

Deep Generative Models

Lecture 9

Roman Isachenko



Ozon Masters

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Recap of previous lecture

ELBO surgery

$$\frac{1}{n} \sum_{i=1}^n \mathcal{L}_i(q, \theta) = \underbrace{\frac{1}{n} \sum_{i=1}^n \mathbb{E}_{q(\mathbf{z}|\mathbf{x}_i)} \log p(\mathbf{x}_i|\mathbf{z}, \theta)}_{\text{Reconstruction loss}} - \underbrace{\mathbb{I}_q[\mathbf{x}, \mathbf{z}]}_{\text{MI}} - \underbrace{KL(q(\mathbf{z})||p(\mathbf{z}))}_{\text{Marginal KL}}$$

Optimal prior

$$KL(q(\mathbf{z})||p(\mathbf{z})) = 0 \quad \Leftrightarrow \quad p(\mathbf{z}) = q(\mathbf{z}) = \frac{1}{n} \sum_{i=1}^n q(\mathbf{z}|\mathbf{x}_i).$$

The optimal prior distribution $p(\mathbf{z})$ is aggregated posterior $q(\mathbf{z})$.

VampPrior

$$p(\mathbf{z}|\boldsymbol{\lambda}) = \frac{1}{K} \sum_{k=1}^K q(\mathbf{z}|\mathbf{u}_k),$$

where $\boldsymbol{\lambda} = \{\mathbf{u}_1, \dots, \mathbf{u}_K\}$ are trainable pseudo-inputs.

Recap of previous lecture

Autoregressive flow prior

$$\log p(\mathbf{z}|\boldsymbol{\lambda}) = \log p(\boldsymbol{\epsilon}) + \log \det \left| \frac{d\boldsymbol{\epsilon}}{d\mathbf{z}} \right|; \quad \mathbf{z} = g(\boldsymbol{\epsilon}, \boldsymbol{\lambda}) = f^{-1}(\boldsymbol{\epsilon}, \boldsymbol{\lambda})$$

Theorem

VAE with the AF prior for latent code \mathbf{z} is equivalent to using the IAF posterior for latent code $\boldsymbol{\epsilon}$.

$$\begin{aligned} \mathcal{L}(q, \theta) &= \mathbb{E}_{\mathbf{z} \sim q(\mathbf{z}|\mathbf{x})} \left[\log p(\mathbf{x}|\mathbf{z}, \theta) + \underbrace{\left(\log p(f(\mathbf{z}, \boldsymbol{\lambda})) + \log \left| \det \frac{\partial f(\mathbf{z}, \boldsymbol{\lambda})}{\partial \mathbf{z}} \right| \right)}_{\text{AF prior}} - \log q(\mathbf{z}|\mathbf{x}) \right] \\ &= \mathbb{E}_{\mathbf{z} \sim q(\mathbf{z}|\mathbf{x})} \left[\log p(\mathbf{x}|\mathbf{z}, \theta) + \log p(f(\mathbf{z}, \boldsymbol{\lambda})) - \underbrace{\left(\log q(\mathbf{z}|\mathbf{x}) - \log \left| \det \frac{\partial f(\mathbf{z}, \boldsymbol{\lambda})}{\partial \mathbf{z}} \right| \right)}_{\text{IAF posterior}} \right] \end{aligned}$$

Recap of previous lecture

LVM

$$p(\mathbf{x}|\theta) = \int p(\mathbf{x}, \mathbf{z}|\theta) d\mathbf{z} = \int p(\mathbf{x}|\mathbf{z}, \theta) p(\mathbf{z}) d\mathbf{z}$$

- ▶ More powerful $p(\mathbf{x}|\mathbf{z}, \theta)$ leads to more powerful generative model $p(\mathbf{x}|\theta)$.
- ▶ Too powerful $p(\mathbf{x}|\mathbf{z}, \theta)$ could lead to posterior collapse: $q(\mathbf{z}|\mathbf{x})$ will not carry any information about \mathbf{x} and close to prior $p(\mathbf{z})$.

Autoregressive decoder

$$p(\mathbf{x}|\mathbf{z}, \theta) = \prod_{i=1}^n p(x_i | \mathbf{x}_{1:i-1}, \mathbf{z}, \theta)$$

- ▶ Global structure is captured by latent variables.
- ▶ Local statistics are captured by limited receptive field autoregressive model.

Recap of previous lecture

Decoder weakening

- ▶ Powerful decoder $p(\mathbf{x}|\mathbf{z}, \boldsymbol{\theta})$ makes the model expressive, but posterior collapse is possible.
- ▶ PixelVAE model uses the autoregressive PixelCNN model with small number of layers to limit receptive field.

KL annealing

$$\mathcal{L}(q, \boldsymbol{\theta}, \beta) = \mathbb{E}_{q(\mathbf{z}|\mathbf{x})} \log p(\mathbf{x}|\mathbf{z}, \boldsymbol{\theta}) - \beta \cdot KL(q(\mathbf{z}|\mathbf{x})||p(\mathbf{z}))$$

Start training with $\beta = 0$, increase it until $\beta = 1$ during training.

Free bits

Ensure the use of less than λ bits of information:

$$\mathcal{L}(q, \boldsymbol{\theta}, \lambda) = \mathbb{E}_{q(\mathbf{z}|\mathbf{x})} \log p(\mathbf{x}|\mathbf{z}, \boldsymbol{\theta}) - \max(\lambda, KL(q(\mathbf{z}|\mathbf{x})||p(\mathbf{z}))).$$

This results in $KL(q(\mathbf{z}|\mathbf{x})||p(\mathbf{z})) \geq \lambda$.

Recap of previous lecture

Disentanglement learning

A disentangled representation is a one where single latent units are sensitive to changes in single generative factors, while being invariant to changes in other factors.

β -VAE

$$\mathcal{L}(q, \theta, \beta) = \mathbb{E}_{q(\mathbf{z}|\mathbf{x})} \log p(\mathbf{x}|\mathbf{z}, \theta) - \beta \cdot KL(q(\mathbf{z}|\mathbf{x})||p(\mathbf{z})).$$

Representations becomes disentangled by setting a stronger constraint with $\beta > 1$. However, it leads to poorer reconstructions and a loss of high frequency details.

ELBO surgery

$$\frac{1}{n} \sum_{i=1}^n \mathcal{L}_i(q, \theta, \beta) = \underbrace{\frac{1}{n} \sum_{i=1}^n \mathbb{E}_{q(\mathbf{z}|\mathbf{x}_i)} \log p(\mathbf{x}_i|\mathbf{z}, \theta)}_{\text{Reconstruction loss}} - \underbrace{\beta \cdot \mathbb{I}_q[\mathbf{x}, \mathbf{z}]}_{\text{MI}} - \underbrace{\beta \cdot KL(q(\mathbf{z})||p(\mathbf{z}))}_{\text{Marginal KL}}$$

DIP-VAE

Disentangled aggregated variational posterior

$$q(\mathbf{z}) = \frac{1}{n} \sum_{i=1}^n q(\mathbf{z}|\mathbf{x}) = \prod_{j=1}^d q(z_j)$$

DIP-VAE Objective

$$\begin{aligned}\mathcal{L}_{\text{DIP}}(q, \theta) &= \frac{1}{n} \sum_{i=1}^n \mathcal{L}_i(q, \theta) - \lambda \cdot KL(q(\mathbf{z})||p(\mathbf{z})) = \\ &= \frac{1}{n} \sum_{i=1}^n [\mathbb{E}_{q(\mathbf{z}|\mathbf{x}_i)} \log p(\mathbf{x}_i|\mathbf{z}, \theta) - KL(q(\mathbf{z}|\mathbf{x}_i)||p(\mathbf{z}))] - \lambda \cdot KL(q(\mathbf{z})||p(\mathbf{z})) = \\ &= \underbrace{\frac{1}{n} \sum_{i=1}^n [\mathbb{E}_{q(\mathbf{z}|\mathbf{x}_i)} \log p(\mathbf{x}_i|\mathbf{z}, \theta)]}_{\text{Reconstruction loss}} - \underbrace{\mathbb{I}_q[\mathbf{x}, \mathbf{z}]}_{\text{MI}} - \underbrace{(1 + \lambda) \cdot KL(q(\mathbf{z})||p(\mathbf{z}))}_{\text{Marginal KL}}\end{aligned}$$

DIP-VAE

$$\mathcal{L}_{\text{DIP}}(q, \theta) = \frac{1}{n} \sum_{i=1}^n \mathcal{L}_i(q, \theta) - \lambda \cdot \underbrace{KL(q(\mathbf{z}) || p(\mathbf{z}))}_{\text{intractable}}$$

Let match the moments of $q(\mathbf{z})$ and $p(\mathbf{z})$:

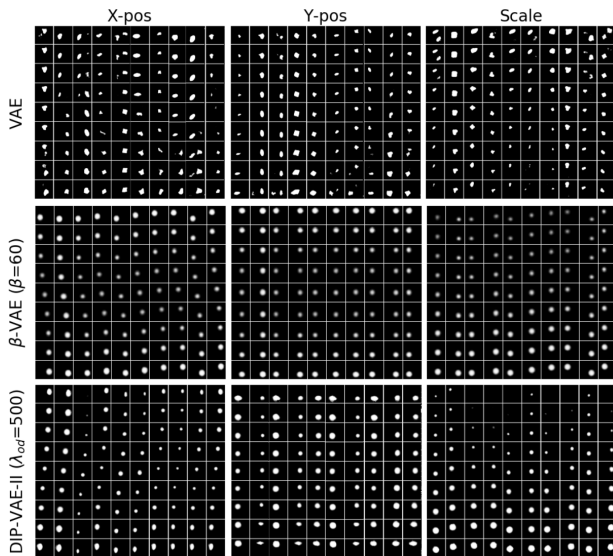
$$\text{cov}_{q(\mathbf{z})}(\mathbf{z}) = \mathbb{E}_{q(\mathbf{z})} \left[(\mathbf{z} - \mathbb{E}_{q(\mathbf{z})}(\mathbf{z}))(\mathbf{z} - \mathbb{E}_{q(\mathbf{z})}(\mathbf{z}))^T \right]$$

DIP-VAE regularizes $\text{cov}_{q(\mathbf{z})}(\mathbf{z})$ to be close to the identity matrix.

Objective

$$\begin{aligned} \max_{q, \theta} & \left[\frac{1}{n} \sum_{i=1}^n \mathcal{L}_i(q, \theta) - \right. \\ & \left. - \lambda_1 \sum_{i \neq j} [\text{cov}_{q(\mathbf{z})}(\mathbf{z})]_{ij}^2 - \lambda_2 \sum_i \left([\text{cov}_{q(\mathbf{z})}(\mathbf{z})]_{ii} - 1 \right)^2 \right] \end{aligned}$$

DIP-VAE



Kumar A., Sattigeri P., Balakrishnan A. *Variational Inference of Disentangled Latent Concepts from Unlabeled Observations*, 2017

Challenging Disentanglement Assumptions

Whether unsupervised disentanglement learning is even possible for arbitrary generative models?

Theorem

For $d > 1$, let $\mathbf{z} \sim P$ denote any distribution which admits a density $p(\mathbf{z}) = \prod_{i=1}^d p(z_i)$. Then, there exists an infinite family of bijective functions $f : \text{supp}(\mathbf{z}) \rightarrow \text{supp}(\mathbf{z})$ such that

- ▶ $\frac{\partial f_i(\mathbf{z})}{\partial z_j} \neq 0$ almost everywhere for all i and j (i.e., \mathbf{z} and $f(\mathbf{z})$ are completely entangled);
- ▶ and $P(\mathbf{z} \leq \mathbf{u}) = P(f(\mathbf{z}) \leq \mathbf{u})$ for all $\mathbf{u} \in \text{supp}(\mathbf{z})$ (i.e., they have the same marginal distribution).

Theorem claims that unsupervised disentanglement learning is impossible for arbitrary generative models with a factorized prior.

Challenging Disentanglement Assumptions

Assume we have $p(\mathbf{z})$ and some $p(\mathbf{x}|\mathbf{z})$ defining a generative model. Consider any unsupervised disentanglement method and assume that it finds a representation that is perfectly disentangled with respect to \mathbf{z} in the generative model.

- ▶ Theorem claims that $\exists \hat{\mathbf{z}} = f(\mathbf{z})$ where $\hat{\mathbf{z}}$ is completely entangled with respect to \mathbf{z} .
- ▶ Since the (unsupervised) disentanglement method only has access to observations \mathbf{x} , it hence cannot distinguish between the two equivalent generative models and thus has to be entangled to at least one of them

$$p(\mathbf{x}) = \int p(\mathbf{x}|\mathbf{z})p(\mathbf{z})d\mathbf{z} = \int p(\mathbf{x}|\hat{\mathbf{z}})p(\hat{\mathbf{z}})d\hat{\mathbf{z}}.$$

Challenging Disentanglement Assumptions

Proof (1)

1. Consider the function $g : \text{supp}(\mathbf{z}) \rightarrow [0, 1]^d$:

$$g_i(\mathbf{u}) = P(z_i \leq u_i), \quad i = 1, \dots, d.$$

- ▶ g is bijective (since $p(\mathbf{z}) = \prod_{i=1}^d p(z_i)$).
- ▶ $\frac{\partial g_i(\mathbf{u})}{\partial u_i} \neq 0$, for all i and $\frac{\partial g_i(\mathbf{u})}{\partial u_j} = 0$ for all $i \neq j$.
- ▶ $g(\mathbf{z})$ is an independent d -dimensional uniform distribution.

2. Consider $h : (0, 1]^d \rightarrow \mathbb{R}^d$

$$h_i(\mathbf{u}) = \psi^{-1}(u_i), \quad i = 1, \dots, d.$$

Here ψ denotes the CDF of a standard normal distribution.

- ▶ h is bijective.
- ▶ $\frac{\partial h_i(\mathbf{u})}{\partial u_i} \neq 0$, for all i and $\frac{\partial h_i(\mathbf{u})}{\partial u_j} = 0$ for all $i \neq j$.
- ▶ $h(g(\mathbf{z}))$ is a d -dimensional standard normal distribution.

Challenging Disentanglement Assumptions

Proof (2)

Let $\mathbf{A} \in \mathbb{R}^{d \times d}$ be an arbitrary orthogonal matrix with $A_{ij} \neq 0$ for all i, j . The family of such matrices is infinite.

- ▶ \mathbf{A} is orthogonal, it is invertible and thus defines a bijective linear operator.
- ▶ $\mathbf{A}h(g(\mathbf{z})) \in \mathbb{R}^d$ is hence an independent, multivariate standard normal distribution.
- ▶ $h^{-1}(\mathbf{A}h(g(\mathbf{z}))) \in \mathbb{R}^d$ is an independent d -dimensional uniform distribution.

Define $f : \text{supp}(\mathbf{z}) \rightarrow \text{supp}(\mathbf{z})$:

$$f(\mathbf{u}) = g^{-1}(h^{-1}(\mathbf{A}h(g(\mathbf{z}))))).$$

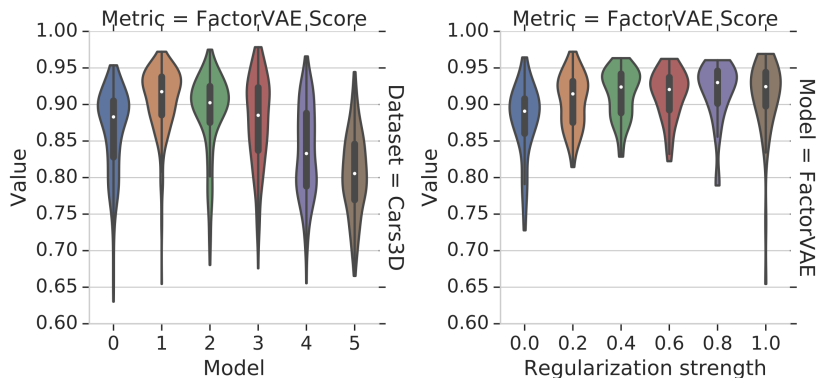
By definition $f(\mathbf{z})$ has the same marginal distribution as \mathbf{z} :

$$P(\mathbf{z} \leq \mathbf{u}) = P(f(\mathbf{z}) \leq \mathbf{u}) \text{ and } \frac{\partial f_i(\mathbf{z})}{\partial z_j} \neq 0.$$

Locatello F. et al. Challenging Common Assumptions in the Unsupervised Learning of Disentangled Representations, 2018

Challenging Disentanglement Assumptions

Importance of different models and hyperparameters for disentanglement



Locatello F. et al. *Challenging Common Assumptions in the Unsupervised Learning of Disentangled Representations*, 2018

Challenging Disentanglement Assumptions

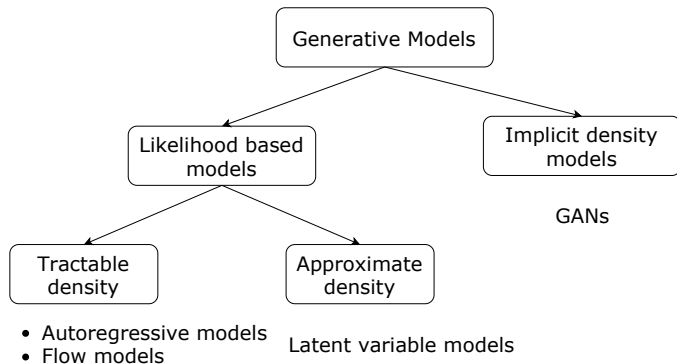
Agreement of different disentanglement metrics

Dataset = Noisy-dSprites

BetaVAE Score (A)	100	80	44	41	46	37
FactorVAE Score (B)	80	100	49	52	25	38
MIG (C)	44	49	100	76	6	42
DCI Disentanglement (D)	41	52	76	100	-8	38
Modularity (E)	46	25	6	-8	100	13
SAP (F)	37	38	42	38	13	100
	(A)	(B)	(C)	(D)	(E)	(F)

Locatello F. et al. *Challenging Common Assumptions in the Unsupervised Learning of Disentangled Representations*, 2018

Generative models zoo



Likelihood based models

Is likelihood a good measure of model quality?

Poor likelihood
Great samples

$$p_1(\mathbf{x}) = \frac{1}{n} \sum_{i=1}^n \mathcal{N}(\mathbf{x}|\mathbf{x}_i, \epsilon \mathbf{I})$$

For small ϵ this model will generate samples with great quality, but likelihood will be very poor.

Great likelihood
Poor samples

$$p_2(\mathbf{x}) = 0.01p(\mathbf{x}) + 0.99p_{\text{noise}}(\mathbf{x})$$

$$\begin{aligned} \log [0.01p(\mathbf{x}) + 0.99p_{\text{noise}}(\mathbf{x})] &\geq \\ &\geq \log [0.01p(\mathbf{x})] = \log p(\mathbf{x}) - \log 100 \end{aligned}$$

Noisy irrelevant samples, but for high dimensions $\log p(\mathbf{x})$ becomes proportional to m .

Likelihood-free learning

- ▶ Likelihood is not a perfect measure quality measure for generative model.
- ▶ Likelihood could be intractable.

Where did we start

We would like to approximate true data distribution $\pi(\mathbf{x})$. Instead of searching true $\pi(\mathbf{x})$ over all probability distributions, learn function approximation $p(\mathbf{x}|\theta) \approx \pi(\mathbf{x})$.

Imagine we have two sets of samples

- ▶ $\mathcal{S}_1 = \{\mathbf{x}_i\}_{i=1}^{n_1} \sim \pi(\mathbf{x})$ – real samples;
- ▶ $\mathcal{S}_2 = \{\mathbf{x}_i\}_{i=1}^{n_2} \sim p(\mathbf{x}|\theta)$ – generated (or fake) samples.

Two sample test

$$H_0 : \pi(\mathbf{x}) = p(\mathbf{x}|\theta), \quad H_1 : \pi(\mathbf{x}) \neq p(\mathbf{x}|\theta)$$

Define test statistic $T(\mathcal{S}_1, \mathcal{S}_2)$. The test statistic is likelihood free. If $T(\mathcal{S}_1, \mathcal{S}_2) < \alpha$, then accept H_0 , else reject it.

Likelihood-free learning

Two sample test

$$H_0 : \pi(\mathbf{x}) = p(\mathbf{x}|\boldsymbol{\theta}), \quad H_1 : \pi(\mathbf{x}) \neq p(\mathbf{x}|\boldsymbol{\theta})$$

Desired behaviour

- ▶ $p(\mathbf{x}|\boldsymbol{\theta})$ minimizes the value of test statistic $T(\mathcal{S}_1, \mathcal{S}_2)$.
- ▶ It is hard to find an appropriate test statistic in high dimensions. $T(\mathcal{S}_1, \mathcal{S}_2)$ could be learnable.

GAN objective

$$\min_G \max_D V(G, D) = \min_G \max_D [\mathbb{E}_{\pi(\mathbf{x})} \log D(\mathbf{x}) + \mathbb{E}_{p(\mathbf{z})} \log(1 - D(G(\mathbf{z})))]$$

- ▶ **Generator:** generative model $\mathbf{x} = G(\mathbf{z})$, which makes generated sample more realistic.
- ▶ **Discriminator:** a classifier $D(\mathbf{x}) \in [0, 1]$, which distinguishes real samples from generated samples.

Vanilla GAN optimality

Theorem

The minimax game

$$\min_G \max_D V(G, D) = \min_G \max_D [\mathbb{E}_{\pi(\mathbf{x})} \log D(\mathbf{x}) + \mathbb{E}_{p(\mathbf{z})} \log(1 - D(G(\mathbf{z})))]$$

has the global optimum $\pi(\mathbf{x}) = p(\mathbf{x}|\theta)$, in this case $D^*(\mathbf{x}) = 0.5$.

Proof (fixed G)

$$\begin{aligned} V(G, D) &= \mathbb{E}_{\pi(\mathbf{x})} \log D(\mathbf{x}) + \mathbb{E}_{p(\mathbf{x}|\theta)} \log(1 - D(\mathbf{x})) \\ &= \int \underbrace{[\pi(\mathbf{x}) \log D(\mathbf{x}) + p(\mathbf{x}|\theta) \log(1 - D(\mathbf{x}))]}_{y(D)} d\mathbf{x} \end{aligned}$$

$$\frac{dy(D)}{dD} = \frac{\pi(\mathbf{x})}{D(\mathbf{x})} - \frac{p(\mathbf{x}|\theta)}{1 - D(\mathbf{x})} = 0 \quad \Rightarrow \quad D^*(\mathbf{x}) = \frac{\pi(\mathbf{x})}{\pi(\mathbf{x}) + p(\mathbf{x}|\theta)}$$

Vanilla GAN optimality

Proof continued (fixed $D = D^*$)

$$\begin{aligned} V(G, D^*) &= \mathbb{E}_{\pi(\mathbf{x})} \log \frac{\pi(\mathbf{x})}{\pi(\mathbf{x}) + p(\mathbf{x}|\theta)} + \mathbb{E}_{p(\mathbf{x}|\theta)} \log \frac{p(\mathbf{x}|\theta)}{\pi(\mathbf{x}) + p(\mathbf{x}|\theta)} \\ &= KL\left(\pi(\mathbf{x}) \parallel \frac{\pi(\mathbf{x}) + p(\mathbf{x}|\theta)}{2}\right) + KL\left(p(\mathbf{x}|\theta) \parallel \frac{\pi(\mathbf{x}) + p(\mathbf{x}|\theta)}{2}\right) - 2 \log 2 \\ &= 2JSD(\pi(\mathbf{x}) \parallel p(\mathbf{x}|\theta)) - 2 \log 2. \end{aligned}$$

Jensen-Shannon divergence (symmetric KL divergence)

$$JSD(\pi(\mathbf{x}) \parallel p(\mathbf{x}|\theta)) = \frac{1}{2} \left[KL\left(\pi(\mathbf{x}) \parallel \frac{\pi(\mathbf{x}) + p(\mathbf{x}|\theta)}{2}\right) + KL\left(p(\mathbf{x}|\theta) \parallel \frac{\pi(\mathbf{x}) + p(\mathbf{x}|\theta)}{2}\right) \right]$$

Could be used as a distance measure!

$$V(G^*, D^*) = -2 \log 2, \quad \pi(\mathbf{x}) = p(\mathbf{x}|\theta).$$

Vanilla GAN optimality

Theorem

The minimax game

$$\min_G \max_D V(G, D) = \min_G \max_D [\mathbb{E}_{\pi(\mathbf{x})} \log D(\mathbf{x}) + \mathbb{E}_{p(\mathbf{z})} \log(1 - D(G(\mathbf{z})))]$$

has the global optimum $\pi(\mathbf{x}) = p(\mathbf{x}|\theta)$, in this case $D^*(\mathbf{x}) = 0.5$.

Proof

for fixed G :

$$D^*(\mathbf{x}) = \frac{\pi(\mathbf{x})}{\pi(\mathbf{x}) + p(\mathbf{x}|\theta)}$$

for fixed $D = D^*$:

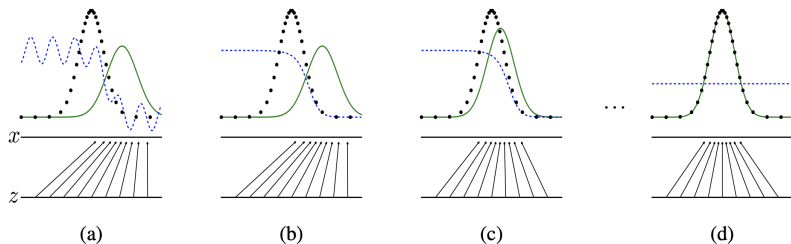
$$\min_G V(G, D^*) = \min_G [2JSD(\pi||p) - \log 4] = -\log 4, \quad \pi(\mathbf{x}) = p(\mathbf{x}|\theta).$$

If the generator could be any function and the discriminator is optimal at every step, then the generator is guaranteed to converge to the data distribution.

Vanilla GAN

Objective

$$\min_G \max_D V(G, D) = \min_G \max_D [\mathbb{E}_{\pi(\mathbf{x})} \log D(\mathbf{x}) + \mathbb{E}_{p(\mathbf{z})} \log(1 - D(G(\mathbf{z})))]$$



- ▶ Generator updates are made in parameter space.
- ▶ Discriminator is not optimal at every step.
- ▶ Generator and discriminator loss keeps oscillating during GAN training.

Summary

- ▶ Majority of disentanglement learning models use heuristic objective or regularizers to achieve the goal, but the task itself could not be solved without good inductive bias.
- ▶ Likelihood is not a perfect criteria to measure quality of generative model.
- ▶ Adversarial learning suggest to solve minimax problem to match the distributions.
- ▶ Vanilla GAN tries to optimize Jensen-Shannon divergence (in theory).