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STEKLOV'S OPERATOR TECHNIQUE IN COUPLED DYNAMIC THERMOELASTICITY

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Abstract

In this paper we study non-smooth solutions of coupled non-stationary problems in thermoelasticity. Since the classical tools based on the Taylor's expansion of unknown functions may not be appropriate in obtaining a measure of quality for numerical methods, we apply the averaging Steklov operators and the technique based on the Bramble-Hilbert lemma in order to establish the convergence result for a class of generalized solutions. Effective explicit numerical schemes and error estimates are presented.

Key words: Coupled field theory, dynamic thermoelasticity, Steklov's averaging technique, a-priori and a-posteriori estimates, Sobolev spaces.

1 Coupled Field Theory and Hyperbolic Modes in Dynamics.

All real processes, dynamic systems and phenomena describe a transformation of different types of *energy*. This implies that in general, mathematical models applied to them should have integral rather than differential features. However, when engaged in mathematical modelling, we can often observe a gap between theoretical assumptions about a solution's smoothness and the actual smoothness of the solution in a practical problem. Moreover, including additional *information* about the process, system or phenomenon into the model (for example, by an improved physical parameterization or by additional relations between system parameters) ultimately leads to a change of solution regularity. The process of model improvement may continue indefinitely, and hence it is important to find a balance between the energetic and informational parts of the model complexity [23, 14]. Coupling, which is a natural way of reflecting additional information about a process, system, phenomenon, requires the relaxation of traditional regularity assumptions. Otherwise, a-priori estimates of solutions may become meaningless [10, 13].

Non-smoothness of solutions is a typical feature of many important problems in structural mechanics where a structure may be subjected to extreme mechanical and/or thermal loads. A lack of regularity for the solutions of thermoelasticity problems is also typical in many other areas of application. The study of weak solutions in thermoelasticity is especially important when the coupling between thermal and mechanical fields is relatively strong and has to be considered in the dynamics. Coupled problems in computational physics and other sciences is a two-way dynamic interaction between physically distinct components [5]. Such components may be mechanical, thermal, electromagnetic, biological in nature, but at least one component of the system as a whole has the *hyperbolic mode*. This leads to mathematical challenges in the investigation of such problems which have to be addressed. Computationally we also have a challenging problem because the states of all components should be considered when integrating over time. It is often inappropriate to approximate the mathematical models in coupled field theory using the arguments of "parabolization" [6]. The latter may lead to a distorted picture in the description of real objects by mathematical models. While for coupled mathematical models L^2-type estimates may provide an important characterization of unknown solutions, after the "parabolic" approximation such estimates may be completely inadequate to grasp changes in the solution behaviour [13].

Thermoelasticity is one of the first areas in coupled field theory that attracted the attention of mathematicians. Nevertheless there are still many problems in this field that have to be addressed. One of them is considered in this paper.

The remaining part of the paper is organized as follows.

- Section 2 provides the reader with the basic mathematical and numerical models that are investigated in the paper.
- In Section 3 we consider a more general operator-difference scheme, prove its stability and obtain a new estimate for its solution. Convergence results for the classical case follow easily from our consideration.
- Section 4 deals with the case of generalized solutions. Using Steklov's operator technique and the Bramble-Hilbert lemma, we prove the convergence result when the solution of the problem is from the Sobolev class $W_2^2(Q_T)$.
- Future directions of present work are addressed in Section 5.

2 Mathematical Model of Coupled Thermoelasticity in Nonstationary case.

Exact solutions of initial-boundary value problems in thermoelasticity, obtainable by analytic technique, are known for a relatively narrow class of problems, mainly for problems in uncoupled thermoelasticity. In general, the application of analytic procedures to nonstationary thermoelasticity problems and the necessity of taking into consideration the coupling phenomenon lead to serious mathematical difficulties. As a result, the development of numerical methods for coupled nonstationary problems of thermoelasticity is a fruitful area of investigation abounding in new mathematical ideas.

We limit ourselves in this paper to the investigation of a one-dimensional system of coupled thermoelasticity for a homogeneous isotropic body. This system after an appropriate scaling has the following form

$$\begin{cases} \dfrac{\partial^2 s}{\partial t^2} - \dfrac{\partial^2 s}{\partial x^2} + \dfrac{\partial^2 \Theta}{\partial t^2} = f_1(x,t), \\ (1+\epsilon)\dfrac{\partial \Theta}{\partial t} - a\dfrac{\partial^2 \Theta}{\partial x^2} + \epsilon\dfrac{\partial s}{\partial t} = f_2(x,t). \end{cases} \tag{2.1}$$

Here s denotes stresses, Θ denotes temperature, ϵ is the coupling parameter between thermal and mechanical fields, and a is the coefficient of thermal conductivity [20, 11]. The system (2.1) holds in the space-time domain $Q_T = \{(x,t) : \ 0 < x < 1, 0 < t \le \bar{T}\}$, thus the problem is considered in $\bar{Q}_T = Q_T \cup \Gamma \cup \bar{\Upsilon}$, where $\Gamma = \{(x,t) : \ x = 0, x = 1; 0 < t \le \bar{T}\}$, $\bar{\Upsilon} = \{(x,t) : \ 0 \le x \le 1, t = 0\}$. The system (2.1) is supplemented by the initial and boundary conditions

$$\Theta(x,0) = \Theta_0(x), \ s(x,0) = s_0(x), \ \frac{\partial s(x,0)}{\partial t} = \bar{s}_0(x), \text{ for } t = 0, \tag{2.2}$$

$$s(x_i,t) = s_i(t), \ \frac{\partial \Theta(x_i,t)}{\partial x} = \Theta_i(t), \text{ for } x_i \in \Gamma, \tag{2.3}$$

where $i = 0, 1, \ x_0 = 0, \ x_1 = 1$.

The problem in stresses (2.1) – (2.3) is equivalent to the problem in deformations. The former can be reduced to the latter by the change of variables $r = s + \Theta$

$$\begin{cases} \dfrac{\partial^2 r}{\partial t^2} = \dfrac{\partial^2 r}{\partial x^2} - \dfrac{\partial^2 \Theta}{\partial t^2} = f_1(x,t), \\ \dfrac{\partial \Theta}{\partial t} + \epsilon\dfrac{\partial r}{\partial t} = -a\dfrac{\partial^2 \Theta}{\partial x^2} = f_2(x,t). \end{cases} \tag{2.4}$$

with the corresponding initial and boundary conditions. The variable r in the system (2.4) denotes deformations, which at the initial moment of time will be denoted by r_0. The rate of change of deformations at the initial moment of time will be denoted by $\bar{r}_0$ (defined more precisely below) and other notation will be as for the problem (2.1)–(2.3). The main reason for the introduction of the model (2.4) lies in the fact that numerical schemes for this model are more easily realizable algorithmically than numerical schemes for the model (2.1).

We note that the above models of thermoelasticity do not belong to any of the classical type of partial differential equations. Such a situation is typical in coupled field theory. We have mixed equations which contain different modes: hyperbolic, parabolic, elliptic. Such mathematical models are important for both mathematical theory and application [16, 4]. In fact, in the case of thermoelasticity we can see that one of the equations of the system contains a hyperbolic type operator, whereas the other contains a parabolic type operator. There is a coupling effect between these equations by the parameter ϵ, which in the case of the formulation in deformations is amplified by non-homogeneous boundary conditions.

Using the basic principles of the construction of difference schemes (see, for example [22] and references therein), let us approximate the problem in deformations. We introduce the grid $\bar{\omega}_{h\tau} = \bar{\omega}_h \times \bar{\omega}_\tau$ in $\bar{Q}_T$ where

$$\bar{\omega}_h = \{x_i = ih,\ h = 1/N,\ i = 1, ..., N\},$$

$$\bar{\omega}_\tau = \{t_j = j\tau,\ \tau = T/M,\ j = 1, ..., M\}.$$

Let $(y, \eta) \equiv (y_i^j, \eta_i^j)$ be the difference approximation on this grid for the solution (r, Θ) of the problem in deformations. Then we construct the following difference scheme with respect to (y, η) (without the loss of generality we assume here that $s_i = \Theta_i = 0,\ i = 1, 2$)

$$\begin{cases} y_{\bar{t}t} = \Lambda y - \Lambda \eta + \varphi_1, & (x,t) \in \omega_h \times \omega_\tau, \\ \eta_t = \bar{\Lambda}\eta^{(\sigma)} - \epsilon y_t + \varphi_2, & (x,t) \in \bar{\omega}_h \times \times \omega_\tau, \end{cases} \tag{2.5}$$

$$y = \eta,\ x \in \Gamma;\ y = r_0(x),\ \eta = \Theta_0(x),\ y_t = r_1(x), x \in \omega_h,\ t = 0. \tag{2.6}$$

In the scheme (2.5)–(2.6) we use the following notation:

- φ_1 and φ_2 denote approximations to the functions f_1 and f_2 respectively;
- $r_0(x) = s_0(x) + \Theta_0(x)$;
- $r_1(x) = \bar{r}_0(x) + \tau[r_0'' + \Theta_0'' + f_1(x, 0)]/2$;
- $\bar{r}_0(x) = [f_2(x, 0) + \bar{s}_0(x) + a\Theta_0'']/(1 + \epsilon)$.

The operator Λ denotes the second difference derivatives in space, that is $\Lambda y = y_{\bar{x}x}$, and

$$\bar{\Lambda}\eta = \begin{cases} \Lambda\eta, & \text{when} \quad x \in \omega_h, \\ 2\eta_x/h, & \text{when} \quad x = 0, \\ -2\eta_{\bar{x}/h}, & \text{when} \quad x = 1. \end{cases}$$

We use the difference scheme with weights σ $(0 \leq \sigma \leq 1)$ in order to achieve the second order of approximation in space and time (in the general case we have $\Lambda y^{(\sigma)} = \sigma\Lambda\hat{y} + (1 - \sigma)\Lambda y$, where the hat denotes the value of y taken from the upper time-level). The approximations to f_1 and f_2 are chosen by analogous reasoning, for example, $\varphi_1 = f_1(x_i, t_j)$, $\varphi_2 = f_2(x_i, t_{j+0.5})$ (another way to approximate the functions f_1 and f_2 will be given in Section 4).

In addition to the consistency of difference scheme (2.5), (2.6) we need a stability result for our discrete approximation to ensure that the error of such an approximation is small. However, due to non-homogeneity of the Dirichlet boundary conditions for difference approximations of deformations (which are computed using approximate values of temperature!) analysis of the stability of difference scheme (2.5)–(2.6) is hampered. To overcome this difficulty we propose to return to the formulation in stresses for *the discrete rather than for the continuous problem*. This idea is intrinsic in the analysis of stability and in obtaining a new *a-priori estimate* in the next section. The accuracy of *a-posteriori error estimates* will depend on the accuracy of the recovered stresses which are computed directly from a difference scheme that we propose in the next section. The implicit balance between a-priori and a-posteriori estimates provide a foundation for effective adaptive computational procedures [10].

3 Stable Operator-Difference Scheme for Problems of Thermoelasticity.

Let us introduce a *discrete* function $v = y - \eta$ that gives an approximation to the function of stresses. Then the system of difference equations (2.5)–(2.6) may be rewritten in terms of (v, η) as follows

$$\begin{cases} v_{\bar{t}t} = \Lambda v - \eta_{\bar{t}t} + \varphi_1, \\ (1+\epsilon)\eta_t = \bar{\Lambda}\eta^{(\sigma)} - \epsilon v_t + \varphi_t + \varphi_2, \end{cases} \tag{3.1}$$

$$v = 0,\ x \in \Gamma; \quad v = s_0(x),\ v_t = s_1(x),\ \eta = \Theta_0(x), t = 0, \tag{3.2}$$

where

$$s_1(x) = (1-\epsilon)r_1(x) + \bar{\Lambda}\eta^{(\sigma)}\big|_{t=0} + \varphi_2(x, 0).$$

We introduce two sets of discrete functions

$$H_1 = \{v(x) :\ x \in \bar{\omega}_h\}, \quad H_2 = \{\eta(x) :\ x \in \bar{\omega}_h\}$$

with the scalar product

$$(y, z) = \sum_{x \in \bar{\omega}_h} \hbar y(x) z(x)$$

($y \in H_i$, and $z \in H_i$, for $i = 1, 2$ respectively), where

$$\hbar = \begin{cases} h, & \text{when } x \in \omega_h, \\ h/2, & \text{when } x \in \bar{\omega}_h/\omega_h. \end{cases}$$

Instead of the difference scheme (3.1)–(3.2) we shall investigate a more general operator-difference scheme. All results for the scheme (3.1)–(3.2) will follow as a special case of our construction.

Let us define operators from

$$\overset{0}{H}_1 = v(x) :\ x \in \bar{\omega}_h; v = 0\ x = 0, 1$$

into H_1 as follows

$$D_1 = I - \frac{\sigma_1 + \sigma_2}{2}\tau^2\Lambda,\ B_1 = -(\sigma_1 - \sigma_2)\tau\Lambda,\ A_1 = -\Lambda,\ C_1 = I, \tag{3.3}$$

and the operators from H_2 into H_2 by the formulas

$$B_2 = (1+\epsilon)I + (1-\sigma_3)\tau\bar{\Lambda}, \quad A_2 = -\bar{\Lambda}, \quad C_2 = \epsilon I, \tag{3.4}$$

where σ_i, $i = 1, 2, 3$ are weights taking values between zero and one, and I is the identity operator. Let us further assume that

$$\sigma_3 \geq \frac{1+\alpha}{2} - \frac{h^2(1-\beta)}{4\alpha\tau}, \quad \frac{\sigma_1 + \sigma_2}{2} \geq \frac{1+\gamma}{4} - \frac{h^2}{4\tau^2}, \tag{3.5}$$

where

$$\alpha, \gamma > 0,\ 0 < \beta < 1,\ 0 \leq \sigma_i \leq 1,\ i = 1, 2, 3.$$

We note that if the conditions (3.5) are satisfied then the operators defined by the formulas (3.3), (3.4) exist, they are positive definite and selfadjoint. Now we are in the position to consider the following operator-difference generalization of the scheme (3.1)–(3.2)

$$\begin{cases} D_1 v_{\bar{t}t} + B_1 v_{\overset{\circ}{t}} + A_1 v + C_1 \eta_{\bar{t}t} = \varphi_1, \\ B_2 \eta_t + A_2 \hat{\eta} + C_2 v_t = \varphi_2, \end{cases} \tag{3.6}$$

where $v_{\overset{\circ}{t}}$ denotes the central difference derivative (that is $v_{\overset{\circ}{t}} = (v_i^{j+1} - v_i^{j-1})/(2\tau)$). For norms of our discrete functions we introduce the following notation

$$\|y(t)\|^2 = (y(t), y(t)), \quad \|y(t)\|^2_{A_1} = (A_1 y(t), y(t)),$$

$$\|y(t)\|^2_{(1)} = \|y(t)\|^2, \quad \|y(t)\|^2_{(2)} = \|\sum_{t'=0}^{t} A_2 y(t')\|^2.$$

We formulate the main result of this section as follows

Theorem 3.1. *The operator-difference scheme (3.6), (3.2) is stable with respect to initial data and right-hand side if the conditions (3.5) are satisfied. For the discrete approximation of the problem (2.1)–(2.3) by the scheme (3.6), (3.2) the following estimate*

$$\|v(t_1)\|_{(1)} + \|\eta(t_1)\|_{(2)} \leq M\{\|v(0)\|_{\bar{D}_1} + \|A_2 v_t(0)\| + \|A_2 \eta(0)\| + \sum_{t'=0}^{T} \tau \|A_2(\kappa_1 + \kappa_2)\| + \sum_{t'=0}^{T} \tau \|\xi_1 + \xi_2\| + \sum_{t'=\tau}^{T} \tau \|(\kappa_1 + \kappa_2)_{\bar{t}})\|, \tag{3.7}$$

holds, where

$$\bar{D}_1 = (B_2 + \frac{\tau}{2} A_2) D_1 - C_2, \quad \varphi_1 = (\xi_1)_{\bar{t}} + (\xi_2)_{\bar{x}}, \quad \varphi_2 = (\kappa_1)_{\bar{t}} + (\kappa_2)_{\bar{x}}.$$

Proof. First, we apply the operator $\bar{B}_2 = B_2 + \tau A_2/2$ to both parts of the first equation of the system (3.6). Then, using the expression for the second difference derivative $\eta_{\bar{t}t}$ from the second equation of the system (3.6), and substituting its value into the resulting first equation, we have

$$\bar{D}_1 v_{\bar{t}t} + \bar{B}_2 B_1 v_{\overset{\circ}{t}} + \bar{B}_2 A_1 v - A_2 \eta_{\overset{\circ}{t}} = \bar{B}_2 \varphi_1 - (\varphi_2)_{\bar{t}}, \tag{3.8}$$

where $\bar{D}_1 = \bar{B}_2 D_1 - C_2$.

Now we apply

- the operator C_2 to both parts of the equation (3.8), and
- the operator A_2 to the following consequence of the second equation in (3.6)

$$B_2 \eta_{\overset{\circ}{t}} + A_2(\hat{\eta} + \eta)/2 + C_2 v_{\overset{\circ}{t}} = \varphi_2 + \check{\varphi}_2,$$

where the above-check denotes values taken from the previous time layer. As a result we obtain the following system of operator-difference equations

$$
\begin{cases}
C_2\bar{D}_1 v_{\bar{t}t} + C_2\bar{B}_2 B_1 v_{\underset{t}{0}} + C_2\bar{B}_2 A_1 v - C_2 A_2 \eta_{\underset{t}{0}} = \\
\qquad C_2\bar{B}_2\varphi_1 - \check{C}_2(\varphi_2)_{\bar{t}}, \\
A_2 B_2 \eta_{\underset{t}{0}} + (A_2)^2(\hat{\eta} + \eta)/2 + A_2 C_2 v_{\underset{t}{0}} = A_2(\varphi_2 + \check{\varphi}_2).
\end{cases}
\tag{3.9}
$$

We note that the operators A_2 and C_2 are commutative. Then we perform the following two operations:

- we multiply both parts of the first equation in equation (3.9) by the function of the discrete argument $t \in \omega_\tau$

$$w(t) = \sum_{t'=t+\tau}^{t_1} \tau[v(t') + v(t' - \tau)],$$

  which has the following properties [19]

$$w_{\bar{t}} = -(v + \check{v})/2, \quad w(t) = 0, \ \ t \geq t_1,$$

  and

- multiply both parts of the second equation in equation (3.9) by the function of the discrete argument $t \in \omega_\tau$

$$\zeta(t) = \sum_{t'=t+\tau}^{t_1} \tau[\eta(t') + \eta(t' - \tau)].$$

Then the resulting equalities are added together, the result is multiplied by 2τ, and is summed up in t from τ to t_1. Using a technique developed for hyperbolic equations (see [19]) we can get an energy identity. Assuming that

$$\varphi_1 = (\xi_1)_{\bar{t}} + (\xi_2)_{\bar{x}}, \quad \varphi_2 = (\kappa_1)_{\bar{t}} + (\kappa_2)_{\bar{x}},$$

we use the conditions (3.5) and apply Cauchy-Schwarz inequality and the discrete analogue of the Gronwall lemma [7] to ensure the validity of the estimate (3.7). ∎

Convergence results under classical assumptions on solution smoothness follow easily from Theorem 3.1. For example, from the analysis of approximation error using the Taylor formula and the a-priori estimate (3.7) we can readily come to the following conclusion.

Corollary 3.1. *If conditions (3.5) are satisfied, then the solution (v, η) of the scheme (3.6), (3.2) converges to the solution of the problem (2.1)–(2.3) $(s, \Theta) \in C^{4,4}(\bar{Q}_T) \times C^{4,3}(\bar{Q}_T)$, and the error of the scheme is characterized by the following estimate*

$$\|v^j - s^j\|_{(1)} \leq M_1\Phi_1(h, \tau), \quad \|\eta^j - \Theta^j\|_{(1)} \leq M_2\Phi_1(h, \tau),$$

where j is the index of the current time layer, M_1 and M_2 are constants that do not depend on h and τ, and $\Phi_1(h, \tau) = h^2 + \tau^2 + (\sigma_1 - \sigma_2)\tau + (\sigma_3 - 0.5)\tau$.

In the next section Theorem 3.1 enables us to prove convergence of difference approximations under relaxed smoothness assumptions that are typical in many applications of thermoelasticity.

4 Convergence of the Operator-Difference Scheme on Weak Solutions of Thermoelasticity Problems.

It is well-known that the analysis of error approximation using the technique of the classical Taylor formula leads to *excessive requirements* on the smoothness of the sought-for solution [16, 21]. Below we shall show how such requirements can be relaxed using the Steklov operators.

Let us assume that conditions under which the solution of the problem (2.1)–(2.3) belongs to the class $W_2^2(Q_T)$ are satisfied (see [9, 12] and references therein). That is, we assume that $(s, \Theta) \in W_2^2(Q_T) \times W_2^2(Q_T)$.

We consider the operator-difference scheme (3.6), (3.2) with the following source terms

$$\varphi_1 = S^x \otimes S^{t_1} f_1(x,t), \quad \varphi_2 = S^x \otimes S^{t_2} f_2(x,t),$$

where $S^x \otimes S^{t_i}$, $i = 1,2$ denote the composition of averaging Steklov's operator acting in space and time. The operators S^x and S^{t_i} acting on a function $u(x,t)$ are introduced as follows

$$S^x u(x,t) = \begin{cases} 2\displaystyle\int_0^{h/2} u(\xi,t)d\xi/h, & \text{when} \quad x = 0, \\ \displaystyle\int_{x-h/2}^{x+h/2} u(\xi,t)d\xi/h & \text{when} \quad 0 < x < 1, \\ 2\displaystyle\int_{1-h/2}^{1} u(\xi,t)d\xi/h & \text{when} \quad x = 1, \end{cases}$$

$$S^{t_1} u(x,t) = \begin{cases} \displaystyle\int_{t-\tau/2}^{t+\tau/2} u(x,\mu)d\mu/\tau, & \text{when} \quad t > 0, \\ 2\displaystyle\int_0^{\tau/2} u(x,\mu)d\mu/h & \text{when} \quad t = 0, \end{cases}$$

$$S^{t_2} u(x,t) = \begin{cases} \displaystyle\int_{t}^{t+\tau} u(x,\mu)d\mu/\tau, & \text{when} \quad t > 0, \\ 2\displaystyle\int_0^{\tau/2} u(x,\mu)d\mu/h & t = 0. \end{cases}$$

Then the scheme error

$$z_1 = v - s, \quad z_2 = \eta - \Theta$$

is the solution of the following problem

$$\begin{cases} D_1(z_1)_{\bar{t}t} + B_1(z_1)_{\overset{\circ}{t}} + A_1 z_1 + C_1(z_2)_{\bar{t}t} = \psi_1, \\ \bar{B}_2(z_2)_t + A_2(\hat{z}_2 + z_2)/2 + C_2(z_1)_t = \psi_2, \end{cases}$$

$$z_1 = 0, \text{ when } x = 0, 1; \quad z_1 = 0, \ (z_1)_t = \psi_1 \text{ when } t = 0, \tag{4.1}$$

where

$$\psi_1 = \varphi_1 - [D_1 s_{\bar{t}t} + B_1 s_{\underset{t}{0}} + A_1 s + C_1 \Theta_{\bar{t}t}], \text{ if } t \in \omega_\tau,$$

$$\psi_2 = \varphi_2 - [\bar{B}_2 \Theta_t + A_2(\hat{\Theta} + \Theta)/2 + C_2 s_t], \text{ if } t \in \omega_\tau, \text{ and}$$

$$\psi_1 = \bar{r}_0 + \tau[r_0'' - \Theta_0'' + \varphi_1(x, 0)]/2 - s_t(x, 0), \text{ if } t = 0.$$

We apply the composition of the operators $S^x \otimes S^{t_1}$ and $S^x \otimes S^{t_2}$ to the first and the second equations of the system (2.1) respectively. Then we use the basic properties of Steklov's operators as follows

$$S^x \frac{\partial u}{\partial x} = \frac{1}{h}[u(x + h/2, t) - u(x - h/2, t)] = (u^{(-0.5)})_x,$$

$$S^{t_1} \frac{\partial u}{\partial t} = (\check{\bar{u}})_t, \quad S^{t_2} \frac{\partial u}{\partial t} = u_t,$$

where $\bar{u} = u(x, t + \tau/2)$. It is straightforward to deduce a representation for the scheme error, for example, for internal nodes we have

$$\psi_1 = (\eta_1)_t + \frac{\sigma_1 + \sigma_2}{2} \tau^2 (\eta_2)_t + (\eta_3)_x + (\sigma_1 - \sigma_2)\tau(\eta_5)_t,$$

where the functionals $\eta_i, \ i = 1, ..., 5$ are defined as follows

$$\eta_1 = S^x \left(\frac{\partial \check{\bar{s}}}{\partial t} \right), \ \eta_2 = \Lambda s_{\bar{t}}, \ \eta_3 = s_{\bar{x}} - S^{t_1} \left((\frac{\partial s}{\partial x})^{(-0.5)} \right),$$

$$\eta_4 S^x \left(\frac{\partial \check{\bar{\Theta}}}{\partial t} \right) - \Theta_{\bar{t}}, \ \eta_5 = \Lambda(s - \tau s_{\bar{t}}/2),$$

and similarly,

$$\psi_2 = (1 + \epsilon)(\eta_6)_t + \epsilon(\eta_7)_t - (\sigma_3 - 0.5)\tau(\eta_8)_t + a(\eta_9)_x,$$

where the functionals $\eta_i, \ i = 6, 7, 8, 9$ are defined by the formulas

$$\eta_6 = S^x \Theta - \Theta, \ \eta_7 = S^x s - s, \quad \eta_8 = A_2 \Theta,$$

$$\eta_9 = \left(\frac{\hat{\Theta} + \Theta}{2} \right)_{\bar{x}} - S^{t_2} \left((\frac{\partial \Theta}{\partial x})^{(-0.5)} \right).$$

Analogous functions are present in the representations of error approximation of

- boundary conditions for temperature, and
- initial conditions of the problem.

Functionals $\eta_i, \ i = 1, ...9$ are estimated using the Bramble-Hilbert lemma (see [16] and references therein). For example, it is easy to see that the linear functional η_3 is bounded in the space $W_2^2(Q_T)$, therewith

$$|\eta_3| \leq M h^{(-1)} ||s||_{W_2^2(e)},$$

where

$$e = \{(x', t') : \quad x - h < x' < x, \ t - \tau/2 < t' < t + \tau/2\}.$$

By a standard linear change of variables we can pass from the domain e to the domain

$$E = \{(u_1, u_2): \quad -1 < u_1 < 0,\ -0.5 < u_2 < 0.5\}.$$

Since a linear change of variables does not change the class of functions, we have

$$|\eta_3| \le M h^{(-1)} ||s||_{W_2^2(E)}.$$

Now it is easy to verify that the functional

$$\eta_3 = \frac{1}{2h}\{\bar{s}(0,0) - \bar{s}(-1,0) - \int_{-0.5}^{0.5} \frac{\partial \bar{s}(-0.5, u_2)}{\partial u_1} du_2\}$$

(where $\bar{s}(u) = s(x(\xi_1), t(\xi_2)))$ is zero for polynomials up to first degree inclusive. Therefore, from the Bramble-Hilbert lemma we have

$$|\eta_3| \le M h^{-1} |\bar{s}|_{W_2^2(E)},$$

and passing to the variables (x, t) we finally get

$$|\eta_3| \le M \frac{h^2 + \tau^2}{h} (h\tau)^{(-1/2)} |s|_{W_2^2(E)}.$$

Using the technique of the Bramble-Hilbert lemma for other functionals , and applying the estimate (3.7), we come to the following result

Theorem 4.1. *The solution of the operator-difference scheme (3.6), (3.2) with $\varphi_i = S^x \otimes S^{t_i} f_i$, $i = 1, 2$ converges to the generalized solution of the problem (2.1)–(2.3) $(s, \Theta) \in W_2^2(Q_T) \times W_2^2(Q_T)$ if the stability conditions (3.5) are satisfied. Therewith the following accuracy estimate*

$$||v^j - s^j||_{(1)} \le M_1 \Phi_2(h, \tau), \quad ||\eta^j - \Theta^j||_{(1)} \le M_2 \Phi_2(h, \tau), \tag{4.2}$$

where j is the index of the current time layer, M_1 and M_2 are constants that do not depend on h and τ, and $\Phi_2(h, \tau) = h + \tau$. holds.

Remark 4.1. In the case of uncoupled thermoelasticity (when $\epsilon = 0$) and homogeneous boundary conditions for deformations the accuracy estimate (4.2) can be improved using the result of Theorem 3.1.

5 Future directions.

The effective computational procedures developed for problems in coupled thermoelasticity can be used as building blocks for the development of refined models in other areas of coupled field theory, for example, in coupled thermo-electroelasticity. Interest in piezoelectrics has been revitalized [1] by its significance for smart materials [17] and the importance of piezoelectricity in biopolymers [8] . Mathematical models of dynamic electroelasticity have been extensively studied in the literature (see [16] and references therein). However, notwithstanding the coupled treatment of electro-mechanical fields in such models, for many practical applications it is important to take into consideration thermal effects as temperature has a profound influence on the properties of piezoelectrics.

To be competitive numerical codes in coupled field theory have to be adaptive. One of the major goal of adaptive computational schemes is to control the computational process. The success of this process often hinges on how well the numerical error can be estimated. This implies that in the general case we need a combination of a-priori and a-posteriori estimates [10]. The major difficulty in obtaining such a combination stems from the fact that the error needs to be integrated *with respect to time* which complicates numerical analysis in the nonstationary case. Computationally we are inevitably led to an optimization problem and the whole computational process can be seen as a problem of *optimal error control* [15].

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