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ANALYSIS OF CONSERVATIVE DIFFERENCE SCHEMES FOR MATERIALS WITH MEMORY

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ABSTRACT. In this paper we analyse a new conservative scheme for the description of dynamic behaviour of materials with memory, in particular shape memory alloys. The main result of the present paper is the establishing of unconditional convergence of the conservative scheme in a discrete semi-norm W_2^1.

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1. INTRODUCTION

Numerical schemes that inherit energy conservation or dissipation properties from the original differential models play a fundamental role in many applications. Although the energy methodology has been developed since the seminal work by Courant, Friedrichs, and Lewy [3], a majority of the contributions to the field deal with linear models. The construction and analysis of conservative numerical schemes for nonlinear problems of mathematical physics are among most important tasks in theory and applications of mathematical modelling tools. The interest to the methodology based on energy inequalities has recently been renewed in the context of nonlinear problems and some interesting results have been achieved for the KdV, Cahn-Hilliard and some other models (e.g., [4, 1, 5]). This methodology can be used effectively to analyse convergence properties of constructed difference schemes. However, its standard application leads to typically restrictive assumptions on the grid size. Such assumptions are very undesirable, in particular in problems involving phase transformations. In this paper we analyse a new conservative scheme proposed recently for the description of dynamic behaviour of materials with memory, in particular shape memory alloys. We show that the scheme is unconditional convergent in a semi-norm W_2^1.

2. MODEL FOR DYNAMIC BEHAVIOUR OF MATERIALS WITH SHAPE MEMORY EFFECT

It has been recently shown [11] that the general model describing the dynamics of shape memory alloys (e.g., [8])

$$
\begin{aligned}
&\rho\frac{\partial^2 u_i}{\partial t^2} = \nabla_x \cdot \vec{\sigma} + f_i, \quad i,j = 1,2, \\
&\rho\frac{\partial e}{\partial t} - \vec{\sigma}^T : (\nabla \mathbf{v}) + \nabla \cdot \mathbf{q} = g,
\end{aligned} \tag{1}
$$

can be reduced in a number of practically interesting cases (e.g., in the case of square-to-rectangular transformations) to a simpler model which can be written in the Falk form

$$
(2)\qquad \begin{aligned}
&\rho\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(k_1(\theta-\theta_1)\frac{\partial u}{\partial x} - k_2\left(\frac{\partial u}{\partial x}\right)^3 + k_3\left(\frac{\partial u}{\partial x}\right)^5\right) + F,\\
&c_v\frac{\partial \theta}{\partial t} = k\frac{\partial^2 \theta}{\partial x^2} + k_1\theta\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + G.
\end{aligned}
$$

We use the following notation in writing the above systems: ρ is the density of the material, $\mathbf{u} = \{u_i\}|_{i=1,2}$ is the displacement vector, $\mathbf{v}$ is the velocity, $\vec{\sigma} = \{\sigma_{ij}\}$ is the stress tensor, $\mathbf{q}$ is the heat flux, θ is the temperature, e is the internal energy, $\mathbf{f} = (f_1, f_2)^T$ and g are mechanical and thermal loadings, respectively, c_v is the specific heat constant, θ_0 is the martensite transition temperature, k_1, k_2,k_3, c_v and k are normalised material-specific constants, and F and G are normalised distributed mechanical and thermal loadings. We assume that the strain can be modelled sufficiently accurately by the Cauchy-Lagrangian strain tensor

$$
(3)\qquad \eta_{ij}(\mathbf{x},t) = \left(\frac{\partial u_i(\mathbf{x},t)}{\partial x_j} + \frac{\partial u_j(\mathbf{x},t)}{\partial x_i}\right)\Big/ 2\ , \qquad i,j = 1,2,
$$

where $\boldsymbol{x}$ is the coordinates of a material point in the domain of interest, and the relationships between the stress and strain and between the internal energy e and the free energy ϕ of the system are given in the form

$$
(4)\qquad \vec{\sigma} = \rho\frac{\partial \phi}{\partial \vec{\eta}}, \quad e = \phi - \theta\frac{\partial \phi}{\partial \theta}.
$$

System (2), supplemented by appropriate initial and boundary conditions, is in the focus of our analysis in the present paper. Before proceeding further, we notice that the model can be re-written as follows:

$$
(5)\qquad \begin{aligned}
&\frac{\partial \epsilon}{\partial t} = \frac{\partial v}{\partial x}, \quad \rho\frac{\partial v}{\partial t} = \frac{\partial s}{\partial x} + F,\\
&C_v\frac{\partial \theta}{\partial t} = k\frac{\partial^2 \theta}{\partial^2 x} + k_1\theta\epsilon\frac{\partial v}{\partial x} + G, \quad s = k_1(\theta-\theta_1)\epsilon - k_2\epsilon^3 + k_3\epsilon^5.
\end{aligned}
$$

3. CONSERVATIVE DIFFERENCE SCHEME

In the present paper we analyse the following difference scheme proposed recently in [9]

$$
(6)\qquad \begin{aligned}
&\frac{\epsilon_i^{n+1} - \epsilon_i^n}{\tau} = \frac{v_{i+1/2}^{\sigma_1} - v_{i-1/2}^{\sigma_1}}{h}, \quad i = 1, 2, \cdots, N-1,\\
&\rho\frac{v_{i+1/2}^{n+1} - v_{i+1/2}^{n}}{\tau} = \frac{s_{i+1}^{n+1} - s_i^{n+1}}{h}, \quad i = 0, 1, 2, \cdots, N-1,\\
&C_v\frac{\theta_i^{n+1} - \theta_i^n}{\tau} = k\frac{\theta_{i+1}^{\sigma_3} - 2\theta_i^{\sigma_3} + \theta_{i-1}^{\sigma_3}}{h^2} + k_1\theta_i^{\sigma_3}\epsilon_i^{\sigma_2}\frac{v_{i+1/2}^{\sigma_1} - v_{i-1/2}^{\sigma_1}}{h},\\
&i = 1, 2, \cdots, N-1,\\
&s_i^{n+1} = k_1(\theta_i^{\sigma_3} - \theta_1)\epsilon_i^{\sigma_2} - \frac{k_2}{4}g_1(\epsilon_i^n, \epsilon_i^{n+1}) + \frac{k_3}{6}g_2(\epsilon_i^n, \epsilon_i^{n+1}),
\end{aligned}
$$

where the discrete analogues of the Steklov averaging are defined as follows

$$(7) \qquad g_1(\epsilon,\bar{\epsilon}) = \frac{\bar{\epsilon}^4-\epsilon^4}{\bar{\epsilon}-\epsilon} = \sum_{k=0}^{3} \bar{\epsilon}^{3-k}\epsilon^k, \quad g_2(\epsilon,\bar{\epsilon}) = \frac{\bar{\epsilon}^6-\epsilon^6}{\bar{\epsilon}-\epsilon} = \sum_{k=0}^{5} \bar{\epsilon}^{5-k}\epsilon^k,$$

and, as usual,

$$(8) \qquad y^\sigma = \sigma y^{n+1} + (1-\sigma)y^n, \quad 0 \le \sigma \le 1.$$

The bar above a variable indicates that the value is taken at the flux point $\bar{x}_i = (i+\frac{1}{2})h$, $i = 0,1,2,\cdots,N-1$. Steps of discretisation in space and time are denoted here by h and τ, respectively. It has been shown in [9] that in the absence of thermomechanical loading the scheme preserves energy conservation properties that are fulfilled for the original differential model. However, no analysis of convergence of the scheme has been carried out so far. This analysis is the subject of the present paper.

The fully conservative scheme is obtained with $\sigma_3 = 1$ (see [9]). Other two weight parameters are chosen here as $\sigma_1 = 0.5$ and $\sigma_2 = 1$. In this case, our scheme can be written in the index-free notation (e.g., [10]) as follows

$$(9) \qquad \begin{aligned} &(\epsilon_h)_t = (\bar{v}_h)_{\bar{x}}^{(0.5)}, \quad \rho(\bar{v}_h)_t = (\hat{s}_h)_x, \\ &\hat{s}_h = k_1(\hat{\theta}_h - \theta_1)\hat{\epsilon}_h - k_2\hat{\epsilon}_h^3 + k_3\hat{\epsilon}_h^5, \\ &c_v(\theta_h)_t = k(\hat{\theta}_h)_{\bar{x}x} + k_1\hat{\theta}_h\hat{\epsilon}_h(\epsilon_h)_t, \end{aligned}$$

where $y = (\epsilon_h, \bar{v}_h, \theta_h)$ is a vector of the approximate solution (a vector of discrete functions of the grid size), obtained with scheme (9). As we have already mentioned, we use the bar to indicate the value calculated at the flux point (that is $\bar{v}_h = v_h(x_{i+1/2}, t_n)$), and the exact solution calculated at the same grid point will be denoted by $u = (\epsilon, \bar{v}, \theta)$. Hence, we are in a position to introduce the error of approximation as $z = y - u = (\Delta\epsilon, \Delta v, \Delta\theta)$, where

$$(10) \qquad \Delta\epsilon = \epsilon_h - \epsilon, \quad \Delta v = \bar{v}_h - \bar{v}, \quad \Delta\theta = \theta_h - \theta.$$

We substitute functions ϵ_h, $\bar{v}_h$, and θ_h, found from (10), into the scheme (9). We have immediately that

$$(11) \qquad \Delta\epsilon_t = \Delta v_{\bar{x}}^{(0.5)} + \psi_1^{(0.5)}, \quad \rho\Delta v_t = \Delta\hat{s}_x + \hat{\psi}_2,$$

where

$$(12) \quad \psi_1^{(0.5)} = -\epsilon_t + \bar{v}_{\bar{x}}^{(0.5)} = \mathbb{O}(h^2+\tau^2), \quad \hat{\psi}_2 = -\rho\bar{v}_t + \hat{s}_x = \mathbb{O}(h^2+\tau).$$

After some tedious transformations we obtain the following equation for the error of approximation $\Delta\theta$:

$$(13) \qquad \begin{aligned} &c_v\Delta\theta_t = k\Delta\hat{\theta}_{\bar{x}x} + k_1\hat{\epsilon}\epsilon_t\Delta\hat{\theta} + k_1\hat{\theta}\epsilon_t\Delta\hat{\epsilon} + \\ &k_1\hat{\theta}\hat{\epsilon}\Delta\epsilon_t + k_1\hat{\theta}\Delta\hat{\epsilon}\Delta\epsilon_t + k_1\hat{\epsilon}\Delta\hat{\theta}\Delta\epsilon_t + k_1\epsilon_t\Delta\hat{\theta}\Delta\hat{\epsilon} + \\ &k_1\Delta\hat{\theta}\Delta\hat{\epsilon}\Delta\epsilon_t + \hat{\psi}_3, \end{aligned}$$

where

$$(14) \qquad \hat{\psi}_3 = -c_v\theta_t + k\hat{\theta}_{\bar{x}x} + k_1\hat{\theta}\hat{\epsilon}\epsilon_t = \mathbb{O}(h^2+\tau).$$

Initial and boundary conditions of the problem for the error approximation are homogeneous

$$\Delta\epsilon^0 = \Delta\theta^0 = \Delta v^0 = 0, \tag{15}$$

$$\Delta\epsilon_0^{n+1} = \Delta\epsilon_0^{n+1} = 0, \quad \Delta\theta_0^{n+1} = \Delta\theta_0^{n+1} = 0. \tag{16}$$

Hence, the problem for the error of approximation of our scheme (9) is completely defined by (11)–(16).

4. ENERGY INEQUALITIES AND CONVERGENCE

The analysis of convergence of solutions of difference schemes for nonlinear problems of mathematical physics is an important and difficult task. One of the methodologies that has been applied to this analysis is the so-called ν-method (e.g., [2, 6]). This methodology works well for a wide class of nonlinear problems, but in its standard implementation it requires quite restrictive assumptions on the grid size (typically such as $\tau = h^\kappa$, $\kappa > 1$). Since conservative schemes are known for their robustness even on crude grids, it is desirable to remove such assumptions when the convergence of the scheme is analysed. One way to do that is to use an assumption of the increased solution smoothness. However, in the problems like ours it is not an option since we have to deal with steep gradients in the solution (e.g., [8, 11]). A two-stage methodology that does not require excessive smoothness assumptions has been originally proposed in [6]. It rests on the analysis of the difference solution in discrete norms L_2 and W_2^1. More precisely, first we analyse the difference solution in the grid norm L_2 and C under the condition $\tau \le \alpha_0 h$ with given $\alpha_0 = const > 0$ (e.g., $\alpha_0 = 1$), and then we carry out the analysis of the solution in the discrete norm W_2^1 in the case $\tau \ge \alpha_0 h$. Combining these two results yields to unconditional convergence of the difference solution in the uniform norm. The success of the methodology ultimately rests on the embedding theorems. Indeed, if one analyses the difference solution in L_2, the assumption $\tau \le \alpha_0 h$ is a consequence of the embedding theorem $\|y\|_C^2 \le h^{-1}\|y\|_{L_2}^2$. In what follows we consider in some details only one of the above two cases.

The discrete C (uniform) and L_2 norms are defined in a standard manner

$$\begin{gathered} \|v\|_C = \max_{x\in\omega_h} |v(x)|, \quad \|v]|_C = \max_{x\in\omega_h^-} |v(x)|, \ \omega_h^- = \omega_h \cup \{x_0 = 0\}, \\ \|u\| = \sqrt{(u,u)},\ (u,v) = \sum_{x\in\omega_h} huv, \quad \|u]| = \sqrt{(u,u]},\ (u,v] = \sum_{x\in\omega_h^+} huv, \end{gathered} \tag{17}$$

and norms $\|v]|_C$ and $|[u\|$ are defined analogously.

4.1. Estimate Involving ψ_1. Our main result in this subsection is the following estimate:

$$\begin{gathered} -2\tau(\Delta v_{t\bar{x}}, \Delta v_{\bar{x}}^{(0.5)} + \psi_1^{(0.5)}] \le -(1-0.5\tau)\|\Delta\hat{v}_{\bar{x}} + \hat{\psi}_1]|^2 + \\ (1+0.5\tau)\|\Delta v_{\bar{x}} + \psi_1]|^2 + \tau\|(\psi_1)_t]|^2. \end{gathered} \tag{18}$$

To prove (18), we multiply the first equation in (11) by $-2\tau\Delta v_{t\bar{x}}$ in the inner product sense (summing the respective terms from i=1 to i=N). We analyse each term separately, starting with

$$-2\tau(\delta v_{t\bar{x}}, \Delta v_{\bar{x}}^{(0.5)}] = -\|\Delta\hat{v}_{\bar{x}}]|^2 - \|\Delta v_{\bar{x}}\|^2. \tag{19}$$

Then, by using

$$(fg)_t = f^{(0.5)} g_t + g^{(0.5)} f_t, \tag{20}$$

we transform the second term in the left-hand side of (18). We sum the obtained two equalities up, and use a consequence of the so-called ϵ-inequality (e.g., [10])

$$|ab| \le \epsilon a^2 + \frac{1}{4\epsilon} b^2, \quad \epsilon > 0. \tag{21}$$

In particular, for $\epsilon = 0.5$ we have

$$\tau(\Delta\hat{v}_{\bar{x}} + \hat{\psi}_1, (\psi_1)_t] \le 0.5\tau||\Delta\hat{v}_{\bar{x}} + \hat{\psi}_1]|^2 + 0.5\tau||(\psi_1)_t]|^2, \tag{22}$$

with a similar expression for $\tau(\Delta v_{\bar{x}} + \psi_1, (\psi_1)_t]$. Note further that

$$||\Delta\epsilon_t||^2 \le \frac{1}{2}(||\Delta\hat{v}_x + \hat{\psi}_1||^2 + ||\Delta v_x + \psi_1||^2), \tag{23}$$

which follows immediately from the equation for the error of approximation

$$\Delta\epsilon_t = \Delta v_{\bar{x}}^{(0.5)} + \psi_1^{(0.5)} \tag{24}$$

and the application of the ϵ-inequality (21). Combining these results leads immediately to (18).

4.2. **Estimate Involving ψ_2.** Obtaining an estimate for the second equation in (11) is much more involved. The final result can be presented in the following form:

$$-2\tau\rho^{-1}(\Delta\epsilon_t, (\hat{\psi}_2)_{\bar{x}}) \ge -\tau c||\Delta\epsilon_t||^2 - \tau c||(\hat{\psi}_2)_{\bar{x}}||^2, \tag{25}$$

where c is a constant that does not dependent on grid steps τ and h, or approximate solution ϵ_h, $\bar{v}_h$, and θ_h, or the error of approximation $\Delta\epsilon_h$, $\Delta\bar{v}_h$, and $\Delta\theta_h$.

In what follows we highlight the main steps of obtaining this estimate. By using the second equation in (11), we obtain the following equality

$$-2\tau(\Delta\epsilon_t, \Delta v_{t\bar{x}}) = -2\tau\rho^{-1}(\Delta\epsilon_t, (\hat{l}_1\Delta\hat{\epsilon} + k_1(\hat{\epsilon}\Delta\hat{\theta} + \Delta\hat{\theta} + \Delta\hat{\theta}\Delta\hat{\epsilon}) + \sum_{k=2}^{5} \hat{l}_k \Delta\hat{\epsilon}^k)_{x\bar{x}} + (\hat{\psi}_2)_{\bar{x}}). \tag{26}$$

Then we analyse each term in (26). The technique used in the analysis requires additional identities and embedding theorems. In particular, we make use of the first Green's difference formula (e.g., [10]):

$$(z, (ay_x)_{\bar{x}} = [z_x, ay_x), \quad z_0 = z_N = 0, \tag{27}$$

the equality

$$\Delta\hat{\epsilon} = \Delta\epsilon^{(0.5)} + 0.5\tau\Delta\epsilon_t, \tag{28}$$

the Cauchy inequality (a direct consequence of the ϵ inequality (21)):

$$|(u,v)| \le ||u||||v|| \le \epsilon||u||^2 + \frac{1}{4\epsilon}||v||^2, \tag{29}$$

and the following embedding theorems

$$||u||^2 \le \frac{l^2}{8}||u_{\bar{x}}]|^2,\ u_0 = u_N = 0, \quad ||\Delta\theta||_C \le \frac{\sqrt{l}}{2}||\Delta\theta_{\bar{x}}]|,\ \theta_0 = \theta_N = 0, \tag{30}$$

$$||\Delta\theta_{\bar{x}}]| \le \frac{l}{2\sqrt{2}}||\Delta\theta_{\bar{x}x}||,\ \theta_0 = \theta_N = 0, \quad ||\Delta\theta|| \le \frac{l^2}{8}||\Delta\theta_{\bar{x}x}||. \tag{31}$$

Finally, we apply the following formulae for difference differentiation

$$(32)\qquad (\Delta\hat{\epsilon}^k)_{\bar{x}} = \Delta\hat{\epsilon}_{\bar{x}} \sum_{m=0}^{k-1} \Delta\hat{\epsilon}^{k-1-m}\Delta\hat{\epsilon}^m_{(-1)}, \quad v_{(\pm 1)} = v(x \pm h) = v_{i\pm 1},$$

$$(33)\qquad (\Delta\epsilon^\alpha)_t = \Delta\epsilon_t \sum_{m=0}^{\alpha-1} \Delta\hat{\epsilon}^{\alpha-1-m}\Delta\epsilon^m$$

in order to arrive at (25).

4.3. **Estimate Involving ψ_3.** The estimate for the error $\Delta\theta$

$$2\tau(\Delta\theta_t - \Delta\hat{\theta}_{\bar{x}x}, k_1(\hat{\theta}\Delta\hat{\epsilon}\Delta\epsilon_t + \hat{\epsilon}\Delta\hat{\theta}\Delta\epsilon_t + \epsilon_t\Delta\hat{\theta}\Delta\hat{\epsilon} + \Delta\hat{\theta}\Delta\hat{\epsilon}\Delta\epsilon_t) + \hat{\psi}_3) \le$$

$$(34)\qquad \tau c_v\epsilon_2\|\Delta\theta_t\|^2 + \tau k\epsilon_1\|\Delta\hat{\theta}_{\bar{x}x}\|^2 + \tau c\Big(\|\Delta\hat{\theta}\|^2_C\|\Delta\hat{\epsilon}\|^2_C\|\Delta\epsilon_t\|^2_C +$$

$$(\|\Delta\hat{\epsilon}\|^2_C + \|\Delta\hat{\theta}\|^2_C)\|\Delta\epsilon_t\|^2 + \|\Delta\hat{\theta}\|^2_C\|\Delta\hat{\theta}_x\|^2_1\Big) + \tau c\|\hat{\psi}_3\|^2.$$

is also obtained by using the technique described above. In (34) ϵ_i, $i = 1, 2$ are positive constants resulted from the application of the ϵ-inequality. A key starting point in obtaining (34) is the multiplication of (13) by $2\tau(\theta_t - \Delta\hat{\theta}_{\bar{x}x})$ in the scalar product sense.

4.4. **Main Energy Inequality.** In what follows, we use the following notations:

$$l_0 = const, \quad l_1 = k_1(\theta - \theta_1) - 3k_2\epsilon^2 + 5k_3\epsilon^4 \ge l_0 > 0,$$

$$(35)\qquad l_2 = -3k_2\epsilon + 10k_3\epsilon^3, \quad l_3 = 10k_3\epsilon^2 - k_2, \quad l_4 = 5k_3\epsilon, \quad l_5 = k_3,$$

$$l_1 + k_1\Delta\theta + \Phi_1^*(\Delta\epsilon) \ge \bar{l}_0 > 0,$$

where

$$(36)\qquad \Phi_1^*(\Delta\epsilon) = \sum_{k=1}^{4}\sum_{m=0}^{k} \Delta\epsilon^{k-m}_{(+1)}\Delta\epsilon^m l_{k+1}.$$

Then, summarising the results obtained in Sections 4.1 - 4.3, and using (23), we arrive at the following energy estimate for the error approximation obtained with conservative scheme (9):

$$(1 - \tau c_1)\|\hat{z}\|^2_1 - \tau c\Big[\Big(\|\Delta\hat{\theta}_x\|^2_C + \sum_{k=1}^{4}\|\Delta\hat{\epsilon}\|^{2k}_C + \|\Delta\hat{\theta}\|^2_C\Big)\|\Delta\hat{\epsilon}_x\|^2_1 +$$

$$(37)\qquad (\|\Delta\hat{\theta}\|^2_C + \|\Delta\hat{\theta}\|^2_C\|\Delta\hat{\epsilon}\|^2_C + \|\Delta\hat{\epsilon}\|^2_C)\|\Delta\hat{v}_{\bar{x}} + \hat{\psi}_1]\|\Big] + \tau c_v\|\Delta\theta_t\| +$$

$$\tau k\|\Delta\hat{\theta}_{\bar{x}x}\|^2 + I(z) \le (1 + \tau c)\|z\|^2_1 + \tau c(\|\Delta\epsilon_x\|^2_C +$$

$$f_1(\Delta\epsilon))\|\Delta\epsilon_x\|^2_1 + \tau c\|\hat{\psi}\|^2,$$

where

$$I(z) = \tau k(1 - (2\epsilon_1 + \epsilon_1\|\Delta\hat{\epsilon}\|^2_C))\|\Delta\hat{\theta}_{\bar{x}x}\|^2 +$$

$$(38)\qquad \tau^2\rho^{-1}[l_1 + k_1\Delta\hat{\theta} + \Phi_1^*(\Delta\hat{\epsilon}), \Delta\epsilon^2_{tx}) +$$

$$\tau c_v(1 - 3\epsilon_2)\|\Delta\theta_t\|^2 + \tau^2(k + c_v)\|\Delta\theta_{tx}]\|^2 \ge 0,$$

$$f_1(\Delta\epsilon) = (1 + \sum_{k=1}^{3} \|\Delta\epsilon\|_C^{2k})\|\Delta\epsilon_x\|_C^2 + \sum_{k=1}^{3} \|\Delta\epsilon\|_C^k, \tag{39}$$

$$\|\hat{\psi}\|^2 = \|(\psi_1)_t]|^2 + \|(\hat{\psi}_2)_{\bar{x}}\|^2 + \|(\hat{\psi}_3)\|^2. \tag{40}$$

Norm $\|\cdot\|_1$ in (37),

$$\|z\|_1^2 = \rho^{-1}[l_1 + k_1\Delta\theta + \Phi_1^*(\Delta\epsilon), \Delta\epsilon_{\bar{x}}^2) + \|\Delta v_{\bar{x}} + \psi_1]|^2 + (c_v + k)\|\Delta\theta_{\bar{x}}]|^2, \tag{41}$$

is the norm used in the analysis of convergence in the case $\tau \leq h$. Note (e.g., [6]) that condition $\tau \leq h$ is a consequence of the embedding theorem $\|y\|_C^2 \leq h^{-1}\|y\|_{L_2}^2$ connecting the uniform and L_2 discrete norms.

4.5. **Convergence in the semi-norm W_2^1.** Our procedure here follows the line of reasoning proposed originally in [7]. It is a two-stage procedure that has been applied to analyse convergence of difference schemes for nonlinear problems of mathematical physics. Recall that the ν-method cannot be applied in a straightforward manner to these problems [6]. Indeed, by estimating the error of the first difference derivative in the uniform metric, we would need to impose a restrictive condition like $\tau = h^\kappa$, $\kappa \geq 1$ which we would like to avoid. Hence, the idea here is to estimate first the difference derivative in the L_2 norm for $\tau \leq h$ and then to estimate it in the W_2^1 norm. By combining such two estimates we aim at obtaining an unconditional bound for the accuracy of the scheme in the uniform metric. Each case separately can be analysed with the ν-method. At the first step it is straightforward to obtain a rough estimate of the error

$$\|z_n\|^2 \leq \nu^2(h^2 + \tau), \tag{42}$$

where the norm in (42) is defined as

$$\|z\|^2 = \rho^{-1}\bar{l}_0\|[\Delta\epsilon_x\|^2 + \|\Delta v_{\bar{x}}]|^2 + (c_v + k)\|\Delta\theta_{\bar{x}}]|^2, \tag{43}$$

and $\bar{l}_0$ is defined as a constant that satisfies the following inequality: $l_1 + k_1\Delta\theta + \Phi_1^*(\Delta\epsilon) \geq \bar{l}_0 > 0$.

Then, we can derive an estimate for the error of the approximation at the $(n+1)$st time layer by considering a corresponding iterative process in a way similar to that described in [6]. A chain of recurrent relationships leads to the following estimate:

$$\|z^{n+1}\|^2 + \sum_{k=0}^{n} \tau(c_v\|\Delta(\theta_t)_n\|^2 + k\|\Delta(\theta_{\bar{x}x})^{n+1}\|^2) \leq f_1(\nu)(h^2 + \tau), \tag{44}$$

where $f_1(\nu)$ is a bounded function for sufficiently small step sizes τ and h. This estimate is then improved by using the main energy inequality (37). In particular, we obtain that

$$\|\Delta\epsilon\|_C,\ \|\Delta v\|_C,\ \|\Delta\theta\|_C \leq \nu_1(h^2 + \tau). \tag{45}$$

At the second step of our procedure, we analyse convergence of our difference scheme in the discrete norm W_2^1. This is done for the case of $\tau \geq h$ and hence the embedding theorems (30) are in use. Similarly to the above, we obtain that

$$\|\Delta\epsilon_x\|_C, \|\Delta\theta_x\|_C \leq \nu_2(h^2 + \tau). \tag{46}$$

The norm used for this analysis is

$$\|z\|_2^2 = \rho^{-1}[l_1 + k_1\Delta\theta, \Delta\epsilon_{\bar{x}}^2) + \|\Delta v_{\bar{x}} + \psi_1]|^2 + (c_v + k)\|\Delta\theta_{\bar{x}}]|^2. \tag{47}$$

Finally, the convergence in the semi-norm W_2^1 is obtained as a result of the combination of the above two results by choosing as ν the maximum of the two values of ν_1 and ν_2:

$$\Big\{ \|\Delta\epsilon_{\bar{x}}\|^2 + \|\Delta v_{\bar{x}}\|^2\} + \|\Delta\theta_{\bar{x}}\|^2 + \sum_{t\in\omega_\tau} \tau\|\Delta\theta_t\|^2 + \sum_{t\in\omega_\tau} \tau\|\Delta\hat{\theta}_{\bar{x}x}\|^2 \Big\}^{1/2} \le \nu(h^2+\tau). \tag{48}$$

5. CONCLUSIONS

In this paper we have analysed convergence of a recently proposed conservative difference scheme describing the thermomechanical dynamics of shape memory materials. It has been shown that the scheme is unconditionally convergent in a discrete semi-norm W_2^1.

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