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# EXISTENCE AND UNIQUENESS THEOREMS OF THE GENERALIZED SOLUTION FOR A CLASS OF NON-STATIONARY PROBLEM OF COUPLED ELECTROELASTICITY

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In [1] a non-stationary problem of coupled electroelasticity for cored infinite piezoelectric ceramic radially prepolarized cylinders has been considered. The second-order accuracy difference scheme has been constructed and justified, and computational experiments were carried out to solve this problem.

In the present article the existence and uniqueness theorems for the generalized solution of the problem are proved by the Faedo-Galerkin method, and the smoothness of the solution is studied.

1. The mathematical model of the considered class of the problems includes the motion equations of piezoelastic continuum and the equations of forced electrostatic of dielectrics [2]:

$$\rho\,\frac{\partial^2 u}{\partial t^2}=\frac{1}{r}\,\frac{\partial}{\partial r}(r\sigma_r)-\frac{\sigma_\theta}{r}+f_1(r,t),\quad R_0<r<R_1,\quad t>0, \tag{1}$$

$$\frac{1}{r}\,\frac{\partial}{\partial r}(rD_r)=f_2(r,t), \tag{2}$$

which are connected by means of characteristic equations

$$\sigma_r=c_{11}\varepsilon_r+c_{12}\varepsilon_\theta-e_{11}E_r,\quad \sigma_\theta=c_{12}\varepsilon_r+c_{22}\varepsilon_\theta-e_{12}E_r,\quad D_r=\varepsilon_{11}E_r+e_{12}\varepsilon_\theta+e_{11}\varepsilon_r \tag{3}$$

(the strain-displacement relations (Cauchy's ones) have the form $\varepsilon_r=\frac{\partial u}{\partial r}$, $\varepsilon_\theta=\frac{u}{r}$) and supplied by boundary and initial conditions

$$\sigma_r=p_0(t)\ \text{and}\ \varphi=V(t),\ \text{when}\ r=R_0;\quad \sigma_r=p_1(t)\ \text{and}\ \varphi=-V(t),\ \text{when}\ r=R_1, \tag{4}$$

$$u(r,0)=u_0(r),\quad \frac{\partial u(r,0)}{\partial t}=u_1(r) \tag{5}$$

an electrostatic potential is introduced for the description of electric field with the help of formula $E_r=-\frac{\partial\varphi}{\partial r}$). Here $u$ means radial displacements; $E_r$ and $D_r$ are radial components of the electric field intensity vector and electric induction vector, respectively; $c_{kl}$ are elasticity moduli; $e_{ij}$ are piezoelectric moduli; $\varepsilon_{11}$ is dielectric constant, $\rho$ is density of piezoelectric ceramics; $f_1$ is density of mass forces; $f_2$ is density of body's charge. We suppose that for any $\xi_1$ and $\xi_2$ the condition:

$$\delta(\xi_1^2+\xi_2^2)\le c_{11}\xi_1^2+2c_{12}\xi_1\xi_2+c_{22}\xi_2^2\ ,\quad \delta>0, \tag{6}$$

is fulfilled, that means a non-negativity of the strain energy.

2. The questions of the solution existence for static problems of electroelasticity have been studied in the articles of L.P.Bitsadze, A.B.Belokon', I.I.Vorovich and others (see [3] and [4]). The existence of the coupled non-stationary electroelasticity problems' solution is not investigated yet.

According to the general approach to the boundary problems of mathematical physics (see [5]), let us call the pair of functions $(u(r,t),\varphi(r,t))\in W_2^1(Q_T)\times L_2(I,\overset{\circ}{W}{}_2^1(G))$ ($u(r,t)$ is equal to $u_0(r)$, when $t=0$), which satisfy the following identities:

$$\int_{Q_T} r\Big(-\rho\frac{\partial u}{\partial t}\,\frac{\partial\eta}{\partial t}+\sigma_r\,\frac{\partial\eta}{\partial r}+\frac{\sigma_\theta}{r}\eta\Big)drdt-\int_{R_0}^{R_1} r\rho u_1(r)\eta(r,0)dr=\int_{Q_T} rf_1\eta drdt\quad \forall\eta\in\hat{W}_2^1(Q_T), \tag{7}$$

$$\int_{R_0}^{R_1}(-\varepsilon_{11}r\frac{\partial\varphi}{\partial r}\frac{\partial\zeta}{\partial r}+e_{11}r\varepsilon_r\frac{\partial\zeta}{\partial r}+e_{12}r\varepsilon_\theta\frac{\partial\zeta}{\partial r})dr=-\int_{R_0}^{R_1}rf_2\zeta dr \quad \forall\zeta\in\mathring{W}_2^1(G) \text{ for a.e. } t\in(0,T). \tag{8}$$

as *the generalized solution of the problem* (1)-(5). To simplify the text, we suppose the homogeneity of the boundary conditions (4). The space $\hat{W}_2^1(Q_T)$ consists of elements of $W_2^1(Q_T)$, which vanish if: $t=T$; also $Q_T=I\times G$, $I=(0,T)$, $G=(R_0,R_1)$.

The aim of this item is to prove the following theorem.

THEOREM 1. *If* $f_1\in W_2^1(I,H)$, $\frac{\partial f_2}{\partial t}\in L_2(I,W_2^{-1})$, $f_2|_{t=0}\in W_2^{-1}(G)$, $u_0\in W_2^1(G)$ *and* $u_1\in L_2(G)$, *then the unique generalized solution of the problem* (1)-(5) *with properties*

$$\frac{\partial u}{\partial t}\in L_2(I,H),\ \frac{\partial^2 u}{\partial t^2}\in L_2(I,(W_2^1)^*),\ \frac{\partial\varphi}{\partial t}\in L_2(I,W_2^{-1}), \tag{9}$$

*exists, too, where* $H=L_2(G)$, *and* $W_2^{-1}(G)=(\mathring{W}_2^1(G))^*$.

Perform the proof in three steps, using the Faedo-Galerkin's method (see [6] and [7]).

1). Let $\{\chi_{1m}\}$ and $\{\chi_{2m}\}$ be complete linearly independent systems of functions in the spaces $W_2^1(G)$ and $\mathring{W}_2^1(G)$, respectively; these systems satisfy the following properties of orthonormalization:

$$(\chi_{1k},\chi_{1l})=\delta_{kl},\quad (\nabla\chi_{2k},\nabla\chi_{2l})=\delta_{kl}, \tag{10}$$

where $(u,v)=\int_{R_0}^{R_1}ruvdr$, $\nabla u=\frac{\partial u}{\partial r}$.

In the expressions for the Galerkin's estimates

$$u^m=\sum_{k=1}^m g_{1k}(t)\chi_{1k}(r),\quad \varphi^m=\sum_{k=1}^m g_{2k}=(t)\chi_{2k}(r) \tag{11}$$

he functions $g_{ik}(t)$, $i=1,2$; $k=\overline{1,m}$, can be found from relations:

$$(\rho\frac{\partial^2u^m}{\partial t^2},\ \chi_{1l})+(\sigma_r^m,\nabla\chi_{1l})+(\sigma_\theta^m,\frac{\chi_{1l}}{r})=(f_1,\chi_{1l}), \tag{12}$$

$$-(D_r^m,\nabla\chi_{2l})=(f_2,\chi_{2l}),\quad l=\overline{1,m}, \tag{13}$$

$$\frac{d}{dt}g_{1k}(t)|_{t=0}=(u_1,\chi_{1k}),\quad g_{1k}(0)=\alpha_{km}, \tag{14}$$

here $\alpha_{km}$ are coefficients of the sums $u_0^m(r)=\sum_{k=1}^m\alpha_{km}\chi_{1k}(r)$. These approximate the unction $u_0(r)$ in norm $W_2^1(G)$, where $m\to\infty$, and the expressions of $\sigma_r^m$, $\sigma_\theta^m$ and $D_r^m$ were btained with the replacement of all $u$ and $\varphi$ by $u^m$ and $\varphi^m$ in the appropriate equaions (3).

By virtue of the choice of the functions $\chi_{2k}$, $k=\overline{1,m}$, from relationship (12) his is a system of $m$ linear algebraic equations for $m$ unknowns $g_{2i}$) we can easily btain:

$$g_{2l}=\frac{1}{\varepsilon_{11}}\Big\{(f_2,\chi_{2l})+e_{12}(\sum_{i=1}^m g_{1i}\frac{\chi_{1i}}{r},\Delta\chi_{2l})+e_{11}(\sum_{i=1}^m g_{1i}\nabla\chi_{1i},\nabla\chi_{2l})\Big\},\quad l=\overline{1,m}. \tag{15}$$

ing (15) for transformation (12), we get an ordinary equation system of the nd order with respect to $t$ for the unknown $g_{1k}$, $k=\overline{1,m}$, in the following form

$$\rho\frac{d^2g_1}{dt^2}+(A+B)g_1=F$$

ovided by the initial conditions (14) and by symmetric matrices of "deform of "electric field" $B$. Here: $g_1=(g_{11},g_{12},\ldots,g_{1m})^T$, $F=(F_1,F_2,\ldots,F_m)^T$, $A=(a$

$B=(b_{ij})$, $i,j=\overline{1,m}$, $a_{kl}=c_{11}(\nabla\chi_{1k},\nabla\chi_{1l})+c_{12}\left[(\frac{\chi_{1k}}{r},\nabla\chi_{1l})+(\nabla\chi_{1k},\frac{\chi_{1l}}{r})\right]+c_{22}(\frac{\chi_{1k}}{r},\frac{\chi_{1l}}{r})$,

$b_{kl}=\sum_{j=1}^{m}\left\{\frac{e_{11}^2}{\varepsilon_{11}}(\nabla\chi_{1l},\nabla\chi_{2j})(\nabla\chi_{1k},\nabla\chi_{2j})+\frac{e_{12}^2}{\varepsilon_{11}}(\frac{\chi_{1l}}{r},\nabla\chi_{2j})(\frac{\chi_{1k}}{r},\nabla\chi_{2j})+\frac{e_{11}e_{12}}{\varepsilon_{11}}\left[(\nabla\chi_{1l},\nabla\chi_{2j})\times\right.\right.$

$\left.\left.\times(\frac{\chi_{1k}}{r},\nabla\chi_{2j})+(\frac{\chi_{1l}}{r},\nabla\chi_{2j})(\nabla\chi_{1k},\nabla\chi_{2j})\right]\right\}$, $F_1=(f_1,\chi_{1l})+\frac{1}{\varepsilon_{11}}\sum_{j=1}^{m}(f_2,\chi_{2j})\left[(e_{11}(\nabla\chi_{2j},\nabla\chi_{1l})+\right.$

$\left.+e_{12}(\nabla\chi_{2j},\frac{\chi_{1l}}{r})\right]$. In analogous to [8] (p.327) way we can show that the system (16) has a unique solution, which belongs to $W_2^2(0,T)$ and satisfies (14), being also such that $\frac{dg_1}{dt}\in W_2^1(0,T)$. Therefore, by virtue of the theorem suppositions and the relation (15) we have that $\frac{dg_{2k}}{dt}\in L_2(0,T)$, $k=\overline{1,m}$, i.e., the system (12)-(14) has a unique solution in the interval $\bar{I}$ and, in view of (11), $u^m$ and $\varphi^m$ are defined uniquely.

2) Let us multiply the equation (12) by function $\frac{dg_{1l}}{dt}$, and the equation (13), after differentiation with respect to $t$, by $g_{2l}$. Let us summarize (with respect to $l$ from 1 to $m$) these results separately and then take a sum of them. Then we receive as a result:

$$\frac{d}{dt}E^m(t)=(f_1,\frac{\partial u^m}{\partial t})+(\frac{\partial f_2}{\partial t},\varphi^m), \tag{17}$$

where

$$E^m(t)=\frac{1}{2}\rho\|\frac{\partial u^m}{\partial t}\|^2+\frac{1}{2}c_{11}\|\varepsilon_r^m\|^2+c_{12}(\varepsilon_r^m,\varepsilon_\theta^m)^2+\frac{1}{2}c_{22}\|\varepsilon_\theta^m\|^2+\frac{1}{2}\varepsilon_{11}\|E_r^m\|^2.$$

Integrating the equation (17) with respect to $t$, estimating the terms on the right-hand side of the obtained identity by the help of the Cauchy-Bunyakovskii generalized inequality, taking into account the Poincare inequality and the condition (16), we have

$$\|\frac{\partial u^m}{\partial t}(t)\|_H^2+\|u^m(t)\|^2_{W_2^1(G)}+\|\varphi^m(t)\|^2_{\overset{\circ}{W}{}_2^1(G)}\le M_1\left\{\|E_r^m(0)\|_H^2+\int_0^t\|u^m(\tau)\|^2_{W_2^1(G)}d\tau+\right.$$

$$\left.+\int_0^t\|\varphi^m(\tau)\|^2_{\overset{\circ}{W}{}_2^1(G)}d\tau+\|u_0^m\|^2_{W_2^1(G)}+\|u_1^m\|_H^2+\|f_1\|^2_{W_2^1(I,H)}+\|\frac{\partial f_2}{\partial t}\|^2_{L_2(I,W_2^{-1})}\right\}. \tag{18}$$

On estimation of the term $\|E_r^m(0)\|^2$ we use the equation (13) when $t=0$; the (13) was scalar multiplied by function $g_{2l}$ and summarized with respect to $l$ from 1 to $m$:

$$\|E_r^m(0)\|_H^2\le M_2(\|\varepsilon_r^m(0)\|^2+\|\varepsilon_\theta^m(0)\|^2+\|f_2(0)\|^2_{W_2^{-1}(G)})\le M_3\|u^m(0)\|^2_{W_2^1(G)}+M_2\|f_2(0)\|^2_{W_2^{-1}(G)}.$$

Introducing the notation:

$$M_4=M_1M_3\|u^m(0)\|^2_{W_2^1(G)}+M_1M_2\|f_2(0)\|^2_{W_2^{-1}(G)}+M_1\left(\|u_0^m\|^2_{W_2^1(G)}+\|u_1^m\|_H^2+\right.$$

$$\left.+\|f_1\|^2_{W_2^1(I,H)}+\|\frac{\partial f_2}{\partial t}\|^2_{L_2(I,W_2^{-1})}\right)$$

and using the Gronwall's inequality [10], we get from (18) a following *a priori* estimation:

$$\|\frac{\partial u^m(t)}{\partial t}\|_H^2+\|u^m(t)\|^2_{W_2^1(G)}+\|\varphi^m(t)\|^2_{\overset{\circ}{W}{}_2^1(G)}\le C(T,M_1,M_4). \tag{19}$$

3) Due to the the inequality (19) the sequences $\{u^m(t)\}$, $\left\{\frac{\partial u^m(t)}{\partial t}\right\}$ and $\{\varphi^m(t)\}$ are bounded in the spaces $L_2(I,W_2^1)$, $L_2(I,H)$ and $L_2(I,H)\cap L_2(I,\overset{\circ}{W}{}_2^1)$, respectively. Furthermore, we can choose from these sequences the following subsequences (denote

them by $\{u^\nu\}$, $\{\frac{\partial u^\nu}{\partial t}\}$ and $\{\varphi^\nu\}$, respectively), which converge weakly in appropriate spaces:

$$u^\nu(t)\longrightarrow z(t) \text{ weakly in } L_2(I,W_2^1),\ \frac{\partial u^\nu}{\partial t}\longrightarrow\frac{\partial z(t)}{\partial t} \text{ weakly in } L_2(I,H) \text{ and}$$
$$\varphi^\nu(t)\longrightarrow y(t) \text{ weakly in } L_2(I,H)\cap L_2(I,\overset{\circ}{W}{}_2^1),\quad \text{if } \nu\longrightarrow\infty. \tag{20}$$

Like in [6] and [7] we conclude that the initial condition $u|_{t=0}=u_0(r)$ is fulfilled due to the convergence of $u^\nu$ to $z$ in $L_2(G)$ and by virtue of $u^\nu(r,0)\longrightarrow u_0(r)$ in $L_2(G)$. Let us select the functions $\xi^l(t)\in W_2^1(0,T)$, $\xi^l(T)=0$ and $\xi_l(t)\in C^\infty(\bar{I})$. We multiply each relationship from (12) by the appropriate function $\xi^l$, summarize the obtained equalities with respect to $l$ and integrate the result with respect to $t$ from zero to $T$. Integrating by parts the obtained result, we transfer the derivative with respect to time from $u^m$ onto $\eta\equiv\sum_{l=1}^m\xi^l(t)\chi_{1l}(r)$. We get an identity as a result:

$$\int_{Q_T} r\Big(-\rho\frac{\partial u^m}{\partial t}\frac{\partial\eta}{\partial t}+\sigma_r^m\frac{\partial\eta}{\partial r}+\frac{\sigma_\theta^m}{r}\eta\Big)drdt-\int_{R_0}^{R_1} r\rho\frac{\partial u^m}{\partial t}\eta\Big|_{t=0}dr=\int_{Q_T} rf_1\eta drdt, \text{ that holds}$$
$$\text{for any } \eta \text{ of the form: } \sum_{l=1}^m\xi^l(t)\chi_{1l}(r). \tag{21}$$

We multiply the relationships (13) by functions $\xi_l$ and summarize the obtained equations with respect to $l$:

$$\int_{R_0}^{R_1}\Big(-\varepsilon_{11}r\frac{\partial\varphi^m}{\partial r}\frac{\partial\zeta}{\partial r}+e_{11}r\varepsilon_r^m\frac{\partial\zeta}{\partial r}+e_{12}r\varepsilon_\theta^m\frac{\partial\zeta}{\partial r}\Big)dr=-\int_{R_0}^{R_1}rf_2\zeta dr\qquad\forall\zeta\equiv\sum_{l=1}^m\xi_l(t)\chi_{2l}(r). \tag{22}$$

Denote by $\pi_m$ the set of functions $\eta$ representable in the form $\sum_{l=1}^m\xi^l(t)\chi_{1l}(r)$, and $\Omega_m$ is a notation for the set of functions $\zeta$ such that they have the form $\sum_{l=1}^m\xi_l(t)\chi_{2l}(r)$. According to (20) we can pass in (21) and (22) to the limits by the above chosen subsequences $\{u^\nu\}$ and $\{\varphi^\nu\}$, with the fixed $\eta\in\pi_{m_i}$ and $\zeta\in\Omega_{m_j}$. That leads to identities (7) and (8) for a pair of the limit functions: $\{z,y\}$ $\forall\eta\in\pi_{m_i}$, $\zeta\in\Omega_{m_j}$. Since $\bigcup_{m=1}^{j}\pi_m$ is dense in $\hat{W}_2^1(Q_T)$ and $\bigcup_{m=1}^{\infty}\Omega_m$ is dense in $L_2(I,\overset{\circ}{W}{}_2^1(G))$ (see [5], p.215, [11], p.39) and also $z\in W_2^1(Q_T)$ and $y\in L_2(I,\overset{\circ}{W}{}_2^1(G))$, we have that (7) and (8) are fulfilled for the pair of functions $\{z,y\}$ when $\forall\eta\in\hat{W}_2^1(Q_T)$ and $\zeta\in\overset{\circ}{W}{}_2^1(G)$ (for almost all $t\in(0,T)$). Using now (20), equation (1) and equation (2) differentiated with respect to $t$, we get

$$\frac{\partial z}{\partial t}\in L_2(I,H),\quad\frac{\partial^2 z}{\partial t^2}\in L_2(I,(W_2^1)^*),\quad\frac{\partial y}{\partial t}\in L_2(I,W_2^{-1}).$$

Supposing the existence of two solutions $(u,\varphi)$ and $(w,\beta)$ of the problem (1)-(5), we notice that their difference $U=u-w$, $\Phi=\varphi-\beta$ satisfy the homogeneous equations system (1)-(2) with homogeneous initial conditions. Repeating the argument of the item 2, we get an *a priori* estimation (19) for the functions $(U,\Phi)$ when $C=0$. It follows that $U=\Phi=0$ (the proof of the uniqueness of the solutions is given also [12], p.22), that complete the proof of the theorem 1.

3.1) Let us study the smoothness of the generalized solution. We show that strengthening of the theorem 1 conditions we can prove the existence of a smooth solution than the solution with the properties (9) (they will coincide virtue of the uniqueness).

THEOREM 2. *If the conditions* $f_1 \in W_2^2(I,H)$, $\frac{\partial f_2}{\partial t} \in W_2^1(I, W_2^{-1}(G))$, $\frac{\partial f_2}{\partial t}|_{t=0} \in W_2^{-1}(G)$, $u_0 \in W_2^2(G)$ *and* $u_1 \in W_2^1(G)$ *are fulfilled, then the unique generalized solution of the problem* (1)-(5) *with properties*

$$\frac{\partial^2 u}{\partial t^2} \in L_2(I,H), \quad \frac{\partial^3 u}{\partial t^3} \in L_2(I,(W_2^1)^*), \quad \frac{\partial^2 \varphi}{\partial t^2} \in L_2(I, W_2^{-1}). \tag{23}$$

*exists.*

PROOF. The equation system

$$\rho \frac{\partial^2 P}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}(r\sigma_r^{(P,S)}) - \frac{\sigma_\theta^{(P,S)}}{r} + \frac{\partial f_1}{\partial t}, \tag{24}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rD_r^{(P,S)}) = \frac{\partial f_2}{\partial t}, \tag{25}$$

$$P(0)=u_1, \ \rho\left(\frac{\partial P}{\partial t}\right)(0) = \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(c_{11}\frac{\partial u_0}{\partial r} + c_{12}\frac{u_0}{r} - e_{11}E_r(0)\right]\right. -$$

$$- \frac{1}{r}\left(c_{12}\frac{\partial u_0}{\partial r} + c_{22}\frac{u_0}{r} - e_{12}E_r(0)\right) + f_1(r,0), \ \sigma_r^{(P,S)}|_{\partial G} = S|_{\partial G} = 0, \tag{26}$$

where $\sigma_r^{(P,S)}$, $\sigma_\theta^{(P,S)}$ and $D_r^{(P,S)}$ are obtained with the replacements of all $u$ by $P$ and $\varphi$ by $S$ in the correspondent relationships, $\partial G = \{R_0, R_1\}$ and the function $E_r(0)$, determined from relationship

$$\frac{\partial}{\partial r}\left[e_{12}u_0 + e_{11}r\frac{\partial u_0}{\partial r} + \varepsilon_{11} r E_r(0)\right] = rf_2(r,0), \tag{27}$$

has, by the theorem 1, the unique generalized solution: $(P,S) \in W_2^1(Q_T) \times L_2(I, \overset{\circ}{W}{}_2^1)$, provided with the properties

$$\frac{\partial P}{\partial t} \in L_2(I,H), \quad \frac{\partial^2 P}{\partial t^2} \in L_2(I,(W_2^1)^*), \quad \frac{\partial S}{\partial t} \in L_2(I, W_2^{-1}). \tag{28}$$

Integrating the equations (24) and (25) along an interval $(0,t)$ and taking into account the conditions (26) and (27), we obtain the following equation system:

$$\rho\frac{\partial^2 w}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}(r\sigma_r^{(w,\eta)}) - \frac{\sigma_r^{(w,\eta)}}{r} + f_1, \quad \frac{1}{r}\frac{\partial}{\partial r}(rD_r^{(w,\eta)}) = f_2,$$

where the functions $w(t)$ and $\eta(t)$ are defined by formulas

$$w(t) = u_0 + \int_0^t P(\tau)d\tau, \quad \eta(t) = \varphi(0) + \int_0^t S(\tau)dr \tag{29}$$

and have the properties

$$\frac{\partial w}{\partial t} = P(t), \ \frac{\partial \eta}{\partial t} = S(t), \ w(0) = u_0, \ \left(\frac{\partial w}{\partial t}\right)(0) = u_1, \ \eta|_{\partial G} = \sigma_r^{(w,\eta)}|_{\partial G} = 0. \tag{30}$$

In accordance with (9) and (28)-(30) we have $\frac{\partial w}{\partial t} \in L_2(I, W_2^1)$, $\frac{\partial^2 w}{\partial t^2} \in L_2(I,H)$, $\frac{\partial^3 w}{\partial t^3} \in L_2(I,(W_2^1)^*)$, $\frac{\partial \eta}{\partial t} \in L_2(I, \overset{\circ}{W}{}_2^1)$ and $\frac{\partial^2 \eta}{\partial t^2} \in L_2(I, W_2^{-1})$, i.e., the pair of functions $(w,\eta)$ is the generalized solution of the problem (1)-(5), satisfying the conditions (9) and (23).

The proof of the solutions' uniqueness is carried out by the well-known methods (see the analogous statement proof for the theorem 1 in [12]) and completes the proof of the theorem 2.

2) Finally we find the conditions, under which the generalized solution of the problem (1)-(5), provided with the properties (23), possesses also the properties:

$$\frac{\partial^2 u}{\partial r^2},\ \frac{\partial^2\varphi}{\partial r^2},\ \frac{\partial^2 u}{\partial t\partial r},\ \frac{\partial^2\varphi}{\partial t\partial r}\in L_2(I,H). \tag{31}$$

e consider the auxiliary equation system for unknown $(\sigma_r,\varphi)$. This system is obtained by differentiation with respect to $r$ of the equation (1) of the original ystem:

$$\rho r\ \frac{\partial^2\sigma_r}{\partial t^2}=a\frac{1}{r}\ \frac{\partial}{\partial r}[r\frac{\partial}{\partial r}(r\sigma_r)]+b\ \frac{1}{r}\ \frac{\partial}{\partial r}(r\sigma_r)-a\frac{1}{r}\ \frac{\partial}{\partial r}(r\sigma_\theta)-b\frac{\sigma_\theta}{r}+F_1\ , \tag{32}$$

$$\frac{1}{r}\ \frac{\partial}{\partial r}[r(k_1\sigma_r+k_2\sigma_\theta+k_3E_r)]=f_2, \tag{33}$$

here $\sigma_r$, $\sigma_\theta$ and $E_r$ are connected by the condition:

$$c_{11}\ \frac{\partial}{\partial r}(r\sigma_\theta)+c_{12}\sigma_\theta=c_{12}\ \frac{\partial}{\partial r}(r\sigma_r)+c_{22}\sigma_r+k_2'\ \frac{\partial}{\partial r}(rE_r)+k_1'E_r; \tag{34}$$

$=c_{11}-e_{11}^2/\varepsilon_{11}$, $b=c_{12}-e_{11}e_{12}/\varepsilon_{11}$, $F=a\frac{1}{r}\ \frac{\partial}{\partial r}(r^2f_1)+bf_1$, $k_1'=c_{22}e_{11}-c_{12}e_{12}$,

$'_2=e_{11}c_{12}-e_{12}c_{11}$, $q=c_{12}^2-c_{11}c_{22}$, $k_1=-k_1'/q$, $k_2=-k_2'/q$, $k_3=\varepsilon_{11}+(c_{11}e_{12}^2-c_{22}e_{11}^2)/q$.

he initial boundary conditions have the form:

$$\sigma_r|_{\partial G}=\varphi|_{\partial G}=0,\quad \sigma_r|_{t=0}=\sigma_0,\quad \frac{\partial\sigma_r}{\partial t}|_{t=0}=\sigma_1, \tag{35}$$

here $\sigma_0$ and $\sigma_1$ are the radial stress and the rate of change of $\sigma_r$ in the zero ime.

We say that the pair of functions $(\sigma_r(r,t),\varphi(r,t))\in W^1_{2,0}(Q_T)\times L_2(I,\overset{\circ}{W}{}^1_2(G))$ $\sigma_r(r,t)$ equal to $\sigma_0(r)$ when $t=0$) is a generalized solution f the problem (33)-(35) if the identity

$$\int_{Q_T} r\left(-\rho r\frac{\partial\sigma_r}{\partial t}\ \frac{\partial\eta}{\partial t}\ +\ a\frac{\partial}{\partial r}(r\sigma_r)\frac{\partial\eta}{\partial r}\ +\ b\sigma_r\ \frac{\partial\eta}{\partial r}\right)drdt\ -\ \int_{R_0}^{R_1}\rho r\sigma_1\eta(r,0)dr\ =$$

$$=\int_{Q_T} r(a\sigma_\theta\ \frac{\partial\eta}{\partial r}\ -b\frac{\sigma_\theta}{r}\eta+F_1\eta)drdt\quad \forall\eta\in\hat{W}^1_{2,0}(Q_T),$$

$$\int_{R_0}^{R_1}\left[-k_3r\frac{\partial\varphi}{\partial r}\ \frac{\partial\zeta}{\partial r}\ +k_1r\sigma_r\ \frac{\partial\zeta}{\partial r}\ +k_2r\sigma_\theta\ \frac{\partial\zeta}{\partial r}\right]dr\ =\ \int_{R_0}^{R_1}rf_2\zeta dr$$

olds for any $\zeta\in\overset{\circ}{W}{}^1_2(G)$ and for almost all $t\in(0,T)$. Here $W^1_{2,0}(Q_T)$ is the subspace of he space $W^1_2(Q_T)$, in which the smooth functions, vanishing near $r=R_0$ and $r=R_1$ epresents a dense set (see [5],p.24),

$$\sigma_\theta=\delta_1\ \frac{\partial}{\partial r}(r\sigma_r)+\delta_2\sigma_r+\delta_3\ \frac{\partial}{\partial r}(rE_r)+\delta_4E_r, \tag{36}$$

here $\sigma_1=1+\ \dfrac{c_{11}k_1}{c_{12}k_2}$ , $\delta_2=c_{22}/c_{11}$, $\delta_3=\ \dfrac{1}{c_{12}}\left[k_2'\ +\ \dfrac{c_{11}k_3}{k_2}\right]$, $\delta_4=k_1'/c_{12}$ (the condition (36) ollows from (34) in view of (33)).

Let $\{\chi_{1m}\}$ and $\{\chi_{2m}\}$ be linearly independent functions set in the space $\overset{\circ}{W}{}^1_2(G)$ atisfying the orthonormality conditions (10). Let us define the Galerkin's approximations:

$$\sigma_r^m=\sum_{j=1}^m g^1_{mj}(t)\chi_{1j}\quad\text{and}\quad \varphi^m=\sum_{j=1}^m g^2_{mj}(t)\chi_{2j}.$$

he unknown functions $g^i_{mj}$, $i=1,2$, are defined from the relationships:

$$\rho r\ \frac{\partial^2\sigma_r^m}{\partial t^2}\ ,\ \chi_{11})+a(\frac{\partial}{\partial r}(r\sigma_r^m),\nabla\chi_{11})+b(\sigma_r^m,\nabla\chi_{11})=a(\sigma_\theta^m,\nabla\chi_{11})-b(\frac{\sigma_\theta^m}{r}\ ,\ \chi_{11})+(F_1,\chi_{11}), \tag{37}$$

$$k_1\sigma_r^m+k_2\sigma_\theta^m+k_3E_r^m,\nabla\chi_{21})=(f_2,\chi_{21}),\ l=\overline{1,m}\ ,\ \frac{dg^1_{mj}}{dt}\Big|_{t=0}=(\sigma_1,\chi_{1j}),\ g^1_{mj}(0)\ =\alpha_j^m,$$

where $\alpha_j^m$ are coefficients of the expansion $\sigma_0^m(r)=\sum_{j=1}^{m}\alpha_j^m\chi_{1j}(r)$, which approximates for $m\to\infty$ the function $\sigma_0(r)$ in norm $\overset{\circ}{W}{}_2^1(G)$;

$$\sigma_\theta^m=\delta_1\frac{\partial}{\partial r}(r\sigma_r^m)+\delta_2\sigma_r^m+\delta_3\frac{\partial}{\partial r}(rE_r^m)+\delta_4E_r^m.$$

One can show, as it was done in the proof of theorem 1, that the system (37) and (38) has an unique solution in the interval $I$ and, in addition, $\frac{dg_{mj}^1}{dt}\in W_2^1(0,T)$, $g_{mj}^2\in L_2(0,T)$. It means that we can define uniquely $\sigma_r^m$ and $\varphi^m$ from this system.

We multiply the first equation of the system (37) by the function $\frac{dg_{mj}^1}{dt}$, the second equation by $g_{mj}^2$, summarize the results with respect to $l$ from 1 to $m$ and take the sum of them. Using the same methods of an *a priori* estimation like in the proof of the theorem 1, we obtain

$$\rho\|r\frac{\partial\sigma_r^m(t)}{\partial t}\|_H^2+\|\frac{\partial}{\partial r}(r\sigma_r^m)(t)\|_H^2+\int_0^t\|E_r^m(t)\|_H^2dt\le M_1+M_2\int_0^t\|\frac{\partial}{\partial r}(r\sigma_\theta^m(t)\|_H^2dt+M_3\int_0^t\|\sigma_\theta^m(t)\|_H^2\,dt. \tag{39}$$

Taking into account the condition (34) and equation (3), we can show that

$$\alpha[\frac{\partial}{\partial r}(r\sigma_\theta)]^2+\beta(\sigma_\theta)^2\le[\gamma_1\frac{\partial}{\partial r}(r\sigma_r)+\gamma_2\sigma_r]^2,\ \alpha,\beta,\gamma_i>0,\ i=1,2.$$

Therefore we obtain from (39) the *a priori* estimation for the Galerkin's approximations $\sigma_r^m$, $\varphi^m$:

$$\|r\frac{\partial\sigma_r^m(t)}{\partial t}\|_H^2+\|\frac{\partial}{\partial r}(r\sigma_r^m)(t)\|_H^2+\int_0^t\|E_r^m(t)\|_H^2dt\le C. \tag{40}$$

From the inequality (40) it follows that the sequences $\{\sigma_r^m(t)\}$, $\{\frac{\partial}{\partial r}(r\sigma_r^m)(t)\}$ and $\{\frac{\partial}{\partial r}\sigma_r^m(t)\}$ are bounded in the spaces $L_2(I,H)\cap L_2(I,\overset{\circ}{W}{}_2^1)$, $L_2(I,H)$ and $L_2(I,H)$, respectively. Following the scheme of the theorem 1 proof, we obtain the next result:

LEMMA. *If $F_1\in W_2^1(I,H)$, $f_2\in L_2(I,W_2^{-1})$, $\sigma_0\in\overset{\circ}{W}{}_2^1(G)$ and $\sigma_1\in L_2(G)$, then there exists the unique generalized solution of the problem (32)-(35) with the properties $\frac{\partial\sigma_r}{\partial t}\in L_2(I,H)$ and $\frac{\partial^2\sigma_r}{\partial t^2}\in L_2(I,W_2^{-1})$.*

To formulate the existence and uniqueness theorem for generalized solution of the problem (1)-(5), which would have the properties (23) and (31), let us find the explicit form of the concordance condition:

$$\sigma|_{t=0}=\sigma_0|_{\partial G}=[c_{11}\frac{\partial u_0}{\partial r}+c_{12}\frac{u_0}{r}-e_{11}E_r(0)]|_{\partial G}=0.$$

For this purpose the value $E_r(0)$ can be found from the relationship (27) after integrating from $R_0$ up to $r$ ($R_0<r\le R_1$), and substituting $E_r(0)$ into the latter relation. In consequence, the concordance condition between the initial boundary conditions and the right-hand side of the equation (2) will take the form

$$[(c_{11}+\frac{e_{11}^2}{\varepsilon_{11}})r\frac{\partial u_0(r)}{\partial r}+(c_{12}+\frac{e_{11}e_{12}}{\varepsilon_{11}})u_0(r)]\Big|_{R_0}^{R_1}=\frac{e_{11}}{\varepsilon 11}\int_{R_0}^{R_1}r'f_2(r',0)dr'. \tag{41}$$

Thus, taking into account the definiens for the electrostatic potential, characteristic equations (3) and the Cauchy relations, we have the following

THEOREM 3. *If the conditions*

$f_1\in W_2^2(I,H)$, $f_2\in L_2(I,W_2^{-1})$, $\frac{\partial f_2}{\partial t}\in W_2^1(I,W_2^{-1})$, $\frac{\partial f_2}{\partial t}|_{t=0}\in W_2^{-1}(G)$; $u_0\in W_2^2(G)$ *and* $u_1\in W_2^1(G)$

*and concordance conditions (41) hold, then there exists the unique solution of the* *problem* (1)-(5), *provided with the properties* (23) and (31).

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