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A new method of solution of the Wetterich equation and its applications

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Abstract

The known approximation schemes for the solution of the Wetterich exact renormalization group (RG) equation are critically reconsidered, and a new truncation scheme is proposed. In particular, the equations of the derivative expansion up to the ∂^2 order for a scalar model are derived in a suitable form, clarifying the role of the off-diagonal terms in the matrix of functional derivatives. The natural domain of validity of the derivative expansion appears to be limited to small values of q/k in the calculation of the critical two-point correlation function, depending on the wave-vector magnitude q and the infrared cut-off scale k. The new approximation scheme has the advantage to be valid for any q/k, and, therefore, it can be auspicious in many current and potential applications of the celebrated Wetterich equation and similar models. Contrary to the derivative expansion, derivatives are not truncated at a finite order in the new scheme. The RG flow equations in the first approximation of this new scheme are derived and approximately solved as an example. It is shown that the derivative expansion up to the ∂^2 order is just the small-q approximation of our new equations at the first order of truncation.

Keywords: functional renormalization, exact renormalization group equations, Wetterich equation, derivative expansion, off-diagonal terms, non-perturbative approaches, quantum and statistical field theories

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1. Introduction

In various studies of critical phenomena [1, 2], the renormalization group (RG) approach is the most extensively used general method. Particularly, the perturbative RG approach is well known [3–6]. However, the perturbative approach suffers from some problems [7]. Moreover, the vicinity of the critical point is not a natural domain of validity of any perturbation theory, non-perturbative approaches are to be preferred. Non-perturbative effects can be very important in some specific cases, as has been recently demonstrated in [8] for the hierarchical Edwards–Anderson model of spin glasses. These effects can be important more generally, even in the 3D Ising model, as argued in [9].

Hence, it is important to look for a non-perturbative RG approach. Historically, non-perturbative RG equations have been developed in parallel to the perturbative ones. These are so called exact RG equations (ERGE), used in functional renormalization. The oldest non-perturbative equation of this kind is the Wegner–Houghton equation [10]. The method of its derivation is close in spirit to Wilson's famous approach, where the basic idea is to integrate out the short-wave fluctuations corresponding to the wave vectors within $\Lambda/s < q < \Lambda$ with the upper cutoff parameter Λ and the renormalization scale $s > 1$. A sharp momentum cut-off is used in Wilson's case. Later, a similar equation with smooth momentum cutoff was proposed by Polchinski [11]. The RG equations of this class are reviewed in [12], and there is also a very recent paper on this topic [13]. The exactness of the Wegner–Houghton equation, however, has been questioned in [14].

Here we focus on the ERGE of another type, proposed by Wetterich [15] and reviewed in [16]. This is called the Wetterich equation, which is discussed in detail in section 2. The underlying method is quite different from Wilson's renormalization, since it deals with the effective average action in the presence of external sources and infrared (lower) cut-off rather than with the integration over short-wave fluctuations. A conventional statement is that a certain link between these two methods exists (recently reviewed, e.g. in [17], a master thesis from the Wetterich lab, see especially point 6 at the end of section 2.1), although the known physical interpretations of this issue seem to be somewhat confusing. In particular, in section 2.1 of [16] one can find the following interpretation: 'the average action Γ_k can be viewed as the effective action for averages of fields over a volume with size k^{-d} and is similar in spirit to the action for block-spins on the sites of a coarse lattice'. Here k is the infrared cut-off parameter, d is the spatial dimensionality, and the 'averages of fields' are denoted by $\phi(\mathbf{x})$ in the Wetterich equation. It would allow to link $\phi(\mathbf{x})$ to the block-averages of the real-space renormalization in the Wilson's approach. However, such an interpretation does not naturally emerge from the non-perturbative derivation of the Wetterich equation (see the related discussion in the introduction part of [14]), where $\phi(\mathbf{x})$ is uniquely determined by external sources $J(\mathbf{x})$. The only correct interpretation within this derivation, obviously, is that $\phi(\mathbf{x})$ is just an ensemble-average bearing a one-to-one correspondence to $J(\mathbf{x})$. The block-average would be performed in addition. Here we rely on the non-perturbative derivation of the Wetterich equation and useful mathematical relations given in [15, 16]. From this perspective, the Wetterich equation could serve as a perfect mathematical tool for a non-perturbative analysis of critical phenomena.

The Wetterich equation has been already used to study the critical phenomena in the Ginzburg–Landau model [18–21], as well as in the QCD model with chiral phase transition [22], applying traditional approximation or truncation schemes, such as the local potential approximation (LPA) and the derivative expansion. It is widely used for quantum systems [23–25]. For example, it has been applied to the quantum gravity problem [24, 25]. Proposed more than 25 years ago [15], the Wetterich equation has continued to generate considerable interest in many areas of science in general and physics in particular. This reflects the fundamental importance of developing efficient tools for the interpolation between microscopic and macroscopic scales. The functional-renormalization-group approach, which is at the heart of such models as the Wetterich equation, provides an appropriate framework for the analysis of a range of central problems in such areas as quantum gravity (e.g. asymptotic safety), Yang–Mills theories, and non-relativistic quantum systems, to name just a few. Recently, focusing on the Wetterich equation, the cases where the LPA equations are well-posed or ill-posed within the usual schemes of functional truncations have been investigated in [26]. Furthermore, quantum gravity and the asymptotic safety of the related field theories have been recently studied, based on Wetterich and Polchinski equations [27–34]. The recent progress in solving the non-perturbative RG for tensorial group field theory can be found in [35].

In the current paper, we have critically reconsidered the derivative expansion, proposing a radically different method instead. Correspondingly, the main result of our paper is represented by equation (33) along with the method of derivation of the RG flow equations for the functions $\theta_k^{(m)}$ in (33). We have derived the RG flow equations in the first approximation of the proposed new truncation scheme, providing also an approximate solution as an example. The natural domain of validity of the derivative expansion appears to be limited to small q/k values in the calculation of the critical two-point correlation function $G_k(q)$. Our new approach allows to extend this region of validity to arbitrary q/k values. This is the main advantage of our new method. Hence, it can be useful not only in calculations of the critical exponents, but also in other possible applications, not necessarily related to critical phenomena (see section 12). We also note that the derivative expansion shows up just as a small-q approximation within our approach, which we show in section 10.

As a secondary issue, we have clarified the role of the off-diagonal terms in the matrix of functional derivatives, contained in the Wetterich equation.

Our paper is organized as follows. In sections 2 and 3, we review the Wetterich equation in relation to the critical phenomena. In section 4, we clarify the conditions at which the matrix of functional derivatives, contained in the Wetterich equation, is exactly diagonal. The known approximation schemes—the LPA and the derivative expansion, are reviewed and reconsidered in sections 5 and 6. Relevant matrix calculations are provided in section 7, which are further used in our new approximation scheme introduced in section 8, deriving the RG flow equations at the first order of truncation within this scheme. In section 9, these equations are reformulated in a more convenient way and transformed in a scaled form, suitable for applications in critical phenomena. The relation of our new equations to the derivative expansion is shown in section 10. In section 11, an approximate solution of the new equations is provided as an example, finding the fixed-point solution and determining the critical exponents η and ν. The advantages of our new method are shown in section 12. Finally, the summary and conclusions are provided in section 13.

2. The Wetterich equation

Models of statistical physics are routinely described by the action S, which is defined as $S = H/T$, where H is the Hamiltonian and T is temperature in energy units. In the Wetterich approach [15], the effective average action $\Gamma_k[\phi]$ is introduced, which is related to the action $S[\chi]$, external sources and infrared cut-off, as defined by equations (2.5)–(2.9) in [16]. It obeys the Wetterich equation

$$\frac{\partial}{\partial k}\Gamma_k[\phi] = \frac{1}{2}\ \mathrm{Tr}\left\{\left[\Gamma_k^{(2)}[\phi] + R_k\right]^{-1}\frac{\partial}{\partial k}R_k\right\}, \tag{1}$$

where $\Gamma_k[\phi]$ depends on the averaged order parameter $\phi(\mathbf{x})$ with components $\phi_j(\mathbf{x}) = \langle \chi_j(\mathbf{x}) \rangle$, $j = 1, \ldots, N$, where N is the dimensionality of the vectors $\phi(\mathbf{x})$ and $\chi(\mathbf{x})$. Here, the averaging of the original order parameter χ is performed in the presence of external field $J(\mathbf{x})$, so that ϕ is determined by J. This functional relation is dependent on k, i.e. $\phi = \phi_k[J]$, and there exists also the inverse relation $J = J_k[\phi]$. The effective action contains a smooth infrared (lower) cut-off of fluctuations with wave vector magnitude $q \lesssim k$, represented by the term R_k in (1). The upper cut-off at $q = \Lambda$ is also included. The Wetterich equation (1) describes the variation of $\Gamma_k[\phi]$ depending on the running cut-off scale parameter k. In the wave-vector space, the quantity $\Gamma_k^{(2)}[\phi]$ is a matrix with elements

$$\left(\Gamma_k^{(2)}\right)_{ij}(\mathbf{q}, \mathbf{q}') = \frac{\delta^2 \Gamma_k[\phi]}{\delta\phi_i(-\mathbf{q})\, \delta\phi_j(\mathbf{q}')}. \tag{2}$$

The cut-off R_k is a diagonal matrix with elements

$$R_{k,ij}(\mathbf{q}, \mathbf{q}') = R_k(q)\, \delta_{ij}\, (2\pi)^d\, \delta(\mathbf{q} - \mathbf{q}'), \tag{3}$$

where $q = |\mathbf{q}|$. We found it very convenient to consider the wave vectors $\mathbf{q}$ as discrete quantities at the beginning at finite volume V, treating the thermodynamic limit $V \to \infty$ afterward, and replacing the sums over $\mathbf{q}$ by integrals where appropriate. In this case R_k is represented as

$$R_{k,ij}(\mathbf{q}, \mathbf{q}') = R_k(q)\, \delta_{ij}\, \delta_{\mathbf{q},\mathbf{q}'}. \tag{4}$$

The trace of matrix 'Tr' (the sum of diagonal elements) in (1) includes the summation over all order-parameter components j and wave vectors $\mathbf{q}$. The infrared cut-off function $R_k(q)$ must satisfy $R_k(q) \to 0$ at $k \to 0$ and $R_k(q) \to \infty$ at $k \to \infty$ for a fixed q. An appropriate choice for application in critical phenomena is [16, 20, 21]

$$R_k(q) = \frac{\alpha\, Z_k q^2}{\mathrm{e}^{q^2/k^2} - 1}, \tag{5}$$

where Z_k is a renormalization constant, which scales as $Z_k \sim k^{-\eta}$ at $k \to 0$ (at the critical temperature $T = T_\mathrm{c}$) with the critical exponent η. The parameter α is used for optimization [20, 21].

Derivatives of $\Gamma_k[\phi]$ are related to correlation functions. It allows us to extract the physics from this approach. In particular,

$$\left(\left[\Gamma_k^{(2)}[\phi] + R_k\right]^{-1}\right)_{ij}(\mathbf{q}, \mathbf{q}') = G_{ij}(\mathbf{q}, \mathbf{q}') = \langle \chi_i(-\mathbf{q}) \chi_j(\mathbf{q}') \rangle \tag{6}$$

holds, where $G_{ij}(\mathbf{q}, \mathbf{q}')$ is the two-point correlation function (the average propagator) in the wave-vector space for the original order parameter χ.

In the applications to quantum systems [23, 24], the Wetterich equation has been modified as compared to its original version in [15, 16], in such a way that ϕ is replaced by $\bar{\phi} + \phi$, where $\bar{\phi}$ is a background term and ϕ describes fluctuations around this background. Moreover, the second derivative $\Gamma_k^{(2)}[\bar{\phi} + \phi]$ is taken with respect to ϕ. The independence on the choice of $\bar{\phi}$ has been demonstrated in [24]. We will focus on the application of the Wetterich equation to the classical (not quantum) Ginzburg–Landau model. In this case, the background $\bar{\phi}$ can be identified with the spontaneous magnetization. Currently, we will consider only the case above the critical temperature, where $\bar{\phi} = 0$ is a natural choice. Thus, we use the original form of the Wetterich equation (1).

Equation (1) contains the inverse matrix $\left(\Gamma_k^{(2)}[\phi] + R_k\right)^{-1}$ on the right-hand side. The inversion of the $\Gamma_k^{(2)}[\phi] + R_k$ matrix for a quantum problem has been considered in [24], where this is a block-matrix. For the Ginzburg–Landau model at finite volume V, this matrix contains nonzero off-diagonal terms in general, making its inversion nontrivial. Nevertheless, in the thermodynamic limit $V \to \infty$, it is often represented as diagonal in the $\mathbf{q}$-variable, setting the matrix elements (6) with $\mathbf{q} \neq \mathbf{q}'$ equal to zero—see equation (29) in [15]. This is a great simplification. Moreover, it leads to just a diagonal matrix in the one-component case of $N = 1$. Unfortunately, such a representation appears to be somewhat confusing, as it turns out that the off-diagonal terms are relevant, and they have been taken into account in the standard scheme of the derivative expansion [18], further discussed in section 6.1.

3. The choice of initial effective action $\Gamma_{k_0}[\phi]$ and universality

It is well known that the critical behavior is universal for a wide class (i.e. universality class) of actions S with a given symmetry. In particular, the action S can be relatively simple, like that of the $O(N)$ or φ^4 (or χ^4) model:

$$S[\chi] = \int \left(r_0 \chi^2(\mathbf{x}) + c(\nabla \chi(\mathbf{x}))^2 + u\left(\chi^2(\mathbf{x})\right)^2 \right) \mathrm{d}\mathbf{x}. \tag{7}$$

We assume periodic boundary conditions, in which case the Fourier representation is

$$S[\chi] = \sum_{l,\mathbf{q}} \left(r_0 + c\,\mathbf{q}^2\right) |\chi_l(\mathbf{q})|^2 + uV^{-1} \sum_{l,j,\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3} \chi_l(\mathbf{q}_1)\chi_l(\mathbf{q}_2)\chi_j(\mathbf{q}_3)\chi_j(-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3), \tag{8}$$

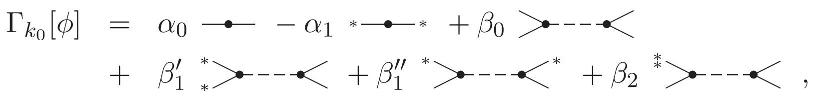
where $\chi(\mathbf{q}) = V^{-1/2} \int \chi(\mathbf{x}) \exp(-\mathrm{i}\mathbf{q}\mathbf{x}) \mathrm{d}\mathbf{x}$ and $\chi(\mathbf{x}) = V^{-1/2} \sum_{q<\Lambda} \chi(\mathbf{q}) \exp(\mathrm{i}\mathbf{q}\mathbf{x})$. Moreover, the only allowed configurations of $\chi(\mathbf{x})$ are those for which $\chi(\mathbf{q}) = 0$ holds at $q > \Lambda$. For simplicity of notations, in this paper (like also in [15, 16]), the same symbol (χ or ϕ) is used for the local order parameter in coordinate space and for its Fourier transform, since these two can be distinguished by means of their arguments.

According to [16], $\Gamma_\Lambda \approx S$ is expected. Hence, it should be enough to take a simple form like (7) as the initial condition for $\Gamma_k[\phi]$ at $k = \Lambda$ or at $k = k_0 \lesssim \Lambda$ to get the correct universal critical exponents via integration of (1) up to a small enough value of k. It is fully consistent with the discussion in [16] (see text below equation (2.12) in [16]).

As a relevant example, we consider the form

$$\begin{aligned}\Gamma_{k_0}[\phi] = \int \Big(&\alpha_0 \phi^2(\mathbf{x}) + \alpha_1 (\nabla\phi(\mathbf{x}))^2 + \beta_0 \left(\phi^2(\mathbf{x})\right)^2 \\ &- \beta_1' (\nabla\phi(\mathbf{x}))^2 \phi^2(\mathbf{x}) - \beta_1''(\phi(\mathbf{x})\nabla\phi(\mathbf{x}))^2 - \beta_2(\phi(\mathbf{x})\Delta\phi(\mathbf{x}))\phi^2(\mathbf{x}) \Big) \mathrm{d}\mathbf{x}. \end{aligned} \tag{9}$$

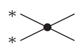
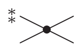
It is purposeful to represent (9) in wave-vector space by Feynman diagrams as



(10)

noting that $\phi_j(\mathbf{x}) = V^{-1/2} \sum_{\mathbf{q}} \phi_j(\mathbf{q}) \mathrm{e}^{\mathrm{i}\mathbf{q}\mathbf{x}}$ and $\phi_j(\mathbf{q}) = V^{-1/2} \int \phi_j(\mathbf{x}) \mathrm{e}^{-\mathrm{i}\mathbf{q}\mathbf{x}} \mathrm{d}\mathbf{x}$ hold, where $\phi_j(\mathbf{x})$ and $\phi_j(\mathbf{q})$ are components of the vectors $\phi(\mathbf{x})$ and $\phi(\mathbf{q})$, respectively. We have introduced here some specific notations: if a line is marked by an asterisk, it means that the wave vector corresponding to this line shows up as a factor. Two asterisks mean that the square of this wave vector is included as a factor. Besides, a vertex with $2n$ lines contains a factor V^{1-n} and, as usual for Feynman diagrams, summation over all possible wave vectors, associated with the lines and factors $\phi_j(\mathbf{q})$, is performed. Moreover, there is a constraint that the sum of wave vectors for lines entering a node is zero. The dashed line is used to separate two sub-vertices of a fourth-order vertex, where factors $\phi_j(\mathbf{q}_1)$ and $\phi_j(\mathbf{q}_2)$ are attributed to the solid lines of one sub-vertex, performing summation over the values of j.

In a particular case of scalar order parameter ϕ, i.e. at $N = 1$, the diagrammatic representation (10) can be simplified as

$\Gamma_{k0}[\phi] = \alpha_0$ $- \alpha_1$ $+ \beta_0$  $+ \beta_1$  $+ \beta_2$  , (11)

where $\beta_1 = \beta_1' + \beta_1''$.

4. The conditions of diagonality of the matrix $\Gamma_k^{(2)}[\phi]$

In this section, we clarify the cases where the matrix $\Gamma_k^{(2)}[\phi]$ is exactly diagonal. Here we call a term 'diagonal' only if it is diagonal with respect to both wave vectors $\mathbf{q}$ and indexes j. All other terms are off-diagonal terms of some kind.

The elements of the inverse matrix $\left[\Gamma_k^{(2)}[\phi] + R_k\right]^{-1}$ are given by the correlation function $G_{ij}(\mathbf{q}, \mathbf{q}') = \langle \chi_i(-\mathbf{q})\chi_j(\mathbf{q}')\rangle$ in accordance with (6). Using the Fourier representation $\chi_j(\mathbf{q}) = V^{-1/2}\int \chi_j(\mathbf{x})\exp(-\mathrm{i}\mathbf{qx})\mathrm{d}\mathbf{x}$, we obtain

$$G_{jl}(\mathbf{q}, \mathbf{q}') = V^{-1}\int \left[\int \langle \chi_j(\mathbf{x})\chi_l(\mathbf{x}+\mathbf{y})\rangle \mathrm{e}^{-\mathrm{i}\mathbf{q}'\mathbf{y}}\mathrm{d}\mathbf{y}\right] \mathrm{e}^{-\mathrm{i}\mathbf{px}}\mathrm{d}\mathbf{x}, \tag{12}$$

where $\mathbf{p} = \mathbf{q}' - \mathbf{q}$. The expression in square brackets is the Fourier transform $\mathcal{G}_{\mathbf{x}}^{jl}(\mathbf{q}')$ of the two-point correlation function $\langle \chi_j(\mathbf{x})\chi_l(\mathbf{x}+\mathbf{y})\rangle$ over the variable $\mathbf{y}$. It depends on the wave vector $\mathbf{q}'$, as well as on the coordinate $\mathbf{x}$. For a homogeneous external field and periodic boundary conditions, it does not depend on $\mathbf{x}$ and, therefore, the integral over $\mathbf{x}$ in (12) vanishes at $\mathbf{p} \neq \mathbf{0}$. In the case of $N = 1$, the condition $\mathbf{p} \neq \mathbf{0}$ holds for all the off-diagonal terms. It means that for $N = 1$, the matrix G is exactly diagonal at a homogeneous external field. For $N > 1$, there are also off-diagonal terms with $\mathbf{p} = \mathbf{0}$ and $j \neq l$. At a homogeneous field, oriented along one of the N axes, $\langle \chi_j(\mathbf{x})\chi_l(\mathbf{x}+\mathbf{y})\rangle$ is zero for $j \neq l$ because of the $\chi_l \to -\chi_l$ symmetry for the transverse directions. Therefore, the corresponding terms of the propagator matrix G vanish for such a homogeneous field, and G is exactly diagonal in this case.

Summarizing the analysis of (12), the matrix G and, therefore, also its inverse matrix $\Gamma_k^{(2)}[\phi] + R_k$ and $\Gamma_k^{(2)}[\phi]$ are exactly diagonal matrices at a homogeneous external field for $N = 1$. It is true also for $N > 1$, if the field is oriented along one of the N axes.

5. The local potential approximation

Here we consider the known LPA [16]. The LPA is obtained by evaluating the Wetterich equation (1) at a homogeneous external field, which is oriented along one of the N axes in the general N-component case, assuming a certain form of the effective average action, i.e.

$$\Gamma_k[\phi] = \int \left[U_k(\rho(\mathbf{x})) + \alpha(\nabla\phi(\mathbf{x}))^2\right] \mathrm{d}\mathbf{x}, \tag{13}$$

where $\rho(\mathbf{x}) = \phi^2(\mathbf{x})/2$. A homogeneous external field implies a homogeneous ϕ configuration $\phi_j^2(\mathbf{x}) = 2\delta_{j,1}\,\rho$. By considering only such configurations, we certainly lose some information. In particular, we cannot prove by such a treatment that the solution of (1) has the form (13). In fact, it is not proven in this way, but is justified by general physical arguments—$O(N)$ symmetry and local interaction [16].

The matrix $\Gamma_k^{(2)}[\phi]$ is exactly diagonal at $\phi_j^2(\mathbf{x}) = 2\delta_{j,1}\,\rho$—see section 4. The calculation of the diagonal elements is straightforward, using periodic boundary conditions and the Fourier representation $\rho(\mathbf{x}) = \frac{1}{2}V^{-1}\sum_{j,\mathbf{q}_1,\mathbf{q}_2}\phi_j(\mathbf{q}_1)\phi_j(\mathbf{q}_2)\mathrm{e}^{\mathrm{i}(\mathbf{q}_1+\mathbf{q}_2)\mathbf{x}}$ and $\int(\nabla\phi(\mathbf{x}))^2\mathrm{d}\mathbf{x} = \sum_{j,\mathbf{q}}\mathbf{q}^2\phi_j(\mathbf{q})\phi_j(-\mathbf{q})$, where $\phi_j(\mathbf{q}) = V^{-1/2}\int\phi_j(\mathbf{x})\mathrm{e}^{-\mathrm{i}\mathbf{q}\mathbf{x}}\mathrm{d}\mathbf{x}$ (see section 3). It yields

$$\left.\frac{\delta^2\int U_k(\rho(\mathbf{x}))\mathrm{d}\mathbf{x}}{\delta\phi_j(-\mathbf{q})\delta\phi_j(\mathbf{q})}\right|_{\phi_j^2(\mathbf{x})=2\delta_{j,1}\,\rho} = U_k'(\rho) + 2\rho\,U_k''(\rho)\,\delta_{j,1}, \tag{14}$$

where primes denote the derivatives with respect to ρ. This is the contribution to $\left(\Gamma_k^{(2)}\right)_{jj}(\mathbf{q},\mathbf{q})$ provided by the $\int U_k(\rho(\mathbf{x}))\mathrm{d}\mathbf{x}$ term. The remaining contribution is just $2\alpha q^2$. Note that the gradient term in (13) cancels at a homogeneous ϕ configuration, therefore such a treatment gives no information about its renormalization. It is assumed to be unrenormalized in the LPA. In the thermodynamic limit $V\to\infty$, the sum over $\mathbf{q}$ in (1) is replaced by the integral according to the well known rule $\sum_{\mathbf{q}}\to V(2\pi)^{-d}\int\mathrm{d}^dq$. In summary, we obtain the following LPA equation for U_k:

$$\frac{\partial U_k}{\partial k} = \frac{K_d}{2}\int_0^\Lambda\frac{\partial R_k}{\partial k}\left(\frac{1}{U_k'(\rho)+2\rho\,U_k''(\rho)+2\alpha q^2+R_k(q)} + \frac{N-1}{U_k'(\rho)+2\alpha q^2+R_k(q)}\right)q^{d-1}\mathrm{d}q, \tag{15}$$

where $K_d = S(d)/(2\pi)^d$, $S(d)$ being the surface of unit sphere in d dimensions. This equation is a particular case of equation (31) in [15]. We can generalize (13), including other gradient terms representable by vertices with asterisks, like in (10), but including all the even-order vertices with even number of asterisks. It leads to a generalized equation, where $2\alpha q^2$ in (15) is replaced by $Z_k^{\parallel}\left(\rho,q^2\right)q^2$ in the first fraction and by $Z_k^{\perp}\left(\rho,q^2\right)q^2$ in the second fraction. Besides, we have $Z_k^{\parallel}\left(0,q^2\right) = Z_k^{\perp}\left(0,q^2\right)$, since only the second-order vertices contribute to these terms, providing this equality. Such an equation has the same form as (31) with insertions defined by (32) in [15].

6. The derivative expansion

6.1. The RG flow equations at $N = 1$

In this subsection, we review and reconsider the known method of the derivative expansion for $N = 1$, deriving the RG flow equations at the ∂^2 order in a form, which is particularly suitable for comparison with our new method further introduced in section 8. Moreover, we provide important details of derivation of the RG flow equations, which are not easy to find in the published literature, as well as clarify the role of the off-diagonal terms in $\Gamma_k^{(2)}$.

In the derivative expansion up to order ∂^4, the effective action is represented as [21]

$$\Gamma_k[\phi] = \int\mathrm{d}^dx\left\{U_k(\rho) + \frac{1}{2}\left[Z_k(\rho)(\nabla\phi)^2 + w_k^a(\rho)(\Delta\phi)^2 + w_k^b(\rho)(\nabla\phi)^2(\phi\Delta\phi)^2 + w_k^c(\rho)\left((\nabla\phi)^2\right)^2\right]\right\}. \tag{16}$$

The RG flow equation for U_k is obtained directly from the Wetterich equation, evaluating it at a homogeneous ϕ configuration $\phi(\mathbf{x}) = \sqrt{2\rho}$. Performing similar calculations as in the case of LPA (section 5), one obtains

$$\frac{\partial U_k}{\partial k} = \frac{K_d}{2} \int_0^{\Lambda} \frac{\partial R_k}{\partial k} \left(\frac{1}{U_k'(\rho) + 2\rho\, U_k''(\rho) + Z_k(\rho) q^2 + R_k(q)} \right) q^{d-1} \mathrm{d}q \tag{17}$$

at the ∂^2 order, where only the terms with $U_k(\rho)$ and $Z_k(\rho)$ are included in (16). Here, equation (14) at $N = 1$ (where $j = 1$) is used. The term $Z_k(\rho)q^2$ is easily obtained, using the Fourier representation of $(\nabla\phi)^2$ and noting that terms with Z_k' and Z_k'' in $\Gamma_k^{(2)}(\mathbf{q}, \mathbf{q})$ vanish at $\phi(\mathbf{x}) = \sqrt{2\rho}$, i.e. at $\phi(\mathbf{q}) = V^{1/2}\sqrt{2\rho}\,\delta_{\mathbf{q},\mathbf{0}}$.

Following the method of [21], the derivative terms in (16) are linked to a specific wave-vector dependence of the effective action, e.g.

$$Z_k(\rho) = \lim_{p \to 0} \partial_{\mathbf{p}^2} \left. \frac{\delta^2 \Gamma_k}{\delta\phi(\mathbf{p})\delta\phi(-\mathbf{p})} \right|_{\phi(\mathbf{x})=\sqrt{2\rho}}. \tag{18}$$

Similar relations for $Z_k(\rho)$, $w_k^a(\rho)$, $w_k^b(\rho)$ and $w_k^c(\rho)$ are given in [21]. The validity of (18) can be easily verified, representing the gradient term $\int Z_k(\rho)(\nabla\phi)^2 \mathrm{d}^d x$, where $Z_k(\rho) = a_0 + a_2\phi^2 + a_4(\phi^2)^2 + \ldots$, by the corresponding vertices of Feynman diagrams. Note that the operator $\phi\Delta\phi$ can be used instead of $(\nabla\phi)^2$, obtaining an equivalent representation of Γ_k, which is reducible to (16) using integration by parts. The wave-vector dependence of Γ_k is independent of the representation. Therefore, (18) gives a uniquely defined $Z_k(\rho)$, which always corresponds to the coefficient at $(\nabla\phi)^2/2$ in (16).

Furthermore, the obvious way to obtain the RG flow equation for $Z_k(\rho)$ (and similarly for $w_k^a(\rho)$, $w_k^b(\rho)$, $w_k^c(\rho)$) in this approach is to perform the derivative with respect to k in (18) and then insert here $\partial\Gamma_k[\phi]/\partial k$ from the Wetterich equation (1). It yields

$$\frac{\partial Z_k(\rho)}{\partial k} = \lim_{p \to 0} \partial_{\mathbf{p}^2} \frac{\delta^2}{\delta\phi(\mathbf{p})\delta\phi(-\mathbf{p})} \left[\frac{1}{2} \ \mathrm{Tr} \left\{ \left[\Gamma_k^{(2)}[\phi] + R_k \right]^{-1} \frac{\partial}{\partial k} R_k \right\} \right], \tag{19}$$

evaluated at $\phi(\mathbf{x}) = \sqrt{2\rho}$. This equation is similar to (6.3) in [18] with the only difference that the latter one refers to the $N \geqslant 2$ case. Equation (6.4) in [18] contains a prescription how such an expression can be calculated, relating it to 3-point and 4-point functions.

To reveal important details of such calculations, we first consider the expansion

$$(A + B)^{-1} = A^{-1} - A^{-1}BA^{-1} + A^{-1}BA^{-1}BA^{-1} - A^{-1}BA^{-1}BA^{-1}BA^{-1} + \cdots, \tag{20}$$

where A and B are matrices and B is a small perturbation. It is a formally correct expansion of the inverse matrix $(A + B)^{-1}$, since the unity matrix is obtained order by order when this expansion is multiplied by $A + B$. In fact, the functional derivative in (19) can be calculated non-perturbatively based on the derivative formula

$$\frac{\partial}{\partial\mu} \left(A^{-1} \right) = \lim_{\mathrm{d}\mu \to 0} \frac{\left(A + \frac{\partial A}{\partial\mu}\mathrm{d}\mu \right)^{-1} - A^{-1}}{\mathrm{d}\mu} = -A^{-1} \frac{\partial A}{\partial\mu} A^{-1}, \tag{21}$$

where μ is a parameter on which the matrix A depends. This formula is a consequence of (20) at an infinitesimal perturbation $B = \frac{\partial A}{\partial\mu}\mathrm{d}\mu$ for $\mathrm{d}\mu \to 0$.

In our calculation, equation (21) is used in (19) with $\mu = \phi(-\mathbf{p})$ and $\mu = \phi(\mathbf{p})$. It yields

$$\frac{\partial Z_k(\rho)}{\partial k} = \frac{1}{2}\lim_{p\to 0}\partial_{\mathbf{p}^2}\sum_{\mathbf{q}}\left\{\left(-\frac{\tilde{\Gamma}_k^{(4)}(-\mathbf{p},\mathbf{p},-\mathbf{q},\mathbf{q})}{\left(\Gamma_k^{(2)}(\mathbf{q},\mathbf{q})+R_k(q)\right)^2}\right.\right.$$
$$\left.\left.+\frac{2\,\tilde{\Gamma}_k^{(3)}(-\mathbf{p},-\mathbf{q},\mathbf{q}+\mathbf{p})\,\tilde{\Gamma}_k^{(3)}(\mathbf{p},-\mathbf{p}-\mathbf{q},\mathbf{q})\,\theta(\Lambda-|\mathbf{q}+\mathbf{p}|)}{\left(\Gamma_k^{(2)}(\mathbf{q},\mathbf{q})+R_k(q)\right)^2\left(\Gamma_k^{(2)}(\mathbf{q}+\mathbf{p},\mathbf{q}+\mathbf{p})+R_k(|\mathbf{q}+\mathbf{p}|)\right)}\right)\frac{\partial}{\partial k}R_k(q)\right\}\Bigg|_{\phi(\mathbf{x})=\sqrt{2\rho}}, \quad (22)$$

where

$$\tilde{\Gamma}_k^{(n)}(\mathbf{q}_1,\mathbf{q}_2,\ldots,\mathbf{q}_n) = \frac{\delta^n\Gamma_k[\phi]}{\delta\phi(\mathbf{q}_1)\delta\phi(\mathbf{q}_2)\cdots\delta\phi(\mathbf{q}_n)} \quad (23)$$

is the n-point function, noting that $\Gamma_k^{(2)}(\mathbf{q}_1,\mathbf{q}_2) = \tilde{\Gamma}_k^{(2)}(-\mathbf{q}_1,\mathbf{q}_2)$. The details of derivation of (22) are given in the appendix A. This equation coincides with the expression in the lines 2–5 of (6.4) in [18], noting that the arguments of the n-point functions can be exchanged. In distinction from (6.4) in [18], our equation is derived for $N = 1$, but not for $N \geqslant 2$. A minor difference is that the cutoff, represented here by the θ-function, has not been included explicitly in [18]. The first fraction in the sum of (22) comes from the functional derivatives of the diagonal elements of $\Gamma_k^{(2)}$, whereas the second fraction emerges from those of the off-diagonal elements in this matrix. Thus, the off-diagonal terms have been taken into account both here and in [18], despite a seemingly confusing fact that only the diagonal elements of $\Gamma_k^{(2)}$ appear in the first line of equation (6.4) in [18].

At the ∂^2 order, the terms with $w_k^a(\rho)$, $w_k^b(\rho)$ and $w_k^c(\rho)$ are omitted in (16). Using (22) with such a truncated expression for $\Gamma_k[\phi]$, we obtain the equation

$$\frac{\partial Z_k(\rho)}{\partial k} = -\frac{Z_k'(\rho)+2\rho Z_k''(\rho)}{2\,(2\pi)^d}\int_{q<\Lambda}\frac{1}{\mathcal{A}_k^2(\rho,q)}\frac{\partial}{\partial k}R_k(q)d^dq + \frac{2\rho}{(2\pi)^d}\lim_{p\to 0}\partial_{\mathbf{p}^2}$$
$$\times\int_{q<\Lambda}\frac{\left(W_k'(\rho)+\frac{1}{2}Z_k'(\rho)\left[p^2+q^2+|\mathbf{q}+\mathbf{p}|^2\right]\right)^2\theta(\Lambda-|\mathbf{q}+\mathbf{p}|)}{\mathcal{A}_k^2(\rho,q)\,\mathcal{A}_k(\rho,|\mathbf{q}+\mathbf{p}|)}\frac{\partial}{\partial k}R_k(q)\mathbf{d}^dq, \quad (24)$$

where $W_k(\rho) = U_k'(\rho) + 2\rho U_k''(\rho)$ and $\mathcal{A}_k(\rho,q) = W_k(\rho) + Z_k(\rho)q^2 + R_k(q)$. The derivation of this equation is also given in the appendix A. The terms in the second line of (24) represent the contribution of the off-diagonal elements of $\Gamma_k^{(2)}[\phi]$. This contribution is given in a raw form, so that one would need to apply the operator $\lim_{p\to 0}\partial_{\mathbf{p}^2}$ for specific calculations. This, however, is a straightforward procedure of extraction of the coefficient at p^2 for $p\to 0$. The actual compact form (24) is particularly suitable for the comparison with our new equations, introduced in section 8. The expected consistency in the small wave-vector approximation is further established in section 10. This agreement of equations, obtained by completely different methods, cannot be accidental. Thus, it is clear that (24) is the correct equation for $Z_k(\rho)$ at the ∂^2 order.

6.2. The convergence and the natural domain of validity

Consider now the general equation for the potential (at $N = 1$)

$$\frac{\partial U_k}{\partial k} = \frac{K_d}{2}\int_0^{\Lambda}\frac{\partial R_k}{\partial k}\frac{q^{d-1}\mathbf{d}q}{U_k'(\rho)+2\rho U_k''(\rho)+Y_k(\rho,q^2)+R_k(q)} \quad (25)$$

consistent with the consideration given below equation (15). Here $Y_k(\rho,q^2)$ is the q-dependent part of $\Gamma_k^{(2)}(\mathbf{q},\mathbf{q})$. In the derivative expansion, $Y_k(\rho,q^2)$ is expanded in powers of q^2 around $q=0$ [21]. A standard argument to justify such an expansion is the following: the critical singularity builds up gradually as $k\to 0$, and the function $Y_k(\rho,q^2)$ is smooth around $q\approx k$, allowing also the expansion around $q=0$ [16, 21].

It is commonly accepted to write the RG flow equations in a scaled form, as proposed, e.g. in [20]. Following this idea, we use the transformations

$$U_k(\rho)=k^d u_k(\tilde\rho), \quad \text{where} \quad \tilde\rho = Z_k k^{2-d}\rho, \tag{26}$$

$$R_k(q)=Z_k q^2 r(y), \quad \text{where} \quad y=q^2/k^2, \tag{27}$$

$$Y_k\left(\rho,q^2\right)=Z_k q^2 f_k(\tilde\rho,y) \tag{28}$$

to rewrite equation (25) in a dimensionless form

$$\begin{aligned}\frac{\partial u_k(\tilde\rho)}{\partial t} &= -d\,u_k(\tilde\rho)+(d-2+\eta(k))\tilde\rho\,u_k'(\tilde\rho)\\ &\quad -\frac{K_d}{4}\int_0^{\Lambda^2/k^2}\frac{y^{\frac{d}{2}-1}\left(2y^2r'(y)+\eta y\,r(y)\right)\,\mathrm{d}y}{u_k'(\tilde\rho)+2\tilde\rho u_k''(\tilde\rho)+y[f_k(\tilde\rho,y)+r(y)]},\end{aligned} \tag{29}$$

where $t=\ln(k/\Lambda)$, $r'(y)=\mathrm{d}r/\mathrm{d}y$ and $\eta(k)=-\frac{\mathrm{d}}{\mathrm{d}t}\ \ln\ Z_k$.

In the derivative expansion, the function $f_k(\tilde\rho,y)$ is expanded in powers of y. According to (28), it means that $Y_k(\rho,q^2)/q^2$ is expanded in powers of q^2/k^2. In the derivative expansion up to the ∂^2 order, we have $f_k(\tilde\rho,y)=z_k(\tilde\rho)$ and $Y_k(\rho,q^2)=Z_k(\rho)q^2$, where $Z_k(\rho)=Z_k\,z_k(\tilde\rho)$ holds, in accordance with (17). Hence, the expansion in powers of $y=q^2/k^2$ really shows up in approximations beyond the ∂^2 order. The natural domain of validity of such an expansion is $y\to 0$. The opposite limit of large y, obviously, is not its natural domain of validity. Therefore, the convergence of the derivative expansion at large q/k values is dubious. In any case, it is clear that the derivative expansion can only be either asymptotically slowly convergent (as explained below) or divergent at large $y=q^2/k^2$ values, as any expansion in positive integer powers of y at large y.

If a series $\mathcal{S}=\sum_{n=1}^{\infty}a_n y^n$ converges at large y, we call such convergence 'asymptotically slow'. It is because the given accuracy, i.e. $\left|\mathcal{S}-\sum_{n=1}^{m}a_n y^n\right|<\varepsilon$ for $m\geqslant M$ at a small positive ε, can be reached only at $M\to\infty$ for $y\to\infty$. Obviously, $|\sum_{n=1}^{M}a_n y^n|\to\infty$ holds at $y\to\infty$ for any fixed $M\geqslant n_0$, where a_{n_0} is the first non-vanishing coefficient in the series. Therefore, at large y, the series $\sum_{n=1}^{\infty}a_n y^n$ can only be either divergent or asymptotically slowly convergent in the sense explained above.

The problem of asymptotically slow convergence (in the best case) or divergence (in the worst case) at large q/k values is overcome, going beyond the derivative expansion within the new method proposed in section 8. The advantages of this new method are further discussed in section 12.

7. Matrix calculations for some special ϕ configurations

Here we consider the $\Gamma_k^{(2)}[\phi]$ matrix for some special ϕ configurations at $N=1$, which are relevant for our further analysis and derivation of the new equations in section 8. The Wetterich equation can be solved by calculating the diagonal elements $\left(\Gamma_k^{(2)}[\phi]+R_k\right)^{-1}(\mathbf{q},\mathbf{q})$ of the inverse matrix $\left(\Gamma_k^{(2)}[\phi]+R_k\right)^{-1}$. It is a very difficult problem in general. However, this calculation can be done in a case, where ϕ contains only a finite number of modes, in particular, for $\phi(\mathbf{x})=\phi_0\,(1+h\ \cos(\bar{\mathbf{q}}\mathbf{x}))$ at small h. In this case there are modes with $\mathbf{q}=\mathbf{0}$ and $\mathbf{q}=\pm\bar{\mathbf{q}}$ with amplitudes $V^{1/2}\phi_0$ and $\frac{1}{2}V^{1/2}h\phi_0$, respectively, according to $\phi(\mathbf{x})=V^{-1/2}\sum_{\mathbf{q}}\phi(\mathbf{q})\exp(\mathrm{i}\mathbf{q}\mathbf{x})$. For $\Gamma_k[\phi]$ in the form of (13) or in a refined form, used in the first approximation of section 8, elements of the matrix $\Gamma_k^{(2)}[\phi]+R_k$ at $\phi(\mathbf{x})=\phi_0\,(1+h\ \cos(\bar{\mathbf{q}}\mathbf{x}))$ can be written as (see appendix B)

$$\left(\Gamma_k^{(2)}+R_k\right)(\mathbf{q},\mathbf{q}')=\left(\mathcal{A}_k(\rho_0,\mathbf{q})+h^2\mathcal{B}_k(\rho_0,\bar{\mathbf{q}},\mathbf{q})\right)\delta_{\mathbf{q}',\mathbf{q}}+a_k(\rho_0,\bar{\mathbf{q}};\mathbf{q},\mathbf{q}')\,h\left(\delta_{\mathbf{q}',\mathbf{q}+\bar{\mathbf{q}}}+\delta_{\mathbf{q}',\mathbf{q}-\bar{\mathbf{q}}}\right)$$
$$+\ b_k(\rho_0,\bar{\mathbf{q}};\mathbf{q},\mathbf{q}')\,h^2\left(\delta_{\mathbf{q}',\mathbf{q}+2\bar{\mathbf{q}}}+\delta_{\mathbf{q}',\mathbf{q}-2\bar{\mathbf{q}}}\right)+\mathcal{O}\left(h^3\right), \tag{30}$$

where $\rho_0=\phi_0^2/2$. The quantities $\mathcal{A}_k$ and $\mathcal{B}_k$ will be given in equations (37) and (39). The resulting matrix is real and symmetric. It can be easily generalized to the case of $\phi(\mathbf{x})=\phi_0\left(1+\frac{h}{2}\ \exp(\mathrm{i}\bar{\mathbf{q}}\mathbf{x})+\frac{h^*}{2}\ \exp(-\mathrm{i}\bar{\mathbf{q}}\mathbf{x})\right)$ with real $\phi(\mathbf{x})$ and complex Fourier amplitudes. The matrix is Hermitian in this case. It has the form

$$\left(\Gamma_k^{(2)}+R_k\right)(\mathbf{q},\mathbf{q}')=\left(\mathcal{A}_k(\rho_0,\mathbf{q})+|h|^2\mathcal{B}_k(\rho_0,\bar{\mathbf{q}},\mathbf{q})\right)\delta_{\mathbf{q}',\mathbf{q}}$$
$$+\,a_k(\rho_0,\bar{\mathbf{q}};\mathbf{q},\mathbf{q}')\left(h^*\delta_{\mathbf{q}',\mathbf{q}+\bar{\mathbf{q}}}+h\delta_{\mathbf{q}',\mathbf{q}-\bar{\mathbf{q}}}\right)$$
$$+\,b_k(\bar{\rho},\bar{\mathbf{q}};\mathbf{q},\mathbf{q}')\left(h^{*2}\delta_{\mathbf{q}',\mathbf{q}+2\bar{\mathbf{q}}}+h^2\delta_{\mathbf{q}',\mathbf{q}-2\bar{\mathbf{q}}}\right)+\mathcal{O}\left(|h|^3\right). \tag{31}$$

Our aim is to evaluate the diagonal elements of the inverse matrix, including terms up to $\mathcal{O}\left(|h|^2\right)$ order at $|h|\to 0$. The result is given by

$$\left(\Gamma_k^{(2)}+R_k\right)^{-1}(\mathbf{q},\mathbf{q})=\frac{1}{\left(\Gamma_k^{(2)}+R_k\right)(\mathbf{q},\mathbf{q})}$$
$$\times\left(1+\sum_{\mathbf{q}'=\mathbf{q}\pm\bar{\mathbf{q}}}\frac{\theta(\Lambda-|\mathbf{q}'|)\left|\left(\Gamma_k^{(2)}\right)(\mathbf{q},\mathbf{q}')\right|^2}{\left(\Gamma_k^{(2)}+R_k\right)(\mathbf{q}',\mathbf{q}')\times\left(\Gamma_k^{(2)}+R_k\right)(\mathbf{q},\mathbf{q})}\right)$$
$$+\ \mathcal{O}\left(|h|^4\right),\qquad |\mathbf{q}|<\Lambda, \tag{32}$$

which is true both for (30) and (31) at $\bar{\mathbf{q}}\neq\mathbf{0}$. The formal derivation of (32), using matrix algebra, is given in the appendix C. The sum over $\mathbf{q}'$ in (32) represents the contribution of the off-diagonal terms, which is a small correction of order $\mathcal{O}\left(|h|^2\right)$ at $h\to 0$.

8. New approximation scheme: beyond the derivative expansion

Next, we will consider the effective action of the $O(N)$ symmetric model at $N=1$ in the following general form

$$\Gamma_k[\phi]=\int\Bigg(U_k(\rho(\mathbf{x}))+V^{-1}\sum_{\mathbf{q}_1,\mathbf{q}_2}\left[\theta_k^{(1)}\left(\rho(\mathbf{x});\mathbf{q}_1\right)+\theta_k^{(2)}\left(\rho(\mathbf{x});\mathbf{q}_1,\mathbf{q}_2\right)\right]\phi(\mathbf{q}_1)\phi(\mathbf{q}_2)\mathrm{e}^{\mathrm{i}(\mathbf{q}_1+\mathbf{q}_2)\mathbf{x}}$$
$$+\,V^{-2}\sum_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3,\mathbf{q}_4}\left[\theta_k^{(3)}\left(\rho(\mathbf{x});\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3\right)+\theta_k^{(4)}\left(\rho(\mathbf{x});\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3,\mathbf{q}_4\right)\right]$$
$$\times\,\phi(\mathbf{q}_1)\phi(\mathbf{q}_2)\phi(\mathbf{q}_3)\phi(\mathbf{q}_4)\mathrm{e}^{\mathrm{i}(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3+\mathbf{q}_4)\mathbf{x}}\ +\cdots\Bigg)\,\mathrm{d}\mathbf{x}, \tag{33}$$

where $\theta_k^{(m)}\left(\rho(\mathbf{x});\mathbf{q}_1,\mathbf{q}_2,\ldots,\mathbf{q}_m\right)=0$, if $\mathbf{q}_j=0$ holds for any of $j\in[1,m]$. This equation along with the method of derivation of the RG flow equations for $\theta_k^{(m)}$ is the central result of our paper. The term with $\theta_k^{(m)}$ includes in a closed form all relevant (corresponding to the symmetry of the model) terms of the kind $\phi^\ell \frac{\partial^{\alpha_1}\phi}{\partial\mathbf{x}^{\alpha_1}}\frac{\partial^{\alpha_2}\phi}{\partial\mathbf{x}^{\alpha_2}}\cdots\frac{\partial^{\alpha_m}\phi}{\partial\mathbf{x}^{\alpha_m}}$, where $\alpha_j=(\alpha_{j1},\alpha_{j2},\ldots,\alpha_{jd})$ (with d being the spatial dimensionality) is the multi-index in the standard notations of the functional analysis. The representation (33) is obtained by Fourier-transforming $\frac{\partial^{\alpha_1}\phi}{\partial\mathbf{x}^{\alpha_1}}\frac{\partial^{\alpha_2}\phi}{\partial\mathbf{x}^{\alpha_2}}\cdots\frac{\partial^{\alpha_m}\phi}{\partial\mathbf{x}^{\alpha_m}}$ for even m and doing the same with $\phi\frac{\partial^{\alpha_1}\phi}{\partial\mathbf{x}^{\alpha_1}}\frac{\partial^{\alpha_2}\phi}{\partial\mathbf{x}^{\alpha_2}}\cdots\frac{\partial^{\alpha_m}\phi}{\partial\mathbf{x}^{\alpha_m}}$ for odd m. The derivative expansion is recovered from (33) by expanding the functions $\theta_k^{(m)}$ in small q_j limit. In (33), however, the magnitudes of wave vectors q_j need not to be small and it includes all terms of the gradient expansion summed up with arbitrary weight coefficients. The effective action in such a general form can be obtained as a solution of the Wetterich equation. Due to the symmetry of the model, the functions $\theta_k^{(m)}$ have a certain form. In particular, $\theta_k^{(1)}$ depends on $|\mathbf{q}_1|^2$, the function $\theta_k^{(2)}$ depends on $|\mathbf{q}_1|$, $|\mathbf{q}_2|$ and the angle made by these two vectors, and so on.

Some formal similarity of our method with the expansion in powers of fields [16], known also as the vertex expansion, can be mentioned. The vertex expansion is represented by equation (2.34) in [16]. It contains n-point functions in the coordinate space, i.e. $\Gamma_k^{(n)}(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n)$, or the corresponding n-point functions in the wave-vector space. The expansion (33) contains quantities $\theta_k^{(n)}\left(\rho(\mathbf{x});\mathbf{q}_1,\mathbf{q}_2,\ldots,\mathbf{q}_n\right)$, which are even more complex functions, as they depend on n wave vectors (like n-point functions in the wave-vector space), as well as on the coordinate variable $\rho(\mathbf{x})$.

We propose an approximation scheme, where terms $\theta_k^{(j)}$ with $j\leqslant m$ are included in the mth approximation. It is a truncation. However, at any m, the order of derivatives included is not limited. This is a salient difference from a truncated derivative expansion. As a result, the new approximation scheme proposed here can, in principle, provide better results than the derivative expansion. Particularly, it refers to the calculation of the critical two-point correlation function $G_k(\mathbf{q})$ at $q\gg k$, further considered in section 12. For this calculation, one needs to find the function $Y_k(\rho,q^2)$ in (25). In this case, the derivative expansion represents an expansion of $Y_k(\rho,q^2)/q^2$ in powers of q^2/k^2, just as in the example of section 6.2. To the contrary, our equations provide a solution with the expected stable power-law asymptotic $G_k(\mathbf{q})\sim q^{-2+\eta}$ at $q\gg k$, as further shown in section 11.2. It is true because our method allows to find approximations for functions like $Y_k(\rho,q^2)/q^2$ in a closed form, i.e. not expanding in powers of q^2/k^2.

Returning to the representation (33), one has to note that it is not unique because some terms can be reduced to others via integration by parts. However, if we find some procedure which allows to define uniquely the functions $U_k(\rho)$ and $\theta_k^{(j)}$ $(j=1,2,\ldots,m)$ at any approximation order m, then, in fact, we choose a specific representation among many equivalent ones. The uniqueness is reached by requiring that $\theta_k^{(j)}$ are continuous functions of the wave vectors, as shown in the appendix D.

In general, the RG flow equations in the mth approximation can be obtained by evaluating the Wetterich equation at $\phi(\mathbf{x})=\phi_0\left(1+\sum_{j=1}^m h_j\ \cos(\bar{\mathbf{q}}_j\mathbf{x})\right)$ and requiring asymptotic consistency of terms at $h_j\to 0$. It leads to the RG flow equations, which do not contain h_j, and thus are valid for arbitrary field configurations with any h_j values. This issue is discussed in the appendix E. In this case the consistency of terms at $\phi(\mathbf{x})=\phi_0\left(1+\sum_{j=1}^n h_j\ \cos(\bar{\mathbf{q}}_j\mathbf{x})\right)$ with $n<m$ will be reached, as well, due to contributions containing a subset of only n modes from the total number of m modes. In fact, the consistency of terms, containing n modes with $n=1,2,\ldots,m$ will give m independent equations for determination of $\partial\theta_k^{(j)}/\partial k$. The $\bar{\mathbf{q}}_j$-independent terms will give the equation for $\partial U_k/\partial k$. Certainly, $\phi(\mathbf{x})=$

$\phi_0 \left(1 + \sum_{j=1}^{m} \left(h_j \ \exp(\mathrm{i}\bar{\mathbf{q}}_j\mathbf{x}) + h_j^* \ \exp(-\mathrm{i}\bar{\mathbf{q}}_j\mathbf{x})\right)\right)$ can be used for generalization, but it, likely, will give no extra independent equations (otherwise the RG flow equations would be over-defined). In particular, it is easy to verify that the dependence on h_j^2 is simply replaced by the same dependence on $h_j h_j^* = |h_j|^2$ in the first approximation $m = 1$. We expect this to be true also for $m > 1$.

Consider now the RG flow equations at the order $m = 1$ of our new truncation scheme. In this case we re-denote $\theta_k^{(1)}(\rho;\mathbf{q})$ by $\theta_k(\rho;\mathbf{q})$ for simplicity. The details of derivation are given in the appendix B. These equations read:

$$\frac{\partial U_k(\rho)}{\partial k} = \frac{1}{2\,(2\pi)^d} \int\limits_{q<\Lambda} \frac{\frac{\partial}{\partial k} R_k(q)\mathrm{d}^d q}{\mathcal{A}_k(\rho,\mathbf{q})}, \tag{34}$$

$$\frac{\partial \theta_k(\rho;\mathbf{q})}{\partial k} = \frac{1}{2\sqrt{\rho}} \int_0^{\rho} \rho_1^{-3/2} \left(C_k(\rho_1,\mathbf{q}) - C_k(\rho_1,\mathbf{0})\right) \mathrm{d}\rho_1, \tag{35}$$

where

$$C_k(\rho,\mathbf{q}) = \frac{1}{2\,(2\pi)^d} \int\limits_{q'<\Lambda} \left\{ -\frac{\grave{\mathcal{B}}_k(\rho,\mathbf{q})}{\mathcal{A}_k^2(\rho,\mathbf{q}')} + \sum_{\mathbf{p}=\pm\mathbf{q}} \frac{\theta(\Lambda - |\mathbf{q}'+\mathbf{p}|)\, a_k^2(\rho,\mathbf{q};\mathbf{q}',\mathbf{q}'+\mathbf{p})}{\mathcal{A}_k(\rho,\mathbf{q}'+\mathbf{p})\mathcal{A}_k^2(\rho,\mathbf{q}')} \right\}$$
$$\times \frac{\partial}{\partial k} R_k(q')\mathrm{d}^d q', \tag{36}$$

$$\mathcal{A}_k(\rho,\mathbf{q}) = W_k(\rho) + 2\Psi_k(\rho;\mathbf{q}) + R_k(q), \tag{37}$$

$$a_k(\rho,\mathbf{q};\mathbf{q}_1,\mathbf{q}_2) = \rho\, \left(W_k'(\rho) + \Psi_k'(\rho;\mathbf{q}) + \Psi_k'(\rho;\mathbf{q}_1) + \Psi_k'(\rho;\mathbf{q}_2)\right), \tag{38}$$

$$\grave{\mathcal{B}}_k(\rho,\mathbf{q}) = 3\rho\,\theta_k'(\rho;\mathbf{q}) + 12\rho^2\,\theta_k''(\rho;\mathbf{q}) + 4\rho^3\,\theta_k'''(\rho;\mathbf{q}). \tag{39}$$

Here primes denote the derivatives with respect to ρ for the ρ-dependent quantities. The notations

$$W_k(\rho) = U_k'(\rho) + 2\rho\, U_k''(\rho), \tag{40}$$

$$\Psi_k(\rho;\mathbf{q}) = \theta_k(\rho;\mathbf{q}) + 2\rho\,\theta_k'(\rho;\mathbf{q}) \tag{41}$$

are introduced to make the writing of (38) more compact and elegant. Note that the term with $\sum_{\mathbf{p}=\pm\mathbf{q}}$ in (36) is the contribution of the off-diagonal terms. Due to the symmetry of the model, we have a solution, where the wave-vector dependence reduces to the wave-vector-modulus dependence, e.g. the dependence on q, q' and $|\mathbf{q}'+\mathbf{p}|$ in (36).

9. Transformed and scaled equations

In this section, we will transform the RG flow equations of section 8 into a more convenient form for further analysis of the critical behavior. First, we note that these equations are simplified, if $\Psi_k(\rho;\mathbf{q})$ is used as an independent variable instead of $\theta_k(\rho;\mathbf{q})$ in accordance with (41). Then we have

$$\frac{\partial \Psi_k(\rho;\mathbf{q})}{\partial k} = \frac{\partial \theta_k(\rho;\mathbf{q})}{\partial k} + 2\rho \frac{\partial}{\partial \rho} \frac{\partial \theta_k(\rho;\mathbf{q})}{\partial k} = 2\rho^{1/2} \frac{\partial}{\partial \rho} \left(\rho^{1/2} \frac{\partial \theta_k(\rho;\mathbf{q})}{\partial k}\right). \tag{42}$$

According to (35), it gives the RG flow equation

$$\frac{\partial \Psi_k(\rho;\mathbf{q})}{\partial k} = \frac{1}{\rho}\left[C_k(\rho;\mathbf{q}) - C_k(\rho;\mathbf{0})\right] \tag{43}$$

for $\Psi_k(\rho;\mathbf{q})$. This expression is finite at $\rho \to 0$, since $C_k(\rho;\mathbf{q})$ is proportional to ρ in this limit. Furthermore, the equation (39) reduces to

$$\dot{\mathcal{B}}_k(\rho,\mathbf{q}) = \rho\,\Psi_k'(\rho;\mathbf{q}) + 2\rho^2\,\Psi_k''(\rho;\mathbf{q}) \tag{44}$$

in agreement with (41). It means that the RG flow equations of section 8 are completely reformulated in variables $U_k(\rho)$ and $\Psi_k(\rho;\mathbf{q})$.

Usually, the RG flow equations are represented in a certain scaled form to find the fixed point and critical exponents. For this purpose, we use the already considered scaling transformations (26) and (27) completed by

$$\Psi_k(\rho;\mathbf{q}) = \frac{1}{2}Z_k q^2 f_k(\tilde{\rho};y), \tag{45}$$

where Z_k is defined as

$$Z_k = \lim_{q\to 0}\left(\frac{2}{q^2}\Psi_k(0;\mathbf{q})\right). \tag{46}$$

We define the exponent $\eta(k)$ by the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\ \ln\ Z_k = -\eta(k), \tag{47}$$

where $t = \ln(k/\Lambda)$. According to (45) and (46), we have

$$f_k(0;0) = 1, \tag{48}$$

$$\frac{\partial}{\partial k}f_k(0;0) = 0. \tag{49}$$

In fact, the RG flow equation (49) is further used for the calculation of the exponent $\eta(k)$. The renormalization factor Z_k can then be determined from (47). The universal critical exponent η is equal to the fixed-point value of $\eta(k)$, obtained at $k \to 0$ at the critical point. In this case Z_k scales as $Z_k \sim k^{-\eta}$, according to (47).

The above transformations lead to the following RG flow equations:

$$\frac{\partial u_k(\tilde{\rho})}{\partial t} = -d\,u_k(\tilde{\rho}) + (d-2+\eta(k))\tilde{\rho}\,u_k'(\tilde{\rho}) - \frac{K_d}{4}\int_0^{\Lambda^2/k^2}\frac{y^{\frac{d}{2}-1}\zeta_k(y)\,\mathrm{d}y}{\mathcal{P}_k(\tilde{\rho},y)}, \tag{50}$$

$$\begin{aligned}\frac{\partial f_k(\tilde{\rho};y)}{\partial t} &= \eta(k)f_k(\tilde{\rho};y) + \tilde{\rho}(d-2+\eta(k))f_k'(\tilde{\rho};y) + 2y\frac{\partial f_k(\tilde{\rho};y)}{\partial y}\\ &\quad + \frac{K_d}{4}\left(f_k'(\tilde{\rho};y) + 2\tilde{\rho}f_k''(\tilde{\rho};y)\right)\int_0^{\Lambda^2/k^2}\frac{y_1^{\frac{d}{2}-1}\zeta_k(y_1)\,\mathrm{d}y_1}{\mathcal{P}_k^2(\tilde{\rho},y_1)}\\ &\quad - y^{-1}\left[\hat{C}_k(\tilde{\rho},y) - \hat{C}_k(\tilde{\rho},0)\right],\end{aligned} \tag{51}$$

where

$$\zeta_k(y) = 2y^2 r'(y) + \eta(k)\, y r(y), \tag{52}$$

$$w_k(\tilde{\rho}) = u_k'(\tilde{\rho}) + 2\tilde{\rho}\, u_k''(\tilde{\rho}), \tag{53}$$

$$\mathcal{P}_k(\tilde{\rho}, y) = w_k(\tilde{\rho}) + y[f_k(\tilde{\rho}; y) + r(y)] \tag{54}$$

and

$$\hat{C}_k(\tilde{\rho}, y) = \tilde{\rho}\, \tilde{K}_d \int_0^{\Lambda^2/k^2} \int_0^{\pi} \zeta_k(y_1)\, \Theta\left(\frac{\Lambda^2}{k^2} - Y\right) y_1^{\frac{d}{2}-1} (\sin\ \theta)^{d-2}$$
$$\times \frac{\left(w_k'(\tilde{\rho}) + \frac{1}{2}\left[y f_k'(\tilde{\rho}, y) + y_1 f_k'(\tilde{\rho}, y_1) + Y f_k'(\tilde{\rho}, Y)\right]\right)^2}{\mathcal{P}_k(\tilde{\rho}, Y)\, \mathcal{P}_k^2(\tilde{\rho}, y_1)}\, \mathrm{d}y_1 \mathrm{d}\theta, \tag{55}$$

where $Y = y + y_1 + 2\sqrt{y y_1}\ \cos\ \theta$. Here primes denote derivatives with respect to $\tilde{\rho}$, whereas $r'(y) = \mathrm{d}r/\mathrm{d}y$, as before. Furthermore, Θ is the Heaviside theta function and θ is the angle made by vectors $\mathbf{q}'$ and $\mathbf{p}$ in the original expression (36). We have used the standard integration procedure in spherical coordinates in d dimensions, assuming that $\mathbf{q}$ is oriented along the polar axis. Here $\tilde{K}_d = S(d-1)/(2\pi)^d$, where $S(d-1)$ is the area of the unit sphere in $d-1$ dimensions, and Y is $|\mathbf{q}' + \mathbf{p}|^2$ in the scaled variables. The contribution of the off-diagonal terms is represented just by the $y^{-1}\left[\hat{C}_k(\tilde{\rho}, y) - \hat{C}_k(\tilde{\rho}, 0)\right]$ term in (51). It comes from the sum $\sum_{\mathbf{p}=\pm\mathbf{q}}$ in (36) with $\mathbf{p} = \mathbf{q}$ and $\mathbf{p} = -\mathbf{q}$ giving equal contributions. The second line of (51) comes from the term with $\grave{\mathcal{B}}(\rho, \mathbf{q})$ in (36). The terms in the first line of (51) are produced by the transformation of $\partial\Psi_k(\rho, \mathbf{q})/\partial k$ into new variables. Similarly, the non-integral terms in (50) appear due to the transformation of $\partial U_k(\rho)/\partial k$.

The equation for $\eta(k)$ follows from (48), (49) and (51), taking into account that $\hat{C}_k(0, y) = 0$, i.e.

$$\eta(k) = -\frac{K_d}{4} f_k'(0; 0) \int_0^{\Lambda^2/k^2} \frac{y^{\frac{d}{2}-1} \zeta_k(y)\, \mathrm{d}y}{\mathcal{P}_k^2(0, y)}. \tag{56}$$

Note that the integral in this expression depends on $\eta(k)$ via $\zeta_k(y)$.

10. Relation to the derivative expansion

In the limit $q \to 0$, both quantities $\theta_k(\rho, \mathbf{q})$ and $\Psi_k(\rho, \mathbf{q})$ have a certain meaning, related to the derivative expansion. We have

$$\theta_k(\rho; \mathbf{q}) \approx \frac{1}{2} \tilde{Z}_k(\rho)\, q^2, \tag{57}$$

$$\Psi_k(\rho; \mathbf{q}) \approx \frac{1}{2} Z_k(\rho)\, q^2 \tag{58}$$

at $q \to 0$ in accordance with the scaled form of the equations in section 9. The coefficients $\tilde{Z}_k(\rho)$ and $Z_k(\rho)$ are related to each other via

$$Z_k(\rho) = \tilde{Z}_k(\rho) + 2\rho\, \tilde{Z}_k'(\rho) \tag{59}$$

according to (41). Moreover, $Z_k(0) = Z_k$ holds according to (46). From (33), where $\theta_k^{(1)} = \theta_k$, we find that $\tilde{Z}_k(\rho)$ is the coefficient at the $-\phi\Delta\phi/2$ term in the expression of the average effective action, approximated as

$$\Gamma_k[\phi] = \int \left\{ U_k(\rho(\mathbf{x})) - \frac{1}{2}\tilde{Z}_k(\rho(\mathbf{x}))\phi\Delta\phi \right\} \mathrm{d}\mathbf{x}. \tag{60}$$

In fact, this approximation is obtained when (57) is used for all $\mathbf{q}$. Applying the integration by parts at periodic boundary conditions, we obtain

$$\int \tilde{Z}_k(\rho(\mathbf{x}))\phi\Delta\phi \mathrm{d}\mathbf{x} = -\int Z_k(\rho(\mathbf{x}))(\nabla\phi)^2 \mathrm{d}\mathbf{x}, \tag{61}$$

provided that (59) holds. It means that $Z_k(\rho)$ is the coefficient at the $(\nabla\phi)^2/2$ term in the expression of $\Gamma_k[\phi]$, approximated as

$$\Gamma_k[\phi] = \int \left\{ U_k(\rho(\mathbf{x})) + \frac{1}{2}Z_k(\rho(\mathbf{x}))(\nabla\phi)^2 \right\} \mathrm{d}\mathbf{x}, \tag{62}$$

corresponding to $\Psi_k(\rho;\mathbf{q}) = \frac{1}{2}Z_k(\rho)\,q^2$. The expression (62) is commonly used in the derivative expansion up to the ∂^2 order [20, 21]. Hence, it should be possible to recover the results of the derivative expansion at this order from our equations, based on the representation of section 9 with Ψ_k and U_k as independent variables, using the small-q approximation (58) for all $\mathbf{q}$.

In the approximation (58), equation (37) acquires the form

$$\mathcal{A}_k(\rho,\mathbf{q}) = W_k(\rho) + Z_k(\rho)q^2 + R_k(q) \tag{63}$$

and, taking into account (40), the equation (34) for $U_k(\rho)$ reduces to (17). It represents the known result of the derivative expansion up to the ∂^2 order [16].

Furthermore, equations (44) and (38) reduce to

$$\grave{\mathcal{B}}_k(\rho,\mathbf{q}) = \frac{\rho}{2}\left(Z_k'(\rho) + 2\rho Z_k''(\rho)\right) q^2, \tag{64}$$

$$a_k(\rho,\mathbf{q};\mathbf{q}_1,\mathbf{q}_2) = \rho\left(W_k'(\rho) + \frac{1}{2}Z_k'(\rho)\left(q^2 + q_1^2 + q_2^2\right)\right) \tag{65}$$

in this approximation. Noting that $Z_k(\rho) = 2\,\lim_{q\to 0}\,\partial_{\mathbf{q}^2}\Psi_k(\rho;\mathbf{q})$ holds according to (58), the RG flow equation for $Z_k(\rho)$ is obtained from (43), using the approximations (63)–(65) for calculation of $C_k(\rho,\mathbf{q})$ given by (36). Thus, we have

$$\frac{\partial Z_k(\rho)}{\partial k} = 2\rho^{-1}\lim_{q\to 0}\partial_{\mathbf{q}^2}C_k(\rho,\mathbf{q}), \tag{66}$$

where $C_k(\rho,\mathbf{q})$ is defined by (36), whereas the quantities $\mathcal{A}_k$, $\grave{\mathcal{B}}_k$ and a_k in this expression are given by (63)–(65). Two terms of the sum in (36) give equal contributions, therefore we take only one of them (choosing '+' from '$\pm$') with the weight coefficient 2. Finally, we re-denote $\mathbf{q}\to\mathbf{p}$, $\mathbf{q}'\to\mathbf{q}$. It leads straightforwardly to the RG flow equation (24), obtained within the derivative expansion up to the ∂^2 order.

11. Approximate solution of the RG flow equations

11.1. A polynomial approximation

Here we consider as an example a simple approximation of our scaled (dimensionless) equations, using polynomials of $\tilde{\rho}$. In particular, $u_k(\tilde{\rho})$ is approximated by a ϕ^4 potential and

$f_k(\tilde{\rho}; y)$—by a linear function of $\tilde{\rho}$, i.e.

$$u_k(\tilde{\rho}) \approx u_{0,k} + u_{1,k}\,\tilde{\rho} + u_{2,k}\,\tilde{\rho}^2, \tag{67}$$

$$f_k(\tilde{\rho}; y) \approx f_{0,k}(y) + f_{1,k}(y)\,\tilde{\rho}. \tag{68}$$

This is, probably, the simplest possible approximation, which gives nontrivial results. In this case, the RG flow equations for the quantities $u_{n,k}$ and $f_{n,k}(y)$ are obtained by expanding (50) and (51) in powers of $\tilde{\rho}$. It yields

$$\frac{\partial u_{n,k}}{\partial t} = [(d-2+\eta(k))n - d]\; u_{n,k} - (-1)^n L_{n,k}, \quad n = 0, 1, 2, \tag{69}$$

$$\begin{aligned} \frac{\partial f_{n,k}(y)}{\partial t} &= [(d-2+\eta(k))n + \eta(k)]\; f_{n,k}(y) + 2y\frac{\partial f_{n,k}(y)}{\partial y} \\ &\quad + (-1)^n f_{1,k}(y) M_{n,k} - n\mathcal{F}_k(y), \quad n = 0, 1 \end{aligned} \tag{70}$$

where

$$L_{n,k} = \frac{K_d}{4}\int_0^{\Lambda^2/k^2} \frac{y^{\frac{d}{2}-1}\zeta_k(y)\big(6u_{2,k} + y f_{1,k}(y)\big)^n\,\mathrm{d}y}{[\mathcal{P}_{0,k}(y)]^{n+1}}, \tag{71}$$

$$M_{n,k} = (n+1)\,\frac{K_d}{4}\int_0^{\Lambda^2/k^2} \frac{y^{\frac{d}{2}-1}\zeta_k(y)\big(6u_{2,k} + y f_{1,k}(y)\big)^n\,\mathrm{d}y}{[\mathcal{P}_{0,k}(y)]^{n+2}}, \tag{72}$$

$$\mathcal{P}_{0,k}(y) = u_{1,k} + y[f_{0,k}(y) + r(y)], \tag{73}$$

$$\mathcal{F}_k(y) = y^{-1}[F_k(y) - F_k(0)], \tag{74}$$

$$\begin{aligned} F_k(y) &= \tilde{\rho}\,\tilde{K}_d \int_0^{\Lambda^2/k^2}\int_0^{\pi} \zeta_k(y_1)\,\Theta\left(\frac{\Lambda^2}{k^2} - Y\right) y_1^{\frac{d}{2}-1}(\sin\ \theta)^{d-2} \\ &\quad \times \frac{\left(6u_{2,k} + \frac{1}{2}\left[y f_{1,k}(y) + y_1 f_{1,k}(y_1) + Y f_{1,k}(Y)\right]\right)^2}{\mathcal{P}_{0,k}(Y)\,\mathcal{P}_{0,k}^2(y_1)}\;\mathrm{d}y_1\mathrm{d}\theta \end{aligned} \tag{75}$$

with $Y = y + y_1 + 2\sqrt{y y_1}\ \cos\ \theta$ in the latter equation. The equation (56) for $\eta(k)$ in this approximation reduces to

$$\eta(k) = -f_{1,k}(0)\,M_{0,k}, \tag{76}$$

and is consistent with the fact that $f_{0,k}(0) \equiv 1$ holds.

11.2. An iterative method of finding the fixed-point solution

Here we propose a semi-analytic method for finding the stationary or fixed-point solution of the equations of section 11.1, based on a certain iteration scheme. We do not consider the renormalization of the constant $u_{0,k}$, since a constant shift in the potential is unimportant. Noting that the fixed point is reached at $k \to 0$, the upper integration limits for y and y_1 are $\Lambda^2/k^2 = \infty$ in this case. Moreover, we set $\eta(k) = \eta$ at the fixed point. Assuming $\eta = 0$, $f_{0,k}(y) = 1$ and $f_{1,k}(y) = 0$ as the initial approximation, the iterations proceed as follows:

(a) Solve numerically (e.g. by Newton's iterations) the system of equations $\partial u_{n,k}/\partial t = 0$ for $n = 1, 2$ and find the nontrivial solution for $u_{n,k}$, as well as $M_{1,k}$, at fixed η and $f_{n,k}(y)$.

(b) Calculate $\mathcal{F}_k(y)$ at given η, $u_{n,k}$ and $f_{n,k}(y)$.

(c) Solve the equation $\partial f_{1,k}(y)/\partial t = 0$ at given η, $M_{1,k}$ and $\mathcal{F}_k(y)$. This differential equation with respect to $f_{1,k}(y)$ is solved analytically by the method of variation of constant. The general solution reads

$$\tilde{f}_{1,k}(y) = \frac{1}{2} y^{-\frac{a}{2}} \left(\int_0^y x^{\frac{a}{2}-1} \mathcal{F}_k(x)\, \mathrm{d}x + C \right), \tag{77}$$

where $a = d - 2 + 2\eta - M_{1,k}$ and C is a constant. Since $a > 0$ holds, the physically meaningful solution is obtained, requiring that $\tilde{f}_{1,k}(y)$ is finite at $y \to 0$. Hence, we have $C = 0$ and

$$\tilde{f}_{1,k}(y) = \frac{1}{2} y^{-\frac{a}{2}} \int_0^y x^{\frac{a}{2}-1} \mathcal{F}_k(x)\, \mathrm{d}x, \quad \tilde{f}_{1,k}(0) = \frac{\mathcal{F}_k(0)}{a}. \tag{78}$$

Now we update the function $f_{1,k}(y)$ as follows:

$$\big(f_{1,k}(y)\big)_{\text{new}} = \kappa \tilde{f}_{1,k}(y) + (1-\kappa)\big(f_{1,k}(y)\big)_{\text{old}}, \tag{79}$$

where the subscript 'new' stands for the updated function and 'old'—for the old one, and κ is an optimization parameter for speeding up the convergence.

(d) Solve the equation $\eta = -f_{1,k}(0) M_{0,k}$ with respect to η at fixed $u_{n,k}$ and $f_{n,k}(y)$. In this case, $M_{0,k}$ depends linearly on η through $\zeta_k(y) = 2y^2 r'(y) + \eta y r(y)$. Thus we find the updated value of η, as well as the value of $M_{0,k}$, corresponding to this η.

(e) Update the function $f_{0,k}(y)$ by solving the equation $\partial f_{0,k}(y)/\partial t = 0$ at fixed η, $f_{1,k}(y)$ and $M_{0,k}$, using the method of step (c):

$$f_{0,k}(y) = -\frac{1}{2} y^{-\frac{\eta}{2}} M_{0,k} \int_0^y x^{\frac{\eta}{2}-1} f_{1,k}(x)\, \mathrm{d}x. \tag{80}$$

We have $f_{0,k}(0) = 1$, because $\eta = -f_{1,k}(0) M_{0,k}$ holds according to step (d). After this step we return to the step (a).

We have used this iteration scheme to find the fixed-point solution and the critical exponent η at $d = 3$ for the known [20, 21] cut-off function, corresponding to

$$r(y) = \frac{\alpha}{\mathrm{e}^y - 1}, \tag{81}$$

depending on the optimization parameter α. The iterations do not converge well at $\kappa = 1$, where oscillations appear. This problem has been solved by setting $\kappa = 0.5$. In this case, the accuracy of 10^{-12} (the maximal discrepancy) is typically reached in 32 iterations. The details of numerical calculations within this iteration scheme are given in appendix F.

The results for η vs α are shown in figure 1. This plot has a minimum near $\alpha = 2$, where $\eta \approx 0.0161$. Formally, this could be considered as the best value according to the known principle of the minimal sensitivity (PMS) [20, 21], since η is the least dependent on α around $\alpha = 2$. On the other hand, this is just a minimum of the plot. Since the actually obtained approximate η values are clearly underestimated (the values around 0.033 or 0.036 are usually expected [5, 16]) the PMS does not work well in this example, i.e. the minimal value is the worst rather

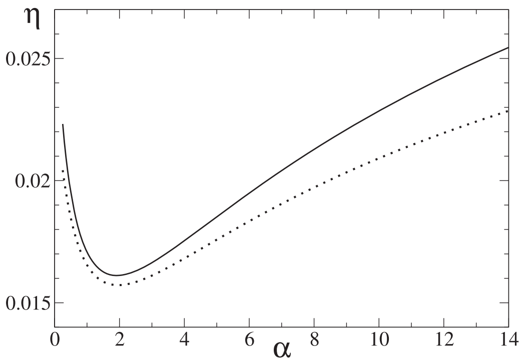


Figure 1. The critical exponent η depending on the optimization parameter α at $d = 3$. The results of our new method at the $m = 1$ order with extra polynomial approximations (67) and (68) are shown by a solid line. The results of the derivative expansion at the ∂^2 order with the corresponding polynomial approximations are shown by a dotted line for comparison.

than the best choice in this case. However, this still does not rule out a possibility that the PMS could work well for refined approximations.

We have considered how these results are changed if only the diagonal contribution, coming just from the diagonal elements of the matrix $\Gamma_k^{(2)}[\phi]$, is retained in our RG flow equations. It means that the term $n\mathcal{F}_k(y)$ is omitted in (70) in this diagonal approximation. Within the actual iterative scheme, at least, it leads to the LPA fixed-point solution with $\eta = 0$. Indeed, setting $f_{0,k}(y) = 1$ and $f_{1,k}(y) = 0$ as the initial approximation, equations (69) and (70) with $\mathcal{F}_k(y) \to 0$ ensure that this simple form of functions $f_{n,k}(y)$ is conserved, and this is just the LPA. We have also compared our results with the derivative expansion at the ∂^2 order, noting that the latter is obtained by formally setting $f_k(\tilde{\rho}; y) \to f_k(\tilde{\rho})$, where $f_k(\tilde{\rho}) = \lim_{y\to 0} f_k(\tilde{\rho}; y)$, in accordance with the consideration in section 10. For a fair comparison, we have used the same polynomial approximation (67) for the potential and the corresponding to (68) approximation $f_k(\tilde{\rho}) \approx f_{0,k} + f_{1,k}\,\tilde{\rho}$ for the function $f_k(\tilde{\rho})$. Moreover, we have $f_{0,k} = 1$ in accordance with $f_k(0; 0) = 1$. Based on these relations, we have adapted our iteration scheme for calculation of η within the considered here derivative expansion. The results are shown in figure 1 by a dotted line. As we can see, our new equations at the first order of truncation ($m = 1$) give a slightly larger η (solid line), i.e. slightly improve the estimates of the derivative expansion at the ∂^2 order for the actually considered functional truncations, represented by equations (67), (68) and $f_k(\tilde{\rho}) \approx 1 + f_{1,k}\,\tilde{\rho}$.

The scaling functions $f_{n,k}(y)$ at $\alpha = 2$ and $\alpha = 10$ are shown in figure 2 in the log–log scale. These log-log plots are asymptotically linear for large y, in agreement with the power-law behaviors $f_{0,k}(y) \propto y^{-\eta/2}$ and $f_{1,k}(y) \propto y^{-a/2}$ following from (80) and (78), taking into account that the integrals in these expressions converge to constant values at $y \to \infty$.

11.3. Small deviations from the fixed-point solution and the critical exponent ν

The fixed-point solution is reached at $k \to 0$ only if the renormalization is started at such parameters of the average effective action that correspond to the critical temperature. At a small deviation ΔT from the critical temperature, the RG flow is essentially affected by this deviation at a scale of $k \sim 1/\xi \sim (\Delta T)^\nu$. It means that the deviation from the fixed-point solution

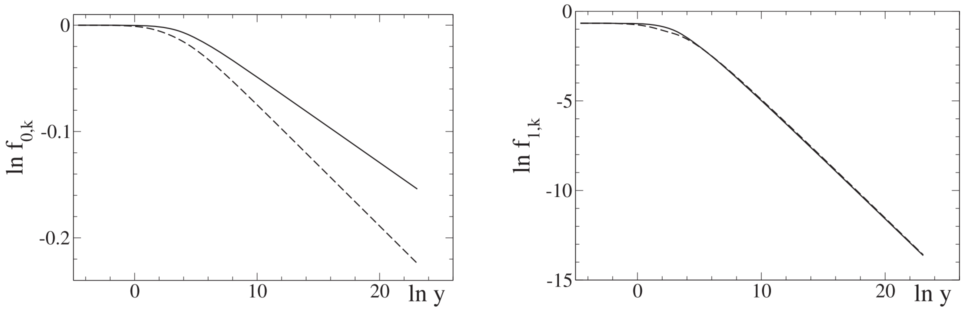


Figure 2. The log–log plots of the scaling functions $f_{0,k}(y)$ (left) and $f_{1,k}(y)$ (right) for $\alpha = 2$ (solid lines) and $\alpha = 10$ (dashed lines) in three dimensions ($d = 3$).

grows as $\Delta T k^{-1/\nu} \propto \Delta T \mathrm{e}^{-\lambda t}$ at $k \to 0$ or $t \to -\infty$, where $\lambda = 1/\nu$. According to this, the critical exponent ν can be obtained from the linear stability analysis, considering small deviations from the fixed-point solution and finding the maximal (having the largest real part) Lyapunov exponent λ. In fact, this exponent appears to be real and, therefore, just equal to $1/\nu$.

We consider the deviations $\Delta u_{n,k} = u_{n,k} - u^*_{n,k}$ and $\Delta f_{n,k}(y) = f_{n,k}(y) - f^*_{n,k}(y)$ from the fixed-point values. In this analysis, all the fixed-point values are marked by an asterisk, except for the universal fixed-point value of $\eta(k)$, which is just the critical exponent η. An appropriate form of the RG flow equations for these deviations are obtained by a formal subtraction $\partial \Delta u_{n,k}/\partial t = (\partial u_{n,k}/\partial t) - (\partial u^*_{n,k}/\partial t)$, and similarly for $\Delta f_{n,k}(y)$. It yields

$$\begin{aligned} \frac{\partial}{\partial t}\Delta u_{n,k} &= n\,(\eta\,(k) - \eta)\,u^*_{n,k} + (-1)^n\left(L^*_{n,k} - L_{n,k}\right) \\ &\quad + \left[(d-2+\eta\,(k))\,n - d\,\right]\Delta u_{n,k}, \quad n = 1,2 \end{aligned} \tag{82}$$

$$\begin{aligned} \frac{\partial}{\partial t}\Delta f_{n,k}\,(y) &= \left[(d-2+\eta\,(k))\,n + \eta\,(k)\,\right]\Delta f_{n,k}\,(y) + (n+1)\,(\eta\,(k)-\eta)\,f^*_{n,k}\,(y) \\ &\quad + 2y\frac{\partial}{\partial y}\Delta f_{n,k}\,(y) + (-1)^n\left[M_{n,k}\Delta f_{1,k}\,(y) + \left(M_{n,k} - M^*_{n,k}\right)f^*_{1,k}\,(y)\right] \\ &\quad - n\left(\mathcal{F}_k\,(y) - \mathcal{F}^*_k\,(y)\right), \quad n = 0,1. \end{aligned} \tag{83}$$

Formally, we can consider the solution vector $\vec{F} = (\Delta u_{1,k}, \Delta u_{2,k}, \Delta f_{0,k}(y), \Delta f_{1,k}(y))$. In this case, $\eta(k)$, $L_{n,k}$, $M_{n,k}$ and $\mathcal{F}_k(y)$ are considered as dependent quantities, calculated from $\Delta u_{n,k}$ and $\Delta f_{n,k}(y)$. The general solution can be represented as $\vec{F} = \sum_i C_i \vec{F}_i\ \exp(-\lambda_i t)$, where $\vec{F}_i$ are the eigenvectors and λ_i—the eigenvalues, i.e. the Lyapunov exponents. These are found by solving the eigenvalue problem $\partial\vec{F}/\partial t = -\lambda\vec{F}$. Note that the eigenvector is defined up to a constant factor, therefore, we have found it convenient to assign $\Delta u_{1,k}$ a small fixed value (such as $\Delta u_{1,k} = 10^{-5}$) in a numerical solution. Similarly as in section 11.2, here we propose the following iteration scheme:

(a) Solve the system of equations $(\partial\Delta u_{n,k}/\partial t) + \lambda\Delta u_{n,k} = 0$ for $n = 1,2$ with respect to $\Delta u_{2,k}$ and λ at a fixed small value of $\Delta u_{1,k}$ and fixed values of $\Delta f_{n,k}$ (setting $\Delta f_{n,k} = 0$ for the first iteration). Here $\eta(k)$, $L_{n,k}$ and $M_{n,k}$ are dependent quantities, and the equations are linearized with respect to $\Delta u_{2,k}$. In the limit of small deviations from the fixed point, it yields the maximal eigenvalue and $\Delta u_{2,k}$:

$$\lambda = -\frac{A_1 + B_2}{2} + \sqrt{\left(\frac{A_1 + B_2}{2}\right)^2 + A_2 B_1 - A_1 B_2}, \tag{84}$$

$$\Delta u_{2,k} = -\Delta u_{1,k}\,(A_1 + \lambda)/B_1, \tag{85}$$

where

$$A_1 = \frac{\eta(k) - \eta}{\Delta u_{1,k}}\, u^*_{1,k} + \frac{\left(L_{1,k} - L^*_{1,k}\right)_{\Delta u_{2,k}=0}}{\Delta u_{1,k}} + \eta - 2, \tag{86}$$

$$A_2 = 2\,\frac{\eta(k) - \eta}{\Delta u_{1,k}}\, u^*_{2,k} - \frac{\left(L_{2,k} - L^*_{2,k}\right)_{\Delta u_{2,k}=0}}{\Delta u_{1,k}}, \tag{87}$$

$$B_1 = \left(\partial L_{1,k}/\partial u_{2,k}\right)^*, \tag{88}$$

$$B_2 = -\left(\partial L_{2,k}/\partial u_{2,k}\right)^* + d - 4 + 2\eta. \tag{89}$$

Note that B_1 and B_2 are constants, which are determined at the fixed point and which do not change in the iteration process. The quantities $L_{n,k} - L^*_{n,k}$ are evaluated at $\Delta u_{2,k} = 0$, whereas $\eta(k)$ does not depend on $\Delta u_{2,k}$. Simultaneously with λ and $\Delta u_{2,k}$, the corresponding values of $\eta(k)$ and $M_{n,k}$ are calculated at this step.

(b) Calculate $\mathcal{F}_k(y)$ at given $\eta(k)$, $\Delta u_{n,k}$ and $\Delta f_{n,k}(y)$.

(c) Solve the equation $(\partial \Delta f_{1,k}(y)/\partial t) + \lambda \Delta f_{1,k}(y) = 0$ with respect to $\Delta f_{1,k}(y)$ at given λ, $\eta(k)$, $M_{1,k}$ and $\mathcal{F}_k(y)$. In this case, we use the same method as in step (c) of the iteration scheme of section 11.2. It yields the solution

$$\widetilde{\Delta f}_{1,k}(y) = \frac{1}{2} y^{-\frac{a+\lambda}{2}} \int_0^y x^{\frac{a+\lambda}{2}-1} \mathcal{Q}_{1,k}(x)\,\mathrm{d}x, \quad \widetilde{\Delta f}_{1,k}(0) = \frac{\mathcal{Q}_{1,k}(0)}{a+\lambda}, \tag{90}$$

where $a = d - 2 + 2\eta(k) - M_{1,k}$ and $\mathcal{Q}_{1,k}(y) = \mathcal{F}_k(y) - \mathcal{F}^*_k(y) + (M_{1,k} - M^*_{1,k} - 2[\eta(k) - \eta])f^*_{1,k}(y)$. We update the function $\Delta f_{1,k}(y)$ as follows:

$$\left(\Delta f_{1,k}(y)\right)_{\text{new}} = \kappa \widetilde{\Delta f}_{1,k}(y) + (1-\kappa)\left(\Delta f_{1,k}(y)\right)_{\text{old}}, \tag{91}$$

where the subscript 'new' stands for the updated function and 'old'—for the old one, and κ is an optimization parameter for speeding up the convergence.

(d) Recalculate $\eta(k)$ and $M_{0,k}$ by solving the equation $\eta(k) = -(f^*_{1,k}(0) + \Delta f_{1,k}(0))M_{0,k}$ at fixed $\Delta u_{n,k}$ and $\Delta f_{n,k}(y)$, taking into account that $M_{0,k}$ depends linearly on $\eta(k)$.

(e) Update the function $\Delta f_{0,k}(y)$ by solving the equation $(\partial \Delta f_{0,k}(y)/\partial t) + \lambda \Delta f_{0,k}(y) = 0$ at fixed λ, $\eta(k)$, $\Delta f_{1,k}(y)$ and $M_{0,k}$, using the method of step (c):

$$\Delta f_{0,k}(y) = -\frac{1}{2} y^{-\frac{\eta+\lambda}{2}} \int_0^y x^{\frac{\eta+\lambda}{2}-1} \mathcal{Q}_{0,k}(x)\,\mathrm{d}x, \tag{92}$$

where $\mathcal{Q}_{0,k}(y) = (\eta(k) - \eta) f^*_{0,k}(y) + (M_{0,k} - M^*_{0,k}) f^*_{1,k}(y) + M_{0,k}\,\Delta f_{1,k}(y)$. We have $\Delta f_{0,k}(0) = 0$, because $\mathcal{Q}_{0,k}(0) = 0$ holds according to step (d), taking into account that $f^*_{0,k}(0) = 1$ and $\eta = -f^*_{1,k} M^*_{0,k}$. After this step we return to the step (a).

This iteration scheme converges at $\kappa = 1$ in the actually considered case of $d = 3$, but the convergence is faster at a smaller κ value. We have chosen $\kappa = 0.7$ as a nearly optimal value. In this case, the accuracy of 10^{-7} in ν is reached in 7 iterations at $\alpha = 1$. We have used the same numerical procedures for calculation of the integrals as in the iteration scheme of section 11.2—see appendix F.

One of our aims is to compare the exponent ν, obtained by our method, with the value of the LPA. Moreover, it is meaningful to compare the results for the same approximation of the potential (67). In this case, the LPA is obtained by setting $f_{0,k}(y) = 1$ and $f_{1,k}(y) = 0$ in our equations. It means that $\eta = 0$ holds, and ν is obtained by solving the eigenvalue problem $(\partial \Delta u_{n,k}/\partial t) + \lambda \Delta u_{n,k} = 0$ for $n = 1, 2$ at $\Delta u_{nk} \to 0$ with $u^*_{n,k}$ being determined at $f_{0,k}(y) = 1$ and $f_{1,k}(y) = 0$. It reduces to $\nu = 1/\lambda$, where λ is given by (84) with $A_1 = \left(\partial L_{1,k}/\partial u_{1,k}\right)^* - 2$, $A_2 = -\left(\partial L_{2,k}/\partial u_{1,k}\right)^*$, $B_1 = \left(\partial L_{1,k}/\partial u_{2,k}\right)^*$ and $B_2 = -\left(\partial L_{2,k}/\partial u_{2,k}\right)^* + d - 4$.

Like with the exponent η in section 11.2, we have considered also the diagonal approximation, as well as the derivative expansion at the ∂^2 order for the exponent ν. Based on the same arguments as in section 11.2, we conclude that the diagonal approximation with $\mathcal{F}_k(y) \to 0$ gives the same results as the LPA within the considered here iteration scheme, at least. As described in section 11.2, the derivative expansion at the ∂^2 order is obtained by setting $f_k(\tilde{\rho}; y) \to f_k(\tilde{\rho}) = \lim_{y \to 0} f_k(\tilde{\rho}; y)$. For a fair comparison between the derivative expansion and the other our results, we have adapted our iteration scheme for approximate calculations at the ∂^2 order of the derivative expansion, using similar functional truncations of a polynomial form (equation (67) and $f_k(\tilde{\rho}) \approx 1 + f_{1,k}\, \tilde{\rho}$) as in the other cases.

The plot of the critical exponent $\nu = 1/\lambda$, calculated by our iteration scheme, depending on the optimization parameter α is shown in figure 3 by a solid line. For comparison, the corresponding LPA values are shown by a dashed line, whereas those of the derivative expansion—by a dotted line. The actual LPA gives underestimated values of ν, according to the existing more accurate estimations, which usually give $\nu \approx 0.63$ [16]. The correction, provided by our method, slightly improves the results. There is only a tiny difference between the current results of our new method (solid line in figure 3) and the obtained here results of the derivative expansion (dotted line in figure 3). However, one can see that the result of our method is slightly better (ν is larger) within a range of α values, where ν is varied relatively slowly. The optimal parameter α could be chosen from this range according to the already mentioned criterion of PMS. The reported in literature critical exponents of the derivative expansion (see e.g. [20, 21]) are more accurate, but these refer to refined functional truncations.

Obviously, the expansion of the potential around $\tilde{\rho} = 0$, truncated as in (67), is a too rough approximation for an accurate estimation of the exponent ν. Our aim here is only to give an example of calculation, showing how our equations can be solved, as well as to show that, in principle, our method works and gives reasonable results, which can be compared with those obtained by standard techniques at similar functional truncations. The considered here very simple functional truncations (67) and (68) serve well for this purpose. A refining of them is a challenge for further investigations.

11.4. The Gaussian fixed point

Up to now we were looking only for a nontrivial fixed-point solution. However, there exists also an in some sense trivial solution $u_{n,k} = 0$, $f_{0,k}(y) = 1$ and $f_{1,k}(y) = 0$. It corresponds to the Gaussian fixed point. In this case we have $\eta = -f_{1,k}(0) M_{0,k} = 0$ and $\nu = 1/2$. The latter value follows from the fact that $A_1 = -2$, $A_2 = 0$ and $B_2 = d - 4$ hold for this solution, and it yields $\lambda = 2$ according to (84). As it is well known [4], the Gaussian fixed point with $\eta = 0$ and $\nu = 1/2$ represents the physical solution above the upper critical dimension, i.e. at $d > 4$. Therefore, the existence of such a solution is an important fact, which confirms the general validity of our method.

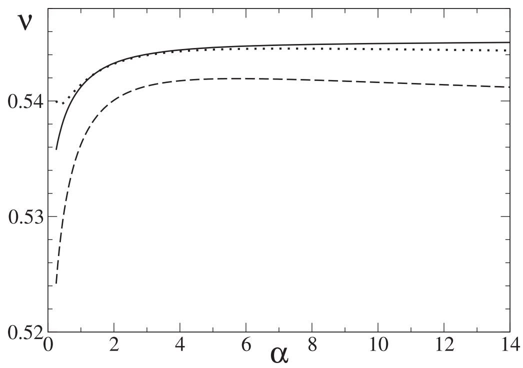


Figure 3. The critical exponent ν depending on the optimization parameter α in three dimensions ($d = 3$). The results of our new method at the $m = 1$ order with extra polynomial approximations (67) and (68) are shown by a solid line. The results of LPA with the same approximation (67) for the potential are shown by a dashed line, and the results of the derivative expansion at the ∂^2 order with the corresponding polynomial approximations are shown by a dotted line for comparison. The diagonal approximation, where $\mathcal{F}_k(y)$ is set to zero, gives the same results as the LPA (dashed line).

12. Calculation of the critical two-point correlation function: advantages of our new method

Here we consider the critical two-point correlation function of the scalar model ($N = 1$) at vanishing external field (at $\rho(\mathbf{x}) \equiv 0$), where the $\Gamma_k^{(2)}$ matrix is exactly diagonal. Hence, using (6), as well as (45) and (68), we obtain

$$G_k(\mathbf{q}) = \langle|\chi(\mathbf{q})|^2\rangle = \left[\Gamma_k^{(2)}(\mathbf{q},\mathbf{q}) + R_k(q)\right]^{-1} \approx \left[U_k'(0) + 2\Psi_k(0;\mathbf{q}) + R_k(q)\right]^{-1}$$
$$\approx \left[U_k'(0) + Z_k q^2 f_{0,k}\left(q^2/k^2\right) + R_k(q)\right]^{-1}. \tag{93}$$

This is the general equation for the two-point correlation function $G_k(\mathbf{q})$ at $\rho(\mathbf{x}) \equiv 0$, obtained within our new truncation scheme in the approximation considered in section 11. Using the fixed-point solution found in section 11.2, this equation allows us to calculate the critical two-point correlation function at arbitrary values of $y = q^2/k^2$.

According to this solution, $u_k'(0)$ has a finite value at the fixed point, which is reached at $k \to 0$ at the critical temperature. Hence, $U_k'(0)$ behaves as $\sim Z_k k^2$ (see equation (26)) for small k at the critical point, whereas $R_k(q)$ is by a factor of $\sim \mathrm{e}^{-q^2/k^2}$ smaller than $Z_k q^2 f_{0,k}\left(q^2/k^2\right)$ at small k/q. It means that, for the critical two-point correlation function,

$$G_k(\mathbf{q}) \approx \frac{1}{Z_k q^2 f_{0,k}\left(q^2/k^2\right)} \approx \frac{A}{k^\eta Z_k}\, q^{-2+\eta} \tag{94}$$

holds at small k and large q/k (small k/q), according to the $f_{0,k}(y) \propto y^{-\eta/2}$ behavior of the scaling function $f_{0,k}(y)$ at $y \to \infty$, where $A^{-1} = \lim_{y\to\infty} y^{\eta/2} f_{0,k}(y)$. This scaling function has been found in section 11.2 for the three-dimensional case ($d = 3$) as an example. According to (94), our new method allows us to calculate the asymptotic behavior of the critical two-point correlation $G_k(\mathbf{q})$, thus identifying not only the critical exponent η, but also the amplitude $A\, k^{-\eta} Z_k^{-1}$.

This amplitude hardly can be calculated by the traditional method of the derivative expansion. Note that $Z_k f_{0,k}(y)$ corresponds to $Y_k(0, q^2)/q^2$, and $Y_k(0, q^2)/q^2$ is expanded in powers of q^2/k^2 in the derivative expansion—see section 6.2, where the function $Y_k(\rho, q^2)$ is introduced as the q-dependent part of $\Gamma_k^{(2)}(\mathbf{q}, \mathbf{q})$. In fact, $Y_k(0, q^2)$ is the q-dependent part of $\Gamma_k^{(2)}(\mathbf{q}, \mathbf{q})$ in (93), where $\rho = 0$. One needs to consider the limit $y = q^2/k^2 \to \infty$ for the calculation of the amplitude. Even if the derivative expansion would converge at large q/k values, one would need to take into account an infinitely large number of expansion terms in powers of $y = q^2/k^2$ to reach a given accuracy at $y \to \infty$, according to the property of the asymptotically slow convergence explained in section 6.2. In any case, large q/k is not the natural domain of validity of the expansion in powers of q^2/k^2, generated by the derivative expansion. Therefore, our method clearly provides a more elegant way of treating this case. The extension of the natural domain of validity to arbitrary q/k values is the main advantage of our new method. Hence, apart from calculations of the critical exponents (where the derivative expansion is a standard technique), it can be useful in other possible applications, even in those ones (e.g. in the derivation of a coarse-grained theory), which are not necessarily related to critical phenomena.

Another advantage of our new method is its internal consistency, as explained below. There are at least two methods for the critical exponent η to be determined from the scaling. First, it can be obtained from the $G_k(\mathbf{q}) \propto q^{-2+\eta}$ asymptotic scaling of the critical two-point correlation function at $k/q \to 0$ and $q \to 0$. Second, it can be obtained from the $Z_k \propto k^{-\eta}$ scaling of the renormalization constant Z_k at the critical point at $k \to 0$. Our new approach gives consistent results: we have $Z_k \sim k^{-\eta}$ according to (47), and the same exponent η shows up in (94).

For comparison, note that only one of these two methods, i.e. that one which uses the scaling of Z_k, works properly within the derivative expansion and gives $\eta > 0$ in three dimensions—see, e.g. [20, 21] for a review of η values, obtained by different functional truncations within the derivative expansion. In fact, the other method (which uses the scaling of $G_k(\mathbf{q})$) is based on first principles, but it does not work with the derivative expansion just because of the convergence problems discussed in detail earlier. In the derivative expansion up to the ∂^2 order, the q-dependence of $\Gamma_k^{(2)}(\mathbf{q}, \mathbf{q})$ is given by $Y_k(\rho, q^2) = Z_k(\rho)q^2$, where $Z_k(\rho)$ is a q-independent factor—see section 6.2. Consequently, the asymptotic scaling of $G_k(\mathbf{q})$ is just $G_k(\mathbf{q}) \propto q^{-2}$. Thus, it formally gives $\eta = 0$.

Finally, one has to note that the derivative expansion shows up as a small-q approximation within our approach, as shown in section 10. From this aspect, our approach provides a framework for deriving practically solvable RG flow equations, which are more accurate than those of the derivative expansion.

13. Summary and conclusions

The Wetterich equation and the problem of extraction of critical exponents from this equation have been discussed in sections 1–3. Further on (sections 4–6), the known approximation schemes for its solution have been critically considered, clarifying the role of the off-diagonal terms in the $\Gamma_k^{(2)}[\phi]$ matrix. This consideration includes the LPA for general N, as well as the derivative expansion at $N = 1$. We have revealed important details of calculations and have demonstrated how the RG flow equations at the ∂^2 order of the derivative expansion are derived in a compact form (see section 6.1 and appendix A), suitable for a comparison with our new equations in section 8. The problem of convergence of the derivative expansion has been discussed in section 6.2, showing that, in the best case, it is asymptotically slowly convergent at large q/k values in calculations of the q-dependent part of the $\Gamma_k^{(2)}(\mathbf{q}, \mathbf{q})$ elements of $\Gamma_k^{(2)}[\phi]$. In section 8, a new approximation (truncation) scheme has been proposed, allowing to extend the natural domain of validity in such calculations to arbitrary q/k values. The truncation is performed in equation (33) in this new scheme. This equation along with the method of derivation of the RG flow equations for $\theta_k^{(m)}$ in (33) is the main result of our paper. This method requires the inversion of the $\Gamma_k^{(2)}[\phi]$ matrix for special ϕ configurations, as described in section 7.

The RG flow equations in the first approximation of our new scheme are derived and presented at the end of section 8, and the details of the derivation are given in the appendix B. A radical difference from the derivative expansion is that the derivatives are not truncated at a finite order. In other words, instead of formal expansions in powers of q^2/k^2, the new equations describe the wave-vector (q) dependencies in a closed form. As a result, the new method is potentially better than the derivative expansion.

In section 9, our equations of the $m = 1$ order have been reformulated in a more convenient way and further transformed into a scaled form, suitable for the analysis of critical phenomena. It is shown in section 10 that the derivative expansion at the ∂^2 order is obtained as a small-q approximation of our new equations at the first order of truncation.

An example of calculation in three dimensions with the optimized cut-off function (81) has been provided in section 11, using the truncated expansions (67) and (68) for an approximate solution of our equations. The fixed-point solution of these equations and the critical exponent η (see figure 1) have been found in section 11.2, using a semi-analytic iteration scheme. Small deviations from the fixed-point solution have been considered in section 11.3 by solving the related eigenvalue problem and finding the critical exponent ν. The solution, again, has been found based on a semi-analytic iteration scheme.

It has been found that the results for both η and ν, obtained by our iteration schemes, reduce to those of the LPA (giving $\eta = 0$) in the diagonal approximation, where only the diagonal elements of the matrix $\Gamma_k^{(2)}[\phi]$ are taken into account. The results for ν have been compared with those of the LPA, assuming the same approximate form (67) of the potential. Our method slightly improves the estimates of LPA—see figure 3. In addition, we have compared the results of our new equations at the first order of truncation with those of the derivative expansion at the ∂^2 order, based on the relation established in section 10. For a fair comparison, we have used functional truncations like (67) and (68) in both cases. At this choice, our new method provides slightly better results for η (see figure 1), whereas those for ν are very similar (see figure 3). Certainly, the state-of-the-art techniques of the derivative expansion provide more accurate values of the critical exponents η and ν than those shown in figures 1 and 3, e.g. $\eta = 0.0443$ and $\nu = 0.6281$ reported in [20] for the ∂^2 order. These values, however, have been obtained by more sophisticated functional truncations than those used in our example calculations. Refined functional truncations can be used in our equations, as well. It would allow for a fair comparison on a refined level. This is a challenge for further investigations.

The approximations (67) and (68) appear to be too rough for an accurate estimation of the critical exponents. Nevertheless, our example calculation shows how our equations can be solved. Moreover, it shows that our method works and gives reasonable results. It is confirmed also by the fact that our equations have a nontrivial solution with $\eta > 0$, as well as a very simple solution, corresponding to the Gaussian fixed point—see section 11.4.

A remarkable fact is that our example solution demonstrates a stable behavior of the scaling functions $f_{n,k}(y)$ at any y, including very large values, where $f_{0,k}(y) \sim y^{-\eta/2}$ holds—see equation (80) and figure 2. It means that our solution correctly describes the behavior of the critical two-point correlation function $G_k(\mathbf{q})$ for all q/k values. As a consequence, our method allows to calculate not only the critical exponent η, but also the amplitude of the critical correlation function, which is hardly possible by the standard techniques of the derivative expansion—see section 12. In distinction from the latter, our new method allows to find the exponent η directly from the scaling of this correlation function, and not only from the scaling of the renormalization constant Z_k. Thus, it exhibits a distinct internal consistency (section 12). The extension of the natural domain of validity to arbitrary q/k values is the main advantage of our new method, allowing for possible wider applications, not restricted to the calculation of the critical exponents.

Since the derivative expansion shows up as a small-q approximation within our method, this newly developed approach provides a framework for deriving practically solvable RG flow equations of a higher accuracy than those of the derivative expansion.

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Appendix A. Derivation of equations (22) and (24)

The RG flow equation (19) can be written as

$$\frac{\partial Z_k(\rho)}{\partial k} = \frac{1}{2} \lim_{p \to 0} \partial_{\mathbf{p}^2} \operatorname{Tr} \left\{ \frac{\delta^2 \left[\Gamma_k^{(2)} + R_k \right]^{-1}}{\delta\phi(\mathbf{p})\delta\phi(-\mathbf{p})} \frac{\partial R_k}{\partial k} \right\} \Bigg|_{\phi(\mathbf{x})=\sqrt{2\rho}} . \tag{95}$$

According to the derivative formula (21), we have

$$\frac{\delta}{\delta\phi(-\mathbf{p})} \left[\Gamma_k^{(2)} + R_k \right]^{-1} = -\left[\Gamma_k^{(2)} + R_k \right]^{-1} \frac{\delta\Gamma_k^{(2)}}{\delta\phi(-\mathbf{p})} \left[\Gamma_k^{(2)} + R_k \right]^{-1} \tag{96}$$

and

$$\begin{aligned} &\frac{\delta^2}{\delta\phi(\mathbf{p})\delta\phi(-\mathbf{p})} \left[\Gamma_k^{(2)} + R_k \right]^{-1} \\ &\quad = -\left[\Gamma_k^{(2)} + R_k \right]^{-1} \frac{\delta^2\Gamma_k^{(2)}}{\delta\phi(\mathbf{p})\delta\phi(-\mathbf{p})} \left[\Gamma_k^{(2)} + R_k \right]^{-1} \\ &\qquad + \left[\Gamma_k^{(2)} + R_k \right]^{-1} \frac{\delta\Gamma_k^{(2)}}{\delta\phi(\mathbf{p})} \left[\Gamma_k^{(2)} + R_k \right]^{-1} \frac{\delta\Gamma_k^{(2)}}{\delta\phi(-\mathbf{p})} \left[\Gamma_k^{(2)} + R_k \right]^{-1} \\ &\qquad + \left[\Gamma_k^{(2)} + R_k \right]^{-1} \frac{\delta\Gamma_k^{(2)}}{\delta\phi(-\mathbf{p})} \left[\Gamma_k^{(2)} + R_k \right]^{-1} \frac{\delta\Gamma_k^{(2)}}{\delta\phi(\mathbf{p})} \left[\Gamma_k^{(2)} + R_k \right]^{-1}, \end{aligned} \tag{97}$$

where $\frac{\delta^2\Gamma_k^{(2)}}{\delta\phi(\mathbf{p})\delta\phi(-\mathbf{p})}$ is a matrix with elements

$$\left[\frac{\delta^2\Gamma_k^{(2)}}{\delta\phi(\mathbf{p})\delta\phi(-\mathbf{p})} \right] (\mathbf{q}, \mathbf{q}') = \tilde{\Gamma}_k^{(4)}(\mathbf{p}, -\mathbf{p}, -\mathbf{q}, \mathbf{q}') \equiv \tilde{\Gamma}_k^{(4)}(-\mathbf{p}, \mathbf{p}, -\mathbf{q}, \mathbf{q}') \tag{98}$$

and $\frac{\delta\Gamma_k^{(2)}}{\delta\phi(\mathbf{p})}$ is a matrix with elements

$$\left[\frac{\delta\Gamma_k^{(2)}}{\delta\phi(\mathbf{p})}\right]\left(\mathbf{q},\mathbf{q}'\right) = B_p(\mathbf{q},\mathbf{q}') = \tilde{\Gamma}_k^{(3)}(\mathbf{p},-\mathbf{q},\mathbf{q}'). \tag{99}$$

Here the notation $B_{\mathbf{p}}(\mathbf{q},\mathbf{q}')$ is introduced for brevity. The elements of matrix $B_{\mathbf{p}}$ have the property

$$B_{\mathbf{p}}(\mathbf{q},\mathbf{q}')|_{\phi(\mathbf{x})=\sqrt{2\rho}} \neq 0 \quad \text{only if} \quad \mathbf{p}-\mathbf{q}+\mathbf{q}'=\mathbf{0}. \tag{100}$$

To prove this, we note that $\Gamma_k[\phi]$ can be represented as a linear combination of vertices

$$W^{(n)} = V^{1-\frac{n}{2}} \sum_{\substack{\mathbf{q}_1,\mathbf{q}_2,\ldots,\mathbf{q}_n \\ \mathbf{q}_1+\mathbf{q}_2+\cdots+\mathbf{q}_n=\mathbf{0}}} \phi(\mathbf{q}_1)\phi(\mathbf{q}_2)\cdots\phi(\mathbf{q}_n)\, Q(\mathbf{q}_1,\mathbf{q}_2,\ldots,\mathbf{q}_n) \tag{101}$$

A contribution of $W^{(n)}$ to $B_{\mathbf{p}}(\mathbf{q},\mathbf{q}')$ is given by $\frac{\delta^{(3)}W^{(n)}}{\delta\phi(\mathbf{p})\delta\phi(-\mathbf{q})\delta\phi(\mathbf{q}')}$. Calculating this functional derivative, we obtain a vertex with $n-3$ lines, a factor $\widetilde{Q}(\mathbf{p},\mathbf{q},\mathbf{q}';\mathbf{q}_1,\ldots,\mathbf{q}_{n-3})$ instead of $Q(\mathbf{q}_1,\mathbf{q}_2,\ldots,\mathbf{q}_n)$, and a constraint $\mathbf{q}_1+\mathbf{q}_2+\cdots+\mathbf{q}_{n-3}=-\mathbf{p}+\mathbf{q}-\mathbf{q}'$ for the wave vectors associated with $n-3$ lines of the vertex. Obviously, this vertex vanishes at $\phi(\mathbf{x})=\sqrt{2\rho}$ or $\phi(\mathbf{q})=\sqrt{2\rho}\,V^{1/2}\,\delta_{\mathbf{q},\mathbf{0}}$ due to this constraint unless $\mathbf{p}-\mathbf{q}+\mathbf{q}'=\mathbf{0}$ holds. It proves the condition defined in (100).

At $\phi(\mathbf{x})=\sqrt{2\rho}$, the matrix $\Gamma_k^{(2)}$ is exactly diagonal (see section 4), so that the diagonal elements of the matrix product in the second line of (97) contains only the diagonal elements of $\frac{\delta^2\Gamma_k^{(2)}}{\delta\phi(\mathbf{p})\delta\phi(-\mathbf{p})}$, coming from the functional derivatives of the diagonal elements of $\Gamma_k^{(2)}$. The corresponding contribution to the RG flow equation for $Z_k(\rho)$ reads

$$\left(\frac{\partial Z_k(\rho)}{\partial k}\right)_{\text{diag}} = -\frac{1}{2}\lim_{p\to 0}\partial_{\mathbf{p}^2}\sum_{\mathbf{q}}\left\{\frac{\tilde{\Gamma}_k^{(4)}(-\mathbf{p},\mathbf{p},-\mathbf{q},\mathbf{q})}{\left(\Gamma_k^{(2)}(\mathbf{q},\mathbf{q})+R_k(q)\right)^2}\,\frac{\partial R_k}{\partial k}\right\}\Bigg|_{\phi(\mathbf{x})=\sqrt{2\rho}}. \tag{102}$$

The terms in the following lines of (97) provide contributions, coming from the functional derivatives of the off-diagonal elements of the $\Gamma_k^{(2)}$ matrix at $\mathbf{p}\neq\mathbf{0}$ due to the condition (100). To evaluate these contributions, first we need to find the diagonal elements of the matrix products in the third and the fourth lines of (97) at $\phi(\mathbf{x})=\sqrt{2\rho}$. Let us denote $\widetilde{B}_{\mathbf{p}} = \left(\Gamma_k^{(2)}+R_k\right)^{-1}B_{\mathbf{p}}$. The elements of this matrix are

$$\widetilde{B}_{\mathbf{p}}(\mathbf{q}',\mathbf{q}) = \left(\Gamma_k^{(2)}(\mathbf{q}',\mathbf{q}')+R_k(q')\right)^{-1}B_{\mathbf{p}}(\mathbf{q}',\mathbf{q}) \tag{103}$$

at $\phi(\mathbf{x})=\sqrt{2\rho}$, since $\Gamma_k^{(2)}+R_k$ is a diagonal-matrix in this case. Now we are ready to evaluate the diagonal elements of the matrix product $\frac{\delta\Gamma_k^{(2)}}{\delta\phi(\mathbf{p})}\left[\Gamma_k^{(2)}+R_k\right]^{-1}\frac{\delta\Gamma_k^{(2)}}{\delta\phi(-\mathbf{p})} = B_{\mathbf{p}}\widetilde{B}_{-\mathbf{p}}$ at a $\phi(\mathbf{x})=\sqrt{2\rho}$. These are

$$\left[B_{\mathbf{p}}\widetilde{B}_{-\mathbf{p}}\right](\mathbf{q},\mathbf{q}) = \sum_{\mathbf{q}'} B_{\mathbf{p}}(\mathbf{q},\mathbf{q}')\widetilde{B}_{-\mathbf{p}}(\mathbf{q}',\mathbf{q}) = B_{\mathbf{p}}(\mathbf{q},\mathbf{q}-\mathbf{p})\widetilde{B}_{-\mathbf{p}}(\mathbf{q}-\mathbf{p},\mathbf{q})\,\theta(\Lambda-|\mathbf{q}-\mathbf{p}|)$$
$$= \frac{\tilde{\Gamma}_k^{(3)}(\mathbf{p},-\mathbf{q},\mathbf{q}-\mathbf{p})\tilde{\Gamma}_k^{(3)}(-\mathbf{p},\mathbf{p}-\mathbf{q},\mathbf{q})\,\theta(\Lambda-|\mathbf{q}-\mathbf{p}|)}{\Gamma_k^{(2)}(\mathbf{q}-\mathbf{p},\mathbf{q}-\mathbf{p})+R_k(|\mathbf{q}-\mathbf{p}|)} \tag{104}$$

according to (99), (100), (103) and the condition $q' < \Lambda$. $\Gamma_k^{(2)} + R_k$ is a diagonal-matrix at $\phi(\mathbf{x}) = \sqrt{2\rho}$. Therefore, in this case, the diagonal elements of the whole product of matrices in the third line of (97) are obtained when (104) is multiplied by $\left(\Gamma_k^{(2)}(\mathbf{q},\mathbf{q}) + R_k(q)\right)^{-2}$. Concerning the fourth line of (97), the corresponding expression is obtained by $\mathbf{p} \to -\mathbf{p}$ replacement, so that it provides the same contribution to the trace in (95). Hence, using one of the symmetric representations (where $\mathbf{p} \to -\mathbf{p}$ in (104)), the total contribution of the third and the fourth lines of (97) to $\partial Z_k(\rho)/\partial k$ reads

$$\left(\frac{\partial Z_k(\rho)}{\partial k}\right)_{\text{offd}} = \lim_{p\to 0} \partial_{\mathbf{p}^2} \sum_{\mathbf{q}} \times \left\{ \frac{\tilde{\Gamma}_k^{(3)}(-\mathbf{p},-\mathbf{q},\mathbf{q}+\mathbf{p})\tilde{\Gamma}_k^{(3)}(\mathbf{p},-\mathbf{p}-\mathbf{q},\mathbf{q})\,\theta(\Lambda-|\mathbf{q}+\mathbf{p}|)}{\left(\Gamma_k^{(2)}(\mathbf{q},\mathbf{q})+R_k(q)\right)^2\left(\Gamma_k^{(2)}(\mathbf{q}+\mathbf{p},\mathbf{q}+\mathbf{p})+R_k(|\mathbf{q}+\mathbf{p}|)\right)} \times \frac{\partial R_k(q)}{\partial k} \right\}\Bigg|_{\phi(\mathbf{x})=\sqrt{2\rho}}. \tag{105}$$

The subscript 'offd' points to its off-diagonal origin. Summing up the two contributions (102) and (105), we finally arrive at the RG flow equation (22).

In the following, we will evaluate (22) at

$$\Gamma_k[\phi] = \int \left(U_k(\rho) + \frac{1}{2}Z_k(\rho)(\nabla\phi)^2\right) \mathrm{d}^d x \tag{106}$$

to derive equation (24). The term with the potential $U_k(\rho)$, obviously, gives only $\mathbf{p}$-independent contributions to the 4-point function evaluated at $\phi(\mathbf{x}) = \text{const}$. Therefore, only the part

$$\Delta\Gamma_k[\phi] = \int \left(\frac{1}{2}Z_k(\rho)(\nabla\phi)^2\right) \mathrm{d}^d x$$
$$= -\frac{1}{2\,\mathrm{V}} \int \sum_{\mathbf{q}_1,\mathbf{q}_2} Z_k(\rho(\mathbf{x}))\,\mathbf{q}_1\mathbf{q}_2\,\phi(\mathbf{q}_1)\phi(\mathbf{q}_2)\mathrm{e}^{\mathrm{i}(\mathbf{q}_1+\mathbf{q}_2)\mathbf{x}}\,\mathrm{d}^d x \tag{107}$$

of $\Gamma_k[\phi]$ is relevant for the 4-point function contained in (102). The Fourier transformation $\phi(\mathbf{x}) = V^{-1/2}\sum_{\mathbf{q}}\phi(\mathbf{q})\mathrm{e}^{\mathrm{i}\mathbf{q}\mathbf{x}}$ is used in the second line of (107). The corresponding part $\Delta\Gamma_k^{(2)}$ of $\Gamma_k^{(2)}$ is calculated straightforwardly from this equation. It yields

$$\Delta\Gamma_k^{(2)}(\mathbf{q},\mathbf{q}') = \frac{\delta^2\Delta\Gamma_k[\phi]}{\delta\phi(-\mathbf{q})\delta\phi(\mathbf{q}')} = V^{-1}\int Z_k(\rho(\mathbf{x}))\,\mathbf{q}\mathbf{q}'\,\mathrm{e}^{\mathrm{i}(\mathbf{q}'-\mathbf{q})\mathbf{x}}\,\mathrm{d}^d x$$
$$- V^{-1}\int \sum_{\mathbf{q}_1} Z_k'(\rho(\mathbf{x}))\,\frac{\delta\rho(\mathbf{x})}{\delta\phi(-\mathbf{q})}\,\mathbf{q}_1\mathbf{q}'\,\phi(\mathbf{q}_1)\,\mathrm{e}^{\mathrm{i}(\mathbf{q}_1+\mathbf{q}')\mathbf{x}}\,\mathrm{d}^d x$$

$$+ V^{-1} \int \sum_{\mathbf{q}_1} Z_k'(\rho(\mathbf{x})) \frac{\delta \rho(\mathbf{x})}{\delta \phi(\mathbf{q}')} \mathbf{q}_1 \mathbf{q}\, \phi(\mathbf{q}_1)\, \mathrm{e}^{\mathrm{i}(\mathbf{q}_1 - \mathbf{q})\mathbf{x}}\, \mathrm{d}^d x$$
$$- \frac{1}{2} V^{-1} \int \sum_{\mathbf{q}_1, \mathbf{q}_2} Z_k''(\rho(\mathbf{x})) \frac{\delta \rho(\mathbf{x})}{\delta \phi(-\mathbf{q})} \frac{\delta \rho(\mathbf{x})}{\delta \phi(\mathbf{q}')} \mathbf{q}_1 \mathbf{q}_2\, \phi(\mathbf{q}_1) \phi(\mathbf{q}_2)\, \mathrm{e}^{\mathrm{i}(\mathbf{q}_1 + \mathbf{q}_2)\mathbf{x}}\, \mathrm{d}^d x$$
$$- \frac{1}{2} V^{-1} \int \sum_{\mathbf{q}_1, \mathbf{q}_2} Z_k'(\rho(\mathbf{x})) \frac{\delta^2 \rho(\mathbf{x})}{\delta \phi(-\mathbf{q}) \delta \phi(\mathbf{q}')} \mathbf{q}_1 \mathbf{q}_2\, \phi(\mathbf{q}_1) \phi(\mathbf{q}_2)\, \mathrm{e}^{\mathrm{i}(\mathbf{q}_1 + \mathbf{q}_2)\mathbf{x}}\, \mathrm{d}^d x. \quad (108)$$

One further needs to calculate $\tilde{\Gamma}_k^{(4)}(-\mathbf{p}, \mathbf{p}, -\mathbf{q}, \mathbf{q})$, defined in (98), and $\Gamma_k^{(2)}(\mathbf{q}, \mathbf{q})$. Note that $\mathbf{q}\phi(\mathbf{q}) = 0$ holds at $\phi(\mathbf{x}) = \sqrt{2\rho}$, i.e. at $\phi(\mathbf{q}) = \sqrt{2\rho}\, V^{1/2} \delta_{\mathbf{q},\mathbf{0}}$. This property makes it obvious that only the first line of (108) contributes to $\Delta\Gamma_k^{(2)}(\mathbf{q}, \mathbf{q}) = Z_k(\rho) q^2$ at $\phi(\mathbf{x}) = \sqrt{2\rho}$. Including also the contribution, coming from the $U_k(\rho)$ term in (106) (see equation (14) at $N = 1$ or equation (140) in appendix B), we obtain

$$\Gamma_k^{(2)}(\mathbf{q}, \mathbf{q}) \Big|_{\phi(\mathbf{x}) = \sqrt{2\rho}} = W_k(\rho) + Z_k(\rho) q^2, \quad (109)$$

where

$$W_k(\rho) = U_k'(\rho) + 2\rho\, U_k''(\rho). \quad (110)$$

We set $\mathbf{q}' = \mathbf{q}$ in (108) and apply the operator $\frac{\delta^2}{\delta \phi(-\mathbf{p}) \delta \phi(\mathbf{p})}$ to this expression for the calculation of $\tilde{\Gamma}_k^{(4)}(-\mathbf{p}, \mathbf{p}, -\mathbf{q}, \mathbf{q})$. The integral in the first line of (108) generates a $\mathbf{p}$-independent contribution at $\phi(\mathbf{x}) = \sqrt{2\rho}$, which is not relevant. The contributions of the second and the third lines vanish at $\phi(\mathbf{x}) = \sqrt{2\rho}$. Indeed, one derivative must refer to $\phi(\mathbf{q}_1)$ for non-vanishing partial contributions at $\phi(\mathbf{x}) = \sqrt{2\rho}$ (because $\mathbf{q}\phi(\mathbf{q}) = 0$ holds in this case), but the two partial contributions with $\mathbf{q}_1 = \pm \mathbf{p}$ have opposite signs and thus cancel. In these calculations, the relations

$$\frac{\delta^2 \rho(\mathbf{x})}{\delta \phi(\mathbf{p}_1) \delta \phi(\mathbf{p}_2)} = V^{-1}\ \mathrm{e}^{\mathrm{i}(\mathbf{p}_1 + \mathbf{p}_2)\mathbf{x}}, \quad (111)$$

$$\frac{\delta \rho(\mathbf{x})}{\delta \phi(\mathbf{p}_1)} \frac{\delta \rho(\mathbf{x})}{\delta \phi(\mathbf{p}_2)} \Big|_{\phi(\mathbf{x}) = \sqrt{2\rho}} = 2\rho V^{-1}\ \mathrm{e}^{\mathrm{i}(\mathbf{p}_1 + \mathbf{p}_2)\mathbf{x}} \quad (112)$$

are used, which follow from the Fourier transformation

$$\rho(\mathbf{x}) = \frac{1}{2\,\mathrm{V}} \sum_{\mathbf{q}_1, \mathbf{q}_2} \phi(\mathbf{q}_1) \phi(\mathbf{q}_2) \mathrm{e}^{\mathrm{i}(\mathbf{q}_1 + \mathbf{q}_2)\mathbf{x}}. \quad (113)$$

Only the last two lines of (108) appear to be relevant in the calculation of $\tilde{\Gamma}_k^{(4)}(-\mathbf{p}, \mathbf{p}, -\mathbf{q}, \mathbf{q})$. Considering the derivatives with respect to $\phi(\mathbf{p})$ and $\phi(-\mathbf{p})$ of these expressions at $\phi(\mathbf{q}) = \sqrt{2\rho}\, V^{1/2} \delta_{\mathbf{q},\mathbf{0}}$, one has to note that only those terms do not vanish, for which both of these derivatives refer to the $\phi(\mathbf{q}_1)\phi(\mathbf{q}_2)$ term. It is true because $\mathbf{q}\phi(\mathbf{q}) = 0$ holds at $\phi(\mathbf{q}) \propto \delta_{\mathbf{q},\mathbf{0}}$. According to (111) and (112), $\frac{\delta^2 \rho(\mathbf{x})}{\delta \phi(-\mathbf{q}) \delta \phi(\mathbf{q})} = V^{-1}$ and $\frac{\delta \rho(\mathbf{x})}{\delta \phi(-\mathbf{q})} \frac{\delta \rho(\mathbf{x})}{\delta \phi(\mathbf{q})} = 2\rho V^{-1}$ hold at $\phi(\mathbf{x}) = \sqrt{2\rho}$. It leads to the expression $V^{-1} \left\{ Z_k'(\rho) + 2\rho\, Z_k''(\rho) \right\} p^2$ for the $\mathbf{p}$-dependent part of $\tilde{\Gamma}_k^{(4)}(-\mathbf{p}, \mathbf{p}, -\mathbf{q}, \mathbf{q})$, evaluated at $\phi(\mathbf{x}) = \sqrt{2\rho}$. Inserting this expression and (109) into (102), we obtain

$$\left(\frac{\partial Z_k(\rho)}{\partial k} \right)_{\mathrm{diag}} = - \frac{Z_k'(\rho) + 2\rho Z_k''(\rho)}{2\,(2\pi)^d} \int\limits_{q < \Lambda} \frac{1}{\mathcal{A}_k^2(\rho, q)} \frac{\partial}{\partial k} R_k(q) \mathrm{d}^d q, \quad (114)$$

where

$$\mathcal{A}_k(\rho, q) = \Gamma_k^{(2)}(\mathbf{q}, \mathbf{q})\Big|_{\phi(\mathbf{x})=\sqrt{2\rho}} + R_k(q) = W_k(\rho) + Z_k(\rho)q^2 + R_k(q). \tag{115}$$

Equation (108) is also used for the evaluation of a particular contribution to the 3-point function, coming from $\Delta\Gamma_k^{(2)}$, i.e.

$$\Delta\tilde{\Gamma}_k^{(3)}(-\mathbf{p}, -\mathbf{q}, \mathbf{q}+\mathbf{p}) = \frac{\delta}{\delta\phi(-\mathbf{p})} \left(\Delta\Gamma_k^{(2)}(\mathbf{q}, \mathbf{q}+\mathbf{p})\right). \tag{116}$$

The integral term in the first line of (108) gives the contribution

$$\begin{aligned}\left(\Delta\tilde{\Gamma}_k^{(3)}(-\mathbf{p}, -\mathbf{q}, \mathbf{q}+\mathbf{p})\right)_1 &= \frac{\delta}{\delta\phi(-\mathbf{p})} \left(V^{-1} \int Z_k(\rho(\mathbf{x}))\, \mathbf{q}(\mathbf{q}+\mathbf{p})\, \mathrm{e}^{\mathrm{i}\mathbf{p}\mathbf{x}} \mathrm{d}^d x\right) \\ &= V^{-1} \int Z_k'(\rho(\mathbf{x}))\, \frac{\delta\rho(\mathbf{x})}{\delta\phi(-\mathbf{p})}\, \mathbf{q}(\mathbf{q}+\mathbf{p})\, \mathrm{e}^{\mathrm{i}\mathbf{p}\mathbf{x}} \mathrm{d}^d x. \end{aligned} \tag{117}$$

Taking into account that $\frac{\delta\rho(\mathbf{x})}{\delta\phi(-\mathbf{p})} = V^{-1/2}\sqrt{2\rho}\, \mathrm{e}^{-\mathrm{i}\mathbf{p}\mathbf{x}}$ holds at $\phi(\mathbf{q}) = \sqrt{2\rho}\, V^{1/2}\delta_{\mathbf{q},\mathbf{0}}$ in accordance with (113), we obtain

$$\left(\Delta\tilde{\Gamma}_k^{(3)}(-\mathbf{p}, -\mathbf{q}, \mathbf{q}+\mathbf{p})\right)_1\Big|_{\phi(\mathbf{x})=\sqrt{2\rho}} = V^{-1/2}\sqrt{2\rho}\, Z_k'(\rho)\, \mathbf{q}(\mathbf{q}+\mathbf{p}). \tag{118}$$

For similar calculations of the contributions coming from the second and the third lines of (108), one has to note that only those terms remain after the evaluation at $\phi(\mathbf{q}) = \sqrt{2\rho}\, V^{1/2}\delta_{\mathbf{q},\mathbf{0}}$ (i.e. $\phi(\mathbf{x}) = \sqrt{2\rho}$), for which the derivative refers to the $\phi(\mathbf{q}_1)$ factor. Indeed, other terms contain the factor $\mathbf{q}_1\phi(\mathbf{q}_1)$, which vanishes at $\phi(\mathbf{q}) = \sqrt{2\rho}\, V^{1/2}\delta_{\mathbf{q},\mathbf{0}}$. Hence, the calculation of these contributions at $\phi(\mathbf{x}) = \sqrt{2\rho}$ yields

$$\left(\Delta\tilde{\Gamma}_k^{(3)}(-\mathbf{p}, -\mathbf{q}, \mathbf{q}+\mathbf{p})\right)_2\Big|_{\phi(\mathbf{x})=\sqrt{2\rho}} = V^{-1/2}\sqrt{2\rho}\, Z_k'(\rho)\, \mathbf{p}(\mathbf{q}+\mathbf{p}), \tag{119}$$

$$\left(\Delta\tilde{\Gamma}_k^{(3)}(-\mathbf{p}, -\mathbf{q}, \mathbf{q}+\mathbf{p})\right)_3\Big|_{\phi(\mathbf{x})=\sqrt{2\rho}} = -V^{-1/2}\sqrt{2\rho}\, Z_k'(\rho)\, \mathbf{p}\mathbf{q}, \tag{120}$$

where the subscript indexes '2' and '3' indicate that these contributions refer to the second and the third line of (108), respectively. The last two lines of (108) do not contribute to $\Delta\tilde{\Gamma}_k^{(3)}(-\mathbf{p}, -\mathbf{q}, \mathbf{q}+\mathbf{p})$ at $\phi(\mathbf{x}) = \sqrt{2\rho}$. Indeed, after taking the derivative with respect to $\phi(-\mathbf{q})$, these expressions still contain at least one of the factors $\mathbf{q}_1\phi(\mathbf{q}_1)$ or $\mathbf{q}_2\phi(\mathbf{q}_2)$, which vanish at $\phi(\mathbf{q}) = \sqrt{2\rho}\, V^{1/2}\delta_{\mathbf{q},\mathbf{0}}$. Hence, the final expression for $\Delta\tilde{\Gamma}_k^{(3)}(-\mathbf{p}, -\mathbf{q}, \mathbf{q}+\mathbf{p})$ at $\phi(\mathbf{x}) = \sqrt{2\rho}$ is obtained by summing up the contributions (118)–(120). It yields

$$\begin{aligned}\left(\Delta\tilde{\Gamma}_k^{(3)}(-\mathbf{p}, -\mathbf{q}, \mathbf{q}+\mathbf{p})\right)\Big|_{\phi(\mathbf{x})=\sqrt{2\rho}} &= V^{-1/2}\sqrt{2\rho}\, Z_k'(\rho)\, \left(\mathbf{q}(\mathbf{q}+\mathbf{p}) + \mathbf{p}^2\right) \\ &= V^{-1/2}\sqrt{2\rho}\, Z_k'(\rho)\, \frac{1}{2}\left(\mathbf{p}^2 + \mathbf{q}^2 + |\mathbf{q}+\mathbf{p}|^2\right). \end{aligned} \tag{121}$$

Equation (105) contains 3-point functions $\tilde{\Gamma}_k^{(3)}(-\mathbf{p}, -\mathbf{q}, \mathbf{q}+\mathbf{p})$ and $\tilde{\Gamma}_k^{(3)}(\mathbf{p}, -\mathbf{p}-\mathbf{q}, \mathbf{q})$. The calculations of these two functions are symmetric ($\mathbf{q} \to -\mathbf{q}$, $\mathbf{p} \to -\mathbf{p}$ symmetry), so that $\Delta\tilde{\Gamma}_k^{(3)}(\mathbf{p}, -\mathbf{p}-\mathbf{q}, \mathbf{q}) = \Delta\tilde{\Gamma}_k^{(3)}(-\mathbf{p}, -\mathbf{q}, \mathbf{q}+\mathbf{p})$ holds at $\phi(\mathbf{x}) = \sqrt{2\rho}$.

The term with potential U_k in (106) also contributes to these 3-point functions. Denoting such a contribution to $\tilde{\Gamma}_k^{(3)}(-\mathbf{p},-\mathbf{q},\mathbf{q}+\mathbf{p})$ by $\grave{\Gamma}_k^{(3)}(-\mathbf{p},-\mathbf{q},\mathbf{q}+\mathbf{p})$, we have

$$\begin{aligned}
\grave{\Gamma}_k^{(3)}(-\mathbf{p},-\mathbf{q},\mathbf{q}+\mathbf{p}) &= \frac{\delta}{\delta\phi(-\mathbf{p})}\left(\frac{\delta^2\int U_k(\rho(\mathbf{x}))\mathrm{d}^d x}{\delta\phi(-\mathbf{q})\delta\phi(\mathbf{q}+\mathbf{p})}\right) \\
&= \int\Bigg\{U_k'(\rho(\mathbf{x}))\,\frac{\delta^3\rho(\mathbf{x})}{\delta\phi(-\mathbf{p})\delta\phi(-\mathbf{q})\delta\phi(\mathbf{q}+\mathbf{p})} + U_k''(\rho(\mathbf{x})) \\
&\quad\times\left[\frac{\delta\rho(\mathbf{x})}{\delta\phi(-\mathbf{p})}\,\frac{\delta^2\rho(\mathbf{x})}{\delta\phi(-\mathbf{q})\delta\phi(\mathbf{q}+\mathbf{p})}\right. \\
&\quad\left.+\frac{\delta\rho(\mathbf{x})}{\delta\phi(-\mathbf{q})}\,\frac{\delta^2\rho(\mathbf{x})}{\delta\phi(-\mathbf{p})\delta\phi(\mathbf{q}+\mathbf{p})} + \frac{\delta\rho(\mathbf{x})}{\delta\phi(\mathbf{q}+\mathbf{p})}\,\frac{\delta^2\rho(\mathbf{x})}{\delta\phi(-\mathbf{p})\delta\phi(-\mathbf{q})}\right] \\
&\quad+U_k'''(\rho(\mathbf{x}))\,\frac{\delta\rho(\mathbf{x})}{\delta\phi(-\mathbf{p})}\,\frac{\delta\rho(\mathbf{x})}{\delta\phi(-\mathbf{q})}\,\frac{\delta\rho(\mathbf{x})}{\delta\phi(\mathbf{q}+\mathbf{p})}\Bigg\}\,\mathrm{d}^d x.
\end{aligned} \tag{122}$$

The factor at U_k' in (122) is just zero according to (113). The factor at U_k'' is $3\,\mathrm{V}^{-3/2}\sqrt{2\rho}$ and the factor at U_k''' is $2\,\mathrm{V}^{-3/2}\rho\sqrt{2\rho}$ at $\phi(\mathbf{x})=\sqrt{2\rho}$, i.e. at $\phi(\mathbf{q})=\sqrt{2\rho}\,V^{1/2}\delta_{\mathbf{q},\mathbf{0}}$, in accordance with (111)–(113). Inserting these constant factors in (122), we obtain

$$\grave{\Gamma}_k^{(3)}(-\mathbf{p},-\mathbf{q},\mathbf{q}+\mathbf{p})\Big|_{\phi(\mathbf{x})=\sqrt{2\rho}} = V^{-1/2}\sqrt{2\rho}\left(3U_k''(\rho)+2\rho U_k'''(\rho)\right) = V^{-1/2}\sqrt{2\rho}\,W_k'(\rho), \tag{123}$$

where $W_k(\rho)$ is defined by (110). Summing up this result with (121), we obtain the final expression for $\tilde{\Gamma}_k^{(3)}(-\mathbf{p},-\mathbf{q},\mathbf{q}+\mathbf{p})$ at $\phi(\mathbf{x})=\sqrt{2\rho}$. Moreover, the symmetric calculation for $\tilde{\Gamma}_k^{(3)}(\mathbf{p},-\mathbf{q}-\mathbf{p},\mathbf{q})$ gives the same result. Thus, we have

$$\begin{aligned}
\tilde{\Gamma}_k^{(3)}(-\mathbf{p},-\mathbf{q},\mathbf{q}+\mathbf{p})\Big|_{\phi(\mathbf{x})=\sqrt{2\rho}} &= \tilde{\Gamma}_k^{(3)}(\mathbf{p},-\mathbf{q}-\mathbf{p},\mathbf{q})\Big|_{\phi(\mathbf{x})=\sqrt{2\rho}} \\
&= V^{-1/2}\sqrt{2\rho}\left(W_k'(\rho)+\frac{1}{2}Z_k'(\rho)\left[\mathbf{p}^2+\mathbf{q}^2+|\mathbf{q}+\mathbf{p}|^2\right]\right).
\end{aligned} \tag{124}$$

Inserting (124) into (105) and using the definition (115), we obtain the off-diagonal term in the RG flow equation for $Z_k(\rho)$:

$$\begin{aligned}
\left(\frac{\partial Z_k(\rho)}{\partial k}\right)_{\mathrm{offd}} &= \frac{2\rho}{(2\pi)^d}\,\lim_{p\to 0}\,\partial_{\mathbf{p}^2} \\
&\quad\times\int\limits_{q<\Lambda}\frac{\left(W_k'(\rho)+\frac{1}{2}Z_k'(\rho)\left[p^2+q^2+|\mathbf{q}+\mathbf{p}|^2\right]\right)^2\theta(\Lambda-|\mathbf{q}+\mathbf{p}|)}{\mathcal{A}_k^2(\rho,q)\,\mathcal{A}_k(\rho,|\mathbf{q}+\mathbf{p}|)}\,\frac{\partial}{\partial k}R_k(q)\mathrm{d}^d q.
\end{aligned} \tag{125}$$

Finally, summing up the diagonal contribution (114) and the off-diagonal contribution (125), we arrive at the RG flow equation (24).

Appendix B. Derivation of the RG flow equations at $m = 1$

In the considered approximation of $m = 1$, the effective average action is

$$\Gamma_k[\phi] = \int \left(U_k(\rho(\mathbf{x})) + V^{-1} \sum_{\mathbf{q}_1,\mathbf{q}_2} \theta_k \left(\rho(\mathbf{x}); \mathbf{q}_1\right) \phi(\mathbf{q}_1)\phi(\mathbf{q}_2) \mathrm{e}^{\mathrm{i}(\mathbf{q}_1+\mathbf{q}_2)\mathbf{x}} \right) \mathrm{d}\mathbf{x}. \tag{126}$$

We consider the field configuration

$$\phi(\mathbf{x}) = \phi_0(1 + h \ \cos(\bar{\mathbf{q}}\mathbf{x})) \tag{127}$$

with periodic boundary conditions at $h \to 0$. Consequently,

$$\rho(\mathbf{x}) = \rho_0 \left(1 + 2h \ \cos(\bar{\mathbf{q}}\mathbf{x})) + h^2 \ \cos^2(\bar{\mathbf{q}}\mathbf{x}))\right), \tag{128}$$

where $\rho_0 = \phi_0^2/2$, and

$$\rho(\mathbf{x}) - \rho_0 = \rho_0 \left[h \left(\mathrm{e}^{\mathrm{i}\bar{\mathbf{q}}\mathbf{x}} + \mathrm{e}^{-\mathrm{i}\bar{\mathbf{q}}\mathbf{x}}\right) + \frac{h^2}{4} \left(\mathrm{e}^{2\mathrm{i}\bar{\mathbf{q}}\mathbf{x}} + 2 + \mathrm{e}^{-2\mathrm{i}\bar{\mathbf{q}}\mathbf{x}}\right) \right]. \tag{129}$$

The Fourier transform of (127) reads

$$\phi(\mathbf{q}) = V^{-1/2} \int \phi(\mathbf{x}) \mathrm{e}^{-\mathrm{i}\mathbf{q}\mathbf{x}} \mathrm{d}\mathbf{x} = V^{1/2} \phi_0 \left[\delta_{\mathbf{q},\mathbf{0}} + \frac{h}{2} \left(\delta_{\mathbf{q},\bar{\mathbf{q}}} + \delta_{\mathbf{q},-\bar{\mathbf{q}}} \right) \right]. \tag{130}$$

Let us now calculate the left-hand side of the Wetterich equation. It consists of the $\bar{\mathbf{q}}$-independent part

$$\left(\frac{\partial \Gamma_k}{\partial k} \right)_0 = V \left(\frac{\partial U_k(\rho_0)}{\partial k} + \mathcal{O}\left(h^2\right) \right), \tag{131}$$

evaluated at $\bar{\mathbf{q}} \to \mathbf{0}$, and the $\bar{\mathbf{q}}$-dependent part, which vanishes at $\bar{\mathbf{q}} = \mathbf{0}$ and reads

$$\left(\frac{\partial \Gamma_k}{\partial k} \right)_1 = V \left(h^2 \frac{\partial}{\partial k} \left[\rho_0 \, \theta_k(\rho_0; \bar{\mathbf{q}}) + 2\rho_0^2 \, \theta_k'(\rho_0; \bar{\mathbf{q}}) \right] + o\left(h^2\right) \right) \tag{132}$$

$$= V \left(2h^2 \rho_0^{3/2} \frac{\partial}{\partial \rho_0} \left[\rho_0^{1/2} \frac{\partial \theta_k(\rho_0; \bar{\mathbf{q}})}{\partial k} \right] + o\left(h^2\right) \right). \tag{133}$$

Equation (131) is simply obtained, applying the operator $\frac{\partial}{\partial k}$ to the right-hand side of (126). In this case, we expand $U_k(\rho(\mathbf{x}))$ around $\rho(\mathbf{x}) = \rho_0$ and integrate over $\mathbf{x}$, taking into account (129) and the property

$$\int \mathrm{e}^{\mathrm{i}\mathbf{q}\mathbf{x}} \mathrm{d}\mathbf{x} = V \delta_{\mathbf{q},\mathbf{0}}. \tag{134}$$

In addition, we use the fact that the term with θ_k in (126) vanishes at $\bar{\mathbf{q}} \to \mathbf{0}$ because equation (130) and $\theta_k(\rho; \mathbf{0}) = 0$ hold. Equation (132) is obtained, applying the operator $\frac{\partial}{\partial k}$ to the therm with θ_k in (126), which is represented by the expansion

$$\theta_k(\rho(\mathbf{x}); \mathbf{q}) = \theta_k(\rho_0; \mathbf{q}) + \theta_k'(\rho_0; \mathbf{q}) \left(\rho(\mathbf{x}) - \rho_0\right) + \mathcal{O}\left(h^2\right). \tag{135}$$

The result (132) follows from a straightforward calculation, using the property $\theta_k(\rho; \mathbf{0}) = 0$, as well as the equations (129), (130) and (134).

For the calculation of the right-hand side of the Wetterich equation, we need to find the matrix elements $\left(\Gamma_k^{(2)}[\phi]\right)(\mathbf{q},\mathbf{q}')$, applying the operator $\frac{\delta^2}{\delta\phi(-\mathbf{q})\delta\phi(\mathbf{q}')}$ to the terms in (126) in accordance with (2). For this purpose, first we derive some useful general relations. Using, the Fourier-representation $\phi(\mathbf{x}) = V^{-1/2}\sum_{\mathbf{q}}\phi(\mathbf{q})\mathrm{e}^{\mathrm{i}\mathbf{q}\mathbf{x}}$ of the field $\phi(\mathbf{x})$, we easily find the corresponding representation for $\rho(\mathbf{x}) = \phi^2(\mathbf{x})/2$, i.e.

$$\rho(\mathbf{x}) = \frac{1}{2}V^{-1}\sum_{\mathbf{q}_1,\mathbf{q}_2}\phi(\mathbf{q}_1)\phi(\mathbf{q}_2)\mathrm{e}^{\mathrm{i}(\mathbf{q}_1+\mathbf{q}_2)\mathbf{x}}. \tag{136}$$

From this we obtain

$$\frac{\delta\rho(\mathbf{x})}{\delta\phi(\mathbf{q}')} = V^{-1}\sum_{\mathbf{q}_1}\phi(\mathbf{q}_1)\mathrm{e}^{\mathrm{i}(\mathbf{q}_1+\mathbf{q}')\mathbf{x}}, \tag{137}$$

$$\frac{\delta^2\rho(\mathbf{x})}{\delta\phi(-\mathbf{q})\delta\phi(\mathbf{q}')} = V^{-1}\ \mathrm{e}^{\mathrm{i}(\mathbf{q}'-\mathbf{q})\mathbf{x}}, \tag{138}$$

$$\frac{\delta\rho(\mathbf{x})}{\delta\phi(-\mathbf{q})}\ \frac{\delta\rho(\mathbf{x})}{\delta\phi(\mathbf{q}')} = V^{-2}\sum_{\mathbf{q}_1,\mathbf{q}_2}\phi(\mathbf{q}_1)\phi(\mathbf{q}_2)\mathrm{e}^{\mathrm{i}(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}'-\mathbf{q})\mathbf{x}}. \tag{139}$$

Now we calculate

$$\begin{aligned}\frac{\delta^2\int U_k(\rho(\mathbf{x}))\mathrm{d}\mathbf{x}}{\delta\phi(-\mathbf{q})\delta\phi(\mathbf{q}')} &= \int\left[U_k''(\rho(\mathbf{x}))\frac{\delta\rho(\mathbf{x})}{\delta\phi(-\mathbf{q})}\ \frac{\delta\rho(\mathbf{x})}{\delta\phi(\mathbf{q}')} + U_k'(\rho(\mathbf{x}))\frac{\delta^2\rho(\mathbf{x})}{\delta\phi(-\mathbf{q})\delta\phi(\mathbf{q}')}\right]\mathrm{d}\mathbf{x} \\ &= \int\left[U_k''(\rho(\mathbf{x}))\,V^{-2}\sum_{\mathbf{q}_1,\mathbf{q}_2}\phi(\mathbf{q}_1)\phi(\mathbf{q}_2)\mathrm{e}^{\mathrm{i}(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}'-\mathbf{q})\mathbf{x}}\right. \\ &\qquad\left. + U_k'(\rho(\mathbf{x}))\,V^{-1}\mathrm{e}^{\mathrm{i}(\mathbf{q}'-\mathbf{q})\mathbf{x}}\right]\mathrm{d}\mathbf{x} \\ &= V^{-1}\int\left\{2\rho(\mathbf{x})\,U_k''(\rho(\mathbf{x})) + U_k'(\rho(\mathbf{x}))\right\}\mathrm{e}^{\mathrm{i}(\mathbf{q}'-\mathbf{q})\mathbf{x}}\mathrm{d}\mathbf{x}.\end{aligned} \tag{140}$$

Equations (138) and (139) are used to obtain the second and the third lines of (140), whereas equation (136) is used to obtain the fourth line. Further on, we use the expansions

$$U_k'(\rho(\mathbf{x})) = U_k'(\rho_0) + U_k''(\rho_0)\,(\rho(\mathbf{x})-\rho_0) + \mathcal{O}\left(h^2\right), \tag{141}$$

$$U_k''(\rho(\mathbf{x})) = U_k''(\rho_0) + U_k'''(\rho_0)\,(\rho(\mathbf{x})-\rho_0) + \mathcal{O}\left(h^2\right) \tag{142}$$

and (129) to evaluate (140) at (127), taking into account (134). It yields

$$\begin{aligned}\frac{\delta^2\int U_k(\rho(\mathbf{x}))\mathrm{d}\mathbf{x}}{\delta\phi(-\mathbf{q})\delta\phi(\mathbf{q}')} &= \left(U_k'(\rho_0) + 2\rho_0\,U_k''(\rho_0) + \tilde{a}_k(\rho_0)h^2 + o\left(h^2\right)\right)\delta_{\mathbf{q}',\mathbf{q}} \\ &\quad + \left(3\rho_0\,U_k''(\rho_0) + 2\rho_0^2\,U_k'''(\rho_0)\right)h\left(\delta_{\mathbf{q}',\mathbf{q}+\bar{\mathbf{q}}} + \delta_{\mathbf{q}',\mathbf{q}-\bar{\mathbf{q}}}\right) \\ &\quad + \tilde{b}_k(\rho_0)\,h^2\left(\delta_{\mathbf{q}',\mathbf{q}+2\bar{\mathbf{q}}} + \delta_{\mathbf{q}',\mathbf{q}-2\bar{\mathbf{q}}}\right) + \mathcal{O}\left(h^3\right).\end{aligned} \tag{143}$$

Here the terms with $\tilde{a}_k(\rho_0)$ and $\tilde{b}_k(\rho_0)$ are irrelevant in the asymptotic equations at $h\to 0$. These are included here without detailed deciphering only to show the general structure of the equation.

Let us now apply the operator $\frac{\delta^2}{\delta\phi(-\mathbf{q})\delta\phi(\mathbf{q}')}$ to the second term in brackets of (126), denoting it by $\mathcal{W}_k$ for convenience. Thus, we have

$$\frac{\delta \mathcal{W}_k}{\delta\phi(\mathbf{q}')} = V^{-1}\left[\sum_{\mathbf{q}_1,\mathbf{q}_2}\theta_k'(\rho(\mathbf{x});\mathbf{q}_1)\,\frac{\delta\rho(\mathbf{x})}{\delta\phi(\mathbf{q}')}\,\phi(\mathbf{q}_1)\phi(\mathbf{q}_2)\mathrm{e}^{\mathrm{i}(\mathbf{q}_1+\mathbf{q}_2)\mathbf{x}}\right.$$

$$\left.+\sum_{\mathbf{q}_2}\theta_k(\rho(\mathbf{x});\mathbf{q}')\phi(\mathbf{q}_2)\mathrm{e}^{\mathrm{i}(\mathbf{q}'+\mathbf{q}_2)\mathbf{x}}+\sum_{\mathbf{q}_1}\theta_k(\rho(\mathbf{x});\mathbf{q}_1)\phi(\mathbf{q}_1)\mathrm{e}^{\mathrm{i}(\mathbf{q}_1+\mathbf{q}')\mathbf{x}}\right] \tag{144}$$

and

$$V\,\frac{\delta^2\mathcal{W}_k}{\delta\phi(-\mathbf{q})\delta\phi(\mathbf{q}')} = \sum_{\mathbf{q}_1,\mathbf{q}_2}\left(\theta_k''(\rho(\mathbf{x});\mathbf{q}_1)\,\frac{\delta\rho(\mathbf{x})}{\delta\phi(-\mathbf{q})}\,\frac{\delta\rho(\mathbf{x})}{\delta\phi(\mathbf{q}')}+\theta_k'(\rho(\mathbf{x});\mathbf{q}_1)\,\frac{\delta^2\rho(\mathbf{x})}{\delta\phi(-\mathbf{q})\delta\phi(\mathbf{q}')}\right)$$

$$\times\ \phi(\mathbf{q}_1)\phi(\mathbf{q}_2)\mathrm{e}^{\mathrm{i}(\mathbf{q}_1+\mathbf{q}_2)\mathbf{x}}+\sum_{\mathbf{q}_2}\theta_k'(\rho(\mathbf{x});-\mathbf{q})\,\frac{\delta\rho(\mathbf{x})}{\delta\phi(\mathbf{q}')}\,\phi(\mathbf{q}_2)\mathrm{e}^{\mathrm{i}(\mathbf{q}_2-\mathbf{q})\mathbf{x}}$$

$$+\sum_{\mathbf{q}_1}\theta_k'(\rho(\mathbf{x});\mathbf{q}_1)\,\frac{\delta\rho(\mathbf{x})}{\delta\phi(\mathbf{q}')}\,\phi(\mathbf{q}_1)\mathrm{e}^{\mathrm{i}(\mathbf{q}_1-\mathbf{q})\mathbf{x}}$$

$$+\sum_{\mathbf{q}_2}\theta_k'(\rho(\mathbf{x});\mathbf{q}')\,\frac{\delta\rho(\mathbf{x})}{\delta\phi(-\mathbf{q})}\,\phi(\mathbf{q}_2)\mathrm{e}^{\mathrm{i}(\mathbf{q}'+\mathbf{q}_2)\mathbf{x}}$$

$$+\sum_{\mathbf{q}_1}\theta_k'(\rho(\mathbf{x});\mathbf{q}_1)\,\frac{\delta\rho(\mathbf{x})}{\delta\phi(-\mathbf{q})}\,\phi(\mathbf{q}_1)\mathrm{e}^{\mathrm{i}(\mathbf{q}_1+\mathbf{q}')\mathbf{x}}$$

$$+\left(\theta_k(\rho(\mathbf{x});\mathbf{q}')+\theta_k(\rho(\mathbf{x});-\mathbf{q})\right)\mathrm{e}^{\mathrm{i}(\mathbf{q}'-\mathbf{q})\mathbf{x}} \tag{145}$$

Using (137)–(139), this equation is reduced to

$$\frac{\delta^2\mathcal{W}_k}{\delta\phi(-\mathbf{q})\delta\phi(\mathbf{q}')} = V^{-3}\sum_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3,\mathbf{q}_4}\theta_k''(\rho(\mathbf{x});\mathbf{q}_1)\,\phi(\mathbf{q}_1)\phi(\mathbf{q}_2)\phi(\mathbf{q}_3)\phi(\mathbf{q}_4)\mathrm{e}^{\mathrm{i}(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3+\mathbf{q}_4+\mathbf{q}'-\mathbf{q})\mathbf{x}}$$

$$+\ V^{-2}\sum_{\mathbf{q}_1,\mathbf{q}_2}\left(3\theta_k'(\rho(\mathbf{x});\mathbf{q}_1)+\theta_k'(\rho(\mathbf{x});\mathbf{q}')+\theta_k'(\rho(\mathbf{x});-\mathbf{q})\right)$$

$$\times\ \phi(\mathbf{q}_1)\phi(\mathbf{q}_2)\mathrm{e}^{\mathrm{i}(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}'-\mathbf{q})\mathbf{x}}$$

$$+\ V^{-1}\left(\theta_k(\rho(\mathbf{x});\mathbf{q}')+\theta_k(\rho(\mathbf{x});-\mathbf{q})\right)\mathrm{e}^{\mathrm{i}(\mathbf{q}'-\mathbf{q})\mathbf{x}}. \tag{146}$$

Based on this equation, we evaluate $\int \frac{\delta^2\mathcal{W}_k}{\delta\phi(-\mathbf{q})\delta\phi(\mathbf{q}')}\mathrm{d}\mathbf{x}$ at $\phi(\mathbf{x}) = \phi_0(1+h\ \cos(\bar{\mathbf{q}}\mathbf{x}))$, using (130), (134), the property $\theta_k(\rho;\mathbf{0}) = 0$, as well as (135) and two additional expansion formulas

$$\theta_k'(\rho(\mathbf{x});\mathbf{q}) = \theta_k'(\rho_0;\mathbf{q})+\theta_k''(\rho_0;\mathbf{q})\,(\rho(\mathbf{x})-\rho_0)+\mathcal{O}\left(h^2\right), \tag{147}$$

$$\theta_k''(\rho(\mathbf{x});\mathbf{q}) = \theta_k''(\rho_0;\mathbf{q})+\theta_k'''(\rho_0;\mathbf{q})\,(\rho(\mathbf{x})-\rho_0)+\mathcal{O}\left(h^2\right), \tag{148}$$

with $\rho(\mathbf{x})-\rho_0$ being given by (129). It yields

$$
\begin{aligned}
\int \frac{\delta^2 \mathcal{W}_k}{\delta\phi(-\mathbf{q})\delta\phi(\mathbf{q}')} \mathrm{d}\mathbf{x} = & \left[2\theta_k(\rho_0;\mathbf{q}) + 4\rho_0\,\theta_k'(\rho_0;\mathbf{q}) + \hat{a}_k(\rho_0,\mathbf{q})h^2\right. \\
& \left. + h^2\left(3\rho_0\,\theta_k'(\rho_0;\bar{\mathbf{q}}) + 12\rho_0^2\,\theta_k''(\rho_0;\bar{\mathbf{q}}) + 4\rho_0^3\,\theta_k'''(\rho_0;\bar{\mathbf{q}})\right) + o\left(h^2\right)\right]\delta_{\mathbf{q}',\mathbf{q}} \\
& + \left[3\rho_0\left(\theta_k'(\rho_0;\bar{\mathbf{q}}) + \theta_k'(\rho_0;\mathbf{q}) + \theta_k'(\rho_0;\mathbf{q}')\right)\right. \\
& \left. + 2\rho_0^2\left(\theta_k''(\rho_0;\bar{\mathbf{q}}) + \theta_k''(\rho_0;\mathbf{q}) + \theta_k''(\rho_0;\mathbf{q}')\right)\right] h\left(\delta_{\mathbf{q}',\mathbf{q}+\bar{\mathbf{q}}} + \delta_{\mathbf{q}',\mathbf{q}-\bar{\mathbf{q}}}\right) \\
& + \hat{b}_k(\rho_0,\bar{\mathbf{q}};\mathbf{q},\mathbf{q}')\,h^2\left(\delta_{\mathbf{q}',\mathbf{q}+2\bar{\mathbf{q}}} + \delta_{\mathbf{q}',\mathbf{q}-2\bar{\mathbf{q}}}\right) + \mathcal{O}\left(h^3\right). \qquad (149)
\end{aligned}
$$

The symmetry property $\theta_k(\rho;\mathbf{q}) = \theta_k(\rho;-\mathbf{q})$ (and similarly for the derivatives of θ_k) has been used here to obtain this expression. Like in (143), it contains two irrelevant at $h \to 0$ terms, i.e. those with $\hat{a}_k$ and $\hat{b}_k$, to show the general structure.

From the structure of (143) and (149) we can already see that the elements of the matrix $\Gamma_k^{(2)} + R_k$ are consistent with (30). The relevant terms in (30) are represented by the coefficients $\mathcal{A}_k(\rho_0,\mathbf{q})$, $a_k(\rho_0,\bar{\mathbf{q}};\mathbf{q},\mathbf{q}')$ and $\mathcal{B}_k(\rho_0,\bar{\mathbf{q}},\mathbf{q})$. Summing up (143) and (149) and adding the diagonal contribution of R_k, we find that the first two of these coefficients are given by (37) and (38), respectively. Concerning the third coefficient, we omit the irrelevant contribution $\tilde{a}_k + \hat{a}_k$, which does not depend on $\bar{\mathbf{q}}$, and denote the remaining term by $\grave{\mathcal{B}}_k(\rho_0,\bar{\mathbf{q}})$. According to the second line in (149), it is given by (39).

Next, we calculate the diagonal elements of the inverse matrix $\left[\Gamma_k^{(2)} + R_k\right]^{-1}$ using (32) and keeping only the terms, which contribute to the final RG flow equations. It means that we set $\left(\Gamma_k^{(2)} + R_k\right)(\mathbf{q},\mathbf{q}) = \mathcal{A}_k(\rho_0,\mathbf{q}) + h^2\grave{\mathcal{B}}_k(\rho_0,\bar{\mathbf{q}})$ and $\left(\Gamma_k^{(2)} + R_k\right)(\mathbf{q},\mathbf{q}') = h\,a_k(\rho_0,\bar{\mathbf{q}};\mathbf{q},\mathbf{q}')$ for $\mathbf{q}' \neq \mathbf{q}$. Moreover, the correction term $h^2\grave{\mathcal{B}}_k(\rho_0,\bar{\mathbf{q}})$ is negligible in the second line of (32) at $h \to 0$. Consequently, we find

$$
\begin{aligned}
\left(\Gamma_k^{(2)} + R_k\right)^{-1}(\mathbf{q},\mathbf{q}) \approx & \left(\frac{1}{\mathcal{A}_k(\rho_0,\mathbf{q})} - h^2\,\frac{\grave{\mathcal{B}}_k(\rho_0,\bar{\mathbf{q}})}{\mathcal{A}_k^2(\rho_0,\mathbf{q})}\right) \\
& \times\left(1 + h^2\sum_{\mathbf{q}'=\mathbf{q}\pm\bar{\mathbf{q}}}\frac{\theta(\Lambda - q')\,a_k^2(\rho_0,\bar{\mathbf{q}};\mathbf{q},\mathbf{q}')}{\mathcal{A}_k(\rho_0,\mathbf{q}')\mathcal{A}_k(\rho_0,\mathbf{q})}\right). \qquad (150)
\end{aligned}
$$

In this approximation, a $\bar{\mathbf{q}}$-dependent term of the order $o\left(h^2\right)$ and a $\bar{\mathbf{q}}$-independent term of the order $\mathcal{O}\left(h^2\right)$ are neglected in the first brackets, and a correction $o\left(h^2\right)$ is neglected in the second brackets on the right-hand side of (150). From this we calculate the constant contribution (obtained at $\bar{\mathbf{q}} \to \mathbf{0}$) to the right-hand side of the Wetterich equation

$$
\left(\frac{1}{2}\,\mathrm{Tr}\left\{\left[\Gamma_k^{(2)}[\phi] + R_k\right]^{-1}\frac{\partial}{\partial k}R_k\right\}\right)_0 = V\left(\frac{1}{2\,(2\pi)^d}\int\limits_{q<\Lambda}\frac{\frac{\partial}{\partial k}R_k(q)\mathrm{d}^d q}{\mathcal{A}_k(\rho_0,\mathbf{q})} + \mathcal{O}\left(h^2\right)\right), \qquad (151)
$$

as well as the $\bar{\mathbf{q}}$-dependent contribution

$$
\left(\frac{1}{2}\,\mathrm{Tr}\left\{\left[\Gamma_k^{(2)}[\phi] + R_k\right]^{-1}\frac{\partial}{\partial k}R_k\right\}\right)_1 = V\left(h^2[C_k(\rho_0,\bar{\mathbf{q}}) - C_k(\rho_0,\mathbf{0})] + o\left(h^2\right)\right), \qquad (152)
$$

where

$$C_k(\rho_0, \bar{\mathbf{q}}) = \frac{1}{2\,(2\pi)^d} \int\limits_{q<\Lambda} \left\{ -\frac{\grave{\mathcal{B}}_k(\rho_0, \bar{\mathbf{q}})}{\mathcal{A}_k^2(\rho_0, \mathbf{q})} + \sum_{\mathbf{p}=\pm\bar{\mathbf{q}}} \frac{\theta(\Lambda - |\mathbf{q}+\mathbf{p}|)\, a_k^2(\rho_0, \bar{\mathbf{q}}; \mathbf{q}, \mathbf{q}+\mathbf{p})}{\mathcal{A}_k(\rho_0, \mathbf{q}+\mathbf{p})\mathcal{A}_k^2(\rho_0, \mathbf{q})} \right\}$$
$$\times \frac{\partial}{\partial k} R_k(q) \mathrm{d}^d q \tag{153}$$

Now we equate the constant contributions on both sides of the Wetterich equation, given by equations (131) and (151). More precisely, we equate the expansion terms at the leading power of h, which gives exactly the h-independent RG flow equation for $U_k(\rho_0)$. For its final form, we re-denote ρ_0 by ρ. The resulting equation is given by (34). Similar procedure for the $\bar{\mathbf{q}}$-dependent terms (132) and (152) gives the equation

$$2\rho_0^{3/2} \frac{\partial}{\partial \rho_0} \left[\rho_0^{1/2} \frac{\partial \theta_k(\rho_0; \bar{\mathbf{q}})}{\partial k} \right] = C_k(\rho_0; \bar{\mathbf{q}}) - C_k(\rho_0; \mathbf{0}). \tag{154}$$

Denoting $X_k(\rho_0; \bar{\mathbf{q}}) = \rho_0^{1/2} \partial\theta_k(\rho_0; \bar{\mathbf{q}})/\partial k$, this equation becomes

$$\frac{\partial}{\partial \rho_0} X_k(\rho_0; \bar{\mathbf{q}}) = \frac{1}{2} \rho_0^{-3/2} [C_k(\rho_0; \bar{\mathbf{q}}) - C_k(\rho_0; \mathbf{0})]. \tag{155}$$

The solution is found by integrating (155) with the initial condition $X_k(0; \mathbf{q}) = 0$. The latter condition is consistent with the fact that $\partial\theta_k(0; \bar{\mathbf{q}})/\partial k$ is finite for all k. Thus, we obtain

$$X_k(\rho_0; \bar{\mathbf{q}}) = \frac{1}{2} \int_0^{\rho_0} \rho_1^{-3/2} \left(C_k(\rho_1, \bar{\mathbf{q}}) - C_k(\rho_1, \mathbf{0}) \right) \mathrm{d}\rho_1 = \rho_0^{1/2} \frac{\partial \theta_k(\rho_0; \bar{\mathbf{q}})}{\partial k}. \tag{156}$$

The condition $X_k(0; \bar{\mathbf{q}}) = 0$ is satisfied, since $C_k(\rho; \bar{\mathbf{q}}) \propto \rho$ holds at $\rho \to 0$, so that the value of the integral in (156) tends to zero at $\rho_0 \to 0$. From (156) we obtain the RG flow equation for $\theta_k(\rho, \mathbf{q})$, given by (35) and (36). Note only that we have re-denoted ρ_0 by ρ, $\bar{\mathbf{q}}$ by $\mathbf{q}$ and the integration variable $\mathbf{q}$ by $\mathbf{q}'$ to obtain the final form of these equations.

Appendix C. Inversion of the $\Gamma_k^{(2)}$ matrix of a special form

We consider a sparse matrix with elements given either by (30) or (31). According to matrix algebra (Cramer's rule), the diagonal elements of the inverse matrix are $A^{-1}(\mathbf{q}, \mathbf{q}) = D_{\mathbf{q}}/D$, where D is the determinant of matrix A, and $D_{\mathbf{q}}$ is the $(\mathbf{q}, \mathbf{q})$-minor of matrix A—the determinant of a reduced matrix, obtained by deleting from A the row and column associated with $\mathbf{q}$. One of the terms in D is the product of diagonal elements. Other terms are obtained by choosing appropriate off-diagonal elements instead of some diagonal ones, taking into account also the sign, in accordance with the definition of the determinant. Let us first skip the elements with $\delta_{\mathbf{q}', q\pm 2\bar{q}}$ in our sparse matrix. Then we can choose either $A(\mathbf{q}, \mathbf{q}+\bar{\mathbf{q}})$ or $A(\mathbf{q}, \mathbf{q}-\bar{\mathbf{q}})$ instead of $A(\mathbf{q}, \mathbf{q})$. In this sense, we will say that $A(\mathbf{q}, \mathbf{q})$ is replaced by $A(\mathbf{q}, \mathbf{q}+\bar{\mathbf{q}})$ or by $A(\mathbf{q}, \mathbf{q}-\bar{\mathbf{q}})$, respectively. If $A(\mathbf{q}, \mathbf{q})$ is replaced by $A(\mathbf{q}, \mathbf{q}+\bar{\mathbf{q}})$, then the diagonal element $A(\mathbf{q}+\bar{\mathbf{q}}, \mathbf{q}+\bar{\mathbf{q}})$ appears to be in the same column as the newly chosen off-diagonal element $A(\mathbf{q}, \mathbf{q}+\bar{\mathbf{q}})$ and therefore must be replaced either by $A(\mathbf{q}+\bar{\mathbf{q}}, \mathbf{q}+2\bar{\mathbf{q}})$ or by $A(\mathbf{q}+\bar{\mathbf{q}}, \mathbf{q})$, and so on. A minimal replacement series is when $A(\mathbf{q}, \mathbf{q})$ and $A(\mathbf{q}+\bar{\mathbf{q}}, \mathbf{q}+\bar{\mathbf{q}})$ are replaced by $A(\mathbf{q}, \mathbf{q}+\bar{\mathbf{q}})$ and $A(\mathbf{q}+\bar{\mathbf{q}}, \mathbf{q})$. Algebraically, it means that $A(\mathbf{q}, \mathbf{q})A(\mathbf{q}+\bar{\mathbf{q}}, \mathbf{q}+\bar{\mathbf{q}})$ is replaced by $-A(\mathbf{q}, \mathbf{q}+\bar{\mathbf{q}})A(\mathbf{q}+\bar{\mathbf{q}}, \mathbf{q}) = -|A(\mathbf{q}, \mathbf{q}+\bar{\mathbf{q}})|^2$. Let us Ω be a subset of elements $A(\mathbf{q}+m\bar{\mathbf{q}}, \mathbf{q}+m\bar{\mathbf{q}})$, $A(\mathbf{q}+m_1\bar{\mathbf{q}}, \mathbf{q}+m_1\bar{\mathbf{q}}+\bar{\mathbf{q}})$ and $A(\mathbf{q}+m_2\bar{\mathbf{q}}, \mathbf{q}+m_2\bar{\mathbf{q}}-\bar{\mathbf{q}})$ with any integer m, m_1 and m_2, for which the magnitudes of wave vectors are within $[0, \Lambda]$. There are infinitely many such separate subsets Ω_j, labeled by index j, in the thermodynamic limit $V \to \infty$. However, the number of elements in each of them is always finite at finite values of $|\bar{\mathbf{q}}|$ and Λ. Obviously, the replacements in different subsets are independent of each other. Hence, we have $D = \prod_j D_j$, where D_j is the determinant of the matrix A_j composed of the elements of subset Ω_j. Let us $A(\mathbf{q}, \mathbf{q}) \in \Omega_{j^*}$ holds for a given $\mathbf{q}$. The above property ensures that $A^{-1}(\mathbf{q}, \mathbf{q}) = D^*_{\mathbf{q}}/D^*$ holds, where D^* is the determinant of matrix A_{j^*} and $D^*_{\mathbf{q}}$ is its $(\mathbf{q}, \mathbf{q})$-minor.

Our aim is to evaluate $A^{-1}(\mathbf{q}, \mathbf{q})$ for $\bar{\mathbf{q}} \neq \mathbf{0}$, including terms up to $\mathcal{O}\left(|h|^2\right)$ order at $|h| \to 0$. In this case, we need to consider only the contribution of diagonal elements and sum up it with small corrections of order $\mathcal{O}\left(|h|^2\right)$, provided by the above mentioned minimal replacements in the calculation of D^*. Since the number of replacements is finite, the sum of these corrections is also a finite quantity of order $\mathcal{O}\left(|h|^2\right)$. Furthermore, the calculation of $D^*_{\mathbf{q}}$ differs only in that the diagonal element $A(\mathbf{q}, \mathbf{q})$ together with two related to it minimal replacements are cancelated. The terms with $\delta_{\mathbf{q}', q\pm 2\bar{q}}$ can be included in this scheme too. However, this gives only a correction of order $\mathcal{O}\left(|h|^4\right)$. Following the above described method, a straightforward calculation leads to (32) for the diagonal elements of the inverse matrix in the cases, where the original matrix is given by (30) or (31) at $\bar{\mathbf{q}} \neq \mathbf{0}$.

This result can be related to inversions of certain 2×2 matrices. Namely, if only the term with $\mathbf{q}' = \mathbf{q} + \bar{\mathbf{q}}$ is retained in (32) and $|\mathbf{q}'| < \Lambda$ holds, then it corresponds to the inversion of the matrix $A = \begin{pmatrix} A(\mathbf{q}, \mathbf{q}) & A(\mathbf{q}, \mathbf{q} + \bar{\mathbf{q}}) \\ A(\mathbf{q} + \bar{\mathbf{q}}, \mathbf{q}) & A(\mathbf{q} + \bar{\mathbf{q}}, \mathbf{q} + \bar{\mathbf{q}}) \end{pmatrix}$ and calculation of $A^{-1}(\mathbf{q}, \mathbf{q})$. Similarly, if only the term $\mathbf{q}' = \mathbf{q} - \bar{\mathbf{q}}$ with $|\mathbf{q}'| < \Lambda$ is retained, then equation (32) gives $A^{-1}(\mathbf{q}, \mathbf{q})$ for $A = \begin{pmatrix} A(\mathbf{q} - \bar{\mathbf{q}}, \mathbf{q} - \bar{\mathbf{q}}) & A(\mathbf{q} - \bar{\mathbf{q}}, \mathbf{q}) \\ A(\mathbf{q}, \mathbf{q} - \bar{\mathbf{q}}) & A(\mathbf{q}, \mathbf{q}) \end{pmatrix}$. The off-diagonal contribution in (32) is obtained by summing up the off-diagonal contributions to $A^{-1}(\mathbf{q}, \mathbf{q})$ in these two cases at $h \to 0$.

Appendix D. Uniqueness of the representation

Here we discuss the issue concerning the non-uniqueness of the representation (33). Within the derivative expansion, at least, the integration by parts allows us to include the term with $\theta_k^{(1)}$ into the $\theta_k^{(2)}$-term. However, we find it more convenient to keep all $\theta_k^{(j)}$ terms within $j \in [1, m]$, the uniqueness being reached by requiring continuity of the functions $\theta_k^{(j)}$. In particular, the subsum with $\mathbf{q}_2 = -\mathbf{q}_1$ in $\sum_{\mathbf{q}_1, \mathbf{q}_2}$ with the term $\theta_k^{(2)}$ gives a special contribution, which is of order $\mathcal{O}\left(h^2\right)$ at $h \to 0$, the remaining terms providing a contribution of order $\mathcal{O}\left(h^4\right)$, if evaluated at $\phi(\mathbf{x}) = \phi_0\left(1 + h \cos(\bar{\mathbf{q}}\mathbf{x})\right)$. According to this, it is meaningful to represent $\theta_k^{(2)}$ as

$$\theta_k^{(2)}(\rho; \mathbf{q}_1, \mathbf{q}_2) = \delta_{\mathbf{q}_1, -\mathbf{q}_2} e_k(\rho; \mathbf{q}_1) + \left(1 - \delta_{\mathbf{q}_1, -\mathbf{q}_2}\right) g_k(\rho; \mathbf{q}_1, \mathbf{q}_2). \tag{157}$$

Let us denote by $(\partial \Gamma_k/\partial k)_1$ the $\bar{\mathbf{q}}$-dependent part of $\partial \Gamma_k/\partial k$ on the left-hand side of the Wetterich equation. Keeping only the $\mathcal{O}\left(h^2\right)$ terms at the $m = 2$ order of our truncation scheme and using (157) to evaluate $(\partial \Gamma_k/\partial k)_1$ at $\phi(\mathbf{x}) = \phi_0\left(1 + h \cos(\bar{\mathbf{q}}\mathbf{x})\right)$, we obtain

$$\left(\frac{\partial \Gamma_k}{\partial k}\right)_1 = V\left(\rho_0 h^2 \frac{\partial}{\partial k}\left[\Psi_k(\rho_0; \bar{\mathbf{q}}) + e_k(\rho_0; \bar{\mathbf{q}})\right] + o\left(h^2\right)\right), \tag{158}$$

where $\Psi_k(\rho_0; \bar{\mathbf{q}})$ is defined by (41). The equation (158) is obtained by the same method as (132) in the appendix B. Note that the quantity Ψ_k is used as an independent variable instead of $\theta_k^{(1)}$ in section 9 and in the following sections. According to (158), it is not possible to derive the RG flow equations separately for the quantities Ψ_k and e_k, but only for their sum at the truncation order of $m = 2$.

Hence, there is some ambiguity in the representation of the solution, as it is expected from the already mentioned fact that some terms can be reduced to other terms, using the integration by parts. The ambiguity appears in the e_k values. It means that different representations are possible, where the values of $\theta_k^{(2)}(\rho; \mathbf{q}_1, \mathbf{q}_2)$ at $\mathbf{q}_1 + \mathbf{q}_2 = \mathbf{0}$ are not related to those at very small but positive $|\mathbf{q}_1 + \mathbf{q}_2|$. Therefore, we require the continuity, i.e. $\theta_k^{(2)}(\rho; \mathbf{q}_1, -\mathbf{q}_1) = \lim_{\mathbf{q}_2 \to -\mathbf{q}_1} \theta_k^{(2)}(\rho; \mathbf{q}_1, \mathbf{q}_2)$, and by this also reach the uniqueness, because now $e_k(\rho; \mathbf{q}_1)$ is defined as the $g_k(\rho; \mathbf{q}_1, \mathbf{q}_2)$ value at $\mathbf{q}_2 \to -\mathbf{q}_1$. The above example refers to the approximation of the order $m = 2$. A similar treatment applies to any higher-order approximation.

Appendix E. Consistency of terms in the Wetterich equation

In our method, the RG flow equations are obtained by requiring the consistency of terms on both sides of the Wetterich equation, evaluated at $\phi(\mathbf{x}) = \phi_0 \left(1 + \sum_{j=1}^{m} h_j \ \cos(\bar{\mathbf{q}}_j \mathbf{x})\right)$. It means that we equate the terms having the same powers of h_j and the same wave-vector dependence in the sense of an analytic continuation from small-q domain. In the latter case, the wave-vector dependence can be represented by an expansion in powers of wave-vector components. In order to obtain the equations for $\theta_k^{(m)}\left(\rho; \mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_m\right)$, we need to identify the corresponding terms on both sides of the Wetterich equation. For example, in the first approximation $m = 1$, we obtain a contribution of the form $h^2 Q_k(\rho_0, \bar{\mathbf{q}})$ on the right hand side of the equation by evaluating it at $\phi(\mathbf{x}) = \phi_0 \left(1 + h \ \cos(\bar{\mathbf{q}}\mathbf{x})\right)$ and expanding $\rho(\mathbf{x})$ around $\rho_0 = \phi_0^2/2$. This contribution $h^2 Q_k(\rho_0, \bar{\mathbf{q}})$ can be expanded in powers of $\bar{q}^2$ in the small-$\bar{q}$ limit, and it includes also the zeroth power. On the other hand, $\theta_k^{(1)}(\rho_0; \bar{\mathbf{q}})$ contains only positive powers of $\bar{q}^2$, since it vanishes at $\bar{\mathbf{q}} = \mathbf{0}$. Therefore, we need to subtract the constant or zeroth-power contribution from $h^2 Q_k(\rho_0, \bar{\mathbf{q}})$ to obtain a term

$$h^2 Q_k^*(\rho_0, \bar{\mathbf{q}}) = h^2 \left[Q_k(\rho_0, \bar{\mathbf{q}}) - Q_k(\rho_0, \mathbf{0})\right] \tag{159}$$

entering the equation for $\theta_k^{(1)}(\rho_0; \bar{\mathbf{q}})$. All terms in this equation are proportional to h^2 in the asymptotic case of $h \to 0$, so that the common factor h^2 is finally removed.

Similar, but more complicated procedure applies to a contribution of the form $h_1^2 h_2^2 Q_k(\rho_0, \bar{\mathbf{q}}_1, \bar{\mathbf{q}}_2)$, which appears at the $m = 2$ order of our truncation scheme when the Wetterich equation is evaluated at $\phi(\mathbf{x}) = \phi_0 \left(1 + h_1 \ \cos(\bar{\mathbf{q}}_1 \mathbf{x}) + h_2 \ \cos(\bar{\mathbf{q}}_2 \mathbf{x})\right)$. In this case, we single out a term $h_1^2 h_2^2 Q_k^*(\rho_0, \bar{\mathbf{q}}_1, \bar{\mathbf{q}}_2)$ in the equation for $\theta_k^{(2)}(\rho_0; \bar{\mathbf{q}}_1, \bar{\mathbf{q}}_2)$, applying the following subtraction procedure:

$$\begin{aligned} &h_1^2 h_2^2 Q_k^*(\rho_0, \bar{\mathbf{q}}_1, \bar{\mathbf{q}}_2) \\ &\quad = h_1^2 h_2^2 \left[Q_k(\rho_0, \bar{\mathbf{q}}_1, \bar{\mathbf{q}}_2) - \widetilde{Q}_k(\rho_0, \mathbf{0}, \bar{\mathbf{q}}_2) - \widetilde{Q}_k(\rho_0, \bar{\mathbf{q}}_1, \mathbf{0}) - Q_k(\rho_0, \mathbf{0}, \mathbf{0})\right], \end{aligned} \tag{160}$$

where $\widetilde{Q}_k(\rho_0, \mathbf{0}, \bar{\mathbf{q}}_2) = Q_k(\rho_0, \mathbf{0}, \bar{\mathbf{q}}_2) - Q_k(\rho_0, \mathbf{0}, \mathbf{0})$ and $\widetilde{Q}_k(\rho_0, \bar{\mathbf{q}}_1, \mathbf{0}) = Q_k(\rho_0, \bar{\mathbf{q}}_1, \mathbf{0}) - Q_k(\rho_0, \mathbf{0}, \mathbf{0})$. Such a subtraction is necessary, since $\theta_k^{(2)}(\rho_0; \bar{\mathbf{q}}_1, \bar{\mathbf{q}}_2)$ contains no zeroth power of any of the wave-vector components in agreement with the fact that $\theta_k^{(2)}(\rho_0; \mathbf{0}, \bar{\mathbf{q}}_2) = \theta_k^{(2)}(\rho_0; \bar{\mathbf{q}}_1, \mathbf{0}) = \theta_k^{(2)}(\rho_0; \mathbf{0}, \mathbf{0}) = 0$ holds. In these expressions, quantities marked by tilde include terms with zeroth power of any of $\bar{\mathbf{q}}_1$ component and only positive powers of $\bar{\mathbf{q}}_2$ components, or vice versa. The subtraction formulas (159) and (160) are valid for arbitrary wave vectors in the sense of an analytic continuation from small-q domain. Such formulas can be found at any approximation order m.

We are looking for the asymptotic consistency at $h_j \to 0$. It means that, among terms with similar dependence on wave vectors, we keep only those ones which have the leading order of h. Formally, we have an expansion in powers of h_j. However, the final RG flow equations are obtained by equating the terms with the same powers of h_j, therefore they do not contain h_j anymore. Since the Wetterich equation is an exact equation, we can evaluate it at any field configuration $\phi(\mathbf{x})$ and extract an exact information from this. We have chosen $\phi(\mathbf{x}) = \phi_0 \left(1 + \sum_{j=1}^{m} h_j \cos(\bar{\mathbf{q}}_j \mathbf{x})\right)$ at $h_j \to 0$. Since the obtained in this way RG flow equations do not contain h_j, these are obviously true for any h_j.

Appendix F. Details of numerical calculations in the iterative scheme of section 11.2

For the numerical calculations of the integrals, we have set a finite, but large enough upper integration limit $y_{\max}$ for y and y_1. The functions $f_{n,k}(y)$ and $\mathcal{F}_k(y)$ have been calculated on a grid of y values within $y \in [0, Y_{\max}]$, using the quadratic Newton's interpolation polynomial for evaluation of these functions between the grid points. This polynomial has been also used to obtain the $\mathcal{F}_k(0)$ value by extrapolation. Note that $Y = y + y_1 + 2\sqrt{yy_1}\ \cos\ \theta$ belongs to the interval $[0, 4Y_{\max}]$ if $y \in [0, Y_{\max}]$ and $y_1 \in [0, Y_{\max}]$, therefore we need an appropriate extrapolation of $f_{n,k}(Y)$ to $Y > Y_{\max}$. We have used the power-law asymptotic $f_{0,k}(y) \propto y^{-\eta/2}$ and $f_{1,k}(y) \propto y^{-a/2}$ at $y \to \infty$, given by (80) and (78), for this purpose. The grid has been used, where, starting with a small first step $(\Delta y)_1 = 0.01$, each next step is by a factor of $1 + \varepsilon$ (with $\varepsilon = 0.2$) larger than the previous one, continuing up to $y = Y_{\max} = 10^5$, or $y = Y_{\max} = 10^{10}$ in some calculations. The integration over y has been performed by Simpson formula, setting $y_{\max} = 25$ and using a non-uniform grid, where the step size increases like described above (with $(\Delta y)_1 = 0.001$ and $\varepsilon = 0.2$), but not exceeding certain maximal value (0.4 in our case). The integration over θ has been performed by the Simpson formula with the step of $\pi/10$. The numerical errors are surprisingly small for such not really small step sizes. For example, the error for η at $\alpha = 0.25$ is about 7×10^{-5}, as we have verified it by refined calculations. This error becomes even much smaller for larger α values.

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References

[1] Amit D J 1984 *Field Theory, The Renormalization Group, and Critical Phenomena* (Singapore: World Scientific)
[2] Sornette D 2000 *Critical Phenomena in Natural Sciences* (Berlin: Springer)
[3] Wilson K G and Fisher M E 1972 *Phys. Rev. Lett.* **28** 240
[4] Ma S-K 1976 *Modern Theory of Critical Phenomena* (New York: Benjamin)
[5] Zinn-Justin J 1996 *Quantum Field Theory and Critical Phenomena* (Oxford: Clarendon)
[6] Kleinert H and Schulte-Frohlinde V 2001 *Critical Properties of ϕ^4 Theories* (Singapore: World Scientific)
[7] Kaupužs J 2012 *Int. J. Mod. Phys.* A **27** 1250114
[8] Castellana M and Parisi G 2015 *Sci. Rep.* **5** 12367
[9] Kaupužs J and Melnik R V N 2018 Non-perturbative approaches in nanoscience and corrections to finite-size scaling *Coupled Mathematical Models for Physical and Biological Nanoscale Systems*

and Their Applications (*Springer Proceedings in Mathematics & Statistics* vol 232) (Berlin: Springer) pp 65pp –73
[10] Wegner F and Houghton A 1973 *Phys. Rev.* A **8** 401
[11] Polchinski J 1984 *Nucl. Phys.* B **231** 269
[12] Bagnuls C and Bervillier C 2001 *Phys. Rep.* **348** 91
[13] Helias M 2020 arXiv:2004.00924v1 [cond-mat.stat-mech]
[14] Kaupužs J 2009 *Int. J. Mod. Phys.* B **23** 5935
[15] Wetterich C 1993 *Phys. Lett.* B **301** 90
[16] Berges J, Tetradis N and Wetterich C 2002 *Phys. Rep.* **363** 223
[17] Riebesell J 2018 Functional renormalization analytically continued *Master Thesis* ITP Heidelberg (arXiv:1712.09863v2 [hep-th])
[18] Papenbrock T and Wetterich C 1995 *Z. Phys.* C **65** 519
[19] Litim D 2001 *Phys. Rev.* D **64** 105007
[20] Canet L, Delamotte B, Mouhanna D and Vidal J 2002 *Phys. Rev.* D **67** 065004
[21] Canet L, Delamotte B, Mouhanna D and Vidal J 2003 *Phys. Rev.* B **68** 064421
[22] Berges J, Jungnickel D-U and Wetterich C 1999 *Phys. Rev.* D **59** 034010
[23] Schütz F and Kopietz P 2006 *J. Phys. A: Math. Gen.* **39** 8205
[24] Benedetty D, Groh K, Machado P F and Saueressig F 2011 *J. High Energy Phys.* JHEP06(2011)079
[25] Demmel M, Saueressig F and Zanusso O 2014 *J. High Energy Phys.* JHEP06(2014)026
[26] Bender C M and Sarkar S 2018 *J. Phys. A: Math. Theor.* **51** 225202
[27] Wetterich C and Yamada M 2019 *Phys. Rev.* D **100** 066017
[28] Platania A B 2018 *Asymptotically Safe Gravity: From Spacetime Foliation to Cosmology* (*Springer Theses-Recognizing Outstanding PhD Research*) (Berlin: Springer) pp 29pp –46
[29] Wetterich C 2017 *Phys. Rev.* D **95** 123525
[30] Eichhorn A 2019 *Frontiers Astron. Space Sci.* **5** 47
[31] Alwis S P 2018 *J. High Energy Phys.* JHEP03(2018)118
[32] Litim D F and Trott M J 2018 *Phys. Rev.* D **98** 125006
[33] Bond A D and Litim D F 2019 *Phys. Rev. Lett.* **122** 211601
[34] Falls K G, Litim D F and Schroder J 2019 *Phys. Rev.* D **99** 126015
[35] Lahoche V and Samary D O 2019 *Universe* **5** 86