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Optimal-by-order quadrature formulae for fast oscillatory functions with inaccurately given a priori information

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Abstract

In this article, the authors construct optimal-by-order quadrature formulae for integration of fast oscillatory functions in interpolational classes $C^1_{1,L,N}$ and $C^1_{1,L,N,\varepsilon}$. The construction of efficient formulae for numerical integration of fast oscillatory functions is based on the application of the residual method and the method of quasi-solutions. Both cases, weak and strong oscillations, are considered. Results of numerical examples are presented.

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1. Introduction

The problem of computing finite integrals with oscillatory functions arises in many areas of mathematics. In mathematical literature some of the most frequently cited examples of this problem are connected with the computation of Fourier transformations and the solution of boundary value problems for partial differential equations. In applications we often come to the above problem when modelling optical and automated control systems, constructing direction diagrammes of antennas, solving problems in radioastronomy, crystallography, signal processing and image recognition and when statistically processing experimental data [8,12,17].

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From the mathematical point of view, many of these applied problems can be reduced to the computation of integrals

$$I^n(f)=\int_{a_1}^{b_1}\dots\int_{a_n}^{b_n} f(x_1,\dots,x_n)\varphi_1(x_1)\dots\varphi_n(x_n)\,\mathrm{d}x_1\dots\mathrm{d}x_n, \tag{1.1}$$

where $\varphi_k(x_k)$, $k=1,\dots,n$ are known integrable oscillatory functions, and $f(x_1,\dots,x_n)$ belongs to a predefined functional class F. In this paper we consider a special case of (1.1) ($n=1$)

$$I^1(f)=\int_a^b f(x)\varphi(x)\,\mathrm{d}x \tag{1.2}$$

with a given oscillatory function $\varphi(x)$ such as $\sin(\omega x)$, $\cos(\omega x)$, $\exp(-\omega x)$ and with possibly very large values of ω. Large values of ω that are typical, for example, when processing high-frequency signals, lead to major problems in computing (1.2). Indeed, using standard approaches such as the Gaussian method, even if $f(x)$ is a smooth function, one has to choose a very high degree polynomial approximating $f(x)$. The degree of such a polynomial has to substantially exceed $[\omega[b-a]/\pi]$. This may result in computational instability [13].

In order to overcome the above difficulties, coefficients of quadrature formulae have to include the dependence on ω. The classical formulae with such a dependence is the Filon formula and its modifications [10,11,19]. The further development of the Filon method has been connected with an approximation of $f(x)$ by an interpolating polynomial and the consequent integration, wherein $\varphi(x)$ has been treated as a weight function. Substantial contributions to the topics of numerical integration of fast oscillatory functions were made by Collatz, Erugin, Sobolev, Krylov, Nikolskii and many other outstanding mathematicians (some references can be found, for example, in [7,9,2,5,18,33,31]). Due to its practical importance, much efforts in this field have been concentrated on the development of algorithms for computing specific type integrals such as Fourier and Bessel integrals [24–26,13,34]. However, the constructive procedures for obtaining optimal-by-order, rather than optimal-by-accuracy, quadrature formulae (or formulae close to them in a certain sense) for the integration of fast oscillatory functions are still lacking in the literature. It is especially true in the case when the problem is considered from the point of view of the total error, taking into account inaccuracy of a priori available information [4,32,20,30]. Such a consideration brings us closer to the real situation when information about integrands are taken from measurements. This case is in the main focus of our paper.

In the construction of methods for numerical integration, there is a natural contradiction between the desire to choose a wider functional class and the desire to better describe the problem and take into account many properties of the problem (that, in turn, leads to narrowing of the functional class). However, if the functional class is chosen, then all a priori information available for the construction of an algorithm for numerical integration is contained in the inclusion $f\in F$ [27]. In this paper we assume that function $f(x)\in F$ is given by a fixed table of its values f_i, $i=1,\dots,N$ in N fixed points $\{x_i\}_{i=1}^N$ from its domain of definition. Although such a definition leads to a considerable narrowing of the corresponding class F to an interpolational class F_N, it allows us the maximal use of available information about the function and, as a result, it allows us to improve the quality of quadrature formulae. Since in practice instead of exact input data $\{f_i\}_{i=1}^N$ we often know only approximate values $\{\tilde f\}_{i=1}^N$, we also consider the case of approximate definition of input data where values $\{f_i\}_{i=1}^N$ are taken from the domain defined by inequalities $|\tilde f_i - f_i|\leqslant\varepsilon_i$, $i=1,\dots,N$.

In this case we shall say that $f(x) \in F_{N,\varepsilon}$ with a fixed vector $\varepsilon = (\varepsilon_1, \dots, \varepsilon_N)$. We specify two main interpolational classes, considered in this paper, as follows.

- $C^1_{1,L,N}$ is the class of continuous functions defined on the interval $[a,b]$, which have bounded (by constant L) first derivatives and take fixed values $f(x_1) = f_1, \dots, f(x_N) = f_N$ at fixed nodes of arbitrary grid, $x_1, \dots, x_N$;
- $C^1_{1,L,N,\varepsilon}$ is the class of continuous on $[a,b]$ functions which have bounded (by L) first derivatives and take values from the intervals $[f_i - \varepsilon_i, f_i + \varepsilon_i]$, $\varepsilon_i \geqslant 0$, $i = 1, \dots, N$ at fixed nodes of arbitrary grid, $x_1, \dots, x_N$.

In what follows, we assume that these functional classes are nonempty. The interest to such interpolational classes and other functional classes that satisfy different forms of the Lipschitz condition (or more generally defined by their quasi-metrics [27]) has recently dramatically increased in the context of optimisation problems.

Many problems of applied and computational mathematics, including those of numerical integration, can be described in the following generic form. We have to solve a problem $P(I,S)$ with a set of initial data $I \in (\mathscr{M}_1, \rho_1)$ and a solution (or a set of solutions) $S \in (\mathscr{M}_2, \rho_2)$ where $\mathscr{M}_1$ and $\mathscr{M}_2$ are certain metric spaces with metrics ρ_1 and ρ_2, respectively. Since exact initial data are typically unavailable, instead of I we are typically given another set $\tilde{I}$ of approximate data. This set is the origin of an *uncertainty domain of input data* (see also [27]), denoted further by $U_1 \in \mathscr{M}_1$. In turn, the uncertainty domain U_1 gives rise to an *uncertainty domain of the solution* denoted further by $U_2 \in \mathscr{M}_2$. It is this set U_2 that determines properties of our solution in a sense that any element $R \in U_2$ can be formally considered as a solution of the problem.

If U_2 is a bounded set, then the Chebyshev center of U_2 is taken as the *optimal solution* of the problem. In other words, we choose an element $x_0 \in \mathscr{M}_2$ such that

$$\sup_{y \in U_1} \rho_2(x_0, y) = \inf_{x \in \mathscr{M}_2} \sup_{y \in U_1} \rho_2(x, y). \tag{1.3}$$

The quantity defined by (1.3) gives the least possible error of the problem solution under given data, and is called the Chebyshev radius of the set U_2. If $\mathscr{M}_2$ is a Banach space that is uniformly convex in every direction, then U_2 has at least one Chebyshev center.

If U_2 is unbounded (which is the case, for example, for many ill-posed problems), then more a priori information have to be used to locate the solution in S. Assume, for example, that the solution belongs to a subdomain G of S ($G \subset S$). Then, as the optimal solution of the problem we take the Chebyshev center of the set $F = G \cap U_2$. This case is much more difficult for investigation compared to the case of bounded U_2. However, instead of finding the Chebyshev center of F, it is often more efficient to use other elements of F. Indeed, it is known that an arbitrary element of F represents this set with the accuracy not exceeding two times of the accuracy of representation of F by its Chebyshev radius [30,31]. In this paper, we use two elements of F that have advantages over the Chebyshev center in the case when a priori information is given inaccurately. These elements can be defined as follows.

- We consider a point S_1 for which the distance from the given point of S (say zero point) is minimal compared to the distance from other points from F. The method of finding the point S_1 is known as *the residual method* (see [21,28,29] and references therein).

- The other our choice, which is especially efficient if F is compact, is the point S_2, for which the corresponding point of the uncertainty domain U_1 is the least remote from a given point. The method of finding the point S_2 is known as *the method of quasi-solutions* (see [14,15,28,29] and references therein).

The use of these two methods is the basis for our constructions of optimal (and close to them) quadrature formulae in interpolational classes $C^1_{1,L,N}$ and $C^1_{1,L,N,\varepsilon}$.

The paper is organized as follows.

- In Section 2, using the residual method and the method of quasi-solutions, we obtain quadrature formulae for the numerical solution of problem (1.2).
- In Section 3 we derive error estimates for these formulae applied to computing (1.2) with $\varphi(x)=\sin(\omega x)$ and $\varphi(x)=\cos(\omega x)$ in interpolational class $C^1_{1,L,N}$. We consider two principally different cases, the case of weak oscillations and the case of strong oscillations of the integrand.
- In Section 4 we generalise the results obtained in Section 3 to class $C^1_{1,L,N,\varepsilon}$.
- In Section 5 we consider the problem of computing estimates of the Fourier transforms when a priori information is given approximately. Algorithms and some numerical examples are also presented in this section.
- Conclusions and future directions are discussed in Section 6.

2. Construction of optimal quadrature formulae in interpolational classes

Let $f\in F_N$ with F_N be an interpolational class and let $\mathscr{M}$ be a set of integration algorithms. Then the accuracy of integration of a certain function f by algorithm A has to be chosen according to a certain criterion by which we can estimate the quality of the algorithm in terms of its error function $v(F_N,A,f)$. For the construction of quadrature formulae for (1.2), we use *the method of limit functions* which consists of the construction of the best algorithm for the worst function in the class [31,27]. This minimax approach to the solution of problems in theory of numerical methods goes back to Chebyshev's works and with respect to the problems of optimal quadratures was first formulated by A.N. Kolmogorov (see [22] and references therein). Since that time, one may trace a close connection of efficient quadrature formulae with spline-approximations [1,6,16,23]. The idea of spline approximation is used also in this paper. As the worst function in class F_N we take a function that provides $\sup_{f\in F_N} v(F_N,A,f)$ for the given algorithm.

We introduce the following characteristic:

$$\delta(F_N)=\inf_{A\in M}\sup_{f\in F_N} v(F_N,A,f), \tag{2.1}$$

where

$$v(F_N,A,f)=|I^1(f)-r(F_N,A,f)|, \tag{2.2}$$

$r(F_N,A,f)$ is the result of application of algorithm A to function f, $\mathscr{M}$ is the set of all quadrature formulae that use information consisting of the definition of class F_N. For the general integral (1.1) the above characteristic is introduced in the same way.

Definition 2.1. A quadrature formula on which $\delta(F_N)$ is achieved (assuming that such a limit exists) is called optimal-by-accuracy for the given class. If there exists a quadrature formulae A^0 such that

$$v(F_N, A^0, f) \leqslant \delta(F_N) + \eta, \quad \eta \geqslant 0, \tag{2.3}$$

then A^0 is called optimal quadrature formula on the class F_N with accuracy up to η. If $\eta = \mathrm{o}(\delta(F_N))$ or $\eta = \mathrm{O}(\delta(F_N))$, then A^0 is called asymptotically optimal or optimal-by-order quadrature formula.

For the construction of optimal-by-order and close to them quadrature formulae in interpolational classes F_N (say, $C^1_{1,L,N}$) and $F_{N,\varepsilon}$ (say, $C^1_{1,L,N,\varepsilon}$), we follow a general procedure, known in literature as the method of limit functions (see [31] and references therein).

Let consider a functional class F_N of functions defined in a given domain.

Definition 2.2. Function $f^{\pm}(x)$ is called majorant (minorant) of class F_N if both of the following two conditions:

- $f^+(x) \geqslant f(x) (f^-(x) \leqslant f(x))\ \forall f \in F_N$ and
- $f^+ \in F_N\ (f^- \in F_N)$.

are satisfied.

We construct upper and lower limits of the set of all possible values of integrals (1.2) in the domain of integration on functions of class F_N as follows:

$$I^+(F_N) = \sup_{f \in F_N} I^1(f), \quad I^-(F_N) = \inf_{f \in F_N} I^1(f). \tag{2.4}$$

Quantities $I^{\pm}(F_N)$ in (2.4) are achieved on $f^{\pm}(x) \in F_N$, which are referred to as majorant and minorant of class F_N, respectively. Then, the Chebyshev center $I^*(F_N)$ and the Chebyshev radius $\delta^*(F_N)$ of the uncertainty domain of values $I^1(f)$ on class F_N are determined as follows (see also [20,27] and references therein):

$$I^*(F_N) = \tfrac{1}{2}(I^+(F_N) + I^-(F_N)), \quad \delta^*(F_N) = \tfrac{1}{2}(I^+(F_N) - I^-(F_N)). \tag{2.5}$$

As follows from the above discussion and Definition 2.1, a quadrature formula that computes $I^*(F_N)$ is called *optimal-by-accuracy*. If the domain of values of integral $I^1(f)$ is D, then the quantity $\delta^*(F_N)$ gives the error of representation of D by $I^*(F_N)$. Furthermore, a quadrature formula for computing $\bar{I}^1(f)$ such that

$$\sup_{f \in F_N} |\bar{I}^1(f) - I^1(f)| \leqslant \delta^* + \eta, \quad \eta > 0 \tag{2.6}$$

with $\eta = \mathrm{o}(\delta^*), \mathrm{O}(\delta^*), \delta^* \to 0$ is called, respectively, *asymptotically optimal* and *optimal-by-order*.

We note that under the given information about the problem any other quadrature formula do not give accuracy less than δ^*. We also note that for interpolational class F_N Chebyshev radius δ^* coincides with the optimal estimate. When $F_N = C^1_{1,L,N}$ or $F_{N,\varepsilon} = C^1_{1,L,N,\varepsilon}$ optimal-by-accuracy algorithms for the numerical evaluation of integral (1.2) with $\varphi(x) = \sin(\omega x)$ and $\varphi(x) = \cos(\omega x)$ (where ω is a real number that determines an oscillatory factor of the integrand such that $|\omega| \geqslant 2\pi/(b-a)$ was investigated in [3,31] (see also references therein). Although such algorithms are suitable for a wide range of oscillatory patterns (with the assumption that the values of L and ε used in these algorithms are given accurately), in many practical cases it is not possible to apply these algorithms. Indeed, in practice it is typical that numerical a priori information (used for the definition of functional classes) is inaccurate. Hence, instead of exact values of L and ε, we rather have some estimations of these values. Below we propose efficient algorithms for such situations. We construct optimal-by-order (with a constant not exceeding 2), rather than optimal-by-accuracy, algorithms that are based on methods of quasi-solutions and the residual minimization.

In numerical integration algorithms, constructed on the basis of quasi-solutions, integrand $f(x)$ is approximated by a function which is the solution of the following problem [3,14,15,28,29]:

$$\min_{f\in F}\max_{i=1,\dots,N}\varepsilon_i \quad \text{where } \varepsilon_i=|f(x_i)-\tilde{f}_i|. \tag{2.7}$$

The method of quasi-solutions consists of the determination of such a function that least deviates from the given set of points $(x_i,\tilde{f}_i)$, $i=1,\dots,N$. It is known [3], that the solution of (2.7) in such classes as $C^1_{1,L,N,\varepsilon}$ (and as a special case in $C^1_{1,L,N}$) is a linear spline $S(x,L)$ for which maximal deviation from the given points $(x_i,\tilde{f}_i)$, $i=1,\dots,N$ is minimal, i.e.,

$$S(x,L)=\hat{f}_i+\frac{x-x_i}{x_{i+1}-x_i}(\hat{f}_{i+1}-\hat{f}_i), \quad x\in[x_i,x_{i+1}],\ i=1,\dots,N-1, \tag{2.8}$$

where

$$\hat{f}_i=(\tilde{f}_i^{+}+\tilde{f}_i^{-})/2, \quad \tilde{f}_i^{\pm}=\pm\max_{1\leqslant j\leqslant N}(\pm(\tilde{f}_j\mp L|x_j-x_i|)), \quad i=1,\dots,N. \tag{2.9}$$

It is often the case that a priori information that defines the class F is given in the form of certain constraints on a functional $\Phi(f)$. For functional classes $C^1_{1,L,N}$ and $C^1_{1,L,N,\varepsilon}$ this functional is the uniform norm of the derivative. In quadrature formulae constructed on the basis of the residual method [21,28,29], integrand $f(x)$ is approximated by a function which is the solution of the following problem:

$$\min_{f\in F}\Phi(f). \tag{2.10}$$

The solution of (2.10) is a linear spline $S(x,M)$ which is defined by (2.8), (2.9) with constant L changed for constant M where

$$M=\max_{1\leqslant i\leqslant N}\left(0,\max_{j>i}\frac{|\tilde{f}_j-\tilde{f}_i|-\varepsilon_j-\varepsilon_i}{x_j-x_i}\right). \tag{2.11}$$

The quadrature formulae, constructed on the basis of the method of quasi-solutions and the residual method, have the following forms, respectively:

$$\bar{R}(\varphi,S)=\int_a^b S(x,L)\varphi(x)\,\mathrm{d}x, \tag{2.12}$$

$$\bar{\bar{R}}(\varphi,S)=\int_a^b S(x,M)\varphi(x)\,\mathrm{d}x. \tag{2.13}$$

Formulae (2.12) and (2.13), used for computing (1.2), are optimal-by-order with a constant that does not exceed 2 (even compared with the case of exactly given L and ε) [30]. Typically, both the residual method and the method of quasi-solutions, are directed to a more precise recovery of available a priori information. Therefore, the application of formulae (2.12), (2.13) are the most appropriate for the case of inaccurately given a priori information.

3. Error estimates for optimal-by-order quadrature formulae in class $C^1_{1,L,N}$

For the integrals

$$I_2^1(\omega, f) = \int_a^b f(x)\sin(\omega x)\,\mathrm{d}x, \tag{3.1}$$

$$I_3^1(\omega, f) = \int_a^b f(x)\cos(\omega x)\,\mathrm{d}x \tag{3.2}$$

with a real number ω such that $|\omega| \geqslant 2\pi/(b-a)$, quadrature formulae (2.12) and (2.13) have the following forms:

$$R_2(\omega, S) = \sum_{i=1}^{N-1} \left(\frac{\hat{f}_i'}{\omega^2} (\sin(\omega x_{i+1}) - \sin(\omega x_i)) \right) - \frac{1}{\omega} (\hat{f}_N \cos(\omega x_N) - \hat{f}_1 \cos(\omega x_1)), \tag{3.3}$$

$$R_3(\omega, S) = \sum_{i=1}^{N-1} \left(\frac{\hat{f}_i'}{\omega^2} (\cos(\omega x_{i+1}) - \cos(\omega x_i)) \right) + \frac{1}{\omega} (\hat{f}_N \sin(\omega x_N) - \hat{f}_1 \sin(\omega x_1)), \tag{3.4}$$

where $\hat{f}_i' = (\hat{f}_{i+1} - \hat{f}_i)/(x_{i+1} - x_i)$ ($\hat{f}_i$, $i = 1, \ldots, N$ are defined by (2.9)).

In order to obtain error estimates for optimal quadrature formulae, constructed for computing (3.1), (3.2), we require a special mutual arrangement of points x_i, $i = 1, \ldots, N$ and zeros of $\sin(\omega x)$ (or $\cos(\omega x)$) on $[a, b]$ [20]. We note that in class $C^1_{1,L,N}$ the solutions of problems (2.7) and (2.10) coincide and the linear spline $S(x, L)$ is a broken line that joins points (x_i, f_i), $i = 1, \ldots, N$

$$S(x, L) = f_i + \frac{x - x_i}{x_{i+1} - x_i} (f_{i+1} - f_i), \quad x \in [x_i, x_{i+1}], \ \ i = 1, \ldots, N-1. \tag{3.5}$$

Indeed, if $f(x) \in C^1_{1,L,N}$ then $\hat{f}_i = f_i$. Let us consider the second formula in (2.9). Since $\tilde{f}_i = f_i$, $i = 1, \ldots, N$, it may be rewritten in the following form:

$$f_i^{\pm} = \pm \max_{1 \leqslant j \leqslant N} (\pm(f_j \mp L|x_j - x_i|)), \quad i = 1, \ldots, N. \tag{3.6}$$

It is obvious that for any function $f(x) \in C^1_{1,L,N}$

$$|f_j - f_i| \leqslant L|x_j - x_i|, \quad \text{i.e.,} \ -L|x_j - x_i| \leqslant f_j - f_i \leqslant L|x_j - x_i| \tag{3.7}$$

and

$$-f_j - L|x_j - x_i| \leqslant -f_i \leqslant -f_j + L|x_j - x_i|, \quad i, j = 1, \ldots, N. \tag{3.8}$$

Since

$$f_i \geqslant f_j - L|x_j - x_i| \quad \text{and} \quad -f_i \geqslant -f_j - L|x_j - x_i|, \tag{3.9}$$

using (3.6) and the chain of inequalities (3.7), (3.8) we have that

$$\max_{1\leqslant j\leqslant N}(f_j - L|x_j - x_i|) = f_i \quad \text{and} \quad \max_{1\leqslant j\leqslant N}(-f_j - L|x_j - x_i|) = -f_i \tag{3.10}$$

from which formula (3.5) immediately follows.

Let us first consider the case when $N \geqslant |\omega|$. Computing (3.1), we assume that $[(|\omega|/\pi)(b-a)]+1$ *zeros of function* $\sin\omega x$ (or $\cos\omega x$ *for computing* (3.2)) *are included in the number of nodes* x_i, $i=1,\dots,N$. We refer to this condition as *Condition* C1. Then the following result holds.

Theorem 3.1. *Let* $f(x)\in C^1_{1,L,N}$, $N\geqslant|\omega|$, *condition* C1 *satisfied and* $\{f_i\}_{i=1,\dots,N}$ *are given on* $[a,b]$ *in the nodes* x_i, $i=1,\dots,N$ *of the grid with fixed ends* $x_1=a$, $x_{N+1}=b$. *Then quadrature formula* (3.3) *for the computation of integral* (3.1) *is optimal-by-order with a constant that does not exceed* 2. *This result holds with the following error estimate*:

$$\begin{aligned} v(C^1_{1,L,N}, &R_2(\omega,S), f)\\ &\leqslant \frac{L}{\omega}\left(\sum_{i=1}^{N-1}\left|\sin\left(\frac{\omega}{2}(x_{i+1}+x_i)\right)\right|\left(\frac{4}{\omega}\left(\sin^2\frac{\omega\Delta x_i}{4} - \sin^2\frac{\omega|\Delta f_i|}{4L}\right)\right)\right.\\ &\quad + \frac{2}{\omega}\left|\sin\frac{\omega\Delta x_n}{2}\cos\left(\omega\left(b-\frac{\Delta x_N}{2}\right)\right) - \Delta x_N\cos(\omega b)\right|\\ &\quad \left. + \frac{2}{\omega}\left|\sum_{i=1}^{N-1}\ \mathrm{sign}\,(\Delta f_i)\cos\left(\frac{\omega}{2}(x_{i+1}+x_i)\right)\left(\sin\frac{\omega|\Delta f_i|}{2L} - \frac{|\Delta f_i|}{L\Delta x_i}\sin\frac{\omega\Delta x_i}{2}\right)\right|\right), \end{aligned} \tag{3.11}$$

where $\Delta f_i = f_{i+1}-f_i$, $\Delta x_i = x_{i+1}-x_i$, $i=1,\dots,N.$

Proof. It is obvious that $S(x_i,L)=f_i$, $i=1,\dots,N$, $|S'(x,L)|\leqslant L$. Hence, $S(x,L)\in C^1_{1,L,N}$. Since

$$\inf_{\psi\in C^1_{1,L,N}} I^1_2(\psi)\leqslant R_2(\omega,S)\leqslant \sup_{\psi\in C^1_{1,L,N}} I^1_2(\psi), \tag{3.12}$$

then the error estimate for (3.3) can be obtained on the basis of the following inequality:

$$|R_2(\omega,S) - I^1_2(f)|\leqslant \max(I^+_2(f) - R_2(\omega,S), R_2(\omega,S) - I^-_2(f)), \tag{3.13}$$

where $I^-_2(f)$ is the lower and $I^+_2(f)$ is the upper limits of the set of all possible values of integral $I^1_2(f)$ on $C^1_{1,L,N}$.

Taking into account oscillations of $\sin(\omega x)$, the limit functions of the class $C^1_{1,L,N}$, $f^+(x)$ and $f^-(x)$, on each elementary segment $[x_i,x_{i+1}]$ can be written in the explicit form.

(a) For $x\in[x_i,\bar{x}_i]$, $\bar{x}_i = (x_i+x_{i+1})/2 - |\Delta f_i|/(2L)$, $\Delta f_i = f_{i+1}-f_i$ $(\Delta x_i = x_{i+1}-x_i)$ we have

$$f^+(x) = f_i + L(x-x_i)\,\mathrm{sign}\,(\sin(\omega x_i)), \quad f^-(x) = f_i - L(x-x_i)\,\mathrm{sign}\,(\sin(\omega x_i)). \tag{3.14}$$

(b) For $x\in[\bar{x}_i,\bar{\bar{x}}_i]$, $\bar{\bar{x}}_i = (x_i+x_{i+1})/2 + |\Delta f_i|/(2L)$ we have

$$\begin{aligned} f^+(x) &= \tfrac12(1+\mathrm{sign}(\Delta f_i)\,\mathrm{sign}\,(\sin(\omega x_i)))(f_i + L(x-x_i)\,\mathrm{sign}\,(\sin(\omega x_i)))\\ &\quad + \tfrac12(1-\mathrm{sign}\,(\Delta f_i)\,\mathrm{sign}\,(\sin(\omega x_i)))(f_{i+1} + L(x_{i+1}-x)\,\mathrm{sign}\,(\sin(\omega x_i))),\\ f^-(x) &= \tfrac12(1-\mathrm{sign}\,(\Delta f_i)\,\mathrm{sign}\,(\sin(\omega x_i)))(f_i - L(x-x_i)\,\mathrm{sign}\,(\sin(\omega x_i)))\\ &\quad + \tfrac12(1+\mathrm{sign}\,(\Delta f_i)\,\mathrm{sign}\,(\sin(\omega x_i)))(f_{i+1} - L(x_{i+1}-x)\,\mathrm{sign}\,(\sin(\omega x_i))). \end{aligned} \tag{3.15}$$

(c) For $x \in [\bar{\bar{x}}_i, x_{i+1}]$ we have

$$f^+(x) = f_{i+1} + L(x_{i+1} - x)\,\mathrm{sign}\,(\sin(\omega x_i)), \quad f^-(x) = f_{i+1} - L(x_{i+1} - x)\,\mathrm{sign}\,(\sin(\omega x_i)). \tag{3.16}$$

As usual, in (3.14)–(3.16) the signum of function $\sin(\omega x)$ on $[x_i, x_{i+1}]$ $(i=1,\dots,N-1)$ is denoted by $\mathrm{sign}\,(\sin(\omega x_i))$. We choose an approximation $f^*(x)$ for integrand $f(x)$ in the following form:

$$f^*(x) = \begin{cases} S(x,L), \; x \in [x_i, x_{i+1}], & i = 1,\dots,N-1, \\ f_N, \qquad x \in [x_N, x_{N+1}]. \end{cases} \tag{3.17}$$

Let us obtain an estimate from above for $v(C^1_{1,L,N}, R_2(\omega,S), f)$. We have

$$v(C^1_{1,L,N}, R_2(\omega,S), f) \leqslant \max\left(\sum_{i=1}^{N} \int_{x_i}^{x_{i+1}} (f^+(x) - f^*(x)) \times \sin(\omega x)\,\mathrm{d}x, \; \sum_{i=1}^{N} \int_{x_i}^{x_{i+1}} (f^*(x) - f^-(x))\sin(\omega x)\,\mathrm{d}x\right). \tag{3.18}$$

The next step is to calculate explicitly integrals in (3.18). Taking into account the explicit forms of the majorant of class $C^1_{1,L,N}$ (see (3.14)–(3.16)) we get

$$\begin{aligned}
&\sum_{i=1}^{N} \int_{x_i}^{x_{i+1}} (f^+(x) - f^*(x))\sin(\omega x)\,\mathrm{d}x \\
&= \sum_{i=1}^{N-1} \Bigg(\left(L\,\mathrm{sign}(\sin(\omega x_i)) - \frac{\Delta f_i}{\Delta x_i} \right) \int_{x_i}^{\bar{x}_i} (x - x_i)\sin(\omega x)\,\mathrm{d}x \\
&\quad + \frac{1}{2}\left(L - \frac{|\Delta f_i|}{\Delta x_i} \right) \mathrm{sign}(\sin(\omega x_i)) \int_{\bar{x}_i}^{\bar{\bar{x}}_i} ((1 + \mathrm{sign}(\Delta f_i)\mathrm{sign}(\sin(\omega x_i)))(x - x_i) \\
&\quad + (1 - \mathrm{sign}(\Delta f_i)\mathrm{sign}(\sin(\omega x_i)))(x_{i+1} - x))\sin(\omega x)\,\mathrm{d}x + (L\,\mathrm{sign}(\sin(\omega x_i)) \\
&\quad + \frac{|\Delta f_i|}{\Delta x_i} \Bigg) \int_{\bar{\bar{x}}_i}^{x_{i+1}} (x_{i+1} - x)\sin(\omega x)\,\mathrm{d}x \Bigg) + L\,\mathrm{sign}(\sin \omega x_N) \int_{x_N}^{x_{N+1}} (x - x_N)\sin(\omega x)\,\mathrm{d}x \\
&= \frac{L}{\omega} \left(\sum_{i=1}^{N-1} \left| \sin\left(\frac{\omega}{2}(x_{i+1} + x_i) \right) \right| \times \left(\frac{4}{\omega} \left(\sin^2 \frac{\omega \Delta x_i}{4} - \sin^2 \frac{\omega |\Delta f_i|}{4L} \right) \right) \right. \\
&\quad + \frac{2}{\omega} \sum_{i=1}^{N-1} \mathrm{sign}(\Delta f_i)\cos\left(\frac{\omega}{2}(x_{i+1} + x_i) \right) \times \left(\sin\frac{\omega |\Delta f_i|}{2L} - \frac{|\Delta f_i|}{L \Delta x_i}\sin\frac{\omega \Delta x_i}{2} \right) \\
&\quad \left. + \frac{2}{\omega} \left| \sin\frac{\omega \Delta x_{N-1}}{2} \cos\left(\omega\left(b - \frac{\Delta x_{N-1}}{2} \right) \right) - \Delta x_{N-1}\cos(\omega b) \right| \right).
\end{aligned} \tag{3.19}$$

In a similar way, taking into account the explicit representation of the minorant of class $C^1_{1,L,N}$, we obtain

$$\sum_{i=1}^{N}\int_{x_i}^{x_{i+1}}(f^*(x)-f^-(x))\sin(\omega x)\,\mathrm{d}x$$

$$=\sum_{i=1}^{N-1}\left(\left(L\,\mathrm{sign}(\sin(\omega x_i))-\frac{\Delta f_i}{\Delta x_i}\right)\int_{x_i}^{\bar{x}_i}(x-x_i)\sin(\omega x)\,\mathrm{d}x\right.$$

$$+\frac{1}{2}\left(L-\frac{|\Delta f_i|}{\Delta x_i}\right)\mathrm{sign}(\sin(\omega x_i))\int_{\bar{x}_i}^{\bar{\bar{x}}_i}((1-\mathrm{sign}(\Delta f_i)\mathrm{sign}(\sin(\omega x_i)))(x-x_i)$$

$$+(1+\mathrm{sign}(\Delta f_i)\mathrm{sign}(\sin(\omega x_i)))(x_{i+1}-x))\sin(\omega x)\,\mathrm{d}x+\left(L\,\mathrm{sign}(\sin(\omega x_i))\right.$$

$$\left.\left.-\frac{|\Delta f_i|}{\Delta x_i}\right)\int_{\bar{\bar{x}}_i}^{x_{i+1}}(x_{i+1}-x)\sin(\omega x)\,\mathrm{d}x\right)+L\,\mathrm{sign}(\sin(\omega x_N))\int_{x_N}^{x_{N+1}}(x-x_N)\sin(\omega x)\,\mathrm{d}x$$

$$=\frac{L}{\omega}\left(\sum_{i=1}^{N-1}\left|\sin\left(\frac{\omega}{2}(x_{i+1}+x_i)\right)\right|\times\left(\frac{4}{\omega}\left(\sin^2\frac{\omega\Delta x_i}{4}-\sin^2\frac{\omega|\Delta f_i|}{4L}\right)\right)\right.$$

$$-\frac{2}{\omega}\sum_{i=1}^{N-1}\mathrm{sign}(\Delta f_i)\cos\left(\frac{\omega}{2}(x_{i+1}+x_i)\right)\times\left(\sin\frac{\omega|\Delta f_i|}{2L}-\frac{|\Delta f_i|}{L\Delta x_i}\sin\frac{\omega\Delta x_i}{2}\right)$$

$$\left.+\frac{2}{\omega}\left|\sin\frac{\omega\Delta x_{N-1}}{2}\cos\left(\omega\left(b-\frac{\Delta x_{N-1}}{2}\right)\right)-\Delta x_N\cos(\omega b)\right|\right). \tag{3.20}$$

Therefore, substituting (3.19) and (3.20) into (3.18) we derive that

$$v(C^1_{1,L,N},R_2(\omega,S),f)\leqslant\frac{L}{\omega}\left(\sum_{i=1}^{N-1}\left|\sin\left(\frac{\omega}{2}(x_{i+1}+x_i)\right)\right|\left(\frac{4}{\omega}\left(\sin^2\frac{\omega\Delta x_i}{4}-\sin^2\frac{\omega|\Delta f_i|}{4L}\right)\right)\right.$$

$$+\frac{2}{\omega}\left(\sin\frac{\omega\Delta x_N}{2}\cos\left(\omega\left(b-\frac{\Delta x_N}{2}\right)\right)-\Delta x_N\cos(\omega b)\right)$$

$$+\frac{2}{\omega}\left|\sum_{i=1}^{N-1}\mathrm{sign}(\Delta f_i)\cos\left(\frac{\omega}{2}(x_{i+1}+x_i)\right)\right.$$

$$\left.\left.\times\left(\sin\frac{\omega|\Delta f_i|}{2L}-\frac{|\Delta f_i|}{L\Delta x_i}\sin\frac{\omega\Delta x_i}{2}\right)\right|\right), \tag{3.21}$$

which completes the proof. □

Remark 3.2. An analogous result takes place for quadrature formula (3.4) when computing (3.2) in class $C^1_{1,L,N}$, provided condition C1 is satisfied. More precisely, we have the following estimate:

$$
\begin{aligned}
&v(C^1_{1,L,N}, R_3(\omega,S), f)\\
&\quad\leqslant \frac{L}{\omega}\left(\sum_{i=1}^{N-1}\left|\cos\left(\frac{\omega}{2}(x_{i+1}+x_i)\right)\right|\left(\frac{4}{\omega}\left(\sin^2\frac{\omega\Delta x_i}{4}-\sin^2\frac{\omega|\Delta f_i|}{4L}\right)\right)\right.\\
&\qquad\left.+\frac{2}{\omega}\left|\Delta x_N\sin(\omega b)-\sin\frac{\omega\Delta x_N}{2}\times\cos\left(\omega\left(b-\frac{\Delta x_N}{2}\right)\right)\right|\right)\\
&\qquad+\frac{2L}{\omega^2}\left|\sum_{i=1}^{N-1}\operatorname{sign}(\Delta f_i)\sin\left(\frac{\omega}{2}(x_{i+1}+x_i)\right)\left(\sin\frac{\omega|\Delta f_i|}{2L}-\frac{|\Delta f_i|}{L\Delta x_i}\sin\frac{\omega\Delta x_i}{2}\right)\right|.
\end{aligned}
\tag{3.22}
$$

The above results were obtained in the case of weak oscillations (see Condition C1). Now let us consider the case of strong oscillations. Let $f(x)\in C^1_{1,L,N}$ and let us assume that N nodes x_i, $i=1,\dots,N$ are included in the number of zeros of $\sin(\omega x)$ ($\cos(\omega x)$), i.e. $N\gg([(|\omega|/\pi)(b-a)]+1)$. We refer to this condition as Condition C2. We also assume that there are k_i oscillations of function $\sin(\omega x)$ ($\cos(\omega x)$), $i=1,\dots,N$ on segments $[x_i,x_{i+1}]$. Then the following result holds.

Theorem 3.3. *Let $f(x)\in C^1_{1,L,N}$, $N<|\omega|$, condition C2 satisfied and $\{f_i\}_{i=1,\dots,N}$ are given on $[a,b]$ at nodes x_i, $i=1,\dots,N$ of the grid with fixed ends $x_1=a, x_{N+1}=b$. Then quadrature formula* (3.3) *for computing integral* (3.1) *is optimal-by-order with a constant not exceeding* 2. *This result holds with the following estimate*:

$$
\begin{aligned}
&\bar{v}(C^1_{1,L,N}, R_2(\omega,S), f)\\
&\quad\leqslant \frac{2L}{\omega^2}\left(\left[\frac{|\omega|}{\pi}\right]-\sum_{i=1}^{N-1}\left(\left[\frac{|\omega|}{\pi}x_{i+1}\right]-\left[\frac{|\omega|}{\pi}x_i\right]\right)\sin^2\frac{\pi|\Delta f_i|}{4L\Delta x_i}\right)\\
&\qquad+\frac{L}{|\omega|}\left|\frac{2}{\omega}\sin\left(\frac{\omega}{2}\left(b-\left[\frac{|\omega|}{\pi}\right]\frac{\pi}{|\omega|}\right)\right)\cos\left(\frac{\omega}{2}\left(2+\left[\frac{|\omega|}{\pi}\right]\frac{\pi}{|\omega|}-b\right)\right)\right.\\
&\qquad\left.-\left(1-\left[\frac{|\omega|}{\pi}\right]\frac{\pi}{|\omega|}\right)\cos(\omega b)\right|+\frac{2L}{\omega^2}\left|\sum_{i=1}^{N-1}\operatorname{sign}(\Delta f_i)\right.\\
&\qquad\left.\times\sum_{k=0}^{k_i-1}\cos\left(\frac{\omega}{2}(x_{i,k+1}+x_{i,k})\right)\left(\sin\left(\frac{\omega}{2}\frac{\pi|\Delta f_i|}{L\Delta x_i|\omega|}\right)-\frac{|\Delta f_i|}{L\Delta x_i}\sin\frac{\omega\pi}{2|\omega|}\right)\right|,
\end{aligned}
\tag{3.23}
$$

where $\Delta f_i=f_{i+1}-f_i$, $\Delta x_i=x_{i+1}-x_i$, $k_i=[|\omega|x_{i+1}/\pi]-[|\omega|x_i/\pi]$, $x_{i,k}=(\pi/2|\omega|)(2[(|\omega|/\pi)x_i+\frac{1}{2}]+2k+1)$ are zeros of function $\sin(\omega x)$ on $[x_i,x_{i+1}]$, $x_{i,0}=x_i$, $x_{i,k_i}=x_{i+1}$, $k=0,\dots,k_i$, $i=1,\dots,N$.

Proof. We showed (see the proof of Theorem 3.1) that $S(x,L)\in C^1_{1,L,N}$. Therefore, spline $S(x,L)$ satisfies inequalities (3.12) and (3.13). Let $x_N=x_{N-1,[\omega/\pi]}$ be the last zero of function $\sin(\omega x)$ on $[x_{N-1},x_{N+1}]$. Then, taking into account oscillations of function $\sin(\omega x)$, the limit functions of class

$C^1_{1,L,N}$, $f^+(x)$ and $f^-(x)$, will have the form

$$f^+(x) = \bigcup_{i,k} f^+_{i,k}(x), \quad f^-(x) = \bigcup_{i,k} f^-_{i,k}(x), \tag{3.24}$$

where $f^+_{i,k}(x)$, $f^-_{i,k}(x)$ are defined as follows.

(a) For $x \in [x_{i,k}, \bar{x}_{i,k}]$, $\bar{x}_{i,k} = (x_{i,k} + x_{i,k+1})/2 - |\Delta f_i|\pi/(2L\Delta x_i|\omega|)$ we have

$$\begin{aligned} f^+_{i,k}(x) &= f_i + \frac{\Delta f_i}{\Delta x_i}(x_{i,k} - x_i) + L(x - x_{i,k})\mathrm{sign}(\sin(\omega x_{i,k})), \\ f^-_{i,k}(x) &= f_i + \frac{\Delta f_i}{\Delta x_i}(x_{i,k} - x_i) - L(x - x_{i,k})\mathrm{sign}(\sin(\omega x_{i,k})). \end{aligned} \tag{3.25}$$

(b) For $x \in [\bar{x}_{i,k}, \bar{\bar{x}}_{i,k}]$, $\bar{\bar{x}}_{i,k} = (x_{i,k} + x_{i,k+1})/2 + |\Delta f_i|\pi/(2L\Delta x_i|\omega|)$ we have

$$\begin{aligned} f^+_{i,k}(x) = {} & \frac{1}{2}(1 + \mathrm{sign}(\Delta f_i)\mathrm{sign}(\sin(\omega x_{i,k})))\left(f_i + \frac{\Delta f_i}{\Delta x_i}(x_{i,k} - x_i)\right. \\ & \left. + L(x - x_{i,k})\mathrm{sign}(\sin(\omega x_{i,k}))\right) + \frac{1}{2}(1 - \mathrm{sign}(\Delta f_i)\mathrm{sign}(\sin(\omega x_{i,k}))) \\ & \times \left(f_i + \frac{\Delta f_i}{\Delta x_i}(x_{i,k+1} - x_i) + L(x_{i,k+1} - x)\mathrm{sign}(\sin(\omega x_{i,k}))\right), \\ f^-_{i,k}(x) = {} & \frac{1}{2}(1 - \mathrm{sign}(\Delta f_i)\mathrm{sign}(\sin(\omega x_{i,k})))\left(f_i + \frac{\Delta f_i}{\Delta x_i}(x_{i,k} - x_i)\right. \\ & \left. - L(x - x_{i,k})\mathrm{sign}(\sin(\omega x_{i,k}))\right) + \frac{1}{2}(1 + \mathrm{sign}(\Delta f_i)\mathrm{sign}(\sin(\omega x_{i,k}))) \\ & \times \left(f_i + \frac{\Delta f_i}{\Delta x_i}(x_{i,k+1} - x_i) - L(x_{i,k+1} - x)\mathrm{sign}(\sin(\omega x_{i,k}))\right). \end{aligned} \tag{3.26}$$

(c) For $x \in [\bar{\bar{x}}_{i,k}, x_{i,k}]$ we have

$$\begin{aligned} f^+_{i,k}(x) &= f_i + \frac{\Delta f_i}{\Delta x_i}(x_{i,k+1} - x_i) + L(x_{i,k+1} - x)\mathrm{sign}(\sin(\omega x_{i,k})), \\ f^-_{i,k}(x) &= f_i + \frac{\Delta f_i}{\Delta x_i}(x_{i,k+1} - x_i) - L(x_{i,k+1} - x)\mathrm{sign}(\sin(\omega x_{i,k})). \end{aligned} \tag{3.27}$$

In formulae (3.25)–(3.27) $\mathrm{sign}(\sin(\omega x_{i,k}))$ denotes the signum of function $\sin(\omega x)$ on intervals $[x_{i,k}, x_{i,k+1}]$, $k=0,\ldots,k_i-1$, $i=1,\ldots,N$. If further we chose an approximation $f^*_{i,k}(x)$ in the following form:

$$f^*_{i,k}(x) = \begin{cases} S(x,L), & x \in [x_{i,k}, x_{i,k+1}],\ k=0,\ldots,k_i-1,\ i=1,\ldots,N, \\ f_N, & x \in [x_{N-1,[|\omega|/\pi]}, x_{N+1}], \end{cases} \tag{3.28}$$

then in the case of strong oscillations, the error of quadrature formula (3.3) for computing integral (3.1) in class $C^1_{1,L,N}$ can be estimated as follows:

$$\bar{v}(C^1_{1,L,N}, R_2(\omega,S), f) \leqslant \max\left\{\sum_{i=1}^{N+1}\sum_{k=0}^{k_i-1}\int_{x_{i,k}}^{x_{i,k+1}}(f^+_{i,k}(x)-f^*_{i,k}(x))\sin(\omega x)\,\mathrm{d}x,\right.$$
$$\left.\sum_{i=1}^{N+1}\sum_{k=0}^{k_i-1}\int_{x_{i,k}}^{x_{i,k+1}}\left(f^*_{i,k}(x)-f^-_{i,k}(x)\right)\sin(\omega x)\,\mathrm{d}x\right\}. \tag{3.29}$$

Using (3.25)–(3.28) integrals in (3.29) can be found in the explicit form. For the first component under the maximum sign in (3.29) we have:

$$\sum_{i=1}^{N+1}\sum_{k=0}^{k_i-1}\int_{x_{i,k}}^{x_{i,k+1}}(f^+_{i,k}(x)-f^*_{i,k}(x))\sin(\omega x)\,\mathrm{d}x$$
$$=\sum_{i=1}^{N+1}\sum_{k=0}^{k_i-1}\left(\int_{x_{i,k}}^{\bar{x}_{i,k}}\left(f_i+\frac{\Delta f_i}{\Delta x_i}(x_{i,k}-x_i)\right.\right.$$
$$\left.+L(x-x_{i,k})\mathrm{sign}(\sin(\omega x_{i,k}))-f_i-\frac{\Delta f_i}{\Delta x_i}(x-x_i)\right)\sin(\omega x)\,\mathrm{d}x+\int_{\bar{x}_{i,k}}^{\bar{\bar{x}}_{i,k}}\frac{1}{2}((1+\mathrm{sign}(\Delta f_i)$$
$$\times\,\mathrm{sign}(\sin(\omega x_{i,k})))\left(f_i+\frac{\Delta f_i}{\Delta x_i}(x_{i,k}-x_i)+L(x-x_{i,k})\mathrm{sign}(\sin(\omega x_{i,k}))-f_i-\frac{\Delta f_i}{\Delta x_i}(x-x_i)\right)$$
$$+(1-\mathrm{sign}(\Delta f_i)\mathrm{sign}(\sin(\omega x_{i,k})))\left(f_i+\frac{\Delta f_i}{\Delta x_i}(x_{i,k+1}-x_i)+L(x_{i,k+1}-x)\mathrm{sign}(\sin(\omega x_{i,k}))\right.$$
$$\left.\left.-f_i-\frac{\Delta f_i}{\Delta x_i}(x-x_i)\right)\right)\sin(\omega x)\,\mathrm{d}x+\int_{\bar{\bar{x}}_{i,k}}^{x_{i,k+1}}\left(f_i+\frac{\Delta f_i}{\Delta x_i}(x_{i,k+1}-x_i)+L(x_{i,k+1}-x)\right.$$
$$\left.\times\,\mathrm{sign}(\sin(\omega x_{i,k}))-f_i-\frac{\Delta f_i}{\Delta x_i}(x-x_i)\right)\sin(\omega x)\,\mathrm{d}x+\int_{x_{N-1,[|\omega|/\pi]}}^{x_{N+1}}L(x-x_{N-1,[|\omega|/\pi]})$$
$$\left.\times\,\mathrm{sign}(\sin(\omega x_{N-1,[|\omega|/\pi]}))\sin(\omega x)\,\mathrm{d}x\right)$$
$$=\frac{2L}{\omega^2}\left(\left[\frac{|\omega|}{\pi}\right]-\sum_{i=0}^{N-1}\left(\left[\frac{|\omega|}{\pi}x_{i+1}\right]-\left[\frac{|\omega|}{\pi}x_i\right]\right)\right.$$
$$\times\sin^2\frac{|\Delta f_i|\pi}{4L\Delta x_i}+\sum_{i=1}^{N-1}\mathrm{sign}(\Delta f_i)\times\sum_{k=0}^{k_i-1}\cos\left(\frac{\omega}{2}(x_{i,k+1}+x_{i,k})\right)\left(\sin\frac{\omega|\Delta f_i|\pi}{2L\Delta x_i|\omega|}-\frac{|\Delta f_i|}{L\Delta x_i}\right.$$
$$\left.\left.\times\,\sin\left(\frac{\omega\pi}{2|\omega|}\right)\right)\right)+\frac{L}{|\omega|}\left|\frac{2}{\omega}\sin\left(\frac{\omega}{2}\left(b-\left[\frac{|\omega|}{\pi}\right]\frac{\pi}{|\omega|}\right)\right)\cos\left(\frac{\omega}{2}\times\left(2+\left(\left[\frac{|\omega|}{\pi}\right]\frac{\pi}{|\omega|}-b\right)\right)\right)\right.$$
$$\left.-\left(1-\left[\frac{|\omega|}{\pi}\right]\frac{\pi}{|\omega|}\right)\cos(\omega b)\right|. \tag{3.30}$$

Using the explicit form of minorant function given by (3.25)–(3.27) and equality (3.28), we transform the second component under the maximum sign in (3.29)

$$\sum_{i=1}^{N+1}\sum_{k=0}^{k_i-1}\int_{x_{i,k}}^{x_{i,k+1}}(f_{i,k}^{*}(x)-f_{i,k}^{-}(x))\sin(\omega x)\,\mathrm{d}x$$

$$=\sum_{i=1}^{N}\sum_{k=0}^{k_i-1}\left(\int_{x_{i,k}}^{\bar{x}_{i,k}}\left(f_i+\frac{\Delta f_i}{\Delta x_i}(x-x_i)-f_i-\frac{\Delta f_i}{\Delta x_i}(x_{i,k}-x_i)\right.\right.$$

$$\left.+L(x-x_{i,k})\mathrm{sign}(\sin(\omega x_{i,k}))\right)\sin(\omega x)\,\mathrm{d}x+\frac{1}{2}\int_{\bar{x}_{i,k}}^{\bar{\bar{x}}_{i,k}}((1-\mathrm{sign}(\Delta f_i)\mathrm{sign}(\sin(\omega x_{i,k})))$$

$$\times\left(f_i+\frac{\Delta f_i}{\Delta x_i}(x-x_i)-f_i-\frac{\Delta f_i}{\Delta x_i}(x_{i,k}-x_i)+L(x-x_{i,k})\mathrm{sign}(\sin(\omega x_{i,k}))\right)$$

$$+(1+\mathrm{sign}(\Delta f_i)\mathrm{sign}(\sin(\omega x_{i,k})))\left(f_i+\frac{\Delta f_i}{\Delta x_i}(x-x_i)-f_i-\frac{\Delta f_i}{\Delta x_i}(x_{i,k+1}-x_i)\right.$$

$$\left.+L(x_{i,k+1}-x)\mathrm{sign}(\sin(\omega x_{i,k}))\right))\sin(\omega x)\,\mathrm{d}x+\int_{\bar{\bar{x}}_{i,k}}^{x_{i,k}}\left(f_i+\frac{\Delta f_i}{\Delta x_i}(x-x_i)-f_i\right.$$

$$\left.\left.-\frac{\Delta f_i}{\Delta x_i}(x_{i,k+1}-x_i)+L(x_{i,k}-x)\mathrm{sign}(\sin(\omega x_{i,k}))\right)\sin(\omega x)\,\mathrm{d}x\right)$$

$$+\int_{x_{N-1,[|\omega|/\pi]}}^{x_{N+1}}L(x-x_{N-1,[|\omega|/\pi]})\mathrm{sign}(\sin(\omega x_{N-1,[|\omega|/\pi]}))\sin(\omega x)\,\mathrm{d}x$$

$$=\frac{2L}{\omega^2}\left(\left[\frac{|\omega|}{\pi}\right]-\sum_{i=1}^{N-1}\left(\left[\frac{|\omega|}{\pi}x_{i+1}\right]-\left[\frac{|\omega|}{\pi}x_i\right]\right)\sin^2\frac{|\Delta f_i|\pi}{4L\Delta x_i}\right.$$

$$-\sum_{i=1}^{N-1}\mathrm{sign}(\Delta f_i)\times\sum_{k=0}^{k_i-1}\cos\left(\frac{\omega}{2}(x_{i,k+1}+x_{i,k})\right)$$

$$\times\left(\sin\left(\frac{\omega|\Delta f_i|\pi}{2L\Delta x_i|\omega|}\right)-\frac{|\Delta f_i|}{L\Delta x_i}\times\sin\left(\frac{\omega}{2}\frac{\pi}{|\omega|}\right)\right)\Bigg)+\frac{L}{|\omega|}\left|\frac{2}{\omega}\sin\left(\frac{\omega}{2}\left(b-\left[\frac{|\omega|}{\pi}\right]\frac{\pi}{|\omega|}\right)\right)\right.$$

$$\left.\times\cos\left(\frac{\omega}{2}\left(2+\left(\left[\frac{|\omega|}{\pi}\right]\frac{\pi}{|\omega|}-b\right)\right)\right)-\left(1-\left[\frac{|\omega|}{\pi}\right]\frac{\pi}{|\omega|}\right)\cos(\omega b)\right|. \tag{3.31}$$

Substituting resulting equalities (3.30) and (3.31) into (3.29) yields

$$\bar{v}(C_{1,L,N}^{1},R_2(\omega,S),f)$$

$$\leqslant\frac{2L}{\omega^2}\left(\left[\frac{|\omega|}{\pi}\right]-\sum_{i=1}^{N-1}\left(\left[\frac{|\omega|}{\pi}x_{i+1}\right]-\left[\frac{|\omega|}{\pi}x_i\right]\right)\sin^2\frac{\pi|\Delta f_i|}{4L\Delta x_i}\right.$$

$$\left.+\left|\sum_{i=1}^{N-1}\mathrm{sign}(\Delta f_i)\sum_{k=0}^{k_i-1}\cos\left(\frac{\omega}{2}\,(x_{i,k+1}+x_{i,k})\right)\left(\sin\left(\frac{\omega}{2}\frac{\pi|\Delta f_i|}{L\Delta x_i|\omega|}\right)-\frac{|\Delta f_i|}{L\Delta x_i}\sin\left(\frac{\omega\pi}{2|\omega|}\right)\right)\right|\right)$$

$$+\frac{L}{|\omega|}\left|\frac{2}{\omega}\sin\left(\frac{\omega}{2}\left(b-\left[\frac{|\omega|}{\pi}\right]\frac{\pi}{|\omega|}\right)\right)\cos\left(\frac{\omega}{2}\left(2+\left(\left[\frac{|\omega|}{\pi}\right]\frac{\pi}{|\omega|}-b\right)\right)\right)\right.$$
$$\left.-\left(1-\left[\frac{|\omega|}{\pi}\right]\frac{\pi}{|\omega|}\right)\cos(\omega b)\right|, \tag{3.32}$$

that completes the proof. □

Remark 3.4. If condition C2 is satisfied, then for quadrature formula (3.4) computing (3.2) in class $C^1_{1,L,N}$ we have the following estimate:

$$\bar{v}(C^1_{1,L,N}, R_3(\omega,S), f)$$
$$\leqslant \frac{2L}{\omega^2}\left(\left[\frac{|\omega|}{\pi}\right]-\sum_{i=1}^{N-1}\left(\left[\frac{|\omega|}{\pi}x_{i+1}\right]-\left[\frac{|\omega|}{\pi}x_i\right]\right)\cos^2\frac{|\Delta f_i|\pi}{4L\Delta x_i}\right)$$
$$+\frac{L}{|\omega|}\left|\frac{2}{\omega}\cos\left(\frac{\omega}{2}\left(b-\left[\frac{|\omega|}{\pi}\right]\frac{\pi}{|\omega|}\right)\right)\times\cos\left(\frac{\omega}{2}\left(2+\left(\left[\frac{|\omega|}{\pi}\right]\frac{\pi}{|\omega|}-b\right)\right)\right)\right.$$
$$\left.-\left(1-\left[\frac{|\omega|}{\pi}\right]\frac{\pi}{|\omega|}\right)\sin(\omega b)\right|+\frac{2L}{\omega^2}\left|\sum_{i=1}^{N-1}\operatorname{sign}(\Delta f_i)\sum_{k=0}^{k_i-1}\sin\left(\frac{\omega}{2}(x_{i,k+1}+x_{i,k})\right)\right.$$
$$\left.\times\left(\sin\frac{\omega|\Delta f_i|\pi}{2L\Delta x_i|\omega|}-\frac{|\Delta f_i|}{L\Delta x_i}\sin\frac{\omega\pi}{2|\omega|}\right)\right|, \tag{3.33}$$

where $x_{i,k}=(\pi/|\omega|)(2[(|\omega|/\pi)x_i+1]+2k)$ are zeros of function $\cos(\omega x)$ on $[x_i,x_{i+1}]$, $x_{i,0}=x_i$, $x_{i,k_i}=x_{i+1}$, $k=0,\dots,k_i$, $i=1,\dots,N$.

4. Error estimates for optimal-by-order quadrature formulae in class $C^1_{1,L,N,\varepsilon}$

In this section we generalize results obtained in previous sections into class $C^1_{1,L,N,\varepsilon}$. As above, we deal with the case when a priori information is given approximately. This precludes an efficient application of optimal-by-accuracy quadrature formulae that are based on the exact knowledge of a priori information.

Using the method of quasi-solutions in the construction of quadrature formulae for approximate calculation of integrals (3.1) and (3.2) in class $C^1_{1,L,N,\varepsilon}$, we assume that the Lipschitz constant, L, and a certain accuracy estimate for the definition of function $f(x)$ in N nodes of an arbitrary grid are given. In this case the application of a linear spline $S(x,L)$ as an approximation of the function $f(x)$, allows smoothing input data and defining ε with a higher precision. In some applications the Lipschitz constant is not known, but we can estimate accuracy of the definition of $f(x)$ in nodes of the grid. Then it is reasonable to apply quadrature formulae constructed by the residual method where the value of L is not used.

We note that both quadrature formulae for computing (3.1) (those constructed with the method of quasi-solutions and the residual method), has the form (3.3). The difference consists of the fact that in the method of quasi-solutions values $\hat{f}_i$, $i=1,\dots,N$ are computed with (2.9), making use of constant L. In the residual method, for computing $\hat{f}_i$, $i=1,\dots,N$ we change constant L in (2.9) for constant M defined by (2.11).

Let $\Delta\tilde{f}_i=\tilde{f}_{i+1}-\tilde{f}_i$, $\Delta x_i=x_{i+1}-x_i$ and $Q=\Delta\tilde{f}_i/\Delta x_i$, $i=1,\dots,N-1$. Then we introduce the following notation:

$$\Delta_i=\frac{(\xi_i^+ +\xi_{i+1}^+)+(\xi_i^- +\xi_{i+1}^-)-(\Delta\xi_i^+ -\Delta\xi_i^-)\mathrm{sign}(\sin(\omega x))}{4(Q+L)\mathrm{sign}(\sin(\omega x))}, \tag{4.1}$$

where $\xi_i^\pm, i=1,\dots,N$ are, respectively, maximal and minimal admissible solutions of the system of linear inequalities

$$\begin{aligned} &-\varepsilon_i\leqslant\xi_i\leqslant\varepsilon_i,\quad i=1,\dots,N,\\ &-L\Delta x_i-\Delta\tilde{f}_i\leqslant\xi_{i+1}-\xi_i\leqslant L\Delta x_i-\Delta\tilde{f}_i,\quad i=1,\dots,N-1. \end{aligned} \tag{4.2}$$

Let further

$$n_i=\left[\frac{|\omega|}{\pi}x_{i+1}\right]-\left[\frac{|\omega|}{\pi}x_i\right] \tag{4.3}$$

be the number of oscillations of function $\sin\omega x$ on $[x_i,x_{i+1}]$ and

$$x_{i,k}=\left(\left[\frac{|\omega|}{\pi}x_i\right]+k\right)\frac{\pi}{|\omega|},\quad k=1,\dots,n_i \tag{4.4}$$

be zeros of function $\sin(\omega x)$ on $[x_i,x_{i+1}]$, $x_{i,0}=\tilde{x}_{i,0}=x_i$, $x_{i,n_i+1}=\tilde{x}_{i,n_i+1}=x_{i+1}$, $x^1\in[x_i,x_{i+1}]$, $x^2\in[x_{i,n_i},x_{i+1}]$. Finally, let

$$\begin{aligned} &\tilde{x}_{i,1}=x_{i,1}+\left(x_{i,1}-\frac{x_i+x_{i,2}}{2}\right)\frac{L\,\mathrm{sign}(\sin(\omega x))-Q}{L\,\mathrm{sign}(\sin(\omega x))+Q}+\Delta_i,\\ &\tilde{x}_{i,n_i}=x_{i,n_i}+\left(x_{i,n_i}-\frac{x_{i,n_i-1}+x_{i+1}}{2}\right)\frac{L\,\mathrm{sign}(\sin(\omega x))-Q}{L\,\mathrm{sign}(\sin(\omega x))+Q}+\Delta_i,\quad i=1,\dots,N-1,\\ &\tilde{x}_{i,k}=x_{i,k}+\Delta_i,\quad k=2,\dots,n_i-1, \end{aligned} \tag{4.5}$$

$$\Delta\tilde{f}_{i,k}=Q\Delta\tilde{x}_{i,k},\quad \Delta\tilde{x}_{i,k}=\tilde{x}_{i,k+1}-\tilde{x}_{i,k}, \tag{4.6}$$

$$\hat{x}_{i,k}=\frac{\tilde{x}_{i,k}+\tilde{x}_{i,k+1}}{2}-\delta_{i,k},\quad \hat{\hat{x}}_{i,k}=\frac{\tilde{x}_{i,k}+\tilde{x}_{i,k+1}}{2}+\delta_{i,k}, \tag{4.7}$$

where

$$\begin{aligned} \delta_{i,k}=\frac{1}{4L}(2|\Delta\tilde{f}_{i,k}|&+(\xi_{i+1}^+(1-\mathrm{sign}(x_{i+1}-x_{i,k+1}))\\ &-\xi_i^+(1-\mathrm{sign}(x_{i,k}-x_i)))(\mathrm{sign}(\Delta\tilde{f}_i)+1)+(\xi_{i+1}^-(1-\mathrm{sign}(x_{i+1}-x_{i,k+1}))\\ &-\xi_i^-(1-\mathrm{sign}(x_{i,k}-x_i)))(\mathrm{sign}(\Delta\tilde{f}_i)-1)), \end{aligned} \tag{4.8}$$

$k_i=1,\dots,n_i-1$, $i=1,\dots,N-1$.

Now we are in the position to formulate the main result of this section.

Theorem 4.1. *Let $f(x)\in C^1_{1,L,N,\varepsilon}$, condition C2 satisfied and $\{\tilde{f}_i\}_{i=1,\dots,N}$ are given on $[a,b]$ in x_i, $i=1,\dots,N$ nodes of the grid with fixed ends $x_1=a$, $x_N=b$. Then quadrature formula* (3.3) *for computing* (3.1) *constructed by the residual method or the method of quasi-solutions is optimal-by-order with constant not exceeding* 2. *The following estimate holds*:

$$
\begin{aligned}
v(C^1_{1,L,N,\varepsilon}, R_2(\omega,S), f) \leqslant{}& \frac{L}{\omega^2}\sum_{i=1}^{N-1}\left(2\left|\sin\left(\omega\left(\frac{x_i+\tilde{x}_{i,1}}{2}+\frac{\xi_i^- - \xi_i^+}{2L}\right)\right)\right|\right.\\
&\times\cos\left(\omega\frac{2|\Delta\tilde{f}_{i,0}| - |\xi_i^+ + \xi_i^-|\operatorname{sign}(\Delta\tilde{f}_i)}{2L}\right) - |\sin(\omega x_i)|\\
&+\frac{\omega(\xi_i^+ - \xi_i^-)\operatorname{sign}(\sin(\omega x^1))}{2L}\cos(\omega x_i)\\
&+2\left(\left|\sin\left(\omega\frac{\tilde{x}_{i,1}+\tilde{x}_{i,2}}{2}\right)\right| + (n_i-3)\left|\sin\left(\omega\frac{\tilde{x}_{i,2}+\tilde{x}_{i,3}}{2}\right)\right|\right.\\
&\left.+\left|\sin\left(\omega\frac{\tilde{x}_{i,n_i-1}+x_{i,n_i}}{2}\right)\right|\right)\cos\frac{\pi Q}{2L} - |\sin(\omega x_{i+1})|\\
&-\frac{\omega(\xi_{i+1}^+ - \xi_{i+1}^-)\operatorname{sign}(\sin(\omega x^2))}{2L}\cos(\omega x_{i+1}) + 2\left|\sin\left(\omega\left(\frac{\tilde{x}_{i,n_i}+x_{i+1}}{2}\right.\right.\right.\\
&\left.\left.\left.+\frac{\xi_{i+1}^+ - \xi_{i+1}^-}{2}\right)\right)\right|\left.\cos\left(\omega\frac{2|\Delta\tilde{f}_{i,n_i}| + |\xi_{i+1}^+ + \xi_{i+1}^-|\operatorname{sign}(\Delta\tilde{f}_i)}{2L}\right)\right)\\
&+\left|\frac{1}{\omega^2}\sum_{i=1}^{N-1}\left(\sum_{k=0}^{n_i} L\operatorname{sign}(\Delta\tilde{f}_i)(\sin(\omega\hat{\hat{x}}_{i,k}) - \sin(\omega\hat{x}_{i,k}))\right.\right.\\
&\left.-\frac{\Delta\hat{f}_i}{\Delta x_i}(\sin(\omega x_{i+1}) - \sin(\omega x_i))\right) + \frac{1}{\omega}\left((\hat{f}_N - \tilde{f}_N)\cos(\omega x_{N-1})\right.\\
&\left.\left.-(\hat{f}_1 - \tilde{f}_1)\cos(\omega x_1)\right)\right|. \qquad (4.9)
\end{aligned}
$$

Proof. First, we consider quadrature formula (3.3) constructed by the method of quasi-solutions. From the definition of quantities $\tilde{f}_i^+$, $\tilde{f}_i^-$, $i=1,\dots,N$ follows that $S(x_i,L)\in[\tilde{f}_i-\varepsilon,\tilde{f}_i+\varepsilon]$, $i=1,\dots,N$, and that $|S'(x,L)|\leqslant L$. Indeed, for any $x\in[x_i,x_{i+1}]$, $i=1,\dots,N-1$ we have

$$
\begin{aligned}
|S'(x,L)| &= \left|\frac{\hat{f}_{i+1}-\hat{f}_i}{x_{i+1}-x_i}\right| = \left|\frac{\tilde{f}_{i+1}^+ + \tilde{f}_{i+1}^- - \tilde{f}_i^+ - \tilde{f}_i^-}{2(x_{i+1}-x_i)}\right|\\
&= \frac{1}{2(x_{i+1}-x_i)}\left|\max_{1\leqslant j\leqslant N}(\tilde{f}_j - L|x_j - x_{i+1}|) - \max_{1\leqslant j\leqslant N}(-\tilde{f}_j - L|x_j - x_{i+1}|)\right.\\
&\quad\left. - \max_{1\leqslant j\leqslant N}(\tilde{f}_j - L|x_j - x_i|) + \max_{1\leqslant j\leqslant N}(-\tilde{f}_j - L|x_j - x_i|)\right|\\
&= \frac{1}{2(x_{i+1}-x_i)}\times\left|\tilde{f}_{j_1} - L|x_{j_1} - x_{i+1}| + \tilde{f}_{j_2}L|x_{j_2} - x_{i+1}|\right.\\
&\quad\left. - \tilde{f}_{j_3} + L|x_{j_3} - x_i| - \tilde{f}_{j_4} - L|x_{j_4} - x_i|\right|, \qquad (4.10)
\end{aligned}
$$

where j_k, $k = 1,\dots,4$ are the values of index j on which the following values are achieved $\max_{1\leqslant j\leqslant N}(\tilde f_j - L|x_j - x_{i+1}|)$, $\max_{1\leqslant j\leqslant N}(-\tilde f_j - L|x_j - x_{i+1}|)$, $\max_{1\leqslant j\leqslant N}(\tilde f_j - L|x_j - x_i|)$ and $\max_{1\leqslant j\leqslant N}(-\tilde f_j - L|x_j - x_i|)$. If $(-\tilde f_{j_2} - L|x_{j_2} - x_{j+1}|) = \max_{1\leqslant j\leqslant N}(-\tilde f_j - L|x_j - x_{i+1}|)$, then $f_{j_2} + L|x_{j_2} - x_{i+1}|) = \min_{1\leqslant j\leqslant N}(\tilde f_j + L|x_j - x_{i+1}|)$. Hence changing $(\tilde f_{j_2} + L|x_{j_2} - x_{i+1}|)$ for $(\tilde f_{j_4} + L|x_{j_4} - x_{i+1}|)$, we only increase the value of the expression $(\max_{1\leqslant j\leqslant N}(\tilde f_j - L|x_j - x_{i+1}|) - \max_{1\leqslant j\leqslant N}(-\tilde f_j - L|x_j - x_{i+1}|))$. In a similar way, since $(\tilde f_{j_3} - L|x_{j_3} - x_i| = \max_{1\leqslant j\leqslant N}(\tilde f_j - L|x_j - x_i|))$, we can change it for $(\tilde f_{j_1} - L|x_{j_1} - x_i|)$ increasing the expression $(-\max_{1\leqslant j\leqslant N}(\tilde f_j - L|x_j - x_i|) + \max_{1\leqslant j\leqslant N}(-\tilde f_j - L|x_j - x_i|))$. Therefore, from (4.10) we get

$$\begin{aligned}|S'(x,L)| &\leqslant \frac{1}{2(x_{i+1}-x_i)}|\tilde f_{j_1} - L|x_{j_1} - x_{i+1}| + \tilde f_{j_4} + L|x_{j_4} - x_{i+1}| \\ &\quad - \tilde f_{j_1} + L|x_{j_1} - x_i| - \tilde f_{j_4} - L|x_{j_4} - x_i| \\ &\leqslant \frac{L}{2(x_{i+1}-x_i)}(|x_{j_1} - x_i - x_{j_1} + x_{i+1}| + |x_{j_4} - x_{i+1} - x_{j_4} + x_i|) = L. \end{aligned}\tag{4.11}$$

Hence, $S(x,L) \in C^1_{1,L,N,\varepsilon}$.

Error estimates of quadrature formula (3.3) can be obtained on the basis of inequalities (3.12) and (3.13). Indeed, assume that on interval $[x_i, x_{i+1}]$ there are $n_i = [(|\omega|/\pi)x_{i+1}] - [(|\omega|/\pi)x_i]$ oscillations of function $\sin(\omega x)$, $i = 1,\dots,N-1$. Then the limit functions of class $C^1_{1,L,N,\varepsilon}$ on $[x_{i,k}, x_{i,k+1}]$, $k = 0,\dots,n_i - 1$ can be written in the explicit form as follows:

(a) For $x \in [x_{i,k}, \hat x_{i,k}]$ we have

$$\begin{aligned} f^{\pm}_{i,k}(x) = \tilde f_i + &\left(\frac{\xi_i^+ + \xi_i^-}{2} \pm \frac{\xi_i^+ - \xi_i^-}{2}\operatorname{sign}(\sin(\omega x))\right) \\ &\times(1 - \operatorname{sign}(x_{i,k} - x_i)) + Q(\tilde x_{i,k} - x_i) \pm L(x - x_{i,k})\operatorname{sign}(\sin(\omega x)). \end{aligned}\tag{4.12}$$

(b) For $x \in [\hat x_{i,k}, \hat{\hat x}_{i,k}]$ we have

$$\begin{aligned} f^{\pm}_{i,k}(x) = \tilde f_i + &\frac{1}{2}(1 \pm \operatorname{sign}(\Delta \tilde f_i)\operatorname{sign}(\sin(\omega x)))\left(\left(\frac{\xi_i^+ + \xi_i^-}{2} \pm \frac{\xi_i^+ - \xi_i^-}{2}\operatorname{sign}(\sin(\omega x))\right)\right. \\ &\times (1 - \operatorname{sign}(x_{i,k} - x_i)) + Q(\tilde x_{i,k} - x_i) \pm L(x - x_{i,k})\operatorname{sign}(\sin(\omega x))) \\ &+ \frac{1}{2}(1 \mp \operatorname{sign}(\Delta \tilde f_i)\operatorname{sign}(\sin(\omega x))) \times \left(\left(\frac{\xi_{i+1}^+ + \xi_{i+1}^-}{2} \pm \frac{\xi_{i+1}^+ - \xi_{i+1}^-}{2}\operatorname{sign}(\sin(\omega x))\right)\right. \\ &\times (1 - \operatorname{sign}(x_{i+1} - x_{i,k+1})) + Q(\tilde x_{i,k+1} - x_i) \pm L(x_{i,k+1} - x)\operatorname{sign}(\sin(\omega x))). \end{aligned}\tag{4.13}$$

(c) For $x \in [\hat{\hat x}_{i,k}, x_{i,k+1}]$ we have

$$\begin{aligned} f^{\pm}_{i,k}(x) = \tilde f_i + &\left(\frac{\xi_{i+1}^+ + \xi_{i+1}^-}{2} \pm \frac{\xi_{i+1}^+ - \xi_{i+1}^-}{2}\operatorname{sign}(\sin(\omega x))\right) \\ &\times(1 - \operatorname{sign}(x_{i+1} - x_{i,k+1})) + Q(\tilde x_{i,k+1} - x_i) \pm L(x_{i,k+1} - x)\operatorname{sign}(\sin(\omega x)). \end{aligned}\tag{4.14}$$

We recall that $x_1 = a$ and $x_N = b$. Then the error of quadrature formula (3.3) in class $C^1_{1,L,N,\varepsilon}$ in the case of strong oscillations of $\sin \omega x$ can be estimated as follows:

$$v(C^1_{1,L,N,\varepsilon}, R_2(\omega,S), f) \leqslant \max\left(\sum_{i=1}^{N-1} \sum_{k=0}^{n_i} \int_{x_{i,k}}^{x_{i,k+1}} (f^+_{i,k}(x) - S(x,L)) \sin(\omega x)\,\mathrm{d}x, \right.$$

$$\left. \times \sum_{i=1}^{N-1} \sum_{k=0}^{n_i} \int_{x_{i,k}}^{x_{i,k+1}} (S(x,L) - f^-_{i,k}(x)) \sin(\omega x)\,\mathrm{d}x \right). \tag{4.15}$$

Using (4.12)–(4.14), the integrals in (4.15) can be calculated explicitly. Taking into account explicit expressions for the majorant function (see (4.12)–(4.14)), we have

$$\sum_{i=1}^{N-1} \sum_{k=0}^{n_i} \int_{x_{i,k}}^{x_{i,k+1}} (f^+_{i,k}(x) - S(x,L)) \sin(\omega x)\,\mathrm{d}x$$

$$= \sum_{i=1}^{N-1} \sum_{k=0}^{n_i} \left(\int_{x_{i,k}}^{\hat{x}_{i,k}} \left(\tilde{f}_i - \hat{f}_i + Q(\tilde{x}_{i,k} - x_i) \right.\right.$$

$$\left. - \frac{\Delta \hat{f}_i}{\Delta x_i}(x - x_i) + \frac{\xi^+_i + \xi^-_i}{2}(1 - \mathrm{sign}(x_{i,k} - x_i)) + L(x - x_{i,k})\,\mathrm{sign}(\sin(\omega x)) \right) \sin(\omega x)\,\mathrm{d}x$$

$$+ \frac{1}{2} \int_{\hat{x}_{i,k}}^{\hat{\hat{x}}_{i,k}} \left(\left(\tilde{f}_i - \hat{f}_i + Q(\tilde{x}_{i,k} - x_i) - \frac{\Delta \hat{f}_i}{\Delta x_i}(x - x_i) + \frac{\xi^+_i + \xi^-_i}{2}(1 - \mathrm{sign}(x_{i,k} - x_i)) \right.\right.$$

$$+ L(x - x_{i,k})\,\mathrm{sign}(\sin(\omega x)))\,(1 + \mathrm{sign}(\Delta \tilde{f}_i)\,\mathrm{sign}(\sin(\omega x))) + \left(\tilde{f}_i - \hat{f}_i + Q(\tilde{x}_{i,k+1} - x_i) \right.$$

$$\left. - \frac{\Delta \hat{f}_i}{\Delta x_i}(x - x_i) + \frac{\xi^+_{i+1} + \xi^-_{i+1}}{2}(1 - \mathrm{sign}(x_{i+1} - x_{i,k+1})) + L(x_{i,k} - x)\,\mathrm{sign}(\sin(\omega x)) \right)$$

$$\left. \times (1 - \mathrm{sign}(\Delta \tilde{f}_i)\,\mathrm{sign}(\sin(\omega x))) \right) \sin(\omega x)\,\mathrm{d}x + \int_{\hat{\hat{x}}_{i,k}}^{x_{i,k+1}} \left(\tilde{f}_i - \hat{f}_i + Q(\tilde{x}_{i,k+1} - x_i) \right.$$

$$\left.\left. - \frac{\Delta \hat{f}_i}{\Delta x_i}(x - x_i) + \frac{\xi^+_{i+1} + \xi^-_{i+1}}{2}(1 - \mathrm{sign}(x_{i+1} - x_{i,k+1})) \right) \sin(\omega x)\,\mathrm{d}x \right)$$

$$= \frac{L}{\omega^2} \sum_{i=1}^{N-1} \left(2 \left| \sin\left(\omega \left(\frac{x_i + \tilde{x}_{i,1}}{2} + \frac{\xi^-_i - \xi^+_i}{2L} \right) \right) \right| \cos\left(\omega \frac{2|\Delta \tilde{f}_{i,0}| - |\xi^+_i + \xi^-_i|\,\mathrm{sign}(\Delta \tilde{f}_i)}{2L} \right) \right.$$

$$- |\sin(\omega x_i)| + \frac{\omega(\xi^+_i - \xi^-_i)\,\mathrm{sign}(\sin(\omega x^1))}{2L} \cos(\omega x_i) + 2 \left(\left| \sin\left(\omega \frac{\tilde{x}_{i,1} + \tilde{x}_{i,2}}{2} \right) \right| \right.$$

$$\left. + (n_i - 3) \left| \sin\left(\omega \frac{\tilde{x}_{i,2} + \tilde{x}_{i,3}}{2} \right) \right| + \left| \sin\left(\omega \frac{\tilde{x}_{i,n_i-1} + \tilde{x}_{i,n_i}}{2} \right) \right| \right) \cos \frac{\pi Q}{2L} - |\sin(\omega x_{i+1})|$$

$$
\begin{aligned}
&- \frac{\omega(\xi_{i+1}^+ - \xi_{i+1}^-)\,\mathrm{sign}(\sin(\omega x^2))}{2L}\cos(\omega x_{i+1}) + 2\left|\sin\left(\omega\left(\frac{\tilde{x}_{i,n_i} + x_{i+1}}{2} + \frac{\xi_{i+1}^+ - \xi_{i+1}^-}{2}\right)\right)\right| \\
&\times\cos\left(\omega\frac{2|\Delta\tilde{f}_{i,n_i}| + (\xi_{i+1}^+ + \xi_{i+1}^-)\,\mathrm{sign}(\Delta\tilde{f}_i)}{2L}\right)\right) + \frac{1}{\omega^2}\sum_{i=1}^{N-1}\left(L\,\mathrm{sign}(\Delta\tilde{f}_i)\sum_{k=0}^{n_i}(\sin(\omega\hat{\hat{x}}_{i,k})\right. \\
&\left. - \sin(\omega\hat{x}_{i,k})) - \frac{\Delta\hat{f}_i}{\Delta x_i}(\sin(\omega x_{i+1} - \sin(\omega x_i)))\right) + \frac{1}{\omega}((\hat{f}_N - \tilde{f}_N)\cos(\omega x_N) \\
&- (\hat{f}_1 - \tilde{f}_1)\cos(\omega x_1)). \qquad (4.16)
\end{aligned}
$$

Using the explicit form of the minorant (see (4.12)–(4.14)) we transform the second component under the maximum sign in (4.15) as follows:

$$
\begin{aligned}
&\sum_{i=1}^{N-1}\sum_{k=0}^{n_i}\int_{x_{i,k}}^{x_{i,k+1}} (S(x,L) - f_{i,k}^-(x))\sin(\omega x)\,\mathrm{d}x \\
&= \sum_{i=1}^{N-1}\sum_{k=0}^{n_i}\left(\int_{x_{i,k}}^{\hat{x}_{i,k}}\left(\hat{f}_i - \tilde{f}_i - Q(\tilde{x}_{i,k} - x_i)\right.\right. \\
&\left. + \frac{\Delta\hat{f}_i}{\Delta x_i}(x - x_i) - \frac{\xi_i^+ + \xi_i^-}{2}(1 - \mathrm{sign}(x_{i,k} - x_i)) + L(x - x_{i,k})\mathrm{sign}(\sin(\omega x))\right)\sin(\omega x)\,\mathrm{d}x \\
&+ \frac{1}{2}\int_{\hat{x}_{i,k}}^{\hat{\hat{x}}_{i,k}}\left(\left(\hat{f}_i - \tilde{f}_i - Q(\tilde{x}_{i,k} - x_i) + \frac{\Delta\hat{f}_i}{\Delta x_i}(x - x_i) - \frac{\xi_i^+ + \xi_i^-}{2}(1 - \mathrm{sign}(x_{i,k} - x_i))\right.\right. \\
&- L(x - x_{i,k})\,\mathrm{sign}(\sin(\omega x)))(1 - \mathrm{sign}(\Delta\tilde{f}_i)\,\mathrm{sign}(\sin(\omega x))) + \left(\hat{f}_i - \tilde{f}_i - Q(\tilde{x}_{i,k+1} - x_i)\right. \\
&\left. + \frac{\Delta\hat{f}_i}{\Delta x_i}(x - x_i) - \frac{\xi_{i+1}^+ + \xi_{i+1}^-}{2}(1 - \mathrm{sign}(x_{i+1} - x_{i,k+1})) - L(x_{i,k} - x)\,\mathrm{sign}(\sin(\omega x))\right) \\
&\left.\times(1 + \mathrm{sign}(\Delta\tilde{f}_i)\,\mathrm{sign}(\sin(\omega x)))\right)\sin(\omega x)\,\mathrm{d}x + \int_{\hat{\hat{x}}_{i,k}}^{x_{i,k+1}}\left(\hat{f}_i - \tilde{f}_i - Q(\tilde{x}_{i,k+1} - x_i)\right. \\
&\left.\left. + \frac{\Delta\hat{f}_i}{\Delta x_i}(x - x_i) - \frac{\xi_{i+1}^+ + \xi_{i+1}^-}{2}(1 - \mathrm{sign}(x_{i+1} - x_{i,k+1}))\right)\sin(\omega x)\,\mathrm{d}x\right) \\
&= \frac{L}{\omega^2}\sum_{i=1}^{N-1}\left(2\left|\sin\left(\omega\left(\frac{x_i + \tilde{x}_{i,1}}{2} + \frac{\xi_i^- - \xi_i^+}{2L}\right)\right)\right|\cos\left(\omega\frac{2|\Delta\tilde{f}_{i,0}| - |\xi_i^+ + \xi_i^-|\mathrm{sign}(\Delta\tilde{f}_i)}{2L}\right)\right. \\
&- |\sin(\omega x_i)| + \frac{\omega(\xi_i^+ - \xi_i^-)\mathrm{sign}(\sin(\omega x^1))}{2L}\cos(\omega x_i) + 2\left(\left|\sin\left(\omega\frac{\tilde{x}_{i,1} + \tilde{x}_{i,2}}{2}\right)\right|\right. \\
&\left. + (n_i - 3)\left|\sin\left(\omega\frac{\tilde{x}_{i,2} + \tilde{x}_{i,3}}{2}\right)\right| + \left|\sin\left(\omega\frac{\tilde{x}_{i,n_i-1} + \tilde{x}_{i,n_i}}{2}\right)\right|\right)\cos\frac{\pi Q}{2L} - |\sin(\omega x_{i+1})|
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\omega(\xi_{i+1}^{+}-\xi_{i+1}^{-})\operatorname{sign}(\sin(\omega x^{2}))}{2L}\cos(\omega x_{i+1})+2\left|\sin\left(\omega\left(\frac{\tilde{x}_{i,n_i}+x_{i+1}}{2}+\frac{\xi_{i+1}^{+}-\xi_{i+1}^{-}}{2}\right)\right)\right|\\
&\times\cos\left(\omega\frac{2|\Delta\tilde{f}_{i,n_i}|+(\xi_{i+1}^{+}+\xi_{i+1}^{-})\operatorname{sign}(\Delta\tilde{f}_i)}{2L}\right)\Bigg)+\frac{1}{\omega^{2}}\sum_{i=1}^{N-1}\Bigg(L\operatorname{sign}(\Delta\tilde{f}_i)\sum_{k=0}^{n_i}(\sin(\omega\hat{\hat{x}}_{i,k})\\
&-\sin(\omega\hat{x}_{i,k}))-\frac{\Delta\hat{f}_i}{\Delta x_i}(\sin(\omega x_{i+1})-\sin(\omega x_i))\Bigg)+\frac{1}{\omega}((\hat{f}_1-\tilde{f}_1)\cos(\omega x_1)\\
&-(\hat{f}_N-\tilde{f}_N)\cos(\omega x_N)).
\end{aligned}\tag{4.17}
$$

Substituting the final results obtained in (4.16) and (4.17) into (4.15), estimate (4.9) follows immediately.

Now, let us consider quadrature formula (3.3) constructed by the residual method. Analogous to the procedure used for the quadrature formula constructed by the method of quasi-solutions, for $S(x,M)$ we have

$$
S(x_i,M)\in[f_i-\varepsilon_i,f_i+\varepsilon_i],\quad i=1,\dots,N\quad\text{and}\quad |S'(x,M)|\leqslant M,\tag{4.18}
$$

where

$$
\begin{aligned}
M&=\max_{1\leqslant i\leqslant N}\left(0,\max_{j>i}\frac{|\tilde{f}_j-\tilde{f}_i|-\varepsilon_j-\varepsilon_i}{x_j-x_i}\right)\leqslant\max_{1\leqslant i\leqslant N}\left(0,\max_{j>i}\left(L-\frac{\varepsilon_j+\varepsilon_i}{x_j-x_i}\right)\right)\\
&=\max_{1\leqslant i\leqslant N}\left(0,L-\min_{j>i}\frac{\varepsilon_j+\varepsilon_i}{x_j-x_i}\right)\leqslant L.
\end{aligned}\tag{4.19}
$$

From (4.18), (4.19) follows that $S(x,M)\in C^{1}_{1,L,N,\varepsilon}$. Furthermore, analogously to the reasoning conducted for quadrature formula constructed by the method of quasi-solutions, it can be shown that the error of quadrature formula (3.3) constructed by the residual method, has the form (4.9). □

Remark 4.2. For computing integral (3.2) with quadrature formula (3.4) in class $C^{1}_{1,L,N,\varepsilon}$ we have result analogous to Theorem 4.1:

$$
\begin{aligned}
&v(C^{1}_{1,L,N,\varepsilon},R_3(\omega,S),f)\\
&\quad\leqslant\frac{L}{\omega^{2}}\sum_{i=1}^{N-1}\left(2\left|\cos\left(\omega\left(\frac{x_i+\tilde{x}_{i,1}}{2}+\frac{\xi_i^{-}-\xi_i^{+}}{2}\right)\right)\right|\right.\\
&\qquad\times\cos\omega\frac{2|\Delta\tilde{f}_{i,0}|-(\xi_i^{+}+\xi_i^{-})\operatorname{sign}(\Delta\tilde{f}_i)}{2L}-|\cos(\omega x_i)|+\frac{\omega(\xi_i^{+}-\xi_i^{-})\operatorname{sign}(\cos(\omega x^{1}))}{2L}\cos(\omega x_i)\\
&\qquad+2\left(\left|\cos\left(\omega\frac{\tilde{x}_{i,1}+\tilde{x}_{i,2}}{2}\right)\right|+(n_i-3)\left|\cos\left(\omega\frac{\tilde{x}_{i,2}+\tilde{x}_{i,3}}{2}\right)\right|\right.\\
&\qquad\left.+2\left|\cos\left(\omega\frac{\tilde{x}_{i,n_i-1}+\tilde{x}_{i,n_i}}{2}\right)\right|\right)\cos\frac{\pi Q}{2L}+2\left|\cos\left(\omega\left(\frac{\tilde{x}_{i,n_i-1}+\tilde{x}_{i,n_i}}{2L}+\frac{\xi_{i+1}^{+}-\xi_{i+1}^{-}}{2L}\right)\right)\right|
\end{aligned}
$$

$$\times\cos\left(\omega\left(\frac{2|\Delta f_{i,n_i}|+(\xi_{i+1}^{+}+\xi_{i+1}^{-})\,\mathrm{sign}(\Delta\tilde{f}_i)}{2L}\right)\right)-|\cos(\omega x_{i+1})|$$

$$\left.+\frac{\omega(\xi_{i+1}^{+}-\xi_{i+1}^{-})\mathrm{sign}(\cos(\omega x^2))}{2L}\sin(\omega x_{i+1})\right)$$

$$+\left|\frac{1}{\omega^2}\sum_{i=1}^{N-1}\left(L\,\mathrm{sign}(\Delta\tilde{f}_i)\sum_{k=0}^{n_i}(\cos(\omega\hat{\hat{x}}_{i,k})-\cos(\omega\hat{x}_{i,k}))-\frac{\Delta\hat{f}_i}{\Delta x_i}(\cos(\omega x_{i+1})-\cos(\omega x_i))\right)\right.$$

$$\left.+\frac{1}{\omega}((\hat{f}_N-\tilde{f}_N)\sin(\omega x_N)-(\hat{f}_1-\tilde{f}_1)\sin(\omega x_1))\right|. \tag{4.20}$$

We emphasise that for a specific problem when instead of L and ε known only their estimates, the real error of computing integrals (3.1), (3.2) using quadrature formulae (3.3) and (3.4), may be considerably less than corresponding error estimates obtained under approximate a priori information.

5. Computing estimates of Fourier transforms under approximate a priori information

Let us consider the case when integrand $f(x)$ satisfies the Lipschitz condition, but the Lipschitz constant (denoted by L) is not known. The function $f(x)$ is finite on $[a,b]$, has bounded (by L) first derivative and satisfies the condition:

$$|f(x_i)-\tilde{f}_i|\leqslant\varepsilon_i, \tag{5.1}$$

where x_i are nodes of uniform grid $\Delta=\{a=x_1<\cdots<x_N=b\}$ (i.e. we assume that the accuracy of function definition in N nodes of a uniform grid is given), $\tilde{f}_i$, ε_i, $i=1,\dots,N-1$ are given real numbers, $N=2^m+1$, $m\geqslant 3$ is integer, $\omega_k=2\pi k/(b-a)$, $k=1,\dots,N-1$. We compute N approximate values $\tilde{I}(\omega_k)$ of estimates of sin- and cos-Fourier transforms

$$I_2(\omega_k)=\int_a^b f(x)\sin\omega_k x\,\mathrm{d}x,\quad I_3(\omega_k)=\int_a^b f(x)\cos\omega_k x\,\mathrm{d}x \tag{5.2}$$

in a large range of frequencies $\omega_k\geqslant 2\pi$. We also compute an a priori estimate of the total absolute error of the problem solution

$$E=\max_{1\leqslant k\leqslant N-1}|\tilde{I}(\omega_k)-I(\omega_k)|. \tag{5.3}$$

The algorithm based on the residual method consists of the following steps.

Algorithm 5.1.

1. *input initial data* $N,a,b,\{x_i\}_{i=1,\dots,N}$, $\{\tilde{f}_i\}_{i=1,\dots,N}$, $\{\varepsilon_i\}_{i=1,\dots,N}$;
2. *find the values of frequencies* $\{\omega_k\}_{k=1,\dots,N}$ *using the formula*

$$\omega_k=2\pi k/(b-a),\quad k=1,\dots,N-1;$$

3. *calculate constant* M *as follows*:

$$M=\max_{1\leqslant i\leqslant N}\left(0,\max_{j>i}(|\tilde{f}_j-\tilde{f}_i|-\varepsilon_i-\varepsilon_j)/(x_j-x_i)\right);$$

4. *calculate the grid step as* $h = (b-a)/(N-1)$;
5. *calculate* $\tilde{f}_i^+$, $\tilde{f}_i^-$ *using formulae*

$$\tilde{f}_i^+ = \max_{1\leqslant j\leqslant N}(\tilde{f}_j - M|x_j - x_i|), \quad \tilde{f}_i^- = -\max_{1\leqslant j\leqslant N}(-\tilde{f}_j - M|x_j - x_i|);$$

6. *compute* $\hat{f}_i = (\tilde{f}_i^+ + \tilde{f}_i^-)/2$, $i = 1,\dots,N$;
7. *compute* $\hat{f}_i' = (\hat{f}_{i+1} - \hat{f}_i)/h$, $i = 1,\dots,N-1$;
8. *compute values* $\{\sin(\omega_k ih)\}_{i=1,\dots,N}$, $\{\cos(\omega_k ih)\}_{i=1,\dots,N}$, $k = 1,\dots,N-1$;
9. *compute estimate* $\bar{R} = \{\bar{R}_k\}_{k=1,\dots,N}$ *for sin-Fourier transform*:

$$\bar{R}_k = \frac{1}{\omega_k^2}\left(\sin(\omega_k h)\sum_{i=1}^{N-1}\hat{f}_i'\cos(\omega_k ih) - (1-\cos(\omega_k h))\sum_{i=1}^{N-1}\hat{f}_i'\sin(\omega_k ih)\right)$$
$$-\frac{1}{\omega_k}(\hat{f}_N\cos(\omega_k b) - \hat{f}_1\cos(\omega_k a)), \quad k = 1,\dots,N-1;$$

10. *compute estimate* $\bar{\bar{R}} = \{\bar{\bar{R}}_k\}_{k=1,\dots,N}$ *for cos-Fourier transform*:

$$\bar{\bar{R}}_k = -\frac{1}{\omega_k^2}\left(\sin(\omega_k h)\sum_{i=1}^{N-1}\hat{f}_i'\sin(\omega_k ih) + (1-\cos(\omega_k h))\sum_{i=1}^{N-1}\hat{f}_i'\cos(\omega_k ih)\right)$$
$$+\frac{1}{\omega_k}(\hat{f}_N\sin(\omega_k b) - \hat{f}_1\sin(\omega_k a)), \quad k = 1,\dots,N-1;$$

11. *output data* $\omega = \{\omega_k\}_{k=1,\dots,N}$, $\bar{R} = \{\bar{R}_k\}_{k=1,\dots,N}$, $\bar{\bar{R}} = \{\bar{\bar{R}}_k\}_{k=1,\dots,N}$.

One can also compute a priori estimates $\bar{E}$, $\bar{\bar{E}}$ of the total absolute errors of computing sin- and cos-Fourier transform, respectively,

$$\bar{E} = \Delta_1 + \bar{\Delta}_2 + \bar{\Delta}_3, \quad \bar{\bar{E}} = \Delta_1 + \bar{\bar{\Delta}}_2 + \bar{\bar{\Delta}}_3, \tag{5.4}$$

where Δ_1 is the hereditary error, $\bar{\Delta}_2$, $\bar{\bar{\Delta}}_2$ are errors of the residual method, and $\bar{\Delta}_3$, $\bar{\bar{\Delta}}_3$ are round-off errors of computing estimates of sin- and cos-Fourier transforms, respectively. In the case when $\varepsilon_i = 0$, $i = 1,\dots,N$ the hereditary error is zero. If ε_i, $i = 1,\dots,N$ are different from zero, then for the absolute hereditary error Δ_1 of computing $\bar{R}_k$, and $\bar{\bar{R}}_k$, $k = 1,\dots,N-1$ we have

$$\Delta_1 = \tilde{\varepsilon}(b-a), \tag{5.5}$$

where $\tilde{\varepsilon}$ is the maximal error of the definition $f(x)$ in the nodes x_i, $i = 1,\dots,N$:

$$|f(x_i) - \tilde{f}_i| \leqslant \tilde{\varepsilon}, \quad i = 1,\dots,N. \tag{5.6}$$

The error of the method of computing $\bar{R}_k$, $k = 1,\dots,N-1$ ($\bar{\Delta}_2$) has the form (3.11), and the error of the method of computing $\bar{\bar{R}}_k$, $k = 1,\dots,N-1$ ($\bar{\bar{\Delta}}_2$) has the form (3.22), provided $\varepsilon_i = 0$, $i = 1,\dots,N$, $N \geqslant |\omega|$ and condition C1 is satisfied. In the case when $\varepsilon_i = 0$, $i = 1,\dots,N$ and condition C2 is satisfied, the error $\bar{\Delta}_2$ has the form (3.23), and the error $\bar{\bar{\Delta}}_2$ has the form (3.33). Finally, if ε_i, $i = 1,\dots,N$ are nonzeros, then for $\bar{\Delta}_2$ and $\bar{\bar{\Delta}}_2$ we have estimates (4.9) and (4.20) respectively.

Table 1
$f(x)=x$ (for the residual method and the method of quasi-solutions)

Frequency	RS	RC	ST	CT
7.0685830	−0.86228190	0.093400980	−0.86228190	0.093400980
159.1740000	0.003096217	0.005389304	0.003096217	0.005389304
516.0066000	−0.001371164	0.001373098	−0.001371164	0.001373098
864.9852000	0.000576047	−0.001002487	0.000576047	−0.001002487
4741.7110000	0.000105379	−0.000182761	0.000105379	−0.000182761

We assume that computations are conducted in the floating-point regime with the round-off of arithmetical operations on the basis of the standard rule up to τ binary digits in the normalised mantices of numbers. Then the round-off errors of computing $\bar{R}_k$, $\bar{\bar{R}}_k$, $k=1,\dots,N-1$ have the form [31,27]

$$\bar{\Delta}_3 = \frac{M}{\omega_k^2}(8+1.06(N-1))2^{-\tau}\left(|\sin(\omega_k h)| \max_{1\leqslant i\leqslant N-1} |\cos(\omega_k ih)| \right.$$
$$\left. + |1-\cos(\omega_k h)| \max_{1\leqslant i\leqslant N-1} |\sin(\omega_i ih)|\right) + 5\left|\frac{1}{\omega_k}(\hat{f}_N\cos(\omega_k b) - \hat{f}_1\cos(\omega_k a))\right| 2^{-\tau}, \tag{5.7}$$

$$\bar{\bar{\Delta}}_3 = \frac{M}{\omega_k^2}(8+1.06(N-1))2^{-\tau}\left(|\sin(\omega_k h)| \max_{1\leqslant i\leqslant N-1} |\sin(\omega_k ih)| \right.$$
$$\left. + |1-\cos(\omega_k h)| \max_{1\leqslant i\leqslant N-1} |\cos(\omega_i ih)|\right) + 5\left|\frac{1}{\omega_k}(\hat{f}_N\sin(\omega_k b) - \hat{f}_1\sin(\omega_k a))\right| 2^{-\tau}. \tag{5.8}$$

Example 1. Let $f(x)=x$, values $\tilde{f}_i$ are given with errors ε_i at nodes x_i of a uniform grid on $[0,1]$, $i=1,\dots,2^7+1$. We take $\varepsilon_i = 10^{-2}$ when i is even, and $\varepsilon_i = -2\cdot 10^{-2}$ when i is odd (this rule is also used in further examples). By RS and RC we denote errors of sin- and cos-Fourier transforms, respectively, obtained by the application of the proposed method. By ST and CT we denote the values of integrals $I_2(\omega_k)$ and $I_3(\omega_k)$ (see (5.2)) computed analytically.

The results given in Table 1 confirms the fact that quadrature formulae (3.3), (3.4), constructed by the residual method, are exact on linear functions (taking into account round-off error). Below we provide results on three other illustrative examples of the application of quadrature formulae constructed with the residual method.

Example 2. Let $f(x)=\frac{1}{2}x^2$, values of $\tilde{f}_i$ are given with error ε_i in nodes x_i of a uniform grid on $[1,2]$, $i=1,\dots,2^8+1$. The results of computations are given in Table 2.

Example 3. Let $f(x)=\frac{1}{2}x^3$, values of $\tilde{f}_i$ are given with error ε_i in nodes x_i of a uniform grid on $[1,2]$, $i=1,\dots,2^7+1$. The results are given in Table 3.

Example 4. Let $f(x)=\exp(x)$, values of $\tilde{f}_i$ are given with error ε_i in nodes x_i of a uniform grid on $[0,1]$, $i=1,\dots,2^7+1$. The results of computations are given in Table 4.

Table 2
$f(x) = x^2/2$ (for the residual method)

Frequency	RS	RC	ST	CT
7.0685830	0.076986650	0.238778800	0.073891510	0.217943200
159.1740000	0.004804598	−0.013605410	0.004608938	−0.013621320
516.0066000	0.000682280	0.003166615	0.000690012	0.003188090
864.9852000	0.000924193	0.002433502	0.000870774	0.002502229
4741.7110000	0.000158427	0.000452690	0.000158636	0.000456420

Table 3
$f(x) = x^3/2$ (for the residual method)

Frequency	RS	RC	ST	CT
7.0685830	0.139202300	0.554518200	0.142515500	0.484506100
159.1740000	0.010960590	−0.024687310	0.010736880	−0.024570550
516.0066000	0.000664492	0.007070152	0.000703684	0.007062656
864.9852000	0.002015193	0.004482241	0.002032279	0.004502165
4741.7110000	0.000369616	0.000817584	0.000370015	0.000821442

Table 4
$f(x) = \exp(x)$ (for the residual method)

Frequency	RS	RC	ST	CT
7.0685830	−0.089598670	0.282172300	−0.090178490	0.284681300
159.1740000	0.014738880	0.014555340	0.014913670	0.014695670
516.0066000	−0.001788848	0.003717981	−0.001779803	0.003728426
864.9852000	0.002706216	−0.002713822	0.002724084	−0.002724782
4741.7110000	0.000494088	−0.000494915	0.000497213	−0.000496694

The above examples demonstrate that with increasing frequency ω, accuracy of computations using quadrature formulae (3.3), (3.4) increases. For the same frequency, variation of h does not substantially influence accuracy that confirm the theoretical conclusion that accuracy of quadrature formulae (3.3), (3.4) constructed by the residual method is only weakly dependent on mutual location of nodes of the grid and zeros of oscillating factor ($\sin \omega x$ or $\cos \omega x$, respectively).

Now we consider problem (5.2) with condition (5.6) assuming that $\tilde{f}_i$ are given numbers and the Lipschitz constant L is known. We do not assume as a priori, the accuracy of the definition of $f(x)$ in N nodes of a uniform grid. Then Algorithm 5.1 has to be modified as follows:

Algorithm 5.2.

1. *input* n, a, b, $\{x_i\}_{i=1,\dots,N}$, $\{f_i\}_{i=1,\dots,N}$, L;
2. *compute* h *and frequencies* $\{\omega_k\}$, $k = 1,\dots,N-1$ *as in Algorithm* 5.1;
3. *calculate* $\tilde{f}_i^+$, $\tilde{f}_i^-$ *using formulae*

$$\tilde{f}_i^+ = \max_{1 \leqslant j \leqslant N} (\tilde{f}_j - L|x_j - x_i|), \quad \tilde{f}_i^- = -\max_{1 \leqslant j \leqslant N} (-\tilde{f}_j - L|x_j - x_i|);$$

4. *the rest of this algorithm coincides with the corresponding steps of Algorithm* 5.1.

Table 5
$f(x)=x^2/2$ (the method of quasi-solutions)

Frequency	RS	RC
7.0685830	0.077655620	0.238451700
159.1740000	0.004720655	−0.013705960
516.0066000	0.000684278	0.003185287
864.9852000	0.000870180	0.002504955
4741.7110000	0.000158477	0.000456333

Table 6
$f(x)=x^3/2$ and $f(x)=\exp(x)$ (the method of quasi-solutions)

Frequency	RS ($x^3/2$)	RC ($x^3/2$)	RS ($\exp(x)$)	RC ($\exp(x)$)
7.0685830	0.138503100	0.554798100	−0.089598670	0.282172300
159.1740000	0.011005560	−0.024583290	0.014738880	0.014555340
516.0066000	0.000670536	0.007042121	−0.001788848	0.003717981
864.9852000	0.002016058	0.004461390	0.002706216	−0.002713822
4741.7110000	0.000369576	0.000813939	0.000494088	−0.000494915

Quadrature formulae (3.3), (3.4) constructed with the method of quasi-solutions for $f(x)=x$ with exactly given f_i in nodes x_i of a uniform grid on $[0,1]$ ($i=1,\dots,2^7+1$), $L=1$ give results identical to those in Table 1. This support the conclusion that quadrature formulae (3.3), (3.4) constructed with the method of quasi-solutions are exact on linear functions (taking into account round-off errors). For $f(x)=\frac{1}{2}x^2$ with given f_i in nodes x_i of the uniform grid on $[1,2]$ ($i=1,\dots,2^8+1$), $L=2$ the results are given in Table 5.

With the same data as in Examples 3 and 4 (setting $L=6$ and e, respectively), quadrature formulae (3.3), (3.4) constructed with the method of quasi-solutions give the results presented in Table 6.

6. Conclusions and future directions

We derived and tested optimal-by-order, with constant not exceeding 2, quadrature formulae constructed by the residual method and the method of quasi-solutions. Numerical results support theoretical conclusions of the paper that

- these quadrature formulae are exact for linear functions (taking into account round-off errors);
- when $\varepsilon_i=0$, $i=1,\dots,N$ results obtained by the residual method and the method of quasi-solutions coincide (taking into account round-off errors);
- with increasing frequency ω the accuracy of computations by these formulae increases;
- accuracy of the quadrature formulae practically independent on the mutual arrangement of grid nodes and zeros of oscillating factors.

In order to achieve efficiency in computing $\bar{R}_k$ and $\bar{\bar{R}}_k$, $k = 1,\dots,N-1$ we used the Fast-Fourier-Transform (FFT) algorithm for the following expressions:

$$\bar{S}_k = \frac{1}{\omega_k^2}\left(\sin(\omega_k h)\sum_{i=1}^{N-1}\hat{f}_i'\cos(\omega_k ih) - (1-\cos(\omega_k h))\sum_{i=1}^{N-1}\hat{f}_i'\sin(\omega_k ih)\right),$$

$$\bar{\bar{S}}_k = -\frac{1}{\omega_k^2}\left(\sin(\omega_k h)\sum_{i=1}^{N-1}\hat{f}_i'\sin(\omega_k ih) + (1-\cos(\omega_k h))\sum_{i=1}^{N-1}\hat{f}_i'\cos(\omega_k ih)\right),$$

$k = 1,\dots,N-1$. The computation of $\bar{S}_k$, $\bar{\bar{S}}, k = 1,\dots,N-1$ by the standard procedure requires N^2 operations of additions and multiplications. The application of FFT allows us to speed up calculations by the factor 100 for $N = 2^{10}$ due to its requirement for only $N\log_2 N$ arithmetic operations.

The main emphasis in the paper was given to algorithms that guarantee optimal-by-order, rather than optimal-by-accuracy, solution of the problem of computing integrals with fast oscillatory functions in classes $C^1_{1,L,N}$ and $C^1_{1,L,N,\varepsilon}$. Such algorithms are especially effective in the case when a priori information about the problem is given approximately. This case is typical in majority of applications. However, our algorithms can also be applied in the case when a priori information is assumed to be given precisely. In such a case the error of our results will not exceed optimal-by-accuracy results by more than two times.

The results obtained in this paper has been recently used by authors in the development of algorithms for optimal integration of fast oscillatory functions of two variables (recent survey on the topic may be found in [6]) and the solution of the problem of optimal-by-accuracy recovery of such functions (see, for example, [27]). These issues will be discussed elsewhere.

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