Rotational feedback in ***SELEN***
Calculation scheme and Implementation Details

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1 Introduction

The open source program *SELEN* (acronym of SEa Level EquatioN solver) [Spada and Stocchi, 2007; Spada et al., 2012, 2015] solves the so called Sea Level Equation (SLE) following the classical theory of Farrell and Clark [1976] based on the pseudo-spectral approach of Mitrovica and Peltier [1991]. The version *SELEN* 2.9 solves the SLE for a spherical, layered, **non-rotating** Earth with Maxwell visco-elastic rheology.

This user manual introduces our modification of *SELEN* to account for the **rotational feedback (RFB) of a rotating Earth**, which are available as open source as well. The manual includes the fundamental theoretical aspects of RFB in the SLE and describes our implementation in *SELEN*. The manual also provides an installation guide to update *SELEN* 2.9.12 and 2.9.13 for the RFB and shows the effect on the global maps of the `TEST` run of *SELEN*.

This modification was first used in the publication *in prep.* and the prior thesis at Kiel University (Christian-Albrechts-Universität zu Kiel), Germany and the Centre of Earth Evolution and Dynamics, University of Oslo, Norway. When you use this modification, please cite (additionally to Spada and Stocchi [2007]; Spada et al. [2012, 2015]):
in prep.

The code is available at `https://github.com/r-hartmann/RFBupdate_for_SELEN`.

2 Theoretical background: Rotational feedback (RFB) in the sea level equation (SLE)

2.1 Basic terms

The sea level change (SLC) is defined as the difference between the deformations of the equipotential sea surface (also defined as the Geoid) and the solid surface of the Earth [e.g. Farrell and Clark, 1976; Mitrovica and Peltier, 1991; Milne and Mitrovica, 1998]:

$$S(\vartheta, \varphi, t) = \mathcal{O}(\vartheta, \varphi, t) \cdot [N(\vartheta, \varphi, t) - U(\vartheta, \varphi, t)] \tag{1}$$

Here, N is the deformation of the Geoid, U is the radial displacement of the solid Earth, and S is the resulting SLC. ϑ, φ, and t denote longitude, latitude, and time, respectively. $\mathcal{O}$ is the ocean function [Munk and MacDonald, 1960], which limits the sea level to oceanic regions.

Surface load L, which results from ice-ocean mass exchange, combines ice load and ocean load, and hence depends on the ice thickness variation I and SLC S. In *SELEN* the loading series is discretized in equal time steps Δt over the loading period, so that $t_i = i \cdot \Delta t$ for $i = 0, ..., N_t + 1$:

$$L_i(\vartheta, \varphi) = L(\vartheta, \varphi, t_i) = \sum_{i'=0}^{N_t} [\rho_{ice}(I_{i'+1}(\vartheta, \varphi) - I_{i'}(\vartheta, \varphi)) + \rho_w(S_{i'+1}(\vartheta, \varphi) - S_{i'}(\vartheta, \varphi))] \cdot \Theta(i - i') \tag{2}$$

Here, $\rho_{ice,w}$ are the densities of ice and water and Θ is the Heaviside function.

The software package *SELEN* [Spada and Stocchi, 2007; Spada et al., 2012, 2015] uses the pseudo-spectral sea level equation (SLE) [Mitrovica and Peltier, 1991]. To solve the SLE in a spectral approach, all these quantities $\Psi = \{S, U, N, \mathcal{O}, L, I\}$ can be expressed by their Stokes coefficients $\Psi_{l,m}$ and the spherical harmonic basis functions $Y_{l,m}$ in a typical spherical harmonic expansion [e.g. Milne and Mitrovica, 1998]. In the temporal discretization we have:

$$\Psi_i(\vartheta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \Psi_{l,m,i} \cdot Y_{l,m}(\vartheta, \varphi) \tag{3}$$

The pseudo-spectral approach is based on a Maxwell visco-elastic Earth model. Such models can be expressed in the non-dimensional load Love numbers corresponding to a radial 1D structure of the Earth [Farrell, 1972; Peltier, 1974]. The load Love numbers [Love, 1911] define the general impulse response of a visco-elastic Earth to any surface load. In order to determine the radial displacements of solid Earth surface U, Geoid surface N and the related sea level change S, the required h^{L} and k^{L} load Love numbers are defined as [e.g. Peltier and Andrews, 1976]:

$$\begin{aligned} h_l^{\mathrm{L}}(t) =& h_{l,\mathrm{el}}^{\mathrm{L}}\delta(t) + \sum_{j=1}^{J} h_{l,j}^{\mathrm{L}} \exp(-\frac{t}{\tau_{l,j}}) \\ k_l^{\mathrm{L}}(t) =& k_{l,\mathrm{el}}^{\mathrm{L}}\delta(t) + \sum_{j=1}^{J} k_{l,j}^{\mathrm{L}} \exp(-\frac{t}{\tau_{l,j}}) \end{aligned} \tag{4}$$

Here, l is the spherical harmonic degree, t is the time, and δ is the Dirac distribution. The total Love number includes an instantaneous elastic response $\{h, k\}_{l,\mathrm{el}}^{\mathrm{L}}$ and a viscous response of J modes each characterized by a viscous residual $\{h, k\}_{l,j}^{\mathrm{L}}$ and the relaxation time $\tau_{l,j}$. The superscript L indicates the loading type of the Love numbers.

In addition, the ℓ Love number can be used to approximate elastic and viscous responses for the horizontal displacement of the solid surface V:

$$\ell_l^{\mathrm{L}}(t) = \ell_{l,\mathrm{el}}^{\mathrm{L}}\delta(t) + \sum_{j=1}^{J} \ell_{l,j}^{\mathrm{L}} \exp(-\frac{t}{\tau_{l,j}}) \tag{5}$$

2.2 The SLE on a non-rotating Earth - as in *SELEN*

SELEN 2.9.12 [Spada and Stocchi, 2007; Spada et al., 2012, 2015] includes an iterative solver for the pseudo-spectral form of the SLE on an incompressible non-rotating Earth. It solves the SLE up to spherical harmonic degree l_{max} under the assumption of fixed shorelines ($\mathcal{O}(\vartheta,\varphi,t) = \mathcal{O}_{\mathrm{present}}(\vartheta,\varphi)$), which implies a constant area of the ocean and hence omits the effects of shoreline and grounding line migration [e.g. Kendall et al., 2005]. *SELEN* splits the discretized SLE into the contributions of ice load $\mathcal{A}$ and eustatic SLC S^{eust}, which are analytically solvable, and ocean load $\mathcal{B}$, which has to be solved iteratively. Then, the four deformations $\Psi = \{S, U, N, V\}$ are computed following [Spada and Stocchi, 2007; Spada et al., 2015]:

$$\boxed{\begin{aligned}
&\Psi_{l,m,i} = \mathcal{A}_{l,m,i} + \Psi_{l,m,i}^{eust} + \mathcal{B}_{l,m,i} \\
&\Psi_{l,m,i}^{eust} = \begin{cases} -\frac{\rho_{ice}}{\rho_w}\frac{I_{0,0,i}}{\mathcal{O}_{0,0}}\delta_{l,m=0} & : \Psi = S, N \\ 0 & : \Psi = U, V \end{cases} \\
&\mathcal{A}_{l,m,i} = a_l E_l^{\mathrm{L},\Psi}\rho_{ice} I_{l,m,i} + a_l \sum_{i' \le i} \beta_{l,i-i'}^{\mathrm{L},\Psi}\rho_{ice}(I_{l,m,i-i'} - I_{l,m,i-i'-1}) \\
&\mathcal{B}_{l,m,i} = a_l E_l^{\mathrm{L},\Psi}\rho_w S_{l,m,i} + a_l \sum_{i' \le i} \beta_{l,i-i'}^{\mathrm{L},\Psi}\rho_w(S_{l,m,i-i'} - S_{l,m,i-i'-1})
\end{aligned}} \tag{6}$$

Here, a_l is a general prefactor depending on the spherical harmonic degree l and the Earth density ρ_e:

$$a_l = \frac{3}{\rho_e}\frac{1}{(2l+1)} \tag{7}$$

The elastic responses of loading type $E_l^{\mathrm{L},\Psi}$ are defined as:

$$\begin{aligned}
E_l^{\mathrm{L},N} &= 1 + k_{l,\mathrm{el}}^{\mathrm{L}} \\
E_l^{\mathrm{L},U} &= h_{l,\mathrm{el}}^{\mathrm{L}} \\
E_l^{\mathrm{L},S} &= 1 + k_{l,\mathrm{el}}^{\mathrm{L}} - h_{l,\mathrm{el}}^{\mathrm{L}} \\
E_l^{\mathrm{L},V} &= \ell_{l,\mathrm{el}}^{\mathrm{L}}
\end{aligned} \tag{8}$$

And the viscous responses of loading type $\beta_{l,i}^{\mathrm{L},\Psi}$ are given by:

$$\begin{aligned}
\beta_{l,i}^{\mathrm{L},N} &= \sum_{j=1}^{J} k_{l,j}^{\mathrm{L}} \tau_{l,j} \cdot [1 - \exp(-\frac{i \cdot \Delta t}{\tau_{l,j}})] \\
\beta_{l,i}^{\mathrm{L},U} &= \sum_{j=1}^{J} h_{l,j}^{\mathrm{L}} \tau_{l,j} \cdot [1 - \exp(-\frac{i \cdot \Delta t}{\tau_{l,j}})] \\
\beta_{l,i}^{\mathrm{L},S} &= \sum_{j=1}^{J} (k_{l,j}^{\mathrm{L}} - h_{l,j}^{\mathrm{L}}) \tau_{l,j} \cdot [1 - \exp(-\frac{i \cdot \Delta t}{\tau_{l,j}})] \\
\beta_{l,i}^{\mathrm{L},V} &= \sum_{j=1}^{J} \ell_{l,j}^{\mathrm{L}} \tau_{l,j} \cdot [1 - \exp(-\frac{i \cdot \Delta t}{\tau_{l,j}})]
\end{aligned} \tag{9}$$

2.3 The SLE on a rotating Earth - RFB Extension

Following the derivation of Milne and Mitrovica [1998] the SLE (Eq. 6) extends by a third term $\mathcal{C}$, which involves the perturbations of the rotational potential Λ:

$$\boxed{\begin{aligned}
&\Psi_{l,m,i} = \mathcal{A}_{l,m,i} + \Psi_{l,m,i}^{eust} + \mathcal{B}_{l,m,i} + \mathcal{C}_{l,m,i} \\
&\Psi_{l,m,i}^{eust} = \begin{cases} -\frac{\rho_{ice}}{\rho_w} \frac{I_{0,0,i}}{\mathcal{O}_{0,0}} \delta_{l,m=0} & : \Psi = S, N \\ 0 & : \Psi = U, V \end{cases} \\
&\mathcal{A}_{l,m,i} = a_l E_l^{\mathrm{L},\Psi} \rho_{ice} I_{l,m,i} + a_l \sum_{i' \leq i} \beta_{l,i-i'}^{\mathrm{L},\Psi} \rho_{ice} (I_{l,m,i-i'} - I_{l,m,i-i'-1}) \\
&\mathcal{B}_{l,m,i} = a_l E_l^{\mathrm{L},\Psi} \rho_w S_{l,m,i} + a_l \sum_{i' \leq i} \beta_{l,i-i'}^{\mathrm{L},\Psi} \rho_w (S_{l,m,i-i'} - S_{l,m,i-i'-1}) \\
&\mathcal{C}_{l,m,i} = \frac{1}{g} E_l^{\mathrm{T},\Psi} \Lambda_{l,m,i} + \frac{1}{g} \sum_{i' \leq i} \beta_{l,i-i'}^{\mathrm{T},\Psi} (\Lambda_{l,m,i-i'} - \Lambda_{l,m,i-i'-1})
\end{aligned}} \tag{10}$$

The contribution $\mathcal{C}$ from the rotational potential perturbations follows the same structure as the terms $\mathcal{A}, \mathcal{B}$, but with the elastic and viscous responses of tidal type $E_l^{\mathrm{T},\Psi}, \beta_{l,i}^{\mathrm{T},\Psi}$. Their tidal type is indicated by the superscript T. Also the prefactor is simply the reciprocal surface gravity acceleration g^{-1}. The rotational potential perturbations Λ are discretized in consistency with the loading series L (Eq. 2):

$$\Lambda_i(\vartheta, \varphi) = \Lambda(\vartheta, \varphi, t_i) = \sum_{i'=0}^{N_t} (\Lambda_{i'+1}(\vartheta, \varphi) - \Lambda_{i'}(\vartheta, \varphi)) \cdot \Theta(i - i') \tag{11}$$

2.4 The missing terms

2.4.1 Tidal responses and tidal Love numbers

Analog to the load Love numbers, which describe the general impulse response of a Maxwell visco-elastic Earth model to surface load (Eq. 4), the tidal Love numbers describe the general impulse response of such an Earth to a forcing of a general potential, like a perturbation in the rotational potential [Milne and Mitrovica, 1998]:

$$\begin{aligned}
h_l^{\mathrm{T}}(t) =& h_{l,\mathrm{el}}^{\mathrm{T}}\delta(t) + \sum_{j=1}^{J} h_{l,j}^{\mathrm{T}} \exp(-\frac{t}{\tau_{l,j}}) \\
k_l^{\mathrm{T}}(t) =& k_{l,\mathrm{el}}^{\mathrm{T}}\delta(t) + \sum_{j=1}^{J} k_{l,j}^{\mathrm{T}} \exp(-\frac{t}{\tau_{l,j}}) \\
\ell_l^{\mathrm{T}}(t) =& \ell_{l,\mathrm{el}}^{\mathrm{T}}\delta(t) + \sum_{j=1}^{J} \ell_{l,j}^{\mathrm{T}} \exp(-\frac{t}{\tau_{l,j}})
\end{aligned} \tag{12}$$

The superscript T indicates the tidal type of the total, elastic, and viscous residual Love numbers, which differ from their loading type analogue, whereas the spectrum of relaxation times $\tau_{l,j}$ remains the same for all viscous modes.

For a series of rational potential perturbations (Eq. 11), the elastic and viscous responses $E_l^{\mathrm{T},\Psi}$, $\beta_{l,i}^{\mathrm{T},\Psi}$ for the deformations $\Psi = \{N, U, S\}$ are given in line with the loading type [e.g. Milne and Mitrovica, 1998]:

$$\boxed{\begin{aligned}
E_l^{\mathrm{T},N} &= 1 + k_{l,\mathrm{el}}^{\mathrm{T}} \\
E_l^{\mathrm{T},U} &= h_{l,\mathrm{el}}^{\mathrm{T}} \\
E_l^{\mathrm{T},S} &= 1 + k_{l,\mathrm{el}}^{\mathrm{T}} - h_{l,\mathrm{el}}^{\mathrm{T}} \\
E_l^{\mathrm{T},V} &= \ell_{l,\mathrm{el}}^{\mathrm{T}}
\end{aligned}} \tag{13}$$

$$\boxed{\begin{aligned}
\beta_{l,i}^{\mathrm{T},N} &= \sum_{j=1}^{J} k_{l,j}^{\mathrm{T}}\tau_{l,j} \cdot [1 - \exp(-\frac{i \cdot \Delta t}{\tau_{l,j}})] \\
\beta_{l,i}^{\mathrm{T},U} &= \sum_{j=1}^{J} h_{l,j}^{\mathrm{T}}\tau_{l,j} \cdot [1 - \exp(-\frac{i \cdot \Delta t}{\tau_{l,j}})] \\
\beta_{l,i}^{\mathrm{T},S} &= \sum_{j=1}^{J} (k_{l,j}^{\mathrm{T}} - h_{l,j}^{\mathrm{T}})\tau_{l,j} \cdot [1 - \exp(-\frac{i \cdot \Delta t}{\tau_{l,j}})] \\
\beta_{l,i}^{\mathrm{T},V} &= \sum_{j=1}^{J} \ell_{l,j}^{\mathrm{T}}\tau_{l,j} \cdot [1 - \exp(-\frac{i \cdot \Delta t}{\tau_{l,j}})]
\end{aligned}} \tag{14}$$

2.4.2 Perturbations of the rotational potential

Since the rotational potential is only affecting spherical harmonics at degree $l = 2$, no coefficients of higher degrees need to be considered for its perturbations Λ. In a fixed Cartesian coordinate system with its origin at the Earth's center of mass, where the initial rotation vector is given by $\omega_0 = (0, 0, \Omega)$, the full perturbation on the rotational potential can be expressed only by the spherical harmonic coefficients of degrees $l = 0, 2$ [Lambeck, 1980; Milne and Mitrovica, 1998]:

$$\Lambda(\vartheta, \varphi, t) \approx \Lambda_{0,0}(t) \cdot Y_{0,0}(\vartheta, \varphi) + \sum_{m=-2}^{2} \Lambda_{2,m}(t) \cdot Y_{2,m}(\vartheta, \varphi) \tag{15}$$

The rotational potential perturbations Λ are given in Milne and Mitrovica [1998] and can be approximated by dropping higher order terms of the rotation axis deflection $\partial\vec{\omega}(t)$ [Mitrovica et al., 2001]:

$$
\begin{aligned}
\Lambda_{0,0}(t) &= (r_e\Omega)^2 \frac{1}{3}\left[\partial\omega^2(t) + 2\partial\omega_3(t)\right] && \approx (r_e\Omega)^2 \frac{2}{3}\partial\omega_3(t) \\
\Lambda_{2,0}(t) &= (r_e\Omega)^2 \frac{1}{6\sqrt{5}}\left[\partial\omega_1^2(t) + \partial\omega_2^2(t) - 2\partial\omega_3^2(t) - 4\partial\omega_3(t)\right] && \approx (r_e\Omega)^2 \frac{-2}{3\sqrt{5}}\partial\omega_3(t) \\
\Lambda_{2,1}(t) &= (r_e\Omega)^2 \frac{1}{\sqrt{30}}\left[\partial\omega_1(t)(1+\partial\omega_3(t)) - \mathrm{i}\partial\omega_2(t)(1+\partial\omega_3(t))\right] && \approx (r_e\Omega)^2 \frac{1}{\sqrt{30}}\left[\partial\omega_1(t) - \mathrm{i}\partial\omega_2(t)\right] \\
\Lambda_{2,2}(t) &= (r_e\Omega)^2 \frac{1}{\sqrt{220}}\left[\partial\omega_2^2(t) - \partial\omega_1^2(t) + \mathrm{i}2\partial\omega_1(t)\partial\omega_2(t)\right] && \approx 0 \\
\Lambda_{2,-m}(t) &= (-1)^m \Lambda_{2,m}^{\dagger}(t)
\end{aligned}
\tag{16}
$$

Here, r_e is the Earth radius, Ω is the unperturbed rotation, $\partial\omega_{\{1,2,3\}}(t)$ are the components of the the deflection of the rotation axis (defined by: $\omega_v = \Omega(\delta_{v,3} + \partial\omega_v)$), and i is the imaginary unit. The components of the deflection of the rotation axis are given by [Wu and Peltier, 1984]:

$$
\begin{aligned}
\partial\omega_{\{1,2\}}(t) &= \frac{\Omega}{C_{\mathrm{eq}}\sigma_0}\left[D_1\mathcal{J}_{\{1,2\},3}(t) + D_2\int_0^t \mathcal{J}_{\{1,2\},3}(t')\mathrm{d}t' + \sum_{j=1}^{J-1}\mathcal{E}_j(\mathcal{J}_{\{1,2\},3}(t) * \exp(-\alpha_j t))\right] \\
\partial\omega_3(t) &= \frac{-1}{C_{\mathrm{pol}}}\left[D_1\mathcal{J}_{3,3}(t) + \sum_{j=1}^{J-1} k_{2,j}^{\mathrm{L}}(\mathcal{J}_{i,3}(t) * \exp(-\frac{t}{\tau_{2,j}}))\right]
\end{aligned}
\tag{17}
$$

Here, C_{eq} and C_{pol} are the unperturbed equatorial and polar moment of inertia, σ_0 is the Chandler wobble frequency, $D_1, D_2, \mathcal{E}_j, \alpha_j$ are terms describing elastic and viscous deflections due to the perturbations of the moment of inertia tensor $\mathcal{J}$, $k_{2,j}^{\mathrm{L}}$ the viscous residues of the degree 2 k Love number of loading type, and $\tau_{2,j}$ the relaxation times of degree 2 for the J viscous modes. For a series of equal time steps Δt of Heaviside perturbations in the moment of inertia tensor $\mathcal{J}(t) = \sum_{i=1}^{n} \partial\mathcal{J}_i \cdot \Theta(t - t_i)$ these expressions simplify to:

$$
\begin{aligned}
\partial\omega_{\{1,2\}}(t) &= \frac{\Omega}{C_{\mathrm{eq}}\sigma_0}\sum_{t_{i'}\leq t}\left[D_1\delta(t-t_{i'}) + D_2\Delta t + D_3(t-t_{i'}) - D_3(t-t_{i'}-\Delta t)\right]\cdot\mathcal{J}_{\{1,2\},3}(t_{i'}) \\
&= \frac{\Omega}{C_{\mathrm{eq}}\sigma_0}\left[D_1\delta(t) + D_2\Delta t + D_3(t) - D_3(t-\Delta t)\right] * \mathcal{J}_{\{1,2\},3}(t) \\
\partial\omega_3(t) &= \frac{-1}{C_{\mathrm{pol}}}\sum_{t_{i'}\leq t}\left[D_1\delta(t-t_{i'}) + D_4(t-t_{i'}) - D_4(t-t_{i'}-\Delta t)\right]\cdot\mathcal{J}_{3,3}(t_{i'}) \\
&= \frac{-1}{C_{\mathrm{pol}}}\left[D_1\delta(t) + D_4(t) - D_4(t-\Delta t)\right] * \mathcal{J}_{3,3}(t)
\end{aligned}
\tag{18}
$$

In this case the terms $D_1, D_2, D_3(t-t_{i'}), D_4(t-t_{i'})$ are defined as [Mitrovica et al., 2005]:

$$\begin{aligned}
D_1 &= 1 + k^{\mathrm{L}}_{2,\mathrm{el}} \\
D_2 &= \left[1 + k^{\mathrm{L}}_{2,\mathrm{el}} + \sum_{j=1}^{J} k^{\mathrm{L}}_{2,j}\tau_{2,j}\right] \frac{\prod_{j=1}^{J}\tau_{2,j}^{-1}}{\prod_{j=1}^{J-1}\alpha_j} \\
D_3(t-t_{i'}) &= \sum_{j=1}^{J-1} \frac{\mathcal{E}_j}{\alpha_j}(1-\exp(-\alpha_j(t-t_{i'}))) \\
D_4(t-t_{i'}) &= \sum_{j=1}^{J-1} k^{\mathrm{L}}_{2,j}\tau_{2,j}(1-\exp(-\frac{t-t_{i'}}{\tau_{2,j}}))
\end{aligned} \tag{19}$$

The terms $\mathcal{E}_j$ is given by [Milne and Mitrovica, 1998]:

$$\mathcal{E}_j = -\frac{\frac{1}{\alpha_j}\left[1 + k^{\mathrm{L}}_{2,\mathrm{el}} + \sum\limits_{v=1}^{J} k^{\mathrm{L}}_{2,v}\tau_{2,j}\right]\prod\limits_{v=1}^{J}(\tau_{2,v}^{-1}-\alpha_j) + \sum\limits_{w=1}^{J}\left[k^{\mathrm{L}}_{2,w}\tau_{2,w}\prod\limits_{v\neq w}(\tau_{2,v}^{-1}-\alpha_j)\right]}{\prod\limits_{w\neq j}^{J-1}(\alpha_w-\alpha_j)} \tag{20}$$

The parameters α_j are inverse decay times represented by the $J-1$ roots of the polynomial $Q(x)$ [Milne and Mitrovica, 1998]:

$$Q(x) = \frac{\sum\limits_{j=1}^{J}\left[k^{\mathrm{T}}_{2,j}\tau_{2,j}\prod\limits_{v\neq j}(x+\tau_{2,v}^{-1})\right]}{\sum\limits_{j=1}^{J} k^{\mathrm{T}}_{2,j}\tau_{2,j}} \tag{21}$$

The load induced perturbations in the moment of inertia tensor are given by [Wu and Peltier, 1984]:

$$\begin{aligned}
\mathcal{J}_{1,3}(t) &= \frac{4}{3}\sqrt{\frac{6}{5}}\pi r_e^4 \Re(L_{2,1}(t)) \\
\mathcal{J}_{2,3}(t) &= -\frac{4}{3}\sqrt{\frac{6}{5}}\pi r_e^4 \Im(L_{2,1}(t)) \\
\mathcal{J}_{3,3}(t) &= \frac{8}{3}\pi r_e^4\left[L_{0,0}(t) - \frac{1}{\sqrt{5}}L_{2,0}(t)\right]
\end{aligned} \tag{22}$$

Combining Eq. 16, 18, and 22 yields:

$$\begin{aligned}
\Lambda_{0,0}(t) &= \lambda_{0,0}(t) * \left[L_{0,0}(t) - \frac{1}{\sqrt{5}}L_{2,0}(t)\right] \\
\Lambda_{2,0}(t) &= \lambda_{2,0}(t) * \left[L_{0,0}(t) - \frac{1}{\sqrt{5}}L_{2,0}(t)\right] \\
\Lambda_{2,1}(t) &= \lambda_{2,1}(t) * L_{2,1}(t)
\end{aligned} \tag{23}$$

With the time- and degree-dependent prefactors:

$$\begin{aligned}
\lambda_{0,0}(t) &= -\frac{16}{9}\pi\frac{r_e^6\Omega^2}{C_{\mathrm{pol}}}\left[D_1\delta(t) + D_4(t) - D_4(t-\Delta t)\right] \\
\lambda_{2,0}(t) &= \frac{16}{9\sqrt{5}}\pi\frac{r_e^6\Omega^2}{C_{\mathrm{pol}}}\left[D_1\delta(t) + D_4(t) - D_4(t-\Delta t)\right] \\
\lambda_{2,1}(t) &= -\frac{4}{15}\pi\frac{r_e^6\Omega^3}{C_{\mathrm{eq}}\sigma_0}\left[D_1\delta(t) + D_2\Delta t + D_3(t) - D_3(t-\Delta t)\right]
\end{aligned} \tag{24}$$

In the consistent temporal discretization with separated ice and ocean load, the full form of the rotational potential perturbations are:

$$\boxed{\begin{aligned}
\Lambda_{0,0,i} &= \lambda_{0,0,i} * \left[\rho_{ice}\left(I_{0,0,i} - \frac{1}{\sqrt{5}}I_{2,0,i}\right) + \rho_w\left(S_{0,0,i} - \frac{1}{\sqrt{5}}S_{2,0,i}\right)\right] \\
\Lambda_{2,0,i} &= \lambda_{2,0,i} * \left[\rho_{ice}\left(I_{0,0,i} - \frac{1}{\sqrt{5}}I_{2,0,i}\right) + \rho_w\left(S_{0,0,i} - \frac{1}{\sqrt{5}}S_{2,0,i}\right)\right] \\
\Lambda_{2,1,i} &= \lambda_{2,1,i} * \left[\rho_{ice}I_{2,1,i} + \rho_w S_{2,1,i}\right]
\end{aligned}} \tag{25}$$

$$\boxed{\begin{aligned}
\lambda_{0,0,i} &= -\frac{16}{9}\pi\frac{r_e^6\Omega^2}{C_{\text{pol}}}\left[D_1\delta_i + D_{4,i} - D_{4,i-1}\right] \\
\lambda_{2,0,i} &= \frac{16}{9\sqrt{5}}\pi\frac{r_e^6\Omega^2}{C_{\text{pol}}}\left[D_1\delta_i + D_{4,i} - D_{4,i-1}\right] \\
\lambda_{2,1,i} &= -\frac{4}{15}\pi\frac{r_e^6\Omega^3}{C_{\text{eq}}\sigma_0}\left[D_1\delta_i + D_2\Delta t + D_{3,i} - D_{3,i-1}\right]
\end{aligned}} \tag{26}$$

A simplified expression of the perturbations in terms of ice and ocean load can be given by:

$$\Lambda_{l,m,i} = \lambda_{l,m,i} * \left[\rho_{ice}I^{\Lambda}_{l,m,i} + \rho_w S^{\Lambda}_{l,m,i}\right] \tag{27}$$

Here, $I^{\Lambda}_{l,m,i}$ and $S^{\Lambda}_{l,m,i}$ involve the corresponding combination of ice and ocean loads, respectively.

2.5 The final scheme

Splitting the rotational potential perturbations Λ into the contributions of ice and ocean load (Eq. 27), the rotational feedback $\mathcal{C}$ (Eq. 10) can directly be involved in the ice term $\mathcal{A}$ and the ocean term $\mathcal{B}$ of the SLE:

$$\boxed{\begin{aligned}
&\Psi_{l,m,i} = \mathcal{A}_{l,m,i} + \Psi^{eust}_{l,m,i} + \mathcal{B}_{l,m,i} \\
&\Psi^{eust}_{l,m,i} = \begin{cases} -\frac{\rho_{ice}}{\rho_w}\frac{I_{0,0,i}}{\mathcal{O}_{0,0}}\delta_{l,m=0} & : \Psi = S, N \\ 0 & : \Psi = U, V \end{cases} \\
&\mathcal{A}_{l,m,i} = a_l E^{\text{L},\Psi}_l \rho_{ice} I_{l,m,i} \qquad + a_l \sum_{i' \le i} \beta^{\text{L},\Psi}_{l,i-i'}\rho_{ice}(I_{l,m,i-i'} - I_{l,m,i-i'-1}) \\
&\qquad + \frac{1}{g}E^{\text{T},\Psi}_l \rho_{ice}\lambda_{l,m,i} * I^{\Lambda}_{l,m,i} \qquad + \frac{1}{g}\sum_{i' \le i} \beta^{\text{T},\Psi}_{l,i-i'}\rho_{ice}(\lambda_{l,m,i-i'} * I^{\Lambda}_{l,m,i-i'} - \lambda_{l,m,i-i'-1} * I^{\Lambda}_{l,m,i-i'-1}) \\
&\mathcal{B}_{l,m,i} = a_l E^{\text{L},\Psi}_l \rho_w S_{l,m,i} \qquad + a_l \sum_{i' \le i} \beta^{\text{L},\Psi}_{l,i-i'}\rho_w(S_{l,m,i-i'} - S_{l,m,i-i'-1}) \\
&\qquad + \frac{1}{g}E^{\text{T},\Psi}_l \rho_w\lambda_{l,m,i} * S^{\Lambda}_{l,m,i} \qquad + \frac{1}{g}\sum_{i' \le i} \beta^{\text{T},\Psi}_{l,i-i'}\rho_w(\lambda_{l,m,i-i'} * S^{\Lambda}_{l,m,i-i'} - \lambda_{l,m,i-i'-1} * S^{\Lambda}_{l,m,i-i'-1})
\end{aligned}} \tag{28}$$

3 Implementation details: Inclusion of RFB in *SELEN*

3.1 Original structure of *SELEN* 2.9.12/2.9.13

The original version *SELEN* 2.9.12 (and 2.9.13) is subdivided in several subprograms. The subprogram `sle.f90` is the main part of the solver, which includes the iterative calculation of the pseudo-spectral SLE. It uses the discretized spherical harmonic coefficients of the ice thickness $I_{l,m,i}$ and the loading type responses $E_l^{\mathrm{L},\Psi}$ (Eq. 8) and $\beta_{l,i-i'}^{\mathrm{L}}$ (Eq. 9), which are determined before by `shice.f90` and `tb.F90` (Fig. 1). The subprogram `tb.F90`, which computes the loading type responses $E_l^{\mathrm{L},\Psi}$ and $\beta_{l,i-i'}^{\mathrm{L},\Psi}$, already includes a version of *TABOO* [Spada, 2003], which is able to calculate load and tidal Love numbers from a given stratified Earth structure.

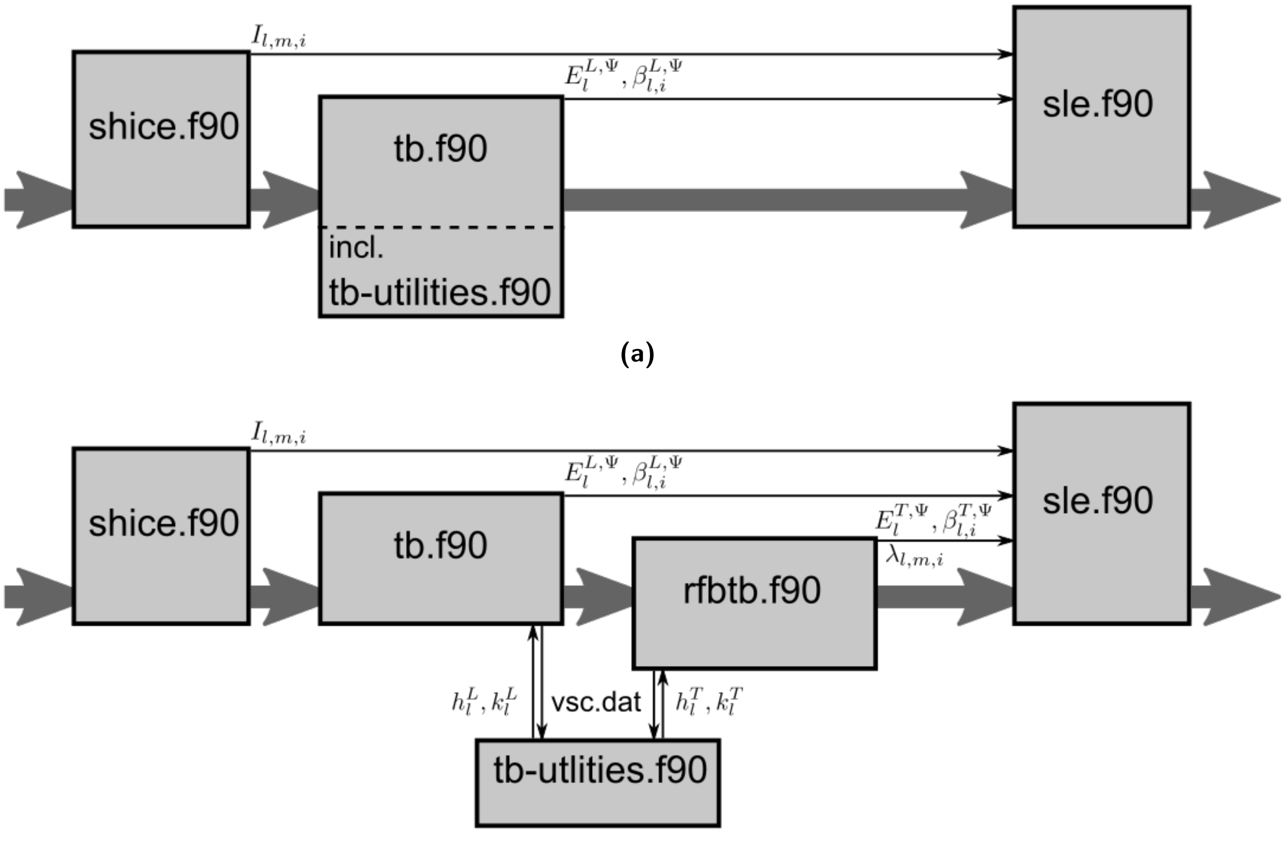

Figure 1
Partial program structure of *SELEN*: (a) Original structure of *SELEN* 2.9.12/2.9.13 for the solution of a non-rotating Earth, (b) Adapted structure with the new subprogram `rfbtb.F90` for the inclusion of RFB. The thick arrows show the sequential order of the different subprograms. The thin arrows show the different inputs respectively outputs of the different subprograms.

3.2 *SELEN* 2.9.12/2.9.13 with RFB

The tidal type responses $E_l^{\mathrm{T},\Psi}$ (Eq. 13) and $\beta_{l,i-i'}^{\mathrm{T},\Psi}$ (Eq. 14), and the coefficients of the rotational potential perturbations $\lambda_{l,m,i}$ (Eq. 26) are computed in a the new subprogram `rfbtb.F90` implemented analog to `tb.F90`. Therefore, the original subprogram `tb.F90` is divided into two parts. The part computing the loading type responses remains as `tb.F90`, but the general code of *TABOO* computing any type of Love numbers is extracted to `tb-utilities.F90`.

The RFB is implemented in the *SELEN* code as an optional feature that the SLE can be solved each time either for a non-rotating or for a rotating Earth. The option to include the computation of RFB in the solution of the SLE has to be set in the configuration file `config.dat`.

In the source code added passages for the RFB are highlighted like:

```
> !  =======================================================
> !  !NEW! ROTATIONAL FEEDBACK !NEW!
> !  =======================================================
> ...
> new code
> ...
> !  =======================================================
> !  =======================================================
```

In the following the adapted subprograms and files are summarized:

config.dat
New line included, option number 131: to set parameter for the inclusion of RFB `'y'`(yes) or `'n'`(no).

Makefile.in
Adapted for the correct compilation of the subprograms `tb.F90`, `tb-utilities.F90`, and `rfbtb.F90` within the new program structure.

config.f90
The program `config.f90` is modified to integrate the new subprogram `rfbtb.F90` in the program structure of *SELEN* (Fig. 1(b)). The parameter of the new RFB option in `config.dat` is now additionally read and stored. If the RFB option is set, `rfbtb.F90` is executed within the sequential order. Also the storage of the new output files from `rfbtb.F90` in the associated output directory `./depot-NAME` is handled in `config.f90`. All files related to the tidal Love numbers and the RFB are saved under `./depot-NAME/Love-Numbers-by-TABOO/tides/`. All additional `.log`-files resulting from `rfbtb.F90` are saved under `./depot-NAME/log/`. Additionally, some extra standard output is added to provide a better report. This includes also that the elapsed run time is displayed for each subprogram.

tb.F90
First part of the original `tb.F90` that computes the arrays of elastic and viscous loading type responses E_l^{L} and $\beta_{l,i-i'}^{\mathrm{L}}$. `tb-utilities.F90` is called to compute the required load Love numbers. Except for the splitting no code is changed.
Output files: `ebs.dat`, `ebu.dat`, `ebn.dat`, `ebv.dat`

tb-utilities.F90
Second part of the original `tb.F90`. This includes all subroutines related to TABOO that computes the load or tidal Love numbers specified by the given input files. Except for the splitting no code is changed.
Output files: `h.dat`, `ih.dat`, `k.dat`, `ik.dat`, `l.dat`, `il.dat`, `spectrum.dat`, `ss.dat`, (optional: `hh.dat`, `ihh.dat`, `kk.dat`, `ikk.dat`, `ll.dat`, `ill.dat`)

rfbtb.F90

New subprogram that computes all load-independent parameters of RFB ($E_l^{\mathrm{T},\Psi}, \beta_{l,i}^{\mathrm{T},\Psi}, \lambda_{l,m,i}$), which are required in the final scheme of the SLE in `sle.f90`. First, the required relaxation spectrum and load Love numbers of degree $l = 2$ are reloaded. Second, the subprogram calls `tb-utilities.F90` for the tidal love Numbers up to degree $l = 2$ and computes the tidal responses $E_l^{\mathrm{T},\Psi}$ and $\beta_{l,i}^{\mathrm{T},\Psi}$ (implemented analog to `tb.F90`). Third, the roots α_j are determined recursively. The recursive search is reported in `lambda_roots.log`. Finally, the time- and degree-dependent prefactors $\lambda_{l,m,i}$ are computed and stored in `rfb.dat`. The algorithm of the new subprogram `rfbtb.F90` is shown in algorithm 1.

Algorithm 1: The algorithm of `rfbtb.F90` to include RFB in the solution of the SLE.

Input : viscosity profile `vsc.dat`
Output: `ebsr.dat`, `ebur.dat`, `ebnr.dat`, `ebvr.dat` ($E_l^{\mathrm{T},\Psi}, \beta_{l,i}^{\mathrm{T},\Psi}$), `rfb.dat` ($\lambda_{l,m,i}$)

if *rfb_opt==y* **then**
  Reload relaxation spectrum and load k Love number of degree $l = 2$: $\tau_{2,j}, k_{2,j}^{\mathrm{L}}, k_{2,\mathrm{el}}^{\mathrm{L}}$;
  Call `tb-utilities.F90` to compute the tidal Love numbers for $l \leq 2$ from `vsc.dat`;
  Build the tidal responses for $\Psi = \{S, U, N, V\}$: $E_l^{\mathrm{T},\Psi}, \beta_{l,i}^{\mathrm{T},\Psi}$;
  Determine the polynomial roots α_j;
  Compute $\mathcal{E}_j$;
  Compute $D_1, D_2, D_{3,i}, D_{4,i}$;
  Compute $\lambda_{l,m,i}$;
  return $E_l^{\mathrm{T},\Psi}, \beta_{l,i}^{\mathrm{T},\Psi}, \lambda_{l,m,i}$;
end

---

sle.f90

Adapted for the final scheme of the SLE (Eq. 28). If the RFB option is set to `'y'`, several modifications are included in the algorithm 2: Before the iterative solver starts, the files including the load-independent parameters of RFB (`ebsr.dat`, `ebur.dat`, `ebnr.dat`, `ebvr.dat` for $E_l^{\mathrm{T},\Psi}, \beta_{l,i}^{\mathrm{T},\Psi}$; `rfb.dat` for $\lambda_{l,m,i}$) are loaded, and the RFB contribution form the ice load $I_{l,m,i}^{\Lambda}$ is calculated. In the iterative solver for the SLC $S_{l,m,i}$ the RFB contribution from ocean load $S_{l,m,i}^{\Lambda}$ is included. After the iterative solver the final RFB contribution from ocean load $S_{l,m,i}^{\Lambda}$ is then used to compute the deformations U, N, V.

Algorithm 2: The algorithm of `sle.f90` with the modifications to include RFB in the solution of the SLE.

Input : $I_{l,m,i}$, $E_l^{\mathrm{L},\Psi}$, $\beta_{l,i}^{\mathrm{L},\Psi}$, $E_l^{\mathrm{T},\Psi}$, $\beta_{l,i}^{\mathrm{T},\Psi}$, $\lambda_{l,m,i}$

Output: `shs.bin`, `shu.bin`, `shn.bin`, `shv.bin` $(\Psi_{l,m,i})$

Load $I_{l,m,i}$, $E_l^{\mathrm{L},\Psi}$, $\beta_{l,i}^{\mathrm{L},\Psi}$;

if *rfb_opt==y* **then**

  Load $E_l^{\mathrm{T},\Psi}$, $\beta_{l,i}^{\mathrm{T},\Psi}$, $\lambda_{l,m,i}$;

  Compute $I_{l,m,i}^{\Lambda}$;

end

Compute $S_{l,m,i}^{\mathrm{eust}}$;

Compute $\mathcal{A}_{l,m,i}$;

if *rfb_opt==y* **then** Add terms of $I_{l,m,i}^{\Lambda}$;

Initialize $S_{l,m,i}^{(\mathrm{it}=0)}$;

if *rfb_opt==y* **then** Initialize $S_{l,m,i}^{\Lambda,(\mathrm{it}=0)}$;

for *iteration* $\mathrm{it} = 1, ..., N_{\mathrm{it}}$ **do**

  Compute $\mathcal{B}_{l,m,i}^{(\mathrm{it}-1)}$;

  if *rfb_opt==y* **then** Add $S_{l,m,i}^{\Lambda,(\mathrm{it}-1)}$;

  Compute $S_{l,m,i}^{(\mathrm{it})} = \mathcal{A}_{l,m,i} + S_{l,m,i}^{\mathrm{eust}} + \mathcal{B}_{l,m,i}^{(\mathrm{it}-1)}$;

  if *rfb_opt==y* **then** Update $S_{l,m,i}^{\Lambda,(\mathrm{it})}$;

end

Compute $U_{l,m,i}$;

if *rfb_opt==y* **then** Add terms of $I_{l,m,i}^{\Lambda}$ and $S_{l,m,i}^{\Lambda}$;

Compute $N_{l,m,i}$;

if *rfb_opt==y* **then** Add terms of $I_{l,m,i}^{\Lambda}$ and $S_{l,m,i}^{\Lambda}$;

Compute $V_{l,m,i}$;

if *rfb_opt==y* **then** Add terms of $I_{l,m,i}^{\Lambda}$ and $S_{l,m,i}^{\Lambda}$;

return $S_{l,m,i}$, $U_{l,m,i}$, $N_{l,m,i}$, and $V_{l,m,i}$;

4 Installation guide: Updated your *SELEN* 2.9 for RFB

1. Download and install *SELEN* 2.9.12 or 2.9.13 from the CIG website, a general installation guide is given in the official *SELEN* user manual:
   https://geodynamics.org/cig/software/selen/

   Note: It is always a good option to create a backup of the original code version of SELEN!!!

2. Copy all `.F90`-files from the repository `src_RFBupdate` into your *SELEN* source folder `./selen2.9.12/src/` (replace original files).

3. Replace the `Makefile.in` in your *SELEN* directory `./selen2.9.12/` with the new `Makefile.in` from the repository.

4. Replace the `config.dat` in your *SELEN* directory `./selen2.9.12/` with the new `config.dat` from the repository.

5. **DONE!** *SELEN* is still executable via `sh configure` and `make run`.

5 A TEST run

The default parameters in the new `config.dat` mirror the `TEST` run of the *SELEN* manual with the RFB option on. The new output in `./depot-TEST/gmaps/` should look like Fig. 2(a), 3(a), 4(a). For comparison the output without RFB (RFB option off - equals original `TEST` run) is shown on the right side (Fig. 2(b), 3(b), 4(b)).

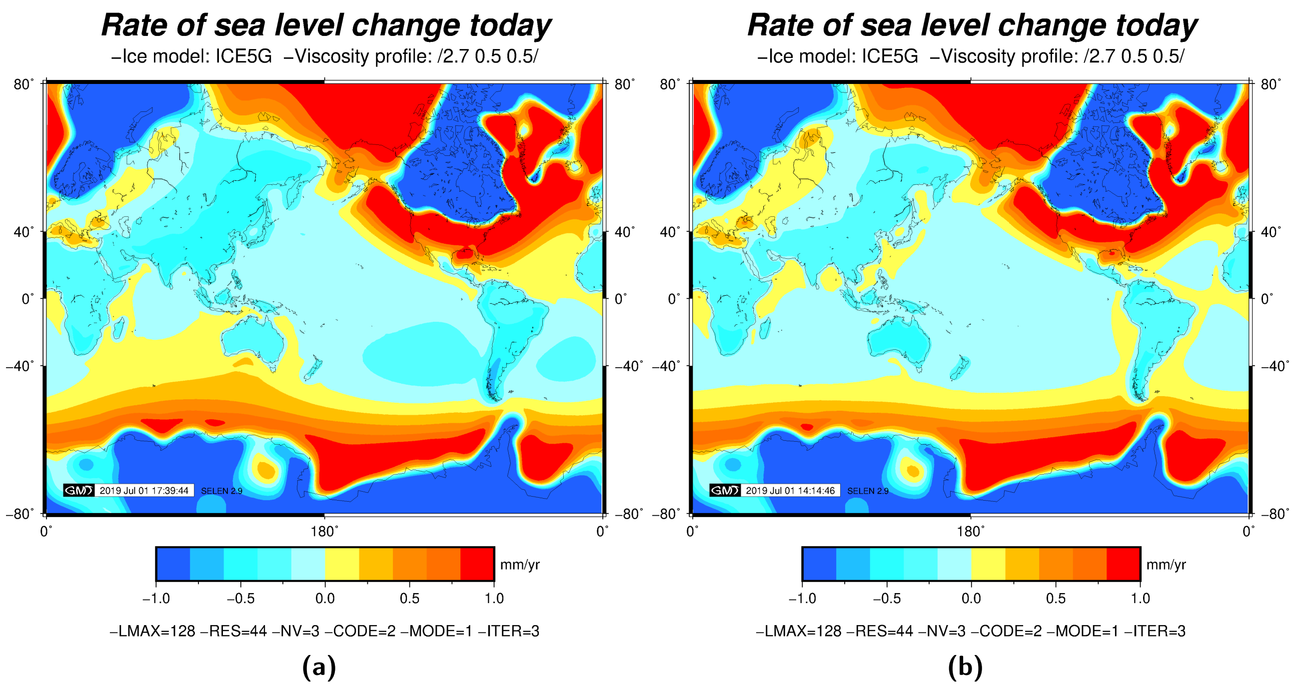

Figure 2
Global maps of present-day rate of sea level change S. Output of the *SELEN* `TEST` run `./depot-TEST/gmaps/sdotmap.pdf`: (a) with rotational feedback, (b) non-rotating case.

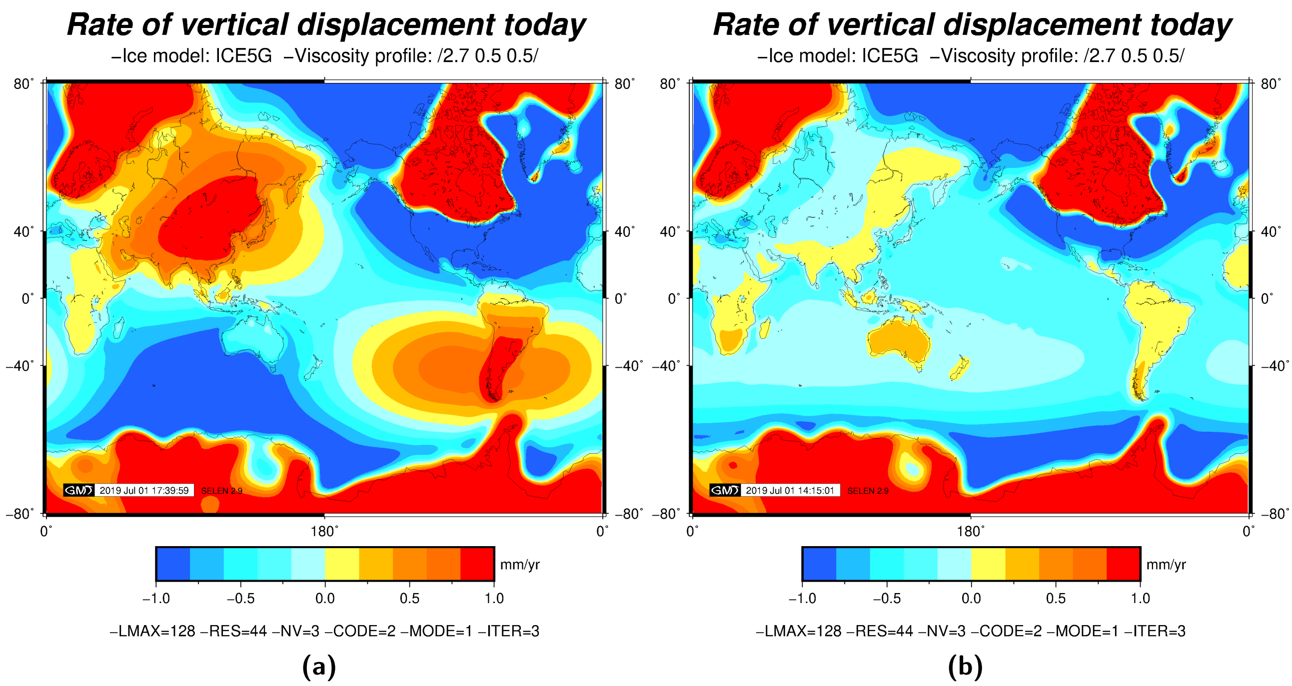

Figure 3
Global maps of present-day rate of solid surface displacement U. Output of the *SELEN* `TEST` run `./depot-TEST/gmaps/udotmap.pdf`: (a) with rotational feedback, (b) non-rotating case.

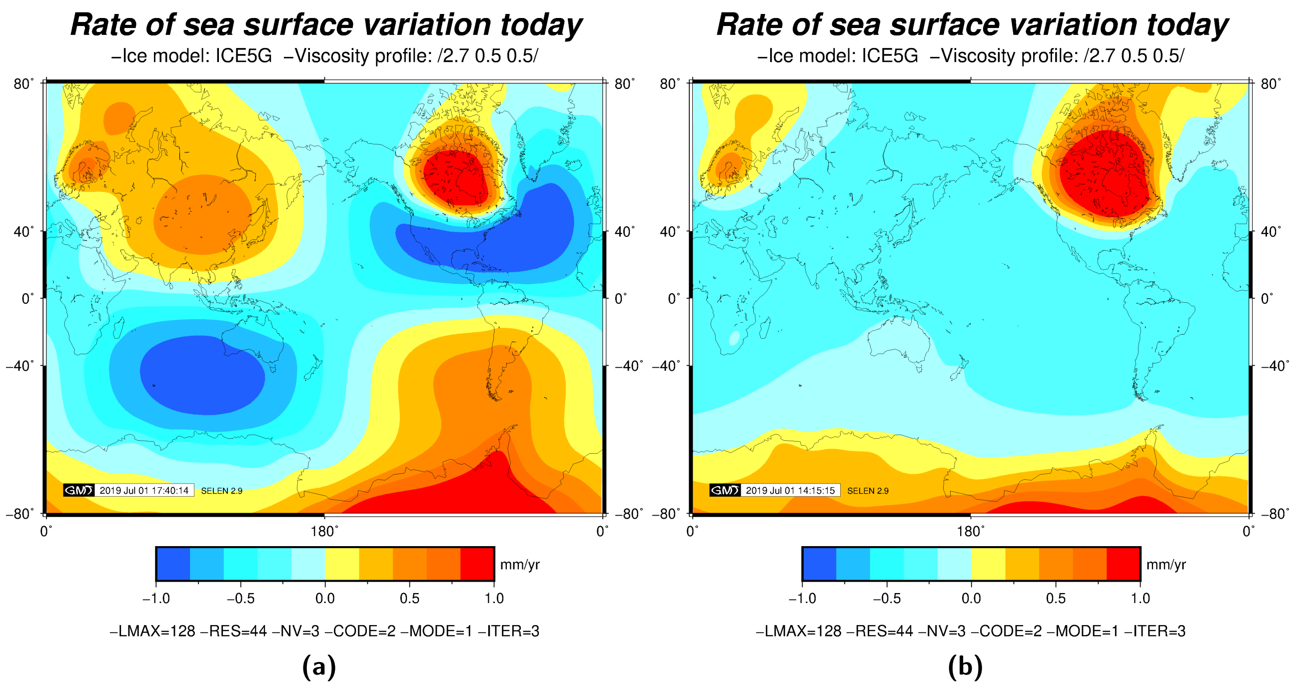

Figure 4
Global maps of present-day rate of sea surface displacement N. Output of the *SELEN* `TEST` run `./depot-TEST/gmaps/ndotmap.pdf`: (a) with rotational feedback, (b) non-rotating case.

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