f, J_2 g \rangle_{L^2} &= \int_{\mathbb{T}} f(x) J_2 g(x) dx = - \int_{\mathbb{T}} f(x) \left( \partial_{xxx} + \frac{2}{3} u \partial_x + \frac{1}{3} u_x \right) g(x) dx \\ &= - \int_{\mathbb{T}} (f g_{xxx} + \frac{2}{3} f u g_x + \frac{1}{3} f u_x g) dx \\ &= - \int_{\mathbb{T}} (-f_{xxx} g - \frac{2}{3} f_x u g - \frac{2}{3} f u_x g + \frac{1}{3} f u_x g) dx = - \langle J_2 f, g \rangle_{L^2}. \end{aligned}$$

Therefore we have a second Poisson structure given by

$$\{F, G\}_2 = \int_{\mathbb{T}} \frac{dF}{du} J_2 \frac{dG}{du} dx. \quad (\text{A.2.8})$$

Again we can write this Poisson bracket in the Fourier setting as

$$\begin{aligned} \{F, G\}_2 &= 2\pi \sum_{k_1+k_2=0} \left( \frac{dF}{du} \right)_{k_1} \left[ ik_2^3 \left( \frac{dG}{du} \right)_{k_2} - \frac{2}{3} \sum_{j+l=k_2} u_{jl} \left( \frac{dG}{du} \right)_l - \frac{1}{3} \sum_{j+l=k_2} i j u_j \left( \frac{dG}{du} \right)_l \right] \\ &= \sum_k \frac{\partial F}{\partial u_k} \left[ \frac{ik^3}{2\pi} \frac{\partial G}{\partial \overline{u_k}} - \frac{i}{6\pi} \sum_{k_1+k_2=k} (2k_2 + k_1) u_{k_1} \frac{\partial G}{\partial \overline{u_{k_2}}} \right]. \end{aligned}$$

Notice that in the Fourier setting

$$\mathcal{H}^{(2)}(\mathbf{u}) = \pi \sum_{k \in \mathbb{Z}} u_k \overline{u_k} \quad (\text{A.2.9})$$

and we can write the KdV equation as

$$\partial_t u_k = \{u_k, \mathcal{H}^{(2)}\}_2 = ik^3 u_k - \frac{ik}{2} \sum_{k_1+k_2=k} u_{k_1} u_{k_2}.$$

We can use this *Bi-Hamiltonian structure* of the KdV equation to find all the conservation laws. The key observation by Lax [25] and Magri [27] was that

$$J_1 \frac{dI_{n+1}}{du} = J_2 \frac{dI_n}{du}, \quad \forall n \geq 0. \quad (\text{A.2.10})$$

Now we reconstruct the proof of (A.2.10) given by Lax. It is useful the following

**Lemma A.2.1** (Lemma 5.8 in [24]). *If  $Q$  is a polynomial in  $u, \partial u, \dots, \partial^n u$  such that*

$$\int_{\mathbb{T}} Q(u)(x) dx = 0$$

*for any  $u \in H^n(\mathbb{T})$ , then there exists a polynomial  $G$  in  $u, \partial u, \dots, \partial^{n-1} u$  such that  $Q = \partial_x G$ .*

*Proof.* We give an explicit construction of the polynomial  $G$ . If we are able to construct

$$G(u, \partial u, \dots, \partial^{n-1} u)$$

for a specific class of periodic functions  $u$ , then automatically we will have  $Q = \partial_x G$ , since  $Q$  and  $G$  are polynomials in  $u$  and its derivatives. Then we consider  $u$  such that

$$u(0) = u'(0) = \dots = u^{(n-1)}(0) = 0,$$

where we are identifying

$$\mathbb{T} \cong [0, 2\pi] / \sim \quad \text{with } 0 \sim 2\pi.$$

Consider  $y \in \mathbb{T} \setminus \{0\}$  and define the new function

$$v(x) = \begin{cases} u(x) & \text{if } 0 \leq x \leq y, \\ q(x) & \text{if } y \leq x \leq y + \varepsilon, \\ 0 & \text{if } x \geq y + \varepsilon, \end{cases}$$

where  $\varepsilon > 0$  and  $q(x)$  is the polynomial of degree  $2n - 1$  which satisfies

$$q(y) = u(y), \quad q'(y) = u'(y), \dots, \quad q^{(n-1)}(y) = u^{(n-1)}(y),$$

$$q(y + \varepsilon) = 0, \quad q'(y + \varepsilon) = 0, \dots, \quad q^{(n-1)}(y + \varepsilon) = 0.$$

The second condition means that  $q(x)$  has a zero of order  $n$  at the point  $x = y + \varepsilon$ , hence we can write

$$q(x) = (y + \varepsilon - x)^n (a_0 + a_1 x + \dots + a_{n-1} x^{n-1}) = (y + \varepsilon - x)^n a(x).$$

Now we can impose the other  $n$  conditions, and we obtain

$$\begin{aligned} \varepsilon^n a(y) &= u(y), \\ n\varepsilon^{n-1} a(y) + \varepsilon^n a'(y) &= u'(y) \\ &\vdots \\ \sum_{k=0}^{n-1} \binom{n-1}{k} n(n-1)\dots(n-k+1) \varepsilon^{n-k} a^{n-k}(y) &= u^{(n-1)}(y). \end{aligned}$$

Since

$$a(x) = a(y) + a'(y)(x - y) + \frac{a''(y)}{2}(x - y)^2 + \dots + \frac{a^{(n-1)}(y)}{(n-1)!}(x - y)^{n-1},$$

we can easily write  $a(x+y)$  as a polynomial in  $x$  whose coefficients depend on  $u(y), u'(y), \dots, u^{(n-1)}(y)$  and  $\varepsilon$ . Notice that  $v \in H^n$ . Moreover the polynomial  $Q$  has not constant terms, otherwise we can evaluate at  $u = 0$  and then  $\int_{\mathbb{T}} Q(u) dx \neq 0$ . Hence, from the hypothesis on  $Q$ , we have

$$\int_{\mathbb{T}} Q(v)(x) dx = \int_0^y Q(u)(x) dx + \int_y^{y+\varepsilon} Q(q)(x) dx = 0,$$

from which we get the equation

$$\int_0^y Q(u)(x) dx = - \int_y^{y+\varepsilon} Q(q)(x) dx = - \int_0^{\varepsilon} Q(q)(x + y) dx. \quad (\text{A.2.11})$$

But

$$q(x+y) = (\varepsilon - x)^n a(x+y) = (\varepsilon - x)^n \left[ a(y) + a'(y)x + \frac{a''(y)}{2}x^2 + \dots + \frac{a^{(n-1)}(y)}{(n-1)!}x^{n-1} \right],$$

therefore the right hand side of (A.2.11) is a polynomial in  $u(y), u'(y), \dots, u^{(n-1)}(y)$  which doesn't depend on  $\varepsilon$ , being equal to the left hand side. Taking the derivative with respect to  $y$  both sides, we obtain the thesis by the fundamental theorem of calculus.  $\square$

**Example A.2.2.** If we take  $Q(u, u_x) = uu_x$ , then a simple computation yields

$$q(x) = \frac{u(y)}{\varepsilon}(y + \varepsilon - x)$$

and so

$$Q(q)(x+y) = \frac{u(y)^2}{\varepsilon^2}(x - \varepsilon).$$

Following the idea of Lemma A.2.1,

$$\int_0^y Q(u) dx = - \int_0^\varepsilon \frac{u(y)^2}{\varepsilon^2}(x - \varepsilon) dx = \frac{u(y)^2}{2}.$$

Therefore  $G = \frac{1}{2}u^2$  and we can observe that  $\partial_x G = Q$ .

We can use this lemma to prove the following proposition.

**Proposition A.2.3.** *There exists a sequence  $(G_n)_{n=-1}^{+\infty}$  of polynomials in  $u$  and its derivatives up to order  $2n$  (with the exception of  $G_{-1} = \text{const.}$ ) such that*

$$J_1 G_{n+1} = J_2 G_n.$$

Moreover each  $G_n$  is uniquely determined if we choose the constants of integration equal to zero.

*Proof.* Let us start with  $G_{-1} = 1$ . Then

$$J_2 G_{-1} = - \left( \partial_{xxx} + \frac{2}{3}u\partial_x + \frac{1}{3}u_x \right) 1 = -\frac{1}{3}u_x = \partial_x \left( -\frac{1}{3}u \right) = J_1 \left( -\frac{1}{3}u \right).$$

Therefore we can set  $G_0 = -\frac{1}{3}u$ . Assume now that we have found all the polynomials up to  $G_k$ . Then, to prove that there exists  $G_{k+1}$  such that

$$J_1 G_{k+1} = \partial_x G_{k+1} = J_2 G_k,$$

by Lemma A.2.1 it is enough to show that

$$\int_{\mathbb{T}} J_2 G_k = 0 \quad \forall u.$$

But

$$\begin{aligned} \int_{\mathbb{T}} J_2 G_k &= \langle J_2 G_k, G_0 \rangle_{L^2} = -\langle G_k, J_2 G_0 \rangle_{L^2} = -\langle G_k, J_1 G_1 \rangle_{L^2} = \langle J_1 G_k, G_1 \rangle_{L^2} \\ &= \langle J_2 G_{k-1}, G_1 \rangle_{L^2} = \dots \end{aligned}$$

and we get

$$\begin{cases} \langle J_2 G_{k/2}, G_{k/2} \rangle_{L^2} & \text{if } k \text{ is even,} \\ \langle J_1 G_{(k+1)/2}, G_{(k+1)/2} \rangle_{L^2} & \text{if } k \text{ is odd.} \end{cases}$$

Since  $J_1$  and  $J_2$  are antisymmetric, we can conclude that

$$\int_{\mathbb{T}} J_2 G_k = 0.$$

In the same way one can prove that

$$\int_{\mathbb{T}} G_k J_r G_l = 0 \quad \forall k, l, \forall r = 1, 2. \quad (\text{A.2.12})$$

□

**Remark A.2.4.** Observe that  $G_0 = G_0(u)$  depends only on  $u$  and not on its derivatives. Assume that  $G_n$  depends on  $u$  and its derivatives up to order  $2n$ . Then, using the recursive formula, we find that  $G_{n+1}$  needs two derivatives more. Sometimes we will require more derivatives as we want to apply some differential operator to  $G_n$ .

**Theorem A.2.5.** (*Theorem 3.3 in [25]*).  $G_n$  is the  $L^2$ -differential of a functional  $F_n$  of the form

$$F_n = \int_{\mathbb{T}} P_n(u) dx$$

where  $P_n(u)$  is a polynomial in  $u$  and its derivatives up to order  $2n$ .

*Proof.* Set

$$\frac{dG}{du}[v] = N(u)v.$$

Since we are working with polynomials, we can assume all the regularity that we need for  $G$ . In particular the Gateaux and the Fréchet differentials coincide, and we can write

$$\frac{d}{d\varepsilon} G(u + \varepsilon v)_{|\varepsilon=0} = N(u)v.$$

The first thing we want to prove is that, if  $G$  were the  $L^2$ -differential of some functional  $F$ , then  $N(u)$  would be symmetric with respect to the  $L^2$ -scalar product. Indeed we can use the symmetry of the derivatives to say that

$$\frac{d^2}{d\varepsilon d\eta} F(u + \varepsilon v + \eta w)_{|\varepsilon=\eta=0} = \frac{d^2}{d\eta d\varepsilon} F(u + \varepsilon v + \eta w)_{|\varepsilon=\eta=0},$$

but this means that

$$\langle N(u)v, w \rangle_{L^2} = \langle N(u)w, v \rangle_{L^2}$$

and we have proved the symmetry of  $N(u)$ . Now we want to prove the converse, *i.e.* that if the differential of  $G$  is symmetric, then  $G$  is the differential of some functional  $F$ , just like in the finite dimensional case. We consider a smooth path

$$u(\varepsilon) : [0, 1] \rightarrow H^s(\mathbb{T}), \quad s \text{ large enough,}$$

connecting  $u_0$  and  $u_1$ . Then, since we want  $F$  to satisfy

$$\frac{d}{d\varepsilon} F(u) = \langle G(u), \partial_\varepsilon u \rangle_{L^2},$$

we set

$$F(u_1) - F(u_0) = \int_0^1 \langle G(u), \partial_\varepsilon u \rangle d\varepsilon. \quad (\text{A.2.13})$$

We want to show that, given the hypothesis that  $G$  has a symmetric differential, then the functional  $F$  defined in (A.2.13) has  $G$  as differential.

**Step 1:** The definition of  $F$  given in (A.2.13) doesn't depend on the choice of the path  $u(\varepsilon)$ . Indeed, if we consider a 1-parameter family  $u(\varepsilon, \eta)$  of smooth paths connecting  $u_0$  and  $u_1$ , then

$$\frac{d}{d\eta} \int_0^1 \langle G(u), \partial_\varepsilon u \rangle d\varepsilon = \int_0^1 \langle G(u), u_{\varepsilon\eta} \rangle d\varepsilon + \int_0^1 \langle N(u)u_\eta, u_\varepsilon \rangle d\varepsilon.$$

But

$$\langle G(u), u_{\varepsilon\eta} \rangle = \frac{d}{d\varepsilon} \langle G(u), u_\eta \rangle - \langle N(u)u_\varepsilon, u_\eta \rangle$$

and, since  $u_\eta(0) = u_\eta(1) = 0$ , we obtain

$$\frac{d}{d\eta} \int_0^1 \langle G(u), \partial_\varepsilon u \rangle d\varepsilon = - \int_0^1 \langle N(u)u_\varepsilon, u_\eta \rangle d\varepsilon + \int_0^1 \langle N(u)u_\eta, u_\varepsilon \rangle d\varepsilon = 0$$

by the symmetry of  $N(u)$ . Once we have proved that the definition (A.2.13) does not depend on the path, we can choose straight lines. Therefore

$$F(u+h) - F(u) - \langle G(u), h \rangle_{L^2} = \int_0^1 \langle G(u+\varepsilon h) - G(u), h \rangle d\varepsilon = o(\|h\|_{H^s})$$

and we can conclude that the  $L^2$ -derivative of  $F$  is  $G$ .

**Step 2:** Now we compute the derivative of our polynomials  $G_n$ . Consider an arbitrary smooth path  $u(\varepsilon)$  connecting  $u_0$  and  $u_1$ . Remember that we have the recursive relation

$$J_1 G_{n+1} = J_2 G_n,$$

with

$$J_1 = \partial_x \quad \text{and} \quad J_2 = - \left( \partial_{xxx} + \frac{2}{3}u\partial_x + \frac{1}{3}u_x \right).$$

We can derive it with respect to  $\varepsilon$  and we obtain

$$\begin{aligned} \partial_\varepsilon (J_1 G_{n+1}) &= J_1 (\partial_\varepsilon G_{n+1}) = J_1 N_{n+1}(u_\varepsilon) \\ &= \partial_\varepsilon (J_2 G_n) = (\partial_\varepsilon J_2) G_n + J_2 N_n(u_\varepsilon). \end{aligned} \quad (\text{A.2.14})$$

But

$$(\partial_\varepsilon J_2) G_n = - \left( \frac{2}{3}u_\varepsilon \partial_x + \frac{1}{3}\partial_\varepsilon u_x \right) G_n = K_n u_\varepsilon$$

where  $K_n = -\frac{1}{3}G_n J_1 - \frac{2}{3}J_1 G_n$ . If we substitute this in (A.2.14) we get

$$J_1 N_{n+1} = K_n + J_2 N_n. \quad (\text{A.2.15})$$

**Step 3:** Next we prove that

$$K_{n-1} J_2 - K_n J_1$$

is self-adjoint. A direct computation shows that

$$\begin{aligned} K_{n-1} J_2 &= \left( -\frac{1}{3}G_n J_1 - \frac{2}{3}J_1 G_n \right) \left( -\partial_{xxx} - \frac{2}{3}u\partial_x - \frac{1}{3}u_x \right) = \\ &= \left( \frac{1}{3}G_{n-1} \right) \partial_{xxxx} + \left( \frac{2}{3}J_1 G_{n-1} \right) \partial_{xxx} + \left( \frac{2}{9}u G_{n-1} \right) \partial_{xx} + \\ &\quad + \left( \frac{1}{3}u_x G_{n-1} + \frac{4}{9}u J_1 G_{n-1} \right) \partial_x + \left( \frac{1}{9}u_{xx} G_{n-1} + \frac{2}{9}u_x J_1 G_{n-1} \right). \end{aligned}$$

Using integration by parts one can show that

$$\begin{aligned} \int_{\mathbb{T}} \frac{1}{3} G_{n-1} f_{xxxx} g \, dx &= \int_{\mathbb{T}} f \left( \frac{1}{3} G_{n-1xxxx} g + \frac{4}{3} G_{n-1xxx} g_x + 2G_{n-1xx} g_{xx} \right. \\ &\quad \left. + \frac{4}{3} G_{n-1x} g_{xxx} + \frac{1}{3} G_{n-1} g_{xxxx} \right) \, dx, \\ \int_{\mathbb{T}} \frac{2}{3} G_{n-1x} f_{xxx} g \, dx &= \int_{\mathbb{T}} f \left( -\frac{2}{3} G_{n-1xxxx} g - 2G_{n-1xxx} g_x - 2G_{n-1xx} g_{xx} - \frac{2}{3} G_{n-1x} g_{xxx} \right) \, dx, \\ \int_{\mathbb{T}} \frac{2}{9} u G_{n-1} f_{xx} g \, dx &= \int_{\mathbb{T}} f \left( \frac{2}{9} u_{xx} G_{n-1} g + \frac{2}{9} u G_{n-1xx} g + \frac{2}{9} u G_{n-1} g_{xx} \right. \\ &\quad \left. + \frac{4}{9} u_x G_{n-1x} g + \frac{4}{9} u_x G_{n-1} g_x + \frac{4}{9} u G_{n-1x} g_x \right) \, dx, \\ \int_{\mathbb{T}} \left( \frac{1}{3} u_x G_{n-1} + \frac{4}{9} u G_{n-1x} \right) f_x g \, dx &= \int_{\mathbb{T}} f \left( -\frac{1}{3} u_{xx} G_{n-1} g - \frac{1}{3} u_x G_{n-1x} g - \frac{1}{3} u_x G_{n-1} g_x \right. \\ &\quad \left. - \frac{4}{9} u_x G_{n-1x} g - \frac{4}{9} u G_{n-1xx} g - \frac{4}{9} u G_{n-1x} g_x \right) \, dx. \end{aligned}$$

Therefore

$$\langle K_{n-1} J_2 f, g \rangle_{L^2} = \langle f, (K_{n-1} J_2)^* g \rangle_{L^2}$$

with

$$\begin{aligned} (K_{n-1} J_2)^* &= \left( \frac{1}{3} G_{n-1} \right) \partial_{xxxx} + \left( \frac{2}{3} G_{n-1x} \right) \partial_{xxx} + \left( \frac{2}{9} u G_{n-1} \right) \partial_{xx} \\ &\quad + \left( -\frac{2}{3} G_{n-1xxx} + \frac{1}{9} u_x G_{n-1} \right) \partial_x + \left( -\frac{1}{3} G_{n-1xxxx} - \frac{2}{9} u G_{n-1xx} - \frac{1}{9} u_x G_{n-1x} \right). \end{aligned}$$

This yields

$$K_{n-1} J_2 - (K_{n-1} J_2)^* = a_{n-1} \partial_x + b_{n-1}$$

where

$$a_{n-1} = \frac{2}{3} G_{n-1xxx} + \frac{4}{9} u G_{n-1x} + \frac{2}{9} u_x G_{n-1} = -\frac{2}{3} J_2 G_{n-1} = -\frac{2}{3} J_1 G_n$$

and

$$b_{n-1} = \frac{1}{3} G_{n-1xxxx} + \frac{2}{9} u G_{n-1xx} + \frac{1}{3} u_x G_{n-1x} + \frac{1}{9} u_{xx} G_{n-1} = -\frac{1}{3} J_1 J_2 G_{n-1} = -\frac{1}{3} J_1^2 G_n$$

using the recursion formula. We have shown that

$$K_{n-1} J_2 - (K_{n-1} J_2)^* = -\frac{2}{3} G_{n_x} \partial_x - \frac{1}{3} G_{n_{xx}}. \quad (\text{A.2.16})$$

On the other hand one can easily show that

$$K_n J_1 = -\frac{1}{3} G_n \partial_{xx} - \frac{2}{3} G_{n_x} \partial_x \quad (\text{A.2.17})$$

$$(K_n J_1)^* = -\frac{1}{3} G_n \partial_{xx} + \frac{1}{3} G_{n_{xx}} \quad (\text{A.2.18})$$

$$K_n J_1 - (K_n J_1)^* = -\frac{2}{3} G_{n_x} \partial_x - \frac{1}{3} G_{n_{xx}}. \quad (\text{A.2.19})$$

Finally (A.2.16) combined with (A.2.19) gives the thesis.

**Step 4:** At this step we prove that, if we assume that  $N_{n-1}$  and  $N_n$  are symmetric, then also

$$J_1 N_{n+1} J_1$$

is symmetric. To this aim we will use induction. Then, in the next step, we will show that under these hypothesis  $N_{n+1}$  is symmetric as well, which yields the inductive step. Remember that  $G_{-1} = 1$  and  $G_0 = Cu$ , so it is clear that they have symmetric derivatives. Let us start with (A.2.15) for  $n$  and  $n + 1$ , which are

$$K_{n-1} + J_2 N_{n-1} = J_1 N_n \quad (\text{A.2.20})$$

$$K_n + J_2 N_n = J_1 N_{n+1}. \quad (\text{A.2.21})$$

If we multiply (A.2.20) by  $J_2$  on the right and (A.2.21) by  $J_1$  on the right, and then we subtract both sides, we obtain

$$J_1 N_{n+1} J_1 = J_1 N_n J_2 + J_2 N_n J_1 - J_2 N_{n-1} J_2 \quad (\text{A.2.22})$$

$$+ K_n J_1 - K_{n-1} J_2. \quad (\text{A.2.23})$$

Then (A.2.23) is symmetric by step 3, while (A.2.22) is symmetric being the sum of symmetric operators. Indeed each of them is the multiplication of a symmetric operator with two antisymmetric operators.

**Step 5:** The fact that

$$J_1 N_{n+1} J_1 = \partial_x N_{n+1} \partial_x$$

is symmetric implies that  $N_{n+1}$  is symmetric on the subspace  $H_0^s(\mathbb{T})$  of periodic functions of mean value 0. Indeed, if  $f_0$  is a periodic function such that

$$\int_0^{2\pi} f_0(x) dx = 0,$$

then we can consider the *periodic* function

$$f(x) = \int_0^y f_0(y) dy$$

and it follows that  $f_0 = \partial_x f$ . Since mean zero functions can be written as derivatives, if we consider  $f_0$  and  $g_0$  with zero mean value, we have that

$$\begin{aligned} \langle f_0, N_{n+1} g_0 \rangle_{L^2} &= \langle \partial_x f, N_{n+1} \partial_x g \rangle_{L^2} = -\langle f, \partial_x N_{n+1} \partial_x g \rangle_{L^2} \\ &= -\langle \partial_x N_{n+1} \partial_x f, g \rangle_{L^2} = \langle N_{n+1} \partial_x f, \partial_x g \rangle_{L^2} = \langle N_{n+1} f_0, g_0 \rangle_{L^2}. \end{aligned}$$

**Step 6:** Finally we prove that  $N_{n+1}$  is symmetric on the whole space. In according with (A.2.13), taking  $u_0 = 0$ ,  $F(0) = 0$  and  $u(\varepsilon) = \varepsilon u$ , we have

$$F_{n+1}(u) = \int_0^1 \langle G_{n+1}(\varepsilon u), u \rangle d\varepsilon. \quad (\text{A.2.24})$$

Since  $G_{n+1}$  is a polynomial in  $u$  and its derivatives,

$$F_{n+1}(u) = \int_{\mathbb{T}} P_{n+1}(u) dx$$

where  $P_{n+1}$  is again a polynomial in  $u$  and its derivatives. But  $F_{n+1}$  is  $L^2$ -differentiable, so let's call  $\tilde{G}_{n+1}$  its  $L^2$ -derivative, *i.e.*

$$\frac{d}{d\varepsilon} F(u + \varepsilon v)|_{\varepsilon=0} = \langle \tilde{G}_{n+1}(u), v \rangle_{L^2}.$$

We know that on the subspace of mean zero functions  $N_{n+1}$  is symmetric, therefore if  $u_0$  and  $v_0$  have zero-mean,

$$\frac{d}{d\varepsilon} F(u_0 + \varepsilon v_0)|_{\varepsilon=0} = \langle G_{n+1}(u_0), v_0 \rangle_{L^2}.$$

Subtracting both sides we obtain

$$\langle G_{n+1}(u_0) - \tilde{G}_{n+1}(u_0), v_0 \rangle_{L^2} = 0 \quad \forall u_0, v_0 \quad \text{with zero mean.}$$

But we have shown that  $v_0 = \partial_x v$ , therefore

$$\left\langle \partial_x [G_{n+1}(u_0) - \tilde{G}_{n+1}(u_0)], v \right\rangle_{L^2} = 0 \quad \forall v \in H^s(\mathbb{T})$$

which yields  $G_{n+1}(u_0) - \tilde{G}_{n+1}(u_0) = \text{const}$ . Remember that  $G_{n+1}$  is a polynomial in  $u$  and its derivatives without constant term, and so is  $\tilde{G}_{n+1}$  by (A.2.24). For this reason we can conclude that  $G_{n+1} = \tilde{G}_{n+1}$  which implies also that  $N_{n+1}$  is symmetric.  $\square$

**Remark A.2.6.** All the quantities involved are well defined if we work in  $H^s(\mathbb{T})$  with  $s \geq 2(n+1)$ . Indeed, we need  $G_n, F_n, G_{n+1}$  to be defined, and we know that  $2(n+1)$  derivatives are enough.

**Remark A.2.7.** Observe that for  $n = 0, 1$  we have that, up to a constant,

$$\begin{aligned} F_n &= \int_{\mathbb{T}} [(\partial^n u)^2 + P_{\geq 3}(u, \partial u, \dots, \partial^{n-1} u)] dx = \\ &= \int_{\mathbb{T}} \left[ (\partial^n u)^2 + \sum_{\alpha: |\alpha| \geq 3} c_\alpha u^{\alpha_0} (\partial u)^{\alpha_1} \dots (\partial^{n-1} u)^{\alpha_{n-1}} \right] dx \end{aligned}$$

where  $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{n-2})$  is a multi-index and  $|\alpha| = \alpha_0 + \alpha_1 + \dots + \alpha_{n-2}$ . In the following theorem we prove that this structure is common to all the conservation laws.

**Theorem A.2.8** (Structure of the integrals). *For any  $n \in \mathbb{N}_0$  we have that, up to a constant,*

$$F_n = \int_{\mathbb{T}} [(\partial^n u)^2 + P_{\geq 3}(u, \partial u, \dots, \partial^{n-1} u)] dx. \quad (\text{A.2.25})$$

*Proof.* We prove this theorem by induction. We already know that it is true for  $n = 0, 1$ . Then suppose it is true for  $n$  and let's prove it for  $n+1$ . We have that

$$\begin{aligned} \frac{dF_n}{du}[f] &= \int_{\mathbb{T}} [2(\partial^n u)(\partial^n f) \\ &\quad + \sum_{\alpha: |\alpha| \geq 3} c_\alpha \sum_{j=0}^{n-1} \alpha_j u^{\alpha_0} (\partial u)^{\alpha_1} \dots (\partial^{j-1} u)^{\alpha_{j-1}} (\partial^j u)^{\alpha_j-1} (\partial^j f) (\partial^{j+1} u)^{\alpha_{j+1}} \dots (\partial^{n-1} u)^{\alpha_{n-1}}] dx \\ &= \int_{\mathbb{T}} [(-1)^n 2(\partial^{2n} u) f \\ &\quad + \sum_{\alpha: |\alpha| \geq 3} (-1)^j \alpha_j \partial^j (u^{\alpha_0} (\partial u)^{\alpha_1} \dots (\partial^{j-1} u)^{\alpha_{j-1}} (\partial^j u)^{\alpha_j-1} (\partial^{j+1} u)^{\alpha_{j+1}} \dots (\partial^{n-1} u)^{\alpha_{n-1}}) f] dx \end{aligned}$$

using integration by parts ( $n$  times for the first addend,  $j$  times for each of the other addends). It is easy to see that the derivation  $\partial_x$  doesn't change the degree of a polynomial in  $u$  and its derivatives. This yields, up to a constant,

$$\frac{dF_n}{du} = \partial^{2n} u + \sum_{\alpha: |\alpha| \geq 2} \sum_{j=0}^{n-1} c_{\alpha,j} \partial^j (u^{\alpha_0} (\partial u)^{\alpha_1} \dots (\partial^{n-1} u)^{\alpha_{n-1}}).$$

If we apply  $\partial_x^{-1} J_2$  we obtain, up to a sign,

$$\begin{aligned}
\frac{dF_{n+1}}{du} &= \partial^{2n+2} u + \sum_{\alpha:|\alpha| \geq 2} \sum_{j=0}^{n-1} c_{\alpha,j} \partial^{j+2} (u^{\alpha_0} (\partial u)^{\alpha_1} \dots (\partial^{n-1} u)^{\alpha_{n-1}}) \\
&\quad + \frac{2}{3} \partial_x^{-1} (u \partial^{2n+1} u) + \frac{2}{3} \partial_x^{-1} \left[ u \sum_{\alpha:|\alpha| \geq 2} \sum_{j=0}^{n-1} c_{\alpha,j} \partial^{j+1} (u^{\alpha_0} (\partial u)^{\alpha_1} \dots (\partial^{n-1} u)^{\alpha_{n-1}}) \right] \\
&\quad + \frac{1}{3} \partial_x^{-1} (u_x \partial^{2n} u) + \frac{1}{3} \partial_x^{-1} \left[ u_x \sum_{\alpha:|\alpha| \geq 2} \sum_{j=0}^{n-1} c_{\alpha,j} \partial^j (u^{\alpha_0} (\partial u)^{\alpha_1} \dots (\partial^{n-1} u)^{\alpha_{n-1}}) \right] \\
&= \partial^{2n+2} u + \sum_{\alpha:|\alpha| \geq 2} \sum_{j=0}^{n-1} c_{\alpha,j} \partial^{j+2} (u^{\alpha_0} (\partial u)^{\alpha_1} \dots (\partial^{n-1} u)^{\alpha_{n-1}}) \\
&\quad + \frac{2}{3} \left[ \sum_{k=0}^{n-1} (-1)^k \partial^k u \partial^{2n-k} u + (-1)^n \frac{(\partial^n u)^2}{2} \right] \\
&\quad + \frac{1}{3} \left[ \sum_{k=1}^{n-1} (-1)^{k+1} \partial^k u \partial^{2n-k} u + (-1)^{n+1} \frac{(\partial^n u)^2}{2} \right] \\
&\quad + \frac{1}{3} \left[ u \sum_{\alpha:|\alpha| \geq 2} \sum_{j=0}^{n-1} c_{\alpha,j} \partial^j (u^{\alpha_0} (\partial u)^{\alpha_1} \dots (\partial^{n-1} u)^{\alpha_{n-1}}) \right] \\
&\quad + \frac{1}{3} \sum_{\alpha:|\alpha| \geq 2} \sum_{j=0}^{n-1} c_{\alpha,j} \sum_{k=1}^j (-1)^{k+1} \partial^k u \partial^{j-k} (u^{\alpha_0} (\partial u)^{\alpha_1} \dots (\partial^{n-1} u)^{\alpha_{n-1}}) \\
&\quad + \frac{1}{3} \sum_{\alpha:|\alpha| \geq 2} \sum_{j=0}^{n-1} c_{\alpha,j} (-1)^j \partial_x^{-1} (\partial^{j+1} u \cdot u^{\alpha_0} (\partial u)^{\alpha_1} \dots (\partial^{n-1} u)^{\alpha_{n-1}}).
\end{aligned}$$

We know that this is the  $L^2$ -derivative of some functional  $F_{n+1}$ . Using (A.2.24), we obtain, after some integration by parts, that this functional is of the form

$$F_{n+1} = \int_{\mathbb{T}} \left[ \frac{1}{2} (\partial^{n+1} u)^2 + \sum_{\alpha:|\alpha| \geq 3} c_{\alpha}''' u^{\alpha_0} (\partial u)^{\alpha_1} \dots (\partial^n u)^{\alpha_n} \right] dx.$$

Indeed each monomial of the functional is, up to a constant, a monomial of its differential multiplied by  $u$ . But by the formula we have found for  $\frac{dF_{n+1}}{du}$ , it is clear that we can move the derivative so that the quadratic term of  $F_{n+1}$  depends on  $\partial^{n+1} u$ , while the other ones depend on derivative of at least one order less. Finally we can notice that, again by induction, the maximum degree of  $F_{n+1}$  is  $n+2$ . This is because  $F_0$  is quadratic, and at each step we earn one power.  $\square$

**Remark A.2.9.** Theorem A.2.8 shows that

$$F_n : H^n(\mathbb{T}) \rightarrow \mathbb{R}.$$

To deal with the  $L^2$ -differentials of these functionals we need more derivatives, as explained in Remark A.2.4.

**Remark A.2.10.** If we use the Poisson brackets

$$\{F, G\}_1 = \int_{\mathbb{T}} \left( \frac{dF}{du} \right) J_1 \left( \frac{dG}{du} \right) dx, \quad (\text{A.2.26})$$

$$\{F, G\}_2 = \int_{\mathbb{T}} \left( \frac{dF}{du} \right) J_2 \left( \frac{dG}{du} \right) dx, \quad (\text{A.2.27})$$

then from (A.2.12) we have that the  $F_n$ 's are in involution with respect to both these Poisson structures. Moreover

$$\frac{d}{dt} F_n(u) = \frac{dF_n}{du} [\partial_t u] = \left\langle \frac{dF_n}{du}, J_1 \frac{dH^{(1)}}{du} \right\rangle_{L^2} = \left\langle \frac{dF_n}{du}, J_2 \frac{dH^{(2)}}{du} \right\rangle_{L^2} = 0.$$

This means that the  $F_n$ 's are conserved quantities along the solutions of the KdV equation. Notice that, to make sense of  $\{F_n, F_m\}_j$  for  $j = 1, 2$ , we need  $u \in H^s(\mathbb{T})$  with  $s \geq 2 \max\{n, m\} + 2$ . Indeed we want

$$G_n J_1 G_m \in L^1(\mathbb{T}), \quad G_n J_2 G_m \in L^1(\mathbb{T}).$$

For the first one it is enough that one between  $G_n(u)$  and  $G_m(u)$  belongs to  $H^1(\mathbb{T})$  and the other one to  $L^2(\mathbb{T})$ , but the worst case is when  $n = m$  where we need  $s \geq 2n + 1$ . For the second one there is a compensation due to integration by parts, so we need that one between  $G_n(u)$  and  $G_m(u)$  belongs to  $H^2(\mathbb{T})$  and the other one to  $H^1(\mathbb{T})$ . In this case we need  $s \geq 2n + 2$  if  $n \geq m$ .

**Remark A.2.11.** We know that the  $L^2$ -norm of a solution of the KdV equation is constant in time. This is, a priori, not true for the  $H^s$ -norm,  $s > 0$ . Assume that  $u(0) \in H^N(\mathbb{T})$  with  $N \in \mathbb{N}$ . Then in Chapter 2 we proved that we have a unique solution  $u(t, x) \in C_t^0 L_x^2(\mathbb{R} \times \mathbb{T}, \mathbb{R})$ . But using the  $N^{\text{th}}$  conservation law, we have that

$$\begin{aligned} \|u(t)\|_{H^N}^2 &= F_N[u(0)] + \int_{\mathbb{T}} \sum_{j=3}^{N+2} P_j(u, u_x, \dots, \partial^{N-1} u) dx \\ &\leq F_N[u(0)] + \sum_{j=3}^{N+2} C_j \|u(t)\|_{H^{N-1}}^j. \end{aligned}$$

Using recursively this formula we find that  $\|u(t)\|_{H^N}$  is uniformly bounded in  $t$ , because we arrive to the  $L^2$ -norm which is conserved. Hence  $u(t) \in H^N(\mathbb{T})$ . Using this argument for any  $N$ , we proved the existence of smooth solutions.

### A.3 Probability

We recall also some classical results in probability. The first one tells us that when we consider the sum of two independent absolutely continuous random variables, we obtain an absolutely continuous random variable whose density is the convolution of the densities. This allows us also to prove that the sum of two independent Gaussian distributions is Gaussian, and it gives a formula for the mean and the variance.

**Theorem A.3.1.** *If  $(\Omega, \mathcal{A}, p)$  is a probability space and  $X, Y : \Omega \rightarrow \mathbb{R}$  are two independent, real valued, absolutely continuous random variables, with densities respectively  $f, g \in L^1(\mathbb{R})$ , then  $X + Y$  is absolutely continuous with density  $f * g$ .*

*Proof.* Let  $A \in \mathbb{R}$  be a Borel set.

$$\begin{aligned}\mathbb{P}(X + Y \in A) &= \int_{\{(x,y) \in \mathbb{R}^2 \mid x+y \in A\}} f(x)g(y) dx dy = \int_{\mathbb{R}} f(x) \int_{A-x} g(y) dy dx \\ &= \int_{\mathbb{R}} f(x) \int_A g(y-x) dy dx = \int_A (f * g)(y) dy.\end{aligned}$$

□

**Corollary A.3.2.** *The sum of two independent real (resp. complex) Gaussian random variables  $X_1$  and  $X_2$ , is a real (resp. complex) random variable. Moreover*

$$\sigma_{X_1+X_2}^2 = \sigma_1^2 + \sigma_2^2.$$

*Proof.* Let us start from the real case. Consider  $X_1, X_2$  two real, independent, absolutely continuous random variables with densities

$$f_j(x) = \frac{1}{\sqrt{2\pi\sigma_j^2}} \exp\left(-\frac{(x - \mu_j)^2}{2\sigma_j^2}\right), \quad j = 1, 2.$$

Then, by Theorem A.3.1,  $X_1 + X_2$  is absolutely continuous with density

$$\begin{aligned}f_{X_1+X_2}(x) &= (f_1 * f_2)(x) \\ &= \frac{1}{2\pi\sigma_1\sigma_2} \int_{\mathbb{R}} \exp\left(-\frac{(y - \mu_1)^2}{2\sigma_1^2} - \frac{(x - y - \mu_2)^2}{2\sigma_2^2}\right) dy \\ &= \frac{1}{2\pi\sigma_1\sigma_2} \exp\left(-\frac{\mu_1^2}{2\sigma_1^2} - \frac{\mu_2^2}{2\sigma_2^2}\right) \\ &\quad \cdot \int_{\mathbb{R}} \exp\left(-\frac{\sigma_2^2 y^2 - 2\mu_1\sigma_2^2 y + \sigma_1^2 y^2 - 2\sigma_1^2 xy + 2\sigma_1^2\mu_2 y}{2\sigma_1^2\sigma_2^2}\right) dy \\ &= \frac{1}{2\pi\sigma_1\sigma_2} \exp\left(-\frac{\mu_1^2}{2\sigma_1^2} - \frac{(x - \mu_2)^2}{2\sigma_2^2} + \frac{(\mu_1\sigma_2^2 + \sigma_1^2 x - \sigma_1^2\mu_2)^2}{2\sigma_1^2\sigma_2^2(\sigma_1^2 + \sigma_2^2)}\right) \\ &\quad \cdot \int_{\mathbb{R}} \exp\left[-\frac{\left(\sqrt{\sigma_1^2 + \sigma_2^2}y - \frac{\mu_1\sigma_2^2 + \sigma_1^2 x - \sigma_1^2\mu_2}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right)^2}{2\sigma_1^2\sigma_2^2}\right] dy \\ &= \frac{1}{\sqrt{2\pi(\sigma_1^2 + \sigma_2^2)}} \exp\left(-\frac{\mu_1^2}{2\sigma_1^2} - \frac{(x - \mu_2)^2}{2\sigma_2^2} + \frac{\mu_1\sigma_2^2 + \sigma_1^2 x - \sigma_1^2\mu_2}{2\sigma_1^2\sigma_2^2(\sigma_1^2 + \sigma_2^2)}\right) \\ &= \frac{1}{\sqrt{2\pi(\sigma_1^2 + \sigma_2^2)}} \exp\left(-\frac{(x - \mu_1 - \mu_2)^2}{2(\sigma_1^2 + \sigma_2^2)}\right).\end{aligned}$$

Now let

$$X_j = \text{Re}(X_j) + i\text{Im}(X_j), \quad j = 1, 2$$

be two independent complex Gaussian random variables, where  $\text{Re}(X_j)$  and  $\text{Im}(X_j)$  are independent, zero-mean, and

$$\text{Var}(\text{Re}(X_j)) = \text{Var}(\text{Im}(X_j)) = \sigma_j^2.$$

Then, by the previous part of the theorem,

$$X_1 + X_2 = [Re(X_1) + Re(X_2)] + i[Im(X_1) + Im(X_2)],$$

where  $[Re(X_1) + Re(X_2)]$  and  $[Im(X_1) + Im(X_2)]$  are two independent, real, Gaussian random variables with the same variance.  $\square$

**Lemma A.3.3.** *Let  $X$  be a real (resp. complex) Gaussian with mean  $\mu$  and variance  $\sigma^2$ . Given  $k > 0$ , then  $kX$  is a real (resp. complex) Gaussian with mean  $k\mu$  and variance  $k^2\sigma^2$ .*

*Proof.* We prove the claim for a real Gaussian, then the complex case follows immediately as in Corollary A.3.2. We have that, if  $A \subseteq \mathbb{R}$  is a Borel set,

$$\begin{aligned} \mathbb{P}(kX \in A) &= \mathbb{P}(X \in k^{-1}A) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{k^{-1}A} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\ &= \frac{1}{\sqrt{2\pi k^2\sigma^2}} \int_A e^{-\frac{(k^{-1}x-\mu)^2}{2\sigma^2}} dx = \frac{1}{\sqrt{2\pi k^2\sigma^2}} \int_A e^{-\frac{(x-k\mu)^2}{2k^2\sigma^2}} dx. \end{aligned}$$

$\square$

**Corollary A.3.4.** *The linear combination of real (resp. complex), independent, Gaussian random variables is a real (resp. complex) Gaussian random variable.*

We need also some results on the characteristic function of a random variable. In particular we need Levy's continuity theorem to find the distribution of an infinite sum of normal random variables in Lemma 4.1.4.

**Definition A.3.5.** Given a random variable  $X : \Omega \rightarrow \mathbb{R}^d$  and  $t \in \mathbb{R}^d$ , we call *characteristic function* of  $X$  the function

$$\phi_X(t) = \mathbb{E}[e^{i\langle t, X \rangle}] = \int_{\mathbb{R}^d} e^{i\langle t, x \rangle} d\mu_X(x)$$

where  $\mu_X$  is the *law* of  $X$ , i.e. for any Borel set  $A \in \mathbb{R}^d$ ,

$$\mu_X(A) = \mathbb{P}(X \in A).$$

**Lemma A.3.6.** *Let  $X \sim \mathcal{N}_{\mathbb{R}}(\mu, \sigma^2)$ . Then*

$$\phi_X(t) = \exp\left(i\mu t - \frac{\sigma^2 t^2}{2}\right).$$

*Proof.* It is a simple application of Cauchy's formula for holomorphic functions.  $\square$

**Theorem A.3.7** (Lévy's continuity). *Suppose we have a sequence  $(X_n)_n$  of random variables with characteristic functions respectively  $(\phi_n)_n$ . If*

$$\phi_n(t) \rightarrow \phi(t) \quad \text{pointwise,}$$

*then  $X_n \rightarrow X$  in distribution if and only if  $\phi$  is the characteristic function of  $X$ .*

*Proof.* See [32, Theorem 18.1].  $\square$

## A.4 Laplace's Method

Laplace's method is a classical strategy to estimate Gaussian integrals with, approximately, the biggest value that the function assume. It can be used, for example, to prove Stirling's formula.

**Theorem A.4.1** (Laplace's method, [23]). *Consider  $f \in C^2((a, b), \mathbb{R})$  with  $-\infty \leq a \leq b \leq +\infty$ . Assume that  $x_0 \in (a, b)$  is the unique maximum point for  $f$ . Assume also that:*

- 1)  $f''(x_0) < 0$
- 2) For any  $\delta > 0$  there exists  $\eta > 0$  such that

$$f(x) \leq f(x_0) - \eta \quad \text{if } |x - x_0| \geq \delta.$$

- 3) There exists  $N \in \mathbb{N}$  such that

$$\int_a^b e^{Nf(x)} dx < +\infty.$$

Then

$$\lim_{n \rightarrow +\infty} \frac{\int_a^b e^{nf(x)} dx}{e^{nf(x_0)} \sqrt{\frac{2\pi}{-nf''(x_0)}}} = 1.$$

If  $(a, b)$  is finite the statement holds even without hypothesis 2 and 3. If it isn't finite, hypothesis 2 and 3 are sufficient, but not necessary.

**Proof. Lower bound:** By Taylor formula with Lagrange remainder, for  $x \in (x_0 - \delta, x_0 + \delta)$  we have

$$f(x) = f(x_0) + \frac{1}{2}f''(\xi)(x - x_0)^2$$

where  $f'(x_0) = 0$  since  $x_0$  is a maximum point and  $\xi$  is a point between  $x$  and  $x_0$ . By the continuity of  $f''$ , for every  $\varepsilon > 0$  we can choose  $\delta > 0$  such that  $f''(\xi) \geq f''(x_0) - \varepsilon$ . Therefore we have

$$\begin{aligned} \int_a^b e^{nf(x)} dx &\geq e^{nf(x_0)} \int_{x_0-\delta}^{x_0+\delta} e^{\frac{n}{2}(f''(x_0)-\varepsilon)(x-x_0)^2} dx \\ &= e^{nf(x_0)} \sqrt{\frac{2\pi}{n(-f''(x_0) + \varepsilon)}} \cdot \frac{1}{2\pi} \int_{-\delta\sqrt{n(-f''(x_0)+\varepsilon)}}^{\delta\sqrt{n(-f''(x_0)+\varepsilon)}} e^{-\frac{y^2}{2}} dy \end{aligned}$$

from which

$$\liminf_{n \rightarrow +\infty} \frac{\int_a^b e^{nf(x)} dx}{e^{nf(x_0)} \sqrt{\frac{2\pi}{-nf''(x_0)}}} \geq \sqrt{\frac{-f''(x_0)}{-f''(x_0) + \varepsilon}}.$$

From the arbitrariness of  $\varepsilon$  we can conclude that

$$\liminf_{n \rightarrow +\infty} \frac{\int_a^b e^{nf(x)} dx}{e^{nf(x_0)} \sqrt{\frac{2\pi}{-nf''(x_0)}}} \geq 1.$$

**Upper bound:** As for the lower bound, we can find  $\delta > 0$  such that, for any  $x \in (x_0 - \delta, x_0 + \delta)$ , we have that

$$f(x) \leq f(x_0) + \frac{1}{2}(f''(x_0) + \varepsilon)(x - x_0)^2,$$

where we choose  $\varepsilon < -f''(x_0)$ . Then we have

$$\begin{aligned} \int_a^b e^{nf(x)} dx &= \int_a^{x_0-\delta} e^{nf(x)} dx + \int_{x_0-\delta}^{x_0+\delta} e^{nf(x)} dx + \int_{x_0+\delta}^b e^{nf(x)} dx \\ &\leq e^{nf(x_0)} \int_{x_0-\delta}^{x_0+\delta} e^{\frac{n}{2}(f''(x_0)+\varepsilon)(x-x_0)^2} dx + \int_a^b e^{Nf(x)} e^{(n-N)(f(x_0)-\eta)} dx \\ &\leq e^{nf(x_0)} \sqrt{\frac{2\pi}{-n(f''(x_0) + \varepsilon)}} + \int_a^b e^{Nf(x)} e^{(n-N)(f(x_0)-\eta)} dx. \end{aligned}$$

Hence

$$\begin{aligned} &\limsup_{n \rightarrow +\infty} \frac{\int_a^b e^{nf(x)} dx}{e^{nf(x_0)} \sqrt{\frac{2\pi}{-n f''(x_0)}}} \\ &= \limsup_{n \rightarrow +\infty} \left[ \sqrt{\frac{-f''(x_0)}{-f''(x_0) - \varepsilon}} + \sqrt{n} e^{-(n-N)\eta} e^{-Nf(x_0)} \sqrt{\frac{-f''(x_0)}{2\pi}} \int_a^b e^{Nf(x)} dx \right] \\ &= \sqrt{\frac{-f''(x_0)}{-f''(x_0) - \varepsilon}} \end{aligned}$$

and from the arbitrariness of  $\varepsilon$  we obtain that

$$\limsup_{n \rightarrow +\infty} \frac{\int_a^b e^{nf(x)} dx}{e^{nf(x_0)} \sqrt{\frac{2\pi}{-n f''(x_0)}}} \leq 1.$$

□

# Bibliography

- [1] Vladimir Igorevich Arnold. *Mathematical methods of classical mechanics*, volume 60. Springer Science & Business Media, 2013.
- [2] Anatoli V. Babin, Alexei A. Ilyin, and Edriss S. Titi. On the regularization mechanism for the periodic Korteweg-de Vries equation. *Comm. Pure Appl. Math.*, 64(5):591–648, 2011.
- [3] P. Baldi. *Stochastic Calculus: An Introduction Through Theory and Exercises*. Universitext. Springer International Publishing, 2017.
- [4] Dario Bambusi. An introduction to birkhoff normal form. *Dipartimento di Matematica*, 2014.
- [5] J. L. Bona and R. Smith. The initial-value problem for the Korteweg-de Vries equation. *Philos. Trans. Roy. Soc. London Ser. A*, 278(1287):555–601, 1975.
- [6] J. L. Bona and F. B. Weissler. Blowup and ill-posedness for the complex, periodic KdV equation. *Commun. Contemp. Math.*, 25(6):Paper No. 2250044, 57, 2023.
- [7] Joseph Boussinesq. Complément à une étude intitulée «essai sur la théorie des eaux courantes»(publiée dans les tomes xxiii, xxiv du «recueil des savants étrangers»), et à un mémoire «sur l'influence des frottements dans les mouvements réguliers des fluides»(inséré au tome xiii du «journal de mathématiques pures et appliquées», 2e série, 1868). *Journal de Mathématiques Pures et Appliquées*, 4:335–376, 1878.
- [8] Haim Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.
- [9] Giovanni Dematteis, Tobias Grafke, Miguel Onorato, and Eric Vanden-Eijnden. Experimental evidence of hydrodynamic instantons: The universal route to rogue waves. *Phys. Rev. X*, 9:041057, Dec 2019.
- [10] Giovanni Dematteis, Tobias Grafke, and Eric Vanden-Eijnden. Rogue waves and large deviations in deep sea. *Proc. Natl. Acad. Sci. USA*, 115(5):855–860, 2018.
- [11] Boris Dubrovin. Appunti sulla meccanica analitica. *Corso di Laurea Magistrale in Matematica*, 2005/2006.
- [12] M. Burak Erdoğan and Nikolaos Tzirakis. *Dispersive partial differential equations*, volume 86 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2016. Wellposedness and applications.
- [13] Antonio Fasano and Stefano Marmi. *Analytical mechanics*. Oxford Graduate Texts. Oxford University Press, Oxford, 2010. An introduction, Translated from the 2002 Italian edition by Beatrice Pelloni, Reprint of the 2006 translation.

- [14] Clifford S. Gardner. Korteweg-de Vries equation and generalizations. IV. The Korteweg-de Vries equation as a Hamiltonian system. *J. Mathematical Phys.*, 12:1548–1551, 1971.
- [15] Miguel Angel Garrido, Ricardo Grande, Kristin M. Kurianski, and Gigliola Staffilani. Large deviations principle for the cubic NLS equation. *Comm. Pure Appl. Math.*, 76(12):4087–4136, 2023.
- [16] Benoit Grebert. Birkhoff normal form and hamiltonian PDEs, 2006.
- [17] Garrett Heller and Chengyang Shao. Strichartz and multi-linear estimates for the one-dimensional periodic dysthe equation, 2021.
- [18] E. Hewitt and K.A. Ross. *Abstract Harmonic Analysis: Volume I Structure of Topological Groups Integration Theory Group Representations*. Grundlehren der mathematischen Wissenschaften. Springer New York, 1994.
- [19] Edwin Hewitt and Karl Stromberg. *Real and abstract analysis. A modern treatment of the theory of functions of a real variable*. Springer-Verlag, New York, 1965.
- [20] Thomas Kappeler and Jürgen Pöschel. *KDV & KAM*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics. Springer Berlin, Heidelberg, 2003.
- [21] Diederik J. Korteweg and Gustav de Vries. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. *Philos. Mag. (5)*, 39(240):422–443, 1895.
- [22] D. Lannes. *The Water Waves Problem: Mathematical Analysis and Asymptotics*. Mathematical Surveys and Monographs. American Mathematical Society, 2013.
- [23] Pierre Simon Laplace. Mémoire sur la probabilité de causes par les événements. *Mémoire de l'académie royale des sciences*, 1774.
- [24] Peter D. Lax. Periodic solutions of the KdV equation. *Comm. Pure Appl. Math.*, 28:141–188, 1975.
- [25] Peter D. Lax. Almost periodic solutions of the KdV equation. *SIAM Rev.*, 18(3):351–375, 1976.
- [26] Jonathan Luk. Introduction to nonlinear wave equations. *Stanford University*.
- [27] Franco Magri. A simple model of the integrable Hamiltonian equation. *J. Math. Phys.*, 19(5):1156–1162, 1978.
- [28] Robert M. Miura, Clifford S. Gardner, and Martin D. Kruskal. Korteweg-de Vries equation and generalizations. II. Existence of conservation laws and constants of motion. *J. Mathematical Phys.*, 9:1204–1209, 1968.
- [29] Walter Rudin. *Principles of mathematical analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill Book Co., New York-Auckland-Düsseldorf, third edition, 1976.
- [30] M. M. Siddiqui. Some problems connected with Rayleigh distributions. *J. Res. Nat. Bur. Standards Sect. D*, 66D:167–174, 1962.

- [31] Terence Tao. *Nonlinear dispersive equations*, volume 106 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006. Local and global analysis.
- [32] David Williams. *Probability with martingales*. Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge, 1991.