COL100 - Assignment 1

RECURSIVE FUNCTIONS

1) To Show that : factorial $(n) = n!$ $\forall\ n \in \mathbb{N}$

Proof (using PMI version 1)

Basis. For $n=0$, factorial$(n) = 1 = 0!$

by def. of program → mathematical defination

IH let for some $n-1 \geq 0$ we have
factorial$(n-1) = (n-1)!$

IS ∴ for factorial(n):

factorial (n) = $n *$ factorial$(n-1)$
$= n * (n-1)!$ (by IH)
$= n!$

Hence the function factorial implements the factorial function $'n!'$. Proved by PMI

2) To show that : pow$(x,n) = x^n$ $\forall\ n \in \mathbb{N}$

Proof (using PMI version 1)

Basis. For $n=0$ pow$(x,n) = 1 = x^0$

by def. of algo → mathematical def.

IH let for some $n-1 \geq 0$ we have (& for all $x \in \mathbb{R}$)
pow$(x, n-1) = x^{(n-1)}$

IS for pow(x,n)

$$\text{pow}(x,n) = x * \cancel{x}\,\text{pow}(x,n-1)$$
$$= x * x^{n-1} \quad (\text{by IH})$$
$$= x^n$$

Hence the function pow implements the power function x^n. Proved by PMI

~~3)~~ To show that:

3) Fibonnacci is a mathematical defination ∴ it does not require a proof as it is defined that way.

4) Proof for square root

To show that $sqrt(n) = m$ where $m^2 \leq n < (m+1)^2$

PMI version 3

Basis: When $n=0$, $sqrt(n) = 0 = m$

$\because 0^2 = 0$

Induction Hypothesis: For $0 \leq k < n$

$sqrt(k) = l$ where $l^2 \leq k < (l+1)^2$

Induction Step:

Let ~~a = root~~ $a = sqrt\left(\left[\frac{n}{4}\right]\right)$ where $[\cdot]$ is the floor function

$$\Rightarrow a^2 \leq \left[\frac{n}{4}\right] < (a+1)^2$$

$$\Rightarrow 4a^2 \leq 4\left[\frac{n}{4}\right] < (2a+2)^2$$

$$\Rightarrow (2a)^2 \leq 4\left[\frac{n}{4}\right] \leq n < (2a+2)^2 \quad — (1)$$

Case I: $(2a+1)^2 > n$

$$sqrt(n) = 2 \times sqrt\left(\left[\frac{n}{4}\right]\right)$$

$$= 2a$$

Since $(2a)^2 \leq n < (2a+1)^2$ (From (1))

This algorithm gives the correct result

Case II $(2a+1)^2 \leq n$

$$sqrt(n) = 2 \times sqrt\left(\left[\frac{n}{4}\right]\right) + 1$$

$$= 2a + 1$$

Since $(2a+1)^2 \leq n < (2a+2)^2$ (From (1))

Hence this algorithm gives the correct answer

2. Proof of Iterative Power function

TST : For all x, n, c such that $n \geq 0$

$$piter(x,n,c) = c * \prod_{i=1}^{n} x$$

Proof : Using PMI version-1 on $n \geq 0$

Basis : for $n = 0$

$$piter(x,n,c) = c = c * \prod_{i=1}^{0} x = c$$

Induction Hypothesis : let for ~~for~~ some $k \geq 0$

$$piter(x,k,c) = c * \prod_{i=1}^{k} x$$

Induction Step : for some $k+1$

$$piter(x, k+1, c)$$

$$= piter(x, k, c*x)$$

$$= c*x*\prod_{i=1}^{k} x = c * \prod_{i=1}^{k+1} x$$

$\therefore$ $piter(x,n,c) = c * \prod_{i=1}^{n} x$ is proved by PMI

$\therefore$ as $ipow(x,n) = piter(x,n,1) = 1 * \prod_{i=1}^{n} x = x^n$

hence $ipow(n)$ is proved

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Proof of iterative process of fibonacci series

~~fun fib(n) =~~

Invariant : $(n \geq 3) \wedge (3 \leq c \leq n) \wedge (a = fib(c-2) \wedge b = fib(c-1))$

Initialisation: for $n=3$, since $c_0 = 3$

$n = c$

hence $a+b = 2 = fib(3)$

$= fib(1) + fib(2)$

Maintainence: Let for some arbitary step 'c', $a = fib(c-2)$

$b = fib(c-1)$

Hence to prove for $(c+1)^{th}$ step, $a = fib(c-1)$

$b = fib(c)$

~~fun~~ fib_iter(n, fib(c-2), fib(c-1), c+1)

= fib_iter(n, fib(c-1), fib(c-1) + fib(c-2), c+2)

By mathematical defination

$fib(c-1) + fib(c-2) = fib(c)$

hence $a = fib(c-1)$, $b = fib(c)$

Hence proved

Final Step: When $c = n$

$a = fib(n-2)$

$b = fib(n-1)$

hence $fib(n) = a+b = fib(n-2) + fib(n-1)$

LHS = RHS

hence, output will be $fib(n) = fib(n-1) + fib(n-2)$

Iterative

Proof of Iterative factorial function

1) TST: For all m, f, c such that $0 \le c \le m$

$$fact\_iter(m, f, c) = f * \prod_{i=c+1}^{m} i$$

Proof: Using PMI version 1 on $(m-c)$

Basis: $(m-c) = 0$

$$fact\_iter(m, f, c) = f = f * \prod_{i=c+1}^{m} i = f * 1$$

Induction Hypothesis = For some $K = (m-c) \geq 0$

$$fact\_iter(m, f, c) = f * \prod_{i=c+1}^{m} i$$

Induction step: Let $(m-c) = K+1 > 0$ Then

$$fact\_iter(m, f, c) = fact\_iter(m, f*(c+1), c+1)$$
$$= f*(c+1) * \prod_{i=c+2}^{m} i \quad \text{(by IH)}$$
$$= f * \prod_{i=c+1}^{m} i$$

$\therefore$ the invariant is proved

$\therefore$ as $ifact(n) = fact\_iter(n, 1, 0) = 1 . \prod_{i=1}^{n} i = n!$

hence ifact(n) is proved.