

Deep Generative Models

Lecture 9

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Recap of previous lecture

Main problems of standard GAN

- ▶ Vanishing gradients (solution: non-saturating GAN);
- ▶ Mode collapse (caused by Jensen-Shannon divergence).

Standard GAN

$$\min_{\theta} \max_{\phi} [\mathbb{E}_{\pi(\mathbf{x})} \log D_{\phi}(\mathbf{x}) + \mathbb{E}_{p(\mathbf{z})} \log(1 - D_{\phi}(\mathbf{G}_{\theta}(\mathbf{z})))]$$

Informal theoretical results

The real images distribution $\pi(\mathbf{x})$ and the generated images distribution $p(\mathbf{x}|\theta)$ are low-dimensional and have disjoint supports.
In this case

$$KL(\pi||p) = KL(p||\pi) = \infty, \quad JSD(\pi||p) = \log 2.$$

Goodfellow I. J. et al. Generative Adversarial Networks, 2014

Arjovsky M., Bottou L. Towards Principled Methods for Training Generative Adversarial Networks, 2017

Recap of previous lecture

Wasserstein distance

$$W(\pi, p) = \inf_{\gamma \in \Gamma(\pi, p)} \mathbb{E}_{(\mathbf{x}, \mathbf{y}) \sim \gamma} \|\mathbf{x} - \mathbf{y}\| = \inf_{\gamma \in \Gamma(\pi, p)} \int \|\mathbf{x} - \mathbf{y}\| \gamma(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y}$$

- ▶ $\gamma(\mathbf{x}, \mathbf{y})$ – transportation plan (the amount of "dirt" that should be transported from point \mathbf{x} to point \mathbf{y}).
- ▶ $\Gamma(\pi, p)$ – the set of all joint distributions $\gamma(\mathbf{x}, \mathbf{y})$ with marginals π and p ($\int \gamma(\mathbf{x}, \mathbf{y}) d\mathbf{x} = p(\mathbf{y})$, $\int \gamma(\mathbf{x}, \mathbf{y}) d\mathbf{y} = \pi(\mathbf{x})$).
- ▶ $\gamma(\mathbf{x}, \mathbf{y})$ – the amount, $\|\mathbf{x} - \mathbf{y}\|$ – the distance.

Theorem (Kantorovich-Rubinstein duality)

$$W(\pi || p) = \frac{1}{K} \max_{\|f\|_L \leq K} [\mathbb{E}_{\pi(\mathbf{x})} f(\mathbf{x}) - \mathbb{E}_{p(\mathbf{x})} f(\mathbf{x})],$$

where $\|f\|_L \leq K$ are K -Lipschitz continuous functions ($f : \mathcal{X} \rightarrow \mathbb{R}$).

Recap of previous lecture

WGAN objective

$$\min_{\theta} W(\pi || p) = \min_{\theta} \max_{\phi \in \Phi} [\mathbb{E}_{\pi(x)} f_{\phi}(x) - \mathbb{E}_{p(z)} f_{\phi}(\mathbf{G}_{\theta}(z))].$$

- ▶ Function f in WGAN is usually called *critic*.
- ▶ If parameters ϕ lie in a compact set $\Phi \in [-c, c]^d$ then $f(x, \phi)$ will be K -Lipschitz continuous function.

$$\begin{aligned} K \cdot W(\pi || p) &= \max_{\|f\|_L \leq K} [\mathbb{E}_{\pi(x)} f(x) - \mathbb{E}_{p(x)} f(x)] \geq \\ &\geq \max_{\phi \in \Phi} [\mathbb{E}_{\pi(x)} f_{\phi}(x) - \mathbb{E}_{p(x)} f_{\phi}(x)] \end{aligned}$$

"Weight clipping is a clearly terrible way to enforce a Lipschitz constraint"

Recap of previous lecture

Theorem

Let $\pi(\mathbf{x})$ and $p(\mathbf{x})$ be two distributions in \mathcal{X} , a compact metric space. Let γ be the optimal transportation plan between $\pi(\mathbf{x})$ and $p(\mathbf{x})$. Then

$$\mathbb{P}_{(\mathbf{y}, \mathbf{z}) \sim \gamma} \left[\nabla f^*(\hat{\mathbf{x}}_t) = \frac{\mathbf{z} - \hat{\mathbf{x}}_t}{\|\mathbf{z} - \hat{\mathbf{x}}_t\|} \right] = 1.$$

Gradient penalty

$$W(\pi || p) = \underbrace{\mathbb{E}_{\pi(\mathbf{x})} f(\mathbf{x}) - \mathbb{E}_{p(\mathbf{x})} f(\mathbf{x})}_{\text{original critic loss}} + \lambda \underbrace{\mathbb{E}_{U[0,1]} \left[(\|\nabla f(\hat{\mathbf{x}})\|_2 - 1)^2 \right]}_{\text{gradient penalty}}.$$

Samples $\hat{\mathbf{x}}_t = t \cdot \mathbf{y} + (1 - t) \cdot \mathbf{z}$ with $t \in [0, 1]$ are uniformly sampled along straight lines between pairs of points: $\mathbf{y} \sim \pi(\mathbf{x})$ and $\mathbf{z} \sim p(\mathbf{x}|\theta)$.

Recap of previous lecture

f-divergence minimization

$$D_f(\pi || p) = \mathbb{E}_{p(\mathbf{x})} f\left(\frac{\pi(\mathbf{x})}{p(\mathbf{x})}\right) \rightarrow \min_p .$$

Here $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a convex, lower semicontinuous function satisfying $f(1) = 0$.

Variational divergence estimation

$$D_f(\pi || p) \geq \sup_{T \in \mathcal{T}} [\mathbb{E}_\pi T(\mathbf{x}) - \mathbb{E}_p f^*(T(\mathbf{x}))],$$

Fenchel conjugate

$$f^*(t) = \sup_{u \in \text{dom}_f} (ut - f(u)), \quad f(u) = \sup_{t \in \text{dom}_{f^*}} (ut - f^*(t))$$

Note: To evaluate lower bound we only need samples from $\pi(\mathbf{x})$ and $p(\mathbf{x})$. Hence, we could fit implicit generative model.

Outline

1. Evaluation of likelihood-free models

Frechet Inception Distance (FID)

Maximum Mean Discrepancy (MMD)

Precision-Recall

2. Langevin dynamic

3. Score matching

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Evaluation of likelihood-free models

How to evaluate generative models?

Likelihood-based models

- ▶ Split data to train/val/test.
- ▶ Fit model on the train part.
- ▶ Tune hyperparameters on the validation part.
- ▶ Evaluate generalization by reporting likelihoods on the test set.

Not all models have tractable likelihoods

- ▶ VAE: compare ELBO values.
- ▶ GAN: ???

Evaluation of likelihood-free models

Let's take some pretrained image classification model to get the conditional label distribution $p(y|x)$ (e.g. ImageNet classifier).

What do we want from samples?

- ▶ Sharpness



The conditional distribution $p(y|x)$ should have low entropy (each image x should have distinctly recognizable object).

- ▶ Diversity



The marginal distribution $p(y) = \int p(y|x)p(x)dx$ should have high entropy (there should be as many classes generated as possible).

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Frechet Inception Distance (FID)

Wasserstein metric

$$W_s(\pi, p) = \inf_{\gamma \in \Gamma(\pi, p)} (\mathbb{E}_{(\mathbf{x}, \mathbf{y}) \sim \gamma} \|\mathbf{x} - \mathbf{y}\|^s)^{1/s}$$

Theorem

If $\pi(\mathbf{x}) = \mathcal{N}(\boldsymbol{\mu}_\pi, \boldsymbol{\Sigma}_\pi)$, $p(\mathbf{y}) = \mathcal{N}(\boldsymbol{\mu}_p, \boldsymbol{\Sigma}_p)$, then

$$W_2^2(\pi, p) = \|\boldsymbol{\mu}_\pi - \boldsymbol{\mu}_p\|_2^2 + \text{tr} \left[\boldsymbol{\Sigma}_\pi + \boldsymbol{\Sigma}_p - 2 \left(\boldsymbol{\Sigma}_\pi^{1/2} \boldsymbol{\Sigma}_p \boldsymbol{\Sigma}_\pi^{1/2} \right)^{1/2} \right]$$

Frechet Inception Distance

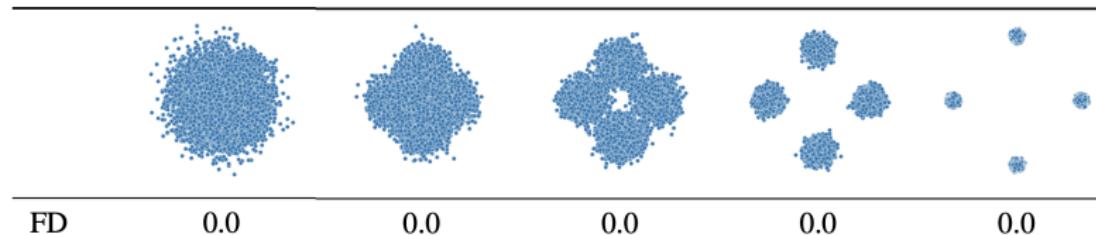
$$\text{FID}(\pi, p) = W_2^2(\pi, p)$$

- ▶ Representations are the outputs of the intermediate layer from the pretrained classification model.
- ▶ $\boldsymbol{\mu}_\pi$, $\boldsymbol{\Sigma}_\pi$ and $\boldsymbol{\mu}_p$, $\boldsymbol{\Sigma}_p$ are the statistics of the feature representations for the samples from $\pi(\mathbf{x})$ and $p(\mathbf{x}|\theta)$.

Frechet Inception Distance (FID)

$$\text{FID}(\pi, p) = \|\boldsymbol{\mu}_\pi - \boldsymbol{\mu}_p\|_2^2 + \text{tr} \left[\boldsymbol{\Sigma}_\pi + \boldsymbol{\Sigma}_p - 2 \left(\boldsymbol{\Sigma}_\pi^{1/2} \boldsymbol{\Sigma}_p \boldsymbol{\Sigma}_\pi^{1/2} \right)^{1/2} \right]$$

- ▶ Needs a large sample size for evaluation.
- ▶ Calculation of FID is slow.
- ▶ High dependence on the pretrained classification model.
- ▶ Uses the normality assumption!



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Maximum Mean Discrepancy (MMD)

Theorem

$\pi(\mathbf{x}) = p(\mathbf{y})$ if and only if $\mathbb{E}_{\pi(\mathbf{x})} f(\mathbf{x}) = \mathbb{E}_{p(\mathbf{y})} f(\mathbf{y})$ for any bounded and continuous f .

$$\text{MMD}(\pi, p) = \sup_f [\mathbb{E}_{\pi(\mathbf{x})} f(\mathbf{x}) - \mathbb{E}_{p(\mathbf{x})} f(\mathbf{x})]$$

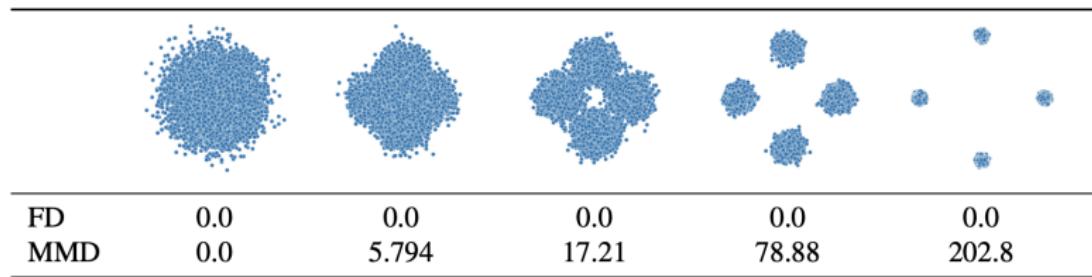
Theorem (Reproducing Kernel Hilbert Space)

$$\text{MMD}^2(\pi, p) = \mathbb{E}_{\substack{\mathbf{x} \sim \pi(\mathbf{x}) \\ \mathbf{x}' \sim \pi(\mathbf{x})}} k(\mathbf{x}, \mathbf{x}') + \mathbb{E}_{\substack{\mathbf{y} \sim p(\mathbf{y}) \\ \mathbf{y}' \sim p(\mathbf{y})}} k(\mathbf{y}, \mathbf{y}') - 2 \mathbb{E}_{\substack{\mathbf{x} \sim \pi(\mathbf{x}) \\ \mathbf{y} \sim p(\mathbf{y})}} k(\mathbf{x}, \mathbf{y})$$

- ▶ $k(\mathbf{x}, \mathbf{y})$ is a positive definite, symmetric kernel function (for example $k(\mathbf{x}, \mathbf{y}) = \frac{\exp(-\|\mathbf{x}-\mathbf{y}\|^2)}{\sigma^2}$);
- ▶ $f(\mathbf{x}) = \langle f, \phi(\mathbf{x}) \rangle_{\mathcal{H}}$;
- ▶ $\phi(\mathbf{x}) = k(\cdot, \mathbf{x})$.

Maximum Mean Discrepancy (MMD)

$$\text{MMD}^2(\pi, p) = \mathbb{E}_{\substack{\mathbf{x} \sim \pi(\mathbf{x}) \\ \mathbf{x}' \sim \pi(\mathbf{x})}} k(\mathbf{x}, \mathbf{x}') + \mathbb{E}_{\substack{\mathbf{y} \sim p(\mathbf{y}) \\ \mathbf{y}' \sim p(\mathbf{y})}} k(\mathbf{y}, \mathbf{y}') - 2\mathbb{E}_{\substack{\mathbf{x} \sim \pi(\mathbf{x}) \\ \mathbf{y} \sim p(\mathbf{y})}} k(\mathbf{x}, \mathbf{y})$$



- ▶ Needs less sample size for evaluation.
- ▶ High dependence on the pretrained classification model.
- ▶ Works with any distribution!

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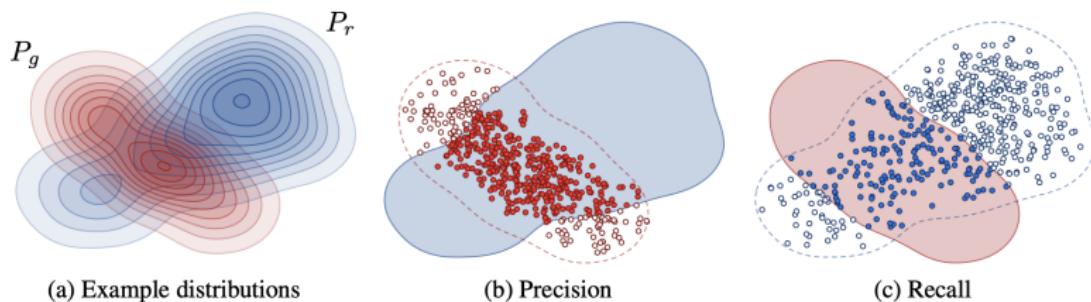
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Precision-Recall

What do we want from samples

- ▶ **Sharpness:** generated samples should be of high quality.
- ▶ **Diversity:** their variation should match that observed in the training set.



- ▶ **Precision** denotes the fraction of generated images that are realistic.
- ▶ **Recall** measures the fraction of the training data manifold covered by the generator.

Precision-Recall

- ▶ $\mathcal{S}_\pi = \{\mathbf{x}_i\}_{i=1}^n \sim \pi(\mathbf{x})$ – real samples;
- ▶ $\mathcal{S}_p = \{\mathbf{x}_i\}_{i=1}^n \sim p(\mathbf{x}|\theta)$ – generated samples.

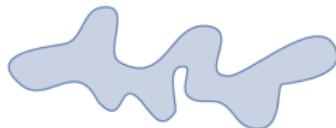
Embed samples using pretrained classifier network (as previously):

$$\mathcal{G}_\pi = \{\mathbf{g}_i\}_{i=1}^n, \quad \mathcal{G}_p = \{\mathbf{g}_i\}_{i=1}^n.$$

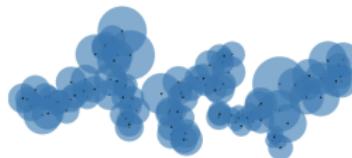
Define binary function:

$$f(\mathbf{g}, \mathcal{G}) = \begin{cases} 1, & \text{if exists } \mathbf{g}' \in \mathcal{G} : \|\mathbf{g} - \mathbf{g}'\|_2 \leq \|\mathbf{g}' - \text{NN}_k(\mathbf{g}', \mathcal{G})\|_2; \\ 0, & \text{otherwise.} \end{cases}$$

$$\text{Precision}(\mathcal{G}_\pi, \mathcal{G}_p) = \frac{1}{n} \sum_{\mathbf{g} \in \mathcal{G}_p} f(\mathbf{g}, \mathcal{G}_\pi); \quad \text{Recall}(\mathcal{G}_\pi, \mathcal{G}_p) = \frac{1}{n} \sum_{\mathbf{g} \in \mathcal{G}_\pi} f(\mathbf{g}, \mathcal{G}_p).$$

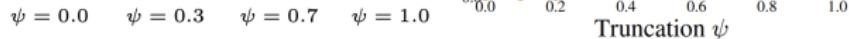
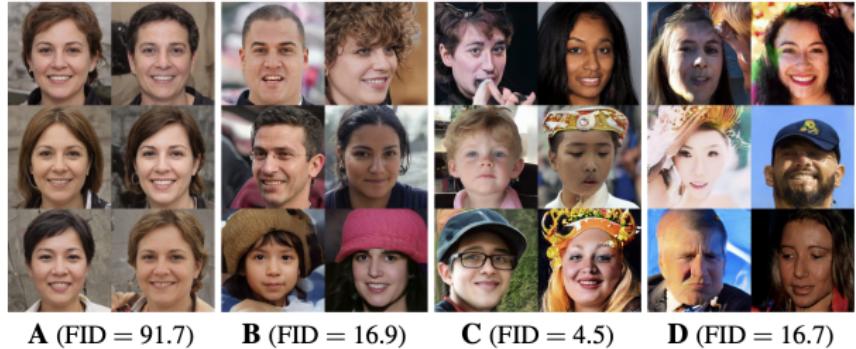
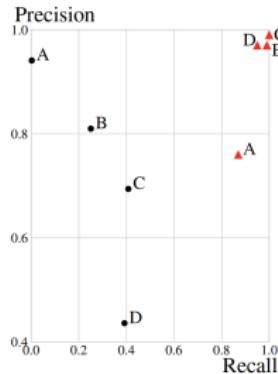


(a) True manifold



(b) Approx. manifold

Precision-Recall



Truncation trick

BigGAN: truncated normal sampling

$$p(\mathbf{z}|\psi) = \mathcal{N}(\mathbf{z}|0, \mathbf{I}) / \int_{-\infty}^{\psi} \mathcal{N}(\mathbf{z}'|0, \mathbf{I}) d\mathbf{z}'$$

Components of $\mathbf{z} \sim \mathcal{N}(0, \mathbf{I})$ which fall outside a predefined range are resampled.

StyleGAN

$$\mathbf{z}' = \hat{\mathbf{z}} + \psi \cdot (\mathbf{z} - \hat{\mathbf{z}}), \quad \hat{\mathbf{z}} = \mathbb{E}_{\mathbf{z}} \mathbf{z}$$

- ▶ Constant ψ is a tradeoff between diversity and fidelity.
- ▶ $\psi = 0.7$ is used for most of the results.

Brock A., Donahue J., Simonyan K. Large Scale GAN Training for High Fidelity Natural Image Synthesis, 2018

Karras T., Laine S., Aila T. A Style-Based Generator Architecture for Generative Adversarial Networks, 2018

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Langevin dynamic

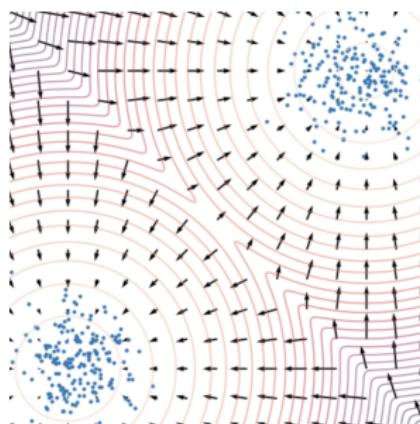
Statement

Let \mathbf{x}_0 be a random vector. Then samples from the following dynamics

$$\mathbf{x}_{l+1} = \mathbf{x}_l + \frac{\eta}{2} \cdot \nabla_{\mathbf{x}_l} \log p(\mathbf{x}_l | \theta) + \sqrt{\eta} \cdot \epsilon, \quad \epsilon \sim \mathcal{N}(0, \mathbf{I}).$$

will come from $p(\mathbf{x} | \theta)$ (under mild regularity conditions, for small enough η and large enough l).

- ▶ Here we assume that we already have some generative model $p(\mathbf{x} | \theta)$.
- ▶ The density $p(\mathbf{x} | \theta)$ is a **stationary** distribution for the Markov chain.
- ▶ What do we get if $\epsilon = \mathbf{0}$?



Energy-based models

Langevin dynamic

$$\mathbf{x}_{I+1} = \mathbf{x}_I + \frac{\eta}{2} \cdot \nabla_{\mathbf{x}_I} \log p(\mathbf{x}_I | \boldsymbol{\theta}) + \sqrt{\eta} \cdot \boldsymbol{\epsilon}, \quad \boldsymbol{\epsilon} \sim \mathcal{N}(0, \mathbf{I}).$$

We are able to sample from the model using Langevin dynamics if we have $\nabla_{\mathbf{x}} \log p(\mathbf{x} | \boldsymbol{\theta})$.

Unnormalized density

$$p(\mathbf{x} | \boldsymbol{\theta}) = \frac{\hat{p}(\mathbf{x} | \boldsymbol{\theta})}{Z_{\boldsymbol{\theta}}}, \quad \text{where } Z_{\boldsymbol{\theta}} = \int \hat{p}(\mathbf{x} | \boldsymbol{\theta}) d\mathbf{x}$$

- ▶ $\hat{p}(\mathbf{x} | \boldsymbol{\theta})$ is any non-negative function.
- ▶ If we use the reparametrization $\hat{p}(\mathbf{x} | \boldsymbol{\theta}) = \exp(-f_{\boldsymbol{\theta}}(\mathbf{x}))$, we remove the non-negativite constraint.

$$\nabla_{\mathbf{x}} \log p(\mathbf{x} | \boldsymbol{\theta}) = \nabla_{\mathbf{x}} \log \hat{p}(\mathbf{x} | \boldsymbol{\theta}) - \nabla_{\mathbf{x}} \log Z_{\boldsymbol{\theta}} = \nabla_{\mathbf{x}} \log \hat{p}(\mathbf{x} | \boldsymbol{\theta})$$

The gradient of the normalized density equals to the gradient of the unnormalized density.

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Score matching

Score function

$$\mathbf{s}_\theta(\mathbf{x}) = \nabla_{\mathbf{x}} \log p(\mathbf{x}|\theta)$$

Langevin dynamic

If we find the score function $\mathbf{s}_\theta(\mathbf{x}) = \nabla_{\mathbf{x}} \log p(\mathbf{x}|\theta)$ we will be able to sample from the model using Langevin dynamic.

$$\mathbf{x}_{I+1} = \mathbf{x}_I + \frac{\eta}{2} \cdot \nabla_{\mathbf{x}_I} \log p(\mathbf{x}_I|\theta) + \sqrt{\eta} \cdot \epsilon = \mathbf{x}_I + \frac{\eta}{2} \cdot \mathbf{s}_\theta(\mathbf{x}_I) + \sqrt{\eta} \cdot \epsilon.$$

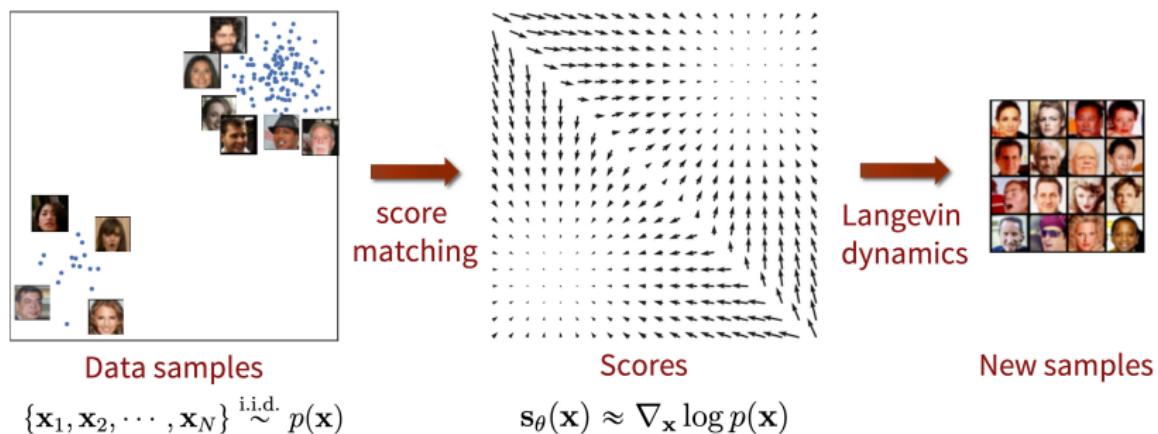
Fisher divergence

$$\begin{aligned} D_F(\pi, p) &= \frac{1}{2} \mathbb{E}_\pi \left\| \nabla_{\mathbf{x}} \log p(\mathbf{x}|\theta) - \nabla_{\mathbf{x}} \log \pi(\mathbf{x}) \right\|_2^2 = \\ &= \frac{1}{2} \mathbb{E}_\pi \left\| \mathbf{s}_\theta(\mathbf{x}) - \nabla_{\mathbf{x}} \log \pi(\mathbf{x}) \right\|_2^2 \rightarrow \min_\theta \end{aligned}$$

Score matching

Fisher divergence

$$D_F(\pi, p) = \frac{1}{2} \mathbb{E}_\pi \| \mathbf{s}_\theta(\mathbf{x}) - \nabla_{\mathbf{x}} \log \pi(\mathbf{x}) \|_2^2 \rightarrow \min_{\theta}$$



Problem: We do not know $\nabla_{\mathbf{x}} \log \pi(\mathbf{x})$.

Summary

- ▶ We need a measure of quality for the implicit models (like GANs).
- ▶ Frechet Inception Distance is the most popular metric for the implicit models evaluation.
- ▶ Maximum Mean Discrepancy tries to fix some of the FID drawbacks.
- ▶ Precision-recall allow to select model that compromises the sample quality and the sample diversity.
- ▶ Truncation tricks help to select model with the compromised samples: diverse and sharp.
- ▶ Langevin dynamics allows to sample from the generative model using the gradient of the log-likelihood.
- ▶ Score matching proposes to minimize the Fisher divergence to get the score function.