

# Deep Generative Models

## Lecture 5

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# Recap of Previous Lecture

## Assumptions

- ▶ Let  $c \sim \text{Cat}(\boldsymbol{\pi})$ , where

$$\boldsymbol{\pi} = (\pi_1, \dots, \pi_K), \quad \pi_k = P(c = k), \quad \sum_{k=1}^K \pi_k = 1.$$

- ▶ Suppose the VAE employs a discrete latent code  $c$ , with prior  $p(c) = \text{Uniform}\{1, \dots, K\}$ .

## ELBO

$$\mathcal{L}_{\phi, \theta}(\mathbf{x}) = \mathbb{E}_{q_\phi(c|\mathbf{x})} \log p_\theta(\mathbf{x}|c) - \text{KL}(q_\phi(c|\mathbf{x}) \| p(c)) \rightarrow \max_{\phi, \theta} .$$

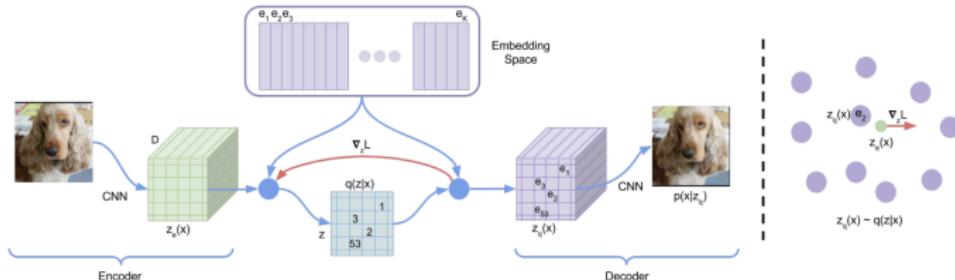
$$\text{KL}(q_\phi(c|\mathbf{x}) \| p(c)) = -H(q_\phi(c|\mathbf{x})) + \log K.$$

## Vector Quantization

Define the codebook  $\{\mathbf{e}_k\}_{k=1}^K$ , where  $\mathbf{e}_k \in \mathbb{R}^L$  and  $K$  is the size of the dictionary.

$$\mathbf{z}_q = \mathbf{q}(\mathbf{z}) = \mathbf{e}_{k^*}, \quad \text{where } k^* = \arg \min_k \|\mathbf{z} - \mathbf{e}_k\|.$$

# Recap of Previous Lecture



## Deterministic Variational Posterior

$$q_\phi(c_{ij} = k^* | \mathbf{x}) = \begin{cases} 1, & \text{if } k^* = \arg \min_k \|[\mathbf{z}_e]_{ij} - \mathbf{e}_k\|; \\ 0, & \text{otherwise.} \end{cases}$$

## ELBO

$$\mathcal{L}_{\phi, \theta}(\mathbf{x}) = \mathbb{E}_{q_\phi(c|\mathbf{x})} \log p_\theta(\mathbf{x}|\mathbf{e}_c) - \log K = \log p_\theta(\mathbf{x}|\mathbf{z}_q) - \log K.$$

## Straight-Through Gradient Estimation

$$\frac{\partial \log p(\mathbf{x}|\mathbf{z}_q, \theta)}{\partial \phi} = \frac{\partial \log p_\theta(\mathbf{x}|\mathbf{z}_q)}{\partial \mathbf{z}_q} \cdot \frac{\partial \mathbf{z}_q}{\partial \phi} \approx \frac{\partial \log p_\theta(\mathbf{x}|\mathbf{z}_q)}{\partial \mathbf{z}_q} \cdot \frac{\partial \mathbf{z}_e}{\partial \phi}$$

# Recap of Previous Lecture

## Theorem

$$\frac{1}{n} \sum_{i=1}^n \text{KL}(q_\phi(\mathbf{z}|\mathbf{x}_i) \| p(\mathbf{z})) = \text{KL}(q_{\text{agg},\phi}(\mathbf{z}) \| p(\mathbf{z})) + \mathbb{I}_q[\mathbf{x}, \mathbf{z}].$$

## ELBO Surgery

$$\frac{1}{n} \sum_{i=1}^n \mathcal{L}_{\phi,\theta}(\mathbf{x}_i) = \underbrace{\frac{1}{n} \sum_{i=1}^n \mathbb{E}_{q_\phi(\mathbf{z}|\mathbf{x}_i)} \log p_\theta(\mathbf{x}_i|\mathbf{z})}_{\text{Reconstruction Loss}} - \underbrace{\mathbb{I}_q[\mathbf{x}, \mathbf{z}]}_{\text{MI}} - \underbrace{\text{KL}(q_{\text{agg},\phi}(\mathbf{z}) \| p(\mathbf{z}))}_{\text{Marginal KL}}$$

## Optimal Prior

$$\text{KL}(q_{\text{agg},\phi}(\mathbf{z}) \| p(\mathbf{z})) = 0 \Leftrightarrow p(\mathbf{z}) = q_{\text{agg}}(\mathbf{z}) = \frac{1}{n} \sum_{i=1}^n q_\phi(\mathbf{z}|\mathbf{x}_i).$$

Thus, the optimal prior distribution  $p(\mathbf{z})$  is the aggregated variational posterior  $q_{\text{agg},\phi}(\mathbf{z})$ .

## Recap of Previous Lecture

- ▶ Standard Gaussian  $p(\mathbf{z}) = \mathcal{N}(0, \mathbf{I}) \Rightarrow$  over-regularization.
- ▶  $p(\mathbf{z}) = q_{\text{agg}, \phi}(\mathbf{z}) = \frac{1}{n} \sum_{i=1}^n q_\phi(\mathbf{z}|\mathbf{x}_i) \Rightarrow$  overfitting and extremely high computational cost.

## Revisiting ELBO

$$\frac{1}{n} \sum_{i=1}^n \mathcal{L}_{\phi, \theta}(\mathbf{x}_i) = \text{RL} - \text{MI} - \text{KL}(q_{\text{agg}, \phi}(\mathbf{z}) \parallel p_\lambda(\mathbf{z}))$$

This is the forward KL divergence with respect to  $p_\lambda(\mathbf{z})$ .

## ELBO with Learnable VAE Prior

$$\begin{aligned}\mathcal{L}_{\phi, \theta}(\mathbf{x}) &= \mathbb{E}_{q_\phi(\mathbf{z}|\mathbf{x})} [\log p_\theta(\mathbf{x}|\mathbf{z}) + \log p_\lambda(\mathbf{z}) - \log q_\phi(\mathbf{z}|\mathbf{x})] \\ &= \mathbb{E}_{q_\phi(\mathbf{z}|\mathbf{x})} \left[ \log p_\theta(\mathbf{x}|\mathbf{z}) + \underbrace{\left( \log p(f_\lambda(\mathbf{z})) + \log |\det(\mathbf{J}_f)| \right)}_{\text{flow-based prior}} - \log q_\phi(\mathbf{z}|\mathbf{x}) \right] \\ \mathbf{z} &= \mathbf{f}_\lambda^{-1}(\mathbf{z}^*) = \mathbf{g}_\lambda(\mathbf{z}^*), \quad \mathbf{z}^* \sim p(\mathbf{z}^*) = \mathcal{N}(0, \mathbf{I})\end{aligned}$$

# Outline

1. Likelihood-Free Learning
2. Generative Adversarial Networks (GAN)
3. Wasserstein Distance
4. Wasserstein GAN

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# Likelihood-Based Models

Poor Likelihood  
High-Quality Samples

$$p_1(\mathbf{x}) = \frac{1}{n} \sum_{i=1}^n \mathcal{N}(\mathbf{x} | \mathbf{x}_i, \epsilon \mathbf{I})$$

If  $\epsilon$  is very small, this model produces excellent, sharp samples but achieves poor likelihoods on test data.

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High Likelihood  
Poor Samples

$$\begin{aligned} p_2(\mathbf{x}) &= 0.01p(\mathbf{x}) + 0.99p_{\text{noise}}(\mathbf{x}) \\ \log [0.01p(\mathbf{x}) + 0.99p_{\text{noise}}(\mathbf{x})] &\geq \\ \geq \log [0.01p(\mathbf{x})] &= \log p(\mathbf{x}) - \log 100 \end{aligned}$$

This model contains mostly noisy, irrelevant samples; for high dimensions,  $\log p(\mathbf{x})$  scales linearly with  $m$ .

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This model contains mostly noisy, irrelevant samples; for high dimensions,  $\log p(\mathbf{x})$  scales linearly with  $m$ .

- ▶ Likelihood isn't always a suitable metric for evaluating generative models.
- ▶ Sometimes, the likelihood function can't even be computed exactly.

# Likelihood-Free Learning

## Motivation

We're interested in approximating the true data distribution  $p_{\text{data}}(\mathbf{x})$ . Instead of searching over all distributions, let's learn a model  $p_{\theta}(\mathbf{x}) \approx p_{\text{data}}(\mathbf{x})$ .

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Suppose we have two sets of samples:

- ▶  $\{\mathbf{x}_i\}_{i=1}^{n_1} \sim p_{\text{data}}(\mathbf{x})$  — real data;
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Define a discriminative model (classifier):

$$p(y = 1 | \mathbf{x}) = P(\mathbf{x} \sim p_{\text{data}}(\mathbf{x})); \quad p(y = 0 | \mathbf{x}) = P(\mathbf{x} \sim p_{\theta}(\mathbf{x}))$$

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## Assumption

The generative model  $p_{\theta}(\mathbf{x})$  matches  $p_{\text{data}}(\mathbf{x})$  if a discriminative model  $p(y|\mathbf{x})$  can't distinguish between them — that is, if  $p(y = 1|\mathbf{x}) = 0.5$  for every  $\mathbf{x}$ .

## Generative Adversarial Networks (GAN)

- ▶ The more expressive the discriminator, the closer we get to the optimal  $p_\theta(\mathbf{x})$ .
- ▶ Standard classifiers are trained by minimizing cross-entropy loss  $-\mathbb{E}_{\hat{p}(\mathbf{x}, y)} \log p(y|\mathbf{x})$  with  
$$\hat{p}(\mathbf{x}, y) = \frac{1}{2}[y = 1]p_{\text{data}}(\mathbf{x}) + \frac{1}{2}[y = 0]p_\theta(\mathbf{x}).$$

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## Cross-Entropy for Discriminator

$$\min_{p(y|\mathbf{x})} \left[ -\mathbb{E}_{p_{\text{data}}(\mathbf{x})} \log p(y=1|\mathbf{x}) - \mathbb{E}_{p_\theta(\mathbf{x})} \log p(y=0|\mathbf{x}) \right]$$
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## Generative Model

Suppose  $p_\theta(\mathbf{x}, \mathbf{z}) = p_\theta(\mathbf{x}|\mathbf{z})p(\mathbf{z})$ , where  $p(\mathbf{z})$  is a base distribution, and  $p_\theta(\mathbf{x}|\mathbf{z}) = \delta(\mathbf{x} - \mathbf{G}_\theta(\mathbf{z}))$  is deterministic.

# Generative Adversarial Networks (GAN)

## Cross-Entropy for Discriminative Model

$$\max_{p(y|x)} [\mathbb{E}_{p_{\text{data}}(x)} \log p(y=1|x) + \mathbb{E}_{p_{\theta}(x)} \log p(y=0|x)]$$

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- ▶ **Discriminator:** A classifier  $p_{\phi}(y=1|x) = D_{\phi}(x) \in [0, 1]$ , distinguishing real and generated samples. The discriminator aims to **maximize** cross-entropy.
- ▶ **Generator:** The generative model  $x = \mathbf{G}_{\theta}(z)$ ,  $z \sim p(z)$ , seeks to fool the discriminator. The generator aims to **minimize** cross-entropy.

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## GAN Objective

$$\min_G \max_D [\mathbb{E}_{p_{\text{data}}(x)} \log D(x) + \mathbb{E}_{p_{\theta}(x)} \log(1 - D(x))]$$

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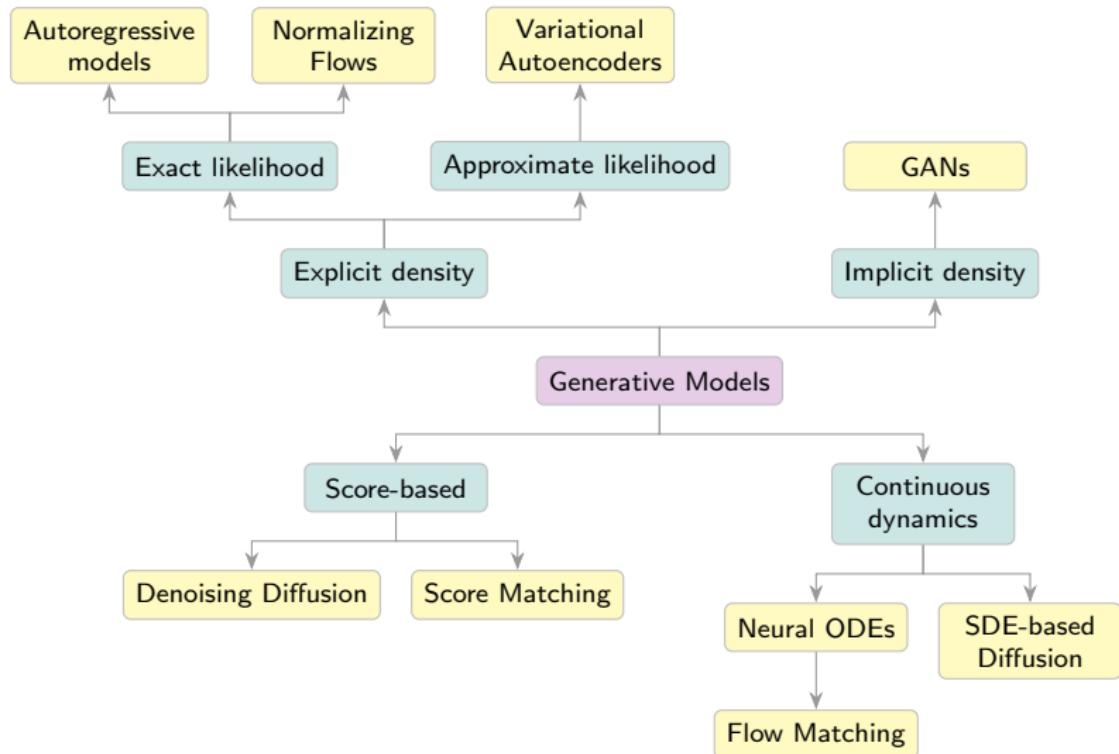
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# Outline

1. Likelihood-Free Learning
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# Generative Models Taxonomy



# GAN Optimality

## Theorem

The minimax game

$$\min_G \max_D \underbrace{\left[ \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \log D(\mathbf{x}) + \mathbb{E}_{p(\mathbf{z})} \log(1 - D(\mathbf{G}(\mathbf{z}))) \right]}_{V(G,D)}$$

achieves its global optimum when  $p_{\text{data}}(\mathbf{x}) = p_\theta(\mathbf{x})$ , and  $D^*(\mathbf{x}) = 0.5$ .

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## Proof (Fixed $G$ )

$$V(G, D) = \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \log D(\mathbf{x}) + \mathbb{E}_{p_\theta(\mathbf{x})} \log(1 - D(\mathbf{x}))$$

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$$\frac{dy(D)}{dD} = \frac{p_{\text{data}}(\mathbf{x})}{D(\mathbf{x})} - \frac{p_\theta(\mathbf{x})}{1 - D(\mathbf{x})} = 0 \quad \Rightarrow \quad D^*(\mathbf{x}) = \frac{p_{\text{data}}(\mathbf{x})}{p_{\text{data}}(\mathbf{x}) + p_\theta(\mathbf{x})}$$

# GAN Optimality

Proof Continued (Fixed  $D = D^*$ )

$$V(G, D^*) = \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \log \left( \frac{p_{\text{data}}(\mathbf{x})}{p_{\text{data}}(\mathbf{x}) + p_{\theta}(\mathbf{x})} \right) + \mathbb{E}_{p_{\theta}(\mathbf{x})} \log \left( \frac{p_{\theta}(\mathbf{x})}{p_{\text{data}}(\mathbf{x}) + p_{\theta}(\mathbf{x})} \right)$$

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Proof Continued (Fixed  $D = D^*$ )

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Jensen-Shannon Divergence (Symmetric KL Divergence)

$$\text{JSD}(p_{\text{data}}(\mathbf{x}) \parallel p_{\theta}(\mathbf{x})) = \frac{1}{2} [\text{KL}(p_{\text{data}}(\mathbf{x}) \parallel \star) + \text{KL}(p_{\theta}(\mathbf{x}) \parallel \star)]$$

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This can be regarded as a proper distance metric!

$$V(G^*, D^*) = -2 \log 2, \quad p_{\text{data}}(\mathbf{x}) = p_{\theta}(\mathbf{x}), \quad D^*(\mathbf{x}) = 0.5.$$

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The following minimax game

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If the generator can express **any** function and the discriminator is **optimal** at every step, the generator **will converge** to the target distribution.

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## Reality

- ▶ Generator updates are performed in parameter space, and the discriminator is often imperfectly optimized.
- ▶ Generator and discriminator losses typically oscillate during GAN training.

# GAN Training

Assume both generator and discriminator are parametric models:

$D_\phi(\mathbf{x})$  and  $\mathbf{G}_\theta(\mathbf{z})$ .

## Objective

$$\min_{\theta} \max_{\phi} \left[ \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \log D_\phi(\mathbf{x}) + \mathbb{E}_{p(\mathbf{z})} \log(1 - D_\phi(\mathbf{G}_\theta(\mathbf{z}))) \right]$$

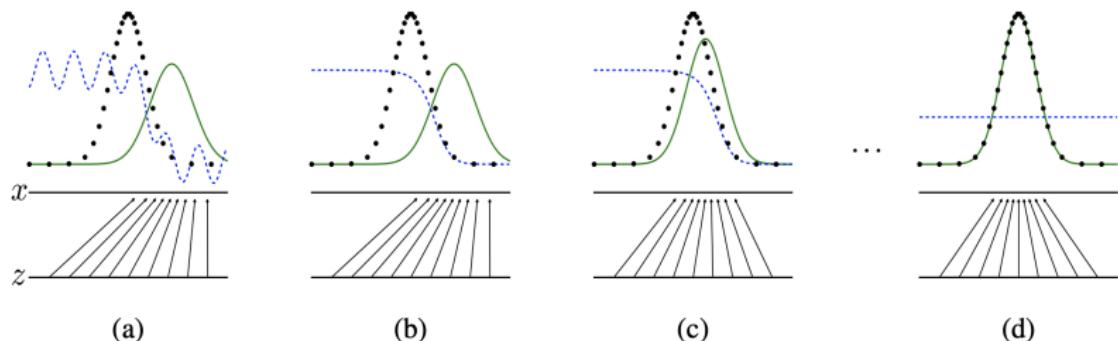
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$$\min_{\theta} \max_{\phi} [\mathbb{E}_{p_{\text{data}}(\mathbf{x})} \log D_\phi(\mathbf{x}) + \mathbb{E}_{p(\mathbf{z})} \log(1 - D_\phi(\mathbf{G}_\theta(\mathbf{z})))]$$

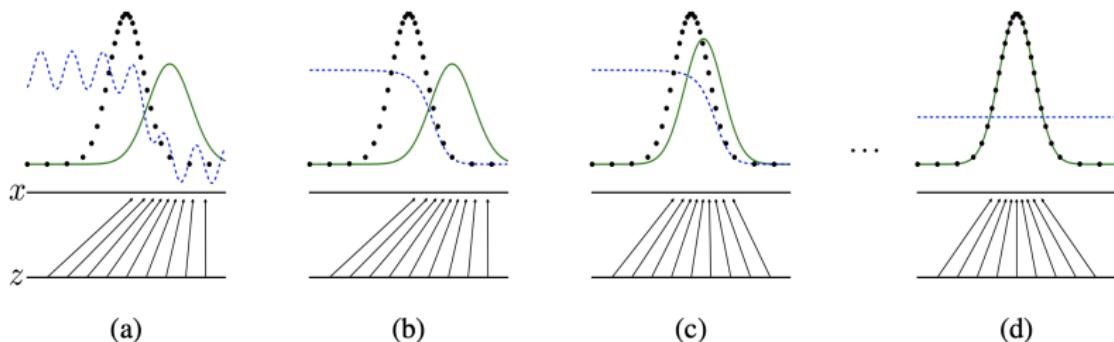


# GAN Training

Assume both generator and discriminator are parametric models:  
 $D_\phi(\mathbf{x})$  and  $\mathbf{G}_\theta(\mathbf{z})$ .

## Objective

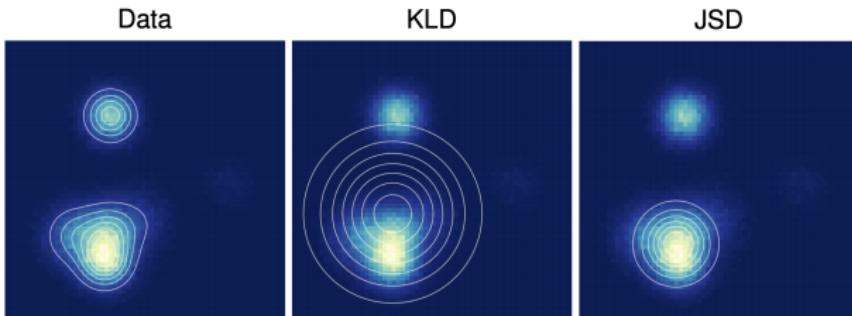
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- ▶  $\mathbf{z} \sim p(\mathbf{z})$  is a latent variable.
- ▶  $p_\theta(\mathbf{x}|\mathbf{z}) = \delta(\mathbf{x} - \mathbf{G}_\theta(\mathbf{z}))$  serves as a deterministic decoder (like normalizing flows).
- ▶ There is no encoder present.

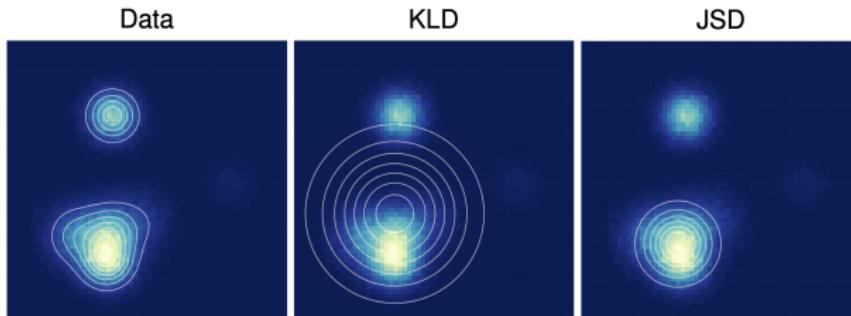
# Mode Collapse

Mode collapse refers to the phenomenon where the generator in a GAN produces only one or a few different modes of the distribution.



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Numerous methods have been proposed to tackle mode collapse: changing architectures, adding regularization terms, injecting noise.

*Goodfellow I. J. et al. Generative Adversarial Networks, 2014*

*Metz L. et al. Unrolled Generative Adversarial Networks, 2016*

## Jensen-Shannon vs Kullback-Leibler Divergences

- ▶  $p_{\text{data}}(\mathbf{x})$  is a fixed mixture of two Gaussians.
- ▶  $p(\mathbf{x}|\mu, \sigma) = \mathcal{N}(\mu, \sigma^2)$ .

### Mode Covering vs. Mode Seeking

$$\text{KL}(\pi \| p) = \int \pi(\mathbf{x}) \log \frac{\pi(\mathbf{x})}{p(\mathbf{x})} d\mathbf{x}, \quad \text{KL}(p \| \pi) = \int p(\mathbf{x}) \log \frac{p(\mathbf{x})}{\pi(\mathbf{x})} d\mathbf{x}$$

$$\text{JSD}(\pi \| p) = \frac{1}{2} \left[ \text{KL}\left(\pi(\mathbf{x}) \| \frac{\pi(\mathbf{x}) + p(\mathbf{x})}{2}\right) + \text{KL}\left(p(\mathbf{x}) \| \frac{\pi(\mathbf{x}) + p(\mathbf{x})}{2}\right) \right]$$

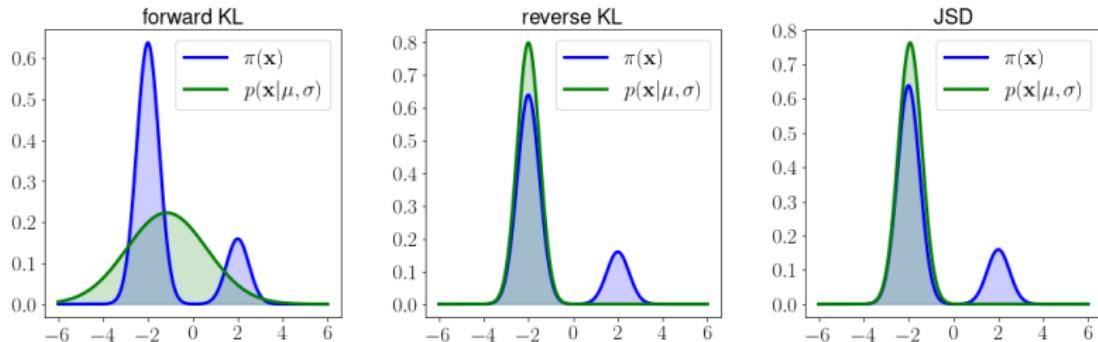
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# Outline

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2. Generative Adversarial Networks (GAN)
3. Wasserstein Distance
4. Wasserstein GAN

## Theoretical Results

- ▶ The dimensionality of  $\mathbf{z}$  is less than that of  $\mathbf{x}$ , so  $p_{\theta}(\mathbf{x})$  with  $\mathbf{x} = \mathbf{G}_{\theta}(\mathbf{z})$  lives on a low-dimensional manifold.

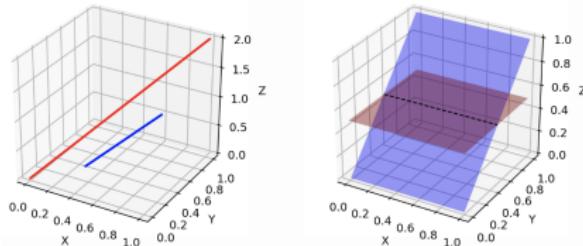
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Weng L. From GAN to WGAN, 2019

Arjovsky M., Bottou L. Towards Principled Methods for Training Generative Adversarial Networks, 2017

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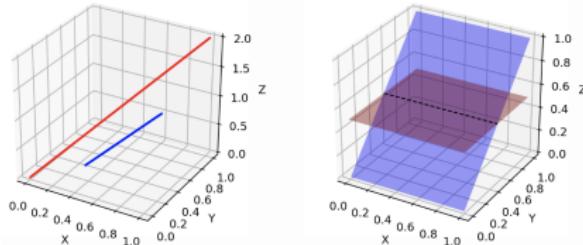


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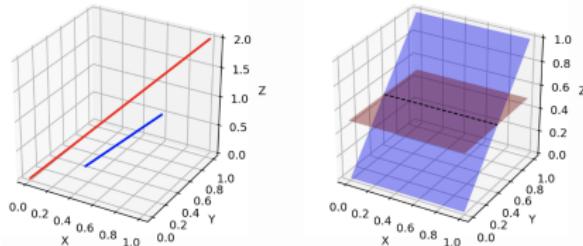
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- ▶ If  $p_{\text{data}}(\mathbf{x})$  and  $p_\theta(\mathbf{x})$  are disjoint, a smooth optimal discriminator can exist!
- ▶ For such low-dimensional, disjoint manifolds:

$$\text{KL}(p_{\text{data}} \parallel p_\theta) = \text{KL}(p_\theta \parallel p_{\text{data}}) = \infty, \quad \text{JSD}(p_{\text{data}} \parallel p_\theta) = \log 2$$

---

Weng L. From GAN to WGAN, 2019

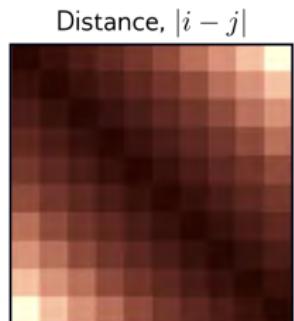
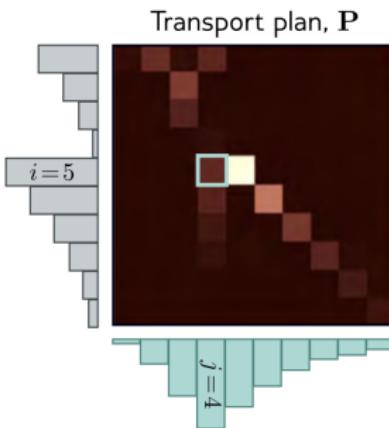
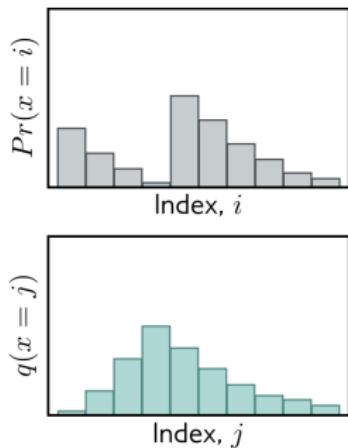
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# Wasserstein Distance (Discrete)

Also known as the **Earth Mover's Distance**.

## Optimal Transport Formulation

The minimum cost of moving and transforming a pile of "dirt" shaped like one probability distribution to match another.



$$\begin{aligned}\text{Wasserstein distance} \\ = \sum \mathbf{P} \cdot |i - j|\end{aligned}$$

## Wasserstein Distance (Continuous)

$$W(\pi \| p) = \inf_{\gamma \in \Gamma(\pi, p)} \mathbb{E}_{(\mathbf{x}_1, \mathbf{x}_2) \sim \gamma} \|\mathbf{x}_1 - \mathbf{x}_2\| = \inf_{\gamma \in \Gamma(\pi, p)} \int \|\mathbf{x}_1 - \mathbf{x}_2\| \gamma(\mathbf{x}_1, \mathbf{x}_2) d\mathbf{x}_1 d\mathbf{x}_2$$

- ▶  $\gamma(\mathbf{x}_1, \mathbf{x}_2)$  is the transport plan: the amount of “dirt” assigned from  $\mathbf{x}_1$  to  $\mathbf{x}_2$ .

$$\int \gamma(\mathbf{x}_1, \mathbf{x}_2) d\mathbf{x} = p(\mathbf{x}_2); \quad \int \gamma(\mathbf{x}_1, \mathbf{x}_2) d\mathbf{x}_2 = \pi(\mathbf{x}_1).$$

- ▶  $\Gamma(\pi, p)$  denotes the set of all joint distributions  $\gamma(\mathbf{x}_1, \mathbf{x}_2)$  with marginals  $\pi$  and  $p$ .
- ▶  $\gamma(\mathbf{x}_1, \mathbf{x}_2)$  is the mass,  $\|\mathbf{x}_1 - \mathbf{x}_2\|$  is the distance.

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## Wasserstein Metric

$$W_s(\pi, p) = \inf_{\gamma \in \Gamma(\pi, p)} \left( \mathbb{E}_{(\mathbf{x}_1, \mathbf{x}_2) \sim \gamma} \|\mathbf{x}_1 - \mathbf{x}_2\|^s \right)^{1/s}$$

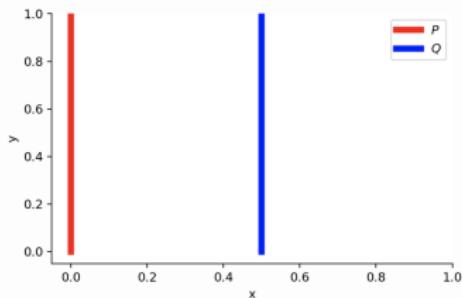
In our setting,  $W(\pi \| p) = W_1(\pi, p)$ , which is the transport cost formulation.

# Wasserstein Distance vs KL vs JSD

Consider two-dimensional distributions:

$$p_{\text{data}}(x, y) = (0, U[0, 1])$$

$$p_{\theta}(x, y) = (\theta, U[0, 1])$$

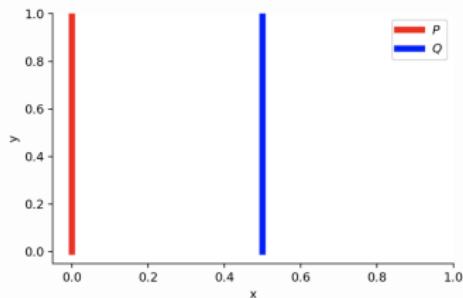


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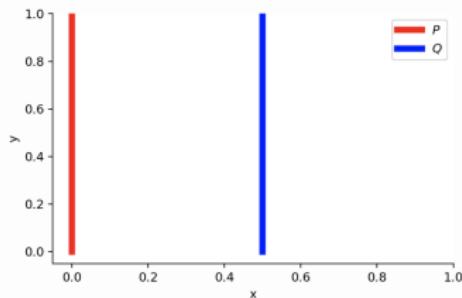
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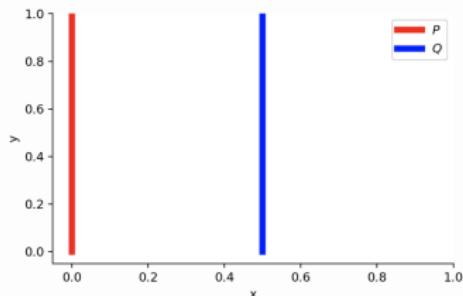
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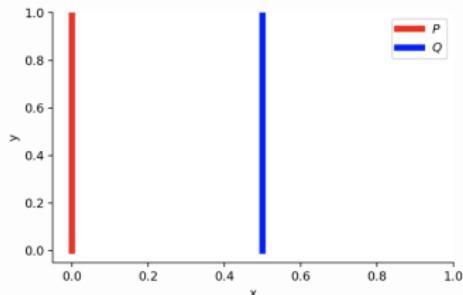
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$$W(p_{\text{data}} \| p_{\theta}) = |\theta|$$

# Wasserstein Distance vs KL vs JSD

## Theorem 1

Let  $\mathbf{G}_\theta(\mathbf{z})$  be (almost) any feedforward neural network, and  $p(\mathbf{z})$  a prior over  $\mathbf{z}$  such that  $\mathbb{E}_{p(\mathbf{z})}\|\mathbf{z}\| < \infty$ . Then  $W(p_{\text{data}}\|p_\theta)$  is continuous everywhere and differentiable almost everywhere.

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## Theorem 2

Let  $\pi$  be a distribution on a compact space  $\mathcal{X}$  and let  $\{p_t\}_{t=1}^\infty$  be a sequence of distributions on  $\mathcal{X}$ .

$$\text{KL}(\pi \| p_t) \rightarrow 0 \quad (\text{or } \text{KL}(p_t \| \pi) \rightarrow 0) \tag{1}$$

$$\text{JSD}(\pi \| p_t) \rightarrow 0 \tag{2}$$

$$W(\pi \| p_t) \rightarrow 0 \tag{3}$$

In summary, as  $t \rightarrow \infty$ , (1)  $\Rightarrow$  (2), and (2)  $\Rightarrow$  (3).

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# Wasserstein GAN

## Wasserstein Distance

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## Theorem (Kantorovich-Rubinstein Duality)

$$W(\pi \| p) = \frac{1}{K} \max_{\|f\|_L \leq K} \left[ \mathbb{E}_{\pi(\mathbf{x})} f(\mathbf{x}) - \mathbb{E}_{p(\mathbf{x})} f(\mathbf{x}) \right]$$

where  $f : \mathbb{R}^m \rightarrow \mathbb{R}$  is  $K$ -Lipschitz ( $\|f\|_L \leq K$ ):

$$|f(\mathbf{x}_1) - f(\mathbf{x}_2)| \leq K \|\mathbf{x}_1 - \mathbf{x}_2\|, \quad \forall \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X}.$$

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We can thus estimate  $W(\pi \| p)$  using only samples and a function  $f$ .

# Wasserstein GAN

## Theorem (Kantorovich-Rubinstein Duality)

$$W(p_{\text{data}} \| p_{\theta}) = \frac{1}{K} \max_{\|f\|_L \leq K} \left[ \mathbb{E}_{p_{\text{data}}(\mathbf{x})} f(\mathbf{x}) - \mathbb{E}_{p_{\theta}(\mathbf{x})} f(\mathbf{x}) \right]$$

- ▶ We must ensure that  $f$  is  $K$ -Lipschitz continuous.
- ▶ Let  $f_{\phi}(\mathbf{x})$  be a feedforward neural network parameterized by  $\phi$ .
- ▶ If the weights  $\phi$  are restricted to a compact set  $\Phi$ , then  $f_{\phi}$  is  $K$ -Lipschitz.

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- ▶ If the weights  $\phi$  are restricted to a compact set  $\Phi$ , then  $f_{\phi}$  is  $K$ -Lipschitz.
- ▶ Clamp weights within the box  $\Phi = [-c, c]^d$  (e.g.  $c = 0.01$ ) after each update.

$$\begin{aligned} K \cdot W(p_{\text{data}} \| p_{\theta}) &= \max_{\|f\|_L \leq K} \left[ \mathbb{E}_{p_{\text{data}}(\mathbf{x})} f(\mathbf{x}) - \mathbb{E}_{p_{\theta}(\mathbf{x})} f(\mathbf{x}) \right] \geq \\ &\geq \max_{\phi \in \Phi} \left[ \mathbb{E}_{p_{\text{data}}(\mathbf{x})} f_{\phi}(\mathbf{x}) - \mathbb{E}_{p_{\theta}(\mathbf{x})} f_{\phi}(\mathbf{x}) \right] \end{aligned}$$

# Wasserstein GAN

## Standard GAN Objective

$$\min_{\theta} \max_{\phi} \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \log D_{\phi}(\mathbf{x}) + \mathbb{E}_{p(\mathbf{z})} \log(1 - D_{\phi}(\mathbf{G}_{\theta}(\mathbf{z})))$$

## WGAN Objective

$$\min_{\theta} W(p_{\text{data}} \| p_{\theta}) \approx \min_{\theta} \max_{\phi \in \Phi} \left[ \mathbb{E}_{p_{\text{data}}(\mathbf{x})} f_{\phi}(\mathbf{x}) - \mathbb{E}_{p(\mathbf{z})} f_{\phi}(\mathbf{G}_{\theta}(\mathbf{z})) \right]$$

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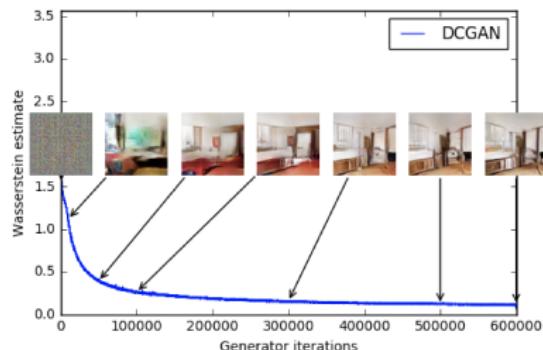
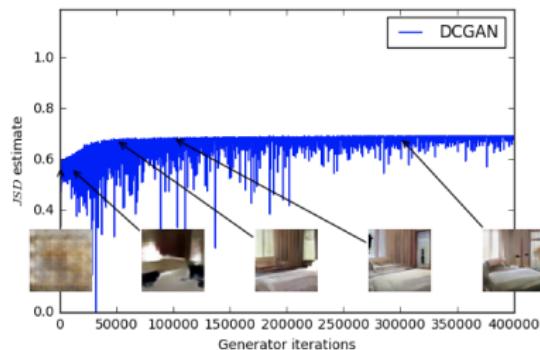
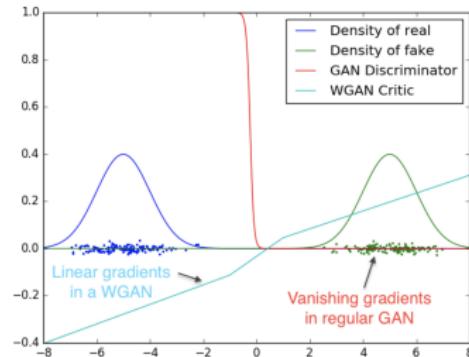
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- ▶ The discriminator  $D$  is replaced by function  $f$ : in WGAN, it is known as the **critic**, which is *not* a classifier.
- ▶ "*Weight clipping is a clearly terrible way to enforce a Lipschitz constraint.*"
  - ▶ If  $c$  is large, optimizing the critic is hard.
  - ▶ If  $c$  is small, gradients may vanish.

# Wasserstein GAN

- ▶ WGAN provides nonzero gradients even if distributions' supports are disjoint.
- ▶  $JSD(p_{\text{data}} \| p_{\theta})$  is poorly correlated with sample quality and remains near its maximum value  $\log 2 \approx 0.69$ .
- ▶  $W(p_{\text{data}} \| p_{\theta})$  is tightly correlated with quality.



## Summary

- ▶ Likelihood is not a reliable metric for generative model evaluation.
- ▶ Adversarial learning casts distribution matching as a minimax game.
- ▶ GANs, in theory, optimize the Jensen-Shannon divergence.
- ▶ KL and JS divergences fail as objectives when the model and data distributions are disjoint.
- ▶ The Earth Mover's (Wasserstein) distance provides a more meaningful loss for distribution matching.
- ▶ Kantorovich-Rubinstein duality allows us to compute the EM distance using only samples.
- ▶ Wasserstein GAN enforces the Lipschitz condition on the critic through weight clipping—although better alternatives exist.