JDEAEA 4(2) 95–214 (1998)
ISSN: 1023-6198

Journal of Difference Equations and Applications

Volume 4, Number 2

Gordon and Breach
Science Publishers

Journal of Difference Equations and Applications

Volume 4, Number 2 (1998)

TABLE OF CONTENTS

Journal of Difference Equations and Applications,
1998, Vol. 4, pp. 185-212
Reprints available directly from the publisher
Photocopying permitted by license only

Convergence of the Operator-Difference Scheme to Generalized Solutions of a Coupled Field Theory Problem

R.V.N. MELNIK*

Centre for Industrial and Applied Mathematics, School of Mathematics, University of South Australia, The Levels Campus, The Levels, S.A. 5095, Australia

(Received 18 January 1996; In final form 29 October 1996)

In this paper the operator-difference scheme for the numerical solution of a problem arising from coupled field theory is thoroughly investigated for the case when the classical assumptions of sufficient smoothness cannot be applied. Such a situation, being typical in many applications, is considered for the problem of nonstationary electroelasticity.

A new *a-priori* estimation for the numerical solution of the problem has been obtained. A scale of accuracy results for generalized solutions of the problem has been derived, and the convergence theorem has been proved. Applications of the theory are considered and computational results are discussed.

Keywords: operator-difference scheme; coupled field theory; generalized solutions; CFL condition for electroelastic waves.

Classification categories: 35A40, 35Q72, 65M06, 70G99.

1. INTRODUCTION

When engaged in mathematical modeling, we can often observe a gap between imposed theoretical assumptions on a solution's smoothness and an actual smoothness of the solution in a real practical problem.

Typically, relaxation of such assumptions is necessary in many problems where interconnection of *physical fields of different nature* is essential in obtaining a plausible picture of the phenomenon under consideration. Coupled field theory problems are of this type. One of the classical examples of this is thermoelasticity [15,

* Present address. Department of Mathematics and computing, University of Southern Queensland, QLD 4350, Australia. Email: melnik@usq.edu.au.

27], where the elastic and thermal fields combine into a unified whole which in general cannot be separated. An efficient way to solve the problem in such cases includes generalized solutions which is intrinsic to coupled field theory and computational models used for the solution of problems arising therein [29, 33].

In fact, all real processes, dynamic systems and phenomena describe a transformation of different types of energy which imply that, in general, mathematical models applied to them should have integral rather than differential features. Clearly, for example, a border between two different media might not be described by any differential equation due to a jump of physical parameters. A similar situation arises when we try to describe a nonhomogeneous medium. Probably one of the most demonstrative examples of difficulties involved in mathematical modelling of such media are the non-local models. Along with classical applications of such models in climate modelling and semiconductor device simulation [16], [20, 24] non-local type models are typical when we address physical problems of mathematical modelling using approaches of extended thermodynamics [9], [26]. In general, many problems in coupled field theory* do not obtain an adequate description in mathematical models if a-priori assumptions of excessive† smoothness are imposed on their solutions. Of course, such assumptions are questionable when solutions exhibit discontinuities or steep gradients in their behaviour.

Typically, coupled field theory in the nonstationary case deals with systems of partial differential equations (PDE) which do not belong to any classical type of PDEs, yet at least one of the equations of such a system is a PDE of *hyperbolic* type. Since there is a connection between the hyperbolic (in general, dissipative) equation with PDE of parabolic (for thermoelasticity) or elliptic (for electroelasticity) type‡, analytical solutions of such problems are quite exceptional. This leads to a situation where numerical methods become the natural and the most efficient way of solving problems arising from coupled field theory. Mathematical challenges and the practical importance of the problems stimulate interest in them from mathematicians, engineers and scientists.

In this paper we deal with a nonstationary problem of coupled electroelasticity, the solution of which is of great importance for reliability of many technical devices such as piezovibrators, different types of transmitters, generators etc (see, for example, [1], [17], [28]). Applications of piezoelectric components in intelligent structures and problems in biophysics give an additional stimulus developing efficient numerical procedures in coupled dynamic electroelasticity. It is believed that the technique proposed in this paper is applicable to a much wider class of problems arising from coupled field theory.

* arising from studying microstructures as well as macrosystems.
† with respect to real solutions.
‡ obviously, there are much more complicated cases.

The paper is organized as follows.

- In Section 2 we formally state the differential formulation of the problem and point out the difficulties involved in its solution.
- Section 3 addresses issues related to generalized solutions of the problem. In this section we recall a computational procedure developed in [17] on the basis of the variational approach.
- In Section 4 we consider a more general operator-difference scheme for solution of the problem. We explicitly derive a stability condition for such a scheme which can be seen as a generalization of the classical Courant-Friedrichs-Lewy (CFL) condition for the case of coupled electro-elastic waves. A new *a-priori* estimation is also obtained in this section.
- Section 5 is devoted to questions of convergence where we derive a scale of accuracy estimations for the difference problem (considered in section 3) when the solution of the differential problem is from defined generalized classes.
- In Section 6 some examples from applications are given and numerical results are discussed.
- Concluding remarks and future directions are addressed in Section 7.

2. MATHEMATICAL MODEL OF COUPLED ELECTROELASTICITY IN THE NONSTATIONARY CASE

Let us consider a mathematical model of electroelasticity where the investigation of coupled electrical and elastic fields under nonstationary conditions is essential to obtain a plausible quantitative (as well as qualitative) picture of physical phenomena in a piezoceramic solid. The existence and uniqueness issues for the mathematical models of this type were addressed in [14], [22].

The process of coupled electroelastic nonstationary oscillations of a piezoceramic cylinder can be described by the system of partial differential equations in the time-space region $Q_T = \{(r, t) : R_0 < r < R_1, 0 < t \le T\}$ (see [17] and the references therein):

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}(r\sigma_r) - \frac{\sigma_\theta}{r} + f_1(r, t) \tag{2.1}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rD_r) = f_2(r, t) \tag{2.2}$$

which should be completed by initial conditions

$$u(r, 0) = u_0(r), \quad \frac{\partial u(r, 0)}{\partial t} = u_1(r), \tag{2.3}$$

and boundary conditions

$$\sigma_r = p_1(t),\ \varphi = V(t) \text{ for } r = R_0, \text{ and } \sigma_r = p_2(t),\ \varphi = -V(t) \text{ for } r = R_1. \tag{2.4}$$

The most difficult (yet the most interesting for practical reasons) case is that of radial preliminary polarization. The connection between electric and elastic fields in this case is fairly strong (see section 6 for details):

$$\begin{cases} \sigma_r = c_{11}\epsilon_r + c_{12}\epsilon_\theta - e_{11}E_r, \\ \sigma_\theta = c_{12}\epsilon_r + c_{22}\epsilon_\theta - e_{12}E_r, \\ D_r = \epsilon_{11}E_r + e_{12}\epsilon_\theta + e_{11}\epsilon_r. \end{cases} \tag{2.5}$$

There remain only Cauchy relations and the formula for electrostatic potential φ to be added to complete the problem formulation:

$$\epsilon_r = \frac{\partial u}{\partial r}, \quad \epsilon_\theta = \frac{u}{r}, \quad E_r = -\frac{\partial\varphi}{\partial r} \tag{2.6}$$

In (2.1)-(2.6) we use the following notations: u is the radial displacement, E_r and D_r are radial components of electric field strength and electric induction respectively. It is assumed that elastic moduli (c_{kl}), piezomoduli (e_{ij}), the dielectric permittivity (ϵ_{11}), and the density of piezoceramic material (ρ) are given constants; whereas f_1, f_2 are given functions of r and t for the density of mass forces and electric charge density of the solid respectively. We also assume non-negativeness of potential energy of deformation, i.e. $\exists\ \delta > 0$ such that $\forall \xi_1$, ξ_2 the following holds

$$(\xi_1^2 + \xi_2^2) \le c_{11}\xi_1^2 + 2c_{12}\xi_1\xi_2 + c_{22}\xi_2^2,$$

Therefore we have a strongly coupled system of partial differential equations of the type

$$\rho\frac{\partial^2 u}{\partial t^2} = \frac{1}{r}c_{11}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - c_{22}\frac{u}{r^2} + \frac{1}{r}e_{11}\frac{\partial^2\varphi}{\partial r^2} - \frac{1}{r}e_{12}\frac{\partial\varphi}{\partial r} + f_1(r,t), \tag{2.7}$$

$$-\epsilon_{11}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) + e_{11}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + e_{12}\frac{1}{r}\frac{\partial u}{\partial r} = f_2(r,t). \tag{2.8}$$

In fact, we have the second derivative of φ with respect to r in equation (2.7) (this equation is mainly responsible for the elastic field) and the second derivative of u with respect to r in equation (2.8) (this equation is mainly responsible for the electric field). This connection of the equations is amplified by the boundary conditions for stresses (2.4).

In many cases efficient finite difference schemes for the solution of coupled field theory problems can be obtained by the use of variational principles. For example, the Biot variational principle can be of great help in the coupled thermoelasticity, whereas in coupled electroelasticity the conservation energy law for the whole electromechanical system plays a similar role [23]. Earlier these ideas were applied to derive difference schemes for coupled nonstationary thermo- and electroelasticity problems [15], [17]. In this paper we are interested in solutions that do not possess such high smoothness (for example, $C^4(Q_T)$) as is often assumed.

In the next sections we develop a technique that allows us to derive efficient computational procedures for the investigation of coupling effects in dynamic electroelasticity problems.

3. GENERALIZED SOLUTIONS OF COUPLED ELECTROELASTICITY AND THE VARIATIONAL APPROACH FOR NUMERICAL PROCEDURES

3.1. Weak formulation of the problem

Following the general approach to the evolutionary problems of mathematical physics [6,11,29,33] we recall that a pair of functions

$(u(r,t), \phi(r,t) \in W_2^1(Q_T) \times L^2(I, \overset{0}{W}{}_2^1(G))$, where $Q_T = G \times I$, $G = (R_0, R_1)$, $I = (0,T)$ ($u(r,t)$ equals to $u_0(r)$ for $t = 0$) is called a generalized solution of the coupled problem of dynamic electroelasticity (2.1)-(2.6) if it satisfies the following integral identifies [17]:

$$\int_{Q_T} r\left(-\rho \frac{\partial u}{\partial t}\frac{\partial \eta}{\partial t} + \sigma_r \frac{\partial \eta}{\partial r} + \frac{\sigma_\theta}{r}\eta\right) dr\,dt - \int_{R_0}^{R_1} r\rho u_1(r)\eta(r,0)\,dr = \int_{Q_T} r f_1 \eta\, dr\,dt \quad \forall \eta \in \hat{W}_2^1(Q_T), \tag{3.1}$$

$$\int_{R_0}^{R_1}\left(\epsilon_{11} r \frac{\partial \phi}{\partial r}\frac{\partial \zeta}{\partial r} + e_{11} r \epsilon_r \frac{\partial \zeta}{\partial r} + e_{12} r \epsilon_\theta \frac{\partial \zeta}{\partial r}\right) dr = \int_{R_0}^{R_1} r f_2 \zeta\, dr \quad \forall \zeta \in W_2^1(G) \tag{3.2}$$

almost everywhere in I. Here $\hat{W}_2^1(Q_T)$ stands for a subspace of $W_2^1(Q_T)$ that consists of all elements of $W_2^1(Q_T)$ which equal zero when $t = T$, and for simplicity we set $p_i(t) = V(t) = 0$, $i = 1, 2$,

This definition reflects the fact that differential equations of electroelasticity are a partial case of a more general variational formulation [17]. The use of variational formulations of the problem is more appropriate to derive computational models appropriate for further numerical algorithmization.

3.2. Energy balance equation

For purposes of clarity, in the next subsections we recall the procedure of the construction of efficient difference schemes for the solution of problem (2.1)-(2.6) [17]. Assume that generalized second derivatives of the solution are square integrable functions from L^2 (existence and uniqueness of such generalized solutions was proved in [14]. Then the solution $(u(r,t),\phi(r,t))$ satisfies the initial system (2.1)-(2.6) in the sense of integral identities (3.1), (3.2) and the following integral identity

$$\int_{Q_T} r\left(\rho\frac{\partial^2 u}{\partial t^2}\eta + \sigma_r\frac{\partial \eta}{\partial r} + \frac{\sigma_\theta}{r}\eta\right) drdt = \int_{Q_T} rf_1\eta drdt, \tag{3.3}$$

where η is an arbitrary element from $W_2^{1,0}(Q_T)^*$. Let us choose the function $\eta(r,t)$ in (3.3) as follows

$$\eta(r,t) \equiv \begin{cases} 0 & \text{for } t \in [t_1, T], \\ \dfrac{\partial u}{\partial t} & \text{for } t \in [0, t_1) \end{cases}$$

Then taking into consideration that[†]

$$\int_{Q_T}\left(\sigma_r\frac{\partial \epsilon_r}{\partial t} + \sigma_\theta\frac{\partial \epsilon_\theta}{\partial t}\right) drdt = \int_{Q_T}\left[c_{11}\epsilon_r\frac{\partial \epsilon_r}{\partial t} + c_{12}\left(\frac{\partial \epsilon_r}{\partial t}\epsilon_\theta + \frac{\partial \epsilon_\theta}{\partial t}\epsilon_r\right) + c_{22}\epsilon_\theta\frac{\partial \epsilon_\theta}{\partial t} + \epsilon_{11}\frac{\partial E_r}{\partial t}E_r - \frac{\partial D_r}{\partial t}E_r\right] drdt$$

we obtain the following integral equality to characterize energy change in the electromechanical system:

$$\int_0^{t_1}\frac{d\varepsilon}{dt}dt = \int_{Q_{t_1}} rf_1\frac{\partial u}{\partial t}drdt + \int_{Q_{t_1}} r\frac{\partial D_r}{\partial t}E_r drdt + \int_0^{t_1}\left[R_1 p_1\frac{\partial u(R_1,t)}{\partial t} - R_0 p_0\frac{\partial u(R_0,t)}{\partial t}\right]dt$$

Here $Q_{t_1} = \{(r,t) : R_0 < r < R_1, 0 < t < t_1\}$ and we have denoted the total energy of the electromechanical system by ε. The latter quantity can be written as a sum $\varepsilon = K + W + P$, where

$$K = \frac{\rho}{2}\int_{R_0}^{R_1} r\left(\frac{\partial u}{\partial t}\right)^2 dr$$

* this is a Hilbert space that consists of elements $u(r,t) \in L^2(Q_T)$ which have square summable generalized derivatives $\partial u/\partial r$.

† this equality is obtained from state equations (2.5).

is the kinetic energy,

$$W = \frac{1}{2}\int_{R_0}^{R_1} r[c_{11}\epsilon_r^2 + 2c_{12}\epsilon_r\epsilon_\theta + c_{22}\epsilon_\theta^2]dr$$

is the energy of elastic deformation, and

$$P = \frac{\epsilon_{11}}{2}\int_{R_0}^{R_1} rE_r^2$$

is the energy of the electric field of the system.

To find the integral $\int_{Q_{t_1}} r\frac{\partial D_r}{\partial t}E_r drdt$ we use identity (3.2) integrated in t from 0 to t_1 where we set

$$\zeta(r,t) \equiv \begin{cases} 0 & \text{for } t \in [t_1, T], \\ \dfrac{\partial\phi}{\partial t} & \text{for } t \in [0, t_1). \end{cases}$$

After simple transformations we have *:

$$-\int_{Q_{t_1}} r\frac{\partial D_r}{\partial t}E_r drdt = -\int_{Q_{t_1}} r\frac{\partial f_2}{\partial t}\phi drdt + \int_0^{t_1} r\frac{\partial D_r}{\partial t}\phi\Big|_{R_0}^{R_1}dt \tag{3.4}$$

Taking into consideration that identities (3.4) and (3.5) are satisfied for any $t_1 \in \bar{I}$, we obtain the equation for energy balance for a piezoelectric solid:

$$\frac{d\mathcal{E}}{dt} = \left[R_1 p_1 \frac{\partial u(R_1,t)}{\partial t} - R_0 p_0 \frac{\partial u(R_0,t)}{\partial t}\right] + \int_{R_0}^{R_1} rf_1\frac{\partial u}{\partial t}dr +$$

$$\int_{R_0}^{R_1} r\phi\frac{\partial f_2}{\partial t}dr + V(t)\left[\frac{\partial D_r(R_1,t)}{\partial t}R_1 + \frac{\partial D_r(R_0,t)}{\partial t}R_0\right]. \tag{3.5}$$

The right hand part of (3.5) contains those sources that causes dynamic behaviour, i.e. loads on the surface of the body, mass forces and surface charges. It is easy to see that from the relationship (3.5) we can obtain the equation of motion (2.1), the Maxwell equation† (2.2) and *natural* boundary conditions of the problem (2.1)-(2.6).

3.3. Derivation of difference schemes

Using a standard technique, we introduce a difference grid covering the region Q_T

$$\bar{\omega}_{h\tau} = \bar{\omega}_h \times \bar{\omega}_\tau$$

---

* using (3.2) for $\zeta = \phi$, and $t = 0, t_1$.

† more precisely, it is the equation of forced electrostatics of piezoelectrics which is a consequence of the Maxwell equation.

where

$$\bar{\omega}_h = \left\{ r_i = R_0 + ih, \quad h = \frac{R_1 - R_0}{N}, \quad i = 0, 1, \dots, N \right\},$$

$$\bar{\omega}_\tau = \{ t_j = j\tau, \quad \tau = T/L, \quad j = 0, 1, \dots, L \}.$$

Let the functions y and μ be functions of two discrete variables defined on this grid which approximate the functions of displacement $u(r,t)$ and electrostatic potential $\varphi(r,t)$ respectively. For each $t \in \bar{\omega}_\tau$ these functions are elements of Hilbert spaces

$$H_1 = \{ y(r) : r \in \bar{\omega}_h \}, \quad \overset{0}{H}_2 = \{ \mu(r) : r \in \omega_h;\ \mu = 0, \ r = R_0, R_1 \},$$

with the scalar product $(y,v) = \sum_{\bar{\omega}_h} \bar{h} y v$, where $\bar{h} = \frac{h}{2}$ for $i = 0, N$ and $\bar{h} = h$ for $i = 1, \dots, N-1$. Let also

$$\omega_h^+ = \{ r_i = R_0 + ih, \ i = 1, \dots, N \}, \quad \omega_h^- = \{ r_i = R_0 + ih, \ i = 0, 1, \dots, N-1 \}.$$

The first backward-difference, the first forward-difference and the second central difference approximation of the function α with respect to r shall be denoted as $\alpha_{\bar{r}}$, α_r, $\alpha_{\bar{r}r}$ respectively. Analogous notations shall be used for difference approximations with respect to t. Notations with a tilde we reserve for functions of the discrete variable $r \in \bar{\omega}_h$ and continuous $t \in \bar{I} = [0, T]$.

Our computational model can be derived in two stages [17].

First we approximate the integral of kinetic energy by the composite trapezoidal rule in the space variable r, that is

$$K^h = \frac{\rho}{2} \sum_{\bar{\omega}_h} \bar{h} r \left(\frac{d\tilde{u}}{dt} \right)^2,$$

where $K = K^h + O(h^2)$, whereas the integrals of elastic deformation and electric field are approximated by the composite rectangular rule:

$$W^h + P^h = \frac{1}{2} \sum_{\omega_h^+} h\bar{r} [c_{11} \tilde{\epsilon}_r^2 + 2 c_{12} \tilde{\epsilon}_r \tilde{\epsilon}_\theta + c_{22} \tilde{\epsilon}_\theta^2 + \epsilon_{11} \tilde{E}_r^2],$$

where $W + P = W^h + P^h + O(h^2)$, and

$$\tilde{\epsilon}_r = \tilde{u}_{\bar{r}}, \ \tilde{\epsilon}_\theta = (\tilde{u} + \tilde{u}^{(-1)})/(2\bar{r}), \ \tilde{E}_r = -\tilde{\phi}_{\bar{r}}, \ \tilde{u}^{(\pm 1)} = \tilde{u}(r \pm h, t), \ \bar{r} = r - h/2 \ r \in \omega_h.$$

Then, we approximate the left hand part of (3.5) as follows:

$$\frac{d\tilde{\varepsilon}}{dt} = \rho \sum_{\bar{\omega}_h} \bar{h} r \tilde{v} \frac{d\tilde{v}}{dt} + \sum_{\omega_h^+} h\bar{r} \left[\frac{\partial \tilde{\epsilon}_r}{\partial t} \tilde{\sigma}_r + \frac{\partial \tilde{\epsilon}_\theta}{\partial t} \tilde{\sigma}_\theta \right] + \sum_{\omega_h^+} h\bar{r} \tilde{E}_r \frac{\partial \tilde{D}_r}{\partial t},$$

where $\tilde{\varepsilon}$ is a differential-difference analog of the total energy of the electromechanical system, $\tilde{v} = d\tilde{u}/dt$, $\tilde{\sigma}_r = c_{11}\tilde{\epsilon}_r + c_{12}\tilde{\epsilon}_\theta - e_{11}\tilde{E}_r$, $\tilde{\sigma}_\theta = c_{12}\tilde{\epsilon}_r + c_{22}\tilde{\epsilon}_\theta - e_{12}\tilde{E}_r$. Now after simple transformations* we obtain a differential-difference analog of the energy identity (3.5):

$$\rho\sum_{\bar{\omega}_h} hr\tilde{v}\frac{d\tilde{v}}{dt} - \sum_{\omega_h} r\tilde{v}h\frac{1}{r}(r\tilde{\sigma}_r)_r + \sum_{\omega_h^+} rh\tilde{v}\frac{\tilde{\sigma}_\theta}{2r} + \sum_{\omega_h^-} rh\tilde{v}\frac{\tilde{\sigma}_\theta^{(+1)}}{2r} + \sum_{\omega_h^+} rh\tilde{E}_r\frac{\partial\tilde{D}_r}{\partial t} =$$

$$[R_1 p_1\tilde{v}(R_1,t) - R_0 p_0\tilde{v}(R_0,t)] - \tilde{v}_N\bar{r}_N(\tilde{\sigma}_r)_N + \tilde{v}_0\bar{r}_1(\tilde{\sigma}_r)_1 + \sum_{\omega_h} hr\tilde{v}f_1 +$$

$$\sum_{\omega_h} rh\tilde{\phi}\frac{\partial f_2}{\partial t} + V(t)\left[\bar{R}_1\frac{\partial\tilde{D}_r(R_1,t)}{\partial t} + \bar{R}_0^{(+1)}\frac{\partial\tilde{D}_r(\bar{R}_0^{(+1)},t)}{\partial t}\right]. \quad (3.6)$$

- Assuming that $\tilde{v} \neq 0$ identically when $\partial\tilde{D}_r/\partial t = \partial f_2/\partial t = 0$ we can derive from (3.6) the differential-difference analogue of the equation for continuum medium movement as well as boundary conditions for stresses in (2.4) (since the latter are natural boundary conditions).
- On the other hand, assuming that $\partial\tilde{D}_r/\partial t \neq 0$ identically when $\tilde{v} = f_1 = 0$ we can obtain the differential-difference analogue of the Maxwell equation:

$$\sum_{\omega_h^+} rh\tilde{E}_r\frac{\partial\tilde{D}_r}{\partial t} = \sum_{\omega_h} rh\tilde{\phi}\frac{\partial f_2}{\partial t} + V(t)\left[\bar{R}_1\frac{\partial\tilde{D}_r(R_1,t)}{\partial t} + \bar{R}_0^{(+1)}\frac{\partial\tilde{D}_r(\bar{R}_0^{(+1)},t)}{\partial t}\right].$$

Of course, if we take into consideration that $\tilde{E}_r = -\tilde{\phi}_{\bar{r}}$ and set†

$$\tilde{\phi} = V(t) \text{ when } r = R_0, \text{ and } \tilde{\phi} = -V(t) \text{ when } r = R_1,$$

then from the last equality we have

$$\left(\tilde{\phi}, \left(\bar{r}\frac{\partial\tilde{D}_r}{\partial t}\right)_r\right) = \left(\tilde{\phi}, r\frac{\partial f_2}{\partial t}\right).$$

Second stage of our derivation consists of time-discretisation of the differential-difference scheme obtained on the first stage. Finally, we obtain the difference scheme for the solution of the problem (2.1)-(2.6) that consists of

- approximation of the equation of motion and boundary conditions for stresses for $t \in \bar{\omega}_\tau$

* using grid formulas for summation by parts.

† since boundary conditions for potential are main boundary conditions, they do not follow directly from (3.7).

$$\rho y_{\bar{t}t} = \begin{cases} \dfrac{1}{r}(\bar{r}\bar{\sigma}_r)_r - \dfrac{\bar{\sigma}_\theta^{(+1)} + \bar{\sigma}_\theta}{2r} + f_1, & \text{for } r \in \omega_h, \\ \dfrac{2}{h}\dfrac{1}{r}\bar{r}^{(+1)}\bar{\sigma}_r^{(+1)} - \dfrac{\bar{\sigma}_\theta^{(+1)}}{r} + f_1 - \dfrac{2}{h}p_0, & \text{for } r = R_0, \\ -\dfrac{2}{h}\dfrac{1}{r}\bar{r}\bar{\sigma}_r - \dfrac{\bar{\sigma}_\theta}{r} + f_1 + \dfrac{2}{h}p_1, & \text{for } r = R_1; \end{cases} \tag{3.7}$$

- approximation of the Maxwell equation for piezoelectrics and the relationship for electric potential:

$$\frac{1}{r}(\bar{r}\bar{D}_r)_r = f_2, \quad \bar{E}_r = -\mu_{\bar{r}}, \tag{3.8}$$

- approximation of the state equations:

$$\begin{aligned} \bar{\sigma}_r &= c_{11}\bar{\epsilon}_r + c_{12}\bar{\epsilon}_\theta - e_{11}\bar{E}_r, \\ \bar{\sigma}_\theta &= c_{12}\bar{\epsilon}_r + c_{22}\bar{\epsilon}_\theta - e_{12}\bar{E}_r, \\ \bar{D}_r &= \epsilon_{11}\bar{E}_r + e_{12}\bar{\epsilon}_\theta + e_{11}\bar{\epsilon}_r, \end{aligned} \tag{3.9}$$

where we approximate the Cauchy relations as follows

$$\bar{\epsilon}_r = (y - y^{(-1)})/h, \quad \bar{\epsilon}_\theta = (y + y^{(-1)})/(2\bar{r}); \tag{3.10}$$

- exact boundary conditions for the potential function on the grid

$$\mu = V(t) \text{ for } r = R_0 \text{ and } \mu = -V(t) \text{ for } r = R_1 \tag{3.11}$$

- and the first initial condition:

$$y(r, 0) = u_0(r). \tag{3.12}$$

- The second initial condition is approximated by the central difference derivative with subsequent elimination of (-1)-fictitious time layer for $t = 0$, i.e. we have

$$\rho y_t = \rho u_1(r) + \frac{\tau}{2}\begin{cases} \dfrac{1}{r}(\bar{r}\bar{\sigma}_r)_r - \dfrac{\bar{\sigma}_\theta^{(+1)} + \bar{\sigma}_\theta}{2r} + f_1 & \text{for } r \in \omega_h, \\ \dfrac{2}{h}\dfrac{1}{r}\bar{r}^{(+1)}\bar{\sigma}_r^{(+1)} - \dfrac{\bar{\sigma}_\theta^{(+1)}}{r} + f_1 - \dfrac{2}{h}p_0 & \text{for } r = R_0, \\ -\dfrac{2}{h}\dfrac{1}{r}\bar{r}\bar{\sigma}_r - \dfrac{\bar{\sigma}_\theta}{r} + f_1 + \dfrac{2}{h}p_1 & \text{for } r = R_1. \end{cases} \tag{3.13}$$

The stability condition for the scheme is provided by the requirement of non-negativeness of the difference analogue of the energy integral (3.5) [17]. Moreover, using a standard technique based on the Taylor expansion it can be proved that the difference scheme (3.8)-(3.14) has the second order of approximation with respect to space and time variables provided the solution of the problem belongs to the class $C^4(Q_T)$. Of course, one can use the Lax theorem relating consistency and stability of this scheme to the convergence. However, the rate of convergence is virtually defined by the order of approximation which depends on *a-priori* assumptions for the solution smoothness. Because in many practical applications the solution does not posses such high smoothness as mentioned above we need a scale of *a-priori* estimations relating the rate of convergence of our scheme to the smoothness of the "exact" solution. In a natural way such a consideration implies the notion of generalized solutions which we introduced at the beginning of this section. In fact, in many applied problems of electroelasticity* solutions can exhibit wave discontinuities as well as sharp increases in the amplitude of oscillations subject to initial data. In such cases it is important to construct numerical procedures and investigate their convergence for problems with generalized solutions. Such investigations become an intrinsic part of the justification of the model.

4. DIFFERENCE SCHEMES AND A NEW A-PRIORI ESTIMATION FOR THE NUMERICAL SOLUTION

4.1. Operator-difference form of the discretized problem

A discrete space-time approximation (3.7)-(3.13) for the problem (2.1)-(2.6) obtained in section 3 is a partial case of more general operator-difference scheme:

$$D_1 y_{\bar{t}t} + A_1 y + C_1 \mu = \varphi_1, \tag{4.1}$$

$$A_2 \mu + C_2 y = \varphi_2, \tag{4.2}$$

$$y = y_0, \;\; D_1 y_t = y_1, \;\; t = 0, \tag{4.3}$$

where operators of this scheme are defined in the following way:

$$A_1 y = \begin{cases} -\frac{2}{h}\bar{r}^{(+1)}\bar{\sigma}_r^{(+1)} + \bar{\sigma}_\theta^{(+1)}, & r = R_0 \\ -(\bar{r}\bar{\sigma}_r)_r + \dfrac{\bar{\sigma}_\theta^{(+1)} + \bar{\sigma}_\theta}{2}, & R_0 < r < R_1 \\ \frac{2}{h}\bar{r}\bar{\sigma}_r + \bar{\sigma}_\theta, & r = R_1 \end{cases}$$

$$C_1 \mu = \begin{cases} \dfrac{2e_{11}}{h}\bar{r}^{(+1)}\bar{E}_r^{(+1)} - e_{12}\bar{E}_r^{(+1)}, & r = R_0 \\ e_{11}(\bar{r}\bar{E}_r)_r - e_{12}\dfrac{\bar{E}_r^{(+1)} + \bar{E}_r}{2}, & R_0 < r < R_1 \\ -\dfrac{2e_{11}}{h}\bar{r}\bar{E}_r + e_{12}\bar{E}_r, & r = R_1 \end{cases}$$

* some examples of this type are given in the section 6

$$D_1 y = r\rho y,\ A_2\mu = \epsilon_{11}(\bar{r}E_r)_r,\ C_2 y = [\bar{r}(e_{12}\bar{\epsilon}_\theta + e_{11}\bar{\epsilon}_r)]_r,\ \varphi_1 = S^r S^t(rf_1),\ \varphi_2 = S^r(rf_2).$$

Here S^r and S^t are the averaging Steklov operators defined by the formulae:

$$S^r u(r,t) = \begin{cases} \dfrac{2}{h}\displaystyle\int_{R_0}^{R_0+\frac{h}{2}} u(\xi,t)d\xi, & r = R_0 \\ \dfrac{1}{h}\displaystyle\int_{r-\frac{h}{2}}^{r+\frac{h}{2}} u(\xi,t)d\xi, & R_0 < r < R_1 \\ \dfrac{2}{h}\displaystyle\int_{R_1-\frac{h}{2}}^{R_1} u(\xi,t)d\xi, & r = R_1 \end{cases}$$

$$S^t v(r,t) = \begin{cases} \dfrac{1}{\tau}\displaystyle\int_{t-\frac{\tau}{2}}^{t+\frac{\tau}{2}} v(r,\mu)d\mu, & t > 0 \\ \dfrac{2}{\tau}\displaystyle\int_{0}^{\frac{\tau}{2}} v(r,\mu)d\mu, & t = 0. \end{cases}$$

The particular form of approximation for the right hand parts of the scheme (4.1)-(4.3) is problem-specific and depends on the boundary conditions for the problem.

Due to the presence of a hyperbolic-type operator in the original model, mathematical justification of the scheme (4.1)-(4.3) in applications to many practically important problems can be provided by obtaining a-priori estimations of difference solutions in negative norms of the right hand parts. However, the latter is complicated by the strong electromechanical coupling in the original system. Hence, to obtain a plausible picture of the propagation of the mixed electro-elastic waves we should deal with coupling from the beginning of the numerical analysis stage of our investigations.

4.2. Stability and the energy identity in the difference case

Let us introduce the following notations for norms and semi-norms of discrete functions y and μ:

$$\|y(t)\|^2 = (y(t), y(t)),\ \|y(t)\|_A^2 = (Ay(t), y(t)),\ \|y(t)\|_0^2 = \sum_{t=0}^{T} \tau\|y(t)\|^2,$$

$$\|y\|^2_{(1)} = \|y\|_{D_1-\frac{\tau^2}{4}A_1} - \tau^2 R_1\left(\frac{e_{11}^2}{\epsilon_{11}}\|\bar{\epsilon}_r\|^2 + \frac{e_{12}^2}{\epsilon_{11}}\|\bar{\epsilon}_\theta\|^2\right) + \left\|\sum_{t'=0}^{t} \tau y(t')\right\|^2_{A_1},$$

$$\|\mu\|^2_{(2)} = \left\|\sum_{t'=0}^{t} \tau\mu(t')\right\|^2_{A_2}.$$

The semi-norm $\|\cdot\|_{(1)}$ exists if the following condition

$$\left(\left(D_1 - \frac{\tau^2}{4}A_1\right)y, y\right) - \tau^2 R_1\left(\frac{e_{11}^2}{\epsilon_{11}}\|\bar{\epsilon}_r\|^2 + \frac{e_{12}^2}{\epsilon_{11}}\|\bar{\epsilon}_\theta\|^2\right) > 0 \tag{4.4}$$

holds $\forall y \in \bar{\omega}_{h\tau}$. This condition allows us to derive the stability condition of the scheme. In fact, taking into consideration easily proved inequalities:

$$\|\bar{\varepsilon}_r\|^2 = \sum_{\omega_h^+} h(y_{\bar{r}})^2 \le \frac{4}{h^2}\sum_{\bar{\omega}_h} \bar{h}y^2,$$

$$\|\bar{\varepsilon}_\theta\|^2 = \sum_{\omega_h^+} h\left(\frac{y + y^{(-1)}}{2\bar{r}}\right)^2 \le \frac{1}{R_0^2}\sum_{\bar{\omega}_h} \bar{h}y^2,$$

$$\sum_{\omega_h^+} h\bar{\varepsilon}_r\bar{\varepsilon}_\theta = \frac{1}{h}\sum_{\omega_h^+} h\frac{y^2 - (y^{(-1)})^2}{2\bar{r}} \le \frac{1}{2R_0 h}\sum_{\bar{\omega}_h} \bar{h}y^2,$$

as well as the equality:

$$(A_1 y, y) = r(\bar{\sigma}_r, \bar{\epsilon}_r) + r(\bar{\sigma}_\theta, \bar{\epsilon}_\theta) = r(c_{11}\|\bar{\epsilon}_r\|^2 + 2c_{12}(\bar{\epsilon}_r, \bar{\epsilon}_\theta) + c_{22}\|\bar{\epsilon}_\theta\|^2),$$

it is not difficult to conclude that the condition (4.4) will be satisfied if the following inequality is fulfilled:

$$\tau \le \frac{h}{c}\left\{\left(1 - \frac{\epsilon}{\rho}\right) \Big/ \left[1 + \frac{3\delta}{1+\delta} + \frac{c_{12}}{4R_0 c_{11}(1+\delta)}h + \frac{c_{22} + 4e_{12}^2/\epsilon_{11}}{4R_0^2 c_{11}(1+\delta)}h^2\right]\right\}^{\frac{1}{2}}, \tag{4.5}$$

where $c = \sqrt{c_{11}(1+\delta)/\rho}$ is the velocity of the mixed electro-elastic wave propagation and $\delta = e_{11}^2/(\epsilon_{11}c_{11})$ is the coupling coefficient of the electromechanical system, $\epsilon > 0$.

To derive a new a-priori estimation for the solution of (4.1)-(4.3) we shall take the scalar product of the equation (4.1) and $\tau w(t)$, where $w(t)$ is defined as in [25]:

$$w(t) = \sum_{t'=t+\tau}^{t_1} \tau(y(t') + \check{y}(t')), \qquad \check{y}(t') = y(t' - \tau).$$

In a similar manner we define the function $w_1(t)$:

$$w_1(t) = \sum_{t'=\tau}^{t} \tau(y(t') + \check{y}(t')).$$

The reason for using the functions $w(t)$ and $w_1(t)$ in obtaining a-priori estimation for the problem (4.1)-(4.3) is because of their properties:

$$w_{\bar{t}} = -(y + \check{y}) \quad \text{for all} \quad 0 \le t < t_1; \quad w(t) = 0, \quad \text{for all} \quad t_1 \le t \le T;$$

$$w(t) = w_1(t_1) - w_1(t), \quad \text{and} \quad w(0) = w_1(t_1).$$

Using the easily verified identity

$$\tau(D_1 y_{\bar{t}t}, w) = (D_1 y_t, w) - (D_1 y_{\bar{t}}, \check{w}) - \tau(D_1 y_{\bar{t}}, w_{\bar{t}}),$$

we get:

$$(D_1 y_t, w) - \tau(D_1 y_{\bar{t}}, w_{\bar{t}}) + \tau(A_1 y, w) + (C_1 \mu, w) = (D_1 y_{\bar{t}}, \check{w}) + \tau(\varphi_1, w).$$

Summing the last identity over t from τ to a certain t_1 ($0 < t_1 \le T$) and taking into account that $w(t_1) = 0$ we come to the **energy identity**:

$$\sum_{t'=\tau}^{t_1} \tau(D_1 y_{\bar{t}}, y + \check{y})(t') + \sum_{t'=\tau}^{t_1} \tau(A_1 y, w)(t') + \sum_{t'=\tau}^{t_1} \tau(C_1 \mu, w)(t') =$$

$$(D_1 y_t, w)(0) + \sum_{t'=\tau}^{t_1} \tau(\varphi_1, w)(t'). \quad (4.6)$$

The latter is a key point in establishing of a new a-priori estimation for the solution of (4.1)-(4.3).

4.3. A-priori estimate for the discretized problem

We need two auxiliary functions:

$$g(t) = \frac{1}{2} w(t) - \frac{\tau}{2} y(t), \quad j(t) = \frac{1}{2} v(t) - \frac{\tau}{2} \mu(t)$$

where the function $v(t)$ is defined as

$$v(t) = \sum_{t'=t+\tau}^{t_1} \tau(\mu(t') + \check{\mu}(t')).$$

The function $v(t)$ has the properties analogous to the properties of the function $w(t)$. Using obvious equalities:

$$y = \frac{1}{2}(y + \check{y}) + \frac{\tau}{2}y_{\bar{t}} = -g_{\bar{t}}, \quad w = g + \check{g},$$

the consequence of the equation (4.2):

$$C_2 w = -A_2 v + \sum_{t' = t+\tau}^{t_1} \tau(\varphi_2 + \check{\varphi}_2)$$

and properties of operators of the scheme (4.1)-(4.3), the second and the third terms of the left hand part of (4.6) can be transformed in the following way:

$$\sum_{t'=\tau}^{t_1} \tau(A_1 y, w)(t') = -\sum_{t'=\tau}^{t_1} \tau(A_1 g_{\bar{t}}, g + \check{g})(t') = -(A_1 g, g)(t_1) + (A_1 g, g)(0) =$$

$$-\frac{\tau^2}{4}(A_1 y, y)(t_1) + (A_1 g, g)(0),$$

$$\sum_{t'=\tau}^{t_1} \tau(C_1 \mu, w) = -\sum_{t'=\tau}^{t_1} \tau(\mu, C_2 w) = \sum_{t'=\tau}^{t_1} \tau(\mu, A_2 v) - \sum_{t'=\tau}^{t_1} \tau\left(\mu, \sum_{t''=t'+\tau}^{t_1} \tau(\varphi_2 + \check{\varphi}_2)\right) =$$

$$-\frac{\tau^2}{4}(A_2 \mu, \mu)(t_1) + (A_2 j, j)(0) - \sum_{t'=\tau}^{t_1} \tau\left(\mu, \sum_{t''=t'+\tau}^{t_1} \tau(\varphi + \check{\varphi}_2)\right).$$

It allows us to rewrite the identity (4.6) in the following form:

$$\left(\left(D_1 - \frac{\tau^2}{4}A_1\right)y, y\right)(t_1) + (A_1 g, g)(0) + (A_2 j, j)(0) -$$

$$\frac{\tau^2}{4}(A_2 \mu, \mu)(t_1) = (D_1 y, y)(0) + (D_1 y_t, w)(0) +$$

$$\sum_{t'=\tau}^{t_1} \tau(\varphi_1, w)(t') + \sum_{t'=\tau}^{t_1} \tau\left(\mu, \sum_{t''=t'+\tau}^{t_1} \tau(\varphi_2 + \check{\varphi}_2)\right). = \quad (4.7)$$

The estimation of the term $-\frac{\tau^2}{4}(A_2 \mu, \mu)(t_1)$ can be performed if we take into consideration the equation (4.2):

$$-\frac{\tau^2}{4}(A_2 \mu, \mu) \geq 3R_1 \frac{\tau^2}{4}\left(\frac{1}{R_0^2 \epsilon_{11}}\|\hat{\lambda}\|^2 + \frac{e_{11}^2}{\epsilon_{11}}\|\bar{\epsilon}_r\|^2 + \frac{e_{12}^2}{\epsilon_{11}}\|\bar{\epsilon}_\theta\|^2\right),$$

where $\hat{\lambda}(t) = \lambda(t+\tau)$, $\varphi_2 = \hat{\lambda}_r$ and $\hat{\lambda} = 0$ for $r = R_0$. From the obvious inequality $(a-b)^2 \geq \frac{1}{2}a^2 - b^2$ the following estimations are implied:

$$\|g(0)\|^2 \geq \frac{1}{8}\|w_1(t_1)\|^2 - \frac{\tau^2}{4}\|y(0)\|^2, \ \|j(0)\|^2 \geq \frac{1}{8}\|v_1(t_1)\|^2 - \frac{\tau^2}{4}\|\mu(0)\|^2,$$

where the function $v_1(t)$ is defined analogously to the function $w_1(t)$ by the replacement of y for μ.

Transforming the remaining terms in the right hand part of (4.7) using the Cauchy-Schwarz inequality*, assuming that $\varphi_1 = (\xi_1)_t + (\xi)_r$, and applying the discrete analogue of the Gronwall lemma [6], we come to the following result:

THEOREM 4.1. *If the condition* (4.5) *is satisfied, then the following a-priori estimation:*

$$\|y(t_1)\|^2_{(1)} + \|\mu(t_1)\|^2_{(2)} \leq M\Bigg\{\left(\left(D_1 + \frac{\tau^2}{4}A_1\right)y, y\right)(0) + \frac{\tau^2}{4}(A_2\mu, \mu)(0) +$$

$$\|D_1 y_t(0)\|^2 + \sum_{t'=0}^{T} \tau(\|\xi_1\|^2 + \|\xi_2\|^2 + \|\hat{\lambda}\|^2 + \|\hat{\lambda}\|^2_0)\Bigg\} \tag{4.8}$$

holds for the solution of the problem (4.1)-(4.3).

Remark 4.1. *In fact the condition (3.5) is the CFL-type stability condition in this case of coupled field theory. It contains the velocity of the coupled electro-elastic wave. If $\delta \to 0$ this stability condition coincides in the dominant part with the stability condition obtained in [21].*

5. DIFFERENCE SCHEME CONVERGENCE IN THE GENERALIZED SOLUTION CLASSES

5.1. Problem for approximation errors

Let us apply the *a-priori* estimation (4.8) in the investigation of convergence of the difference scheme (4.1)-(4.3). We consider here the generalized solutions from

* we also use its direct consequence which is known as the ϵ-inequality: $|(u, v)| \leq \|u\|\|v\| \leq \epsilon\|u\|^2 + \frac{1}{4\epsilon}\|v\|^2 \ \forall \epsilon > 0$.

Sobolev's spaces $W_2^k(Q_T)$, $k = \overline{2,4}$ and the space $V(\overline{Q}_T) = C(\overline{Q}_T) \cap Q_1(\overline{Q}_T)$ of continuous functions with piecewise first derivatives which have square integrable generalized derivatives in the continuity region.

First let us consider an approximation error of the scheme (4.1)-(4.3):

$$z = y - u, \quad \zeta = \mu - \varphi.$$

We note that the approximation error is the solution of the following operator-difference scheme:

$$\begin{cases} D_1 z_{\bar{t}t} + A_1 z + C_1 \zeta = \psi, & t \in \omega_\tau \\ A_2 \zeta + C_2 z = \chi, & t \in \overline{\omega}_\tau \\ z = 0,\ D_1 z_t = \psi, & t = 0. \end{cases}$$

Then we introduce notations:

$$u\left(r \pm \frac{h}{2}, t\right) = u^{(\pm 0.5)},\ u\left(r, t + \frac{\tau}{2}\right) = \bar{u},\ u\left(r, t - \frac{\tau}{2}\right) = \check{u}$$

and consider the error of approximation of equation (2.1) in the inner nodes of the mesh:

$$\psi = \varphi_1 - r\rho u_{\bar{t}t} + c_{11}(\bar{r}u_{\bar{r}})_r - c_{22}\left[\frac{u^{(+1)} + u}{4\bar{r}^{(+1)}} + \frac{u + u^{(-1)}}{4\bar{r}}\right] +$$

$$e_{11}(\bar{r}\varphi_{\bar{r}})_r - e_{12}\frac{\varphi_r + \varphi_{\bar{r}}}{2}. \tag{5.1}$$

We then perform the following operations:

- apply the composition of the averaging operators $S^r S^t$ to the equation (2.1);
- calculate φ_1 from the obtained equality;
- substitute φ_1 into (5.1);
- use the main property of the averaging operators:

$$S^r\frac{\partial u}{\partial r} = \frac{1}{h}[u^{(+0.5)} - u^{(-0.5)}] = (u^{(-0.5)})_r, \quad S^t\frac{\partial u}{\partial t} = \frac{1}{\tau}[\bar{u} - \check{u}] = (\check{u})_t,$$

to get

$$\psi = \rho(\eta_{11})_t + c_{12}(\eta_{12})_r + c_{22}(\eta_{13})_r + e_{11}(\eta_{14})_r + e_{12}(\eta_{15})_r,$$

where

$$\eta_{11} = S^r\left(r\frac{\partial \check{\bar{u}}}{\partial t}\right) - ru_{\bar{r}},\ \eta_{12} = \bar{r}u_{\bar{r}} - S^t\left(\left(r\frac{\partial u}{\partial r}\right)^{(-0.5)}\right),$$

$$(\eta_{13})_{\bar r} = \psi_{13} = S^r S^t\left(\frac{u}{r}\right) - \left(\frac{u^{(+1)} + u}{4\bar r^{(+1)}} + \frac{u + u^{(-1)}}{4\bar r}\right),$$

$$(\eta_{13})_i = \sum_{j=0}^{i-1} h(\psi_{13})_j,\ (\eta_{13})_0 = 0,$$

$$\eta_{14} = \bar r\varphi_{\bar r} - S^t\left(\left(r\frac{\partial\varphi}{\partial r}\right)^{(-0.5)}\right),\ \eta_{15} = S^t((\varphi)^{(-0.5)}) - \frac{\varphi + \varphi^{(-1)}}{2}.$$

5.2. Application of the Bramble-Hilbert technique

Now let us consider the case when the solution of the problem (2.1)-(2.6) belongs to the space $W_2^2(Q_T)$. We shall estimate the functionals η_{1k}, $k = \overline{1, 5}$, using the Bramble-Hilbert lemma [2], [29], [30]. First let us consider the functional η_{12}. It is easy to see that the linear functional η_{12} is bounded in $W_2^2(Q_T)$. Moreover,

$$|\eta_{12}| \le Mh^{-1}\|u\|_{W_2^2(e_1)}$$

The linear substitution $\xi_1 = r + s_1 h$, $\xi_2 = t + s_2\tau$ permits us to transform the region

$$e_1 = \left\{(r', t') : r - h < r' < r, t - \frac{\tau}{2} < t' < t + \frac{\tau}{2}\right\}$$ into the region $E = \{(s_1, s_2) :$

$-1 < s_1 < 0,\ -\frac{1}{2} < s_2 < \frac{1}{2}\}$. It is well-known that a linear substitution does not change the class of functions, and therefore,

$$|\eta_{12}| \le Mh^{-1}\|u\|_{W_2^2(E)}$$

Further one can verify that the functional

$$\eta_{12} = -\frac{1}{2h}\left\{\tilde u(0, 0) - \tilde u(-1, 0) - \int_{-0.5}^{0.5}\frac{\partial\tilde u\left(-\frac{1}{2}, s_2\right)}{\partial s_1}ds_2\right\},\ \text{where } \tilde u(s) = u(r(\xi_1), t(\xi_2))$$

becomes zero for all polynomials up to the first degree inclusively. That is why according to the Bramble-Hilbert lemma we have:

$$|\eta_{12}| \le Mh^{-1}\|\tilde u\|_{W_2^2(E)}.$$

Transforming back to the variables (r, t) we get:

$$|\eta_{12}| \le M\frac{h^2 + \tau^2}{h}(h\tau)^{-\frac{1}{2}}|u|_{W_2^2(e_1)}.$$

In the same way it is not difficult to show that

$$|\eta_{12}| \le M\frac{(h^2+\tau^2)^{\frac{k}{2}}}{h}(h\tau)^{-\frac{1}{2}}|u|_{W_2^k(e_1)}, \text{ where } k = 3, 4. \tag{5.2}$$

Analogously, using the Bramble-Hilbert lemma technique one can obtain estimations for other functionals.

Let us consider now the error of approximation on the boundary. For example if $r = R_0$:

$$\psi|_{R_0} = \varphi_1 - \rho r u_{\bar{t}t} + \frac{2}{h}\bar{r}^{(+1)}\breve{\sigma}_r^{(+1)} - \breve{\sigma}_\theta^{(+1)},$$

where

$$\breve{\sigma}_r = c_{11}u_{\bar{r}} + c_{12}\frac{u+u^{(-1)}}{2r} + e_{11}\varphi_{\bar{r}},\ \breve{\sigma}_\theta = c_{12}u_{\bar{r}} + c_{22}\frac{u+u^{(-1)}}{2r} + e_{12}\varphi_{\bar{r}},$$

it can be shown that

$$\psi|_{R_0} = \rho(\eta'_{11})_{,} + \frac{2}{h}(\eta'_{12})_r + (\eta'_{13})_r,$$

where

$$\eta'_{11} = \frac{2}{h}\int_{R_0}^{R_0+\frac{h}{2}} r\frac{\partial \dot{u}}{\partial t}dr - ru_{\bar{t}},\ (\eta'_{12})_r = \psi'_{12} = \bar{r}^{(+1)}\breve{\sigma}_r^{(+1)} - \frac{1}{\tau}\int_{t-\frac{\tau}{2}}^{t+\frac{\tau}{2}} r\sigma_r\Big|_{R_0+\frac{h}{2}}dt,$$

$$(\eta'_{13})_r = \psi'_{13} = \frac{2}{h\tau}\int_{t-\frac{\tau}{2}}^{t+\frac{\tau}{2}}\int_{R_0}^{R_0+\frac{h}{2}} \sigma_\theta dr dt - \breve{\sigma}_\theta^{(+1)}.$$

We consider for the sake of brevity only the functional ψ'_{13}. If we represent the functional in the form*:

$$\psi'_{13} = \frac{1}{\tau}\int_{t-\frac{\tau}{2}}^{t+\frac{\tau}{2}}\left[\frac{2}{h}\int_{R_0}^{R_0+\frac{h}{2}}\sigma_\theta dr - \sigma_\theta\left(R_0+\frac{h}{2}, t\right)\right]dt + \left[\frac{1}{\tau}\int_{t-\frac{\tau}{2}}^{t+\frac{\tau}{2}}\sigma_\theta\left(R_0+\frac{h}{2}, t\right)dt - \breve{\sigma}_\theta^{(+1)}\right]$$

then its estimation can be performed with the help of the Bramble-Hilbert lemma:

$$|\psi'_{13}| \le M(h^2+\tau^2)^{\frac{p}{2}}(h\tau)^{-\frac{1}{2}}|\sigma_\theta|_{W_2^p(e'_1)},\ p = \overline{1,3}, \tag{5.3}$$

* The error of the right rectangular quadrature formula is recognizable in the brackets of the first term.

where

$$e'_1 = \{(r', t') : R_0 < r' < R_0 + \frac{h}{2}, t - \frac{\tau}{2} < t' < t + \frac{\tau}{2}\}.$$

Finally we should take into consideration the following:

$$(\eta'_{13})_i = \sum_{j=0}^{i-1} h(\psi'_{13})_j, \ (\eta'_{13})_0 = 0, \text{ and } \sigma_\theta = c_{12}\frac{\partial u}{\partial r} + c_{22}\frac{u}{r} + e_{12}\frac{\partial \varphi}{\partial r}.$$

The approximation error of initial conditions has the form:

$$\psi|_{t=0} = \rho r u_1 - S^r\Big(r\rho\frac{\partial u}{\partial t}(0)\Big) + \Big[S^r\Big(r\rho\frac{\partial u}{\partial t}\big(\frac{\tau}{2}\big)\Big) - r\rho u_t\Big] +$$

$$\frac{\tau}{2}\Big[S^t S^r\Big(\frac{\partial}{\partial r}(r\sigma_r) - \sigma_\theta\Big) - A_1 u - C_1\varphi\Big]. \tag{5.4}$$

It has been obtained taking into consideration the equation (2.1), on which we preliminary acted by the composition of operators $S^r S^t$ (where S^t is defined for $t = 0$).

The approximation error for the equation (4.2) is readily obtained if we act on the equation (2.2) by the averaging operator S^r:

$$\chi = -\epsilon_{11}\left[\frac{1}{h}\int_{r-\frac{h}{2}}^{r+\frac{h}{2}} \frac{\partial}{\partial r}\Big(r\frac{\partial \varphi}{\partial r}\Big)dr - (\bar{r}\varphi_{\bar{r}})_r\right] + e_{11}\left[\frac{1}{h}\int_{r-\frac{h}{2}}^{r+\frac{h}{2}} \frac{\partial}{\partial r}\Big(r\frac{\partial u}{\partial r}\Big)dr - (\bar{r}u_{\bar{r}})_r\right] +$$

$$e_{12}\left[\frac{1}{h}\int_{r-\frac{h}{2}}^{r+\frac{h}{2}} \frac{\partial u}{\partial r}dr - \Big(\frac{u + u^{(-1)}}{2}\Big)_r\right]. \tag{5.5}$$

The estimations of the right hand parts of (5.4), (5.5) do not cause any difficulties. They are obtained by the described technique.

5.3. Convergence of the operator-difference scheme

To obtain the estimation of accuracy for the scheme (4.1)-(4.3) in cases where the required solution belongs to Sobolev's spaces $W_2^k(Q_T)$, $k = \overline{2, 4}$, we should take into consideration *a-priori* estimation obtained in the theorem 4.1, which for the approximation error has the form:

$$\|z\|^2_{(1)} + \|\zeta\|^2_{(2)} \le M\left\{\|\psi|_{t=0}\|^2 + \sum_{t'=0}^{T} \tau\big(\|\xi_1\|^2 + \|\xi_2\|^2\big) + \|\bar{\lambda}\|^2 + \|\bar{\lambda}\|_0^2\right\}, \tag{5.6}$$

where

$$\psi = (\bar{\xi}_1)_t + (\bar{\xi}_2)_r, \ \chi = (\bar{\lambda})_r.$$

The right hand side of (5.6) is estimated with the help of inequalities analogous to (5.2), (5.3). For example, using the estimations of the functionals η_{11}, η'_{11} we can get:

$$\left(\sum_{t'=0}^{T} \tau \|\bar{\xi}_1\|^2\right)^{\frac{1}{2}} = \left(\sum_{t'=0}^{T} \tau \sum_{\bar{\omega}_h} \bar{h} \rho^2 |\eta_{11}|^2\right)^{\frac{1}{2}} \le$$

$$M\left(\sum_{t'=0}^{T} \sum_{\bar{\omega}_h} \tau \bar{h} \frac{(h^2+\tau^2)^k}{\tau^2} (h\tau)^{-1} |u|^2_{W_2^k(e_m)}\right)^{\frac{1}{2}} \le M\frac{(h^2+\tau^2)^{\frac{k}{2}}}{\tau} |u|_{W_2^k(Q_T)}, \ k = \overline{2, 4},$$

where

$$e_m = e_2 = \left\{(r', t') : r - \frac{h}{2} < r' < r + \frac{h}{2}, t - \tau < t' < t\right\} \qquad \text{for } r \in \omega_h,$$

$$e_m = e'_1 \qquad \text{for } r = R_0,$$

$$e_m = e'_2 = \{(r', t') : R_1 - \frac{h}{2} < r' < R_1, t - \frac{\tau}{2} < t' < t + \frac{\tau}{2}\} \qquad \text{for } r = R_1.$$

In the same way the norms of the remaining functionals of the right hand side of (5.6) are estimated. The only exception is the estimation of the functional ψ'_{13} . It does not become zero on polynomials of the first degree if the required solution belongs to the space $W_2^3(Q_T)$. However, it can be estimated by the Ilyin inequality that gives an integral estimation on the near-boundary strip of the region [29].

Some applied problems in coupled dynamic electroelasticity are characterized by the fact that the solution derivatives have a discontinuity of the first kind. Thus we shall consider the case when the solution of the problem (2.1)-(2.6) belongs to the space $V(\overline{Q}_T)$. The a-priori estimation (5.6) for the error of the scheme remains true in this case as well, and

$$\sum_{t'=0}^{T} \tau \|\bar{\xi}_i\|^2 = \sum_{t'=0}^{T} \tau \sum_{r' \in \omega_p} h \bar{\xi}_i^2(r', t') + \sum_{t'=0}^{T} \tau \sum_{r' \in \bar{\omega}_h / \omega_p} h \bar{\xi}_i^2(r', t'), \ i = 1, 2,$$

where ω_p are the points of the mesh, the neighborhood of which contain points of discontinuity of the first derivatives of the solution.

In the domain of continuity of the first derivatives, corresponding functionals which occur in the approximation error of ψ and χ have been estimated earlier. In the points of their discontinuity the functional are bounded. So far, as the total number of points ω_p (where the first derivatives have discontinuity) is finite, then

$$\sum_{t'=0}^{T} \tau \sum_{r' \in \omega_p} h \bar{\xi}_i^2(r', t') = O(\tau + h) .$$

As a result we have proved the following

THEOREM 5.1. *Under the stability condition (4.5) the solution of the difference scheme (4.1)-(4.3) converges to the generalized solution of the coupled dynamic problem of electroelasticity at a rate of $O(h^k + \tau^k)$. The following accuracy estimations*

$$\|z\|_{(1)} + \|\zeta\|_{(2)} \le M(h^k + \tau^k) , \tag{5.7}$$

hold for $k = \frac{1}{2}$ *if the solution of the problem* (2.1)-(2.6) *belongs to the space* $V(Q_T)$ *and for* $k = \frac{p}{2}$ *if the solution is from the class* $W_2^p(Q_T)$, $p = \overline{2, 4}$.

Remark 5.1. *When the equations (2.1) and (2.2) are coupled only by the state equations (2.5), but there is no connection through the boundary conditions for stresses, then the accuracy estimation (5.7) can be improved**. *In such a semi-coupled case the convergence of the difference scheme with the second order (to the generalized solution from* $W_2^2(Q_T)$*), can be proved in a weaker than* $L^2(\omega)$ *metric [5]. These issues will be addressed elsewhere.*

Remark 5.2. *Similar results have also been obtained in the coupled theory of thermoelasticity [15] where mixed parabolic and hyperbolic operators are non-separable globally.*

6. APPLICATIONS OF THE OPERATOR-DIFFERENCE SCHEME TO COUPLED PROBLEMS IN DYNAMIC ELECTROELASTICITY

Studies in electromechanical interactions are important in many areas of applications including engineering and biophysics [1], [7], [8]. An increasing range of applications of piezoelectrics in semiconductors and intelligent structures stimulates a greater interest in coupling effects between mechanical and electric fields [3], [4], [10]. However, many problems arising in this field require mathematical tools that allow treatment of steep gradients and even discontinuities of the solutions using efficient numerical procedures.

Since many technical devices work in the regime of steady-state harmonic oscillations, this area of mathematical modelling is well elucidated in the litera-

* Such an improvement is possible, for example, when displacements rather than stresses are given on the boundary.

ture and continues to attract the attention of researchers from around the world. Nevertheless, many applications require by necessity the investigation of coupled electromechanical fields that have nonstationary rather than steady-state character. Such problems are typical in the analysis of transient processes in various technical devices.

The other essential area of study in the coupled electroelasticity field is based on some assumptions simplifying the original coupled problem. Many methods for the solutions of dynamic problems in electroelasticity are based on thickness averaging* and the use of the Kirchhoff-type hypothesizes. Such simplifications may not be appropriate for thin structures which are important in many applications. A typical example of this type is thin hollow piezoceramic cylinders. They are widely used as active elements in many technical devices. Thin hollow cylinders may provide a basis for investigation of electromechanical processes in bones and other biological tissues. Such investigations are extremely important since there is evidence that the piezoelectric effect plays an essential role in a feedback mechanism that controls activity of biological cells (see [7], [8] and references therein). The above examples emphasize that in many applications *consistent* solutions of the coupled nonstationary problem of electroelasticity are required. Moreover, numerical methods provide a natural and the most effective way to find such solutions.

As an example, we consider the process of coupled electroelastic oscillations of a hollow piezoceramic cylinder with radial preliminary polarization. When the thickness-to-length ratio is small the model (2.1)-(2.6) gives an appropriate description of the underlying physical processes. It is assumed that there are no stresses on the exterior and interior surfaces, and a given potential difference $2V$ is maintained. At the initial moment of time we assume the unexcited state of the piezoceramic PZT-4. After scaling we have the following values of coefficients:

$$2V = 1,\ \rho = 1,\ R_1 = 1,\ c_{11} = 0.82734,\ c_{12} = 0.53453,\ c_{22} = 1,$$
$$e_{11} = 0.54027,\ e_{12} = -\,0.18605,\ \epsilon_{11} = 1.$$

The dynamics of stresses and displacements were investigated with respect to cylinder thickness. In all cases the stability condition (4.5) maintained. Principal conclusions from numerical experiments can be summarized as follows.

- Using the described model we computed stresses and displacements for cylinders of small thickness. In general, due to the inconsistency of initial and boundary conditions at the initial moment of time, the stresses function is discontinuous. Discontinuities prevent us from using schemes justified for excessive a-priori smoothness assumptions. Hence, the analysis performed in sections 3-5 becomes a necessary part of the justification of the computational model in this

* Usually, for mechanical components of electroelastic fields.

case. The stresses function reflects the process of contraction (extension) of small-thickness cylinders. Figures 1 and 2 show displacements and stresses for cylinders of different thickness l^* at the moment of scaled time $t = 10$. When the thickness increases, oscillations in stresses become essential. Though in some cases smoothing procedures may be applied, further increase of thickness indicates that the use of the higher dimensional models is required. We note that displacements remain a smooth function when thickness is increased. Qualitatively, quite similar pictures are observed in non-coupled electroelasticity. It should be noted however, that a comparison of absolute values of displacements and stresses for circular and radial polarisations shows that in the latter case they can essentially exceed corresponding values for circular polarization† (see [17, 22] and references therein). This explains our interest in the investigation of coupling effects in the case of radial preliminary polarization.

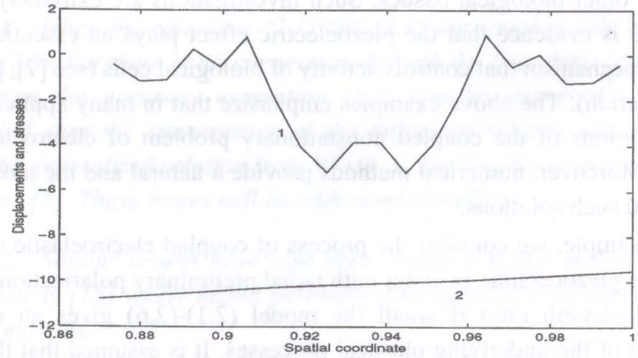

FIGURE 1 Radical stresses (curve 1) and displacements (curve 2) at $t = 10$ (radial preliminary polarisation ; thickness $l = 0.13$)

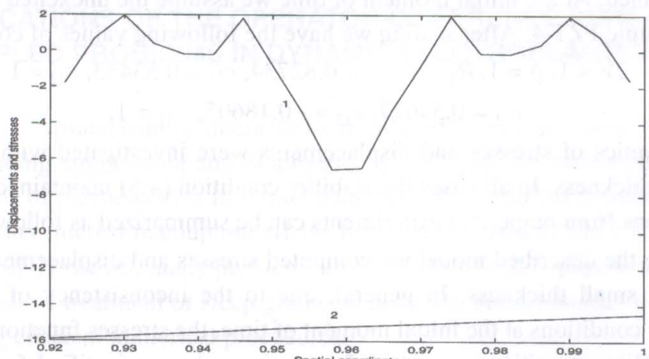

FIGURE 2 Radical stresses (curve 1) and displacements (curve 2) at $t = 10$ (radial preliminary polarisation ; thickness $l = 0.08$)

* Here l denotes scaled thickness of the cylinder.

† For example, for the cylinder with $l = 0.13$ an increase is bigger by a factor 8-9.

• Investigating the dependency of a discontinuous step-function of stresses in time at the middle section $(R_0 + R_1)/2$, we also observe that with decreasing cylinder thickness, the amplitude of thickness oscillation increases (Fig. 3, 4). In a qualitative manner, this effect is expected from uncoupled electroelasticity.

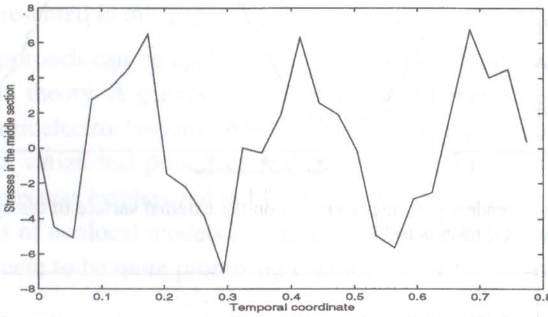

FIGURE 3 Time-dependency of stresses in the middle section of the cylinder (radial preliminary polarization; thickness $l = 0.13$)

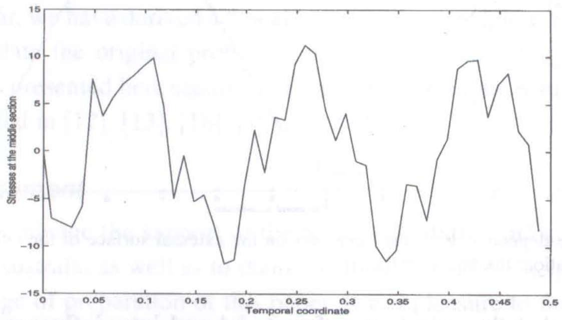

FIGURE 4 Time-dependency of stresses in the middle section of the cylinder (radial preliminary polarization; thickness $l = 0.08$)

- We also analyse the dynamics of displacements in time on the external surface of cylinders (Fig. 5, 6). The underlying processes are important in the design of technical devices which include a cylindrical vibrator*. Figures are presented for the interval of the scaled time $0 \leq 0 \leq 10$. As for stresses, with decreasing thickness, we observe an increase in amplitude of oscillations. As we pointed out, an increase of amplitude for cylinders preliminary polarized radially is much greater [17, 22]. Figures 5 and 6 show a large increase in amplitudes of oscillations for thin cylinders preliminarily polarized radially.

* Hydro-acoustics applications, including cylindrical acoustic vibrators, give typical examples of this type.

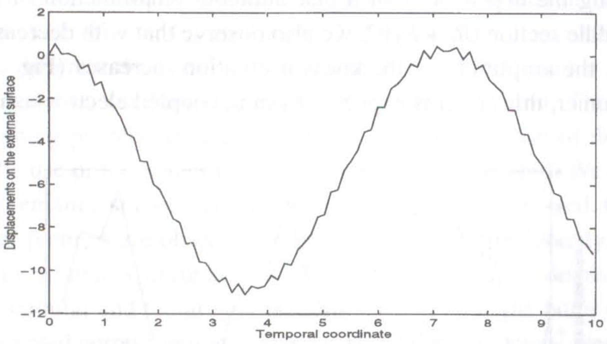

FIGURE 5 Time-dependency of displacements on the external surface of the cylinder (radial preliminary polarization; thickness l=0.13)

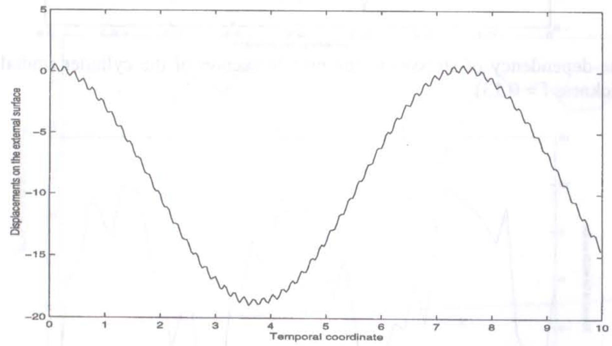

FIGURE 6 Time-dependency of displacements on the external surface of the cylinder (radial preliminary polarization; thickness l = 0.08)

We conclude that the anisotropy of material and inter-influence of elastic and electric fields in the non-stationary case essentially influence the characteristics of designed technical devices. A possibility of discontinuities and rapid changes of computed functions require the development of appropriate mathematical tools in justifications of underlying numerical procedures.

7. CONCLUSIONS AND FUTURE DIRECTIONS

Mathematical modelling in coupled field theory requires approaches which can be applied even if a solution of the problem does not possess an excessive smoothness imposed as an *a-priori* assumption. In this paper we have developed such an approach with respect to problems arising from nonstationary electroelasticity. We explicitly derived the stability condition for the operator-difference scheme applied to the numerical solution of the problem. We also proved convergence of such a solution to a generalized solution of the original problem. Depending on the smoothness of the latter we obtained a scale of accuracy estimations. Such scales give important *a-priori* characteristics of the *computational efficiency* of underlying numerical procedures.

Finally, we would like to mention two directions for possible development of the technique presented in this paper.

- A similar approach can be applied to more general mathematical models of coupled field theory. A generalization might be obtained for dynamic problems of thermoelectroelasticity. Along this line the importance of a connection between variational principles and computational models for new areas of applications was emphasized in [31, 32].
  Applications of nonlocal models in semiconductor device theory and climate modelling seem to be quite promising candidates for future investigations in this field.
- Recently, we attempted to approach some problems in nonsmooth (including stochastic) optimal control theory on the basis of using Steklov's operators. In particular, we have derived a "local" optimality principle which allowed us to reformulate the original problems in such a way that the application of some ideas presented here seems to be encouraging. Some results in this field are published in [12], [13], [18], [19].

Acknowledgement

I wish to acknowledge the support of the School of Mathematics at the University of South Australia as well as to thank Dr John Boland for helpful assistance at the final stage of preparation of this paper. It is a pleasure to thank an anonymous referees for their careful reading and helpful comments.

References

[1] Berlincourt, D.A., Curran, D.R., and Jaffe, H. Piezoelectric and Piezomagnetic Materials and Their Function in Transducers. in *"Physical Acoustics", Ed. W.P. Mason, Vol. IA*, New York and London: Academic Press, 1964, 204-236.

[2] Bramble, J.H. and Hilbert, S.R. Estimation of Linear Functionals on Sobolev Spaces with Application to Fourier Transforms and Spline Interpolation. *SIAM J. Num. Anal.* 7(1970), 113-124.

[3] Carthy M.F. and Tiersten H.F. The Growth of Wave Discontinuities in Piezoelectric Semiconductors. *J. Math. Phys.* V.20 (1979), No.12, 2682-2691.

[4] Daher, N. A continuum Energy Approach of Deformable Piezoelectric and Ferroelectric Semiconductors. *Proc. of 1994 IEEE Ultrasonics Symposium, France, Vol. 2*, (1994), 701-704.

[5] Djuraev, I.N., and Moskalkov, M.N. Convergence of Difference Scheme Solution to the Generalized Solution of the Wave Equation from $W_2^2(Q_T)$. *Differential Equations*, 21(1985), 2145-2152.

[6] Fujita, H. and Suzuki, T. Evolution Problems, *in "Handbook of Numerical Analysis", ed. P.G. Ciarlet & J.L. Lions*, North-Holland, Amsterdam, New York, Oxford, Tokyo, 1991, 789-923.

[7] Fukada E. Piezoelectric Properties of Biological Polymers. *Quart. Rev. Biophys.*, V.16(1983), No.l, 59-87.

[8] Gizdulich, P., Aschero, G. and Romano, S.M. On the electromechanical effect in bones. *Proc. of the 16th Intern. Conf. of the IEEE Engineering in Medicine and Biology*, Vol.2(1994), 744-745.

[9] Jou, D., Casa-Vasquez, J and Lebon, G. *Extended Irreversible Thermodynamics*. Springer-Verlag, 1993.

[10] Kulkarni, G. and Hanagud, S. Modelling Issues in the Vibration Control with Piezoceramic Actuators. *In Smart Structures and Materials*, MD-Vol.24/AMD-Vol 123, ASME, New York, 1991, 7-15.

[11] Ladyzhenskaya, O.A. *Boundary-value problems in Mathematical Physics*. Springer-Verlag, 1985.

[12] Melnik, V.N. Nonconservation Law Equation in Mathematical Modelling: Aspects of Approximation. *Proc. of the International Conf. AEMC'96, Sydney*, 423-430, Eds. W.Y.D. Yuen, P. Broadbridge and J.M. Steiner.

[13] Melnik, V.N. Generalised Solutions of Hamilton-Jacobi-Bellman Equation from Sobolev's Classes. *ANZIAM Conference, Masterton, New Zealand*, 1996.

[14] Melnik, V.N. Existence and Uniqueness Theorems of the Generalized Solution for a Class of Non-Stationary Problems of Coupled Electroelasticity. *S. Mathematics (Iz.VUZ)*, 35(1991), 24-32; *Allerton Press, ISSN 0197-7156*, 1991, 23-30.

[15] Melnik, V.N. Accuracy of Difference Schemes in Coupled Nonstationary Thermoelasticity for Problems in Stresses Formulation. *Mathematical Methods and Physico-Mechanical Fields; ISSN 0130-9420*, 34(1991), 95-99.

[16] Melnik, V.N. Semi-Implicit Difference Schemes with FCT for Quasihydrodynamic Model of Semiconductor Devices. *Engineering Simulation; Gordon and Breach*, 12(1995), 856-865.

[17] Melnik, R.V.N. The Stability Condition and Energy Estimate for Nonstationary Problems of Coupled Electroelasticity. *Mathematics and Mechanics of Solids*, 2(1997), 153-180.

[18] Melnik R.V.N. On Consistent Regularities of Control and Value Functions. *Numer. Func. Anal. and Optimiz.*, 18(384) (1997), 401-426.

[19] Melnik R.V.N. Dynamic System Evolution and Markov Chain Approximation. *Discrete Dynamics in Nature and Society*, to appear (1998).

[20] Melnik R.V.N. Mathematical Models for Climate as a Link Between Coupled Physical Processes and Computational Decoupling. *Engineering Simulation*, to appear (1998).

[21] Melnik, V.N. and Moskalkov, M.N. On the Coupled Non-Stationary Electroelastic Oscillations of a Piezoceramic Cylinder with Radial Polarization. *Computational Mathematics and Mathematical Physics*; 28(1988); *Pergamon Press*, 1990, 109-110.

[22] Melnik, V.N. and Moskalkov, M.N. Difference Schemes for and Analysis of Approximate Solutions of Two-Dimensional Nonstationary Problems in Coupled Electroelasticity. *Differential Equations*, 27(1991), 1220-1229; *Plenum Publishing,C/B, New York*, 1992, 860-867.

[23] Mindlin, R. Equations of High Frequency Vibrations of Thermopiezoelectric Crystal Plates. *Int.J.Solids and Structure*, 10(1974), 625-637.

[24] Monin, A.S. Weather Forecasting as a Problem in Physics. *The Massachusetts Institute of Technology, 1972*.

[25] Moskalkov, M.N. On Accuracy of Difference Schemes for Approximation of Wave Equation with Piecewise Smooth Solutions. *Computational Mathematics and Mathematical Physics*, 14(1974), 390-401.

[26] Muller, I., Ruggeri, T. *Extended Thermodynamics*, Springer-Verlag, 1993.

[27] Nowacki, W. *Dynamic Problems of Thermoelasticity*. Noordhoff International Publishing, Leyden, The Netherland and PWN, 1975.

[28] Nowacki, W. *Electromagnetic Effects in Solids*, Moscow, Mir Publishers, 1986.

[29] Samarskiy, A., Lazarov, R. and Makarov, V. *Difference Schemes for Differential Equations with Generalized Solutions*. Moscow, VS Publishers, 1987.

[30] Strang, G. and Fix, G. *An Analysis of the Finite Element Method*. Prentice-Hall, 1973.

[31] Yang, J.S. and Batra, R.C. Mixed Variational Principles in Non-Linear Electroelasticity. *Int. J. Non-Linear Mechanics*, 30(1995), no.5, 719-725.

[32] Yu Y.Y. Some Recent Advances in Linear and Nonlinear Dynamical Modeling of Elastic and Piezoelectric Plates. *J. of Intelligent Material Systems and Structures*, 6(1995), 237-254.

[33] Zeidler E. *Applied Functional Analysis. Applications to Mathematical Physics*, Springer Verlag, 1995.