

Deep Generative Models

Lecture 4

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Likelihood-based models so far...

Autoregressive models

$$p(\mathbf{x}|\boldsymbol{\theta}) = \prod_{i=1}^m p(x_i|\mathbf{x}_{1:i-1}, \boldsymbol{\theta})$$

- ▶ tractable likelihood,
- ▶ no inferred latent factors.

Latent variable models

$$p(\mathbf{x}|\boldsymbol{\theta}) = \int p(\mathbf{x}, \mathbf{z}|\boldsymbol{\theta}) d\mathbf{z}$$

- ▶ latent feature representation,
- ▶ intractable likelihood.

How to build model with latent variables and tractable likelihood?

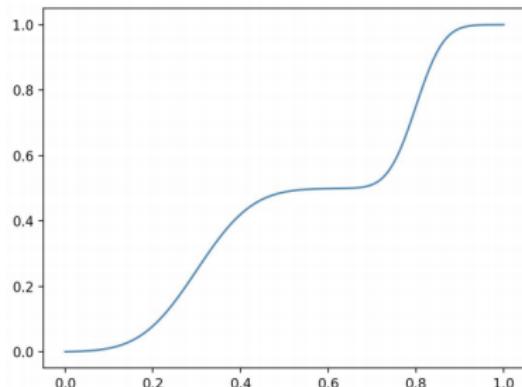
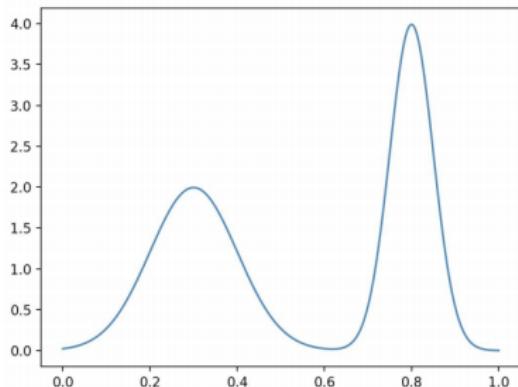
Flows intuition

Let X be a random variable with density $p_X(x)$. Then

$$Z = F(X) = \int_{-\infty}^x p(t)dt \sim U[0, 1].$$

Hence

$$Z \sim U[0, 1]; \quad X = F^{-1}(Z) \quad X \sim p(x).$$



Change of variables

Theorem

Let

- ▶ \mathbf{x} is a random variable,
- ▶ $f : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a differentiable, invertible function,
- ▶ $\mathbf{z} = f(\mathbf{x})$, $\mathbf{x} = f^{-1}(\mathbf{z}) = g(\mathbf{z})$.

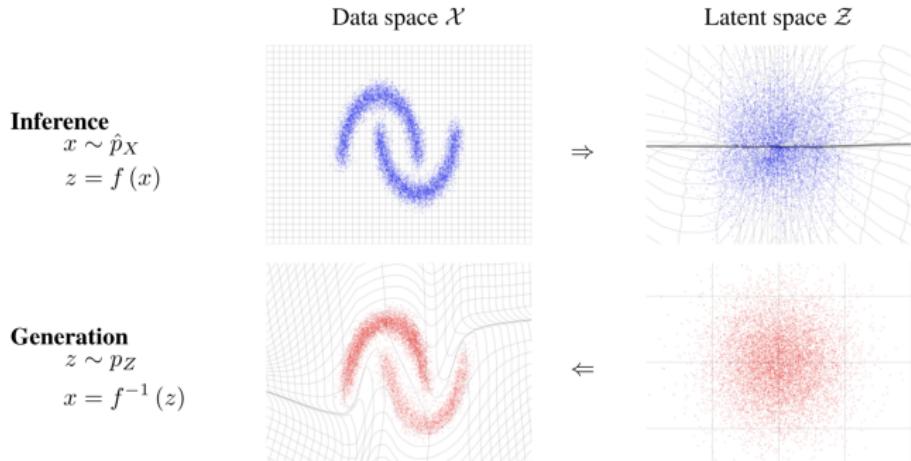
Then

$$p(\mathbf{x}) = p(\mathbf{z}) \left| \det \left(\frac{\partial \mathbf{z}}{\partial \mathbf{x}} \right) \right| = p(f(\mathbf{x})) \left| \det \left(\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \right) \right|.$$

Note

- ▶ \mathbf{x} and \mathbf{z} have the same dimensionality;
- ▶ $\left| \det \left(\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \right) \right| = \left| \det \left(\frac{\partial g^{-1}(\mathbf{x})}{\partial \mathbf{x}} \right) \right| = \left| \det \left(\frac{\partial g(\mathbf{z})}{\partial \mathbf{z}} \right) \right|^{-1}$;
- ▶ $f(\mathbf{x}, \theta)$ could be parametric.

Flows



- ▶ Latent representation is given by $\mathbf{z} = f(\mathbf{x}, \theta)$.
- ▶ Likelihood is given by $\mathbf{z} = f(\mathbf{x}, \theta)$ and change of variables.
- ▶ Sampling of \mathbf{x} is performed by sampling from a base distribution $p(\mathbf{z})$ and applying $\mathbf{x} = f^{-1}(\mathbf{z}, \theta) = g(\mathbf{z}, \theta)$.

Fitting flows

MLE problem

$$\boldsymbol{\theta}^* = \arg \max_{\boldsymbol{\theta}} p(\mathbf{X}|\boldsymbol{\theta}) = \arg \max_{\boldsymbol{\theta}} \prod_{i=1}^n p(\mathbf{x}_i|\boldsymbol{\theta}) = \arg \max_{\boldsymbol{\theta}} \sum_{i=1}^n \log p(\mathbf{x}_i|\boldsymbol{\theta}).$$

Challenge

$p(\mathbf{x}|\boldsymbol{\theta})$ could be intractable.

Fitting flow to solve MLE

$$p(\mathbf{x}|\boldsymbol{\theta}) = p(\mathbf{z}) \left| \det \left(\frac{\partial \mathbf{z}}{\partial \mathbf{x}} \right) \right| = p(f(\mathbf{x}, \boldsymbol{\theta})) \left| \det \left(\frac{\partial f(\mathbf{x}, \boldsymbol{\theta})}{\partial \mathbf{x}} \right) \right|$$

Stacking flows

Let $\mathbf{z} = f(\mathbf{x}) = f_2 \circ f_1(\mathbf{x})$ and f_1, f_2 satisfy conditions of the change of variable theorem (differentiable and invertible).

$$\begin{aligned} p(\mathbf{x}) &= p(\mathbf{z}) \left| \det \left(\frac{\partial \mathbf{z}}{\partial \mathbf{x}} \right) \right| = p(f(\mathbf{x})) \left| \det \left(\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \right) \right| = \\ &= p(f(\mathbf{x})) \left| \det \left(\frac{\partial f_2 \circ f_1(\mathbf{x})}{\partial \mathbf{x}} \right) \right| = p(f(\mathbf{x})) \left| \det \left(\frac{\partial \mathbf{f}_2}{\partial \mathbf{f}_1} \cdot \frac{\partial \mathbf{f}_1}{\partial \mathbf{x}} \right) \right| = \\ &\quad = p(f(\mathbf{x})) \left| \det \left(\frac{\partial \mathbf{f}_2}{\partial \mathbf{f}_1} \right) \right| \cdot \left| \det \left(\frac{\partial \mathbf{f}_1}{\partial \mathbf{x}} \right) \right| \end{aligned}$$

What will we get in the case $\mathbf{z} = f(\mathbf{x}) = f_n \circ \cdots \circ f_1(\mathbf{x})$?

Flows

$$\log p(\mathbf{x}|\boldsymbol{\theta}) = \log p(f(\mathbf{x}, \boldsymbol{\theta})) + \log \left| \det \left(\frac{\partial f(\mathbf{x}, \boldsymbol{\theta})}{\partial \mathbf{x}} \right) \right|$$

Definition

Normalizing flow is a *differentiable, invertible* mapping from data \mathbf{x} to the noise \mathbf{z} .

- ▶ Normalizing - convert the data distribution into a *noise*.
- ▶ Flow - a sequence of such mappings is also a flow

$$\mathbf{z} = f_K \circ \cdots \circ f_1(\mathbf{x}); \quad \mathbf{x} = f_1^{-1} \circ \cdots \circ f_K^{-1}(\mathbf{z}) = g_1 \circ \cdots \circ g_K(\mathbf{z})$$

$$\begin{aligned} p(\mathbf{x}) &= p(f_K \circ \cdots \circ f_1(\mathbf{x})) \left| \det \left(\frac{\partial f_K \circ \cdots \circ f_1(\mathbf{x})}{\partial \mathbf{x}} \right) \right| = \\ &= p(f_K \circ \cdots \circ f_1(\mathbf{x})) \prod_{k=1}^K \left| \det \left(\frac{\partial \mathbf{f}_k}{\partial \mathbf{f}_{k-1}} \right) \right|. \end{aligned}$$

Flows

$$\log p(\mathbf{x}|\boldsymbol{\theta}) = \log p(f(\mathbf{x}, \boldsymbol{\theta})) + \log \left| \det \left(\frac{\partial f(\mathbf{x}, \boldsymbol{\theta})}{\partial \mathbf{x}} \right) \right|$$

What is the complexity of the determinant computation?

What we want

- ▶ Efficient computation of Jacobian $\frac{\partial f(\mathbf{x}, \boldsymbol{\theta})}{\partial \mathbf{x}}$;
- ▶ Efficient sampling from the base distribution $p(\mathbf{z})$;
- ▶ Efficient inversion of $f(\mathbf{x}, \boldsymbol{\theta})$.

Planar Flows

$$g(\mathbf{z}, \theta) = \mathbf{z} + \mathbf{u} h(\mathbf{w}^T \mathbf{z} + b).$$

- ▶ $\theta = \{\mathbf{u}, \mathbf{w}, b\}$;
- ▶ h is a smooth element-wise non-linearity.

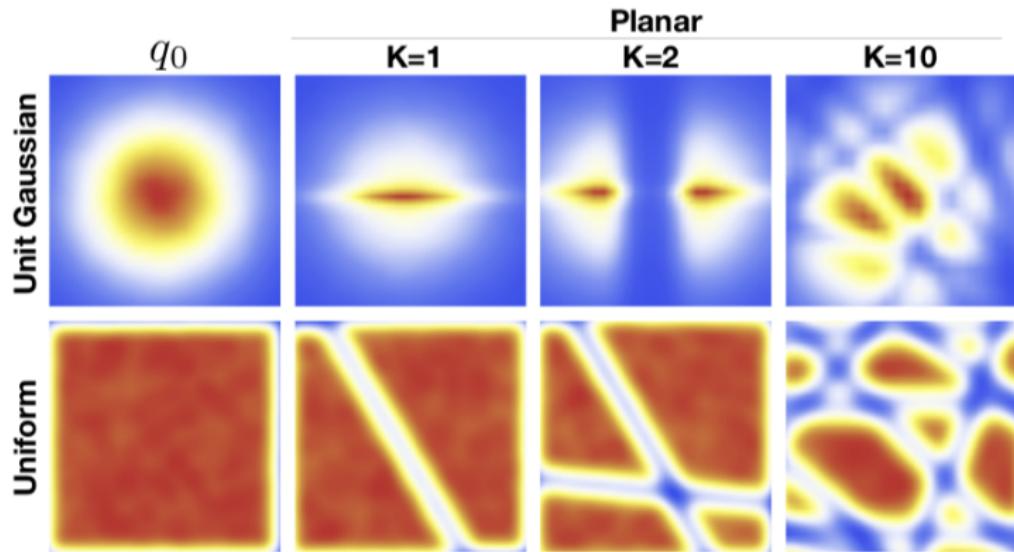
$$\begin{aligned}\left| \det \left(\frac{\partial g(\mathbf{z}, \theta)}{\partial \mathbf{z}} \right) \right| &= \left| \det \left(\mathbf{I} + h'(\mathbf{w}^T \mathbf{z} + b) \mathbf{w} \mathbf{u}^T \right) \right| \\ &= \left| 1 + h'(\mathbf{w}^T \mathbf{z} + b) \mathbf{w}^T \mathbf{u} \right|\end{aligned}$$

The transformation is invertible if (just one of example)

$$h = \tanh; \quad h'(\mathbf{w}^T \mathbf{z} + b) \mathbf{u}^T \mathbf{w} \geq -1.$$

Planar Flows

$$\mathbf{z}_K = g_1 \circ \cdots \circ g_K(\mathbf{z}); \quad g_k = g(\mathbf{z}_k, \theta_k).$$



Jacobian structure

- ▶ What is a determinant of a diagonal matrix?

$$\mathbf{z} = f(\mathbf{x}, \boldsymbol{\theta}) = (f_1(x_1, \boldsymbol{\theta}), \dots, f_m(x_m, \boldsymbol{\theta})).$$

$$\log \left| \det \left(\frac{\partial f(\mathbf{x}, \boldsymbol{\theta})}{\partial \mathbf{x}} \right) \right| = \log \left| \prod_{i=1}^m f'_i(x_i, \boldsymbol{\theta}) \right| = \sum_{i=1}^m \log |f'_i(x_i, \boldsymbol{\theta})|.$$

- ▶ What is a determinant of a triangular matrix?

Let z_i depends only on $\mathbf{x}_{1:i}$ (or without loss of generality x_i depends on $\mathbf{z}_{1:i}$).

What is the inverse of such a transformation?

Coupling layer

$$\begin{cases} \mathbf{z}_{1:d} = \mathbf{x}_{1:d} \\ \mathbf{z}_{d:m} = \tau(\mathbf{x}_{d:m}, c(\mathbf{x}_{1:d})) \end{cases} \quad \begin{cases} \mathbf{x}_{1:d} = \mathbf{z}_{1:d} \\ \mathbf{x}_{d:m} = \tau^{-1}(\mathbf{z}_{d:m}, c(\mathbf{z}_{1:d})) \end{cases}$$

- ▶ $c : \mathbb{R}^d \rightarrow \mathbb{R}^k$ – coupling function;
- ▶ $\tau : \mathbb{R}^{m-d} \times c(\mathbb{R}^d) \rightarrow \mathbb{R}^{m-d}$ – coupling law.
- ▶

$$\det \left(\frac{\partial \mathbf{z}}{\partial \mathbf{x}} \right) = \det \begin{pmatrix} \mathbf{I}_d & \mathbf{0}_{d \times m-d} \\ \frac{\partial \mathbf{z}_{d:m}}{\partial \mathbf{x}_{1:d}} & \frac{\partial \mathbf{z}_{d:m}}{\partial \mathbf{x}_{d:m}} \end{pmatrix} = \det \left(\frac{\partial \mathbf{z}_{d:m}}{\partial \mathbf{x}_{d:m}} \right)$$

Coupling layer

$$\begin{cases} \mathbf{z}_{1:d} = \mathbf{x}_{1:d}; \\ \mathbf{z}_{d:m} = \tau(\mathbf{x}_{d:m}, c(\mathbf{x}_{1:d})); \end{cases} \Rightarrow \begin{cases} \mathbf{x}_{1:d} = \mathbf{z}_{1:d}; \\ \mathbf{x}_{d:m} = \tau^{-1}(\mathbf{z}_{d:m}, c(\mathbf{z}_{1:d})). \end{cases}$$

Coupling function $c(\cdot)$

Any complex function (without restrictions). For example, neural network.

Coupling law $\tau(\cdot, \cdot)$

- ▶ $\tau(x, c) = x + c$ – additive;
- ▶ $\tau(x, c) = x \odot c, c \neq 0$ – multiplicative;
- ▶ $\tau(x, c) = x \odot c_1 + c_2, c_1 \neq 0$ – affine.

To obtain more flexible class of distributions, stack more coupling layers (with different ordering of components!).

NICE

$$\det \left(\frac{\partial \mathbf{z}}{\partial \mathbf{x}} \right) = \det \begin{pmatrix} \mathbf{I}_d & \mathbf{0}_{d \times m-d} \\ \frac{\partial \mathbf{z}_{d:m}}{\partial \mathbf{x}_{1:d}} & \frac{\partial \mathbf{z}_{d:m}}{\partial \mathbf{x}_{d:m}} \end{pmatrix} = \det \left(\frac{\partial \mathbf{z}_{d:m}}{\partial \mathbf{x}_{d:m}} \right)$$

What is the Jacobian for the additive coupling law

$$\tau(x + c) = x + c?$$

In this case the transformation is *volume preserving*.

The last layer is rescaling:

$$z_i = s_i x_i; \quad x_i = z_i / s_i.$$

What is the Jacobian of the last layer?

NICE



(a) Model trained on MNIST



(b) Model trained on **TFD**

RealNVP

Affine coupling law

$$\begin{cases} \mathbf{z}_{1:d} = \mathbf{x}_{1:d}; \\ \mathbf{z}_{d:m} = \mathbf{x}_{d:m} \odot \exp(c_1(\mathbf{x}_{1:d}, \theta)) + c_2(\mathbf{x}_{1:d}, \theta). \end{cases}$$

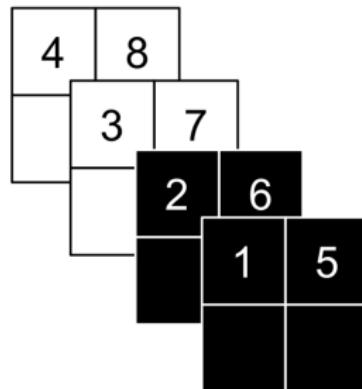
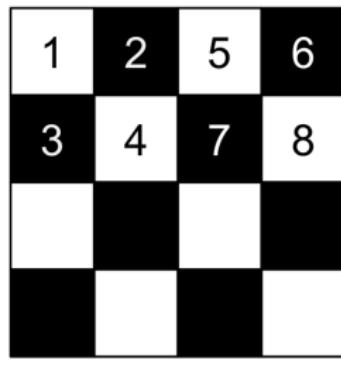
$$\begin{cases} \mathbf{x}_{1:d} = \mathbf{z}_{1:d}; \\ \mathbf{x}_{d:m} = (\mathbf{z}_{d:m} - c_2(\mathbf{x}_{1:d}, \theta)) \odot \exp(-c_1(\mathbf{x}_{1:d}, \theta)). \end{cases}$$

Jacobian

$$\det \left(\frac{\partial \mathbf{z}}{\partial \mathbf{x}} \right) = \det \begin{pmatrix} \mathbf{I}_d & 0_{d \times m-d} \\ \frac{\partial \mathbf{z}_{d:m}}{\partial \mathbf{x}_{1:d}} & \frac{\partial \mathbf{z}_{d:m}}{\partial \mathbf{x}_{d:m}} \end{pmatrix} = \prod_{i=1}^{m-d} \exp(c_1(\mathbf{x}_{1:d}, \theta)_i).$$

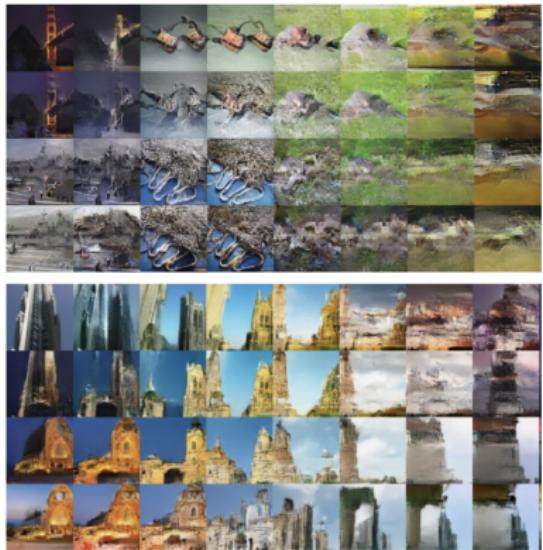
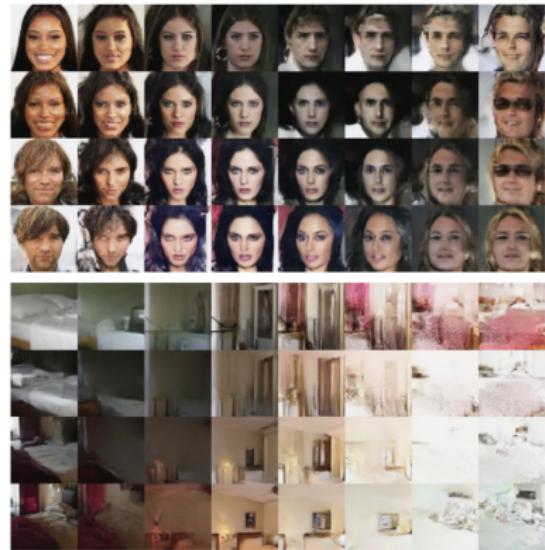
Non-Volume Preserving.

RealNVP

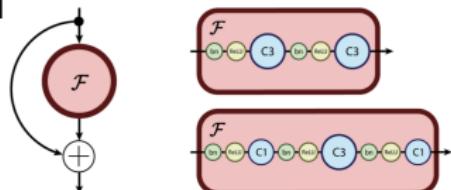


Masked convolutions are used to define ordering.

RealNVP



- ▶ Modern neural networks are trained via backpropagation.
- ▶ Residual networks are state of the art in image classification.
- ▶ Backpropagation requires storing the network activations.



Problem

Storing the activations imposes an increasing memory burden.

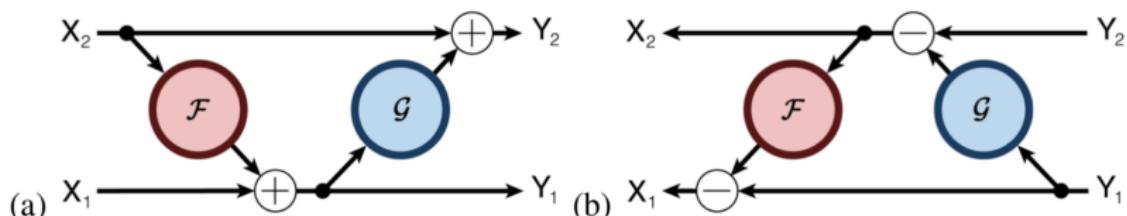
GPUs have limited memory capacity, leading to constraints often exceeded by state-of-the-art architectures (with thousand layers).

NICE

$$\begin{cases} \mathbf{z}_1 = \mathbf{x}_1; \\ \mathbf{z}_2 = \mathbf{x}_2 + \mathcal{F}(\mathbf{x}_1, \theta); \end{cases} \Leftrightarrow \begin{cases} \mathbf{x}_1 = \mathbf{z}_1; \\ \mathbf{x}_2 = \mathbf{z}_2 - \mathcal{F}(\mathbf{z}_1, \theta). \end{cases}$$

RevNet

$$\begin{cases} \mathbf{y}_1 = \mathbf{x}_1 + \mathcal{F}(\mathbf{x}_2, \theta); \\ \mathbf{y}_2 = \mathbf{x}_2 + \mathcal{G}(\mathbf{y}_1, \theta); \end{cases} \Leftrightarrow \begin{cases} \mathbf{x}_2 = \mathbf{y}_2 - \mathcal{F}(\mathbf{y}_1, \theta); \\ \mathbf{x}_1 = \mathbf{y}_1 - \mathcal{G}(\mathbf{x}_2, \theta). \end{cases}$$



Architecture	CIFAR-10 [15]		CIFAR-100 [15]	
	ResNet	RevNet	ResNet	RevNet
32 (38)	7.14%	7.24%	29.95%	28.96%
110	5.74%	5.76%	26.44%	25.40%
164	5.24%	5.17%	23.37%	23.69%

- ▶ If the network contains non-reversible blocks (poolings, strides), activations for these blocks should be stored.
- ▶ To avoid storing activations in the modern frameworks, the backward pass should be manually redefined.

Hypothesis

The success of deep convolutional networks is based on progressively discarding uninformative variability about the input with respect to the problem at hand.

- ▶ It is difficult to recover images from their hidden representations.
- ▶ Information bottleneck principle: an optimal representation must reduce the MI between an input and its representation to reduce uninformative variability + maximize the MI between the output and its representation to preserve each class from collapsing onto other classes.

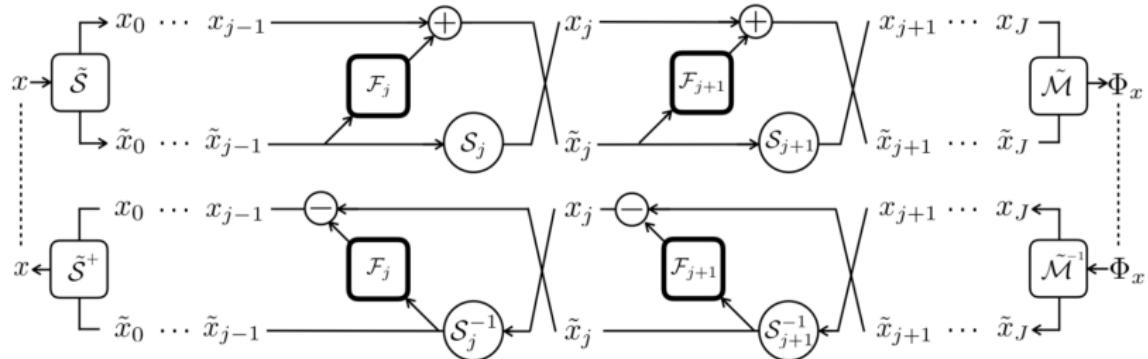
Hypothesis

The success of deep convolutional networks is based on progressively discarding uninformative variability about the input with respect to the problem at hand.

Idea

Build a cascade of homeomorphic layers (i-RevNet), a network that can be fully inverted up to the final projection onto the classes, i.e. no information is discarded.

i-RevNet, 2018



Architecture	Injective	Bijective	Top-1 error	Parameters
ResNet	-	-	24.7	26M
RevNet	-	-	25.2	28M
<i>i</i> -RevNet (a)	yes	-	24.7	181M
<i>i</i> -RevNet (b)	yes	yes	26.7	29M

Summary