PHY407 Lab-08 Report

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[1] *Wrote and solved Q1,Q2*

QUESTION 1 - ELECTROSTATICS AND LAPLACES EQUATION

PART (A)

We wish to simulate an electronic capacitor with potential walls at 0V, and two plates positioned parallel to one another within the box at ± 1.0V. We use the Gauss-Seidel method without over-relaxation in order to simulate the potential inside the box. With only the Gauss-Seidel method, the time to run this program took 104.66 seconds to run.

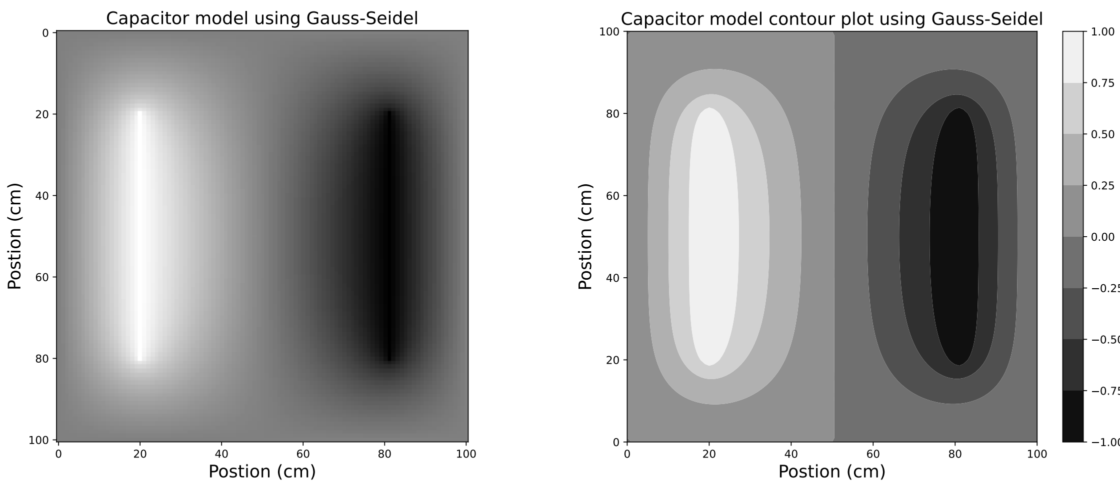


Figure 1: *Left*: The potential of two flat metal plates in a box. On the right we have a plate held at a +1.0V charge, and the right has a -1.0V charge. *Right*; Contour plot of the left figure.

PART (B)

Now we apply over-relaxation with the Gauss-Seidel method in order to speed up the process even further. We can try this with $\omega = 0.1, 0.5$. The time to run the program for $\omega = 0.5$ is 42.18 seconds, and for $\omega = 0.1$ the program took 60.0 seconds.

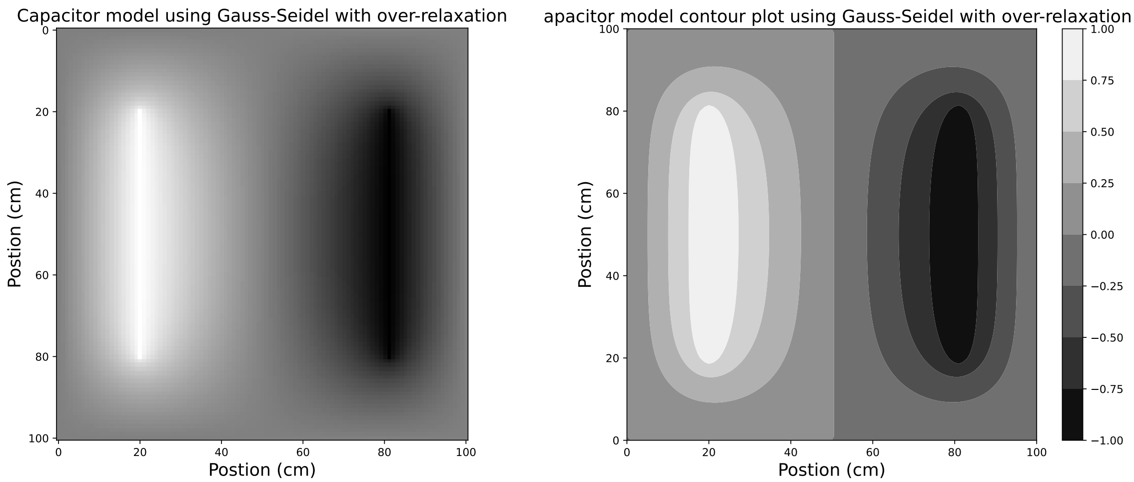


Figure 2: *Left*: The potential of two flat metal plates in a box. We expect this method to result in the same values as using only the Gauss-Seidel, now sped up with the incorporation of over-relaxation. *Right*: The contour plot of the Gauss-Seidel method with over-relaxation.

QUESTION 2 - SIMULATING THE SHALLOW WATER SYSTEM

We use the one-dimensional Naiver=Stokes shallow water equations,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial \eta}{\partial x} \tag{1}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial (uh)}{\partial x} = 0 \tag{2}$$

Where u is the fluid velocity in the x direction, $h = \eta - \eta_b$ is the water column height, η is the altitude of the free surface (water level), η_b is the altitude of the bottom topography, and $g = 9.81\ \mathrm{m/s}^{-1}$ is the acceleration due to gravity.

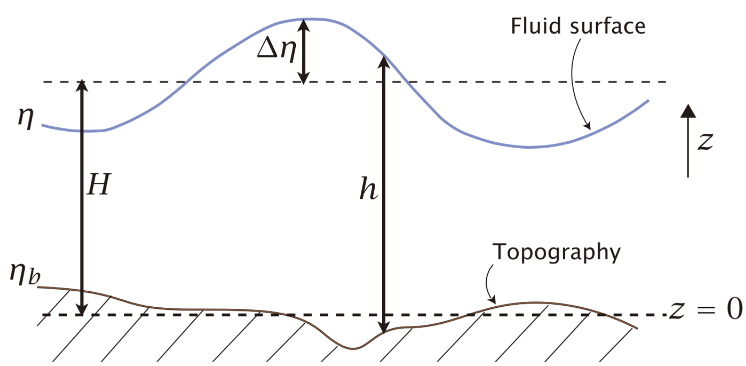


Figure 3: Schematic of shallow water system. The top dashed black line represents the equilibrium position of the fluid surface (solid blue) at $z = \eta(x,t)$ in absence of disturbances. The bottom dashed black line ($z = 0$) represents the fixed bottom topography at $z = \eta_b(x)$. H is then the average in time and space of the water column height and h is the total height between the changing topography of the water and basin surfaces.

PART (A)

We want to show the one-dimensional can be re-written in flux-conservative form,

$$\frac{\partial \vec{u}}{\partial t} = -\frac{\partial \vec{F}(\vec{u})}{\partial x} \tag{3}$$

Where $\vec{u}$ and $\vec{F}$ are vectors consisting of vectors of multiple fields. In our case, $\vec{u} = (u, \eta)$, and therefore $\vec{F} = (u, \eta)$. We note that in flux-conservative form, the left side is an equation with partial derivative with respect to t and the right side is with respect to x. Therefore we solve for the $\partial/\partial t$ term in Equations 1 and 2.

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x} - u\frac{\partial u}{\partial x},$$
$$\frac{\partial \eta}{\partial t} = -\frac{\partial (uh)}{\partial x}$$

We see here by inspection with $h = \eta - \eta_b$, the second line is already in flux-conservative form, so $\vec{F} = (u, \eta) = [u, (\eta - \eta_b)u]$. To find the u component of the $\vec{F}$, we write,

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(g\eta + \frac{1}{2}u^{1/2}\right).$$

And so,

$$\vec{F} = \left[g\eta + \frac{1}{2}u^{1/2}, (\eta - \eta_b)u\right]. \tag{4}$$

We can discretize the shallow water equations using the forward-time centred-space (FTCS) scheme, where one uses a forward difference for the time derivative, and the another uses a centred difference for the spatial derivative. We can write the partials approximately equal to

$$\left.\frac{\partial u}{\partial t}\right|_j^n \approx \frac{1}{\Delta t}(u_j^{n+1} - u_j^n), \tag{5}$$

$$\left.\frac{\partial F}{\partial x}\right|_j^n \approx \frac{1}{2\Delta x}(F_{j+1}^n - F_{j-1}^n). \tag{6}$$

Where n is the time step index, and j is the spatial index. We the substitute the (u, η) components of the $\vec{u}, \vec{F}$ into Equations 5 and 6 and equate them, then solve for the $(n+1)$ terms.

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{2\Delta x}\left[\left(g\eta + \frac{1}{2}u^{1/2}\right)_j^{n+1} - \left(g\eta + \frac{1}{2}u^{1/2}\right)_j^n\right], \tag{7}$$

$$\eta_j^{n+1} = \eta_j^n - \frac{\Delta t}{2\Delta x}\left[\Big((\eta - \eta_b)u\Big)_j^{n+1} - \Big((\eta - \eta_b)u\Big)_j^n\right], \tag{8}$$

are the discrete forms of u, η.

PART (B)

We now implement the shallow water equations using the FTCS scheme. We choose $\eta_b = 0$ the describe the bottom of the basin with domain $L = 0$m to $L = 1$m and $H = 0.1$m. Boundary conditions follow $u(0, t) = u(L, t) = 0$ (rigid walls) and time step $\Delta t = 0.01$ s. The initial conditions are:

$$u(x, 0) = 0, \qquad \eta(x, 0) = H + Ae^{-(x-\mu)2/\sigma^2} - \left\langle Ae^{-(x-\mu)2/\sigma^2}\right\rangle. \tag{9}$$

With $A = 0.002$ m, $\mu = 0.5$ m, $\sigma = 0.05$ m, and the $<>$ is the average operator. We can show the initial conditions of the system by displaying the 0th second.

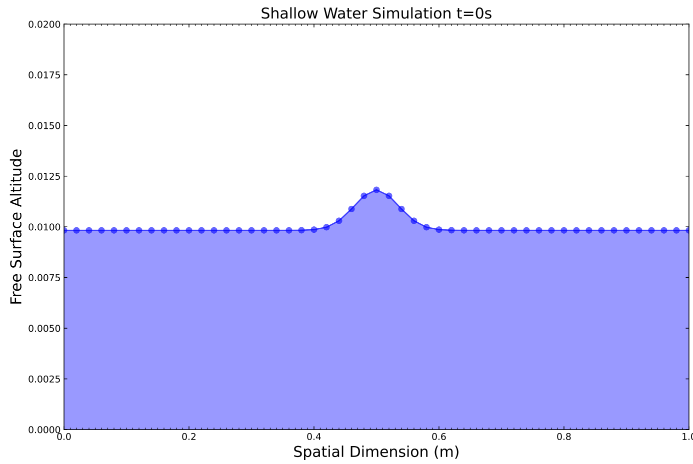
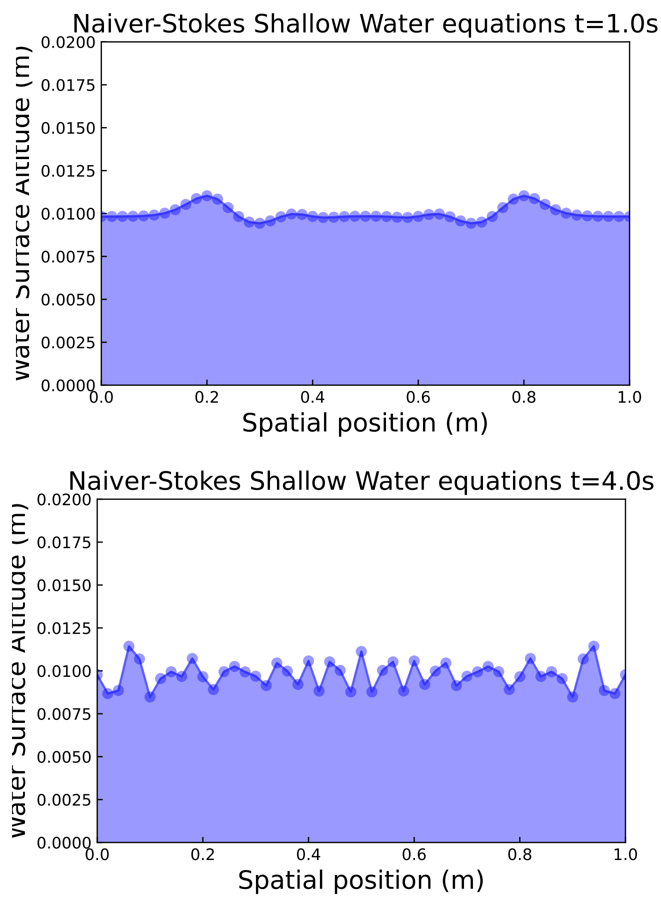


Figure 4: The initial conditions of the shallow water system has a Gaussian-like shape for its initial conditions.



(a) *Top*: The symmetric initial conditions in Figure 4 propagate symmetrically. *Bottom*: By the time out solution reaches the four second mark it it still symmetric, but now with many choppy waves.

PART (C)

Past the four second mark in Figure ??, we saw the behaviour of the solution begin diverging, no longer an accurate depiction of a physical system. We can study the numerical stability of this method using von Neumann stability analysis of the FTCS scheme. To do so, we must must linearize the shallow water equations 1,2 through a Taylor expansion of the first order evaluated at $(0, H)$.

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x}$$
$$\frac{\partial \eta}{\partial t} = -H\frac{\partial \mu}{\partial x}$$

We can express these two equations using the FTCS scheme,

$$u(x, t+\Delta t) = u(x,t) - g\frac{\Delta t}{2\Delta x}(\eta(x+\Delta x, t) - \eta(x-\Delta x, t))$$
$$\eta((x, t+\Delta t) = \eta(x,t) - H\frac{\Delta t}{2\Delta x}(\eta(x+\Delta x, t) - \eta(x-\Delta x, t))$$

The solutions can then be expressed as Fourier coefficients, $C_u(t)e^{ikx}$, which gives,

$$C_u(t+\Delta t) = C_u(t) - g\frac{\Delta t}{\Delta x}C_\eta(t)\sin(f\Delta x)$$
$$C_\eta(t+\Delta t) = C_\eta(t) - H\frac{\Delta t}{\Delta x}C_u(t)\sin(\Delta x)$$

Where the system of equations can be written in matrix form,

$$\vec{C}(t+\Delta) = A\vec{C}(t) \quad \text{with,}$$
$$A = \begin{pmatrix} 1 & -g\frac{\Delta t}{\Delta x}\sin(f\Delta x) \\ -H\frac{\Delta t}{\Delta x}\sin(\Delta x) & 1 \end{pmatrix}$$

From von Neumann analysis, solutions diverge for eigenvalues of the matricx A that are greater than unity. So we calculate the eigenvalues of A.

$$det(A - \lambda I) = (1-\lambda)^2 + gh\left(\frac{\Delta t}{\Delta x}\right)^2 \sin^2(k\Delta x) = 0$$
$$\implies \lambda = 1 \pm i\sqrt{gH\left(\frac{\Delta t}{\Delta x}\right)^2 \sin^2(k\Delta x)}$$
$$|\lambda| = \sqrt{1 + gH\left(\frac{\Delta t}{\Delta x}\right)^2 \sin^2(k\Delta x)} \tag{10}$$

The FTCS scheme on the shallow water system is clearly unstable as it will never be less than one.