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Mixed electroelastic waves and CFL stability conditions in computational piezoelectricity

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Abstract

Modelling vibrational dynamics of piezoelectric materials and structures undergoing mechanical and/or electric loadings is a challenging interdisciplinary area at the interface of applied mathematics, materials science and engineering. When numerical methods are applied to such problems stability criteria play a fundamental role in the success of the entire modelling exercise. The main result of this paper is a complete and rigorous derivation of the generalisation of the classical Courant–Friedrichs–Lewy stability condition to the case of dynamic piezoelectricity for variational difference schemes.

Keywords: Coupled dynamics; Discrete conservation laws; Courant–Friedrichs–Lewy condition; Mixed electroelastic waves

1. Introduction

Many problems in science and engineering require dealing with coupling between two or more different physical fields. Such problems are often termed as multiphysics problems since they require a unification of several physical theories considered separately before [33]. One class of such problems is in the focus of the present paper where we consider coupling between mechanical and electric fields in dynamic piezoelectricity.

The continuous growth of applications of piezoelectric materials in a variety of fields requires more close attention to the development of effective methods for the analysis of vibrations in piezoelectric-based devices and structures. In many cases such analysis has to be conducted for dynamic rather than stationary problems for which the coupling coefficient (a dimensionless measurement of the efficiency in energy conversion) is fairy large. Mathematically, we have to solve a coupled nonstationary problem

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that is described by a mixed-type system of partial differential equations with appropriate boundary and initial conditions. Except for quite special situations the analytical technique has a limited success in the solution of these types of problems. As a result, numerical methods become the most natural and most effective approach for the solution of these problems [20,22,24,30,28,31,29].

Along with the traditional applications of piezoelectrics, new and emerging technologies based on smart structures and piezoelectric biopolymers have reemphasised the importance of piezoelectrics in the aerospace industry, civil engineering, microprocessor design, aeroelastic control, oceanography, biophysics, medical imaging, consumer electronics and in many other areas of human endeavour [2, 7,36,37,40,47,4,16,14]. Piezoelectricity is a fundamental physical phenomenon that occurs in a wide range of applications, from human biological tissues to man-made artificial atoms known as quantum dots [34,39].

Starting with the general formulation of coupled dynamic thermopiezoelectricity, the major emphasis in this paper is paid to the analysis of variational schemes for the problem of electromechanical vibrations of piezoelectric systems of cylindrical shapes with radial preliminary polarisation. In this case one observes a strong coupling between mechanical and electric fields which can be utilized in a range of practical applications. However, from a numerical analysis point of view this is the most difficult case of coupling. While simpler cases of axial preliminary polarisation and weak coupling have already been studied in the literature (e.g., [29]), the case of radial preliminary polarisation has not been analysed in detail in the context of stability of associated numerical approximations. It is the main purpose of this paper to derive rigorously stability conditions for the second order variational numerical approximation obtained from the energy balance formulation.

The paper is organised as follows. In Section 2 we give the general three-dimensional governing equations of coupled problems of dynamic thermopiezoelectricity and, subsequently, electroelasticity. We reduce them to two-dimensional models for cylindrical piezoelectric shells in the case of axial symmetry. We discuss constitutive relationships for different preliminary polarisations. In Section 3 we specify initial and boundary conditions and review main modelling techniques for coupled problems of dynamic piezoelectricity. Section 4 describes the methodology for constructing second order variational schemes based on energy balance equations. Using such schemes as a basis, in Section 5 we derive rigorously the discrete analogue of the conservation law for piezoelectric-based coupled electromechanical systems of interest. In Section 6 we obtain a priori energy estimate for the system under consideration. As a consequence of nonnegativity of the discrete energy operator, in Section 7 we derive constructively a generalisation of the classical Courant–Friedrichs–Lewy stability condition to the case of mixed electromechanical waves. Conclusions are given in Section 8.

2. Mathematical models coupling mechanical and electric fields

In what follows, we assume the existence of three thermodynamic quantities as independent variables, so that the internal energy function of the electroelastic solid can be represented as a function of these variables:

$$\psi = \psi(P, \boldsymbol{\varepsilon}, \theta), \quad \theta > 0, \quad \inf_{(x,t)} \theta = 0, \tag{2.1}$$

where $\boldsymbol{\varepsilon}$ is the strain, θ is the temperature of the system, and P is the electric polarisation. Relationship (2.1) provides a foundation upon which coupling effects between mechanical, electric, and thermal fields are considered. These effects, which may be nonlinear in the general case, have several sources. We start our consideration from the definition of the finite strain which is a quadratic function in the displacement gradient (e.g., [26])

$$\varepsilon_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right], \quad i, j = 1, 2, 3, \tag{2.2}$$

where $\mathbf{x} = (x_1, x_2, x_3)^{\mathrm{T}}$ is the position vector in the Cartesian system of coordinates, and u_i are the components of the elastic displacement. Formula (2.2) is a standard (Lagrange) strain representation where the Einstein summation is used for repeated indices in the same term (in our case "k"). Geometric nonlinearities could be induced by finite strain tensors (accounting for nonlinear contributions in ∇u), while physical (thermomechanical) nonlinearities have their source in the general case from a nonlinear dependency of the stress σ_{ij} on ε_{ij}. In addition to coupling effects resulted from the stress-strain constitutive relation, we need to consider the connection between the electric polarisation and the electric field. In the most general case, this is another source of possible nonlinearities in the system, so that we have

$$\mathbf{E} = \mathbf{E}(\mathbf{P}), \tag{2.3}$$

where $\mathbf{E}$ is a (possibly nonlinear) function of $\mathbf{P}$. Note that the electric field is sometimes treated in the literature as an intensive variable (e.g., [19]), in which case $\mathbf{P}$ becomes an extensive variable. In both cases, however, the electric flux density (the electric displacement) should be treated as an extensive variable. Finally, the coupling between temperature and the entropy density is realised by postulating the internal energy function (2.1) in the following form

$$\psi(\boldsymbol{\varepsilon}, P, \theta) := \Psi(\boldsymbol{\varepsilon}, P, \eta) + \theta\eta, \tag{2.4}$$

where Ψ is a Helmholtz–Gibbs functional invariant under a time shift. Hence, according to the Legendre transformation, we can write down the constitutive equation of our electromechanical system as

$$\theta = \frac{\partial \Psi}{\partial \eta}, \qquad \boldsymbol{\sigma} = \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}}, \qquad \mathbf{D} = -\frac{\partial \Psi}{\partial \mathbf{P}}, \tag{2.5}$$

where $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ are strain and stress tensors, respectively. Thermal effects have been analysed in [33] and in what follows we focus ourselves on the coupling between mechanical and electric fields only. Based on the above relationships, and taking into account that for solids we have

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}, \tag{2.6}$$

where ε_0 is the dielectric permittivity of vacuum, the constitutive relationships of the linear piezoelectricity are obtained in the following form

$$\boldsymbol{\sigma} = \mathbf{c}\boldsymbol{\varepsilon} - \mathbf{e}\mathbf{E}, \qquad \mathbf{D} = \boldsymbol{\varepsilon}\mathbf{E} + \mathbf{e}^{\mathrm{T}}\boldsymbol{\varepsilon}, \tag{2.7}$$

where for mm2 materials the elastic, piezoelastic, and dielectric permittivity tensors can be presented as

$$\mathbf{c} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{pmatrix}, \qquad \mathbf{e} = \begin{pmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{32} \\ 0 & 0 & e_{33} \\ 0 & e_{24} & 0 \\ e_{15} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\boldsymbol{\varepsilon} = \operatorname{diag} \varepsilon_{kk}, \quad k = 1, 2, 3. \tag{2.8}$$

The relationships (2.7) are written for Cartesian coordinates. These constitutive laws couple together mechanical and electric field in an electromechanical system, dynamics of which is described by the equation of motion and the Maxwell equation considered here in the dielectric approximation

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \nabla \cdot \boldsymbol{\sigma} + \mathbf{F}, \qquad \operatorname{div} \mathbf{D} = \mathbf{G}. \tag{2.9}$$

In (2.9) ρ is the density of piezoelectric material under consideration, and **G** and **F** are electric (volume charge) and body forces on the piezoelectric (if any). In what follows, the electric field is assumed to be electrostatic, so that $E_i = -\partial\varphi/\partial x_i,\ i = 1, 2, 3$.

Next, we simplify the above equations to the case of axial symmetry and consider effects of preliminary polarisation on the resulting constitutive equations. Our major interest is in the analysis of piezoelectric elements of cylindrical shape, in particular hollow piezoelectric cylinders. This interest is inspired by the existing and potential applications of miniaturised piezoceramic transducers of hollow cylindrical shape in acoustics, biomedical imaging, sensor and hydrophone design, consumer electronics and other areas [13,35,12].

We consider the process of propagation of electroelastic waves in hollow finite-length piezoceramic cylinders with *radial* preliminary polarisation, the case which is much more difficult to analyse compared to circular or axial preliminary polarisations (see [28,31,3,19] and references therein). Mathematical models for the description of such a process include

• the coupled system of equations of motion and the Maxwell equation for piezoelectrics (in the acoustic range of frequencies, the latter is the forced electrostatic equation of dielectrics)

$$\begin{cases} \rho \frac{\partial^2 u_r}{\partial t^2} = \frac{\partial \sigma_r}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} + f_1, \\ \rho \frac{\partial^2 u_z}{\partial t^2} = \frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\sigma_{rz}}{r} + f_2, \\ \frac{1}{r} \frac{\partial}{\partial r}(r D_r) + \frac{\partial D_z}{\partial z} = f_3, \end{cases} \tag{2.10}$$

where all notations are identical to those explained before with indices r, θ, and z, reflecting that the model (2.10) has been written in cylindrical coordinates.

We aim at the numerical analysis of a procedure for the complete transient solution of model (2.10). It should be noted that there is a substantial body of literature where the solution to (2.10) is sought in the harmonic regime which simplifies the problem substantially. Such an analysis may be appropriate for some special cases only [21], e.g., in some piezoelectric transducer designs. The mathematical foundations of this approach applied to piezoelectric solids have been known for quite some time [44]. In contrast, we are interested in the complete dynamic analysis of model (2.10). This interest has been stimulated by new applications of piezoelectric in smart materials and structures technology [21,46]. In many such new applications the methodologies based on reduction of the problem to harmonic vibrations only is no longer appropriate and the complete nonstationary problem should be solved.

The material of choice in most of these new applications is piezoceramics. Electromechanical properties of piezoceramics are the result of transverse-isotropic solids with the symmetry axis coinciding with the direction of preliminary polarisation. It is technologically possible to prepare piezoceramic cylindrical elements of different polarisations, in particular axial, circular, and radial. It should be emphasised that the piezoeffect depends substantially on the direction of preliminary polarisation. Hence, a special care should be taken when writing the constitutive equations for different types of preliminary polarisations. Indeed, in the general case coefficients in the constitutive relations are coordinatedependent. Hence, it is not always possible to derive constitutive equations with constant coefficients, in particular when curvilinear systems are analysed [34]. Fortunately, having constitutive equations in Cartesian coordinates in the case where OZ coincides with the direction of preliminary polarisation, it is easy to deduce their counterparts for cylindrical bodies with *axial preliminary polarisation* in the form (2.7) with the following transformation of coordinates:

$$(x_1, x_2, x_3) \to (r, \theta, z). \tag{2.11}$$

This case has received the most thorough investigation in the literature and it is not of interest in the present paper. We remark that an essential assumption in (2.7)–(2.8) is that of a homogeneous field of preliminary polarisation, so that the direction of the field does not change inside the body. As soon as the external field of preliminary polarisation becomes a function of spatial coordinates, the task of writing the constitutive equations with constant coefficients is possible to fulfill only if the field has some symmetry properties. Having made this remark, we note further that in cylindrical system of coordinates it is still possible to write the constitutive equations with constant coefficients not only for the axial preliminary polarisation, but also for circular and radial polarisation cases. In particular, the transformation

$$(x_1, x_2, x_3) \to (z, r, \theta), \tag{2.12}$$

will reduce the constitutive equation (2.7) to the case of circular preliminary polarisation (a weakly coupled case). The most interesting but at the same time the most difficult case is the case of *radial preliminary polarisation.* In this case we have a strong coupling between mechanical and electric case, and the constitutive equations can be written as follows

$$\begin{cases} \sigma_r = c_{33}\varepsilon_r + c_{13}(\varepsilon_\theta + \varepsilon_z) - e_{33}E_r, & \sigma_\theta = c_{13}\varepsilon_r + c_{11}\varepsilon_\theta + c_{12}\varepsilon_z - e_{13}E_r, \\ \sigma_z = c_{13}\varepsilon_r + c_{12}\varepsilon_\theta + c_{11}\varepsilon_z - e_{13}E_r, & \sigma_{rz} = c_{44}\varepsilon_{rz} - e_{15}E_z, \\ D_r = e_{33}\varepsilon_r + e_{13}(\varepsilon_\theta + \varepsilon_z) + \varepsilon_{33}E_r, & D_z = 2e_{15}\varepsilon_{rz} + \varepsilon_{11}E_z. \end{cases} \tag{2.13}$$

This case has not been studied in the literature with the vigour it deserves, and it is on the analysis of this case that we put our main efforts in the remainder of this paper.

3. Well-posedness of the model and methodologies for its numerical solution

We need to complete the model formulation by supplementing (2.10)–(2.13) with appropriate boundary and initial conditions. As we have already mentioned in Section 2, we use the electrostatic approximation, so that the function of electrostatic potential, φ, is introduced by the formulae

$$E_r = -\frac{\partial \varphi}{\partial r}, \qquad E_z = -\frac{\partial \varphi}{\partial z}, \tag{3.1}$$

and we assume the Cauchy-type relationship between displacements and deformations

$$\varepsilon_r = \frac{\partial u_r}{\partial r}, \qquad \varepsilon_\theta = \frac{u_r}{r}, \qquad \varepsilon_z = \frac{\partial u_z}{\partial z}, \qquad \varepsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right). \tag{3.2}$$

While the formulation of initial conditions for models of electroelasticity is a simple task, this cannot be said about the boundary conditions. Their formulation require additional assumptions since supply of energy to the piezoelectric solid is usually carried out by covering a part of the whole surface of the body by electrodes. Indeed, the formulation of electrical boundary conditions depends on the character of electric loading and on the location of electrodes on the body surface. Typical assumptions require that such electrodes are ideal conductors and they are sufficiently thin and light [38]. Again, depending on the type of a specific design we may have the values of the potential given on the boundary and/or electrical displacements (based on the fact that dielectric permittivities of the surroundings are often hundred times smaller to that of piezoceramic). Note that the latter condition immediately complicates the problem further since the coupling between mechanical and electrical fields on the boundary becomes an important factor in this case. We include this case into our consideration. In what follows we assume the following initial and boundary conditions

- initial

$$\begin{cases} u_r(r,z,0) = u_r^{(0)}(r,z), & \frac{\partial u_r(r,z,0)}{\partial t} = u_r^{(1)}(r,z), \\ u_z(r,z,0) = u_z^{(0)}(r,z), & \frac{\partial u_z(r,z,0)}{\partial t} = u_z^{(1)}(r,z), \end{cases} \tag{3.3}$$

- and boundary conditions

$$\begin{cases} \sigma_r(R_i,z,t) = p_r^{(i)}(z,t), & \sigma_z(r,Z_i,t) = p_z^{(i)}(r,t), \\ \sigma_{rz}(R_i,z,t) = p_{zt}^{(i)}(z,t), & \sigma_{rz}(r,Z_i,t) = p_{rt}^{(i)}(r,t), \\ \varphi(R_i,z,t) = 0, & D_z(r,Z_i,t) = 0, \quad i = 0,1, \end{cases} \tag{3.4}$$

where $u_r^{(i)}, u_z^{(i)}, p_r^{(i)}, p_z^{(i)}, p_{zt}^{(i)}, p_{rt}^{(i)}$ $(i = 0, 1)$ are given functions. As it is seen, we assume that the lateral surfaces of the cylinder are covered by infinitely thin short circuiting electrodes, and that the dielectric permittivity of the surrounding media is much less than the dielectric permittivity of ceramics. This is true for both vacuum and air. Note further that the homogeneity of the electrical conditions in (3.4) does not restrict generality of the model since problems with nonhomogeneous conditions can be reduced to the homogeneous case by the known procedure [28].

The model (2.10), (2.13), (3.1)–(3.4) is considered in the space–time region $\overline{Q}_T = \overline{G} \times \bar{I}$, where

$$\overline{G} = \big\{(r,z)\colon\ R_0 \leqslant r \leqslant R_1,\ Z_0 \leqslant z \leqslant Z_1\big\}, \qquad \bar{I} = \{t\colon\ 0 \leqslant t \leqslant T\},$$

and the condition of nonnegativity of the potential energy of deformation,

$$\delta_1 \sum_{i=1}^{4} \xi_i^2 \leqslant c_{33}\xi_1^2 + c_{11}\big(\xi_2^2 + \xi_3^2\big) + 2c_{13}(\xi_2\xi_1 + \xi_3\xi_1) + 2c_{12}\xi_3\xi_2 + 2c_{44}\xi_4^2, \quad \delta_1 > 0, \tag{3.5}$$

is assumed.

Although foundations of modelling piezoelectric phenomena was laid by W. Voigt with fundamental contributions made to the field by R.D. Mindlin, N.A. Shulga, G.A. Maugin, and many other researchers (e.g., [44,19,26] and references therein), it was not until late 1980s that well-posedness of the complete coupled system of electroelasticity in the strong coupling case of radial preliminary polarisation has been proven and regularity of generalised solutions have been investigated in detail [27,31]. It was shown that under appropriate assumptions, model (2.10), (2.13), (3.1)–(3.4) has a unique solution. The analysis of this model is far from trivial. Indeed, the strong coupling between electric and elastic fields for the model we are interested in manifests itself not only through the system (2.10) but also through boundary conditions for stresses. This fact complicates mathematical analysis of the model and the development of efficient numerical methodologies for the description of vibrational characteristics of piezoelectric structures.

With a large range of applications of piezoelectrics, there has been extensive work on the development of numerical methodologies for the solution of piezoelectricity problems. Finite element, finite difference, and boundary element methodologies have all been applied to such problems in different contexts and for different application areas. However, most studies performed so far are available for the steady-state or harmonic-oscillation cases only. Indeed, boundary element methodologies applied to problems of piezoelectricity are typically limited to the steady-state applications (e.g., [48,9,11,8,23]). Most results available for finite element methodologies are either for the steady-state case or for harmonic oscillations (e.g., [21,1]). A similar situation is observed for the finite difference methodologies (e.g., [18]).

It should also be noted that there has been little work on systematic studies of models for piezoelectric cylindrical shells with radial preliminary polarisation, in particular in the context of stability of numerical approximations. The stationary problem for studying piezoelectric bodies with radial polarisation was considered in [17,5] under quite simplified assumptions. Approximations were based on the Fourier series expansions which, by using simplified boundary conditions, allowed the authors to apply a separation technique and to reduce the problem to a system of ODEs with respect to one of the coordinates (e.g., radial as in [17]). However, the idea of such expansions, traditionally applied in the analysis of *harmonic* oscillations of piezoelectric bodies [44], cannot be extended in a straightforward manner to the general nonstationary case. Indeed, if such methodologies are to be generalised to nonstationary problems (e.g., by applying the method of lines), a detailed stability analysis is required.

4. Methodology for constructing conservative difference schemes for coupled problems of piezoelectricity

It is well known that in practice, solutions of dynamic problems of electroelasticity do not have to be smooth. They might exhibit steep gradients or even discontinuities. In order to treat these situations numerical methods for the solution of coupled electroelasticity problems were constructed directly from the definition of generalised solutions using the energy balance equation [28,29,31]. The success of the implementation of energy-based methodologies such as that applied in this paper rests on certain invariance properties of the model, so that Noether's theorem can be applied to derive conservation law equations. Then, a key idea is to preserve those conservative properties when constructing numerical approximations. Note that by using this idea in the general nonlinear case one can develop a methodology for constructing difference schemes that inherit energy conservation properties from the original nonlinear PDE-based model, or to inherit energy dissipation properties when it is appropriate (e.g., [15,25]).

When dealing with model (2.10), (2.13), (3.1)–(3.4) we first derive the following energy balance equation by using the energy method. This is done in a standard manner (e.g., [31,15]) and without compromising clarity of further discussion we present here the final result:

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = \iint_{\Omega} r\left[\frac{\partial D_r}{\partial t}E_r + \frac{\partial D_z}{\partial t}E_z\right]\mathrm{d}\Omega + \int_{R_0}^{R_1} r\left[\sigma_{rz}\frac{\partial u_r}{\partial t} + \sigma_z\frac{\partial u_z}{\partial t}\right]\mathrm{d}r\Big|_{Z_0}^{Z_1}$$
$$+ \int_{Z_0}^{Z_1} r\left[\sigma_r\frac{\partial u_r}{\partial t} + \sigma_{rz}\frac{\partial u_z}{\partial t}\right]\mathrm{d}z\Big|_{R_0}^{R_1} + \iint_{\Omega} r\left[f_1\frac{\partial u_r}{\partial t} + f_2\frac{\partial u_z}{\partial t}\right]\mathrm{d}\Omega, \tag{4.1}$$

where $\mathcal{E}$ is the inner energy of the electromechanical system described by the model (2.10), (2.13), (3.1)–(3.4). It consists of the three coupled parts, kinetic energy, the energy of elastic deformation, and the energy of electric field, namely

$$\begin{aligned}\mathcal{E} = {}& \frac{\rho}{2}\iint_{\Omega} r\left\{\left(\frac{\partial u_r}{\partial t}\right)^2\left(\frac{\partial u_z}{\partial t}\right)^2\right\}\mathrm{d}\Omega \\ &+ \frac{1}{2}\iint_{\Omega} r\left\{c_{33}\varepsilon_r^2 + c_{11}\left(\varepsilon_\theta^2+\varepsilon_z^2\right) + 2c_{13}(\varepsilon_\theta\varepsilon_r+\varepsilon_z\varepsilon_r) + 2c_{12}\varepsilon_z\varepsilon_\theta + 2c_{44}\varepsilon_{rz}^2\right\}\mathrm{d}\Omega \\ &+ \frac{\varepsilon_{33}}{2}\iint_{\Omega} rE_r^2\,\mathrm{d}\Omega + \frac{\varepsilon_{11}}{2}\iint_{\Omega} rE_z^2\,\mathrm{d}\Omega. \end{aligned} \tag{4.2}$$

To proceed further we note that the functional (4.2) is bounded, so that, as shown in [31], the following theorem is true.

Theorem 4.1. *If the condition* (3.5) *is fulfilled, then the solution of the problem* (2.10)–(3.2) *satisfies the following energy bound*

$$\begin{aligned}\mathcal{E}(t_1) \leqslant M\Bigg\{ & \rho\iint_{\Omega} r\left[\left(u_r^{(1)}\right)^2 + \left(u_z^{(1)}\right)^2\right]\mathrm{d}\Omega \\ &+ \iint_{\Omega} r\left[c_{33}\varepsilon_r^2 + c_{11}\left(\varepsilon_\theta^2+\varepsilon_z^2\right) + 2c_{13}(\varepsilon_\theta+\varepsilon_z)\varepsilon_r + 2c_{12}\varepsilon_z\varepsilon_\theta + 2c_{44}\varepsilon_{rz}^2\right]\Big|_{t=0}\mathrm{d}\Omega \\ &+ \int_{R_0}^{R_1}\left[\sum_{i,j=0}^{1}\left(\left|p_{rt}^{(i)}(r,t_j)\right|^2 + \left|p_z^{(i)}(r,t_j)\right|^2\right)\right]\mathrm{d}r \\ &+ \int_{Z_0}^{Z_1}\left[\sum_{i,j=0}^{1}\left(\left|p_r^{(i)}(z,t_j)\right|^2 + \left|p_{zt}^{(i)}(z,t_j)\right|^2\right)\right]\mathrm{d}z \\ &+ \int_0^{t_1}\int_{R_0}^{R_1}\sum_{i=0}^{1}\left[\left(\frac{\partial p_{rt}^{(i)}}{\partial t}\right)^2 + \left(\frac{\partial p_z^{(i)}}{\partial t}\right)^2\right]\mathrm{d}r\,\mathrm{d}t \\ &+ \int_0^{t_1}\int_{Z_0}^{Z_1}\sum_{i=0}^{1}\left[\left(\frac{\partial p_r^{(i)}}{\partial t}\right)^2 + \left(\frac{\partial p_{zt}^{(i)}}{\partial t}\right)^2\right]\mathrm{d}z\,\mathrm{d}t + \iint_{\Omega} r\lambda^2|_{t=0}\,\mathrm{d}\Omega \\ &+ \int_0^{t_1}\iint_{\Omega} r\left(f_1^2+f_2^2\right)\mathrm{d}\Omega\,\mathrm{d}t\Bigg\}, \end{aligned} \tag{4.3}$$

where $\mathcal{E}(t)$ is the total energy of the electro-mechanical system at time t, and λ is defined by following relationships

$$\frac{\partial\lambda}{\partial r}+\frac{\partial\lambda}{\partial z}=f_3,\qquad \lambda(R_0,z,t)=\lambda(r,Z_0,t)=0. \tag{4.4}$$

This allows us to obtain the variational difference scheme which is the subject of our analysis in the subsequent sections. In particular, we apply the method of approximating a quadratic functional for spatial approximations, and then perform temporal discretisation which allows us to arrive at the following scheme specified below.

We cover the space–time region $\overline{Q}_T$ with the difference grid $\bar\omega_{h\tau}=\bar\omega_h\times\bar\omega_\tau$, where $\bar\omega_h=\bar\omega_{h_1}\times\bar\omega_{h_2}$ is the spatial grid with $\bar\omega_{h_1}=\{r_i\colon\ r_i=R_0+ih_1,\ i=0,1,\dots,N,\ h_1=(R_1-R_0)/N\}$, $\bar\omega_{h_2}=\{z_j\colon\ z_j=Z_0+jh_2,\ j=0,1,\dots,M,\ h_2=(Z_1-Z_0)/M\}$, and $\bar\omega_\tau=\{t_k\colon\ t_k=k\tau,\ \tau=T/L,\ k=0,1,\dots,L\}$ is the temporal grid. We use the following finite difference notation

$$\begin{aligned}\delta_z^+v&:=\frac{v(x+h)-v(x)}{h},\\ \delta_x^-v&:=\frac{v(x)-v(x-h)}{h},\\ \delta_x^1v&:=\frac{v(x+h)+v(x-h)}{2},\end{aligned} \tag{4.5}$$

which denote forward, backward, and central difference derivatives, respectively [45,43,15]. The second central difference derivative is

$$\delta_x^2v:=\frac{\delta_x^+-\delta_x^-}{h}=\frac{v(x+h)-2v(x)+v(x-h)}{h^2}. \tag{4.6}$$

Further, we denote by $\gamma_1=\{(r,z)\colon\ R_0<r<R_1,\ z=Z_0\}$, $\gamma_2=\{(r,z)\colon\ R_0<r<R_1,\ z=Z_1\}$, $\gamma_3=\{(r,z)\colon\ r=R_0,\ Z_0<z<Z_1\}$, $\gamma_4=\{(r,z)\colon\ r=R_1,\ Z_0<z<Z_1\}$, the boundaries of the spatial region G, and by $\gamma_{13}=\{r=R_0,\ z=Z_0\}$, $\gamma_{23}=\{r=R_0,\ z=Z_1\}$, $\gamma_{24}=\{r=R_1,\ z=Z_1\}$, $\gamma_{14}=\{r=R_1,\ z=Z_0\}$ corner points of this region. Finally, by $\bar r=r-h_1/2$, $\bar z=z-h_2/2$ we denote "flux" nodes where values of deformations and stresses will be determined.

Note that the constitutive relationships can be approximated in a straightforward manner:

$$\begin{cases}\bar\sigma_r=c_{33}\bar\varepsilon_r+c_{33}(\bar\varepsilon_\theta+\bar\varepsilon_z)-e_{33}\overline{E}_r, & \bar\sigma_\theta=c_{13}\bar\varepsilon_r+c_{11}\bar\varepsilon_\theta+c_{12}\bar\varepsilon_z-e_{13}\overline{E}_r,\\ \bar\sigma_z=c_{13}\bar\varepsilon_r+c_{12}\bar\varepsilon_\theta+c_{11}\bar\varepsilon_z-e_{13}\overline{E}_r, & \bar\sigma_{rz}=c_{44}\bar\varepsilon_{rz}-e_{15}\overline{E}_z,\\ \overline{D}_r=\overline{E}_r+e_{33}\bar\varepsilon_r+e_{13}(\bar\varepsilon_\theta+\bar\varepsilon_z), & \overline{D}_z=\varepsilon_{11}\overline{E}_z+2e_{15}\bar\varepsilon_{rz},\end{cases} \tag{4.7}$$

where

$$\begin{aligned}&\overline{E}_r=\frac12\bigl(\delta_r^-\mu+\delta_r^-\bigl(\mu^{(-1_z)}\bigr)\bigr), &&\overline{E}_z=\frac12\bigl(\delta_z^-\mu+\delta_z^-\bigl(\mu^{(-1_r)}\bigr)\bigr),\\ &\bar\varepsilon_r=\frac12\bigl(\delta_r^-y+\delta_r^-\bigl(y^{(-1_z)}\bigr)\bigr), &&\bar\varepsilon_\theta=\frac{1}{4\bar r}\bigl(y+y^{(-1_r)}+y^{(-1_z)}+y^{(-1,-1)}\bigr),\\ &\bar\varepsilon_z=\frac12\bigl(\delta_z^-g+\delta_z^-\bigl(g^{(-1_r)}\bigr)\bigr), &&2\bar\varepsilon_{rz}=\frac12\bigl(\delta_z^-y+\delta_z^-\bigl(y^{(-1_r)}\bigr)+\delta_r^-g+\delta_r^-\bigl(g^{(-1_z)}\bigr)\bigr).\end{aligned}$$

Then, the second order scheme (in space and time) for the solution of problem (2.10), (2.13), (3.1)–(3.4) can be given in the following form:

$$\begin{cases} \rho\delta_t^2 y = \Lambda_1(y,g,\mu) + F_1, \\ \rho\delta_t^2 g = \Lambda_2(y,g,\mu) + F_2, \\ \Lambda_3(y,g,\mu) = F_3, \end{cases} \tag{4.8}$$

where functions y, g and μ are discrete-argument functions that give approximations to the functions $u_r(r,z,t), u_z(r,z,t)$ and $\varphi(r,z,t)$, respectively. The difference operators Λ_i and right sides F_i, $i = 1,2,3$ in (4.8) are defined as follows

$$\Lambda_1(y,g,\mu) = \begin{cases} \frac{1}{r}\delta_r^+\left(\bar r\frac{\bar\sigma_r+\bar\sigma_r^{(+1_z)}}{2}\right) + \frac{1}{r}\delta_z^+\left(\bar r\bar\sigma_{rz} + \bar r^{(+1)}\bar\sigma_{rz}^{(+1_r)}\right) & \\ \quad - \frac{\bar\sigma_\theta+\bar\sigma_\theta^{(+1_r)}+\bar\sigma_\theta^{(+1_z)}+\bar\sigma_\theta^{(+1,+1)}}{4r}, & (r,z)\in\omega_h, \\ \frac{1}{r}\delta_r^+\left(\bar r\bar\sigma_r^{(+1_z)}\right) + \frac{1}{r}\frac{2}{h_2}\left(\frac{\bar r\bar\sigma_{rz}^{(+1_z)}+\bar r^{(+1)}\bar\sigma_{rz}^{(+1,+1)}}{2}\right) & \\ \quad - \frac{\bar\sigma_\theta^{(+1_z)}+\bar\sigma_\theta^{(+1,+1)}}{2r}, & (r,z)\in\gamma_1, \\ \frac{1}{r}\delta_r^+(\bar r\bar\sigma_r) - \frac{1}{r}\frac{2}{h_2}\left(\frac{\bar r\bar\sigma_{rz}+\bar r^{(+1)}\bar\sigma_{rz}^{(+1_r)}}{2}\right) - \frac{\bar\sigma_\theta+\bar\sigma_\theta^{(+1_r)}}{2r}, & (r,z)\in\gamma_2, \\ \frac{1}{r}\bar r^{(+1)}\delta_z^+\left(\bar\sigma_{rz}^{(+1_r)}\right) + \frac{1}{r}\frac{2}{h_1}\bar r^{(+1)}\left(\frac{\bar\sigma_r^{(+1_r)}+\bar\sigma_r^{(+1,+1)}}{2}\right) & \\ \quad - \frac{\bar\sigma_\theta^{(+1_r)}+\bar\sigma_\theta^{(+1,+1)}}{2r}, & (r,z)\in\gamma_3, \\ \frac{1}{r}\bar r\delta_z^+(\bar\sigma_{rz}) - \frac{1}{r}\frac{2}{h_1}\bar r\left(\frac{\bar\sigma_r+\bar\sigma_r^{(+1_z)}}{2}\right) - \frac{\bar\sigma_\theta+\bar\sigma_\theta^{(+1_z)}}{2r}, & (r,z)\in\gamma_4, \\ \frac{1}{r}\frac{2}{h_1}\bar r^{(+1)}\bar\sigma_r^{(+1,+1)} + \frac{1}{r}\frac{2}{h_2}\bar r^{(+1)}\bar\sigma_{rz}^{(+1,+1)} - \frac{\bar\sigma_\theta^{(+1,+1)}}{r}, & (r,z)\in\gamma_{13}, \\ \frac{1}{r}\frac{2}{h_1}\bar r^{(+1)}\bar\sigma_r^{(+1_r)} - \frac{1}{r}\frac{2}{h_2}\bar r^{(+1)}\bar\sigma_{rz}^{(+1_r)} - \frac{\bar\sigma_\theta^{(+1_r)}}{r}, & (r,z)\in\gamma_{23}, \\ -\frac{1}{r}\frac{2}{h_1}\bar r\bar\sigma_r^{(+1_z)} + \frac{1}{r}\frac{2}{h_2}\bar r\bar\sigma_{rz}^{(+1_z)} - \frac{\bar\sigma_\theta^{(+1_z)}}{r}, & (r,z)\in\gamma_{14}, \\ -\frac{1}{r}\frac{2}{h_1}\bar r\bar\sigma_r - \frac{1}{r}\frac{2}{h_2}\bar r\bar\sigma_{rz} - \frac{\bar\sigma_\theta}{r}, & (r,z)\in\gamma_{24}, \end{cases}$$

$$\Lambda_2(y,g,\mu) = \begin{cases} \frac{1}{r}\delta_r^+\left(\bar r\frac{\bar\sigma_{rz}+\bar\sigma_{rz}^{(+1_z)}}{2}\right) + \frac{1}{r}\delta_z^+\left(\bar r\bar\sigma_z + \bar r^{(+1)}\bar\sigma_z^{(+1_r)}\right), & (r,z)\in\omega_h, \\ \frac{1}{r}\delta_r^+\left(\bar r\bar\sigma_{rz}^{(+1_z)}\right) + \frac{1}{r}\frac{2}{h_2}\left(\frac{\bar r\bar\sigma_z^{(+1_z)}+\bar r^{(+1)}\bar\sigma_z^{(+1,+1)}}{2}\right), & (r,z)\in\gamma_1, \\ \frac{1}{r}\delta_r^+(\bar r\bar\sigma_{rz}) - \frac{1}{r}\frac{2}{h_2}\left(\frac{\bar r\bar\sigma_z+\bar r^{(+1)}\bar\sigma_z^{(+1_r)}}{2}\right) - \frac{\bar\sigma_\theta+\bar\sigma_\theta^{(+1_r)}}{2r}, & (r,z)\in\gamma_2, \\ \frac{1}{r}\bar r^{(+1)}\delta_z^+\left(\bar\sigma_z^{(+1_r)}\right) + \frac{1}{r}\frac{2}{h_1}\bar r^{(+1)}\left(\frac{\bar\sigma_{rz}^{(+1_r)}+\bar\sigma_{rz}^{(+1,+1)}}{2}\right), & (r,z)\in\gamma_3, \\ \frac{1}{r}\bar r\delta_z^+(\bar\sigma_z) - \frac{1}{r}\frac{2}{h_1}\bar r\left(\frac{\bar\sigma_{rz}+\bar\sigma_{rz}^{(+1_z)}}{2}\right), & (r,z)\in\gamma_4, \\ \frac{1}{r}\frac{2}{h_2}\bar r^{(+1)}\bar\sigma_z^{(+1,+1)} + \frac{1}{r}\frac{2}{h_1}\bar r^{(+1)}\bar\sigma_{rz}^{(+1,+1)}, & (r,z)\in\gamma_{13}, \\ \frac{1}{r}\frac{2}{h_1}\bar r^{(+1)}\bar\sigma_{rz}^{(+1_r)} - \frac{1}{r}\frac{2}{h_2}\bar r^{(+1)}\bar\sigma_z^{(+1_r)}, & (r,z)\in\gamma_{23}, \\ -\frac{1}{r}\frac{2}{h_1}\bar r\bar\sigma_{rz}^{(+1_z)} + \frac{1}{r}\frac{2}{h_2}\bar r\bar\sigma_z^{(+1_z)}, & (r,z)\in\gamma_{14}, \\ -\frac{1}{r}\frac{2}{h_1}\bar r\bar\sigma_{rz} - \frac{1}{r}\frac{2}{h_2}\bar r\bar\sigma_z, & (r,z)\in\gamma_{24}, \end{cases}$$

$$
\Lambda_3(y,g,\mu)=\begin{cases}
\frac{1}{r}\delta_r^+\Big(\bar r\,\frac{\overline{D}_r+\overline{D}_r^{(+1_z)}}{2}\Big)+\frac{1}{r}\delta_z^+\big(\bar r\,\overline{D}_z+\bar r^{(+1)}\,\overline{D}_z^{(+1_r)}\big), & (r,z)\in\omega_h,\\
\frac{1}{r}\delta_r^+\big(\bar r\,\overline{D}_r^{(+1_z)}\big)+\frac{1}{r}\frac{2}{h_2}\Big(\frac{\bar r\,\overline{D}_z^{(+1_z)}+\bar r^{(+1)}\,\overline{D}_z^{(+1,+1)}}{2}\Big), & (r,z)\in\gamma_1,\\
\frac{1}{r}\delta_r^+\big(\bar r\,\overline{D}_r\big)-\frac{1}{r}\frac{2}{h_2}\Big(\frac{\bar r\,\overline{D}_z+\bar r^{(+1)}\,\overline{D}_z^{(+1_r)}}{2}\Big), & (r,z)\in\gamma_2,\\
\mu, & (r,z)\in\overline{\omega}_h/(\omega_h\cup\gamma_1\cup\gamma_2),
\end{cases}
$$

$$
F_3=\begin{cases} f_3, & (r,z)\in\omega_h\cup\gamma_1\cup\gamma_2,\\ 0, & (r,z)\in\overline{\omega}_h/(\omega_h\cup\gamma_1\cup\gamma_2),\end{cases}
$$

$$
F_1=f_1+\begin{cases}
0, & (r,z)\in\omega_h,\\
-\frac{2}{h_2}p_{rt}^{(0)}, & (r,z)\in\gamma_1,\\
\frac{2}{h_2}p_{rt}^{(1)}, & (r,z)\in\gamma_2,\\
-\frac{2}{h_1}p_{r}^{(0)}, & (r,z)\in\gamma_3,\\
\frac{2}{h_1}p_{rt}^{(1)}, & (r,z)\in\gamma_4,\\
-\frac{2}{h_1}p_{r}^{(0)}-\frac{2}{h_2}p_{rt}^{(0)}, & (r,z)\in\gamma_{13},\\
-\frac{2}{h_1}p_{r}^{(0)}+\frac{2}{h_2}p_{rt}^{(1)}, & (r,z)\in\gamma_{23},\\
\frac{2}{h_1}p_{r}^{(1)}-\frac{2}{h_2}p_{rt}^{(0)}, & (r,z)\in\gamma_{14},\\
\frac{2}{h_1}p_{r}^{(1)}+\frac{2}{h_2}p_{rt}^{(1)}, & (r,z)\in\gamma_{24},
\end{cases}
\qquad
F_2=f_2+\begin{cases}
0, & (r,z)\in\omega_h,\\
-\frac{2}{h_2}p_{z}^{(0)}, & (r,z)\in\gamma_1,\\
\frac{2}{h_2}p_{z}^{(1)}, & (r,z)\in\gamma_2,\\
-\frac{2}{h_1}p_{zt}^{(0)}, & (r,z)\in\gamma_3,\\
\frac{2}{h_1}p_{zt}^{(1)}, & (r,z)\in\gamma_4,\\
-\frac{2}{h_1}p_{zt}^{(0)}-\frac{2}{h_2}p_{z}^{(0)}, & (r,z)\in\gamma_{13},\\
-\frac{2}{h_1}p_{zt}^{(0)}+\frac{2}{h_2}p_{z}^{(1)}, & (r,z)\in\gamma_{23},\\
\frac{2}{h_1}p_{zt}^{(1)}-\frac{2}{h_2}p_{z}^{(0)}, & (r,z)\in\gamma_{14},\\
\frac{2}{h_1}p_{zt}^{(1)}+\frac{2}{h_2}p_{z}^{(1)}, & (r,z)\in\gamma_{24}.
\end{cases}
$$

The approximation of initial conditions is performed as follows

$$
y(r,z,0)=u_r^{(0)}(r,z),\qquad g(r,z,0)=u_z^{(0)}(r,z), \tag{4.9}
$$

$$
\rho\delta_t^+y=\rho u_r^{(1)}+\frac{\tau}{2}\big(F_1+\Lambda_1(y,g,\mu)\big),\qquad \rho\delta_t^+g=\rho u_z^{(1)}+\frac{\tau}{2}\big(F_2+\Lambda_2(y,g,\mu)\big). \tag{4.10}
$$

The discrete analogue of the energy conservation law (4.1) is fundamental in the investigation of the system stability. One of the key factors in this investigation is establishing some bounds for the energy functional at any given moment of time. Ultimately, it is these bounds that allow us to guarantee the stability of the corresponding difference problem (4.8)–(4.10) under certain conditions on the time and space discretisations.

5. Discrete analogue of conservation laws in coupled dynamic piezoelectricity

The Courant–Friedrichs–Lewy condition is a fundamental stability condition in theory of difference schemes for hyperbolic partial differential equations [45,43]. Since it is known that there are no explicit unconditionally stable consistent finite difference schemes for hyperbolic systems of partial differential equations [6,45], it would be beneficial to obtain an analogue of the CFL condition in dimensions higher than one, in particular in the case of *mixed* waves reflecting the dynamics of coupling between several fields of different physical nature. One would expect that such conditions would involve mixed electro-elastic waves, but a detailed proof of this fact has not been given in the literature so far.

As a first step towards achieving this goal, the aim of this section is to establish the analogue of (4.1) for the discrete model (4.8)–(4.10). Several new notation are required for convenience to proceed further. By

$\omega_{h_1}=\{r_i=R_0+ih_1,\ i=1,\dots,N-1\}$, $\omega_{h_1}^{+}=\{r_i=R_0+ih_1,\ i=1,\dots,N\}$, $\omega_{h_1}^{-}=\{r_i=R_0+ih_1,\ i=0,\dots,N-1\}$ we denote auxiliary spatial grids in the r-direction, and in a similar way we define auxiliary spatial grids in the z-direction ($\omega_{h_2},\omega_{h_2}^{+},\omega_{h_2}^{-}$). We will also need difference analogues of conventional functional norms. Namely, by $\|y\|_C=\max_{\overline{\omega}_h}|y(x)|\equiv\max_{0\leqslant i\leqslant N}|y_i|$ we denote a grid analogue of the Chebyshev norm in the functional space C, while by $\|y]|_0^2=\sum_{\omega_h^+}y^2h$ we denote a grid analogue of the norm in the functional space L_2 (note that the norm $\|y\|_0=\sum_{\omega_h}y^2h$ can also be considered as a grid analogue of L_2). The above notations are frequently used in the context of theory of difference schemes [41,42,28,43].

We start from the following identity

$$\sum_{\overline{\omega}_h}\rho r\hbar_1\hbar_2\big[\big(\delta_t^+y-\delta_t^-y\big)\big(\delta_t^+y+\delta_t^-y\big)+\big(\delta_t^+g-\delta_t^-g\big)\big(\delta_t^+g+\delta_t^-g\big)\big]$$
$$+2\tau\Big[\sum_{\omega_h^+}\bar r h_1h_2\big(\bar\sigma_r\delta_t^1(\bar\varepsilon_r)+\bar\sigma_\theta\delta_t^1(\bar\varepsilon_\theta)+\bar\sigma_z\delta_t^1(\bar\varepsilon_z)+2\bar\sigma_{rz}\delta_t^1(\bar\varepsilon_{rz})\big)\Big]$$
$$=2\tau\Big\{\sum_{\overline{\omega}_h}r\hbar_1\hbar_2(f_1v+f_2w)+\sum_{\overline{\omega}_{h_1}}r\hbar_1[\bar\sigma_{rz}v+\bar\sigma_zw]\big|_{Z_0}^{Z_1}+\sum_{\overline{\omega}_{h_2}}r\hbar_2[\bar\sigma_rv+\bar\sigma_{rz}w]\big|_{R_0}^{R_1}\Big\}. \tag{5.1}$$

The identity (5.1) is called the difference energy identity and can be easily obtained using the procedure described in [28] for the one-dimensional case. In (5.1) we have denoted δ_t^1y by v, δ_t^1g by w, and used the equality $\delta_t^-(\overline{D}_r)=\delta_t^-(\overline{D}_z)=0$ (see [28] for details).

Taking into account the state equations (4.7) and the easily verified identities,

$$y=\frac{\hat y+2y+\check y}{4}-\frac{\tau^2}{4}\delta_t^2y,\qquad 2\tau\delta_t^1y=\tau\big(\delta_t^+y+\delta_t^-y\big)=\hat y-\check y,$$

we can derive from (5.1) the discrete analogue of energy conservation law for the electromechanical system described by the model (2.10), (2.13), (3.1)–(3.4). As usual, $\hat y\equiv y(r,z,t+\tau)$ and $\check y\equiv y(r,z,t-\tau)$ denote the discrete function y from the "upper" and "lower" time layer, respectively.

We have

$$\overline{\mathcal E}(t+\tau)=\overline{\mathcal E}(t)+2\tau\sum_{\overline{\omega}_h}r\hbar_1\hbar_2(f_1v+f_2w)$$
$$+2\tau\Big\{\sum_{\overline{\omega}_{h_1}}[\bar\sigma_{rz}v+\bar\sigma_zw]\big|_{Z_0}^{Z_1}+\sum_{\overline{\omega}_{h_2}}r\hbar_2[\bar\sigma_rv+\bar\sigma_{rz}w]\big|_{R_0}^{R_1}\Big\}, \tag{5.2}$$

where $\overline{\mathcal E}(t)$ denotes the discrete analogue of the total energy of the electromechanical system defined as follows [31]

$$\overline{\mathcal E}(t)=\rho\sum_{\overline{\omega}_h}r\hbar_1\hbar_2\big(\big(\delta_t^-y\big)^2+\big(\delta_t^-g\big)^2\big)$$
$$+\sum_{\omega_h^+}\bar r\hbar_1\hbar_2\big\{c_{33}\Phi(\bar\varepsilon_r)+c_{11}\big(\Phi(\bar\varepsilon_\theta)+\Phi(\bar\varepsilon_z)\big)$$
$$+c_{13}\big[\bar\varepsilon_r(\bar\varepsilon_\theta+\bar\varepsilon_z)+\check{\bar\varepsilon}_r(\check{\bar\varepsilon}_\theta+\check{\bar\varepsilon}_z)-\tau^2\delta_t^-(\bar\varepsilon_r)\big(\delta_t^-(\bar\varepsilon_\theta)+\delta_t^-(\bar\varepsilon_z)\big)\big]$$
$$+c_{12}\big[\bar\varepsilon_z\bar\varepsilon_\theta+\check{\bar\varepsilon}_z\check{\bar\varepsilon}_\theta-\tau^2\delta_t^-(\bar\varepsilon_z)\delta_t^-(\bar\varepsilon_\theta)\big]+2c_{44}\Phi(\bar\varepsilon_{rz})+c_{33}\Phi\big(\overline{E}_r\big)+\varepsilon_{11}\Phi\big(\overline{E}_z\big)\big\}, \tag{5.3}$$

and

$$\Phi(y) = \frac{(y+\check{y})^2}{4} - \frac{\tau^2}{4}(\delta_t^- y)^2.$$

We impose the nonnegativeness requirement on the difference analogue of the energy functional (5.3). We shall derive conditions when this requirement is satisfied.

6. A priori estimate for the discrete analogue of the total energy of the piezoelectric system

The purpose of this section is to obtain a priori estimate for the discrete energy.

Summing (5.2) over t from τ to a certain t_1 ($\tau \leqslant t_1 \leqslant T$), we obtain

$$\begin{aligned}\overline{\mathcal{E}}(t_1+\tau) = \overline{\mathcal{E}}(\tau) &+ 2\tau \sum_{t'=\tau}^{t_1} \sum_{\overline{\omega}_h} r\hbar_1\hbar_2(f_1 v + f_2 w) \\ &+ 2\tau \sum_{t'=\tau}^{t_1} \left\{ \sum_{\omega_{h_1}} r\hbar_1[\bar{\sigma}_{rz} v + \bar{\sigma}_z w]|_{Z_0}^{Z_1} + \sum_{\omega_{h_2}} r\hbar_2[\bar{\sigma}_r v + \bar{\sigma}_{rz} w]|_{R_0}^{R_1} \right\}. \end{aligned} \tag{6.1}$$

Additives in the curl brackets of the right-hand side of (6.1) are estimated analogously to each other. As an example we will estimate the second of them. The technique of estimation is similar to that described in [28]. We use an easily proved identity

$$\left(\delta_t^1 u\right) v = \delta_t^1 (uv) - \frac{\hat{u}\delta_t^+ v - \check{u}\delta_t^- v}{2},$$

the Young (ε) inequality [41,42,28]

$$\left|(u, v)\right| \leqslant \|u\| \|v\| \leqslant \varepsilon \|u\|^2 + \frac{1}{4\varepsilon} \|v\|^2,$$

and grid analogues of the Sobolev embedding theorems which we formulate below (e.g., [41,42]).

Lemma 6.1. *For any grid function $y(x)$ given on an arbitrary (possibly nonuniform) grid $\overline{\omega}_h \in [0, L]$ the following inequality*

$$\|y\|_C \leqslant \varepsilon \left\|\delta_x^- y\right]\Big|_0^2 + \left(\frac{1}{\varepsilon} + \frac{1}{L}\right) \|y]|_0^2,$$

holds for any positive constant ε.

Lemma 6.2. *For any grid function $y(x)$ given on an arbitrary (possibly nonuniform) grid $\overline{\omega}_h \in [0, L]$ the following inequality*

$$\left\|\delta_x^- y\right]\Big|_0^2 = \kappa_0 y^2(0) + \kappa_1 y^2(L) \geqslant M_2 \|y]|_0^2,$$

holds for any positive constants κ_0 and κ_1, such that $\kappa_0 + \kappa_1 > 0$ and

$$M_2 = \frac{8(\kappa_0 + \kappa_1 + L\kappa_0\kappa_1)^2}{L(2 + L\kappa_0)(2 + L\kappa_1)(2\kappa_0 + 2\kappa_1 + L\kappa_0\kappa_1)}.$$

If $y_0 = y(L) = 0$, then the last inequality may be simplified to the Friedrichs inequality

$$\left\|\delta_x^- \right]\big|_0 \geqslant \frac{8}{L^2}\|y]|_0^2.$$

This allows us to obtain the following result

$$\sum_{t'=\tau}^{t_1} 2\tau\bar{\sigma}_z\omega|_{Z_0}^{Z_1} \leqslant M_1\overline{\mathcal{E}}(t_1+\tau) + M_2\left\{\max_{t=0,\tau,t_1,t_1+\tau}\sum_{k=0}^{1}\left|p_z^{(k)}\right|^2 + \sum_{t'=\tau}^{t_1}\tau\sum_{k=0}^{1}\left|\delta_t^1\left(p_z^{(k)}\right)\right|^2\right\}$$
$$+ M_3\left\{\sum_{t'=\tau}^{t_1}\tau\max_{t'-\tau,t'+\tau}\sum_{k=0}^{1}\left|g(R_k,Z_0,t')\right|^2 + \sum_{t'=\tau}^{t_1}\tau\max_{t'-\tau,t'+\tau}\sum_{k=0}^{1}\left|g(R_k,Z_1,t')\right|^2\right\}.$$

Taking into account the latter inequality and using a similar technique for estimating other additives in (6.1) it is not difficult to obtain

$$\begin{aligned}
\overline{\mathcal{E}}(t_1+\tau) \leqslant{}& M_4\overline{\mathcal{E}}(\tau) + M_5\overline{\mathcal{E}}(t_1+\tau) \\
&+ M_6\Bigg\{\sum_{\overline{\omega}_{h_1}}\max_{0,\tau,t_1,t_1+\tau}\left[\sum_{k=0}^{1}\left(\left(p_z^{(k)}\right)^2 + \left(p_{rt}^{(k)}\right)^2\right)\right] \\
&\qquad + \sum_{\overline{\omega}_{h_2}} r\hbar_2\max_{0,\tau,t_1,t_1+\tau}\left[\sum_{k=0}^{1}\left(\left(p_r^{(k)}\right)^2 + \left(p_{rz}^{(k)}\right)^2\right)\right]\Bigg\} \\
&+ \sum_{t'=\tau}^{t_1}\tau\Bigg\{\sum_{\overline{\omega}_{h_1}} r\hbar_1\left[\sum_{k=0}^{1}\left(\left|\delta_t^1\left(p_z^{(k)}\right)\right|^2 + \left|\delta_t^1\left(p_{rt}^{(k)}\right)\right|^2\right)\right] \\
&\qquad + \sum_{\overline{\omega}_{h_2}} r\hbar_2\left[\sum_{k=0}^{1}\left(\left|\delta_t^1\left(p_r^{(k)}\right)\right|^2 + \left|\delta_t^1\left(p_{rz}^{(k)}\right)\right|^2\right)\right]\Bigg\} \\
&+ M_7\sum_{t'=\tau}^{t_1}\tau\Bigg\{\max_{t'-\tau,t'+\tau}\sum_{i,j=0}^{1}\left[\left|g(R_i,Z_j,t')\right|^2 + \left|y(R_i,Z_j,t')\right|^2\right] \\
&\qquad + \sum_{\overline{\omega}_h} r\hbar_1\hbar_2\left(\left(\delta_t^1 y\right)^2 + \left(\delta_t^1 g\right)^2\right) + \sum_{\overline{\omega}_h} r\hbar_1\hbar_2\left(f_1^2 + f_2^2\right)\Bigg\}.
\end{aligned} \tag{6.2}$$

Then we use the technique of [31] and apply the discrete analogue of the Gronwall lemma formulated below (see, for example, [41,42]).

Lemma 6.3. *Let $g_j \geqslant 0$, $j = 1,2,\ldots$ and $f_j \geqslant 0$, $j = 0,1,\ldots$ be nonnegative grid functions (for example, $f_j \equiv f(t_j)$, $t_j = j\tau$, $j = 0,1,\ldots,n_0$). If f_j is a nondecreasing function (i.e., $f_{j+1} \geqslant f_j$), then from the inequality*

$$g_{j+1} \leqslant c_0\sum_{k=1}^{j}\tau g_k + f_j, \quad j = 1,2,\ldots,\ g_1 \leqslant f_0,\ c_0 = \text{const} > 0,$$

follows the estimate

$$g_{j+1} \leqslant \exp(c_0 t_j) f_j.$$

Applying Lemma 6.3 from (6.2) we derive the following inequality

$$\begin{aligned}\overline{\mathcal{E}}(t_1+\tau) \leqslant M_8\overline{\mathcal{E}}(\tau) + M_9\Biggl\{ &\sum_{\overline{\omega}_{h_1}} r\hbar_1 \max_{0,\tau,t_1,t_1+\tau}\Biggl[\sum_{k=0}^{1}\bigl(\bigl(p_z^{(k)}\bigr)^2 + \bigl(p_{rt}^{(k)}\bigr)^2\bigr)\Biggr] \\ &+ \sum_{\overline{\omega}_{h_2}} r\hbar_2 \max_{0,\tau,t_1,t_1+\tau}\Biggl[\sum_{k=0}^{1}\bigl(\bigl(p_r^{(k)}\bigr)^2 + \bigl(p_{rz}^{(k)}\bigr)^2\bigr)\Biggr] \\ &+ \sum_{t'=\tau}^{t_1}\tau\Biggl\{\sum_{\overline{\omega}_{h_1}} r\hbar_1\Biggl[\sum_{k=0}^{1}\bigl(\bigl|\delta_t^1\bigl(p_z^{(k)}\bigr)\bigr|^2 + \bigl|\delta_t^1\bigl(p_{rt}^{(k)}\bigr)\bigr|^2\bigr)\Biggr] \\ &\qquad + \sum_{\overline{\omega}_{h_2}} r\hbar_2\Biggl[\sum_{k=0}^{1}\bigl(\bigl|\delta_t^1\bigl(p_r^{(k)}\bigr)\bigr|^2 + \bigl|\delta_t^1\bigl(p_{rz}^{(k)}\bigr)\bigr|^2\bigr)\Biggr]\Biggr\} \\ &+ \sum_{\overline{\omega}_h} r\hbar_1\hbar_2\bigl(f_1^2+f_2^2\bigr)\Biggr\}. \end{aligned} \tag{6.3}$$

In order to obtain a bound on the discrete energy function we have to estimate the quantity $\overline{\mathcal{E}}(\tau)$. The most difficult part of this is the estimation of additives responsible for the electric field. Applying the Cauchy–Schwartz inequality to the difference analogue of the Maxwell equation (forced electrostatic of dielectrics) we arrive at the following result

$$\begin{aligned}&\varepsilon_{33}\sum_{\omega_h^+}\bar r h_1 h_2 \overline{E}_r^2 + \varepsilon_{11}\sum_{\omega_h^+}\bar r h_1 h_2 \overline{E}_z^2 \\ &\leqslant e_{33}\Biggl(\sum_{\omega_h^+}\bar r h_1 h_2 \bar\varepsilon_r^2\Biggr)^{1/2}\Biggl(\sum_{\omega_h^+}\bar r h_1 h_2 \overline{E}_r^2\Biggr)^{1/2} \\ &\quad + e_{13}\Biggl(\sum_{\omega_h^+}\bar r h_1 h_2 (\bar\varepsilon_\theta+\bar\varepsilon_z)^2\Biggr)^{1/2}\Biggl(\sum_{\omega_h^+}\bar r h_1 h_2 \overline{E}_r^2\Biggr)^{1/2} + 2e_{15}\Biggl(\sum_{\omega_h^+}\bar r h_1 h_2 \bar\varepsilon_{rz}^2\Biggr)^{1/2} \\ &\quad \times\Biggl(\sum_{\omega_h^+}\bar r h_1 h_2 \overline{E}_z^2\Biggr)^{1/2} + \Biggl(\sum_{\omega_h^+}\bar r h_1 h_2 \bar\lambda^2\Biggr)^{2}\Biggl(\sum_{\omega_h^+}\bar r h_1 h_2 \overline{E}_r^2\Biggr)^{1/2} \\ &\quad + \Biggl(\sum_{\omega_h^+}\bar r h_1 h_2 \bar\lambda^2\Biggr)^{1/2}\Biggl(\sum_{\omega_h^+}\bar r h_1 h_2 \overline{E}_z^{+}\Biggr)^{1/2}, \end{aligned} \tag{6.4}$$

where the function $\bar\lambda$ is defined by the relationships

$$\delta_r^+\Biggl(\bar r\,\frac{\bar\lambda + \bar\lambda^{(+1_z)}}{2}\Biggr) + \delta_z^+\Biggl(\frac{\bar r\bar\lambda + \bar r^{(+1)}\bar\lambda^{(+1_r)}}{2}\Biggr) = r f_3, \tag{6.5}$$

$$\bar{\lambda}^{(+1_r)} = \overline{D}_r^{(+1_r)} \quad \text{for } r = R_0, \quad \text{and} \quad \bar{\lambda}^{(+1_z)} = \overline{D}_z^{(+1_z)} \quad \text{for } z = Z_0\ \forall t \in \overline{\omega}_\tau. \tag{6.6}$$

Using the Cauchy inequality

$$\left(\sum_{i=1}^{n} a_i b_i\right) \leqslant \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right),$$

from (6.4) we get

$$\begin{aligned}
&\left(\varepsilon_{33}\sum_{\omega_h^+}\bar{r}h_1h_2\overline{E}_r^2 + \varepsilon_{11}\sum_{\omega_h^+}\bar{r}h_1h_2\overline{E}_z^2\right)^2 \\
&\quad\leqslant \left[\frac{e_{33}^2}{\varepsilon_{33}}\sum_{\omega_h^+}\bar{r}h_1h_2\bar{\varepsilon}_r^2 + \frac{e_{13}^2}{\varepsilon_{33}}\sum_{\omega_h^+}\bar{r}h_1h_2(\bar{\varepsilon}_\theta + \bar{\varepsilon}_z)^2 + \frac{4e_{15}^2}{\varepsilon_{33}}\sum_{\omega_h^+}\bar{r}h_1h_2\bar{\varepsilon}_{rz}^2 + 2\sum_{\omega_h^+}\bar{r}h_1h_2\bar{\lambda}^2\right] \\
&\qquad\times\left(3\sum_{\omega_h^+}\bar{r}h_1h_2\overline{E}_r^2 + 2\sum_{\omega_h^2}\bar{r}h_1h_2\overline{E}_z^2\right)\varepsilon_{33}.
\end{aligned} \tag{6.7}$$

It is not difficult to find a constant M_{10} so that the inequality

$$\begin{aligned}
&M_{10}\left(3\varepsilon_{33}\sum_{\omega_h^+}\bar{r}h_1h_2\overline{E}_r^2 + 2\varepsilon_{33}\sum_{\omega_h^2}\bar{r}h_1h_2\overline{E}_z^2\right) \\
&\quad\leqslant \varepsilon_{33}\sum_{\omega_h^+}\bar{r}h_1h_2\overline{E}_r^2 + \varepsilon_{11}\sum_{\omega_h^+}\bar{r}h_1h_2\overline{E}_z^2
\end{aligned} \tag{6.8}$$

holds, for example, by setting

$$M_{10} = \min\left\{\frac{1}{3}, \frac{\varepsilon_{11}}{2\varepsilon_{33}}\right\}.$$

Therefore, using (6.8) from (6.7) we obtain

$$\begin{aligned}
&\varepsilon_{33}\sum_{\omega_h^+}\bar{r}h_1h_2\overline{E}_r^2 + \varepsilon_{11}\sum_{\omega_h^+}\bar{r}h_1h_2\overline{E}_z^2 \\
&\quad\leqslant \frac{1}{M_{10}}\left[\frac{e_{33}^2}{\varepsilon_{33}}\sum_{\omega_h^+}\bar{r}h_1h_2\bar{\varepsilon}_r^2 + \frac{e_{13}^2}{\varepsilon_{33}}\sum_{\omega_h^+}\bar{r}h_1h_2(\bar{\varepsilon}_\theta + \bar{\varepsilon}_z)^2 + \frac{4e_{15}^2}{\varepsilon_{33}}\sum_{\omega_h^+}\bar{r}h_1h_2\bar{\varepsilon}_{rz}^2 + 2\sum_{\omega_h^+}\bar{r}h_1h_2\bar{\lambda}^2\right].
\end{aligned} \tag{6.9}$$

Taking into account (6.9) and the nonnegativity of the potential energy of deformation (see (3.5)) from (6.3) we obtain the following estimate for the solution of the discrete problem (4.8)–(4.10)

$$\begin{aligned}
&\overline{\mathcal{E}}(t_1+\tau) \\
&\quad\leqslant M\left\{\rho\sum_{\overline{\omega}_h} r\hbar_1\hbar_2\left(\left(\delta_t^+ y(0)\right)^2 + \left(\delta_t^+ g(0)\right)^2\right)\right. \\
&\qquad + \sum_{\omega_h^+}\bar{r}h_1h_2\left\{c_{33}\left[\left(\bar{\varepsilon}_r(0)\right)^2 + \frac{\tau^2}{4}\left(\delta_t^+\left(\bar{\varepsilon}_r(0)\right)\right)^2\right]\right.
\end{aligned}$$

$$
\begin{aligned}
&+c_{11}\Big[\big(\bar{\varepsilon}_\theta(0)\big)^2+\big(\bar{\varepsilon}_z(0)\big)^2+\frac{\tau^2}{4}\Big(\big(\delta_t^+\bar{\varepsilon}_\theta(0)\big)^2+\big(\delta_t^+\varepsilon_z(0)\big)^2\Big)\Big]\\
&+c_{13}\Big[\bar{\varepsilon}_r(0)\big(\bar{\varepsilon}_\theta(0)+\bar{\varepsilon}_z(0)\big)+\frac{\tau}{2}\Big(\bar{\varepsilon}_r(0)\big(\delta_t^+\bar{\varepsilon}_\theta(0)+\delta_t^+\bar{\varepsilon}_z(0)\big)+\delta_t^+\bar{\varepsilon}_r(0)\big(\bar{\varepsilon}_\theta(0)+\bar{\varepsilon}_z(0)\big)\Big)\Big]\\
&+c_{12}\Big[\bar{\varepsilon}_z(0)\bar{\varepsilon}_\theta(0)+\frac{\tau}{2}\big(\bar{\varepsilon}_z(0)\delta_t^+\bar{\varepsilon}_\theta(0)+\delta_t^+\bar{\varepsilon}_z(0)\bar{\varepsilon}_\theta(0)\big)\Big]\\
&+2c_{44}\Big[\big(\bar{\varepsilon}_{rz}(0)\big)^2+\frac{\tau^2}{4}\big(\delta_t^+\bar{\varepsilon}_{rz}(0)\big)^2\Big]\Big\}\\
&+\sum_{\overline{\omega}_{h_1}} r\hbar_1\max_{0,\tau,t_1,t_1+\tau}\Big[\sum_{k=0}^{1}\Big(\big(p_z^{(k)}\big)^2+\big(p_{rt}^{(k)}\big)^2\Big)\Big]\\
&+\sum_{\overline{\omega}_{h_2}} r\hbar_2\max_{0,\tau,t_1,t_1+\tau}\Big[\sum_{k=0}^{1}\Big(\big(p_r^{(k)}\big)^2+\big(p_{rz}^{(k)}\big)^2\Big)\Big]\\
&+\sum_{t'=\tau}^{t_1}\Big\{\sum_{\overline{\omega}_{h_1}} r\hbar_1\Big[\sum_{k=0}^{1}\Big(\big|\delta_t^1\big(p_z^{(k)}\big)\big|^2+\big|\delta_t^1\big(p_{rt}^{(k)}\big)\big|^2\Big)\Big]\\
&+\sum_{\overline{\omega}_{h_2}} r\hbar_2\Big[\sum_{k=0}^{1}\Big(\big|\delta_t^1\big(p_r^{(k)}\big)\big|^2+\big|\delta_t^1\big(p_{rz}^{(k)}\big)\big|^2\Big)\Big]\Big\}\\
&+\sum_{\omega_h^+}\bar{r}h_1h_2\big(\bar{\lambda}(0)\big)^2+\sum_{t'=\tau}^{t_1}\tau\sum_{\overline{\omega}_h} r\hbar_1\hbar_2\big(f_1^2+f_2^2\big)\Big\}.
\end{aligned}
\tag{6.10}
$$

7. Stability of variational difference schemes for coupled dynamic piezoelectricity

We recall that the estimate (6.10) has been obtained under the requirement of nonnegativity of the discrete analogue of the energy functional $\overline{\mathcal{E}}(t)$. Such a requirement leads to the stability conditions for the difference scheme (4.8)–(4.10) [28]. Indeed, applying the technique of the derivation of estimate (6.9), we have

$$
\begin{aligned}
&-\varepsilon_{33}\sum_{\omega_h^+}\bar{r}h_1h_2\big(\delta_t^-\overline{E}_r\big)^2-\varepsilon_{11}\sum_{\omega_h^+}\bar{r}h_1h_2\big(\delta_t^-\overline{E}_z\big)^2\\
&\leqslant\frac{1}{\varepsilon^M}\Big[\frac{e_{33}^2}{\varepsilon_{33}}\sum_{\omega_h^+}\bar{r}h_1h_2\big(\delta_t^-\bar{\varepsilon}_r\big)^2+\frac{e_{13}}{\varepsilon_{33}}\sum_{\omega_h^+}\bar{r}h_1h_2\big(\delta_t^-(\bar{\varepsilon}_\theta+\bar{\varepsilon}_z)\big)^2+\frac{4e_{15}^2}{\varepsilon_{33}}\sum_{\omega_h^+}\bar{r}h_1h_2\big(\delta_t^-\bar{\varepsilon}_{rz}\big)^2\Big],
\end{aligned}
\tag{7.1}
$$

where $\varepsilon^M=\min\{\frac{1}{2},\frac{\varepsilon_{11}}{\varepsilon_{33}}\}$. Hence, taking into account condition (3.5) and estimate (7.1) we obtain the following condition for nonnegativity of $\overline{\mathcal{E}}$ (see [28] for the one-dimensional case)

$$
\rho\sum_{\overline{\omega}_h} r\hbar_1\hbar_2\Big(\big(\delta_t^-y\big)^2+\big(\delta_t^-g\big)^2\Big)-\tau^2\left(\frac{c_{33}}{4}+\frac{e_{33}^2}{4\varepsilon_{33}\varepsilon^M}\right)\sum_{\omega_h^+}\bar{r}h_1h_2\big(\delta_t^-\bar{\varepsilon}_r\big)^2
$$

$$
\begin{aligned}
&-\tau^2\left[\frac{c_{11}}{4}+\frac{e_{13}^2}{4\varepsilon_{33}\varepsilon^M}\right]\sum_{\omega_h^+}\bar r h_1 h_2\big(\delta_t^-(\bar\varepsilon_\theta+\bar\varepsilon_z)\big)^2 \\
&-\tau^2\left[\frac{2c_{44}}{4}+\frac{(2e_{15})^2}{4\varepsilon_{33}\varepsilon^M}\right]\sum_{\omega_h^+}\bar r h_1 h_2\big(\delta_t^-\bar\varepsilon_{rz}\big)^2-\frac{\tau^2}{2}c_{13}\sum_{\omega_h^+}\bar r h_1 h_2\delta_t^-\bar\varepsilon_r\big(\delta_t^-\bar\varepsilon_\theta+\delta_t^-\bar\varepsilon_z\big) \\
&-\frac{\tau^2}{2}c_{12}\sum_{\omega_h^+}\bar r h_1 h_2\delta_t^-\bar\varepsilon_z\delta_t^-\bar\varepsilon_\theta\geqslant 0.
\end{aligned}\tag{7.2}
$$

Using the easily proved inequalities

$$
\begin{aligned}
&\sum_{\omega_h^+}\bar r h_1 h_2(\bar\varepsilon_r)^2\leqslant\frac{4}{h_1^2}\left(1+\frac{h_1}{2R_0}\right)\sum_{\overline{\omega}_h}r\hbar_1\hbar_2 y^2,\\
&\sum_{\omega_h^+}\bar r h_1 h_2(\bar\varepsilon_\theta)^2\leqslant\frac{1}{R_0^2}\sum_{\overline{\omega}_h}r\hbar_1\hbar_2 y^2,\\
&\sum_{\omega_h^+}\bar r h_1 h_2(\bar\varepsilon_z)^2\leqslant\frac{4}{h_2^2}\left(1+\frac{h_1}{2R_0}\right)\sum_{\overline{\omega}_h}r\hbar_1\hbar_2 g^2,\\
&\sum_{\omega_h^+}\bar r h_1 h_2\bar\varepsilon_r\bar\varepsilon_\theta\leqslant\frac{1}{2R_0h_1}\sum_{\overline{\omega}_h}r\hbar_1\hbar_2 y^2,\\
&\sum_{\omega_h^+}\bar r h_1 h_2(\bar\varepsilon_{rz})^2\leqslant\frac{1}{2}\left[\frac{4}{h_2^2}\sum_{\overline{\omega}_h}r\hbar_1\hbar_2 y^2\left(1+\frac{h_1}{2R_0}\right)+\frac{4}{h_1^2}\sum_{\overline{\omega}_h}r\hbar_1\hbar_2 g^2\left(1+\frac{h_1}{2R_0}\right)\right],\\
&\sum_{\omega_h^+}\bar r h_1 h_2\bar\varepsilon_z\bar\varepsilon_\theta\leqslant\frac{1}{4R_0h_2}\sum_{\overline{\omega}_h}r\hbar_1\hbar_2\big(g^2+y^2\big),\\
&\sum_{\omega_h^+}\bar r h_1 h_2\bar\varepsilon_r\bar\varepsilon_z\leqslant\frac{1}{2h_1h_2}\sum_{\overline{\omega}_h}r\hbar_1\hbar_2\big(y^2+g^2\big),
\end{aligned}
$$

it can be shown that (7.2) will be satisfied if the inequality

$$
\begin{aligned}
&\sum_{\overline{\omega}_h}r\hbar_1\hbar_2\big(\delta_t^-y\big)^2\Bigg\{\rho-\tau^2\Bigg[\frac{4}{h_1^2}\left(1+\frac{h_1}{2R_0}\right)\left(\frac{c_{33}}{4}+\frac{e_{33}^2}{4\varepsilon_{33}\varepsilon^M}\right)+\frac{1}{R_0^2}\left(\frac{c_{11}}{4}+\frac{e_{13}^2}{4\varepsilon_{33}\varepsilon^M}\right)\\
&\qquad+\frac{4}{h_2^2}\left(1+\frac{h_1}{2R_0}\right)\left(\frac{c_{44}}{4}+\frac{e_{15}^2}{2\varepsilon_{33}\varepsilon^M}\right)+c_{13}\frac{1}{4h_1R_0}+c_{13}\frac{1}{4h_1h_2}+c_{12}\frac{1}{8h_2R_0}\Bigg]\Bigg\}\\
&+\sum_{\overline{\omega}_h}r\hbar_1\hbar_2\big(\delta_t^-g\big)^2\Bigg\{\rho-\tau^2\Bigg[\frac{4}{h_2^2}\left(1+\frac{h_1}{2R_0}\right)\left(\frac{c_{11}}{4}+\frac{e_{13}^2}{4\varepsilon_{33}\varepsilon^M}\right)\\
&\qquad+\frac{4}{h_1^2}\left(1+\frac{h_1}{2R_0}\right)\left(\frac{c_{44}}{4}+\frac{e_{15}^2}{2\varepsilon_{33}\varepsilon^M}\right)+\frac{c_{13}}{4h_1h_2}+\frac{c_{12}}{8h_2R_0}\Bigg]\Bigg\}\geqslant\varepsilon,
\end{aligned}
$$

holds for $\varepsilon>0$.

The latter is satisfied if the following two inequalities

$$\begin{cases} \rho - \varepsilon_1^0 \leqslant \frac{\tau^2}{h_1^2}\Big[\big(1+\frac{h_1}{2R_0}\big)\big(c_{33}+\frac{e_{33}^2}{\varepsilon_{33}\varepsilon^M}\big)+\frac{c_{13}}{8}\frac{h_1}{h_2}+\frac{c_{13}}{4R_0}h_1+\frac{1}{4R_0^2}\big(c_{11}+\frac{e_{13}^2}{\varepsilon_{33}\varepsilon^M}\big)h_1^2\Big] \\ \qquad\qquad + \frac{\tau^2}{h_2^2}\Big[\big(1+\frac{h_1}{2R_0}\big)\big(c_{44}+\frac{2e_{15}^2}{\varepsilon_{33}\varepsilon^M}\big)+\frac{c_{12}}{8R_0}h_2+\frac{c_{13}}{8}\frac{h_2}{h_1}\Big], \\ \rho - \varepsilon_2^0 \geqslant \frac{\tau^2}{h_2^2}\Big[\big(1+\frac{h_1}{2R_0}\big)\big(c_{11}+\frac{e_{13}^2}{\varepsilon_{33}\varepsilon^M}\big)+\frac{c_{12}}{8R_0}h_2+\frac{c_{13}}{8}\frac{h_2}{h_1}\Big] \\ \qquad\qquad + \frac{\tau^2}{h_1^2}\Big[\big(1+\frac{h_1}{2R_0}\big)\big(c_{44}+\frac{2e_{15}^2}{\varepsilon_{33}\varepsilon^M}\big)+\frac{c_{13}}{8}\frac{h_1}{h_2}\Big], \end{cases} \tag{7.3}$$

are satisfied simultaneously for $\varepsilon_i^0 > 0,\ i = 1, 2$.

It is known [3,10] that in the general case in an anisotropic electro-elastic medium there are three plane waves, namely quasi-longitudinal and two quasi-transverse (the latter are usually propagated slower than quasi-longitudinal). Therefore we introduce three quantities

$$c_1 = \sqrt{\frac{c_{33}(1+K_1)}{\rho}}, \qquad c_2 = \sqrt{\frac{c_{44}(1+K_2)}{\rho}}, \qquad c_3 = \sqrt{\frac{c_{11}(1+K_3)}{\rho}}, \tag{7.4}$$

that characterise velocities of these waves, where

$$K_1 = \frac{e_{33}^2}{\varepsilon_{33}c_{33}}, \qquad K_2 = \frac{e_{15}^2}{\varepsilon_{11}c_{44}}, \qquad K_3 = \frac{e_{13}^2}{\varepsilon_{11}c_{11}}, \tag{7.5}$$

are constants of electromechanical coupling. Using (7.3) and the following notations

$$\begin{aligned} V_1 &\equiv V_1\big(h_1, h_1/h_2, h_1^2\big) \\ &= \Bigg[\bigg(1+\frac{h_1}{2R_0}\bigg)\frac{1+K_1/\varepsilon^M}{1+K_1} + \frac{c_{13}}{8c_{33}(1+K_1)}\frac{h_1}{h_2} \\ &\quad + \frac{c_{13}h_1}{4R_0c_{33}(1+K_1)} + \frac{1}{4R_0^2c_{33}(1+K_1)}\bigg(c_{11}+\frac{e_{13}^2}{\varepsilon_{33}\varepsilon^M}\bigg)h_1^2\Bigg], \end{aligned} \tag{7.6}$$

$$\begin{aligned} V_2 &\equiv V_2(h_1, h_2/h_1, h_2) \\ &= \Bigg[\bigg(1+\frac{h_1}{2R_0}\bigg)\frac{1+2K_2/\varepsilon^M}{1+K_2} + \frac{c_{13}}{8c_{44}(1+K_2)}\frac{h_2}{h_1} + \frac{c_{12}}{8R_0c_{44}(1+K_2)}h_2\Bigg], \end{aligned} \tag{7.7}$$

$$\begin{aligned} V_3 &\equiv V_3(h_1, h_2/h_1, h_2) \\ &= \Bigg[\bigg(1+\frac{h_1}{2R_0}\bigg)\frac{1+K_3/\varepsilon^M}{1+K_3} + \frac{c_{13}}{8c_{11}(1+K_3)}\frac{h_2}{h_1} + \frac{c_{12}h_2}{8R_0c_{11}(1+K_1)}\Bigg], \end{aligned} \tag{7.8}$$

$$\begin{aligned} V_4 &\equiv V_4(h_1, h_1/h_2) \\ &= \Bigg[\bigg(1+\frac{h_1}{2R_0}\bigg)\frac{1+2K_2/\varepsilon^M}{1+K_2} + \frac{c_{13}}{8c_{44}(1+K_2)}\frac{h_1}{h_2}\Bigg], \end{aligned} \tag{7.9}$$

we can write down the stability conditions for the difference scheme (4.8)–(4.10)

$$\begin{cases} \frac{\tau^2}{h_1^2}c_1^2V_1 + \frac{\tau^2}{h_2^2}c_2^2V_2 \leqslant 1-\varepsilon_1, \\ \frac{\tau^2}{h_2^2}c_3^2V_3 + \frac{\tau^2}{h_1^2}c_2^2V_4 \leqslant 1-\varepsilon_2, \end{cases} \tag{7.10}$$

where $\varepsilon, \varepsilon_i, \varepsilon_i^0,\ i = 1, 2$ are positive constants that do not depend on steps τ, h_1 and h_2. We have obtained the following result.

Theorem 7.1. *If conditions* (7.10) *are satisfied for the solution of the discrete model* (4.8)–(4.10) *then the estimate* (6.10) *with the discrete energy function defined by* (5.3) *holds for arbitrary* $t_1 > 0$.

The conditions (7.10) play the same role in dynamic electroelasticity applied to problems of materials science and engineering as the Courant–Friedrichs–Lewy stability conditions do in the classical theory of hyperbolic equations [6]. They connect steps τ, h_1 and h_2 with the velocity of mixed electroelastic waves.

8. Concluding remarks

In this paper we have developed the fundamental stability criteria for a large class of variational difference schemes applied to the analysis of coupled electroelastic waves in piezoelectrics.

Accounting for the coupling between electric and elastic fields in anisotropic piezoceramic materials is an important prerequisite for the adequate modelling of piezoelectric structures. In solving coupled problems in science and engineering [31,32] such as the one considered in this paper, the quality of the physical parameterisation of mathematical models may decisively influence the complexity of numerical algorithms. Under such circumstances the investigation of the stability of discrete models becomes the main issue in the success of the entire modelling enterprise, in particular in the case where vibrational characteristics of materials and structures are studied. In this paper we have derived rigorously a generalisation of the classical Courant–Friedrichs–Lewy condition to the case of dynamic piezoelectricity for variational difference schemes.

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