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MODELLING OF RECIPROCAL TRANSDUCER SYSTEM ACCOUNTING FOR NONLINEAR CONSTITUTIVE RELATIONS

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Abstract—**The dynamics of reciprocal transducer systems is modelled accounting for a nonlinear constitutive relation between the electric displacement and the electric field as reported in Refs. [1, 2, 3] using an efficient nonlinear numerical scheme. The two transducers are assumed connected in series with a general electrical impedance. Particular emphasis is given to transmitter and receiver temporal responses in the cases with and without nonlinearities and differences between the two model cases are highlighted.**

Keywords: Polarization switching, ferroelectric, Large strain,

1 Introduction

Mathematical modelling of piezoelectric transducer systems is widely used in, e.g., optimizing transducer material choices, geometry and dimensions of material layers, electrode configuration, electrical impedances connected to the transducer [4, 5, 6, 7, 8]. Many modelling activities rely on the use of linear constitutive relationships between elastic and electric parameters such that equivalent electric circuit (EEC) diagrams and standard Fourier analysis apply. Linear constitutive relations are adequate as long as the transducer is operated near a specified working point such that approximative linear dependences between variables exist. However, in applications with large-amplitude electric fields and/or elastic strains, nonlinear effects are inevitable and important. The general hysteresis curve between, e.g., polarization and electric field serves as a well-known evidence that nonlinear effects are important in the general case of transducer operation [7, 9].

In this work, we consider reciprocal one-dimensional transducer systems consisting of a piezoceramic material layer in series with a general electrical impedance. The transmitter is electrically excited by a voltage generator and the transmitter voltage response is the voltage signal across the transmitter electrodes while the receiver voltage response is the voltage across the receiver electrodes (also equal to the voltage over the electrical impedance connected to the receiver) refer to Figure 1. We consider one-layer PZT5 piezoceramic transducers. A dynamic nonlinear model based on an efficient nonlinear numerical scheme is proposed accounting for nonlinearities in the electric field/electric displacement ($E-D$) relation which is one important cause of nonlinearities in transducer systems [1, 2, 3]. We examine and discuss the importance of the $E-D$ nonlinear constitutive relation for transmitter and receiver dynamic responses subject to two kinds of voltage-generator signals: (a) a 23-period sinus burst with driving frequency at 1 MHz and (b) a step-input signal. Although the nonlinearity is assumed to stem from the $E-D$ relation solely we stress that *any* type of nonlinearity can be handled with the present numerical model.

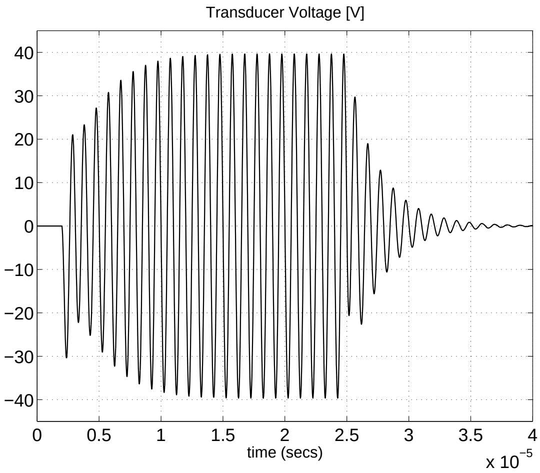

Figure 1: Schematics of reciprocal transducer system

2 Nonlinear $E-D$ Relation

The governing equations for a one-dimensional piezoelectric transducer are Newton's Second Law and the strain-particle velocity relation [8]:

$$\rho \frac{\partial u}{\partial t} = \frac{\partial T}{\partial z}, \tag{1}$$

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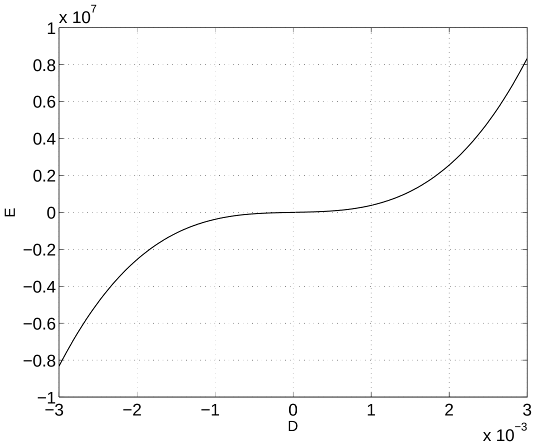

Figure 2: Schematics of the nonlinear $E - D$ relation.

$$\frac{\partial u}{\partial z} = \frac{\partial S}{\partial t}, \tag{2}$$

where u, T, S, ρ, z, and t are the particle velocity, stress, strain, mass density, position, and time, respectively. In addition, we employ the constitutive relations:

$$T = -hD + d^D S, \tag{3}$$

$$E = \beta^S D + K_2 D^3 - hS, \tag{4}$$

where D, E, h, c^D, β^S, and K_2 are the electric displacement, electric field, the piezoelectric h constant, the stiffness, the inverse permittivity, and the nonlinear coefficient in the $E - D$ relation responsible for the superlinear dependence of the absolute value of E with incresing value of D [refer to Figure 2].

As mentioned above, we assume that the transmitter – consisting of a single layer of PZT5 – is connected in series with an electrical impedance Z_{el} and a voltage generator $V(t)$, i.e., Ohm's Law reads:

$$V(t) \quad = Z_{el} I + \int E dz, \tag{5}$$

where I is the electric current and the spatial integration is performed from transmitter electrode 1 to electrode 2. Since drift currents in piezoelectric transducers are insignificant as compared to displacement currents at MHz frequencies, it is a good approximation to write for the current:

$$I = A \frac{\partial D}{\partial t}, \tag{6}$$

with A the cross-sectional area of the the transducer.

Next, employing Equations (3),(4), and (6) in Equation (5), the following differential equation is obtained:

$$\frac{\partial D}{\partial t} = \frac{1}{Z_{el} A} \left[V(t) - \left(\beta^S D + K_2 D^3 - \frac{h^2}{c^D} D \right) L + \int \frac{h}{c^D} T(z) dz \right], \tag{7}$$

with L the length of transducer. Equation (7) in addition to Equation (1) and:

$$\frac{\partial T}{\partial t} = c^D \frac{\partial u}{\partial z} - h \frac{\partial D}{\partial t}, \tag{8}$$

where the latter equation is obtained by combining Equation (2) and Equation (3) constitute the full set of dynamic equations modelled numerically. Thus, the problem involves solving three dynamic equations for u, T, and D.

Boundary conditions imposed everywhere are continuity of stress and particle velocity. At the transducer aperture interfaces facing air and water, respectively, the following boundary conditions hold for the transmitter:

$$\frac{T_{air}^{tra}}{Z_{air}} = +u_{air}^{tra}, \tag{9}$$

$$\frac{T_{water}^{tra}}{Z_{water}} = -u_{water}^{tra}, \tag{10}$$

since stresses are measured positive under compression of the material. The output pressure wave from the transmitter is assumed to impinge on the receiver front face (facing water) neglecting damping in the transmitting medium (water). Hence, continuity of pressure and particle velocity at the receiver interface (facing water) can be formulated as:

$$p_{water}^i + p_{water}^r = T_{water}^{rec}, \tag{11}$$

$$\frac{p_{water}^i}{Z_{water}} - \frac{p_{water}^r}{Z_{water}} = u_{water}^{rec}, \tag{12}$$

where p_{water}^i, p_{water}^r, T_{water}^{rec}, and u_{water}^{rec} are the impinging pressure wave, the reflected pressure wave, the resulting stress at the receiver aperture interface with water, and the particle velocity at the water-receiver interface, respectively. The latter two equations can be used to augment p_{water}^r (an unknown quantity) so as to obtain:

$$\frac{T_{water}^{rec}}{Z_{water}} + u_{water}^{rec} = 2 \frac{p_{water}^i}{Z_{water}}, \tag{13}$$

being the boundary condition for the receiver expressed in terms of the known quantity p_{water}^i. Next, following an analogous procedure for the receiver as compared to the transmitter one obtains the voltage signal generated over the receiver electrodes. Additional details for determining the receiver voltage signal (in the linear-case approximation) can be found in Ref. [8].

3 Differential Algebraic Approach

Because of the nonlinearity in the $E - D$ relationship, the system given by Equations (1), (7), and (8) cannot be solved by transformation methods which is the usual and efficient way for the analysis of linear transducers. Numerical methods, in this case, have to be employed for the nonlinear dynamic analysis instead of linear transformations. Because of its higher accuracy with a relatively smaller number of discretization nodes, the Chebyshev pseudo-spectral method is applied for the numerical analysis presented here. In order to analyse the dynamical behavior of a system given by partial differential equations, a popular approach is the so called "methods of lines". By this method, the whole system is firstly semi-discretized spatially while keeping all derivatives with respect to time unchanged. Obviously, this semi-discretization will convert the original partial differential equations system to a set of ordinary differential equations. The number of ordinary differential equations depends on the number of discretization nodes in space.

In the process of obtaining a unified numerical treatment for both the transmitter and receiver, equations for boundary nodes are replaced by algebraic equations resulting from given boundary conditions. This will convert the ordinary differential equations system into a Differential Algebraic Equations (DAE) system. For the Chebyshev pseudo-spectral approximation, a set of Chebyshev points $\{x_i\}$ are chosen along the thickness direction as follows:

$$x_i = L\left(1-\cos(\frac{\pi i}{N})\right)/2, \quad i=0,1,\ldots,N. \tag{14}$$

Using these nodes, the stress and velocity distributions in the transducer can be expressed in terms of the following linear approximation:

$$f(x) = \sum_{i=0}^{N} f_i \phi_i(x), \tag{15}$$

where $f(x)$ is the stress distribution or velocity distribution, respectively, and f_i is the function value at x_i. $\phi_i(x)$ is the i^{th} interpolating polynomial which has the following property:

$$\phi_i(x_j) = \begin{cases} 1, & i=j, \\ 0, & i\neq j. \end{cases} \tag{16}$$

It is easy to see that the well known Lagrange interpolants satisfy the interpolating requirements. Having obtained $f(x)$ approximately, the derivative $\partial f(x)/\partial x$ can be easily obtained by taking the derivative of the basis functions $\phi_i(x)$ with respect to x:

$$\frac{\partial f}{\partial x} = \sum_{i=1}^{N} f_i \frac{\partial \phi_i(x)}{\partial x}. \tag{17}$$

Following the same idea as given in Ref [12], Equation (17) can be written in a matrix form as:

$$\vec{F}_x = \vec{D}\vec{F}, \tag{18}$$

and the differentiation matrix $\vec{D}$ becomes:

$$\mathrm{D}_{ij} = \begin{cases} \dfrac{2N^2+1}{6} & i=j=0, \\ -\dfrac{2N^2+1}{6} & i=j=N, \\ -\dfrac{x_j}{2(1-x_j^2)} & i=j=1,2,\ldots,N-1, \\ \dfrac{c_i}{c_j}\dfrac{(-1)^{i+j}}{(x_i-x_j)} & i\neq j, \quad i,j=1,2,\ldots,N-1, \end{cases} \tag{19}$$

where

$$c_i = \begin{cases} 2, & i=0,N, \\ 1, & \text{otherwise}. \end{cases} \tag{20}$$

Obviously, $\vec{D}_{ij}$ is an $(N+1)\times(N+1)$ matrix. Here F_x and F are vectors collecting all values of the derivative $\partial f/\partial x$ and the function f at x_i, respectively.

In Equation (7), an integral operator acting on the stress distribution is also involved. This fact makes it necessary to construct a quadrature rule using the same set of points as chosen for the derivative approximation. For the currently chosen discretization nodes, the quadrature can be done by using the Chebyshev-Lobatto rule, which is exact for any polynomials with an order less than $2N-1$. The quadrature rule could be written as:

$$\int_a^b f(x) = \sum_{i=0}^{N} w_i f(x_i), \tag{21}$$

where weight coefficients w_i for the quadrature can be easily obtained using the idea given in [10, 13].

Using the approximations to the derivatives and integral operators, and taking into account the boundary conditions, the system given by partial differential equations will be recasted into the following DAE system:

$$\vec{M}\frac{d\vec{X}}{dt} + \vec{N}\left(t,\vec{X},\vec{U}\right) = \vec{0}, \tag{22}$$

where $\vec{X}$ is a vector with a length of $2\times(N+1)+1$ collecting all the unknowns we are seeking for, including $u(x_i), T(x_i)$, and D. The matrix $\vec{M} = \mathrm{diag}(a_1, a_2, ..., a_{2N+3})$ is a $(2N+3)\times(2N+3)$ matrix having entries "one" for all the differential equations associated with those internal nodes and "zero" for all algebraic equations associated with boundary conditions. Vector $\vec{N}$ is a collection of all the algebraic functions defined by the spatial discretization of the system (linear or nonlinear), and $\vec{U}$ is the input (output) for the transmitter (receiver).

The above system is a stiff system (because matrix $\vec{M}$ is singular), and should be solved by an implicit algorithm.

Here the second order backward differentiation formula method [11] is employed for the purpose. By discretizing the time derivative using a second order approximation, the DAE system becomes the following algebraic system:

$$\vec{M} \quad \left(\tfrac{3}{2}\vec{X}^{n} - 2\vec{X}^{n-1} + \tfrac{1}{2}\vec{X}^{n-2}\right)$$
$$+\Delta t\vec{N}\left(t_n, \vec{X}^{n}, \vec{U}^{n}\right) = 0, \qquad (23)$$

where n denotes the current computational time layer. For each computational time layer, iterations should be carried out using Newton's method for $\vec{X}^{n}$ by use of $\vec{X}^{n-1}$ and $\vec{X}^{n-2}$. Starting from the initial value, the vector of unknowns $\vec{X}$ can be solved for all specified time instances employing this algorithm.

Table 1: Computational parameters for the PZT5 transmitter and receiver

Parameter	Value	Unit
$\omega_0/(2\pi)$ (driving frequency)	$1 \cdot 10^6$	Hz
N (number of periods)	13	
Thickness (piezoceramic)	1.96	mm
Transducer area	380	mm^2
h (piezoceramic)	$1.42 \cdot 10^9$	F^{2}V/m^3
ϵ^S (piezoceramic)	$1440\epsilon_0$	F/m
c^D (piezoceramic)	$1.19 \cdot 10^{11}$	Pa
c^D (water)	$2.25 \cdot 10^9$	Pa
c^D (air)	$1.24 \cdot 10^5$	Pa
ρ (piezoceramic)	7750	kg/m^3
ρ (water)	1000	kg/m^3
ρ (air)	1.29	kg/m^3
Z_{el}	50	Ohm
K_2	$1.3 \cdot 10^5$	m/F

4 Numerical Results and Discussions

In this Section, results are computed for the case where the transmitter and receiver are connected to a 50 Ohm resistor ($Z_{el} = 50$ Ohm). In Figure 3, we show the transmitter response to an input voltage amplitude of 100 V for a 23-period sinus burst corresponding to a driving frequency of 1 MHz employing the nonlinear $E-D$ relation in the computation. The computational parameters for the PZT5 transducer are given in Table 1. Notice that after 13 periods the output signal is almost equivalent to that of a steady-state output signal as the signal amplitude does not change with the further increase in burst periods. This response is similar to what has been reported elsewhere [8]. Our calculations show that despite the high voltage amplitude of 100 V, changes in signal responses caused by nonlinearity in the $E-D$ relation (i.e., caused by changing K_2 from 0 to $1.3 \cdot 10^5$) are hardly visible (except in the signal phase).

The corresponding receiver voltage signal is shown in Figure 4. Again, the conclusion is that significant changes in the response caused by the K_2D^3 nonlinearity term in

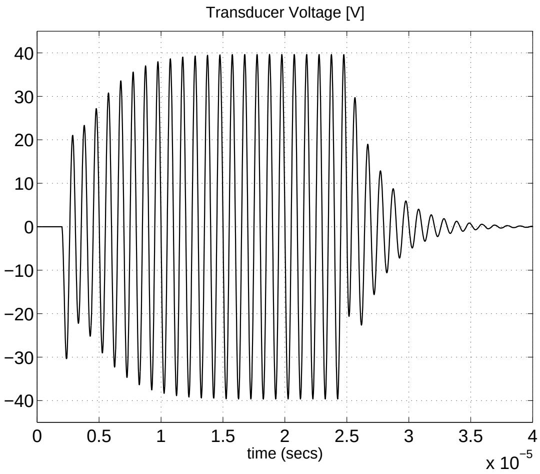

Figure 3: Transmitter voltage signal subject to a generator voltage sinus burst with driving frequency 1 MHz (nonlinear $E-D$ relation is employed in the computation but changes due to the nonlinearity are insignificant). Parameters are given in Table 1

the constitutive relation are hardly visible. Next, we computed the transmitter response to a 100 V step-response signal generated by the voltage generator [Figure 5]. Linear (solid) and nonlinear (dashed) $E-D$ relations are employed in the computations. It is now easily observable that amplitude as well as phase changes in the transmitter voltage signal as a result of the presence of the nonlinear term: K_2D^3 in the $E-D$ relation. In actual fact, the K_2D^3 term is responsible for a significant reduction in the signal amplitude indicating that nonlinear effects are very important at voltages near 100 V.

Finally, we plot in Figure 6 the ratio between the peak-receiver and peak-transmitter voltage signals over time subject to the same sinus burst considered in Figures 3 and 4. Apparently, the response is nonlinear although only slightly since the ratio only changes from approximately 0.11 to 0.115 as the input voltage changes from 0 to 100 V.

Conclusions

A mathematical model for reciprocal transducer systems accounting for nonlinearity in the constitutive relation between the electric field and electric displacement is presented. The model allows for connecting in series a general electrical impedance to the transmitter and the receiver. The transmitter and receiver apertures are assumed coupled to semi-infinite layers of air (one side) and water (other side) such that their influence on the system response is entirely due to their mechanical impedance. Special emphasis is given to differences in transmitter and receiver electrical responses in cases with and without $E-D$ nonlinearity in the model.

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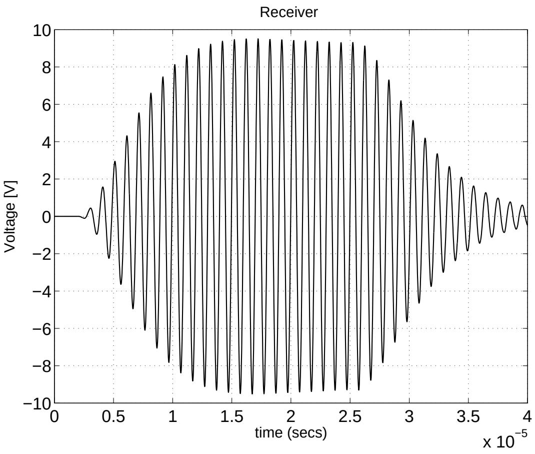

Figure 4: Receiver voltage signal subject to a generator voltage sinus burst with driving frequency 1 MHz (nonlinear $E-D$ relation is employed in the computation but changes due to the nonlinearity are insignificant). Parameters are given in Table 1.

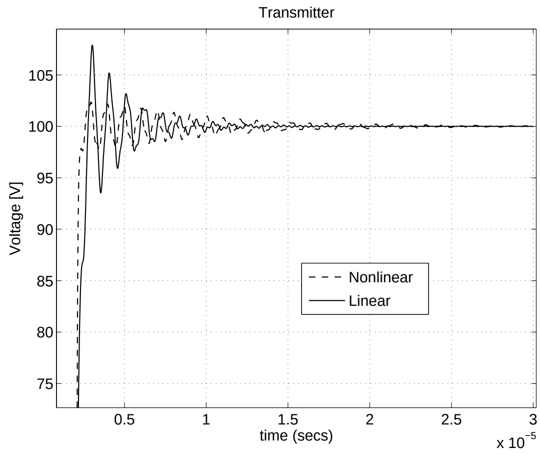

Figure 5: Transmitter voltage signal subject to a generator input-voltage step of 100 V. Linear (solid) and nonlinear (dashed) $E-D$ relations are employed in the computation. Parameters are given in Table 1.

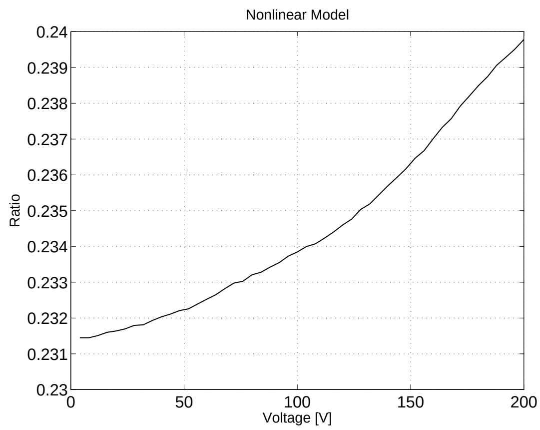

Figure 6: Ratio between the peak receiver and transmitter voltage signals over time subject to the same sinus burst considered in Figures 3 and 4. Parameters are given in Table 1.

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