

Proceedings of  
**Dynamic Systems and  
Applications**

Volume 4

Proceedings of

# **Dynamic Systems and Applications**

**Volume 4**

Proceedings of the Fourth International Conference on Dynamic  
Systems and Applications held at Morehouse College, Atlanta, USA,

May 21-24, 2003.

## **Editors**

**G. S. Ladde,**

University of Texas at Arlington, Arlington, TX, USA

**N. G. Medhin**

North Carolina State University, Raleigh, NC, USA., and

**M. Sambandham**

Morehouse College, Atlanta, GA 30314, USA.

## **Editorial Committee**

**R. P. Agarwal**

**S. Ahmad**

**R. Bozeman**

**C. Y. Chan**

**F. Gazzola**

**J. Graef**

**J. K. Hale**

**D. Kannan**

**V. B. Kolmanovskii**

**V. Lakshmikantham**

**R. E. Mickens**

**A. Msezane**

**S. Pederson**

**C. Peng**

**S. Sathanantham**

**S. Sumner**

**C. P. Tsokos**

**A. S. Vatsala**

**2004**

**Dynamic Publishers, Inc., U.S.A.**

Copyright 2004 by  
Dynamic Publishers, Inc.

All rights reserved.

No part of this journal May be reproduced or transmitted in any form by any means  
without written permission from the publishers.

Dynamic Publishers, Inc.  
P. O. Box 4865  
Atlanta, GA 30362-0654

Library of Congress Catalog Number 2004-132707

Ladde, G.S., Medhin, N. G., and Sambandham, M.  
Proceedings of Dynamic Systems and Applications, Volume 4

**ISBN 1-890888-00-1**

Printed in the United States of America



## ANALYSIS OF CONSERVATIVE DIFFERENCE SCHEMES FOR MATERIALS WITH MEMORY

PETER MATUS AND RODERICK V.N. MELNIK

Institute of Mathematics, National Academy of Sciences, Minsk, 220072, Belarus.  
University of Southern Denmark, MCI, Grundtvigs Alle 150, Sønderborg, DK-6400, Denmark  
and Louisiana Tech University, COES, LA 71272, USA.

**ABSTRACT.** In this paper we analyse a new conservative scheme for the description of dynamic behaviour of materials with memory, in particular shape memory alloys. The main result of the present paper is the establishing of unconditional convergence of the conservative scheme in a discrete semi-norm  $W_2^1$ .

**AMS (MOS) Subject Classification.** 74F05, 65M06.

### 1. INTRODUCTION

Numerical schemes that inherit energy conservation or dissipation properties from the original differential models play a fundamental role in many applications. Although the energy methodology has been developed since the seminal work by Courant, Friedrichs, and Lewy [3], a majority of the contributions to the field deal with linear models. The construction and analysis of conservative numerical schemes for nonlinear problems of mathematical physics are among most important tasks in theory and applications of mathematical modelling tools. The interest to the methodology based on energy inequalities has recently been renewed in the context of nonlinear problems and some interesting results have been achieved for the KdV, Cahn-Hilliard and some other models (e.g., [4, 1, 5]). This methodology can be used effectively to analyse convergence properties of constructed difference schemes. However, its standard application leads to typically restrictive assumptions on the grid size. Such assumptions are very undesirable, in particular in problems involving phase transformations. In this paper we analyse a new conservative scheme proposed recently for the description of dynamic behaviour of materials with memory, in particular shape memory alloys. We show that the scheme is unconditional convergent in a semi-norm  $W_2^1$ .

### 2. MODEL FOR DYNAMIC BEHAVIOUR OF MATERIALS WITH SHAPE MEMORY EFFECT

It has been recently shown [11] that the general model describing the dynamics of shape memory alloys (e.g., [8])

$$(1) \quad \begin{aligned} \rho \frac{\partial^2 u_i}{\partial t^2} &= \nabla_x \cdot \bar{\sigma} + f_i, \quad i, j = 1, 2, \\ \rho \frac{\partial e}{\partial t} - \bar{\sigma}^T : (\nabla \mathbf{v}) + \nabla \cdot \mathbf{q} &= g, \end{aligned}$$

can be reduced in a number of practically interesting cases (e.g., in the case of square-to-rectangular transformations) to a simpler model which can be written in the Falk form

$$(2) \quad \begin{aligned} \rho \frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial x} \left( k_1 (\theta - \theta_1) \frac{\partial u}{\partial x} - k_2 \left( \frac{\partial u}{\partial x} \right)^3 + k_3 \left( \frac{\partial u}{\partial x} \right)^5 \right) + F, \\ c_v \frac{\partial \theta}{\partial t} &= k \frac{\partial^2 \theta}{\partial x^2} + k_1 \theta \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + G. \end{aligned}$$

We use the following notation in writing the above systems:  $\rho$  is the density of the material,  $\mathbf{u} = \{u_i\}_{i=1,2}$  is the displacement vector,  $\mathbf{v}$  is the velocity,  $\bar{\sigma} = \{\sigma_{ij}\}$  is the stress tensor,  $\mathbf{q}$  is the heat flux,  $\theta$  is the temperature,  $e$  is the internal energy,  $\mathbf{f} = (f_1, f_2)^T$  and  $g$  are mechanical and thermal loadings, respectively,  $c_v$  is the specific heat constant,  $\theta_0$  is the martensite transition temperature,  $k_1, k_2, k_3, c_v$  and  $k$  are normalised material-specific constants, and  $F$  and  $G$  are normalised distributed mechanical and thermal loadings. We assume that the strain can be modelled sufficiently accurately by the Cauchy-Lagrangian strain tensor

$$(3) \quad \eta_{ij}(\mathbf{x}, t) = \left( \frac{\partial u_i(\mathbf{x}, t)}{\partial x_j} + \frac{\partial u_j(\mathbf{x}, t)}{\partial x_i} \right) / 2, \quad i, j = 1, 2,$$

where  $\mathbf{x}$  is the coordinates of a material point in the domain of interest, and the relationships between the stress and strain and between the internal energy  $e$  and the free energy  $\phi$  of the system are given in the form

$$(4) \quad \bar{\sigma} = \rho \frac{\partial \phi}{\partial \eta}, \quad e = \phi - \theta \frac{\partial \phi}{\partial \theta}.$$

System (2), supplemented by appropriate initial and boundary conditions, is in the focus of our analysis in the present paper. Before proceeding further, we notice that the model can be re-written as follows:

$$(5) \quad \begin{aligned} \frac{\partial \epsilon}{\partial t} &= \frac{\partial v}{\partial x}, \quad \rho \frac{\partial v}{\partial t} = \frac{\partial s}{\partial x} + F, \\ C_v \frac{\partial \theta}{\partial t} &= k \frac{\partial^2 \theta}{\partial x^2} + k_1 \theta \epsilon \frac{\partial v}{\partial x} + G, \quad s = k_1 (\theta - \theta_1) \epsilon - k_2 \epsilon^3 + k_3 \epsilon^5. \end{aligned}$$

### 3. CONSERVATIVE DIFFERENCE SCHEME

In the present paper we analyse the following difference scheme proposed recently in [9]

$$(6) \quad \begin{aligned} \frac{\epsilon_i^{n+1} - \epsilon_i^n}{\tau} &= \frac{v_{i+1/2}^{\sigma_1} - v_{i-1/2}^{\sigma_1}}{h}, \quad i = 1, 2, \dots, N-1, \\ \rho \frac{v_{i+1/2}^{n+1} - v_{i+1/2}^n}{\tau} &= \frac{s_{i+1}^{n+1} - s_i^{n+1}}{h}, \quad i = 0, 1, 2, \dots, N-1, \\ C_v \frac{\theta_i^{n+1} - \theta_i^n}{\tau} &= k \frac{\theta_{i+1}^{\sigma_3} - 2\theta_i^{\sigma_3} + \theta_{i-1}^{\sigma_3}}{h^2} + k_1 \theta_i^{\sigma_3} \epsilon_i^{\sigma_2} \frac{v_{i+1/2}^{\sigma_1} - v_{i-1/2}^{\sigma_1}}{h}, \\ i &= 1, 2, \dots, N-1, \\ s_i^{n+1} &= k_1 (\theta_i^{\sigma_3} - \theta_1) \epsilon_i^{\sigma_2} - \frac{k_2}{4} g_1(\epsilon_i^n, \epsilon_i^{n+1}) + \frac{k_3}{6} g_2(\epsilon_i^n, \epsilon_i^{n+1}), \end{aligned}$$



where the discrete analogues of the Steklov averaging are defined as follows

$$(7) \quad g_1(\epsilon, \bar{\epsilon}) = \frac{\bar{\epsilon}^4 - \epsilon^4}{\bar{\epsilon} - \epsilon} = \sum_{k=0}^3 \bar{\epsilon}^{3-k} \epsilon^k, \quad g_2(\epsilon, \bar{\epsilon}) = \frac{\bar{\epsilon}^6 - \epsilon^6}{\bar{\epsilon} - \epsilon} = \sum_{k=0}^5 \bar{\epsilon}^{5-k} \epsilon^k,$$

and, as usual,

$$(8) \quad y^\sigma = \sigma y^{n+1} + (1 - \sigma)y^n, \quad 0 \leq \sigma \leq 1.$$

The bar above a variable indicates that the value is taken at the flux point  $\bar{x}_i = (i + \frac{1}{2})h$ ,  $i = 0, 1, 2, \dots, N-1$ . Steps of discretisation in space and time are denoted here by  $h$  and  $\tau$ , respectively. It has been shown in [9] that in the absence of thermomechanical loading the scheme preserves energy conservation properties that are fulfilled for the original differential model. However, no analysis of convergence of the scheme has been carried out so far. This analysis is the subject of the present paper.

The fully conservative scheme is obtained with  $\sigma_3 = 1$  (see [9]). Other two weight parameters are chosen here as  $\sigma_1 = 0.5$  and  $\sigma_2 = 1$ . In this case, our scheme can be written in the index-free notation (e.g., [10]) as follows

$$(9) \quad \begin{aligned} (\epsilon_h)_t &= (\bar{v}_h)_x^{(0.5)}, \quad \rho(\bar{v}_h)_t = (\hat{s}_h)_x, \\ \hat{s}_h &= k_1(\hat{\theta}_h - \theta_1)\hat{\epsilon}_h - k_2\hat{\epsilon}_h^3 + k_3\hat{\epsilon}_h^5, \\ c_v(\theta_h)_t &= k(\hat{\theta}_h)_{xx} + k_1\hat{\theta}_h\hat{\epsilon}_h(\epsilon_h)_t, \end{aligned}$$

where  $y = (\epsilon_h, \bar{v}_h, \theta_h)$  is a vector of the approximate solution (a vector of discrete functions of the grid size), obtained with scheme (9). As we have already mentioned, we use the bar to indicate the value calculated at the flux point (that is  $\bar{v}_h = v_h(x_{i+1/2}, t_n)$ ), and the exact solution calculated at the same grid point will be denoted by  $u = (\epsilon, \bar{v}, \theta)$ . Hence, we are in a position to introduce the error of approximation as  $z = y - u = (\Delta\epsilon, \Delta v, \Delta\theta)$ , where

$$(10) \quad \Delta\epsilon = \epsilon_h - \epsilon, \quad \Delta v = \bar{v}_h - \bar{v}, \quad \Delta\theta = \theta_h - \theta.$$

We substitute functions  $\epsilon_h$ ,  $\bar{v}_h$ , and  $\theta_h$ , found from (10), into the scheme (9). We have immediately that

$$(11) \quad \Delta\epsilon_t = \Delta v_x^{(0.5)} + \psi_1^{(0.5)}, \quad \rho\Delta v_t = \Delta\hat{s}_x + \hat{\psi}_2,$$

where

$$(12) \quad \psi_1^{(0.5)} = -\epsilon_t + \bar{v}_x^{(0.5)} = \mathcal{O}(h^2 + \tau^2), \quad \hat{\psi}_2 = -\rho\bar{v}_t + \hat{s}_x = \mathcal{O}(h^2 + \tau).$$

After some tedious transformations we obtain the following equation for the error of approximation  $\Delta\theta$ :

$$(13) \quad \begin{aligned} c_v\Delta\theta_t &= k\Delta\hat{\theta}_{xx} + k_1\hat{\epsilon}_t\Delta\hat{\theta} + k_1\hat{\theta}_t\Delta\hat{\epsilon} + \\ &k_1\hat{\theta}\hat{\epsilon}\Delta\epsilon_t + k_1\hat{\theta}\Delta\hat{\epsilon}\Delta\epsilon_t + k_1\hat{\epsilon}\Delta\hat{\theta}\Delta\epsilon_t + k_1\epsilon_t\Delta\hat{\theta}\Delta\hat{\epsilon} + \\ &k_1\Delta\hat{\theta}\Delta\hat{\epsilon}\Delta\epsilon_t + \hat{\psi}_3, \end{aligned}$$

where

$$(14) \quad \hat{\psi}_3 = -c_v\theta_t + k\hat{\theta}_{xx} + k_1\hat{\theta}\hat{\epsilon}_t = \mathcal{O}(h^2 + \tau).$$

Initial and boundary conditions of the problem for the error approximation are homogeneous

$$(15) \quad \Delta \epsilon^0 = \Delta \theta^0 = \Delta v^0 = 0,$$

$$(16) \quad \Delta \epsilon_0^{n+1} = \Delta \epsilon_0^{n+1} = 0, \quad \Delta \theta_0^{n+1} = \Delta \theta_0^{n+1} = 0.$$

Hence, the problem for the error of approximation of our scheme (9) is completely defined by (11)–(16).

#### 4. ENERGY INEQUALITIES AND CONVERGENCE

The analysis of convergence of solutions of difference schemes for nonlinear problems of mathematical physics is an important and difficult task. One of the methodologies that has been applied to this analysis is the so-called  $\nu$ -method (e.g., [2, 6]). This methodology works well for a wide class of nonlinear problems, but in its standard implementation it requires quite restrictive assumptions on the grid size (typically such as  $\tau = h^\kappa$ ,  $\kappa > 1$ ). Since conservative schemes are known for their robustness even on crude grids, it is desirable to remove such assumptions when the convergence of the scheme is analysed. One way to do that is to use an assumption of the increased solution smoothness. However, in the problems like ours it is not an option since we have to deal with steep gradients in the solution (e.g., [8, 11]). A two-stage methodology that does not require excessive smoothness assumptions has been originally proposed in [6]. It rests on the analysis of the difference solution in discrete norms  $L_2$  and  $W_2^1$ . More precisely, first we analyse the difference solution in the grid norm  $L_2$  and  $C$  under the condition  $\tau \leq \alpha_0 h$  with given  $\alpha_0 = \text{const} > 0$  (e.g.,  $\alpha_0 = 1$ ), and then we carry out the analysis of the solution in the discrete norm  $W_2^1$  in the case  $\tau \geq \alpha_0 h$ . Combining these two results yields to unconditional convergence of the difference solution in the uniform norm. The success of the methodology ultimately rests on the embedding theorems. Indeed, if one analyses the difference solution in  $L_2$ , the assumption  $\tau \leq \alpha_0 h$  is a consequence of the embedding theorem  $\|y\|_C^2 \leq h^{-1} \|y\|_{L_2}^2$ . In what follows we consider in some details only one of the above two cases.

The discrete  $C$  (uniform) and  $L_2$  norms are defined in a standard manner

$$(17) \quad \|v\|_C = \max_{x \in \omega_h} |v(x)|, \quad \|v\|_C = \max_{x \in \omega_h^-} |v(x)|, \quad \omega_h^- = \omega_h \cup \{x_0 = 0\},$$

$$\|u\| = \sqrt{(u, u)}, \quad (u, v) = \sum_{x \in \omega_h} h u v, \quad \|u\| = \sqrt{(u, u)}, \quad (u, v) = \sum_{x \in \omega_h^+} h u v,$$

and norms  $\|v\|_C$  and  $\|u\|$  are defined analogously.

**4.1. Estimate Involving  $\psi_1$ .** Our main result in this subsection is the following estimate:

$$(18) \quad -2\tau(\Delta v_{tx}, \Delta v_x^{(0.5)} + \psi_1^{(0.5)}) \leq -(1 - 0.5\tau) \|\Delta \hat{v}_x + \hat{\psi}_1\|^2 +$$

$$(1 + 0.5\tau) \|\Delta v_x + \psi_1\|^2 + \tau \|(\psi_1)_t\|^2.$$

To prove (18), we multiply the first equation in (11) by  $-2\tau \Delta v_{tx}$  in the inner product sense (summing the respective terms from  $i=1$  to  $i=N$ ). We analyse each term separately, starting with

$$(19) \quad -2\tau(\delta v_{tx}, \Delta v_x^{(0.5)}) = -\|\Delta \hat{v}_x\|^2 - \|\Delta v_x\|^2.$$



Then, by using

$$(20) \quad (fg)_t = f^{(0.5)} g_t + g^{(0.5)} f_t,$$

we transform the second term in the left-hand side of (18). We sum the obtained two equalities up, and use a consequence of the so-called  $\epsilon$ -inequality (e.g., [10])

$$(21) \quad |ab| \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2, \quad \epsilon > 0.$$

In particular, for  $\epsilon = 0.5$  we have

$$(22) \quad \tau(\Delta \hat{v}_x + \hat{\psi}_1, (\psi_1)_t) \leq 0.5\tau \|\Delta \hat{v}_x + \hat{\psi}_1\|^2 + 0.5\tau \|(\psi_1)_t\|^2,$$

with a similar expression for  $\tau(\Delta v_x + \psi_1, (\psi_1)_t)$ . Note further that

$$(23) \quad \|\Delta \epsilon_t\|^2 \leq \frac{1}{2} (\|\Delta \hat{v}_x + \hat{\psi}_1\|^2 + \|\Delta v_x + \psi_1\|^2),$$

which follows immediately from the equation for the error of approximation

$$(24) \quad \Delta \epsilon_t = \Delta v_x^{(0.5)} + \psi_1^{(0.5)}$$

and the application of the  $\epsilon$ -inequality (21). Combining these results leads immediately to (18).

**4.2. Estimate Involving  $\psi_2$ .** Obtaining an estimate for the second equation in (11) is much more involved. The final result can be presented in the following form:

$$(25) \quad -2\tau\rho^{-1}(\Delta \epsilon_t, (\hat{\psi}_2)_x) \geq -\tau c \|\Delta \epsilon_t\|^2 - \tau c \|(\hat{\psi}_2)_x\|^2,$$

where  $c$  is a constant that does not depend on grid steps  $\tau$  and  $h$ , or approximate solution  $\epsilon_h$ ,  $\bar{v}_h$ , and  $\theta_h$ , or the error of approximation  $\Delta \epsilon_h$ ,  $\Delta \bar{v}_h$ , and  $\Delta \theta_h$ .

In what follows we highlight the main steps of obtaining this estimate. By using the second equation in (11), we obtain the following equality

$$(26) \quad -2\tau(\Delta \epsilon_t, \Delta v_{tx}) = -2\tau\rho^{-1}(\Delta \epsilon_t, (\hat{l}_1 \Delta \hat{\epsilon} + k_1(\hat{\epsilon} \Delta \hat{\theta} + \Delta \hat{\theta} + \Delta \hat{\theta} \Delta \hat{\epsilon}) + \sum_{k=2}^5 \hat{l}_k \Delta \hat{\epsilon}^k)_{xx} + (\hat{\psi}_2)_x).$$

Then we analyse each term in (26). The technique used in the analysis requires additional identities and embedding theorems. In particular, we make use of the first Green's difference formula (e.g., [10]):

$$(27) \quad (z, (ay_x)_x) = [z_x, ay_x], \quad z_0 = z_N = 0,$$

the equality

$$(28) \quad \Delta \hat{\epsilon} = \Delta \epsilon^{(0.5)} + 0.5\tau \Delta \epsilon_t,$$

the Cauchy inequality (a direct consequence of the  $\epsilon$  inequality (21)):

$$(29) \quad |(u, v)| \leq \|u\| \|v\| \leq \epsilon \|u\|^2 + \frac{1}{4\epsilon} \|v\|^2,$$

and the following embedding theorems

$$(30) \quad \|u\|^2 \leq \frac{l^2}{8} \|u_x\|^2, \quad u_0 = u_N = 0, \quad \|\Delta \theta\|_C \leq \frac{\sqrt{l}}{2} \|\Delta \theta_x\|, \quad \theta_0 = \theta_N = 0,$$

$$(31) \quad \|\Delta \theta_x\| \leq \frac{l}{2\sqrt{2}} \|\Delta \theta_{xx}\|, \quad \theta_0 = \theta_N = 0, \quad \|\Delta \theta\| \leq \frac{l^2}{8} \|\Delta \theta_{xx}\|.$$



Finally, we apply the following formulae for difference differentiation

$$(32) \quad (\Delta \epsilon^k)_x = \Delta \hat{\epsilon}_x \sum_{m=0}^{k-1} \Delta \epsilon^{k-1-m} \Delta \hat{\epsilon}_{(-1)}^m, \quad v_{(\pm 1)} = v(x \pm h) = v_{i \pm 1},$$

$$(33) \quad (\Delta \epsilon^\alpha)_t = \Delta \epsilon_t \sum_{m=0}^{\alpha-1} \Delta \epsilon^{\alpha-1-m} \Delta \epsilon^m$$

in order to arrive at (25).

**4.3. Estimate Involving  $\psi_3$ .** The estimate for the error  $\Delta \theta$

$$(34) \quad 2\tau(\Delta \theta_t - \Delta \hat{\theta}_{xx}, k_1(\hat{\theta} \Delta \hat{\epsilon} \Delta \epsilon_t + \hat{\epsilon} \Delta \hat{\theta} \Delta \epsilon_t + \epsilon_t \Delta \hat{\theta} \Delta \hat{\epsilon} + \Delta \hat{\theta} \Delta \hat{\epsilon} \Delta \epsilon_t) + \hat{\psi}_3) \leq$$

$$\tau c_v \epsilon_2 \|\Delta \theta_t\|^2 + \tau k \epsilon_1 \|\Delta \hat{\theta}_{xx}\|^2 + \tau c \left( \|\Delta \hat{\theta}\|_C^2 \|\Delta \hat{\epsilon}\|_C^2 \|\Delta \epsilon_t\|_C^2 + \right.$$

$$\left. (\|\Delta \hat{\epsilon}\|_C^2 + \|\Delta \hat{\theta}\|_C^2) \|\Delta \epsilon_t\|^2 + \|\Delta \hat{\theta}\|_C^2 \|\Delta \hat{\theta}_x\|_1^2 \right) + \tau c \|\hat{\psi}_3\|^2.$$

is also obtained by using the technique described above. In (34)  $\epsilon_i$ ,  $i = 1, 2$  are positive constants resulted from the application of the  $\epsilon$ -inequality. A key starting point in obtaining (34) is the multiplication of (13) by  $2\tau(\theta_t - \Delta \hat{\theta}_{xx})$  in the scalar product sense.

**4.4. Main Energy Inequality.** In what follows, we use the following notations:

$$(35) \quad l_0 = \text{const}, \quad l_1 = k_1(\theta - \theta_1) - 3k_2\epsilon^2 + 5k_3\epsilon^4 \geq l_0 > 0,$$

$$l_2 = -3k_2\epsilon + 10k_3\epsilon^3, \quad l_3 = 10k_3\epsilon^2 - k_2, \quad l_4 = 5k_3\epsilon, \quad l_5 = k_3,$$

$$l_1 + k_1\Delta \theta + \Phi_1^*(\Delta \epsilon) \geq \bar{l}_0 > 0,$$

where

$$(36) \quad \Phi_1^*(\Delta \epsilon) = \sum_{k=1}^4 \sum_{m=0}^k \Delta \epsilon_{(+1)}^{k-m} \Delta \epsilon^m l_{k+1}.$$

Then, summarising the results obtained in Sections 4.1 - 4.3, and using (23), we arrive at the following energy estimate for the error approximation obtained with conservative scheme (9):

$$(37) \quad (1 - \tau c_1) \|\hat{z}\|_1^2 - \tau c \left[ \left( \|\Delta \hat{\theta}_x\|_C^2 + \sum_{k=1}^4 \|\Delta \hat{\epsilon}\|_C^{2k} + \|\Delta \hat{\theta}\|_C^2 \right) \|\Delta \hat{\epsilon}_x\|_1^2 + \right.$$

$$\left. (\|\Delta \hat{\theta}\|_C^2 + \|\Delta \hat{\theta}\|_C^2 \|\Delta \hat{\epsilon}\|_C^2 + \|\Delta \hat{\epsilon}\|_C^2) \|\Delta \hat{v}_x + \hat{\psi}_1\| \right] + \tau c_v \|\Delta \theta_t\| +$$

$$\tau k \|\Delta \hat{\theta}_{xx}\|^2 + I(z) \leq (1 + \tau c) \|z\|_1^2 + \tau c (\|\Delta \epsilon_x\|_C^2 +$$

$$f_1(\Delta \epsilon)) \|\Delta \epsilon_x\|_1^2 + \tau c \|\hat{\psi}\|^2,$$

where

$$(38) \quad I(z) = \tau k (1 - (2\epsilon_1 + \epsilon_1 \|\Delta \hat{\epsilon}\|_C^2)) \|\Delta \hat{\theta}_{xx}\|^2 +$$

$$\tau^2 \rho^{-1} [l_1 + k_1 \Delta \hat{\theta} + \Phi_1^*(\Delta \epsilon), \Delta \epsilon_{tx}^2] +$$

$$\tau c_v (1 - 3\epsilon_2) \|\Delta \theta_t\|^2 + \tau^2 (k + c_v) \|\Delta \theta_{tx}\|^2 \geq 0,$$

$$(39) \quad f_1(\Delta\epsilon) = (1 + \sum_{k=1}^3 \|\Delta\epsilon\|_C^{2k}) \|\Delta\epsilon_x\|_C^2 + \sum_{k=1}^3 \|\Delta\epsilon\|_C^k,$$

$$(40) \quad \|\hat{\psi}\|^2 = \|(\hat{\psi}_1)_t\|^2 + \|(\hat{\psi}_2)_x\|^2 + \|(\hat{\psi}_3)\|^2.$$

Norm  $\|\cdot\|_1$  in (37),

$$(41) \quad \|z\|_1^2 = \rho^{-1} [l_1 + k_1 \Delta\theta + \Phi_1^*(\Delta\epsilon), \Delta\epsilon_x^2] + \|\Delta v_x + \psi_1\|^2 + (c_v + k) \|\Delta\theta_x\|^2,$$

is the norm used in the analysis of convergence in the case  $\tau \leq h$ . Note (e.g., [6]) that condition  $\tau \leq h$  is a consequence of the embedding theorem  $\|y\|_C^2 \leq h^{-1} \|y\|_{L_2}^2$  connecting the uniform and  $L_2$  discrete norms.

**4.5. Convergence in the semi-norm  $W_2^1$ .** Our procedure here follows the line of reasoning proposed originally in [7]. It is a two-stage procedure that has been applied to analyse convergence of difference schemes for nonlinear problems of mathematical physics. Recall that the  $\nu$ -method cannot be applied in a straightforward manner to these problems [6]. Indeed, by estimating the error of the first difference derivative in the uniform metric, we would need to impose a restrictive condition like  $\tau = h^\kappa$ ,  $\kappa \geq 1$  which we would like to avoid. Hence, the idea here is to estimate first the difference derivative in the  $L_2$  norm for  $\tau \leq h$  and then to estimate it in the  $W_2^1$  norm. By combining such two estimates we aim at obtaining an unconditional bound for the accuracy of the scheme in the uniform metric. Each case separately can be analysed with the  $\nu$ -method. At the first step it is straightforward to obtain a rough estimate of the error

$$(42) \quad \|z_n\|^2 \leq \nu^2 (h^2 + \tau),$$

where the norm in (42) is defined as

$$(43) \quad \|z\|^2 = \rho^{-1} \bar{l}_0 [\|\Delta\epsilon_x\|^2 + \|\Delta v_x\|^2 + (c_v + k) \|\Delta\theta_x\|^2],$$

and  $\bar{l}_0$  is defined as a constant that satisfies the following inequality:  $l_1 + k_1 \Delta\theta + \Phi_1^*(\Delta\epsilon) \geq \bar{l}_0 > 0$ .

Then, we can derive an estimate for the error of the approximation at the  $(n+1)$ st time layer by considering a corresponding iterative process in a way similar to that described in [6]. A chain of recurrent relationships leads to the following estimate:

$$(44) \quad \|z^{n+1}\|^2 + \sum_{k=0}^n \tau (c_v \|\Delta(\theta_t)_n\|^2 + k \|\Delta(\theta_{xx})^{n+1}\|^2) \leq f_1(\nu) (h^2 + \tau),$$

where  $f_1(\nu)$  is a bounded function for sufficiently small step sizes  $\tau$  and  $h$ . This estimate is then improved by using the main energy inequality (37). In particular, we obtain that

$$(45) \quad \|\Delta\epsilon\|_C, \|\Delta v\|_C, \|\Delta\theta\|_C \leq \nu_1 (h^2 + \tau).$$

At the second step of our procedure, we analyse convergence of our difference scheme in the discrete norm  $W_2^1$ . This is done for the case of  $\tau \geq h$  and hence the embedding theorems (30) are in use. Similarly to the above, we obtain that

$$(46) \quad \|\Delta\epsilon_x\|_C, \|\Delta\theta_x\|_C \leq \nu_2 (h^2 + \tau).$$

The norm used for this analysis is

$$(47) \quad \|z\|_2^2 = \rho^{-1} [l_1 + k_1 \Delta\theta, \Delta\epsilon_x^2] + \|\Delta v_x + \psi_1\|^2 + (c_v + k) \|\Delta\theta_x\|^2.$$



Finally, the convergence in the semi-norm  $W_2^1$  is obtained as a result of the combination of the above two results by choosing as  $\nu$  the maximum of the two values of  $\nu_1$  and  $\nu_2$ :

$$(48) \quad \left\{ \|\Delta \epsilon_x\|^2 + \|\Delta v_x\|^2 \right\} + \|\Delta \theta_x\|^2 + \sum_{t \in \omega_\tau} \tau \|\Delta \theta_t\|^2 + \sum_{t \in \omega_\tau} \tau \|\Delta \hat{\theta}_{xx}\|^2 \}^{1/2} \leq \nu(h^2 + \tau).$$

## 5. CONCLUSIONS

In this paper we have analysed convergence of a recently proposed conservative difference scheme describing the thermomechanical dynamics of shape memory materials. It has been shown that the scheme is unconditionally convergent in a discrete semi-norm  $W_2^1$ .

## REFERENCES

- [1] R. Abgrall, Towards the ultimate conservative scheme: following the quest, *J. Comput. Phys.*, 167: 277–315, 2001.
- [2] V.N. Abrashin, Difference schemes for nonlinear hyperbolic equations, *Differential Equations*, 11: 294–303, 1975.
- [3] R. Courant, K.O. Friedrichs, and H. Lewy, Über die partiellen differenzengleichungen der mathematischen physik, *Math. Ann.*, 100: 32–74, 1928.
- [4] D. Furihata, Finite difference schemes for  $\partial u / \partial t = (\partial / \partial x)^\alpha \delta G / \delta u$  that inherit energy conservation or dissipation property, *J. Comput. Phys.*, 156: 181–205, 1999.
- [5] D. Furihata, A stable and conservative finite difference scheme for the Cahn-Hilliard equation, *Numer. Math.*, 87: 675–699, 2001.
- [6] P.P. Matus, On unconditional convergence of difference schemes for gas dynamics problems, *Differential Equations*, 21: 1227–1238, 1985.
- [7] Matus, P.P., Stanishevskaya, L.V., Unconditional convergence of difference schemes for nonstationary quasilinear equations of mathematical physics, *Differential Equations*, 27: 1203–1219, 1991.
- [8] R.V.N. Melnik, A.J. Roberts, and K.A. Thomas, Computing dynamics of copper-based SMA via centre manifold reduction of 3D models, *Computational Materials Science*, 18: 255–268, 2000.
- [9] P. Matus, R.V.N. Melnik, and I.V. Rybak, Fully Conservative Difference Schemes for Nonlinear Models Describing Dynamics of Materials with Shape Memory, *Dokl. of the Academy of Sciences of Belarus*, 47: 15–18, 2003.
- [10] A.A. Samarskii, *The Theory of Difference Schemes*, Marcel Dekker, N.Y., 2001.
- [11] L. Wang and R.V.N. Melnik, Nonlinear Coupled Thermomechanical Waves Modelling Shear Type Phase Transformation in Shape Memory Alloys, in *Mathematical and Numerical Aspects of Wave Propagation*, Eds. G.C. Cohen et al, Springer, 723–728, 2003.

of t  
me  
ene

AN

It i  
is  
19  
for  
(B  
for  
sul  
for

Di

wi  
co  
de  
re  
su  
th  
eq  
D

## CONTENTS

Preface	vii
 <b>Dynamic Systems and Control</b>	
Optimal Control of Systems Governed by Impulsive Differential Inclusions <b>N.U.Ahmed</b>	1
Optimal Control of Gantry Cranes for Minimum Payload Oscillations <b>Saroj K. Biswas</b>	12
Weighted Polynomial Approximation and Hilbert Transforms: Their Connections to the Numerical Solution of Singular Integral Equations <b>S.B. Damelin and K. Diethelm</b>	20
White Noise Perturbation in a Model Equation for the Viscoelastic Response of Rubber in Tensile Deformation <b>N. Begashaw, N. G. Medhin, and M. Sambandham</b>	27
 <b>Dynamical Systems and Applications</b>	
An Application of Functional Analysis in a Predator - Prey System <b>Primitivo B. Acosta-Humanez</b>	35
Controlled Differential Equations for Solving Two-person Nonzero-sum Games With Coupled Constraints <b>Anatoly S. Antipin</b>	39
Quasiperiodic Flows and Algebraic Number Fields <b>Lennard F. Bakker</b>	46
New Sufficient Conditions for Global Stability of a Basic Model Describing Population Growth in a Polluted Environment <b>Bruno Buonomo and Deborah Lacitignola</b>	53
Fixed Points and Coincidence Points of Discontinuous Monotone Operators <b>Yong-zhuo Chen</b>	58
Nonlinear Integral Transforms and Dynamical Systems on Hilbert Spaces <b>Agnieszka Kozak and Yuri Kozitsky</b>	63
Polynomial Models of Discrete Time Series over Finite Fields <b>Aihua Li</b>	68
Optimal Control Problems Arising in Marketing Models <b>Luigi De Cesare and Andrea Di Liddo</b>	74
Multiple-period-doubling Bifurcation Route to Chaos in Periodically Pulsed Chaotic Oscillators <b>S. Parthasarathy and K.Srinivasan</b>	80
Homoclinic orbits and bifurcation of entropy-carrying hit-sets <b>Steven M. Pederson</b>	87
The least vulnerable, the most successful and other such extremes <b>Bruce J. West</b>	95



A Geometric Approach to the Legendre Transformation <b>Enrico Pagani</b>	102
 <b>Partial Differential Equations and Applications</b>	
On a Nonlocal Hyperbolic Model Arising from a Reliability System <b>Azmy S. Ackleh, and Keng Deng</b>	113
Blow-Up and Quenching of Solutions due to a concentrated Nonlinear Source <b>C. Y. Chan</b>	121
Blow-Up of Solutions for Degenerate Semilinear Parabolic First Initial-Boundary Value Problems <b>C. Y. Chan and W. Y. Chan</b>	127
The Localization of Shear Bands via the Maximum Principle <b>C. O. Horgan and W. E. Olmstead</b>	133
Response of Dark-adapted Retinal Rod Photoreceptors <b>H. Khanal, V. Alexiades, and E. Dibenedetto</b>	138
Conditions on the Motion of a Heat Source that guarantee a Blow-up Solution of the Two-dimensional Heat Equation <b>C. M. Kirk and W. E. Olmstead</b>	146
Analysis of Conservative Difference Schemes for Materials with Memory <b>Peter Matus and Roderick V.n. Melnik</b>	153
A Nonstandard Problem for the Heat Equation <b>Philip W. Schaefer</b>	161
Diffusional Release of a Dispersed Solute from a Polymeric Matrix: Approximation of the Exact Solutions Using Finite Region Continuity <b>S. Wasuwanich, N. Jinuntuya, P. Petpirom, and R. Collins</b>	167
Numerical Approximations for Thomas-Fermi Model Using Radial Basis Functions <b>S. M. Wong and Y. C. Hon</b>	175
Contraction Mapping and Stability in a Delay-differential Equation <b>Bo Zhang</b>	183
On Maximum Principle and Existence of Positive Solutions for Elliptic Systems <b>A.Yechoui</b>	191
The Dynamics Around the Ground State of the Timoshenko Equation with a Source Term <b>Jorge Alfredo Esquivel-Avila</b>	198
Universal Contingent Claims in a General Market Environment and Multiplicative Measures <b>Valery A. Kholodnyi</b>	206
On the Nonlinear Bi-harmonic Parabolic Equation with Data in $L^p$ Spaces. <b>Tor A. Kwembe</b>	213
Estimates for Solutions to Optimization Problems on Elliptic Equations <b>Fabrizio Cuccu, Monica Marras, and Giovanni Porru</b>	220

## EDITORS

**G. S. Ladde**, Department of Mathematics,  
University of Texas at Arlington, Arlington, TX 76019, USA

**N. G. Medhin**, Department of Mathematics,  
Clark Atlanta University, Atlanta, GA 30314, USA., and

**M. Sambandham**, Department of Mathematics,  
Morehouse College, Atlanta, GA 30314, USA.

## Editorial Committee

- R. P. Agarwal**, Florida Institute of Technology, Melbourne, FL 32901, USA  
**Shair Ahmad**, Dept of Math, University of Texas at San Antonio, TX 78285, USA  
**R. Bozeman**, Dept of Mathematics, Morehouse College, Atlanta, GA 30314, USA  
**C. Y. Chan**, Dept of Math, Univ. of Louisiana at Lafayette, Lafayette, LA 70504, USA  
**Filippo Gazzola**, Dipartimento di Scienze T.A., Italy  
**J. Graef**, Dept of Mathematics, the Univ. of Tennessee, Chattanooga, TN 37403, USA  
**J. K. Hale**, Dept of Math, Georgia Institute of Technology, Atlanta, GA 30332, USA  
**D. Kannan**, Dept of Mathematics, University of Georgia, Athens, GA 30602, USA  
**V. B. Kolmanovskii**, Dept. Automatic Control, Cinvestav-IPN, S. P., Mexico DF  
**V. Lakshmikantham**, Dept of Math, Florida Inst of Tech, Melbourne, FL 32901, USA  
**R. E. Mickens**, Dept of Physics, Clark Atlanta University, Atlanta, GA 30314, USA  
**A. Msezane**, Center for Theor. Stu. of Phy, Clark Atlanta Univ, Atlanta, GA 30314, USA  
**Steve Pederson**, Dept of Mathematics, Morehouse College, Atlanta, GA 30314, USA  
**Chuang Peng**, Dept of Mathematics, Morehouse College, Atlanta, GA 30314, USA  
**S. Sathanantham**, Dept of Math, Tennessee State University, Nashville, TN 37203, USA  
**S. Sumner**, Dept of Math., Mary Washington College, Fredericksburg, VA 22401, USA  
**C. P. Tsokos**, Dept of Mathematics, University of South Florida, Tampa, FL 33620, USA  
**A. S. Vatsala**, Dept of Math, Univ. of Louisiana at Lafayette, Lafayette, LA 70504, USA