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OPTIMAL CUBATURE FORMULAE AND RECOVERY OF FAST-OSCILLATING FUNCTIONS FROM AN INTERPOLATIONAL CLASS *

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Abstract.

The method of limit functions is used to construct optimal-by-accuracy and optimal-by-order (with constant not exceeding two) cubature formulae for the integration of fast oscillatory functions given by their values at a finite number of fixed nodes in a square region. The construction is based on explicit forms of the majorant and minorant in the given interpolational class $C^2_{1,L,N}$ and the solution of the problem of optimal-by-accuracy recovery of functions from this class. It is shown that an appropriate choice of the grid in this interpolational class leads to a substantial reduction in a priori information required for the application of the proposed approach.

AMS subject classification: 65D30, 65D32, 65D07.

Key words: Fast oscillatory functions, numerical integration, optimal recovery, interpolational classes.

1 Introduction.

In this paper we consider the problem of construction of optimal-by-accuracy and optimal-by-order cubature formulae for computing integrals

$$I^2(f) = \int_0^1 \int_0^1 f(x_1, x_2)\varphi_1(x_1)\varphi_2(x_2)dx_1dx_2, \tag{1.1}$$

where $\varphi_1(x_1)$, $\varphi_2(x_2)$ are known integrable functions, and $f(x_1, x_2)$ belongs to a certain given class F_N.

Integrals of the form (1.1) often arise in the context of the Fourier or Fourier–Bessel integral transforms [26] and occur in many applications including signal and image processing, radioastronomy, crystallography and modelling automatic regulation systems. In these applications, functions $\varphi_1(x_1)$ and/or $\varphi_2(x_2)$ in (1.1) often exhibit highly oscillatory behaviour. This leads to the essential mathematical difficulties in computing integrals (1.1) (see [1, 5, 8, 4, 10, 11, 12, 14] and references therein). Such difficulties are encountered even in a relatively simple

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one-dimensional case where we have to integrate the product $f(x)\exp(-i\omega x)$ on an interval (a,b), where $\omega(b-a) \gg 1$ [7, 16]. On the interval (a,b) the functions $\Re e(f(x)\exp(-i\omega x))$ and $\Im m(f(x)\exp(-i\omega x))$ have approximately $\omega(b-a)/\pi$ zeros. Therefore, even if $f(x)$ is a smooth function, in order to achieve an adequate level of approximation we have to choose a polynomial of degree $n \gg \omega(b-a)/\pi$. The use of such a high degree polynomial may eventually lead to instability of computations [11]. The situation becomes more complicated in the two-dimensional case [12, 18, 6, 17]. Moreover, in the majority of practical situations only approximate information about the integrand is given as a result of measurements or physical experiments. Therefore, interpolational classes become the appropriate functional classes for the study of problems in numerical integration of fast oscillatory functions.

Following a general scheme for the construction of numerical algorithms developed in [20] we assume that the only a priori information available about function f is the knowledge that $f \in F$, where F is a given functional class. Since in most practical situations this information is the result of measurements or experiments it is natural to narrow the original class F by assuming only that function $f(X)$ is given by a fixed table of its values $f(X_1) = f_1,\ldots,f(X_N) = f_N$ in N fixed points $\{X_i\}_{i=1}^N$ from its domain of definition D taken here as the square region $\pi_2 = \{X = (x_1,x_2) : 0 \le x_i \le 1, i = 1,2\}$. This approach leads to a narrowing of the corresponding class F to an interpolational class F_N. Therefore, the method can be applied effectively under standard assumptions requiring measurements at pre-defined grid points. However, the proposed methodology goes beyond the standard approaches in a sense that it leads to improving the quality of algorithms for numerical integration via the maximal use of available information about the function [16]. Starting with the pre-defined grid, our approach allows to construct a new grid in π_2 with a smaller number of nodes providing an optimal covering of π_2 and preserving the quality of the numerical solution.

In order to obtain optimal-by-accuracy and optimal-by-order solutions of problem (1.1), we use the method of limit functions [23, 20, 15, 16]. Namely, we define the upper, $I^+(F_N)$, and the lower, $I^-(F_N)$, limits of the set of possible values of the integral (1.1) on functions from class F_N as

$$I^+(F_N) = \sup_{f\in F_N} I^2(f),\ I^-(F_N) = \inf_{f\in F_N} I^2(f), \tag{1.2}$$

and then determine the value

$$I^*(F_N) = \frac{I^+(F_N) + I^-(F_N)}{2}, \tag{1.3}$$

which is taken as the optimal-by-accuracy value of the integral $I^2(f)$. In this case $I^*(F_N)$ is the Chebyshev center of uncertainty domain of values $I^2(f)$ on class F_N [16]. The Chebyshev radius coincides with the error of representation of D by $I^*(F_N)$, $\delta(F_N)$, and is defined as follows:

$$\delta^*(F_N) = \frac{1}{2}\left(I^+(F_N) - I^-(F_N)\right). \tag{1.4}$$

When $\varphi_1(x_1) = \varphi_2(x_2) = 1$, the problem becomes one of computing optimal-by-accuracy value $I_1^*(F_N)$ for integrals

$$I_1^2(f) = \iint_{\pi_2} f(X)dX \tag{1.5}$$

with $f \in F_N$ and $X = (x_1, x_2)$.

In the general case where the accuracy of integration of function $f \in F_N$ by algorithm $\Lambda \in \mathcal{M}$ ($\mathcal{M}$ is the set of all cubature formulae that use information consisting of the definition of class F_N) takes into account the worst function in class F_N (i.e. the function on which $\sup_{f\in F_N} v(F_N, \Lambda, f)$ is achieved), $\delta(F_N)$ can be interpreted as a minimax characteristic of the algorithm quality

$$\delta(F_N) = \inf_{\Lambda\in\mathcal{M}} \sup_{f\in F_N} v(F_N, \Lambda, f), \quad v(F_N, \Lambda, f) = |I^2(f) - r(F_N, \Lambda, f)|, \tag{1.6}$$

where $r(F_N, \Lambda, f)$ is the result of application of algorithm Λ to function f. A cubature formula on which $\delta(F_N)$ is achieved is called optimal-by-accuracy for the given class. If there exists a cubature formula Λ^0 such that $v(F_N, \Lambda^0, f) \le \delta(F_N) + \eta$, $\eta \ge 0$, then Λ^0 is called optimal cubature formula on the class F_N with accuracy up to η. Furthermore, if $\eta = o(\delta(F_N))$ or $\eta = O(\delta(F_N))$, then Λ^0 is called asymptotically optimal or optimal-by-order, respectively. In our specific case where $\delta^*(F_N) \le \delta(F_N) \le \delta^*(F_N) + \eta$ we call a cubature formula $\bar{I}^2(f)$ that satisfies the following inequality:

$$\sup_{f\in F_N} |\bar{I}^2(f) - I^2(f)| \le \delta^* + \eta, \quad \eta \ge 0, \quad \delta^* \to 0 \tag{1.7}$$

with $\eta = o(\delta(F_N))$ or $\eta = O(\delta(F_N))$ asymptotically optimal or optimal-by-order, respectively.

We notice that the problem of optimal-by-accuracy integration on class F_N is closely connected to the problem of optimal-by-accuracy recovery of $f(X) \in F_N$ at point $X = (x_1, x_2) \in \pi_2$ (see, for example, [3, 20] and references therein). In order to explore the connection between the above two problems we recall the definition of majorant (minorants) of functional classes.

Definition 1.1. *A function* $A^+_{F_N}(X)$ ($A^-_{F_N}(X)$) *is called a* majorant (minorant) *of the class* F_N, *if the conditions*

(a) $A^+_{F_N}(X) \ge f(X)(A^-_{F_N}(X) \le f(X))$ *for all* $f \in F_N$, $X = (x_1, x_2) \in D$, *and*

(b) $A^+_{F_N}(X) \in F_N(A^-_{F_N} \in F_N)$

are satisfied.

The value of

$$f^*(X) = \frac{1}{2}\left(A^+_{F_N}(X) + A^-_{F_N}(X)\right) \tag{1.8}$$

(with $A^+_{F_N}(X), A^-_{F_N}(X)$ majorant and minorant of class F_N respectively) is taken as the optimal-by-accuracy recovery of $f(X)$ at $X \in \pi_2$. The error $\bar{\delta}(F_N, X)$ of the recovery of function $f(X) \in F_N$ at point X has the form

$$\bar{\delta}(F_N, X) = \frac{A^+_{F_N}(X) - A^-_{F_N}(X)}{2}. \tag{1.9}$$

Then, the optimal-by-accuracy cubature formulae for computing (1.5) is [23, 20, 17]

$$I_1^*(F_N) = \iint_{\pi_2} f^*(X)\, dX \tag{1.10}$$

with the Chebyshev radius, $\bar{\delta}(F_N)$, of the domain of undefinability of integral (1.5) in the form

$$\bar{\delta}(F_N) = \iint_{\pi_2} \bar{\delta}(F_N, X)\, dX. \tag{1.11}$$

For a constructive solution of problems (1.8), (1.9) and (1.10), (1.11), as well as for the construction of efficient cubature formulae for computing integral (1.1) we have to establish properties of majorants and minorants. Such properties are specific to functional classes under investigation and the main focus of this paper is on the construction of optimal-by-accuracy and optimal-by-order cubature formulae for computing integrals (1.1) in interpolational class $F_N = C^2_{1,L,N}$ and an optimal-by-accuracy recovery $f^*(X)$ at point $X \in \pi_2$ for functions from this class. Recall that the class $C^2_{1,L,N}$ is the class of functions that are defined in the domain π_2, satisfy the Lipschitz condition with constant L,

$$|f(X) - f(Y)| \le L\|X - Y\| = L \max_{i=1,2} |x_i - y_i| \tag{1.12}$$

and take in fixed nodes $X_1, \ldots, X_N$ of arbitrary grid corresponding to fixed values $f(X_1) = f_1, \ldots, f(X_N) = f_N$.

We organise this paper as follows.

- In Sections 2 and 4 we consider the problem of optimal-by-accuracy recovery of a function.
- In Section 3 we deal with problems connected with the choice of the grid in the interpolational class $C^2_{1,L,N}$.
- In Section 5 we construct optimal-by-accuracy cubature formula for computing integral $I_1^2(f)$.
- Finally, in Section 6 we construct optimal-by-order cubature formulae.

2 On majorant and minorant of the functional class $C^2_{1,L,N}$.

The important special cases of integral $I^2(f)$ are integrals of the form

$$I_2^2(f) = \int_0^1 \int_0^1 f(x_1, x_2) \sin \omega_1 x_1 \sin \omega_2 x_2 \, dx_1 \, dx_2, \tag{2.1}$$

$$I_3^2(f) = \int_0^1 \int_0^1 f(x_1, x_2) \cos \omega_1 x_1 \cos \omega_2 x_2 \, dx_1 \, dx_2, \tag{2.2}$$

where $\omega_i, i = 1, 2$ are certain real numbers, $|\omega_i| \geq 2\pi$, $i = 1, 2$. We aim to construct optimal-by-order cubature formulae for computing integrals (2.1), (2.2) and obtain error estimates for these formulae.

For the solution of the above problems we need to know the form and certain properties of the majorant $A^+_{C^2_{1,L,N}}$ and the minorant $A^-_{C^2_{1,L,N}}$ of the class that is being investigated.

It can be shown [13, 2] that

$$A^+_{C^2_{1,L,N}}(X) = \sup_{f \in C^2_{1,L,N}} f(X) = \min_{\mu=1,\dots,N} (f_\mu + L\|X - X_\mu\|_1), \tag{2.3}$$

$$A^-_{C^2_{1,L,N}}(X) = \inf_{f \in C^2_{1,L,N}} f(X) = \max_{\mu=1,\dots,N} (f_\mu - L\|X - X_\mu\|_1), \tag{2.4}$$

where $X = (x_1, x_2)$, $D = \pi_2$ (see Definition 1.1), $\|X\|_1 = \max_{i=1,2}(|x_1|)$. Let us make more precise the notion of the collection of nodes $X_1, \dots, X_N$. For class $C^2_{1,L,N}$ we denote it by $\Delta = \{X_\mu\}_{\mu=1,\dots,N}$, $N = m^2$. For the definition of Δ we consider an auxiliary grid

$$\begin{cases} \Delta' = \{X'_v\}_{v=1,\dots,N'}, \quad N' = (m+1)^2, \quad X'_v = (x'_{1,i}; x'_{2,j}), \\ v = (i-1)(m+1) + j, \quad x_{1,i} = (i-1)\dfrac{1}{m}, \quad x'_{2,j} = (j-1)\dfrac{1}{m}, \end{cases} \tag{2.5}$$

with $i = 1, \dots, m+1, j = 1, \dots, m+1$. The grid Δ' splits the square region π_2 into m^2 equal squares K_μ, $\mu = 1, \dots, m^2$ with sides $h = 1/m$.

Let us consider the grid $\Delta = \{X\}_{\mu=1,\dots,N}$, nodes of which are the centers of squares $K_\mu, \mu = 1, \dots, m^2$. Closest to the sides of K_μ we have rows of nodes of the uniform grid located at distance $1/(2m)$ (in metrics $\|\cdot\|_1$). We note that for any other grid $\tilde{\Delta} = \{X_\mu\}_{\mu=1,\dots,N}$, $N = m^2$ we cannot claim that for any $X \in \pi_2$,

$$\min_{\mu=1,\dots,N} \|X - X_\mu\|_1 \leq \frac{1}{2m}. \tag{2.6}$$

Then we can claim the grid Δ provides an optimal cover of π_2 (see [20] and references therein). Our proofs below will be conducted only for function $A^+_{C^2_{1,L,N}}(X)$. For function $A^-_{C^2_{1,L,N}}(X)$ all proofs are analogous.

For the constructive representation of $A^+_{C^2_{1,L,N}}(X)$ in π_2 we first consider the case when the function is given at nodes

$$\bar{\Delta} = \Delta \cup \Delta', \qquad \bar{\Delta} = \{X_s\}_{s=1,\dots,N}, \quad \bar{N} = N + N'.$$

The grid $\bar{\Delta}$ splits square π_2 into regions of the forms Q and Q' (see Figure 2.1).

Nodes of the grid Δ' are denoted by circles ($\circ$) on Figure 2.1 and nodes of the grid Δ are denoted by stars ($*$). It can be shown (see [19, 20] and references therein), that values of the limit functions from class $C^2_{1,L,\bar{N}}$ in regions Q and Q'

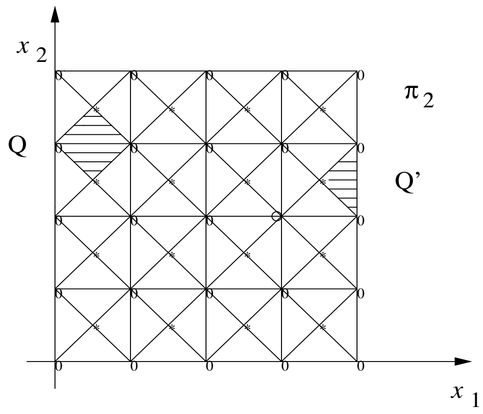

Figure 2.1: Splitting of the domain π_2 by the grid Δ.

are defined by the values of these functions at vertices of the regions Q and Q'. Let, for functions $f(X)$ from class $C^2_{1,L,\bar{N}}$,

$$f(X_\mu) = f_\mu,\ X_\mu \in \Delta \quad \text{and} \quad f(X'_v) = f_v, X'_v \in \Delta'. \tag{2.7}$$

Then the majorant $A^+_{F_N}(X)$ for $F_N = C^2_{1,L,\bar{N}}$ has the form

$$A^+_{C^2_{1,L,N}}(X) = \min_{\mu=1,\dots,m;v=1,\dots,(m+1)^2} (f_\mu + L\|X - X_\mu\|_1, f_v + L\|X - X'_v\|_1). \tag{2.8}$$

Let us single out regions of linearity of $A^+_{C^2_{1,L,N}}(X)$. We split region Q into subregions $\Omega^*_l, l = 1,\dots,4$ (Figure 2.2) and place the origin at the vertex X_{μ_1} of the region Q. Here

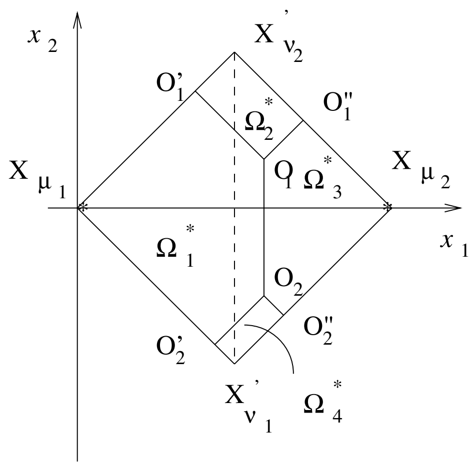

Figure 2.2: Regions of linearity of the domain Q.

$$
\left\{
\begin{array}{l}
X_{\mu_1} = (0;0), \quad X'_{v_1} = (h/2;-h/2), \\
X_{\mu_2} = (h;0), \quad X'_{v_2} = (h/2;h/2), \\
v_1 = (i-1)(m+1)+j, \quad v_2 = (i-1)(m+1)+j+1, \\
\mu_1 = (i-2)m+j, \quad \mu_2 = (i-1)m+j, \\
i = 1,\dots,m+1, \quad j = 1,\dots,m+1.
\end{array}
\right.
\tag{2.9}
$$

For definiteness, let $f_{v_1} + f_{v_2} > f_{\mu_1} + f_{\mu_2}$ and $f_{\mu_2} > f_{\mu_1}$ (see Figure 2.2). Then equations of the lines that split region Q into sub-regions Ω^*_l, $l = 1,\dots 4$, have the form

- for the line through O_1, O_2:

$$x_1 = \frac{f_{\mu_2} - f_{\mu_1}}{2L} + \frac{h}{2};$$

- for O_1, O'_1:

$$x_2 = \frac{f_{v_2} - f_{\mu_1}}{L} + \frac{h}{2} - x_1;$$

- for O_1, O''_1:

$$x_2 = \frac{f_{v_2} - f_{\mu_2}}{L} - \frac{h}{2} + x_1;$$

- for O_2, O''_2:

$$x_2 = \frac{f_{\mu_1} - f_{v_1}}{L} + \frac{h}{2} - x_1;$$

- for O_2, O'_2:

$$x_2 = \frac{f_{\mu_1} - f_{v_1}}{L} - \frac{h}{2} + x_1.$$

Lemma 2.1. *Majorant of class $C^2_{1,L,\bar{N}}$ for $X \in Q = \bigcup_{l=1}^{4} \Omega^*_l$ has the form*

$$
A^+_{C^2_{1,L,\bar{N}}}(X) =
\left\{
\begin{array}{ll}
f_{\mu_1} + L\|X - X_{\mu_1}\|_1, & X \in \Omega^*_1, \\
f_{v_2} + L\|X - X'_{v_2}\|_1, & X \in \Omega^*_2, \\
f_{\mu_2} + L\|X - X_{\mu_2}\|_1, & X \in \Omega^*_3, \\
f_{v_1} + L\|X - X'_{v_1}\|_1, & X \in \Omega^*_4.
\end{array}
\right.
\tag{2.10}
$$

Proof. Let

$$\tilde{g}_{v_1}(X) = f_{v_1} + L\|X - X'_{v_1}\|, \quad g_{\mu_i}(X) = f_{\mu_i} + L\|X - X_{\mu_i}\|_1, \quad i = 1,2.$$

Then it is easy to show that

$$g_{\mu_1}(X) \le g_{\mu_2}(X), \quad g_{\mu_1}(X) \le \tilde{g}_{v_1}(X), \quad l = 1,2, X \in \Omega^*_1.$$

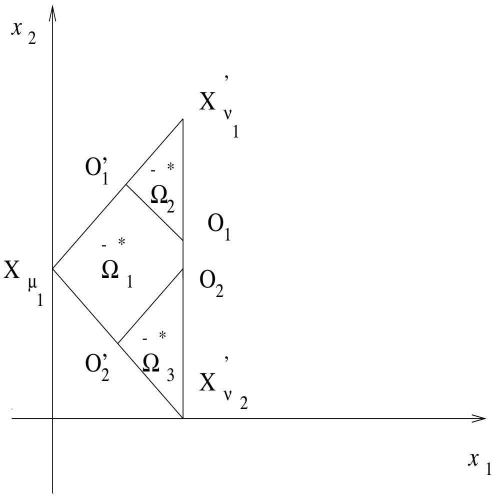

Figure 2.3: Splitting of the domain Q' when $f_{\mu_1} < \frac{1}{2}(f_{v_1} + f_{v_2})$.

Let us prove this inequality for $g_{\mu_2}(X)$, for example. We have

$$\begin{aligned} g_{\mu_1}(X) - g_{\mu_2}(X) &= f_{\mu_1} + L\|X - X_{\mu_1}\|_1 - f_{\mu_2} - L\|X - X_{\mu_2}\|_1 \\ &= f_{\mu_1} - f_{\mu_2} + L|x_1 - x_{1,\mu_1}| - L|x_1 - x_{1,\mu_2}| \\ &= f_{\mu_1} - f_{\mu_2} - Lh \le 0. \end{aligned}$$

Thus, we have shown that the majorant of class $C^2_{1,L,\bar{N}}$ has the form (2.10). We note that our proof is valid for $f_{v_1}+f_{v_2} > f_{\mu_1}+f_{\mu_2}$. In the case $f_{v_1}+f_{v_2} \le f_{\mu_1}+f_{\mu_2}$ the proof is analogous. □

Now we consider the region Q'. We split it into three sub-regions as shown in Figure 2.3 when $f_{\mu_1} < \frac{1}{2}(f_{v_1} + f_{v_2})$, or as shown in Figure 2.4 when $f_{\mu_1} > \frac{1}{2}(f_{v_1} + f_{v_2})$. In the case $f_{\mu_1} = f_{v_1} = f_{v_2}$, the interval O_1O_2 contracts to a point. We have to note that intervals O_1O_1', O_2O_2' can contract to a point under certain relationships between $f_{v_1}, f_{v_2}, f_{\mu_1}$. This remark is also relevant to Figure 2.2.

It is easy to see that the equations of the lines that split the region Q' into $\bar{\Omega}_l^*, l = 1, \dots 3$ in the case $f_{\mu_1} < \frac{1}{2}(f_{v_1} + f_{v_2})$ (see Figure 2.3) have the form

$$x_2 = \frac{f_{v_2} + f_{\mu_1}}{L} + \frac{h}{2} - x_1 \tag{2.11}$$

for the line through points O_1, O_1';

$$x_2 = \frac{f_{\mu_1} + f_{v_1}}{L} + \frac{h}{2} + x_1 \tag{2.12}$$

for O_2, O_2'; and in the case $f_{\mu_1} > \frac{1}{2}(f_{v_1} + f_{v_2})$ (see Figure 2.4) the form

$$x_2 = (f_{v_2} - f_{v_1})/(2L) \tag{2.13}$$

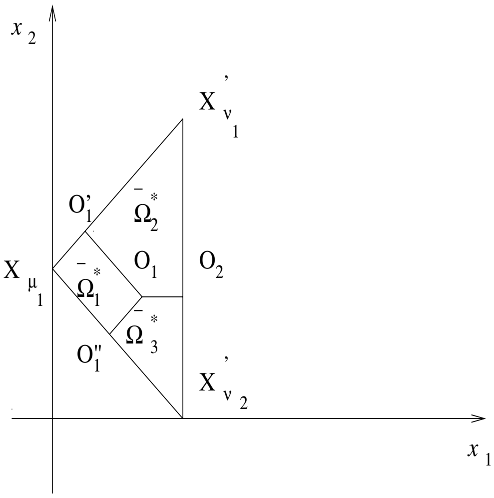

Figure 2.4: Splitting of the domain Q' when $f_{\mu_1} > \frac{1}{2}(f_{v_1} + f_{v_2})$.

for the line through points O_1, O_2; and the form of equations of the lines that pass points O_1, O_1' and O_1, O_1'' as defined by formulae (2.11), (2.12) respectively.

Lemma 2.2. *Majorant $A^+_{C^2_{1,L,\bar{N}}}(X)$ of class $C^2_{1,L,\bar{N}}$ for $X \in Q' = \bigcup_{l=1}^3 \bar{\Omega}^*_l$ has the form*

$$A^+_{C^2_{1,L,\bar{N}}}(X) = \begin{cases} f_{\mu_1} + L\|X - X_{\mu_1}\|_1, & X \in \bar{\Omega}^*_1, \\ f_{v_2} + L\|X - X'_{v_2}\|_1, & X \in \bar{\Omega}^*_2, \\ f_{v_1} + L\|X - X'_{v_1}\|_1, & X \in \bar{\Omega}^*_3. \end{cases} \tag{2.14}$$

Proof. The proof is analogous to the proof of Lemma 2.1. □

Lemmas 2.1 and 2.2 allow us to present majorant $A^+_{C^2_{1,L,\bar{N}}}(X)$ in π_2 in a sufficiently simple form. Indeed, (2.10) and (2.14) can be written as follows:

$$A^+_{C^2_{1,L,\bar{N}}}(X) = \begin{cases} f_{\mu_1} + L|x_1 - x_{1,\mu_1}|, & X \in \Omega^*_1, \\ f_{v_2} + L|x_2 - x'_{2,v_2}|, & X \in \Omega^*_2, \\ f_{\mu_2} + L|x_1 - x_{1,\mu_2}|, & X \in \Omega^*_3, \\ f_{v_1} + L|x_2 - x'_{2,v_1}|, & X \in \Omega^*_4, \end{cases} \tag{2.15}$$

$$A^+_{C^2_{1,L,\bar{N}}}(X) = \begin{cases} f_{\mu_1} + L|x_1 - x_{1,\mu_1}|, & X \in \bar{\Omega}^*_1, \\ f_{v_2} + L|x_2 - x'_{2,v_2}|, & X \in \bar{\Omega}^*_2, \\ f_{v_1} + L|x_2 - x'_{2,v_1}|, & X \in \bar{\Omega}^*_3 \end{cases} \tag{2.16}$$

with $Q = \bigcup_{l=1}^4 \Omega^*_l$ and $Q' = \bigcup_{l=1}^3 \bar{\Omega}^*_l$.

An analogous representation can be obtained for the minorant of this class. Having derived the explicit forms of the majorant and minorant of class $C^2_{1,L,\bar{N}}$ by specifying the regions of linearity of $A^+_{C^2_{1,L,\bar{N}}}(X)$ and $A^-_{C^2_{1,L,\bar{N}}}(X)$, we apply the methodology described in Section 1 (see (1.8)) to solve constructively the problem of optimal-by-accuracy recovery $f^*(X)$ at point $X \in \pi_2$ of the functions from this class. Since an efficient choice of the grid is implied by the form of regions of linearity of majorant and minorant [20], in the next section we focus on this issue in detail on the example of interpolational class $C^2_{1,L,N}$.

3 The choice of the grid in class $C^2_{1,L,N}$.

Let us now consider the case when function values are only given at the nodes of the grid Δ. In this case for the constructive representation of majorant $A^+_{C^2_{1,L,\bar{N}}}(X)$ of class $C^2_{1,L,N}$ we have to perform some additional computations. Let the functions from the class $C^2_{1,L,\bar{N}}$ take fixed values at the nodes of the grid $\bar{\Delta} = \Delta \cup \Delta'$ and let the following relationship hold:

$$\begin{aligned} &C^2_{1,L,\bar{N}} = \{f(X): \ |f(X_1) - f(X_2)| \le L\|X_1 - X_2\|_1, \\ &f(X_\mu) = f(X_\mu), \quad X_\mu \in \Delta, \ f(X'_v) = A^+_{C^2_{1,L,N}}(X), \quad X'_v \in \Delta'\}. \end{aligned} \tag{3.1}$$

Then it is easy to see that

$$A^+_{C^2_{1,L,N}}(X) = A^+_{C^2_{1,L,\bar{N}}}(X), \quad X \in \pi_2 \tag{3.2}$$

and

$$A^-_{C^2_{1,L,N}}(X) \neq A^-_{C^2_{1,L,\bar{N}}}(X). \tag{3.3}$$

Therefore, regions of linearity of the majorant of the class $C^2_{1,L,N}$ can be singled out analogously to class $C^2_{1,L,\bar{N}}$. Let us consider the question of computation of values $A^+_{C^2_{1,L,N}}(X)$ at points $X'_v \in \Delta'$ in detail.

Let $\tilde{\Delta} \subset \Delta'$ be the set of the grid Δ' which lies on the sides of π_2, and $\Delta^* \subset \tilde{\Delta}$ be the set of vertices of π_2. Let us choose in the region π_2 elementary regions $\tilde{K}_p$, $p = 1, \dots, (m_1)^2$, i.e. such squares whose vertices are nodes of the grid Δ. Points $X'_v \in \{\Delta' \backslash \tilde{\Delta}\}$ are centers of these squares (see Figure 3.1). In Figure 3.1 vertices of regions $\tilde{K}_p, p = 1, \dots, (m-1)^2$ are denoted by (x), and points $X'_v \in \{\Delta' \backslash \tilde{\Delta}\}$ by $(\circ)$.

Let $\sigma(\tilde{K}_p) = \{s_l\}_{l=1,\dots,4}$, where $s_1 = (i-1)m + j$, $s_2 = (i-1)m + j + 1$, $s_3 = im + j + 1$, $s_4 = im + j$ be the numbers of nodes X_{s_l}, $l = 1, \dots, 4$ that correspond to the vertices of elementary square $\tilde{K}_p$, $p = (i-1)m + j$, $i = 1, \dots, m$, $j = 1, \dots, m$.

Theorem 3.1. *Let $X'_v \in \{\Delta' \backslash \tilde{\Delta}\}$. Then if $X'_v \in \tilde{K}_p$, we have that*

$$A^+_{C^2_{1,L,N}}(X'_v) = \min_{s \in \sigma(\tilde{K}_p)} f_s + \frac{Lh}{2}, \quad p = 1, \dots, (m-1)^2. \tag{3.4}$$

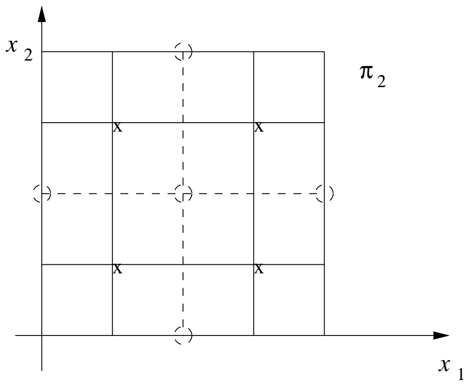

Figure 3.1: Elementary subregions in π_2.

PROOF. Consider the elementary square $\tilde{K}_{\bar{p}}$. Its vertices are the following nodes of the grid Δ:

$$\left\{ \begin{array}{ll} X_{s_1} = ((\bar{i} - \frac{1}{2})h; (\bar{j} - \frac{1}{2})h), & X_{s_2} = ((\bar{i} - \frac{1}{2})h; (\bar{j} + \frac{1}{2})h), \\ X_{s_3} = ((\bar{i} + \frac{1}{2})h; (\bar{j} + \frac{1}{2})h), & X_{s_2} = ((\bar{i} + \frac{1}{2})h; m(\bar{j} - \frac{1}{2})h). \end{array} \right. \tag{3.5}$$

Then the point $X'_{\bar{v}} = (\bar{i}h, \bar{j}h)$ is the center of $\tilde{K}_{\bar{p}}$, $p = 1, \ldots, (m-1)^2$, $\bar{i} = 1, \ldots, m$, $\bar{j} = 1, \ldots, m$. We will show that

$$A^+_{C^2_{1,L,N}}(X'_{\bar{v}}) = \min_{s \in \sigma(\tilde{K}_{\bar{p}})} (f_s + L\|X'_{\bar{v}} - X_s\|_1). \tag{3.6}$$

First, we introduce the following function

$$g_\mu(X) = f_\mu + L\|X - X_\mu\|_1,\ \mu \in \sigma(\Delta) \tag{3.7}$$

and show that

$$g_{\bar{\mu}}(X'_{\bar{v}}) \geq g_{s_l}(X'_{\bar{v}}), \bar{\mu} \in \sigma(\Delta)\ \sigma(\tilde{K}_{\bar{p}}), \quad l = 1, \ldots, 4. \tag{3.8}$$

Let, for example,

$$X_{\bar{\mu}} = ((\bar{i} - \tfrac{1}{2} - k_1)h, \quad (\bar{j} - \tfrac{1}{2} - k_2)h), \quad k_1 = 0, \ldots, \bar{i} - 1, \quad k = 0, \ldots, \bar{j} - 1.$$

Then

$$\begin{aligned} g_{\bar{\mu}}(X'_{\bar{v}}) - g_{s_1}(X'_{\bar{v}}) &= f_{\bar{\mu}} + L\|X'_{\bar{v}} - X_{\bar{\mu}}\|_1 - f_{s_1} - L\|X'_{\bar{v}} - X_{s_1}\|_1 \\ &= f_{\bar{\mu}} - f_{s_1} + L(\max(\bar{i}h - (\bar{i} - \tfrac{1}{2} - k_1)h, \bar{j}h - (\bar{j} - \tfrac{1}{2}k_2)h) \\ &\quad - \max(\bar{i}h - (\bar{i} - \tfrac{1}{2})h, \bar{j}h - (\bar{j}h - (\bar{j} - \tfrac{1}{2})h)) \\ &= f_{\bar{\mu}} - f_{s_1} + Lh\max(k_1, k_2) \\ &\geq -Lh\max(k_1, k_2) + Lh\max(k_1, k_2) = 0. \end{aligned} \tag{3.9}$$

Analogously, it can be shown that $g_{\bar{\mu}}(X'_{\bar{v}}) - g_{s_1}(X'_{\bar{v}}) \geq 0$ for $l = 2, 3, 4$. It is easy to see that this inequality is satisfied for all others $\mu \in \sigma(\Delta)\ \sigma(\tilde{K}_{\bar{p}})$. This means that for all functions $g_\mu(X)$, $\mu \in \sigma(\Delta)\ \sigma(\tilde{K}_{\bar{p}})$ the following inequality holds

$$g_\mu(X'_{\bar{v}}) \geq \min_{s \in \sigma(\tilde{K}_{\bar{p}})} g_s(X'_{\bar{v}}),\ X'_{\bar{v}} \in \tilde{K}_p. \tag{3.10}$$

From (3.10) the statement of the theorem for elementary square $\tilde{K}_{\bar{p}}$, $p = 1, \ldots, (m-1)^2$ follows. □

Corollary 3.1. *Let $X'_v \in \tilde{\Delta}$. Then the following relationship holds:*

$$A^+_{C^2_{1,L,N}}(X'_v) = \min(f_{s_1}, f_{s_2}) + \frac{Lh}{2}, \tag{3.11}$$

where s_1, s_2 are numbers of nodes X_{s_1}, X_{s_2} of the grid Δ for which $\|X'_v - X_{s_1}\|_1 = \frac{h}{2}$, $l = 1, 2$.

Corollary 3.2. *Let $X'_v \in \Delta^*$. Then the following relationship holds:*

$$A^+_{C^2_{1,L,N}}(X'_v) = f_{\bar{s}} + \frac{Lh}{2}, \tag{3.12}$$

where $\bar{s}$ is the number of node $X_{\bar{s}} \in \Delta$ for which $\|X'_v - X_{\bar{s}}\|_1 = \frac{h}{2}$.

Proofs of Corollaries 3.1 and 3.2 are analogous to the proof of Theorem 3.1.

Recall that at the beginning of this section we have shown that the regions of linearity of the majorant of class $C^2_{1,L,N}$ can be singled out analogously to class $C^2_{1,L,\bar{N}}$. Therefore, as follows from Section 2 where properties of class $C^2_{1,L,\bar{N}}$ were investigated in detail, the choice of the grid $\bar{\Delta}$ is dictated by the form of the regions of linearity of functions $A^+_{C^2_{1,L,\bar{N}}}(X)$ and $A^-_{C^2_{1,L,\bar{N}}}(X)$. The difficulty of the realisation of the approach (1.2)–(1.4) lies with the constructive computation of $I_i^+(C^2_{1,L,N})$, $I_i^-(C^2_{1,L,N})$, $i = 1, 2, 3$. In computing $I_i^+(C^2_{1,L,N})$ we extend $f(X) = A^+_{C^2_{1,L,N}}(X)$, $X \in \Delta'$ and for computing $I_i^-(C^2_{1,L,N})$ we extend $f(X) = A^-_{C^2_{1,L,N}}(X)$, $X \in \Delta'$, $i = 1, 2, 3$.

Lemma 3.1. *For the majorant of class $C^2_{1,L,N}$ the following relationship holds:*

$$A^+_{C^2_{1,L,N}}(X) = \min_{\mu=1,\ldots,m; v=1,\ldots,(m+1)^2} (f_\mu + L\|X - X_\mu\|_1, f_v + L\|X - X'_v\|_1),$$

where $f_\mu = f(X_\mu)$ for $X_\mu \in \Delta$ and $f_v = A^+_{C^2_{1,L,N}}(X'_v)$ for $X'_v \in \Delta'$.

An analogous result holds for the minorant of class $C^2_{1,L,N}$:

$$A^-_{C^2_{1,L,N}}(X) = \max_{\mu=1,\ldots,m; v=1,\ldots,(m+1)^2} (f_\mu - L\|X - X_\mu\|_1, f_v - L\|X - X'_v\|_1).$$

The proof of these facts follows the scheme outlined in [17] described in detail for other classes F_N and we shall not concentrate on it here.

The implication of Theorem 3.1, its corollaries and Lemma 3.1 is that they allow us to substantially reduce the amount of *a priori* information necessary for the application of the approach proposed in Section 2. Instead of $2m^2 + 2m + 1$ function values at the nodes of the grid $\bar{\Delta}$ it is sufficient to provide function values only at m^2 nodes of the grid Δ which is the optimal covering of π_2. This leads to an efficient constructive way of solving the problem of optimal-by-accuracy recovery $f(X)$ at point $X \in \pi_2$ of functions from $C^2_{1,L,N}$ by computing the values of majorant and minorant of class $C^2_{1,L,N}$ at nodes of the grid Δ' via the relationships (3.4), (3.11) and (3.12).

4 On optimal integration of product of functions in class $C^2_{1,L,N}$.

Let us consider integrals of the form (1.1) where $f(x_1, x_2) \in X^2_{1,L,N}$ and $\varphi_1(x_1)$, $\varphi_2(x_2)$ are known integrable functions.

In previous sections we studied properties of the functions $A^+_{C^2_{1,L,N}}(X)$, $A^-_{C^2_{1,L,N}}(X)$. Each of the regions of the form Q and Q' were split into sub-regions, in which functions $A^+_{C^2_{1,L,N}}(X)$, $A^-_{C^2_{1,L,N}}(X)$ were linear. We note that in the case when $\varphi_1(x_1)$, $\varphi_2(x_2)$ have simple forms (for example, when $\varphi_1(x_1) = \varphi_2(x_2) = 1$ or $\varphi_1(x_1) = \sin \omega_1 x_1$, $\varphi_2(x_2) = \sin \omega_2 x_2$ or $\varphi_1(x_1) = \cos \omega_1 x_1$, $\varphi_2(x_2) = \cos \omega_2 x_2$), computing $I^+(C^2_{1,L,N})$ and $I^-(C^2_{1,L,N})$ is a relatively easy procedure.

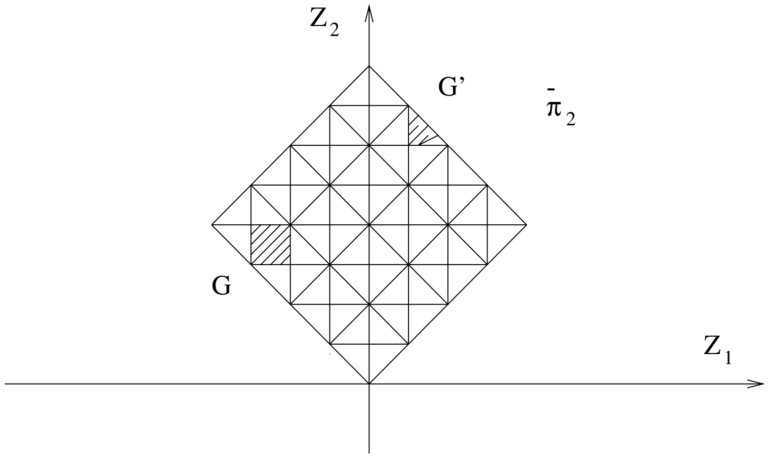

Figure 4.1: Domain π_2 in the new system of coordinates.

In order to simplify computations, we pass from the Cartesian system of coordinates (x_1, x_2) to a new system of coordinates by rotating coordinate axes about the angle $\alpha = -45^0$ (see Figure 4.1):

$$Z_1 = x_1 \cos\alpha + x_2 \sin\alpha, \qquad Z_2 = -x_1 \sin\alpha + x_2 \cos\alpha. \tag{4.1}$$

Therefore

$$\left\{ \begin{aligned} Z_1 &= \frac{\sqrt{2}}{2}(x_1 - x_2), \\ Z_{\overline{\overline{2}}} &= \frac{\sqrt{2}}{2}(x_1 + x_2), \end{aligned} \right. \quad \Rightarrow \quad \left\{ \begin{aligned} x_1 &= \frac{1}{\sqrt{2}}(Z_1 + Z_2), \\ x_2 &= \frac{1}{\sqrt{2}}(Z_2 - Z_1). \end{aligned} \right. \tag{4.2}$$

In the transfer to the new coordinate system, the region π_2 (see Figure 2.1) is transformed into the region $\bar{\pi}_2$ (see Figure 4.1), and regions Q and Q' (Figure 2.1), into regions G and G' respectively (Figure 4.1).

Let

- $\bar{G}_p$, $p = 1, \ldots, m(m-1)$ be regions of the form G that are located on the right of axis Oz_2, and $\hat{G}_p, p = 1, \ldots, m(m-1)$ be regions of the form G that are located on the left of the axis Oz_2;
- $\tilde{G}'_l$, $l = 1, \ldots, 2m$ be regions of the form G' that are located on the right of axis Oz_2, and $\breve{G}''_l$, $l = 1, \ldots, 2m$ be regions of the form G' that are located on the left of the axis Oz_2;

Let also the points $Z_{p_k}, k = 1, \ldots, 4$ be vertices of elementary region $\bar{G}_p$, and f_{p_k}, $k = 1, \ldots, 4$ be values of function $A^+_{C^2_{1,L,N}}(X)$ at these vertices. Then

$$\begin{aligned} Z_{p_1} &= (z_{1,j_1}; z_{2,j_2}), \qquad Z_{p_2} = (z_{1,j_1}; z_{2,j_2+1}),\\ Z_{p_3} &= (z_{1,j_1+1}; z_{2,j_2+1}), \qquad Z_{p_4} = (z_{1,j_1+1}; z_{2,j_2}), \end{aligned}$$

where

$$z_{1,j_1} = (j_1 - 1)h_1, \qquad z_{2,j_2} = (j_2 - 1)h_1, \quad h_1 = h/\sqrt{2}, \tag{4.3}$$

and

$$f_{p_1} = f_{j_1,j_2}, \quad f_{p_2} = f_{j_1,j_2+1}, \quad f_{p_3} = f_{j_1+1,j_2+1}, \quad f_{p_4} = f_{j_1+1,j_2}. \tag{4.4}$$

Here $j_1 = 1, \ldots, j_2 - 1$ for $j_2 = 2, \ldots, m$ and $j_1 = 1, \ldots, 2m - j_2$ for $j_2 = m + 1, \ldots, 2m - 1$, and

$$p = \begin{cases} (j_2 - 2)(j_2 - 1)/2 + j_1, & j_1 = 1, \ldots, j_2 - 1,\\ & j_2 = 2, \ldots, m,\\ (j_2 - 1)(4m - j_2)/2 - m^2 + j_1, & j_1 = 1, \ldots, 2m - j_2,\\ & j_2 = m + 1, \ldots, 2m - 1. \end{cases} \tag{4.5}$$

Setting $h_1 = h/\sqrt{2}$ we analogously determine vertices of the region $\hat{G}_p$, where p is computed by (4.5) ($p = 1, \ldots, m(m-1)$).

Let Z'_{l_k}, $k = 1, 2, 3$, be vertices of the elementary region $\tilde{G}'_l$, and f_{l_k}, $k = 1, 2, 3$ be values of the function $A^+_{C^2_{1,L,N}}(X)$ at these vertices. Then

$$\begin{cases} Z'_{l_1} = (z_{1,j_1}; z_{2,j_2}), \quad Z'_{l_2} = (z_{1,j_1}; z_{2,j_2+1}), \quad Z'_{l_3} = (z_{1,j_1+1}; z_{2,j_2+1}),\\ f_{l_1} = f_{j_1,j_2}, \quad f_{l_2} = f_{j_1,j_2+1}, \quad f_{l_3} = f_{j_1+1,j_2+1} \end{cases} \tag{4.6}$$

for $j_1 = j_2, j_2 = 1, \ldots, m$ and

$$\begin{cases} Z'_{l_1} = (z_{1,j_1}; z_{2,j_2}), \quad Z'_{l_2} = (z_{1,j_1}; z_{2,j_2+1}), \quad Z'_{l_3} = (z_{1,j_1+1}; z_{2,j_2})\\ f_{l_1} = f_{j_1,j_2}, \quad f_{l_2} = f_{j_1,j_2+1}, \quad f_{l_3} = f_{j_1+1,j_2} \end{cases} \tag{4.7}$$

for $j_2 = m + 1, \ldots, 2m$, $j_1 = 2m - j_2 + 1$ and $l = j_2$, $j_2 = 1, \ldots, 2m$.

Setting $h_1 = -h/\sqrt{2}$, in a similar way we determine the vertices of the region $\check{G}'_l$, $l = 1, \dots, 2m$.

We introduce the following notation:

$$
\text{(4.8)}\qquad \begin{cases}
\bar{I}^*_p = \dfrac{1}{2}\displaystyle\iint_{\bar{G}_p} \left(\tilde{A}^+_{C^2_{1,L,N}}(Z) + \tilde{A}^-_{C^2_{1,L,N}}(Z)\right)\tilde{\varphi}_1(z_1)\tilde{\varphi}_2(z_2)dZ,\\
\hat{I}^*_p = \dfrac{1}{2}\displaystyle\iint_{\hat{G}_p} \left(\tilde{A}^+_{C^2_{1,L,N}}(Z) + \tilde{A}^-_{C^2_{1,L,N}}(Z)\right)\tilde{\varphi}_1(z_1)\tilde{\varphi}_2(z_2)dZ,\\
\qquad\qquad\qquad\qquad\qquad p = 1, \dots, m(m-1),\\
\tilde{I}^*_l = \dfrac{1}{2}\displaystyle\iint_{\tilde{G}'_l} \left(\tilde{A}^+_{C^2_{1,L,N}}(Z) + \tilde{A}^-_{C^2_{1,L,N}}(Z)\right)\tilde{\varphi}_1(z_1)\tilde{\varphi}_2(z_2)dZ,\\
\check{I}^*_l = \dfrac{1}{2}\displaystyle\iint_{\check{G}'_l} \left(\tilde{A}^+_{C^2_{1,L,N}}(Z) + \tilde{A}^-_{C^2_{1,L,N}}(Z)\right)\tilde{\varphi}_1(z_1)\tilde{\varphi}_2(z_2)dZ,\\
\qquad\qquad\qquad\qquad\qquad l = 1, \dots, 2m,
\end{cases}
$$

where $\tilde{A}^{\pm}_{C^2_{1,L,N}}(Z)$ are majorant and minorant of the functional class $C^2_{1,L,N}$ in region $\bar{\pi}_2$ and

$$
\tilde{\varphi}_1(z_1)\tilde{\varphi}_2(z_2) = \varphi_1\left(\frac{1}{\sqrt{2}}(z_1+z_2)\right)\varphi_2\left(\frac{1}{\sqrt{2}}(z_2-z_1)\right), \quad (z_1, z_2) \in \bar{\pi}_2.
$$

It is easy to see that

$$
\text{(4.9)}\quad \tilde{A}^+_{C^2_{1,L,N}}(Z) = \begin{cases}
f_{j_1,j_2} + \dfrac{L}{\sqrt{2}}(z_1+z_2-h(j_1+j_2-2)), & Z \in \Gamma^*_1,\\
f_{j_1,j_2+1} + \dfrac{L}{\sqrt{2}}(z_1-z_2+h_1(j_2+1-j_1)), & Z \in \Gamma^*_2,\\
f_{j_1+1,j_2+1} + \dfrac{L}{\sqrt{2}}(h_1(j_1+j_2)-z_1-z_2), & Z \in \Gamma^*_3,\\
f_{j_1+1,j_2} + \dfrac{L}{\sqrt{2}}(z_2-z_1+h_1(j_1+1-j_2), & Z \in \Gamma^*_4,
\end{cases}
$$

where $G = \bigcup_{k=1}^4 \Gamma^*_k$, and Γ^*_k, $k = 1, 2, 3, 4$, are regions of linearity of the function $\tilde{A}^*_{C^2_{1,L,N}}(Z)$ in the region $G \subset \pi_2$ which are defined by transformations of the regions Ω^*_l, $l = 1, 2, 3, 4$, $\bigcup_{l=1}^4 \Omega^*_l = Q$. Further we have

$$
\text{(4.10)}\quad \tilde{A}^+_{C^2_{1,L,N}}(Z) = \begin{cases}
f_{j_1,j_2} + \dfrac{L}{\sqrt{2}}(z_1+z_2-h_1(j_1+j_2-2)), & Z \in \tilde{\Gamma}^*_1,\\
f_{j_1,j_2+1} + \dfrac{L}{\sqrt{2}}(z_1-z_2+h_1(j_2+1-j_1)), & Z \in \tilde{\Gamma}^*_2,\\
f_{j_1+1,j_2+1} + \dfrac{L}{\sqrt{2}}(h_1(j_1+j_2)-z_1-z_2), & Z \in \tilde{\Gamma}^*_3
\end{cases}
$$

with $\bigcup_{k=1}^3 \tilde{\Gamma}^*_k = \tilde{G}'_l, l = 1, \dots, m$ or $\bigcup_{k=1}^3 \tilde{\Gamma}^*_k = \check{G}'_l, l = 1, \dots, m$, and

$$(4.11)\quad \tilde{A}^{+}_{C^2_{1,L,N}}(Z) = \begin{cases} f_{j_1,j_2} + \dfrac{L}{\sqrt{2}}(z_1 + z_2 - h_1(j_1 + j_2 - 2)), & Z \in \check{\Gamma}^*_1, \\ f_{j_1,j_2+1} + \dfrac{L}{\sqrt{2}}(z_1 - z_2 + h_1(j_2 + 1 - j_1)), & Z \in \check{\Gamma}^*_2, \\ f_{j_1+1,j_2} + \dfrac{L}{\sqrt{2}}(z_2 - z_1 + h_1(j_1 + 1 - j_2)), & Z \in \check{\Gamma}^*_3 \end{cases}$$

with $\bigcup_{k=1}^{3} \check{\Gamma}^*_k = \tilde{G}'_l, l = m+1, \ldots, 2m$, or $\bigcup_{k=1}^{3} \check{\Gamma}^*_k = \check{G}'_l, l = m+1, \ldots, 2m$. Here $\tilde{\Gamma}^*_k$, $\check{\Gamma}_k$, $k = 1, 2, 3$, are regions of linearity of the functions $\tilde{A}^{\pm}_{C^2_{1,L,N}}(Z)$ in $\bar{G}'_l$, $\check{G}'_l$, $l = 1, \ldots, 2m$, from $\bar{\pi}_2$, which are determined by transformations of the regions $\bar{\Omega}^*_l$, $l = 1, 2, 3$, $\bigcup_{l=1}^{3} \bar{\Omega}^*_l = Q'$.

Theorem 4.1. *Let $f(X) \in C^2_{1,L,N}$. The optimal-by-accuracy cubature formula for computing integrals* (1.1) *in the case when functions $\tilde{\varphi}_1(z_1)$, $\tilde{\varphi}_2(z_2)$ do not change their signs for $(z_1, z_2) \in \bar{\pi}_2$ has the form*

$$(4.12)\qquad I^*(C^2_{1,L,N}) = \sum_{p=1}^{m(m-1)} (\bar{I}^*_p + \hat{I}^*_p) + \sum_{l=1}^{2m} (\tilde{I}^*_l + \check{I}^*_l),$$

and

$$(4.13)\quad \delta(C^2_{1,L,N}) = \frac{1}{2} \iint_{\bar{\pi}_2} \left(\tilde{A}^{+}_{C^2_{1,L,N}}(Z) - \tilde{A}^{-}_{C^2_{1,L,N}}(Z) \right) |\tilde{\varphi}_1(z_1) \tilde{\varphi}_2(z_2)| \, dZ.$$

Proof. It is clear that for $\tilde{\varphi}_1(z_1)\tilde{\varphi}_2(z_2) > 0$ for all $Z = (z_1, z_2) \in \bar{\pi}_2$ we have

$$(4.14)\qquad I^{\pm}(C^2_{1,L,N}) = \iint_{\bar{\pi}_2} \tilde{A}^{\pm}_{C^2_{1,L,N}}(Z) \tilde{\varphi}_1(z_1) \tilde{\varphi}_2(z_2) \, dZ,$$

and for $\tilde{\varphi}_1(z_1)\tilde{\varphi}_2(z_2) < 0$ we have

$$(4.15)\qquad I^{\pm}(C^2_{1,L,N}) = \iint_{\bar{\pi}_2} \tilde{A}^{\pm}_{C^2_{1,L,N}}(Z) \tilde{\varphi}_1(z_1) \tilde{\varphi}_2(z_2) \, dZ.$$

Taking into consideration (4.8), the statement of the theorem follows from (4.15) $\square$

Corollary 4.1. *Let functions $\tilde{\varphi}_1(z_1), \tilde{\varphi}_2(z_2)$ change their signs for $(z_1, z_2) \in \bar{\pi}_2$. Then the error of the cubature formula* (4.12) *will not exceed more than* 2 *times the optimal.*

The proof of this result follows a standard scheme [21, 22] and can be performed in a way similar to that described in detail in [17] for other classes F_N (see Corollary 3.1).

Remark 4.1. In the new system of coordinates, the majorant of the class $C^2_{1,L,N}$ in the elementary regions $\bar{G}_p$, $\hat{G}_p$, $p = 1, \ldots, m(m-1)$, has a form similar to the forms of majorants of some other classes F_N studied in [17]. This is due to the fact that the equations of the lines that split the regions $\bar{G}_p$, $\hat{G}_p$, $p = 1, \ldots, m(m-1)$

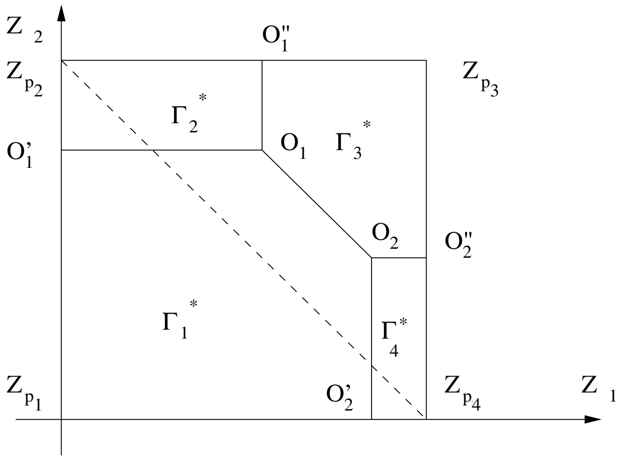

Figure 4.2: Splitting the domain $\bar{G}_p$.

into the regions Γ_k^*, $k = 1, 2, 3, 4$, are analogous to the equations of the lines that split corresponding regions of domains of definition for other classes (for example, where $K_p, p = 1, \ldots, m^2$ from [17] is split into the regions $\Omega_l^+, l = 1, 2, 3, 4$). To obtain such equations in the case of $F_N = C^2_{1,L,N}$, we considered the elementary region $\bar{G}_p$, $p = 1, \ldots, m(m-1)$. We placed the origin at the left lower vertex of this region. Then $Z_{p_1} = (0, 0)$, $Z_{p_2} = (0, h_1)$, $Z_{p_3} = (h_1, h_1)$, $Z_{p_4} = (h_1, 0)$. Let for the sake of definiteness $f_{p_2} + f_{p_4} > f_{p_1} + f_{p_3}$, $f_{p_3} > f_{p_1}$. Then the splitting of the region $\bar{G}_p$ into subregions Γ_k^*, $k = 1, 2, 3, 4$, is performed in the way shown in Figure 4.2, and the equations of the lines that split $\bar{G}_p$ into these sub-regions have the following forms:

(a) $z_2 = -z_1 + \dfrac{f_{p_3} - f_{p_1}}{\sqrt{2}L} + h_1$ for the line through points O_1, O_2;

(b) $z_2 = \dfrac{f_{p_2} - f_{p_1}}{\sqrt{2}L} + \dfrac{h_1}{2}$ for O_1, O_1';

(c) $z_1 = \dfrac{f_{p_3} - f_{p_2}}{\sqrt{2}L} + \dfrac{h_1}{2}$ for O_1, O_1'';

(d) $z_2 = \dfrac{f_{p_3} - f_{p_4}}{\sqrt{2}L} + \dfrac{h_1}{2}$ for O_2, O_2'';

(e) $z_1 = \dfrac{f_{p_4} - f_{p_1}}{\sqrt{2}L} + \dfrac{h_1}{2}$ for O_2, O_2'.

Remark 4.2. Finally, we note that for the construction of cubature formulae for computing integrals $I^2(f)$ in such interpolational classes as $C^2_{1,L,N}$ we can use the method of approximation of integrand by a linear spline [9, 24, 25]. However, if the zeros of the functions $\tilde{\varphi}_1(z_1), \tilde{\varphi}_2(z_2)$ are located in a relatively sparse manner with respect to the nodes of the grid, then the approach described in this paper

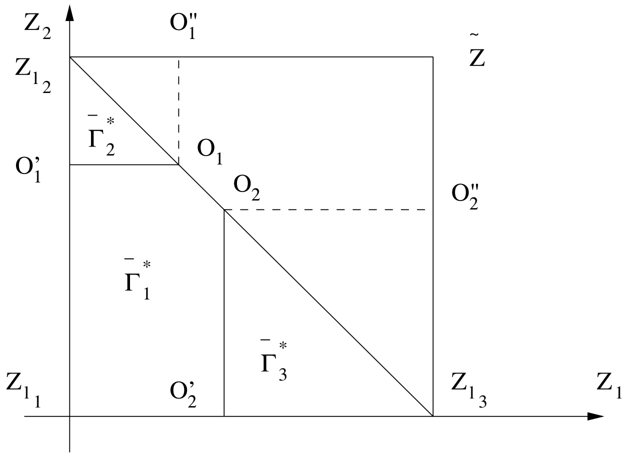

Figure 5.1: Expansion of $\tilde{G}'_l$ when $f_{l_1} < \frac{1}{2}(f_{l_2} + f_{l_3})$.

has advantages. Indeed, in regions of constant sign of functions $\tilde{\varphi}_1(z_1), \tilde{\varphi}_2(z_2)$, formula (4.12) is optimal-by-accuracy. In addition the proposed approach allows us to construct error estimates for formula (4.12)

$$(4.16)\quad v(C^2_{1,L,N}, I^*, \bar{f}) \leq \frac{1}{2}\iint_{\bar{\pi}_2} \left(\tilde{A}^+_{C^2_{1,L,N}}(Z) - \tilde{A}^-_{C^2_{1,L,N}}(Z)\right) |\tilde{\varphi}_1(z_1)\tilde{\varphi}_2(z_2| \, dZ.$$

5 Optimal-by-accuracy cubature formula for functions from class $C^2_{1,L,N}$.

Let us consider the problem of construction of optimal-by-accuracy cubature formula for computing integral (1.5) which is an important special case of the integral $I^2(f)$. As before we perform the rotation of the Cartesian system of coordinates about the angle $\alpha = -45^0$. Then

$$(5.1)\quad I^2_1(f) = \iint_{\bar{\pi}_2} \bar{f}(Z)\, dZ, \qquad \bar{f}(Z) = f\left(\frac{1}{\sqrt{2}}(z_2 + z_1), \frac{1}{\sqrt{2}}(z_2 - z_1)\right).$$

For convenience in computing integrals in regions of the form G' we expand all G' to regions of the form G, setting the value of the majorant in each additional vertex $\tilde{Z}$ equal to the value at that vertex of G' which is symmetric to $\tilde{Z}$ with respect to the center of the region G (see Figures 5.1 and 5.2).

In Figure 5.1 the expansion of elementary region $\tilde{G}'_l, l = m+1, \ldots, 2m$, to the region of the form G is shown in the case when $f_{l_1} < \frac{1}{2}(f_{l_2} + f_{l_3})$. In Figure 5.2 such an expansion is shown in the case when $f_{l_1} > \frac{1}{2}(f_{l_2} + f_{l_3})$, $f_{l_1} = f_{j_1,j_2}$, $f_{l_2} = f_{j_1,j_2+1}$, $f_{l_3} = f_{j_1+1,j_2}$, $j_1 = 2m - j_2 + 1$, $j_2 = m+1, \ldots, 2m$. The expansion of the regions $\tilde{G}'_l$, $l = 1, \ldots, m$, and $\check{G}'_l$, $l = 1, \ldots, 2m$, is performed analogously. Corresponding representations can also be obtained for the minorant of this class.

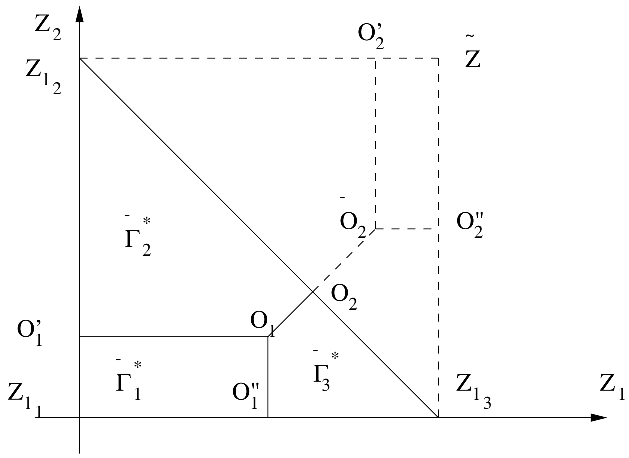

Figure 5.2: Expansion of $\tilde{G}_l'$ when $f_{l_1} > \frac{1}{2}(f_{l_2} + f_{l_3})$.

Let

$$\bar{R}_p^{\pm} = \iint_{\bar{G}_p} \tilde{A}^{\pm}_{C^2_{1,L,N}}(Z)\,dZ, \qquad \hat{R}_p^{\pm} = \iint_{\hat{G}_p} \tilde{A}^{\pm}_{C^2_{1,L,N}}(Z)\,dZ \tag{5.2}$$

where $p = 1, \ldots, m(m+1)$ and

$$\tilde{R}_l^{\pm} = \iint_{\tilde{G}_l'} \tilde{A}^{\pm}_{C^2_{1,L,N}}(Z)\,dZ, \qquad \check{R}_l^{\pm} = \iint_{\check{G}_l'} \tilde{A}^{\pm}_{C^2_{1,L,N}}(Z)\,dZ \tag{5.3}$$

where $l = 1, \ldots, 2m$.

It is easy to see that

$$\tilde{R}_l^{\pm} = \frac{1}{2}\iint_{\tilde{\mathcal{G}}_l'} \tilde{A}^{\pm}_{C^2_{1,L,N}}(Z)\,dZ, \qquad \check{R}_l^{\pm} = \frac{1}{2}\iint_{\check{\mathcal{G}}_l'} \tilde{A}^{\pm}_{C^2_{1,L,N}}(Z)\,dZ \tag{5.4}$$

where $l = 1, \ldots, 2m$, and $\tilde{\mathcal{G}}_l', \check{\mathcal{G}}_l'$ represent regions of the form $\tilde{G}_l', \check{G}_l'$ respectively expanded to the regions of the form G as explained above.

Let

$$\left\{ \begin{aligned} \bar{\xi}_{1,j_1} &= z_{1,j_1} + \frac{h_1}{2} + \sigma\left(\frac{f_{j_1+1,j_2} - f_{j_1,j_2}}{\sqrt{2}L}\mu_1 + \frac{f_{j_1+1,j_2+2} - f_{j_1,j_2+1}}{\sqrt{2}L}\mu\right), \\ \bar{\bar{\xi}}_{1,j_1} &= z_{1,j_1} + \frac{h_1}{2} + \sigma\left(\frac{f_{j_1+1,j_2+2} - f_{j_1,j_2+1}}{\sqrt{2}L}\mu_1 + \frac{f_{j_1+1,j_2} - f_{j_1,j_2}}{\sqrt{2}L}\mu\right), \\ \bar{\xi}_{2,j_2} &= z_{2,j_2} + \frac{h_1}{2} + \sigma\frac{f_{j_1+1,j_2+1} - f_{j_1+1,j_2}}{\sqrt{2}L}, \\ \bar{\bar{\xi}}_{2,j_2} &= z_{2,j_2} + \frac{h_1}{2} + \sigma\frac{f_{j_1,j_2+1} - f_{j_1,j_2}}{\sqrt{2}L}, \end{aligned} \right. \tag{5.5}$$

where

$$\mu_1 = \frac{1}{2}((1-\sigma)\gamma_2 + (1+\sigma)\gamma_1), \quad \mu_2 = \frac{1}{2}((1-\sigma)\gamma_1 + (1+\sigma)\gamma_2), \tag{5.6}$$

and

$$\begin{cases} \gamma_1 = \dfrac{1}{2}(1 - \operatorname{sign}(f_{j_1,j_2} + f_{j_1+1,j_2+1} - f_{j_1,j_2+1} - f_{j_1+1,j_2}), \\ \gamma_2 = \dfrac{1}{2}(1 + \operatorname{sign}(f_{j_1,j_2} + f_{j_1+1,j_2+1} - f_{j_1,j_2+1} - f_{j_1+1,j_2}), \end{cases} \tag{5.7}$$

$\sigma \in \{-1, 1\}$; $j_1 = 1, \dots, j_2 - 1$, for $j_2 = 2, \dots, m$ and $j_1 = 1, \dots, 2m - j_2$, for $j_2 = m+1, \dots, 2m-1$. Then the equation of the line that passes through points $(\bar{\xi}_{1,j_1}, \bar{\xi}_{2,j_2})$, $(\bar{\bar{\xi}}_{1,j_1}, \bar{\bar{\xi}}_{2,j_2})$ has the form

$$\frac{z_1 - \bar{\xi}_{1,j_1}}{\bar{\bar{\xi}}_{1,j_1} - \bar{\xi}_{1,j_1}} = \frac{z_2 - \bar{\xi}_{2,j_2}}{\bar{\bar{\xi}}_{2,j_2} - \bar{\xi}_{2,j_2}}, \tag{5.8}$$

or equivalently

$$(z_2 - \bar{\xi}_{2,j_2})(\bar{\bar{\xi}}_{1,j_1} - \bar{\xi}_{1,j_1}) = (z_1 - \bar{\xi}_{1,j_1})(\bar{\bar{\xi}}_{2,j_2} - \bar{\xi}_{2,j_2}). \tag{5.9}$$

Since

$$\bar{\bar{\xi}}_{1,j_1} - \bar{\xi}_{1,j_1} = \sigma(\mu_1 - \mu_2)\frac{1}{\sqrt{2}L}(f_{j_1+1,j_2+1} - f_{j_1,j_2+1} - f_{j_1+1,j_2} - f_{j_1,j_2}) \tag{5.10}$$

and

$$\bar{\bar{\xi}}_{2,j_2} - \bar{\xi}_{2,j_2} = \sigma(\mu_1 - \mu_2)\frac{1}{\sqrt{2}L}(f_{j_1,j_2+1} + f_{j_1+1,j_2} - f_{j_1+1,j_2+1} - f_{j_1,j_2}), \tag{5.11}$$

then

$$(\bar{\bar{\xi}}_{2,j_2} - \bar{\xi}_{2,j_2})/(\bar{\bar{\xi}}_{1,j_1} - \bar{\xi}_{1,j_1}) = \mu_2 - \mu_1. \tag{5.12}$$

Finally, from (5.12) we get

$$z_2 = \bar{\xi}_{2,j_2} + (\mu_1 - \mu_2)(\bar{\xi}_{1,j_1} - z_1), \tag{5.13}$$

$j_1 = 1, \dots, j_2 - 1$ for $j_2 = 2, \dots, m$ and $j_1 = 1, .., 2m - j_2$ for $j_2 = m+1, \dots, 2m-1$.

It is obvious that equation (5.13) is the equation of the line O_1O_2 for $\sigma = 1$ (see Figure 4.2). It is easy to show that for $\sigma = -1$ points $(\bar{\xi}_{1,j_1}, \bar{\xi}_{2,j_2})$ and $(\bar{\bar{\xi}}_{1,j_1}, \bar{\bar{\xi}}_{2,j_2})$ are the points that determine the splitting of the region of the form G' into regions of linearity of the function $\tilde{A}^{-}_{C^2_{1,L,N}}(Z)$, $j_1 = 1, \dots, j_2 - 1$, for $j_2 = 2, \dots, m$, and $j_1 = 1, \dots, 2m - j_2$ for $j_2 = m+1, \dots, 2m-1$. The explicit forms of (5.2) are given in Appendix A.

Lemma 5.1. *The optimal-by-accuracy cubature formula for computing integral $I_1^2(f)$ in class $C^2_{1,L,N}$ has the form*

$$I_1^*(C^2_{1,L,N}) = \frac{1}{2}\Bigg(\sum_{p=1}^{m(m-1)}(\bar{R}_p^* + \hat{R}_p^*) + \sum_{l=1}^{2m}(\tilde{R}_l^* + \check{R}_l^*)\Bigg), \tag{5.14}$$

where

$$(5.15)\quad \begin{cases} \bar{R}^*_p = \dfrac{1}{2}(\bar{R}^*_p + \bar{R}^-_p), & \hat{R}^*_p = \dfrac{1}{2}(\hat{R}^+_p + \hat{R}^-_p), \quad p = 1, \dots, m(m-1), \\ \tilde{R}^*_l = \dfrac{1}{2}(\tilde{R}^+_l + \tilde{R}^-_l), & \check{R}^*_l = \dfrac{1}{2}(\check{R}^+_l + \check{R}^-_l), \quad l = 1, \dots, 2m, \end{cases}$$

and

$$\bar{\delta}(C^2_{1,L,N}) = \frac{1}{2}\left(\sum_{p=1}^{m(m-1)} (\bar{R}^+_p + \hat{R}^+_p - \bar{R}^-_p - \hat{R}^-_p) + \sum_{l=1}^{2m} (\tilde{R}^+_l - \check{R}^+_l - \tilde{R}^-_l - \check{R}^-_l) \right),$$

and the values $\bar{R}^\pm_p$, $\hat{R}^\pm_p$, $p = 1, \dots, m(m-1)$, *and* $\tilde{R}^\pm_l, \check{R}^\pm_l, l = 1, \dots, 2m$ *are determined by the formulae* (A.1)–(A.4).

Proof. Optimality-by-accuracy of the cubature formula (5.14) follows immediately from Theorem 4.1.

Taking into account the explicit forms of $\bar{R}^\pm_p$ given in Appendix A (in particular, (A.3)), from (4.16), (4.11) it follows that computing $\tilde{R}^\pm_l$, $l = 1, \dots, 2m$, can also be performed using relationships (A.1)–(A.4) for $j_1 = j_2$ when $j_2 = 1, \dots, m$, and for $j_1 = 2m - j_2 + 1$ when $j_2 = m+1, 2m$. Assuming that $h_1 = -h/\sqrt{2}$ from (A.1)–(A.4) we analogously obtain $\hat{R}^\pm_p$, $p = 1, \dots, m(m-1)$, and $\check{R}^\pm_l$, $l = 1, \dots, 2m$. □

6 Optimal-by-order cubature formulae for computing integrals of fast-oscillatory functions in class $C^2_{1,L,N}$.

In this section the problem of computing integrals $I^2_2(f)$, $I^2_3(f)$ is considered in the case when $\omega_1 = \omega_2 = \omega$, $|\omega| \geq 2\pi$. It is easy to see that in the transformation to the new system of coordinates (4.2) we have

$$(6.1)\quad \begin{aligned} I^2_2(f) &= \iint_{\pi_2} f(X) \sin \omega x_1 \sin \omega x_2 \, dX \\ &= \frac{1}{2} \left(\iint_{\bar{\pi}_2} \bar{f}(Z) \cos \bar{\omega} z_1 \, dZ - \bar{f}(Z) \cos \bar{\omega} z_2 \, dZ \right) \end{aligned}$$

and

$$(6.2)\quad \begin{aligned} I^2_3(f) &= \iint_{\pi_2} f(X) \cos \omega x_1 \cos \omega x_2 \, dX \\ &= \frac{1}{2} \left(\iint_{\bar{\pi}_2} \bar{f}(Z) \cos \bar{\omega} z_1 \, dZ + \bar{f}(Z) \cos \bar{\omega} z_2 \, dZ \right), \end{aligned}$$

where

$$(6.3)\quad \bar{f}(Z) = \left(\frac{1}{\sqrt{2}}(z_1 + z_2), \frac{1}{\sqrt{2}}(z_2 - \bar{z}_1) \right), \quad \bar{\omega} = \sqrt{2}\omega.$$

Let

$$(6.4)\quad S(\bar{f}) = \iint_{\bar{\pi}_2} \bar{f}(Z) \cos \bar{\omega} z_1 \, dZ, \quad E(\bar{f}) = \iint_{\bar{\pi}_2} \bar{f}(Z) \cos \bar{\omega} z_2 \, dZ.$$

We note that the problem of computing integrals (6.1), (6.2) reduces to the problem of computing integrals (6.4).

For many classes of problems, for which we have to compute both integrals $I_2^2(f)$ and $I_3^2(f)$, this reduction allows us to substantially increase efficiency of computations. This advantage is especially important in those cases when computing $I_2^2(f)$, I_3^2 have to be performed many times as it is the case in the solution of problems in recognition and classification of images.

Let us construct a cubature formula for computing the first integral in (6.4). Let us introduce the following notation:

$$\bar{S}_p^{\pm} = \iint_{\bar{G}_p} \tilde{A}^{\pm}_{C^2_{1,L,N}}(Z) \cos \bar{\omega} z_1 \, dZ, \quad \hat{S}_p^{\pm} = \iint_{\hat{G}_p} \tilde{A}^{\pm}_{C^2_{1,L,N}}(Z) \cos \bar{\omega} z_1 \, dZ, \tag{6.5}$$

with $p = 1, \dots, m(m-1)$ and

$$\tilde{S}_l^{\pm} = \frac{1}{2} \iint_{\tilde{G}'_l} \tilde{A}^{\pm}_{C^2_{1,L,N}}(Z) \cos \bar{\omega} z_1 \, dZ, \quad \check{S}_l^{\pm} = \frac{1}{2} \iint_{\check{G}'_l} \tilde{A}^{\pm}_{C^2_{1,L,N}}(Z) \cos \bar{\omega} z_1 \, dZ, \tag{6.6}$$

with $l = 1, \dots, 2m$.

LEMMA 6.1. *The cubature formula*

$$\mathcal{S}(\bar{f}) = \sum_{p=1}^{m(m-1)} (\bar{S}'_p + \hat{S}'_p) + \sum_{l=1}^{2m} (\tilde{S}'_l + \check{S}'_l), \tag{6.7}$$

with

$$\begin{cases} \bar{S}'_p = \dfrac{1}{2}(\bar{S}_p^+ + \bar{S}_p^-), & \hat{S}'_p = \dfrac{1}{2}(\hat{S}_p^+ + \hat{S}_p^-), \quad p = 1, \dots, m(m-1), \\ \tilde{S}'_l = \dfrac{1}{2}(\tilde{S}_l^+ + \tilde{S}_l^-), & \check{S}'_l = \dfrac{1}{2}(\check{S}_l^+ + \check{S}_l^-), \quad p = 1, \dots, 2m, \end{cases} \tag{6.8}$$

is optimal-by-order with a constant not exceeding 2. *The error estimate of cubature formula* (6.7) *is determined from the following relationship:*

$$\begin{aligned} &v(C^2_{1,L,N}, \mathcal{S}, \bar{f}) \\ &\quad \le \frac{1}{2}\Big(\sum_{p=1}^{m(m-1)} \Big(\max(\bar{S}_p^+, \bar{S}_p^-) + \max(\hat{S}_p^+, \hat{S}_p^-) - \min(\bar{S}_p^+, \bar{S}_p^-) - \min(\hat{S}_p^+, \hat{S}_p^-) \Big) \\ &\qquad + \sum_{l=1}^{2m} \Big(\max(\tilde{S}_l^+, \tilde{S}_l^-) + \max(\check{S}_l^+, \check{S}_l^-) - \min(\tilde{S}_l^+, \tilde{S}_l^-) - \min(\check{S}_l^+, \check{S}_l^-) \Big) \Big). \end{aligned}$$

The statement of this lemma follows directly from Corollary 3.1. In analogy with relationships (A.1)–(A.4) (see Appendix A) it can be shown that

$$\bar{S}_p^{\pm} = \sum_{k=1}^{3} \bar{P}_k, \quad p = 1, \dots, m(m-1), \tag{6.9}$$

where the explicit forms of $\bar{P}_k$ are given in Appendix B.

Optimal-by-order cubature formulae for computing the second integral in (6.4) are constructed in an analogous way. Indeed, letting

$$\bar{E}_p^{\pm} = \iint_{\bar{G}_p} \tilde{A}^{\pm}_{C^2_{1,L,N}}(Z)\cos\bar{\omega}z_2\,dZ, \qquad \hat{E}_p^{\pm} = \iint_{\hat{G}_p} \tilde{A}^{\pm}_{C^2_{1,L,N}}(Z)\cos\bar{\omega}z_2\,dZ,$$

for $p = 1,\ldots,m(m-1)$, and

$$\tilde{E}_l^{\pm} = \frac{1}{2}\iint_{\tilde{G}'_l} \tilde{A}^{\pm}_{C^2_{1,L,N}}(Z)\cos\bar{\omega}z_2\,dZ, \qquad \check{E}_l^{\pm} = \frac{1}{2}\iint_{\check{G}_l} \tilde{A}^{\pm}_{C^2_{1,L,N}}(Z)\cos\bar{\omega}z_2\,dZ,$$

for $l = 1,\ldots,2m$, we come to the following result:

Lemma 6.2. *The cubature formula*

$$\mathcal{E}(\bar{f}) = \sum_{p=1}^{m(m-1)} (\bar{E}'_p + \hat{E}'_p) + \sum_{l=1}^{2m} (\tilde{E}'_l + \check{E}'_l), \tag{6.10}$$

with

$$\begin{cases} \bar{E}'_p = \frac{1}{2}(\bar{E}_p^+ + \bar{E}_p^-), \quad \hat{E}'_p = \frac{1}{2}(\hat{E}_p^+ + \hat{E}_p^-), \quad p = 1,\ldots,m(m-1), \\ \tilde{E}'_l = \frac{1}{2}(\tilde{E}_l^+ + \tilde{E}_l^-), \quad \check{E}'_l = \frac{1}{2}(\check{E}_l^+ + \check{E}_l^-), \quad p = 1,\ldots,2m, \end{cases} \tag{6.11}$$

is optimal-by-order with a constant not exceeding 2*. The error estimate of cubature formula* (6.10) *is determined from the following relationship:*

$$v(C^2_{1,L,N},\mathcal{E},\bar{f}) \le \frac{1}{2}\Bigg(\sum_{p=1}^{m(m-1)} (|\bar{E}_p^+ - \bar{E}_p^-| + |\hat{E}_p^+ - \hat{E}_p^-|) + \sum_{l=1}^{2m}\Big(|\tilde{E}_l^+ - \tilde{E}_l^-| + |\check{E}_l^+ - \check{E}_l^-|\Big)\Bigg).$$

The statement of the lemma follows directly from Corollary 4.1. In analogy with relationships (4.5)–(4.16) it can be shown that

$$\bar{E}_p^{\pm} = \sum_{k=1}^{3} \bar{Q}_k, \quad p = 1,\ldots,m(m-1), \tag{6.12}$$

where the explicit forms of $\bar{Q}_k$ are given in Appendix B.

7 Conclusions.

In this paper we solved constructively the problem of optimal-by-accuracy recovery of functions from the given interpolational class $C^2_{1,L,N}$. Explicit forms for the majorant and minorant of this class were obtained and their properties were investigated in detail. By using these properties we analysed an intrinsic connection between the problem of optimal-by-accuracy recovery of functions and the problem of optimal-by-accuracy integration. This analysis resulted in the construction of optimal-by-accuracy cubature formulae and optimal-by-order (with a constant not exceeding 2) cubature formulae for computing integrals from fast-oscillatory functions in class $C^2_{1,L,N}$. Along with the derivation of cubature formulae optimal in the specified sense, we also showed that an appropriate choice of the grid in the given interpolational class leads to a substantial reduction in a priori information required for the application of our approach.

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A Appendix

The explicit forms of integrals (5.2) are obtained from the relationships (4.15) and the equations of the lines that split elementary regions in class $F_N = C^2_{1,L,N}$ (see Remark 4.1)

$$
\begin{aligned}
\bar{R}_p^{\pm} &= \int_{z_{1,j_1}}^{\bar{\xi}_{1,j_1}} \int_{z_{2,j_2}}^{\bar{\bar{\xi}}_{2,j_2}} \Big(f_{j_1,j_2} + \frac{\sigma L}{\sqrt{2}}(z_1 - z_{1,j_1} + z_2 - z_{2,j_2}) \Big)\, dz_2\, dz_1 \\
&\quad + \int_{z_{1,j_1}}^{\bar{\xi}_{1,j_1}} \int_{\bar{\bar{\xi}}_{2,j_2}}^{z_{2,j_2+1}} \Big(f_{j_1,j_2+1} + \frac{\sigma L}{\sqrt{2}}(z_1 - z_{1,j_1} + z_{2,j_2+1} - z_2) \Big)\, dz_2\, dz_1 \\
&\quad + \int_{\bar{\xi}_{1,j_1}}^{z_{1,j_1+1}} \int_{\bar{\xi}_{2,j_2}}^{z_{2,j_2+1}} \Big(f_{j_1+1,j_2+2} + \frac{\sigma L}{\sqrt{2}}(z_{1,j_1+1} + z_{2,j_2+1} - z_1 - z_2) \Big)\, dz_2\, dz_1 \\
&\quad + \int_{\bar{\xi}_{1,j_1}}^{z_{1,j_1+1}} \int_{z_{2,j_2}}^{\bar{\xi}_{2,j_2}} \Big(f_{j_1+1,j_2} + \frac{\sigma L}{\sqrt{2}}(z_{1,j_1+1} - z_{2,j_2} - z_1 + z_2) \Big)\, dz_2\, dz_1 \\
&\quad + \int_{\bar{\bar{\xi}}_{1,j_1}}^{\bar{\xi}_{1,j_1}} \int_{z_{2,j_2}}^{\bar{\xi}_{2,j_2} + (\mu_1-\mu_2)(\bar{\xi}_{2,j_2} - z_1)} \Big(\mu_1 f_{j_1,j_2} + \mu_2 f_{j_1+1,j_2} \\
&\qquad\qquad + \frac{\sigma L}{\sqrt{2}}\, ((\mu_1 - \mu_2) z_1 - \mu_1 z_{1,j_1} + \mu_2 z_{1,j_1+1} + z_2 - z_{2,j_2}) \Big)\, dz_2\, dz_1 \qquad \text{(A.1)} \\
&\quad + \int_{\bar{\bar{\xi}}_{1,j_1}}^{\bar{\xi}_{1,j_1}} \int_{\bar{\xi}_{2,j_2} + (\mu_1-\mu_2)(\bar{\xi}_{1,j_1} - z_2)}^{z_{2,j_2+1}} \Big(\mu_1 f_{j_1+1,j_2+1} + \mu_2 f_{j_1,j_2+1} \\
&\qquad\qquad + \frac{\sigma L}{\sqrt{2}}((\mu_2 - \mu_1) z_1 - \mu_2 z_{1,j_1} - \mu_1 z_{1,j_1+1} + z_{2,j_2+1} - z_2) \Big)\, dz_2\, dz_1 \\
&= \sum_{k=1}^{3} \bar{p}_k^{\pm}, \quad p = 1, \ldots, m(m-1),
\end{aligned}
$$

where

$$
\bar{p}_1^{\pm} = (\bar{\bar{\xi}}_{1,j_1} - z_{1,j_1}) \Big(z_{2,j_2+1} f_{j_1,j_2+1} - z_{2,j_2} f_{j_1,j_2} + \frac{\sigma L h_1}{\sqrt{2}}\, \bar{\bar{\xi}}_{1,j_1} \Big)
$$

$$+ (z_{1,j_1+1} - \bar{\xi}_{1,j_1})\Big(z_{2,j_2+1}f_{j_1+1,j_2+1} - z_{2,j_2}f_{j_1+1,j_2} + \frac{\sigma L h_1}{\sqrt{2}}\bar{\xi}_{1,j_1}\Big), \tag{A.2}$$

$$\begin{aligned}\bar{p}_2^{\pm} = {}& (\bar{\xi}_{1,j_1} - \bar{\bar{\xi}}_{1,j_1})(\bar{\xi}_{2,j_2} - z_{2,j_2}) \\ &\times \Big(\mu_1\Big(f_{j_1,j_2} - \frac{\sigma L}{\sqrt{2}}z_{1,j_1}\Big) + \mu_2\Big(f_{j_1+1,j_2} + \frac{\sigma L}{\sqrt{2}}z_{1,j_1+1}\Big)\Big) \\ &+ (z_{2,j_2+1} - \bar{\xi}_{2,j_2})\Big(\mu_1\Big(f_{j_1+1,j_2+1} + \frac{\sigma L}{\sqrt{2}}z_{1,j_1+1}\Big) + \mu_2\Big(f_{j_1,j_2+1} - \frac{\sigma L}{\sqrt{2}}z_{1,j_1}\Big)\Big),\end{aligned} \tag{A.3}$$

$$\begin{aligned}\bar{p}_3^{\pm} = {}& \frac{\sigma L}{2\sqrt{2}}\Big(4(\mu_1 - \mu_2)\bar{\xi}_{1,j_1}\bar{\xi}_{2,j_2} + z_{2,j_2+1}^2 + 2\bar{\xi}_{2,j_2}^2 \\ &- \frac{2}{3}(\bar{\xi}_{1,j_1}^2 + \bar{\xi}_{1,j_1}\bar{\bar{\xi}}_{1,j_1} - 2\bar{\bar{\xi}}_{1,j_1}^2)\Big),\end{aligned} \tag{A.4}$$

$j_1 = 1, \dots, j_2 - 1$, for $j_2 = 2, \dots, m$, and $j_1 = 1, \dots, 2m - j_2$, for $j_2 = m+1, 2m-1$. Setting $\sigma = 1$ and $h_1 = h/\sqrt{2}$ in (A.1)–(A.4), we obtain $\bar{R}_p^+, p = 1, \dots, m(m-1)$, and setting $\sigma = -1$ and $h_1 = h/\sqrt{2}$, we obtain $\bar{R}_p^-$, $p = 1, \dots, m(m-1)$. It is easy to see that values $\hat{R}_p^{\pm}$, $p = 1, \dots, m(m-1)$ can also be determined from formulae (A.1)–(A.4). Indeed, setting $\sigma = 1$, $h = -h_1/\sqrt{2}$ in (A.1)–(A.4), we obtain $\hat{R}_p^+$, and setting $\sigma = -1$, $h = -h_1/\sqrt{2}$ we obtain $\hat{R}_p^-$, $p = 1, \dots, m(m-1)$.

B Appendix

The additives in (6.9) have the following forms:

$$\begin{aligned}\bar{P}_1 = {}& \frac{1}{\omega}\Big\{(\sin\bar{\omega}\bar{\xi}_{1,j_1} - \sin\bar{\omega}z_{1,j_1})\Big(f_{j_1,j_2}(\bar{\bar{\xi}}_{2,j_2} - z_{2,j_2}) + f_{j_1,j_2+1}(z_{2,j_2+1} + \bar{\bar{\xi}}_{2,j_2}) \\ &+ \frac{\sigma L}{\sqrt{2}}\Big(\bar{\bar{\xi}}_{2,j_2}^2 - \bar{\bar{\xi}}_{2,j_2}(z_{2,j_2} + z_{2,j_2+1}) + \frac{1}{2}(z_{2,j_2}^2 + z_{2,j_2+1}^2)\Big)\Big) \\ &+ \big(\sin\bar{\omega}z_{1,j_1+1} - \sin\bar{\omega}\bar{\xi}_{1,j_1}\big) \\ &\times \Big(f_{j_1+1,j_2+1}(z_{2,j_2} - \bar{\xi}_{2,j_2}) + f_{j_1+1,j_2}(\bar{\xi}_{2,j_2} - z_{2,j_2}) \\ &+ \frac{\sigma L}{\sqrt{2}}\Big(\bar{\xi}_{2,j_2}^2 - \bar{\xi}_{2,j_2}(z_{2,j_2} + z_{2,j_2+1}) + \frac{1}{2}(z_{2,j_2}^2 + z_{2,j_2+1}^2)\Big)\Big)\Big\},\end{aligned} \tag{B.1}$$

$$\begin{aligned}\bar{P}_2 = {}& \frac{1}{\omega}(\sin\bar{\omega}\bar{\xi}_{1,j_1} - \sin\omega\,\bar{\bar{\xi}}_{1,j_1})\Big\{z_{2,j_2+1}\Big(\mu_1 f_{j_1+1,j_2+1} + \mu_2 f_{j_1,j_2+1} \\ &+ \frac{\sigma L}{\sqrt{2}}\Big(\mu_1 z_{1,j_1+1} - \mu_2 z_{1,j_1} - (\mu_1 - \mu_2)\bar{\xi}_{1,j_1} - \bar{\xi}_{2,j_2}\Big)\Big) \\ &- z_{2,j_2}\Big(\mu_1 f_{j_1,j_2} + \mu_2 f_{j_1+1,j_2} \\ &+ \frac{\sigma L}{\sqrt{2}}\Big(\mu_1 z_{1,j_1+1} - \mu_2 z_{1,j_1} + (\mu_1 - \mu_2)\bar{\xi}_{1,j_1} + \bar{\xi}_{2,j_2}\Big)\Big) \\ &+ \frac{\sigma L}{\sqrt{2}}\Big(\frac{1}{2}(z_{2,j_2+1}^2 + z_{2,j_2}^2) + (\bar{\xi}_{2,j_2} + (\mu_1 - \mu_2)\bar{\xi}_{1,j_1})^2\Big)\Big\},\end{aligned} \tag{B.2}$$

$$\bar{P}_3 = \frac{1}{\omega}\Big\{\Big(\mu_1(f_{j_1,j_2} - f_{j_1+1,j_2+1}) + \mu_2(f_{j_1+1,j_2} - f_{j_1,j_2+1})$$
$$+ \frac{\sigma L}{\sqrt{2}}(\mu_2 - \mu_1)(z_{1,j_1+1} + z_{1,j_1})\Big)$$
$$\times \Big(\bar{\xi}_{2,j_2} \sin \bar{\omega}\bar{\xi}_{1,j_1} - \bar{\bar{\xi}}_{2,j_2} \sin \bar{\omega}\, \bar{\bar{\xi}}_{1,j_1}$$
$$-\frac{1}{\omega}(\mu_1 - \mu_2)\Big(\cos \bar{\omega}\bar{\xi}_{1,j_1} - \cos \bar{\omega}\, \bar{\bar{\xi}}_{1,j_1}\Big)\Big)$$
$$+ \frac{\sigma L h_1}{\sqrt{2}}\Big(\sin \bar{\omega}\, \bar{\bar{\xi}}_{1,j_1} (\bar{\bar{\xi}}_{1,j_1} - z_{1,j_1}) + (\bar{\xi}_{1,j_1} - z_{1,j_1+1}) \sin \bar{\omega}\bar{\xi}_{1,j_1}$$
$$+ \frac{1}{\omega}\Big(\cos \bar{\omega}\bar{\xi}_{1,j_1} - \cos \bar{\omega} z_{1,j_1} - \cos \bar{\omega} z_{1,j_1+1} + \cos \bar{\omega}\bar{\xi}_{1,j_1}\Big)\Big)\Big\}, \tag{B.3}$$

and $h_1 = h/\sqrt{2}$, $\bar{\xi}_{1,j_1}$, $\bar{\bar{\xi}}_{1,j_1}$, $\bar{\xi}_{2,j_2}$, $\bar{\bar{\xi}}_{2,j_2}$, μ_1, μ_2, are determined from relationships (5.5)–(5.13), where $j_1 = 1, \dots, j_2 - 1$, for $j_2 = 2, \dots, m$, and $j_1 = 1, \dots, 2m - j_2$, for $j_2 = m+1, \dots, 2m-1$. If in (6.9)–(B.3) we set $\sigma = 1$ we obtain $\bar{S}_p^+$, and by setting $\sigma = -1$ we obtain $\bar{S}_p^-$, $p = 1, \dots, m(m-1)$. Calculations of $\tilde{S}_l^{\pm}$, $l = 1, \dots, 2m$ are also performed with the help of relationships (6.9)–(B.3), for $j_1 = j_2$, when $j_2 = 1, \dots, m$ and for $j_1 = 2m - j_2 + 1$, when $j_2 = m+1, \dots, 2m$. Setting $h_1 = -h/\sqrt{2}$ from (6.9)–(B.3) we obtain $\hat{S}_p^{\pm}$, $p = 1, \dots, m(m-1)$ and $\check{S}_l^{\pm}$, $l = 1, \dots, 2m$.

The additives in (6.12) have the following forms

$$\bar{Q}_1 = \frac{1}{\bar{\omega}}\Big\{(\bar{\bar{\xi}}_{1,j_1} - z_{1,j_1})\Big(f_{j_1,j_2}(\sin \bar{\omega}\bar{\xi}_{2,j_2} - \sin \bar{\omega} z_{2,j_2}) + f_{j_1,j_2+1}(\sin \bar{\omega} z_{2,j_2+1}$$
$$- \sin \bar{\omega}\bar{\xi}_{2,j_2}) + \frac{\sigma L}{\sqrt{2}}\Big((2\,\bar{\bar{\xi}}_{2,j_2} - z_{2,j_2} - z_{2,j_2+1}) \sin \bar{\omega}\, \bar{\bar{\xi}}_{2,j_2}$$
$$+ \frac{1}{\bar{\omega}}\big(2 \cos \bar{\omega}\, \bar{\bar{\xi}}_{2,j_2} - \cos \bar{\omega} z_{2,j_2} - \cos \bar{\omega} z_{2,j_2+1}\big)\Big)\Big) + \big(z_{1,j_1+1} - \bar{\xi}_{1,j_1}\big)$$
$$\times \Big(f_{j_1+1,j_2+1}\big(\sin \bar{\omega} z_{2,j_2+1} - \sin \bar{\omega}\bar{\xi}_{2,j_2}\big) + f_{j_1+1,j_2}\big(\sin \bar{\omega}\bar{\xi}_{2,j_2} - \sin \bar{\omega} z_{2,j_2}\big)$$
$$+ \frac{\sigma L}{\sqrt{2}}\Big((2\bar{\xi}_{2,j_2} - z_{2,j_2} - z_{2,j_2+1}) \sin \bar{\omega}\bar{\xi}_{2,j_2}$$
$$+ \frac{1}{\bar{\omega}}\big(2 \cos \bar{\omega}\bar{\xi}_{2,j_2} - \cos \bar{\omega} z_{2,j_2} - \cos \bar{\omega} z_{2,j_2+1}\big)\Big)\Big)\Big\}; \tag{B.4}$$

$$\bar{Q}_2 = \frac{1}{\bar{\omega}}(\bar{\xi}_{1,j_1} - \bar{\bar{\xi}}_{1,j_1})\Big\{\sin \bar{\omega} z_{2,j_2+1}\Big(\mu_1 f_{j_1+1,j_2+1} + \mu_2 f_{j_1,j_2+1}$$
$$+ \frac{\sigma L}{\sqrt{2}}\Big(\mu_1\Big(z_{1,j_1+1} - \frac{1}{2}(\bar{\xi}_{1,j_1} + \bar{\bar{\xi}}_{1,j_1})\Big) - \mu_2(z_{1,j_1} - \frac{1}{2}(\bar{\xi}_{1,j_1} + \bar{\bar{\xi}}_{1,j_1}))\Big)\Big)$$
$$- \sin \bar{\omega} z_{2,j_2}\Big(\mu_1 f_{j_1,j_2} + \mu_2 f_{j_1+1,j_2+1}$$
$$- \frac{\sigma L}{\sqrt{2}}\Big(\mu_1\Big(z_{1,j_1} - \frac{1}{2}(\bar{\xi}_{1,j_1} + \bar{\bar{\xi}}_{1,j_1})\Big) - \mu_2\Big(z_{1,j_1+1} - \frac{1}{2}(\bar{\xi}_{1,j_1} + \bar{\bar{\xi}}_{1,j_1})\Big)\Big)\Big)$$

$$- \frac{\sigma L}{\sqrt{2}\bar{\omega}} \times (\cos\bar{\omega} z_{2,j_2+1} + \cos\bar{\omega} z_{2,j_2}) \Big\}, \tag{B.5}$$

$$\begin{aligned}
\bar{Q}_3 = \frac{1}{\bar{\omega}} \Big\{ \Big(& \mu_1 (f_{j_1,j_2} - f_{j_1+1,j_2+1}) + \mu_2 (f_{j_1+1,j_2} - f_{j_1,j_2+1}) \\
&+ \frac{\sigma L}{\sqrt{2}} ((\mu_2 - \mu_1)(z_{1,j_1+1} + z_{1,j_1}) - z_{2,j_2+1} - z_{2,j_2} \\
&+ 2\big(\bar{\xi}_{2,j_2} + (\mu_1 - \mu_2)\bar{\xi}_{1,j_1}\big)\big((\mu_1 - \mu_2) \\
&\times \cos\bar{\omega}\bar{\xi}_{2,j_2}\big(1 - \cos\bar{\omega}\big(\bar{\xi}_{1,j_1} - \bar{\bar{\xi}}_{1,j_1}\big)\big) + \sin\bar{\omega}\bar{\xi}_{2,j_2} \sin\bar{\omega}(\bar{\xi}_{1,j_1} - \bar{\bar{\xi}}_{1,j_1})) \\
&+ \frac{\sigma L}{\sqrt{2}} \Big(\big(\sin\bar{\omega} z_{2,j_2+1} - \sin\bar{\omega} z_{2,j_2} \big) \Big(\bar{\bar{\xi}}_{1,j_1} \; (\bar{\bar{\xi}}_{1,j_1} - z_{1,j_1}) \\
&+ \bar{\xi}_{1,j_1} (\bar{\xi}_{1,j_1} - z_{1,j_1+1}) + \frac{1}{2}(z_{1,j_1}^2 + z_{1,j_1+1}^2) \Big) \\
&+ \frac{2}{\bar{\omega}^2} (\cos\bar{\omega}\bar{\xi}_{2,j_2} \sin\bar{\omega}\big(\bar{\xi}_{1,j_1} - \bar{\bar{\xi}}_{1,j_1} \,\big) \\
&- (\mu_1 - \mu_2) \sin\bar{\omega}\bar{\xi}_{2,j_2} \big(1 - \cos\bar{\omega}\big(\bar{\xi}_{1,j_1} - \bar{\bar{\xi}}_{1,j_1} \,\big)\big)\Big)\Big) \Big\}
\end{aligned} \tag{B.6}$$

and $h_1 = h/\sqrt{2}$ and $\bar{\xi}_{1,j_1}$, $\bar{\bar{\xi}}_{1,j_1}$, $\bar{\xi}_{2,j_2}$, $\bar{\bar{\xi}}_{2,j_2}$, μ_1, μ_2 are determined from relationships (5.5)–(5.13), $j_1 = 1, \ldots, j_2 - 1$, for $j_2 = 2, \ldots, m$, and $j_1 = 1, \ldots, 2m - j_2$, for $j_2 = m+1, \ldots, 2m-1$. If we set in (6.12)–(B.6) $\sigma = 1$ we obtain $\bar{E}_p^+$, and by setting $\sigma = -1$ we obtain $\bar{E}_p^-$, $p = 1, \ldots, m(m-1)$. Computing $\tilde{E}_l^{\pm}$, $l = 1, \ldots, 2m$, is also performed with the help of relationships (6.12)–(B.6) for $j_1 = j_2$, when $j_2 = 1, \ldots, m$, and for $j_1 = 2m - j_2 + 1$, when $j_2 = m+1, \ldots, 2m$. Setting $h_1 = -h/\sqrt{2}$, from (6.12)–(B.6) we obtain $\hat{E}_p^{\pm}$, $p = 1, \ldots, m(m-1)$ and $\check{E}_l^{\pm}$, $l = 1, \ldots, 2m$.

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