# Deep Generative Models

## Lecture 5

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AI Masters

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# Recap of Previous Lecture

## Assumptions

- Let $c \sim \text{Cat}(\boldsymbol{\pi})$, where
$$\boldsymbol{\pi} = (\pi_1, \dots, \pi_K), \quad \pi_k = P(c = k), \quad \sum_{k=1}^{K} \pi_k = 1.$$
- Suppose the VAE employs a discrete latent code $c$, with prior $p(c) = \text{Uniform}\{1, \dots, K\}$.

## ELBO

$$\mathcal{L}_{\boldsymbol{\phi},\boldsymbol{\theta}}(\mathbf{x}) = \mathbb{E}_{q_{\boldsymbol{\phi}}(c|\mathbf{x})} \log p_{\boldsymbol{\theta}}(\mathbf{x}|c) - \mathrm{KL}(q_{\boldsymbol{\phi}}(c|\mathbf{x}) \,\|\, p(c)) \to \max_{\boldsymbol{\phi},\boldsymbol{\theta}}.$$

$$\mathrm{KL}(q_{\boldsymbol{\phi}}(c|\mathbf{x}) \,\|\, p(c)) = -\mathrm{H}(q_{\boldsymbol{\phi}}(c|\mathbf{x})) + \log K.$$

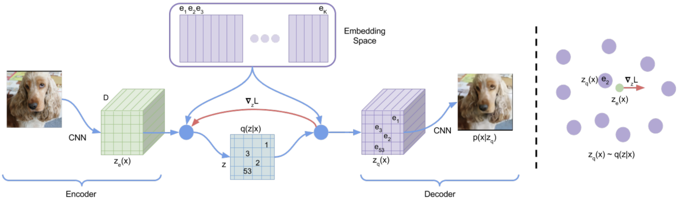
## Vector Quantization

Define the codebook $\{\mathbf{e}_k\}_{k=1}^{K}$, where $\mathbf{e}_k \in \mathbb{R}^L$ and $K$ is the size of the dictionary.

$$\mathbf{z}_q = \mathbf{q}(\mathbf{z}) = \mathbf{e}_{k^*}, \quad \text{where} \;\; k^* = \arg\min_k \|\mathbf{z} - \mathbf{e}_k\|.$$

*Oord A., Vinyals O., Kavukcuoglu K. Neural Discrete Representation Learning, 2017*

# Recap of Previous Lecture




## Deterministic Variational Posterior

$$q_\phi(c_{ij} = k^*|\mathbf{x}) = \begin{cases} 1, & \text{if } \; k^* = \arg\min_k \|[\mathbf{z}_e]_{ij} - \mathbf{e}_k\|; \\ 0, & \text{otherwise.} \end{cases}$$

## ELBO

$$\mathcal{L}_{\phi,\boldsymbol{\theta}}(\mathbf{x}) = \mathbb{E}_{q_\phi(c|\mathbf{x})} \log p_{\boldsymbol{\theta}}(\mathbf{x}|\mathbf{e}_c) - \log K = \log p_{\boldsymbol{\theta}}(\mathbf{x}|\mathbf{z}_q) - \log K.$$

## Straight-Through Gradient Estimation

$$\frac{\partial \log p(\mathbf{x}|\mathbf{z}_q, \boldsymbol{\theta})}{\partial \phi} = \frac{\partial \log p_{\boldsymbol{\theta}}(\mathbf{x}|\mathbf{z}_q)}{\partial \mathbf{z}_q} \cdot \frac{\partial \mathbf{z}_q}{\partial \phi} \approx \frac{\partial \log p_{\boldsymbol{\theta}}(\mathbf{x}|\mathbf{z}_q)}{\partial \mathbf{z}_q} \cdot \frac{\partial \mathbf{z}_e}{\partial \phi}$$

*Oord A., Vinyals O., Kavukcuoglu K. Neural Discrete Representation Learning, 2017*

# Recap of Previous Lecture

## Theorem

$$\frac{1}{n}\sum_{i=1}^{n} \mathrm{KL}(q_\phi(\mathbf{z}|\mathbf{x}_i) \,\|\, p(\mathbf{z})) = \mathrm{KL}(q_{\text{agg},\phi}(\mathbf{z}) \,\|\, p(\mathbf{z})) + \mathbb{I}_q[\mathbf{x}, \mathbf{z}].$$

## ELBO Surgery

$$\frac{1}{n}\sum_{i=1}^{n} \mathcal{L}_{\phi,\theta}(\mathbf{x}_i) = \underbrace{\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}_{q_\phi(\mathbf{z}|\mathbf{x}_i)} \log p_\theta(\mathbf{x}_i|\mathbf{z})}_{\text{Reconstruction Loss}} - \underbrace{\mathbb{I}_q[\mathbf{x}, \mathbf{z}]}_{\text{MI}} - \underbrace{\mathrm{KL}(q_{\text{agg},\phi}(\mathbf{z}) \,\|\, p(\mathbf{z}))}_{\text{Marginal KL}}$$

## Optimal Prior

$$\mathrm{KL}(q_{\text{agg},\phi}(\mathbf{z}) \,\|\, p(\mathbf{z})) = 0 \;\Leftrightarrow\; p(\mathbf{z}) = q_{\text{agg}}(\mathbf{z}) = \frac{1}{n}\sum_{i=1}^{n} q_\phi(\mathbf{z}|\mathbf{x}_i).$$

Thus, the optimal prior distribution $p(\mathbf{z})$ is the aggregated variational posterior $q_{\text{agg},\phi}(\mathbf{z})$.

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*Hoffman M. D., Johnson M. J. ELBO surgery: yet another way to carve up the variational evidence lower bound, 2016*

# Recap of Previous Lecture

- Standard Gaussian $p(\mathbf{z}) = \mathcal{N}(0, \mathbf{I}) \Rightarrow$ over-regularization.
- $p(\mathbf{z}) = q_{\text{agg},\phi}(\mathbf{z}) = \frac{1}{n}\sum_{i=1}^{n} q_\phi(\mathbf{z}|\mathbf{x}_i) \Rightarrow$ overfitting and extremely high computational cost.

## Revisiting ELBO

$$\frac{1}{n}\sum_{i=1}^{n} \mathcal{L}_{\phi,\boldsymbol{\theta}}(\mathbf{x}_i) = \text{RL} - \text{MI} - \text{KL}(q_{\text{agg},\phi}(\mathbf{z}) \,\|\, p_{\boldsymbol{\lambda}}(\mathbf{z}))$$

This is the forward KL divergence with respect to $p_{\boldsymbol{\lambda}}(\mathbf{z})$.

## ELBO with Learnable VAE Prior

$$\mathcal{L}_{\phi,\boldsymbol{\theta}}(\mathbf{x}) = \mathbb{E}_{q_\phi(\mathbf{z}|\mathbf{x})}\left[\log p_{\boldsymbol{\theta}}(\mathbf{x}|\mathbf{z}) + \log p_{\boldsymbol{\lambda}}(\mathbf{z}) - \log q_\phi(\mathbf{z}|\mathbf{x})\right]$$
$$= \mathbb{E}_{q_\phi(\mathbf{z}|\mathbf{x})}\Bigg[\log p_{\boldsymbol{\theta}}(\mathbf{x}|\mathbf{z}) + \underbrace{\Big(\log p(f_{\boldsymbol{\lambda}}(\mathbf{z})) + \log|\det(\mathbf{J}_\mathbf{f})|\Big)}_{\text{flow-based prior}} - \log q_\phi(\mathbf{z}|\mathbf{x})\Bigg]$$

$$\mathbf{z} = \mathbf{f}_{\boldsymbol{\lambda}}^{-1}(\mathbf{z}^*) = \mathbf{g}_{\boldsymbol{\lambda}}(\mathbf{z}^*), \quad \mathbf{z}^* \sim p(\mathbf{z}^*) = \mathcal{N}(0, \mathbf{I})$$

*Chen X. et al. Variational Lossy Autoencoder, 2016*

# Outline

1. Likelihood-Free Learning
2. Generative Adversarial Networks (GAN)
3. Wasserstein Distance
4. Wasserstein GAN

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# Likelihood-Based Models

## Poor Likelihood
## High-Quality Samples

$$p_1(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n}\mathcal{N}(\mathbf{x}|\mathbf{x}_i, \epsilon\mathbf{I})$$

If $\epsilon$ is very small, this model produces excellent, sharp samples but achieves poor likelihoods on test data.

*Theis L., Oord A., Bethge M. A Note on the Evaluation of Generative Models, 2015*

# Likelihood-Based Models

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**High Likelihood**
**Poor Samples**

$$p_2(\mathbf{x}) = 0.01p(\mathbf{x}) + 0.99p_{\text{noise}}(\mathbf{x})$$

$$\log\left[0.01p(\mathbf{x}) + 0.99p_{\text{noise}}(\mathbf{x})\right] \geq$$
$$\geq \log\left[0.01p(\mathbf{x})\right] = \log p(\mathbf{x}) - \log 100$$

This model contains mostly noisy, irrelevant samples; for high dimensions, $\log p(\mathbf{x})$ scales linearly with $m$.

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This model contains mostly noisy, irrelevant samples; for high dimensions, $\log p(\mathbf{x})$ scales linearly with $m$.

- Likelihood isn't always a suitable metric for evaluating generative models.
- Sometimes, the likelihood function can't even be computed exactly.

*Theis L., Oord A., Bethge M. A Note on the Evaluation of Generative Models, 2015*

# Likelihood-Free Learning

## Motivation

We're interested in approximating the true data distribution $p_{\text{data}}(\mathbf{x})$. Instead of searching over all distributions, let's learn a model $p_{\boldsymbol{\theta}}(\mathbf{x}) \approx p_{\text{data}}(\mathbf{x})$.

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Suppose we have two sets of samples:

- $\{\mathbf{x}_i\}_{i=1}^{n_1} \sim p_{\text{data}}(\mathbf{x})$ — real data;
- $\{\mathbf{x}_i\}_{i=1}^{n_2} \sim p_{\boldsymbol{\theta}}(\mathbf{x})$ — generated (fake) data.

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Define a discriminative model (classifier):

$$p(y = 1|\mathbf{x}) = P(\mathbf{x} \sim p_{\text{data}}(\mathbf{x})); \quad p(y = 0|\mathbf{x}) = P(\mathbf{x} \sim p_{\boldsymbol{\theta}}(\mathbf{x}))$$

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## Assumption

The generative model $p_{\boldsymbol{\theta}}(\mathbf{x})$ matches $p_{\text{data}}(\mathbf{x})$ if a discriminative model $p(y|\mathbf{x})$ can't distinguish between them — that is, if $p(y = 1|\mathbf{x}) = 0.5$ for every $\mathbf{x}$.

# Generative Adversarial Networks (GAN)

- The more expressive the discriminator, the closer we get to the optimal $p_{\boldsymbol{\theta}}(\mathbf{x})$.
- Standard classifiers are trained by minimizing cross-entropy loss $-\mathbb{E}_{\hat{p}(\mathbf{x},y)} \log p(y|\mathbf{x})$ with $\hat{p}(\mathbf{x}, y) = \frac{1}{2}[y = 1]p_{\text{data}}(\mathbf{x}) + \frac{1}{2}[y = 0]p_{\boldsymbol{\theta}}(\mathbf{x})$.

*Goodfellow I. J. et al. Generative Adversarial Networks, 2014*

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$\hat{p}(\mathbf{x}, y) = \frac{1}{2}[y = 1]p_{\text{data}}(\mathbf{x}) + \frac{1}{2}[y = 0]p_{\boldsymbol{\theta}}(\mathbf{x})$.

## Cross-Entropy for Discriminator

$$\min_{p(y|\mathbf{x})} \left[-\mathbb{E}_{p_{\text{data}}(\mathbf{x})} \log p(y = 1|\mathbf{x}) - \mathbb{E}_{p_{\boldsymbol{\theta}}(\mathbf{x})} \log p(y = 0|\mathbf{x})\right]$$

$$\max_{p(y|\mathbf{x})} \left[\mathbb{E}_{p_{\text{data}}(\mathbf{x})} \log p(y = 1|\mathbf{x}) + \mathbb{E}_{p_{\boldsymbol{\theta}}(\mathbf{x})} \log p(y = 0|\mathbf{x})\right]$$

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## Generative Model

Suppose $p_{\boldsymbol{\theta}}(\mathbf{x},\mathbf{z}) = p_{\boldsymbol{\theta}}(\mathbf{x}|\mathbf{z})p(\mathbf{z})$, where $p(\mathbf{z})$ is a base distribution, and $p_{\boldsymbol{\theta}}(\mathbf{x}|\mathbf{z}) = \delta(\mathbf{x} - \mathbf{G}_{\boldsymbol{\theta}}(\mathbf{z}))$ is deterministic.

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*Goodfellow I. J. et al. Generative Adversarial Networks, 2014*

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- **Discriminator:** A classifier $p_{\phi}(y = 1|\mathbf{x}) = D_{\phi}(\mathbf{x}) \in [0, 1]$, distinguishing real and generated samples. The discriminator aims to **maximize** cross-entropy.
- **Generator:** The generative model $\mathbf{x} = \mathbf{G}_{\boldsymbol{\theta}}(\mathbf{z})$, $\mathbf{z} \sim p(\mathbf{z})$, seeks to fool the discriminator. The generator aims to **minimize** cross-entropy.

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## GAN Objective

$$\min_{G} \max_{D} \left[ \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \log D(\mathbf{x}) + \mathbb{E}_{p_{\boldsymbol{\theta}}(\mathbf{x})} \log(1 - D(\mathbf{x})) \right]$$

*Goodfellow I. J. et al. Generative Adversarial Networks, 2014*

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$$\min_{G} \max_{D} \left[\mathbb{E}_{p_{\text{data}}(\mathbf{x})} \log D(\mathbf{x}) + \mathbb{E}_{p(\mathbf{z})} \log(1 - D(\mathbf{G}(\mathbf{z})))\right]$$

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*Goodfellow I. J. et al. Generative Adversarial Networks, 2014*

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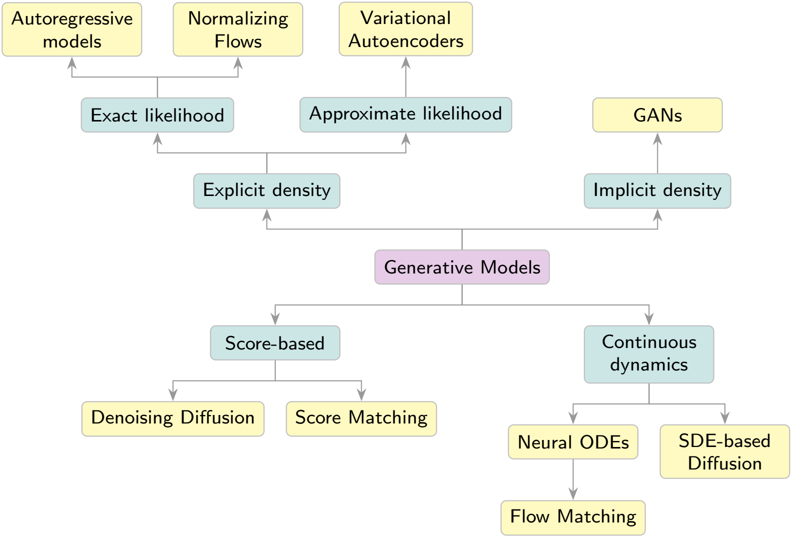
# Generative Models Taxonomy



# GAN Optimality

### Theorem

The minimax game

$$\min_{G} \max_{D} \Big[ \underbrace{\mathbb{E}_{p_{\text{data}}(\mathbf{x})} \log D(\mathbf{x}) + \mathbb{E}_{p(\mathbf{z})} \log(1 - D(\mathbf{G}(\mathbf{z})))}_{V(G,D)} \Big]$$

achieves its global optimum when $p_{\text{data}}(\mathbf{x}) = p_{\boldsymbol{\theta}}(\mathbf{x})$, and $D^*(\mathbf{x}) = 0.5$.

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achieves its global optimum when $p_{\text{data}}(\mathbf{x}) = p_{\boldsymbol{\theta}}(\mathbf{x})$, and $D^*(\mathbf{x}) = 0.5$.

## Proof (Fixed $G$)

$$V(G, D) = \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \log D(\mathbf{x}) + \mathbb{E}_{p_{\boldsymbol{\theta}}(\mathbf{x})} \log(1 - D(\mathbf{x}))$$

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$$\frac{dy(D)}{dD} = \frac{p_{\text{data}}(\mathbf{x})}{D(\mathbf{x})} - \frac{p_{\boldsymbol{\theta}}(\mathbf{x})}{1 - D(\mathbf{x})} = 0 \qquad \Rightarrow \qquad D^*(\mathbf{x}) = \frac{p_{\text{data}}(\mathbf{x})}{p_{\text{data}}(\mathbf{x}) + p_{\boldsymbol{\theta}}(\mathbf{x})}$$

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*Goodfellow I. J. et al. Generative Adversarial Networks, 2014*

# GAN Optimality

## Proof Continued (Fixed $D = D^*$)

$$V(G, D^*) = \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \log \left( \frac{p_{\text{data}}(\mathbf{x})}{p_{\text{data}}(\mathbf{x}) + p_{\boldsymbol{\theta}}(\mathbf{x})} \right) + \mathbb{E}_{p_{\boldsymbol{\theta}}(\mathbf{x})} \log \left( \frac{p_{\boldsymbol{\theta}}(\mathbf{x})}{p_{\text{data}}(\mathbf{x}) + p_{\boldsymbol{\theta}}(\mathbf{x})} \right)$$

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$$= \mathrm{KL}\left( p_{\text{data}}(\mathbf{x}) \,\|\, \frac{p_{\text{data}}(\mathbf{x}) + p_{\boldsymbol{\theta}}(\mathbf{x})}{2} \right) + \mathrm{KL}\left( p_{\boldsymbol{\theta}}(\mathbf{x}) \,\|\, \frac{p_{\text{data}}(\mathbf{x}) + p_{\boldsymbol{\theta}}(\mathbf{x})}{2} \right) - 2 \log 2$$

*Goodfellow I. J. et al. Generative Adversarial Networks, 2014*

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$$= 2\,\mathrm{JSD}(p_{\text{data}}(\mathbf{x}) \,\|\, p_{\boldsymbol{\theta}}(\mathbf{x})) - 2\log 2.$$

*Goodfellow I. J. et al. Generative Adversarial Networks, 2014*

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## Jensen-Shannon Divergence (Symmetric KL Divergence)

$$\mathrm{JSD}(p_{\text{data}}(\mathbf{x}) \| p_{\boldsymbol{\theta}}(\mathbf{x})) = \frac{1}{2} \left[ \mathrm{KL} \left( p_{\text{data}}(\mathbf{x}) \| \star \right) + \mathrm{KL} \left( p_{\boldsymbol{\theta}}(\mathbf{x}) \| \star \right) \right]$$

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## Jensen-Shannon Divergence (Symmetric KL Divergence)

$$\mathrm{JSD}(p_{\text{data}}(\mathbf{x}) \| p_{\boldsymbol{\theta}}(\mathbf{x})) = \frac{1}{2} \left[ \mathrm{KL} \left( p_{\text{data}}(\mathbf{x}) \| \star \right) + \mathrm{KL} \left( p_{\boldsymbol{\theta}}(\mathbf{x}) \| \star \right) \right]$$

This can be regarded as a proper distance metric!

$$V(G^*, D^*) = -2 \log 2, \quad p_{\text{data}}(\mathbf{x}) = p_{\boldsymbol{\theta}}(\mathbf{x}), \quad D^*(\mathbf{x}) = 0.5.$$

---

*Goodfellow I. J. et al. Generative Adversarial Networks, 2014*

# GAN Optimality

### Theorem

The following minimax game

$$\min_{G} \max_{D} \Big[ \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \log D(\mathbf{x}) + \mathbb{E}_{p(\mathbf{z})} \log(1 - D(\mathbf{G}(\mathbf{z}))) \Big]$$

achieves its global optimum precisely when $p_{\text{data}}(\mathbf{x}) = p_{\boldsymbol{\theta}}(\mathbf{x})$, and $D^*(\mathbf{x}) = 0.5$.

### Expectations

If the generator can express **any** function and the discriminator is **optimal** at every step, the generator **will converge** to the target distribution.

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## Expectations

If the generator can express **any** function and the discriminator is **optimal** at every step, the generator **will converge** to the target distribution.

## Reality

- Generator updates are performed in parameter space, and the discriminator is often imperfectly optimized.
- Generator and discriminator losses typically oscillate during GAN training.

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*Goodfellow I. J. et al. Generative Adversarial Networks, 2014*

# GAN Training

Assume both generator and discriminator are parametric models: $D_{\boldsymbol{\phi}}(\mathbf{x})$ and $\mathbf{G}_{\boldsymbol{\theta}}(\mathbf{z})$.

## Objective

$$\min_{\boldsymbol{\theta}} \max_{\boldsymbol{\phi}} \left[ \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \log D_{\boldsymbol{\phi}}(\mathbf{x}) + \mathbb{E}_{p(\mathbf{z})} \log(1 - D_{\boldsymbol{\phi}}(\mathbf{G}_{\boldsymbol{\theta}}(\mathbf{z}))) \right]$$

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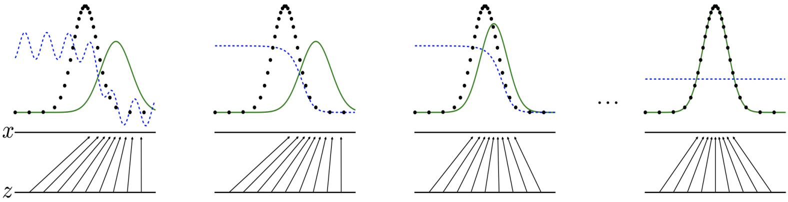


(a) (b) (c) (d)

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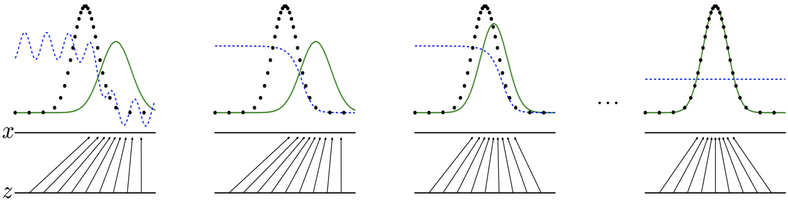


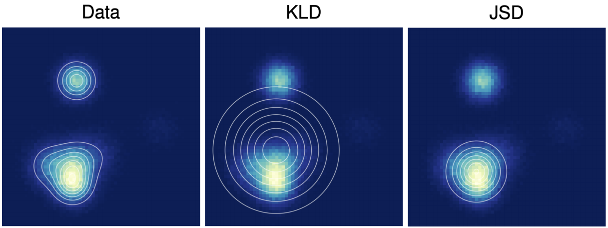

(a) (b) (c) (d)

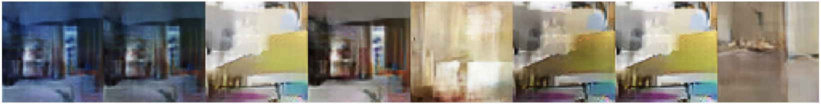
- $\mathbf{z} \sim p(\mathbf{z})$ is a latent variable.
- $p_{\theta}(\mathbf{x}|\mathbf{z}) = \delta(\mathbf{x} - \mathbf{G}_{\theta}(\mathbf{z}))$ serves as a deterministic decoder (like normalizing flows).
- There is no encoder present.

*Goodfellow I. J. et al. Generative Adversarial Networks, 2014*

# Mode Collapse

Mode collapse refers to the phenomenon where the generator in a GAN produces only one or a few different modes of the distribution.



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*Metz L. et al. Unrolled Generative Adversarial Networks, 2016*

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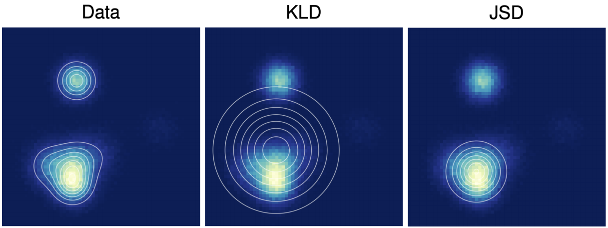


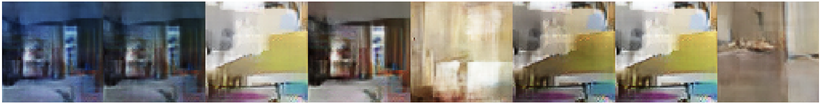

Numerous methods have been proposed to tackle mode collapse: changing architectures, adding regularization terms, injecting noise.

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*Goodfellow I. J. et al. Generative Adversarial Networks, 2014*
*Metz L. et al. Unrolled Generative Adversarial Networks, 2016*

# Jensen-Shannon vs Kullback-Leibler Divergences

- $p_{\text{data}}(\mathbf{x})$ is a fixed mixture of two Gaussians.
- $p(\mathbf{x}|\mu, \sigma) = \mathcal{N}(\mu, \sigma^2)$.

## Mode Covering vs. Mode Seeking

$$\mathrm{KL}(\pi \,\|\, p) = \int \pi(\mathbf{x}) \log \frac{\pi(\mathbf{x})}{p(\mathbf{x})} d\mathbf{x}, \quad \mathrm{KL}(p\|\pi) = \int p(\mathbf{x}) \log \frac{p(\mathbf{x})}{\pi(\mathbf{x})} d\mathbf{x}$$

$$\mathrm{JSD}(\pi \,\|\, p) = \frac{1}{2}\Big[\mathrm{KL}\Big(\pi(\mathbf{x}) \,\|\, \frac{\pi(\mathbf{x}) + p(\mathbf{x})}{2}\Big) + \mathrm{KL}\Big(p(\mathbf{x}) \,\|\, \frac{\pi(\mathbf{x}) + p(\mathbf{x})}{2}\Big)\Big]$$

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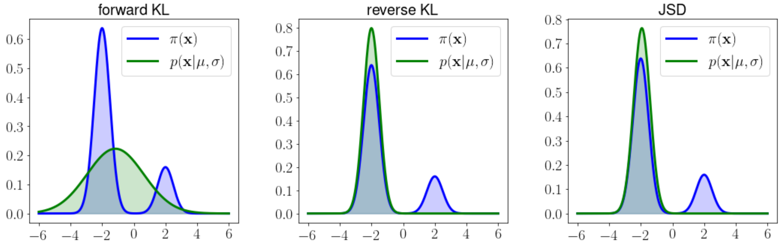

# Outline

1. Likelihood-Free Learning

2. Generative Adversarial Networks (GAN)

3. Wasserstein Distance

4. Wasserstein GAN

# Theoretical Results

- The dimensionality of $\mathbf{z}$ is less than that of $\mathbf{x}$, so $p_{\boldsymbol{\theta}}(\mathbf{x})$ with $\mathbf{x} = \mathbf{G}_{\boldsymbol{\theta}}(\mathbf{z})$ lives on a low-dimensional manifold.

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*Arjovsky M., Bottou L. Towards Principled Methods for Training Generative Adversarial Networks, 2017*

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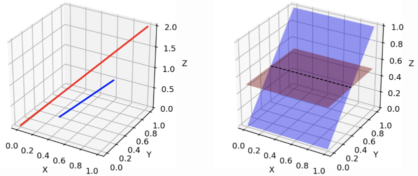


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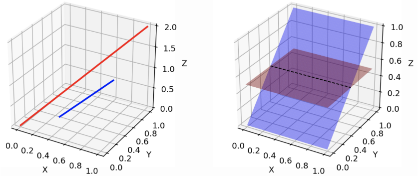


- If $p_{\text{data}}(\mathbf{x})$ and $p_{\boldsymbol{\theta}}(\mathbf{x})$ are disjoint, a smooth optimal discriminator can exist!

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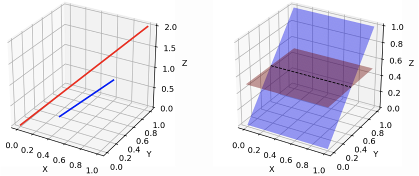


- If $p_{\text{data}}(\mathbf{x})$ and $p_{\boldsymbol{\theta}}(\mathbf{x})$ are disjoint, a smooth optimal discriminator can exist!
- For such low-dimensional, disjoint manifolds:

$$\mathrm{KL}(p_{\text{data}} \| p_{\boldsymbol{\theta}}) = \mathrm{KL}(p_{\boldsymbol{\theta}} \| p_{\text{data}}) = \infty, \quad \mathrm{JSD}(p_{\text{data}} \| p_{\boldsymbol{\theta}}) = \log 2$$

---

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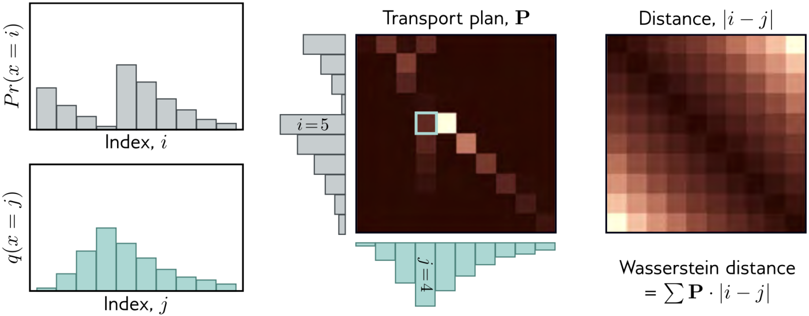
# Wasserstein Distance (Discrete)

Also known as the **Earth Mover's Distance**.

## Optimal Transport Formulation

The minimum cost of moving and transforming a pile of "dirt" shaped like one probability distribution to match another.




Simon J.D. Prince. *Understanding Deep Learning, 2023*

# Wasserstein Distance (Continuous)

$$W(\pi \| p) = \inf_{\gamma \in \Gamma(\pi, p)} \mathbb{E}_{(\mathbf{x}_1, \mathbf{x}_2) \sim \gamma} \|\mathbf{x}_1 - \mathbf{x}_2\| = \inf_{\gamma \in \Gamma(\pi, p)} \int \|\mathbf{x}_1 - \mathbf{x}_2\| \gamma(\mathbf{x}_1, \mathbf{x}_2) d\mathbf{x}_1 d\mathbf{x}_2$$

- $\gamma(\mathbf{x}_1, \mathbf{x}_2)$ is the transport plan: the amount of “dirt” assigned from $\mathbf{x}_1$ to $\mathbf{x}_2$.

$$\int \gamma(\mathbf{x}_1, \mathbf{x}_2) d\mathbf{x} = p(\mathbf{x}_2); \quad \int \gamma(\mathbf{x}_1, \mathbf{x}_2) d\mathbf{x}_2 = \pi(\mathbf{x}_1).$$

- $\Gamma(\pi, p)$ denotes the set of all joint distributions $\gamma(\mathbf{x}_1, \mathbf{x}_2)$ with marginals $\pi$ and $p$.
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## Wasserstein Metric

$$W_s(\pi, p) = \inf_{\gamma \in \Gamma(\pi, p)} \left( \mathbb{E}_{(\mathbf{x}_1, \mathbf{x}_2) \sim \gamma} \|\mathbf{x}_1 - \mathbf{x}_2\|^s \right)^{1/s}$$

In our setting, $W(\pi \| p) = W_1(\pi, p)$, which is the transport cost formulation.

---

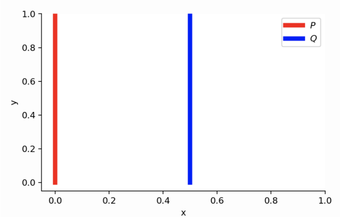
*Arjovsky M., Chintala S., Bottou L. Wasserstein GAN, 2017*

# Wasserstein Distance vs KL vs JSD

Consider two-dimensional distributions:

$$p_{\text{data}}(x, y) = (0, U[0, 1])$$

$$p_\theta(x, y) = (\theta, U[0, 1])$$

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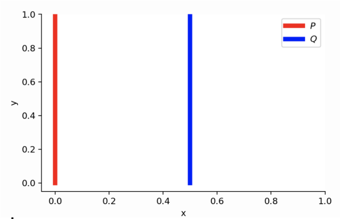


- $\theta = 0$: Both distributions are identical.

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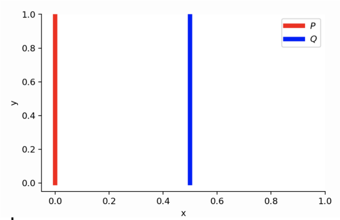


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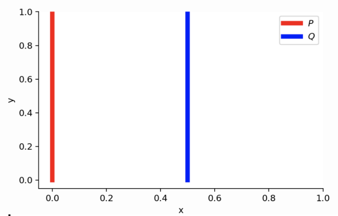


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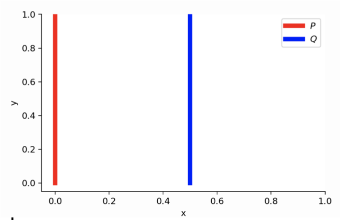


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# Wasserstein Distance vs KL vs JSD

### Theorem 1

Let $\mathbf{G}_{\boldsymbol{\theta}}(\mathbf{z})$ be (almost) any feedforward neural network, and $p(\mathbf{z})$ a prior over $\mathbf{z}$ such that $\mathbb{E}_{p(\mathbf{z})}\|\mathbf{z}\| < \infty$. Then $W(p_{\text{data}}\|p_{\boldsymbol{\theta}})$ is continuous everywhere and differentiable almost everywhere.

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### Theorem 2

Let $\pi$ be a distribution on a compact space $\mathcal{X}$ and let $\{p_t\}_{t=1}^{\infty}$ be a sequence of distributions on $\mathcal{X}$.

$$\mathrm{KL}(\pi\|p_t) \to 0 \quad (\text{or } \mathrm{KL}(p_t \,\|\, \pi) \to 0) \tag{1}$$

$$\mathrm{JSD}(\pi\|p_t) \to 0 \tag{2}$$

$$W(\pi\|p_t) \to 0 \tag{3}$$

In summary, as $t \to \infty$, (1) $\Rightarrow$ (2), and (2) $\Rightarrow$ (3).

*Arjovsky M., Chintala S., Bottou L. Wasserstein GAN, 2017*

# Outline

# Wasserstein GAN

## Wasserstein Distance

$$W(\pi \| p) = \inf_{\gamma \in \Gamma(\pi, p)} \mathbb{E}_{(\mathbf{x}_1, \mathbf{x}_2) \sim \gamma} \|\mathbf{x}_1 - \mathbf{x}_2\| = \inf_{\gamma \in \Gamma(\pi, p)} \int \|\mathbf{x}_1 - \mathbf{x}_2\| \gamma(\mathbf{x}_1, \mathbf{x}_2) \, d\mathbf{x}_1 \, d\mathbf{x}_2$$

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The infimum over all possible $\gamma \in \Gamma(\pi, p)$ is computationally intractable.

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## Theorem (Kantorovich-Rubinstein Duality)

$$W(\pi \| p) = \frac{1}{K} \max_{\|f\|_L \leq K} \left[ \mathbb{E}_{\pi(\mathbf{x})} f(\mathbf{x}) - \mathbb{E}_{p(\mathbf{x})} f(\mathbf{x}) \right]$$

where $f : \mathbb{R}^m \to \mathbb{R}$ is $K$-Lipschitz ($\|f\|_L \leq K$):

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We can thus estimate $W(\pi\|p)$ using only samples and a function $f$.

*Arjovsky M., Chintala S., Bottou L. Wasserstein GAN, 2017*

# Wasserstein GAN

## Theorem (Kantorovich-Rubinstein Duality)

$$W(p_{\text{data}} \| p_{\boldsymbol{\theta}}) = \frac{1}{K} \max_{\|f\|_L \leq K} \left[ \mathbb{E}_{p_{\text{data}}(\mathbf{x})} f(\mathbf{x}) - \mathbb{E}_{p_{\boldsymbol{\theta}}(\mathbf{x})} f(\mathbf{x}) \right]$$

- We must ensure that $f$ is $K$-Lipschitz continuous.
- Let $f_{\phi}(\mathbf{x})$ be a feedforward neural network parameterized by $\phi$.
- If the weights $\phi$ are restricted to a compact set $\boldsymbol{\Phi}$, then $f_{\phi}$ is $K$-Lipschitz.

*Arjovsky M., Chintala S., Bottou L. Wasserstein GAN, 2017*

# Wasserstein GAN

## Theorem (Kantorovich-Rubinstein Duality)

$$W(p_{\text{data}} \| p_{\boldsymbol{\theta}}) = \frac{1}{K} \max_{\|f\|_L \leq K} \left[ \mathbb{E}_{p_{\text{data}}(\mathbf{x})} f(\mathbf{x}) - \mathbb{E}_{p_{\boldsymbol{\theta}}(\mathbf{x})} f(\mathbf{x}) \right]$$

- We must ensure that $f$ is $K$-Lipschitz continuous.
- Let $f_{\phi}(\mathbf{x})$ be a feedforward neural network parameterized by $\phi$.
- If the weights $\phi$ are restricted to a compact set $\boldsymbol{\Phi}$, then $f_{\phi}$ is $K$-Lipschitz.
- Clamp weights within the box $\boldsymbol{\Phi} = [-c, c]^d$ (e.g. $c = 0.01$) after each update.

$$K \cdot W(p_{\text{data}} \| p_{\boldsymbol{\theta}}) = \max_{\|f\|_L \leq K} \left[ \mathbb{E}_{p_{\text{data}}(\mathbf{x})} f(\mathbf{x}) - \mathbb{E}_{p_{\boldsymbol{\theta}}(\mathbf{x})} f(\mathbf{x}) \right] \geq$$
$$\geq \max_{\phi \in \boldsymbol{\Phi}} \left[ \mathbb{E}_{p_{\text{data}}(\mathbf{x})} f_{\phi}(\mathbf{x}) - \mathbb{E}_{p_{\boldsymbol{\theta}}(\mathbf{x})} f_{\phi}(\mathbf{x}) \right]$$

*Arjovsky M., Chintala S., Bottou L. Wasserstein GAN, 2017*

# Wasserstein GAN

## Standard GAN Objective

$$\min_{\boldsymbol{\theta}} \max_{\boldsymbol{\phi}} \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \log D_{\boldsymbol{\phi}}(\mathbf{x}) + \mathbb{E}_{p(\mathbf{z})} \log(1 - D_{\boldsymbol{\phi}}(\mathbf{G}_{\boldsymbol{\theta}}(\mathbf{z})))$$

## WGAN Objective

$$\min_{\boldsymbol{\theta}} W(p_{\text{data}} \| p_{\boldsymbol{\theta}}) \approx \min_{\boldsymbol{\theta}} \max_{\boldsymbol{\phi} \in \boldsymbol{\Phi}} \left[ \mathbb{E}_{p_{\text{data}}(\mathbf{x})} f_{\boldsymbol{\phi}}(\mathbf{x}) - \mathbb{E}_{p(\mathbf{z})} f_{\boldsymbol{\phi}}(\mathbf{G}_{\boldsymbol{\theta}}(\mathbf{z})) \right]$$

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# Wasserstein GAN

## Standard GAN Objective

$$\min_{\boldsymbol{\theta}} \max_{\phi} \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \log D_\phi(\mathbf{x}) + \mathbb{E}_{p(\mathbf{z})} \log(1 - D_\phi(\mathbf{G}_{\boldsymbol{\theta}}(\mathbf{z})))$$

## WGAN Objective

$$\min_{\boldsymbol{\theta}} W(p_{\text{data}} \| p_{\boldsymbol{\theta}}) \approx \min_{\boldsymbol{\theta}} \max_{\phi \in \boldsymbol{\Phi}} \left[ \mathbb{E}_{p_{\text{data}}(\mathbf{x})} f_\phi(\mathbf{x}) - \mathbb{E}_{p(\mathbf{z})} f_\phi(\mathbf{G}_{\boldsymbol{\theta}}(\mathbf{z})) \right]$$

- The discriminator $D$ is replaced by function $f$: in WGAN, it is known as the **critic**, which is *not* a classifier.
- *"Weight clipping is a clearly terrible way to enforce a Lipschitz constraint."*
  - If $c$ is large, optimizing the critic is hard.
  - If $c$ is small, gradients may vanish.

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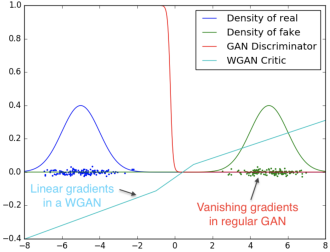
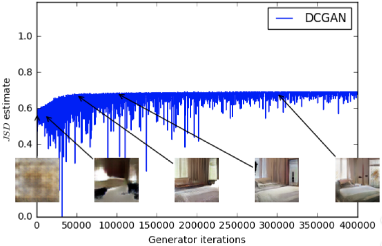
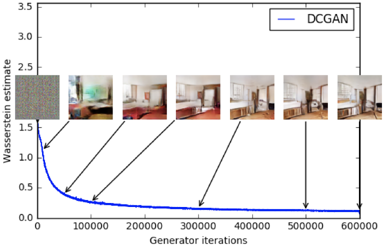
*Arjovsky M., Chintala S., Bottou L. Wasserstein GAN, 2017*

# Wasserstein GAN

- WGAN provides nonzero gradients even if distributions' supports are disjoint.
- $\mathrm{JSD}(p_{\text{data}} \| p_\theta)$ is poorly correlated with sample quality and remains near its maximum value $\log 2 \approx 0.69$.
- $W(p_{\text{data}} \| p_\theta)$ is tightly correlated with quality.

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# Summary

- Likelihood is not a reliable metric for generative model evaluation.
- Adversarial learning casts distribution matching as a minimax game.
- GANs, in theory, optimize the Jensen-Shannon divergence.
- KL and JS divergences fail as objectives when the model and data distributions are disjoint.
- The Earth Mover's (Wasserstein) distance provides a more meaningful loss for distribution matching.
- Kantorovich-Rubinstein duality allows us to compute the EM distance using only samples.
- Wasserstein GAN enforces the Lipschitz condition on the critic through weight clipping—although better alternatives exist.