

Deep Generative Models

Lecture 12

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Recap of previous lecture

ELBO of gaussian diffusion model

$$\begin{aligned}\mathcal{L}(q, \theta) = & \mathbb{E}_{q(\mathbf{x}_1|\mathbf{x}_0)} \log p(\mathbf{x}_0|\mathbf{x}_1, \theta) - KL(q(\mathbf{x}_T|\mathbf{x}_0)||p(\mathbf{x}_T)) - \\ & - \sum_{t=2}^T \underbrace{\mathbb{E}_{q(\mathbf{x}_t|\mathbf{x}_0)} KL(q(\mathbf{x}_{t-1}|\mathbf{x}_t, \mathbf{x}_0)||p(\mathbf{x}_{t-1}|\mathbf{x}_t, \theta))}_{\mathcal{L}_t}\end{aligned}$$

$$q(\mathbf{x}_{t-1}|\mathbf{x}_t, \mathbf{x}_0) = \mathcal{N}(\mathbf{x}_{t-1}|\tilde{\boldsymbol{\mu}}_t(\mathbf{x}_t, \mathbf{x}_0), \tilde{\beta}_t \mathbf{I}),$$

$$p(\mathbf{x}_{t-1}|\mathbf{x}_t, \theta) = \mathcal{N}(\mathbf{x}_{t-1}|\boldsymbol{\mu}_\theta(\mathbf{x}_t, t), \sigma_\theta^2(\mathbf{x}_t, t))$$

Our assumption: $\sigma_\theta^2(\mathbf{x}_t, t) = \tilde{\beta}_t \mathbf{I}$.

$$\mathcal{L}_t = \mathbb{E}_{q(\mathbf{x}_t|\mathbf{x}_0)} \left[\frac{1}{2\tilde{\beta}_t} \|\tilde{\boldsymbol{\mu}}_t(\mathbf{x}_t, \mathbf{x}_0) - \boldsymbol{\mu}_\theta(\mathbf{x}_t, t)\|^2 \right]$$

Recap of previous lecture

$$\mathcal{L}_t = \mathbb{E}_{q(\mathbf{x}_t|\mathbf{x}_0)} \left[\frac{1}{2\tilde{\beta}_t} \left\| \tilde{\mu}_t(\mathbf{x}_t, \mathbf{x}_0) - \mu_\theta(\mathbf{x}_t, t) \right\|^2 \right]$$

Reparametrization

$$\tilde{\mu}_t(\mathbf{x}_t, \mathbf{x}_0) = \frac{1}{\sqrt{\alpha_t}} \mathbf{x}_t - \frac{1 - \alpha_t}{\sqrt{\alpha_t(1 - \bar{\alpha}_t)}} \epsilon$$

$$\mu_\theta(\mathbf{x}_t, t) = \frac{1}{\sqrt{\alpha_t}} \mathbf{x}_t - \frac{1 - \alpha_t}{\sqrt{\alpha_t(1 - \bar{\alpha}_t)}} \epsilon_\theta(\mathbf{x}_t, t)$$

$$\mathcal{L}_t = \mathbb{E}_{\epsilon \sim \mathcal{N}(0, \mathbf{I})} \left[\frac{(1 - \alpha_t)^2}{2\tilde{\beta}_t \alpha_t (1 - \bar{\alpha}_t)} \left\| \epsilon - \epsilon_\theta(\sqrt{\bar{\alpha}_t} \mathbf{x}_0 + \sqrt{1 - \bar{\alpha}_t} \epsilon, t) \right\|^2 \right]$$

At each step of reverse diffusion process we try to predict the noise ϵ that we used in the forward diffusion process!

Simplified objective

$$\mathcal{L}_{\text{simple}} = \mathbb{E}_{t \sim U[2, T]} \mathbb{E}_{\epsilon \sim \mathcal{N}(0, \mathbf{I})} \left\| \epsilon - \epsilon_\theta(\sqrt{\bar{\alpha}_t} \mathbf{x}_0 + \sqrt{1 - \bar{\alpha}_t} \epsilon, t) \right\|^2$$

Recap of previous lecture

Training

1. Get the sample $\mathbf{x}_0 \sim \pi(\mathbf{x})$.
2. Sample timestamp $t \sim U[1, T]$ and the noise $\epsilon \sim \mathcal{N}(0, \mathbf{I})$.
3. Get noisy image $\mathbf{x}_t = \sqrt{\bar{\alpha}_t} \mathbf{x}_0 + \sqrt{1 - \bar{\alpha}_t} \epsilon$.
4. Compute loss $\mathcal{L}_{\text{simple}} = \|\epsilon - \epsilon_{\theta}(\mathbf{x}_t, t)\|^2$.

Sampling

1. Sample $\mathbf{x}_T \sim \mathcal{N}(0, \mathbf{I})$.
2. Compute mean of $p(\mathbf{x}_{t-1} | \mathbf{x}_t, \theta) = \mathcal{N}(\mu_{\theta}(\mathbf{x}_t, t), \tilde{\beta}_t \mathbf{I})$:

$$\mu_{\theta}(\mathbf{x}_t, t) = \frac{1}{\sqrt{\alpha_t}} \mathbf{x}_t - \frac{1 - \alpha_t}{\sqrt{\alpha_t(1 - \bar{\alpha}_t)}} \epsilon_{\theta}(\mathbf{x}_t, t)$$

3. Get denoised image $\mathbf{x}_{t-1} = \mu_{\theta}(\mathbf{x}_t, t) + \sqrt{\tilde{\beta}_t} \cdot \epsilon$, where $\epsilon \sim \mathcal{N}(0, \mathbf{I})$.

Recap of previous lecture

SDE basics

Let define stochastic process $\mathbf{x}(t)$ with initial condition $\mathbf{x}(0) \sim p_0(\mathbf{x})$:

$$d\mathbf{x} = \mathbf{f}(\mathbf{x}, t)dt + g(t)d\mathbf{w},$$

where $\mathbf{w}(t)$ is the standard Wiener process (Brownian motion)

$$\mathbf{w}(t) - \mathbf{w}(s) \sim \mathcal{N}(0, (t-s)\mathbf{I}), \quad d\mathbf{w} = \boldsymbol{\epsilon} \cdot \sqrt{dt}, \text{ where } \boldsymbol{\epsilon} \sim \mathcal{N}(0, \mathbf{I}).$$

Langevin dynamics

Let \mathbf{x}_0 be a random vector. Then under mild regularity conditions for small enough η samples from the following dynamics

$$\mathbf{x}_{t+1} = \mathbf{x}_t + \eta \frac{1}{2} \nabla_{\mathbf{x}_t} \log p(\mathbf{x}_t | \theta) + \sqrt{\eta} \cdot \boldsymbol{\epsilon}, \quad \boldsymbol{\epsilon} \sim \mathcal{N}(0, \mathbf{I}).$$

will come from $p(\mathbf{x} | \theta)$.

The density $p(\mathbf{x} | \theta)$ is a **stationary** distribution for the Langevin SDE.

Outline

1. Score matching
 - Implicit score matching
 - Denoising score matching
2. Noise Conditioned Score Network (NCSN)
3. DDPM vs NCSN

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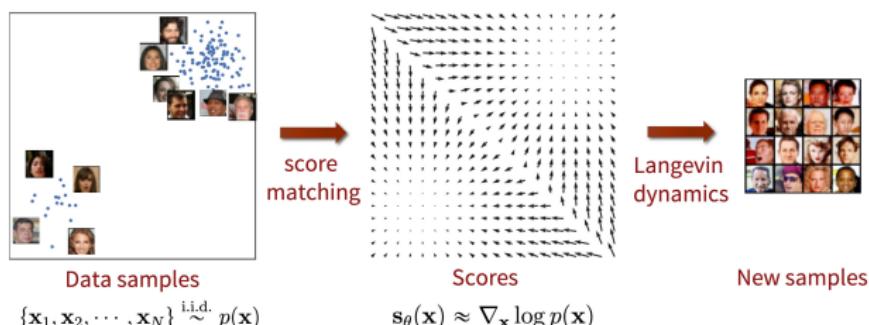
Score matching

We could sample from the model using Langevin dynamics if we have $\nabla_{\mathbf{x}} \log p(\mathbf{x}|\theta)$.

Fisher divergence

$$D_F(\pi, p) = \frac{1}{2} \mathbb{E}_{\pi} \| \nabla_{\mathbf{x}} \log p(\mathbf{x}|\theta) - \nabla_{\mathbf{x}} \log \pi(\mathbf{x}) \|_2^2 \rightarrow \min_{\theta}$$

Let introduce **score function** $s_{\theta}(\mathbf{x}) = \nabla_{\mathbf{x}} \log p(\mathbf{x}|\theta)$.



Problem: we do not know $\nabla_{\mathbf{x}} \log \pi(\mathbf{x})$.

Outline

1. Score matching

Implicit score matching

Denoising score matching

2. Noise Conditioned Score Network (NCSN)

3. DDPM vs NCSN

Implicit score matching

Theorem

Under some regularity conditions, it holds

$$\frac{1}{2} \mathbb{E}_\pi \| \mathbf{s}_\theta(\mathbf{x}) - \nabla_{\mathbf{x}} \log \pi(\mathbf{x}) \|_2^2 = \mathbb{E}_\pi \left[\frac{1}{2} \| \mathbf{s}_\theta(\mathbf{x}) \|_2^2 + \text{tr}(\nabla_{\mathbf{x}} \mathbf{s}_\theta(\mathbf{x})) \right] + \text{const}$$

Proof (only for 1D)

$$\mathbb{E}_\pi \| s(x) - \nabla_x \log \pi(x) \|_2^2 = \mathbb{E}_\pi [s(x)^2 + (\nabla_x \log \pi(x))^2 - 2[s(x) \nabla_x \log \pi(x)]]$$

$$\begin{aligned} \mathbb{E}_\pi [s(x) \nabla_x \log \pi(x)] &= \int \underbrace{\pi(x) s(x)}_g \underbrace{\nabla_x \log \pi(x)}_{\nabla f} dx = \int \underbrace{\nabla_x \log p(x)}_g \underbrace{\nabla_x \pi(x)}_{\nabla f} dx \\ &= \underbrace{\nabla_x \log p(x)}_g \underbrace{\pi(x)}_f \Big|_{-\infty}^{+\infty} - \int \underbrace{\nabla_x (\nabla_x \log p(x))}_{\nabla g} \underbrace{\pi(x)}_f dx \\ &= -\mathbb{E}_\pi \nabla_x s(x) \end{aligned}$$

$$\frac{1}{2} \mathbb{E}_\pi \| s(x) - \nabla_x \log \pi(x) \|_2^2 = \mathbb{E}_\pi \left[\frac{1}{2} s(x)^2 + \nabla_x s(x) \right] + \text{const.}$$

Implicit score matching

Theorem

$$\frac{1}{2} \mathbb{E}_\pi \| \mathbf{s}_\theta(\mathbf{x}) - \nabla_{\mathbf{x}} \log \pi(\mathbf{x}) \|_2^2 = \mathbb{E}_\pi \left[\frac{1}{2} \| \mathbf{s}_\theta(\mathbf{x}) \|_2^2 + \text{tr}(\nabla_{\mathbf{x}} \mathbf{s}_\theta(\mathbf{x})) \right] + \text{const}$$

Here $\nabla_{\mathbf{x}} \mathbf{s}_\theta(\mathbf{x}) = \nabla_{\mathbf{x}}^2 \log p(\mathbf{x}|\theta)$ is a Hessian matrix.

1. The right hand side is complex due to Hessian matrix – **sliced score matching**.
2. The left hand side is intractable due to unknown $\pi(\mathbf{x})$ – **denoising score matching**.

Sliced score matching (Hutchinson's trace estimation)

$$\text{tr}(\nabla_{\mathbf{x}} \mathbf{s}_\theta(\mathbf{x})) = \mathbb{E}_{p(\epsilon)} \left[\boldsymbol{\epsilon}^T \nabla_{\mathbf{x}} \mathbf{s}_\theta(\mathbf{x}) \boldsymbol{\epsilon} \right]$$

Song Y. Sliced Score Matching: A Scalable Approach to Density and Score Estimation, 2019

Song Y. Generative Modeling by Estimating Gradients of the Data Distribution, blog post, 2021

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Denoising score matching

Let perturb original data $\mathbf{x} \sim \pi(\mathbf{x})$ by random normal noise

$$\mathbf{x}' = \mathbf{x} + \sigma \cdot \boldsymbol{\epsilon}, \quad \boldsymbol{\epsilon} \sim \mathcal{N}(0, \mathbf{I}), \quad p(\mathbf{x}' | \mathbf{x}, \sigma) = \mathcal{N}(\mathbf{x}' | \mathbf{x}, \sigma^2 \mathbf{I})$$

$$\pi(\mathbf{x}' | \sigma) = \int p(\mathbf{x}' | \mathbf{x}, \sigma) \pi(\mathbf{x}) d\mathbf{x}.$$

Assumption

The solution of

$$\frac{1}{2} \mathbb{E}_{\pi(\mathbf{x}' | \sigma)} \| \mathbf{s}_\theta(\mathbf{x}', \sigma) - \nabla_{\mathbf{x}'} \log \pi(\mathbf{x}' | \sigma) \|_2^2 \rightarrow \min_{\theta}$$

satisfies $\mathbf{s}_\theta(\mathbf{x}', \sigma) \approx \mathbf{s}_\theta(\mathbf{x}', 0) = \mathbf{s}_\theta(\mathbf{x})$ if σ is small enough.

- ▶ $\mathbf{s}_\theta(\mathbf{x}', \sigma)$ tries to **denoise** a corrupted sample \mathbf{x}' .
- ▶ Score function $\mathbf{s}_\theta(\mathbf{x}', \sigma)$ parametrized by σ .

Denoising score matching

Theorem

$$\begin{aligned}\mathbb{E}_{\pi(\mathbf{x}'|\sigma)} \left\| \mathbf{s}_\theta(\mathbf{x}', \sigma) - \nabla_{\mathbf{x}'} \log \pi(\mathbf{x}'|\sigma) \right\|_2^2 &= \\ &= \mathbb{E}_{\pi(\mathbf{x})} \mathbb{E}_{p(\mathbf{x}'|\mathbf{x}, \sigma)} \left\| \mathbf{s}_\theta(\mathbf{x}', \sigma) - \nabla_{\mathbf{x}'} \log p(\mathbf{x}'|\mathbf{x}, \sigma) \right\|_2^2 + \text{const}(\theta)\end{aligned}$$

Proof

$$\begin{aligned}\mathbb{E}_{\pi(\mathbf{x}'|\sigma)} \left\| \mathbf{s}_\theta(\mathbf{x}', \sigma) - \nabla_{\mathbf{x}'} \log \pi(\mathbf{x}'|\sigma) \right\|_2^2 &= \\ &= \mathbb{E}_{\pi(\mathbf{x}'|\sigma)} \left[\left\| \mathbf{s}_\theta(\mathbf{x}', \sigma) \right\|^2 + \underbrace{\left\| \nabla_{\mathbf{x}'} \log \pi(\mathbf{x}'|\sigma) \right\|_2^2}_{\text{const}(\theta)} - 2 \mathbf{s}_\theta^T(\mathbf{x}', \sigma) \nabla_{\mathbf{x}'} \log \pi(\mathbf{x}'|\sigma) \right] \\ \mathbb{E}_{\pi(\mathbf{x}'|\sigma)} \left\| \mathbf{s}_\theta(\mathbf{x}', \sigma) \right\|^2 &= \int \pi(\mathbf{x}'|\sigma) \left\| \mathbf{s}_\theta(\mathbf{x}', \sigma) \right\|^2 d\mathbf{x}' = \\ &= \int \left(\int p(\mathbf{x}'|\mathbf{x}, \sigma) \pi(\mathbf{x}) d\mathbf{x} \right) \left\| \mathbf{s}_\theta(\mathbf{x}', \sigma) \right\|^2 d\mathbf{x}' = \mathbb{E}_{\pi(\mathbf{x})} \mathbb{E}_{p(\mathbf{x}'|\mathbf{x}, \sigma)} \left\| \mathbf{s}_\theta(\mathbf{x}', \sigma) \right\|^2 d\mathbf{x}'\end{aligned}$$

Denoising score matching

Theorem

$$\begin{aligned}\mathbb{E}_{\pi(\mathbf{x}'|\sigma)} \left\| \mathbf{s}_\theta(\mathbf{x}', \sigma) - \nabla_{\mathbf{x}'} \log \pi(\mathbf{x}'|\sigma) \right\|_2^2 &= \\ &= \mathbb{E}_{\pi(\mathbf{x})} \mathbb{E}_{p(\mathbf{x}'|\mathbf{x}, \sigma)} \left\| \mathbf{s}_\theta(\mathbf{x}', \sigma) - \nabla_{\mathbf{x}'} \log p(\mathbf{x}'|\mathbf{x}, \sigma) \right\|_2^2 + \text{const}(\theta)\end{aligned}$$

Proof (continued)

$$\begin{aligned}\mathbb{E}_{\pi(\mathbf{x}'|\sigma)} [\mathbf{s}_\theta^T(\mathbf{x}', \sigma) \nabla_{\mathbf{x}'} \log \pi(\mathbf{x}'|\sigma)] &= \int \pi(\mathbf{x}'|\sigma) \left[\mathbf{s}_\theta^T(\mathbf{x}', \sigma) \frac{\nabla_{\mathbf{x}'} \pi(\mathbf{x}'|\sigma)}{\pi(\mathbf{x}'|\sigma)} \right] d\mathbf{x}' = \\ &= \int \left[\mathbf{s}_\theta^T(\mathbf{x}', \sigma) \nabla_{\mathbf{x}'} \left(\int p(\mathbf{x}'|\mathbf{x}, \sigma) \pi(\mathbf{x}) d\mathbf{x} \right) \right] d\mathbf{x}' = \\ &= \int \int \pi(\mathbf{x}) [\mathbf{s}_\theta^T(\mathbf{x}', \sigma) \nabla_{\mathbf{x}'} p(\mathbf{x}'|\mathbf{x}, \sigma)] d\mathbf{x}' d\mathbf{x} = \\ &= \int \int \pi(\mathbf{x}) p(\mathbf{x}'|\mathbf{x}, \sigma) [\mathbf{s}_\theta^T(\mathbf{x}', \sigma) \nabla_{\mathbf{x}'} \log p(\mathbf{x}'|\mathbf{x}, \sigma)] d\mathbf{x}' d\mathbf{x} = \\ &= \mathbb{E}_{\pi(\mathbf{x})} \mathbb{E}_{p(\mathbf{x}'|\mathbf{x}, \sigma)} [\mathbf{s}_\theta^T(\mathbf{x}', \sigma) \nabla_{\mathbf{x}'} \log p(\mathbf{x}'|\mathbf{x}, \sigma)]\end{aligned}$$

Denoising score matching

Theorem

$$\begin{aligned}\mathbb{E}_{\pi(\mathbf{x}'|\sigma)} \left\| \mathbf{s}_\theta(\mathbf{x}', \sigma) - \nabla_{\mathbf{x}'} \log \pi(\mathbf{x}'|\sigma) \right\|_2^2 &= \\ &= \mathbb{E}_{\pi(\mathbf{x})} \mathbb{E}_{p(\mathbf{x}'|\mathbf{x}, \sigma)} \left\| \mathbf{s}_\theta(\mathbf{x}', \sigma) - \nabla_{\mathbf{x}'} \log p(\mathbf{x}'|\mathbf{x}, \sigma) \right\|_2^2 + \text{const}(\theta)\end{aligned}$$

Proof (continued)

$$\begin{aligned}\mathbb{E}_{\pi(\mathbf{x}'|\sigma)} \left\| \mathbf{s}_\theta(\mathbf{x}', \sigma) - \nabla_{\mathbf{x}'} \log \pi(\mathbf{x}'|\sigma) \right\|_2^2 &= \\ &= \mathbb{E}_{\pi(\mathbf{x}'|\sigma)} \left[\left\| \mathbf{s}_\theta(\mathbf{x}', \sigma) \right\|^2 - 2 \mathbf{s}_\theta^T(\mathbf{x}', \sigma) \nabla_{\mathbf{x}'} \log \pi(\mathbf{x}'|\sigma) \right] + \text{const}(\theta) = \\ &= \mathbb{E}_{\pi(\mathbf{x})} \mathbb{E}_{p(\mathbf{x}'|\mathbf{x}, \sigma)} \left[\left\| \mathbf{s}_\theta(\mathbf{x}', \sigma) \right\|^2 - 2 \mathbf{s}_\theta^T(\mathbf{x}', \sigma) \nabla_{\mathbf{x}'} \log p(\mathbf{x}'|\mathbf{x}, \sigma) \right] + \text{const}(\theta)\end{aligned}$$

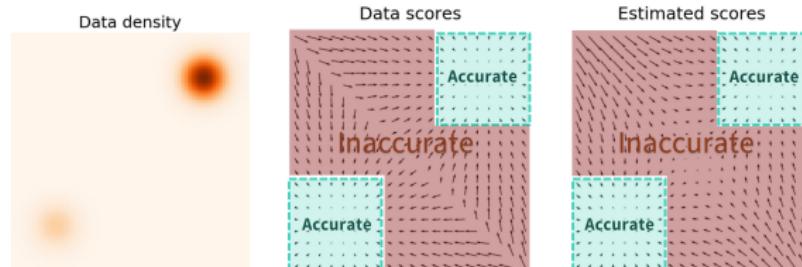
Gradient of the noise kernel

$$\nabla_{\mathbf{x}'} \log p(\mathbf{x}'|\mathbf{x}, \sigma) = \nabla_{\mathbf{x}'} \log \mathcal{N}(\mathbf{x}'|\mathbf{x}, \sigma^2 \mathbf{I}) = -\frac{\mathbf{x}' - \mathbf{x}}{\sigma^2}$$

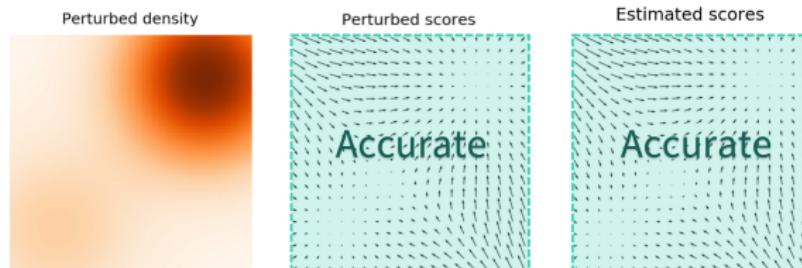
The RHS does not need to compute $\nabla_{\mathbf{x}'} \log \pi(\mathbf{x}'|\sigma)$ and even $\nabla_{\mathbf{x}'} \log \pi(\mathbf{x}')$.

Denoising score matching

- If σ is **small**, the score function is not accurate and Langevin dynamics will probably fail to jump between modes.



- If σ is **large**, it is good for low-density regions and multimodal distributions, but we will learn too corrupted distribution.



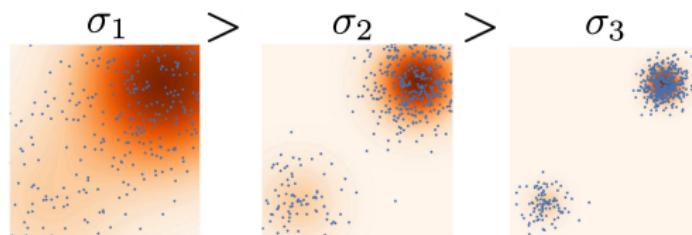
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Noise Conditioned Score Network (NCSN)

- ▶ Define the sequence of the noise levels: $\sigma_1 > \sigma_2 > \dots > \sigma_L$.
- ▶ Perturb the original data with the different noise levels to obtain $\pi(\mathbf{x}'|\sigma_1), \dots, \pi(\mathbf{x}'|\sigma_L)$.
- ▶ Choose σ_1, σ_L such that:

$$\pi(\mathbf{x}'|\sigma_1) \approx \mathcal{N}(0, \sigma_1^2 \mathbf{I}), \quad \pi(\mathbf{x}'|\sigma_L) \approx \pi(\mathbf{x}).$$



Noise Conditioned Score Network (NCSN)

Train the denoising score function $s_\theta(\mathbf{x}', \sigma)$ for each noise level using unified weighted objective:

$$\sum_{l=1}^L \sigma_l^2 \mathbb{E}_{\pi(\mathbf{x})} \mathbb{E}_{p(\mathbf{x}'|\mathbf{x}, \sigma_l)} \|s_\theta(\mathbf{x}', \sigma_l) - \nabla_{\mathbf{x}'} \log p(\mathbf{x}'|\mathbf{x}, \sigma_l)\|_2^2 \rightarrow \min_{\theta}$$

Here $\nabla_{\mathbf{x}'} \log p(\mathbf{x}'|\mathbf{x}, \sigma_l) = -\frac{\mathbf{x}' - \mathbf{x}}{\sigma_l^2} = -\frac{\epsilon}{\sigma_l}$.

Training

1. Get the sample $\mathbf{x}_0 \sim \pi(\mathbf{x})$.
2. Sample noise level $l \sim U[1, L]$ and the noise $\epsilon \sim \mathcal{N}(0, \mathbf{I})$.
3. Get noisy image $\mathbf{x}' = \mathbf{x}_0 + \sigma_l \cdot \epsilon$.
4. Compute loss $\mathcal{L} = \|s_\theta(\mathbf{x}', \sigma_l) + \frac{\epsilon}{\sigma_l}\|^2$.

How to sample from this model?

Noise Conditioned Score Network (NCSN)

Sampling (annealed Langevin dynamics)

- ▶ Sample $\mathbf{x}_0 \sim \mathcal{N}(0, \sigma_1 \mathbf{I}) \approx \pi(\mathbf{x} | \sigma_1)$.
- ▶ Apply T steps of Langevin dynamic

$$\mathbf{x}_t = \mathbf{x}_{t-1} + \frac{1}{2}\eta \mathbf{s}_\theta(\mathbf{x}_{t-1}, \sigma_t) + \sqrt{\eta}\epsilon_t.$$

- ▶ Update $\mathbf{x}_0 := \mathbf{x}_T$ and choose the next σ_t .



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DDPM vs NCSN

NCSN objective

$$\mathbb{E}_{p(\mathbf{x}'|\mathbf{x}, \sigma_I)} \| \mathbf{s}_\theta(\mathbf{x}', \sigma_I) - \nabla_{\mathbf{x}'} \log p(\mathbf{x}'|\mathbf{x}, \sigma_I) \|_2^2 \rightarrow \min_\theta$$

DDPM objective

$$\mathcal{L}_t = \mathbb{E}_{\epsilon \sim \mathcal{N}(0, \mathbf{I})} \left[\frac{(1 - \alpha_t)^2}{2\tilde{\beta}_t \alpha_t} \left\| \frac{\epsilon}{\sqrt{1 - \bar{\alpha}_t}} - \frac{\epsilon_\theta(\mathbf{x}_t, t)}{\sqrt{1 - \bar{\alpha}_t}} \right\|^2 \right]$$

$$q(\mathbf{x}_t | \mathbf{x}_0) = \mathcal{N}(\mathbf{x}_t | \sqrt{\bar{\alpha}_t} \cdot \mathbf{x}_0, (1 - \bar{\alpha}_t) \cdot \mathbf{I})$$

$$\nabla_{\mathbf{x}_t} \log q(\mathbf{x}_t | \mathbf{x}_0) = -\frac{\mathbf{x}_t - \sqrt{\bar{\alpha}_t} \cdot \mathbf{x}_0}{1 - \bar{\alpha}_t} = -\frac{\epsilon}{\sqrt{1 - \bar{\alpha}_t}}.$$

Let reparametrize our model:

$$\mathbf{s}_\theta(\mathbf{x}_t, t) = -\frac{\epsilon_\theta(\mathbf{x}_t, t)}{\sqrt{1 - \bar{\alpha}_t}}.$$

Summary

- ▶ Score matching proposes to minimize Fisher divergence to get score function.
- ▶ Implicit score matching tries to avoid the value of original distribution $\pi(\mathbf{x})$. Sliced score matching makes implicit score matching scalable.
- ▶ Denoising score matching minimizes Fisher divergence on noisy samples.
- ▶ Noise conditioned score network uses multiple noise levels and annealed Langevin dynamics to fit score function.
- ▶ Objective of DDPM is closely related to the noise conditioned score network and score matching.