APPLIED STOCHASTIC MODELS IN BUSINESS AND INDUSTRY
Appl. Stochastic Models Bus. Ind. 2009; **25**:565–582
Published online 19 December 2008 in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/asmb.745

First passage time for multivariate jump-diffusion processes in finance and other areas of applications

Di Zhang and Roderick V. N. Melnik*, †

M²NeT Lab, Wilfrid Laurier University, Waterloo, Ont., Canada N2L 3C5

SUMMARY

The first passage time (FPT) problem is an important problem with a wide range of applications in science, engineering, economics, and industry. Mathematically, such a problem can be reduced to estimating the probability of a stochastic process first to reach a boundary level. In most important applications in the financial industry, the FPT problem does not have an analytical solution and the development of efficient numerical methods becomes the only practical avenue for its solution. Most of our examples in this contribution are centered around the evaluation of default correlations in credit risk analysis, where we are concerned with the joint defaults of several correlated firms, the task that is reducible to a FPT problem. This task represents a great challenge for jump-diffusion processes (JDP). In this contribution, we develop further our previous fast Monte Carlo method in the case of multivariate (and correlated) JDP. This generalization allows us, among other things, to evaluate the default events of several correlated assets based on a set of empirical data. The developed technique is an efficient tool for a number of financial, economic, and business applications, such as credit analysis, barrier option pricing, macroeconomic dynamics, and the evaluation of risk, as well as for a number of other areas of applications in science and engineering, where the FPT problem arises. Copyright © 2008 John Wiley & Sons, Ltd.

Received 27 December 2007; Revised 11 September 2008; Accepted 12 September 2008

KEY WORDS: first passage time problems; stochastic differential equations; jump-diffusion processes; modified Monte Carlo algorithms; default correlations; multiscale problems; Brownian bridge simulations; financial mathematics; mathematical models in industry; interdisciplinary approaches

*Correspondence to: Roderick V. N. Melnik, M²NeT Lab, Wilfrid Laurier University, Waterloo, Ont., Canada N2L 3C5.
†E-mail: rmelnik@wlu.ca

Contract/grant sponsor: NSERC
Contract/grant sponsor: CRC

1. INTRODUCTION

Many applications in science, engineering, finance, economics, and industry require the solution of the first passage problem. Examples range from population genetics and other applications in biology, to neurobiology and neurophysiology, to molecule kinetics and problems in chemistry and physics, to rare events modelling and quantum information, and to aerospace engineering applications (e.g. [1–4]). This problem also arises in financial applications. Although methodologies developed in this paper are applicable to a number of other areas, we exemplify our discussion here with problems from finance.

In the financial world, individual companies are usually linked together via economic conditions, so default correlation, defined as the risk of multiple companies' default together, has been an important area of research in credit analysis with applications to joint defaults, credit derivatives, asset pricing, and risk management.

Currently, there are two dominant groups of theoretical models used in default correlation. One is a reduced form model, such as in [5] that uses a Copula function to parameterize the default correlation. In this context, it is worthwhile noting that Chen and Sopranzetti [6] have translated the joint default probability into a bivariate normal probability function, making the analysis and applications of such models potentially more convenient.

The second group of models for default correlation is a structural form model. Zhou [7] and Hull and White [8] were the first to incorporate default correlation into the Black–Cox first passage structural model. They obtained similar closed-form solutions for two assets. However, their models cannot easily be extended to more than two assets. Furthermore, the developed models do not include jump-diffusion processes (JDP).

Over the recent years, substantial progress has been made in the development of models from both of these groups (see, e.g. [9–12] and references therein) with improved performance of such traditional methodologies such as convolution, transform, and sampling techniques. Nevertheless, the development of efficient computational tools for modeling default correlations in multivariate JDP is lagging behind its practical needs.

As demonstrated in [13], jump risk becomes an important factor in credit risk analysis. It is now widely acknowledged that the standard Brownian motion model for market behavior falls short of explaining empirical observations of market returns and their underlying derivative prices [14]. Among other techniques, multivariate JDP provide a powerful tool for the evaluation of risk in many industrial, business, economic, and financial applications [15–21]. In particular, since the multivariate jump-diffusion model provides a convenient framework for investigating default correlation with jumps, it becomes more readily accepted in the financial world.

One of the major problems in default analysis is to determine when a default will occur within a given time horizon, or in other words, what the default rate is during such a time horizon. This problem is reduced to a first passage time (FPT) problem that can be formalized on the basis of certain stochastic differential equations. It concerns the estimation of the probability density of the time for a random process to cross a specified threshold level. Unfortunately, after including jumps, only special cases have analytical solutions. For most practical cases, closed-form solutions are unavailable and we can only turn to the numerical procedures. Monte Carlo procedures and Markov chain-based approximations have become some of the major tools in addressing complex problems involving uncertainties in a number of application areas [22–26]. Many such problems are multiscale in nature [27–35]. In a number of practically important cases, such multiscale problems require the development of mathematical models that go beyond the description of

DOI: 10.1002/asmb

standard diffusion processes [25, 36, 37]. In addition to problems in science and engineering, such problems also arise in financial applications, in particular in those cases when we have to deal with stochastic processes involving jumps. Monte Carlo simulation is one of the candidates for dealing with such problems, and in what follows we focus on the development of Monte Carlo procedures for solving stochastic differential equations (SDEs) arising in the context of FPT problems. In conventional Monte Carlo (CMC) methods, we need to discretize the time horizon into small enough intervals in order to avoid discretization bias [38–40], and we need to evaluate the processes at each discretized time, which is very time-consuming. Many researchers have contributed to the field and enhanced the efficiency of Monte Carlo simulation. Atiya and Metwally [41, 42] have developed a fast Monte Carlo-type numerical method to solve the FPT problem. Recently, we have reported an extension of this fast Monte Carlo-type method in the context of multiple JDP [20].

In this contribution, we develop further the methodology for solving the FPT problem in the context of multivariate JDP, analyze its efficiency numerically, and provide a number of practical examples from financial applications. The paper is organized as follows. Section 2 provides details of our model followed by a brief description of our computational implementation in Section 3. Section 4 contains the simulation results and discussions, followed by the conclusions given in Section 5.

2. DEVELOPING MULTI-DIMENSIONAL MODELS WITH JUMPS

Mathematical modelling provides a powerful tool in solving a wide range of problems in industrial applications [43–60] and interdisciplinary approaches play an indispensable role in formulating state-of-the-art mathematical models in such applications [61]. For some of these applications it is often the case that a degree of uncertainty about the corresponding system, process, or phenomenon under the study needs to be incorporated into the model. In what follows we exemplify this situation by developing a multivariate model for the analysis of default correlations in financial applications.

2.1. Asset dependence and jump processes

In the market economy, almost all financial applications require a multivariate model with dependence between different assets. In most of these applications, jumps in the price process must be taken into account. A simple method to introduce jumps into a multidimensional model is to suppose that the stock prices do follow a multidimensional Brownian motion such, for example, that this motion has alternating moments and the time intervals between consecutive changes of these moments are described by a known process [62, 63]. This idea has been used in [63] for simulations of FPT. Another natural choice to introduce jumps into a d-dimensional model is to utilize the compound Poisson shocks N_t [62]:

$$\begin{aligned} X_i(t) &= \mu_i t + B_i(t) + Z_i(t), \quad i = 1, 2, \ldots, d \\ Z_i(t) &= \sum_{j=1}^{N_t} Y_{ij} \end{aligned} \tag{1}$$

DOI: 10.1002/asmb

where μ_i are drift term components, $B(t)$ is a d-dimensional Brownian motion with covariance matrix $\sigma=(\sigma_{ij})$ which can be written as

$$B_i(t)=\sum_{j=1}^{d}\sigma_{ij}W_j(t)$$

and $W_j(t)$ is the standard Brownian motion. For the ith process, $\{Y_{ij}\}_{j=1}^{\infty}$ are i.i.d. d-dimensional random vectors that determine the sizes of jumps in each individual process. At the jth shock, the jump-sizes of different processes Y_{ij} may be correlated. Many problems in the financial industry involving JDP can often be reduced to a problem of seeking optimal (or sub-optimal) control policies [64].

2.2. *Default correlations*

Owing to their asset dependence in the market economy, individual companies are inevitably linked together via dynamically changing economic conditions. Take two firms A and B as an example, whose probabilities of default are P_A and P_B, respectively. Then the default correlation can be defined as

$$\rho_{AB}=\frac{P_{AB}-P_AP_B}{\sqrt{P_A(1-P_A)P_B(1-P_B)}} \tag{2}$$

where P_{AB} is the probability of joint default.

Now, we can write P_{AB} as $P_{AB}=P_AP_B+\rho_{AB}\sqrt{P_A(1-P_A)P_B(1-P_B)}$. If we assume $P_A=P_B=p\ll 1$, then we have $P_{AB}\approx p^2+\rho_{AB}p\approx\rho_{AB}p$. Thus, it is apparent that the default correlation ρ_{AB} plays a key role in the joint default with important implications in the field of credit analysis. As already mentioned, Zhou [7], and Hull and White [8] were the first to incorporate default correlation into the Black–Cox first passage structural model. In uncertain environments where a system control is needed, the FPT problem is ubiquitous and many researchers have applied FPT models in different areas of science and engineering [65–69]. Zhou [7] proposed a FPT model to describe default correlations of two firms under the 'bivariate diffusion process'. In particular, if we denote constant drift terms by μ_1 and μ_2, and two independent standard Brownian motions by z_1 and z_2, we have the following system for the two firm asset values V_1 and V_2:

$$\begin{bmatrix}\mathrm{d}\ln(V_1)\\ \mathrm{d}\ln(V_2)\end{bmatrix}=\begin{bmatrix}\mu_1\\ \mu_2\end{bmatrix}\mathrm{d}t+\Omega\begin{bmatrix}\mathrm{d}z_1\\ \mathrm{d}z_2\end{bmatrix} \tag{3}$$

where Ω is a constant 2×2 matrix such that

$$\Omega\cdot\Omega'=\begin{bmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2\\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{bmatrix}$$

while the coefficient ρ is responsible for coupling between the two firms in a sense that it reflects the correlation between the movements in the asset values of these firms. Although this approach allows us to deduce the closed-form solution of default correlations of two assets [7], it does not include possible jumps, a feature that has more significant importance in the default correlation than often perceived. Indeed, simultaneous jumps may enhance the chance of simultaneous defaults which increases the correlation defaults.

DOI: 10.1002/asmb

2.3. Multivariate JDP

For the reason indicated above, we have to consider a more general case. In a complete probability space (Ω, F, P) with information filtration (F_t), suppose that $X_t = \ln(V_t)$ is a Markov process in some state space $D \subset \mathbb{R}^n$, solving the stochastic differential equation [70]

$$\mathrm{d}X_t = \mu(X_t, t)\,\mathrm{d}t + \sigma(X_t, t)\,\mathrm{d}W_t + \mathrm{d}Z_t \tag{4}$$

where W is an (F_t)-standard Brownian motion in $\mathbb{R}^n$; $\mu: D \to \mathbb{R}^n$, $\sigma: D \to \mathbb{R}^{n\times n}$, and Z is a pure jump process whose jumps have a fixed probability distribution v on $\mathbb{R}^n$ such that they arrive with intensity $\{\lambda(X_t): t \geqslant 0\}$, for some $\lambda: D \to [0, \infty)$. Then, based on the theory developed in [70], we can reduce the original model to an affine model. In particular, if

$$\begin{aligned} \mu(X_t, t) &= K_0 + K_1 X_t \\ (\sigma(X_t, t)\sigma(X_t, t)^\top)_{ij} &= (H_0)_{ij} + (H_1)_{ij} X_j \\ \lambda(X_t) &= l_0 + l_1 \cdot X_t \end{aligned} \tag{5}$$

where $K = (K_0, K_1) \in \mathbb{R}^n \times \mathbb{R}^{n\times n}$, $H = (H_0, H_1) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n\times n}$, $l = (l_0, l_1) \in \mathbb{R}^n \times \mathbb{R}^{n\times n}$, and assuming that

- the process W_t in Equation (4) has independent components,
- $K_1 = 0$, $H_1 = 0$ and $l_1 = 0$ in (4) that is the drift term, the diffusion matrix and the arrival intensity are independent of the state vector X_t,
- the jump process Z_t is also independent of X_t,

we can rewrite Equation (4) as a time-homogeneous stochastic differential equation:

$$\mathrm{d}X_t = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t + \mathrm{d}Z_t, \quad \mu = K_0, \quad \sigma\sigma^\top = H_0, \quad \lambda = l_0 \tag{6}$$

At first sight, Equation (6) is similar to Equation (3), but Equation (6) describes a much more general model that can be applied to multiple firms, and where jumps have been taken into account.

2.4. First passage time distribution and kernel estimator

Let us consider a firm $i \in \{1, 2, \ldots, n\}$, as described by Equation (6), such that its state vector X_i satisfies the following SDE:

$$\mathrm{d}X_i = \mu_i\,\mathrm{d}t + \sum_j \sigma_{ij}\,\mathrm{d}W_j + \mathrm{d}Z_i = \mu_i\,\mathrm{d}t + \sigma_i\,\mathrm{d}W_i + \mathrm{d}Z_i \tag{7}$$

where W_i is a standard one-dimensional Brownian motion and σ_i is:

$$\sigma_i = \sqrt{\sum_j \sigma_{ij}^2}$$

As in [20], we assume that in the interval $[0, T]$, the total number of jumps for firm i is M_i. Let the jump instants be $T_1, T_2, \ldots, T_{M_i}$. Let $T_0 = 0$ and $T_{M_i+1} = T$. The quantities τ_j equal interjump times, i.e. $T_j - T_{j-1}$. Let $X_i(T_j^-)$ be the process value immediately before the jth jump, and $X_i(T_j^+)$ be the process value immediately after the jth jump. The jump-size is $X_i(T_j^+) - X_i(T_j^-)$ and we can use such jump-sizes to generate $X_i(T_j^+)$ sequentially.

DOI: 10.1002/asmb

In a structural model, a firm defaults when the firm's assets value V_t falls below a threshold level $D_V(t)$. In this contribution, we use an exponential form, defining the threshold level by $D_V(t)=\kappa\exp(\gamma t)$ as proposed in [7], where γ can be interpreted as the growth rate of the firm's liabilities. Coefficient κ, in front, captures the liability structure of the firm, which is usually defined as a firm's short-term liability plus 50% of the firm's long-term liability. If we take $X_t=\ln(V_t)$, as mentioned before, then the threshold of X_t is $D(t)=\gamma t+\ln(\kappa)$.

Atiya and Metwally [41] have deduced a one-dimensional FPT distribution in time horizon $[0,T]$. Following their idea, generalized in [20], and defining the event that the process X_i crosses the threshold level $D_i(t)$ for the first time in the interval $[t,t+\mathrm{d}t]$, we can obtain the conditional interjump first passage density $g_{ij}(t)$ for $j=1,2,\ldots,M_i$. After getting these results in one interval $[T_{j-1},T_j]$, we combine them to obtain the density for the whole interval $[0,T]$ by using the Brownian bridge concept [20]. For each firm, after generating a series of FPT s_i, we use a kernel density estimator with Gaussian kernel [71] to estimate the first passage time density (FPTD) f. Finally, after obtaining the estimated FPTD, $\widehat{f}$, the cumulative default rates can be written as:

$$P_i(t)=\int_0^t \widehat{f_i}(\tau)\,\mathrm{d}\tau \tag{8}$$

3. COMPUTATIONAL IMPLEMENTATION

In Section 2, we have reduced the solution of the original problem to a multivariate jump-diffusion model as described in Equation (7). As we have already noted, once jumps are included in the process, only for very basic applications closed-form solutions are available [14]; thus in most practically interesting cases, we have to resort to the numerical procedures. Such procedures should be computationally efficient for the following reason: If T_{j-1} and T_j are any two successive jump instants, then, even though there is no jump occurring in the interval $[T_{j-1},T_j]$, in the CMC method we need to evaluate X_i at each discretized time t in $[T_{j-1},T_j]$. This is a very time-consuming procedure, motivating to develop computationally more efficient methodologies.

In [41, 42] two modifications of this conventional procedure have recently been proposed that allow us a potential speed up of the conventional methodology by10–30 times. Both methodologies (based on the uniform sampling (UNIF) method and on the inverse Gaussian density sampling) were developed for the univariate case only. In [20] we have extended the Atiya–Metwally procedure to the multivatiate case. Here, we have developed further the UNIF method applied to multivariate JDP in the context of the evaluation of the default rates of several correlated assets based on a set of empirical data. This method is also called in Section 4 as the multivariate uniform sampling method (MUNIF). As before, the major improvement of the UNIF method stems from the fact that it only evaluates X_i at generated jump instants while between each two jumps the process is a Brownian bridge. In this scenario, the algorithm for multivariate J is reduced to the analysis of the following three steps in a way similar to [20]: (a) first passage occurs inside the interval; (b) first passage does not occur in this interval; (c) first passage occurs at the right boundary of the interval. Prior to this, we generate beforejump and postjump values $X_i(T_j^-)$ and $X_i(T_j^+)$, respectively, for the given interval $[T_{j-1},T_j]$, where as before $j=1,\ldots,M_i$. Finally, we increase j and examine the next interval by analyzing the above three cases for each non-default firm again. After running N times the Monte Carlo cycle, we get the FPTD of firm i as $\widehat{f_i}(t)=(1/N)\sum_{n=1}^{N}\widehat{f_{i,n}}(t)$, as well as the cumulative default rates computed according to formula (8).

DOI: 10.1002/asmb

4. APPLICATIONS AND DISCUSSION

In this section, we will show how our model describes the default correlations of the firms, rated in the same way, via studying the historical data. We will provide details on the calibration of the models applied for this description. We start our discussion from two representative examples with different parameters to compare our methodology with the CMC procedure.

4.1. Credit risk analysis in the two-dimensional case

In what follows, we will provide results of numerical simulations based on the MUNIF method applied to the two-dimensional case. The parameters for the simulations have been taken as follows:

$$X_0 = [2,2]^\top, \quad D(t) = [-0.001t, -0.001t]^\top$$

$$\mu = [-0.001, -0.001]^\top, \quad \sigma = \begin{bmatrix} \sigma_1 & 0.0 \\ \rho\sigma_2 & \sqrt{1-\rho^2}\sigma_2 \end{bmatrix}$$

$$\lambda = 0.1, \quad \mu_T = 1.0$$

$$\mu_J = [\mu_{J_1}, \mu_{J_2}]^\top, \quad \sigma_J = [\sigma_{J_1}, \sigma_{J_2}]^\top$$

where X_0 is the starting value for the process, $D(t)$ is the threshold, μ is the constant instantaneous drift, and σ represents the Brownian motion in which ρ reflects the correlation of diffusion parts of the state vectors of the two processes. λ is the intensity of arrival jumps. The mean value of exponential distribution used for interjump times $(T_j - T_{j-1})$ is μ_T. μ_J and σ_J are the mean and standard deviations of the jump-sizes, respectively.

In the financial industry, we are often concerned with the default correlation $\rho_{12}(t)$ of two processes. It is defined according to (2) with $A=1$ and $B=2$:

$$\rho_{12}(t) = \frac{P_{1\cap 2}(t) - P_1(t)P_2(t)}{\sqrt{P_1(t)[1-P_1(t)]P_2(t)[1-P_2(t)]}} \tag{9}$$

where $P_i(t)$ $(i=1,2)$ is the probability that process X_i crosses the boundary level at time t (known as default rate), $P_{1\cap 2}(t)$ is the probability that both processes cross the level by time t. Default correlation analysis requires the evaluation of $\rho_{12}(t)$ and has many applications in credit analysis, asset pricing, and risk management [7]. In what follows, we focus on how to estimate the default correlation of two correlated processes via our simulations.

In order to judge the validity and efficiency of our methodology, we use two representative examples with different parameters. The simulation was carried out with total Monte Carlo runs $N = 500\,000$ in time period $[0, 20]$. Moreover, we have also carried out the CMC simulation with the same parameters and the discretization size of time period $\Delta = 0.0002$.

4.1.1. Same quality processes. First, we consider two same quality processes, i.e. they possess the same standard deviations σ_1 and σ_2 for Brownian motion and the same distributions of jump-sizes,

$$\sigma_1 = \sigma_2 = 0.09$$

$$\mu_J = [\mu_{J_1}, \mu_{J_2}]^\top = [-0.3, -0.3]^\top$$

$$\sigma_J = [\sigma_{J_1}, \sigma_{J_2}]^\top = [0.6, 0.6]^\top$$

DOI: 10.1002/asmb

During the simulations, we have also tested different parameters ρ. Our results are reported here for ρ ranging from -0.4 to 0.7. In Figure 1(a) and (b), we display the simulated default rate (we only show the results of X_1, since process X_1 is identical to that of X_2) and default correlation with $\rho = 0.4$, respectively. We observe that, except for a small underestimation of default correlation, our method leads to practically identical results compared with the method.

The default correlations with different ρ obtained using the CMC method and MUNIF method are shown in Figure 1(c) and (d), respectively. They are practically identical and show that the default correlations tend to increase with the increase of ρ.

In order to analyze convergence properties of our methodology, we define

$$\Delta_i = \sum_t \left| \frac{P_{i,n}(t) - P_{i,n-1}(t)}{P_{i,n-1}(t)} \right| \quad (i = 1, 2)$$
$$\Delta_{12} = \sum_t \left| \frac{P_{1\cap 2,n}(t) - P_{1\cap 2,n-1}(t)}{P_{1\cap 2,n-1}(t)} \right| \tag{10}$$

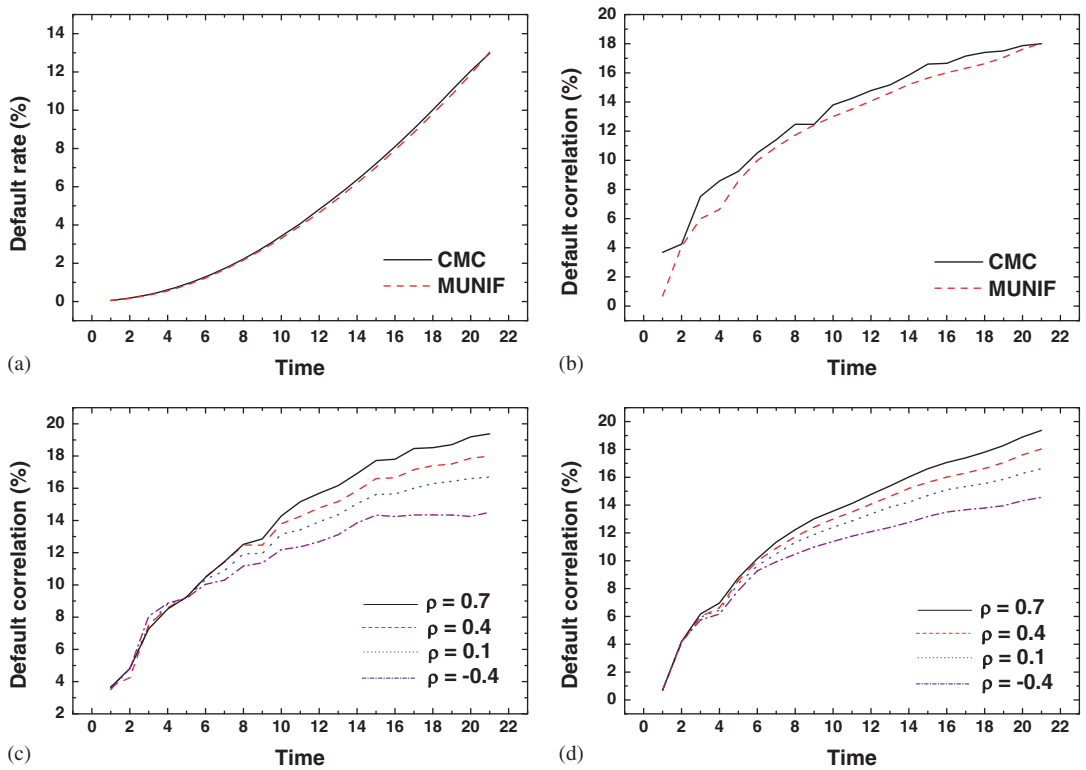

Figure 1. Simulated (a) default rate of X_1 and (b) default correlation with $\rho = 0.4$. The default correlations with different ρ obtained using (c) conventional Monte Carlo method and (d) multivariate uniform sampling (MUNIF) method. For the conventional Monte Carlo method, the simulations were carried out with total Monte Carlo runs $N = 500000$ and discretization step $\Delta = 0.0002$.

DOI: 10.1002/asmb

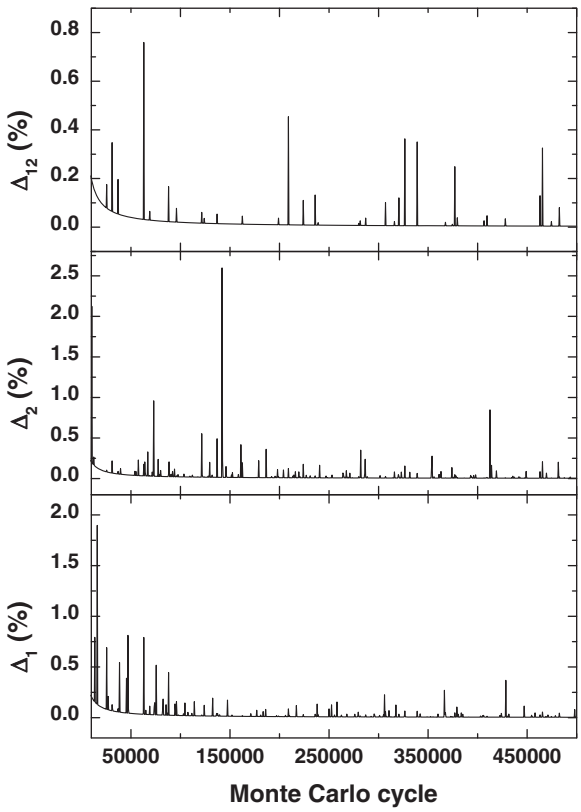

Figure 2. The calculated $\Delta_i\,(i=1,2)$ and Δ_{12} with $\rho=0.4$.

where n is the number of Monte Carlo cycles. $\Delta_i\,(i=1,2)$ and Δ_{12} provide quantitative measures of the convergence. The calculated $\Delta_i\,(i=1,2)$ and Δ_{12} with $\rho=0.4$ are shown in Figure 2. It can be seen that all of them become less than 1% after we carry out 200 000 Monte Carlo runs, so the total Monte Carlo runs $N=500\,000$ is adequate enough for all the reported simulations (Table I).

The calculated optimal bandwidth and the corresponding CPU time are given in Table I. As seen from Table I, the MUNIF approach is much more efficient compared with the conventional methodology.

4.1.2. Different quality processes. Next, we consider two different quality processes. The parameters used in the simulations are,

$$\sigma_1 = 0.09, \sigma_2 = 0.16$$

$$\mu_J = [\mu_{J_1}, \mu_{J_2}]^\top = [-0.3, -0.6]^\top$$

$$\sigma_J = [\sigma_{J_1}, \sigma_{J_2}]^\top = [0.6, 1.6]^\top$$

DOI: 10.1002/asmb

Table I. The optimal bandwidth h_{opt} and CPU time per Monte Carlo run of the simulations.

		Optimal bandwidth		
		X_1	X_2	CPU time
$\rho=-0.4$	CMC	0.730696	0.728936	0.216419
	MUNIF	0.537507	0.537511	0.002011
$\rho=0.1$	CMC	0.730696	0.727525	0.215553
	MUNIF	0.537507	0.537308	0.002208
$\rho=0.4$	CMC	0.730696	0.727186	0.216779
	MUNIF	0.537507	0.537188	0.002330
$\rho=0.7$	CMC	0.730696	0.728070	0.215819
	MUNIF	0.537507	0.536902	0.002388

For conventional Monte Carlo, the simulations were carried out with total Monte Carlo runs $N=500\,000$ and discretization step $\Delta=0.0002$.

We observe that for process X_2, it is easier to cross the boundary level. As before, during the simulations, we have also tested different parameters ρ ranging from -0.4 to 0.7. In Figure 3(a) and (b), we display the simulated default rates of processes X_1 and X_2, and default correlation between them with $\rho=0.4$, respectively. Figure 3(a) also confirms that our methodology gives practically identical results compared with the conventional methodology. As per the simulated default correlation, though our approach underestimates the correlation, yet the difference between them is acceptable in most practical cases.

The default correlations with different ρ obtained using CMC and MUNIF methods are shown in Figure 3(c) and (d), respectively. Similar to the case with same quality processes, with the increase of ρ, the default correlations tend to increase, especially over long time period. The CPU time used during the simulations also confirms the efficiency of the developed methodology. Finally, convergence parameters $\Delta_i (i=1,2)$ and Δ_{12} are shown in Figure 4 with $\rho=0.4$. Again, after 200 000 Monte Carlo runs, $\Delta_i (i=1,2)$ and Δ_{12} become less than 1%.

4.1.3. Discussion. From the above simulations, we can draw several conclusions:

1. The developed methodology is much more efficient compared with the CMC method as demonstrated by a number of representative examples.
2. The developed methodology leads to practically identical results when compared with the CMC method. The deviation arises from the fact that we have used the approximate correlations to generate FPT.
3. Similar to [7], we conclude that the default correlations of same quality processes are usually larger compared with different quality processes. Furthermore, the default correlations tend to increase over long period and may converge to a stable value.
4. The default correlation of two processes is still positive with $\rho=-0.4$, this is because their jump parts are also correlated.
5. The default correlations tend to increase when the parameter ρ increases, especially over long periods. This indicates that default correlations are more sensitive to the correlations of Brownian motion over long period, which should be taken into account in credit analysis.

DOI: 10.1002/asmb

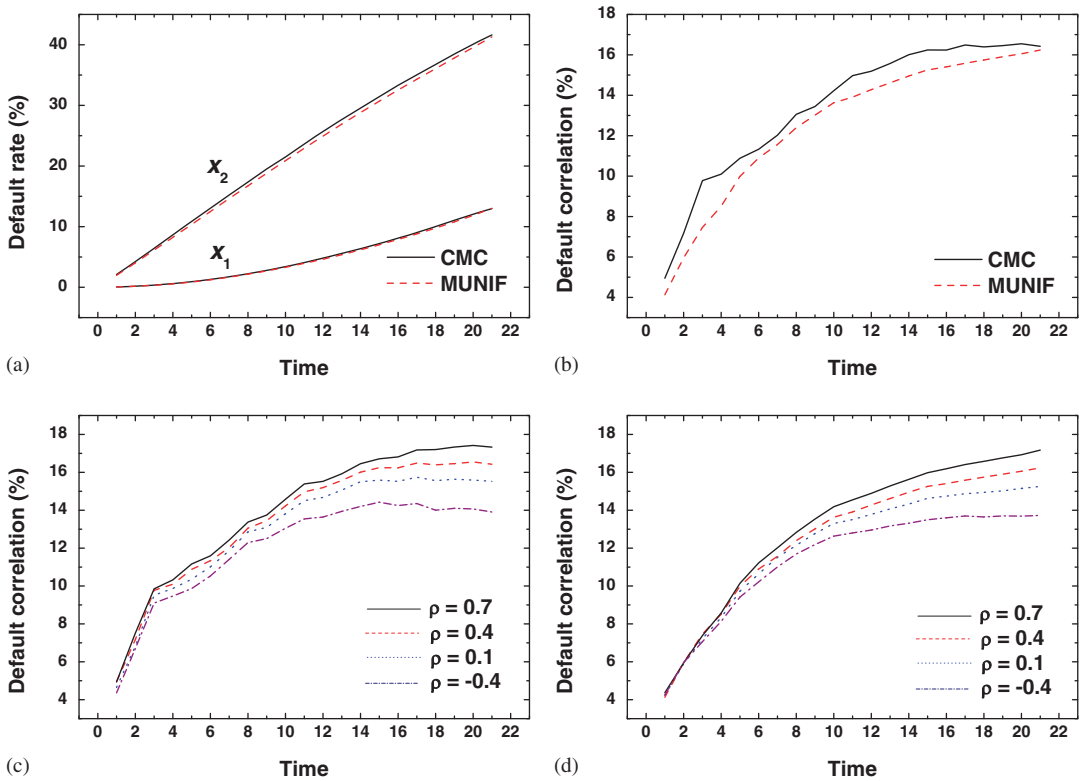

Figure 3. Simulated (a) default rates of X_1 and X_2, (b) default correlation with $\rho = 0.4$. The default correlations with different ρ obtained using (c) conventional Monte Carlo (CMC) method and (d) multivariate uniform sampling (MUNIF) method. For the CMC, the simulations were carried out with total Monte Carlo runs $N = 500\,000$ and discretization step $\Delta = 0.0002$.

4.2. Default rates and historic data

Our next example is based on an available set of historic data. In Figure 5, the black line of square is a set of historical default data of A-rated firm[‡] taken from [7].

First, if we do not consider jumps, as assumed in [7], the firm defaults at time t with probability:

$$P_i(t) = 2 \cdot N\left(-\frac{X_i(0) - \ln(\kappa_i)}{\sigma_i \sqrt{t}}\right) = 2 \cdot N\left(-\frac{Z_i}{\sqrt{t}}\right) \tag{11}$$

where

$$Z_i \equiv \frac{X_i(0) - \ln(\kappa_i)}{\sigma_i}$$

is the standardized distance of firm i to its default point and $N(\cdot)$ denotes the cumulative probability distribution function of a standard normal variable.

[‡]A-rated firm stands for a specific kind of firm following the Moody's Investors Service's definition.

DOI: 10.1002/asmb

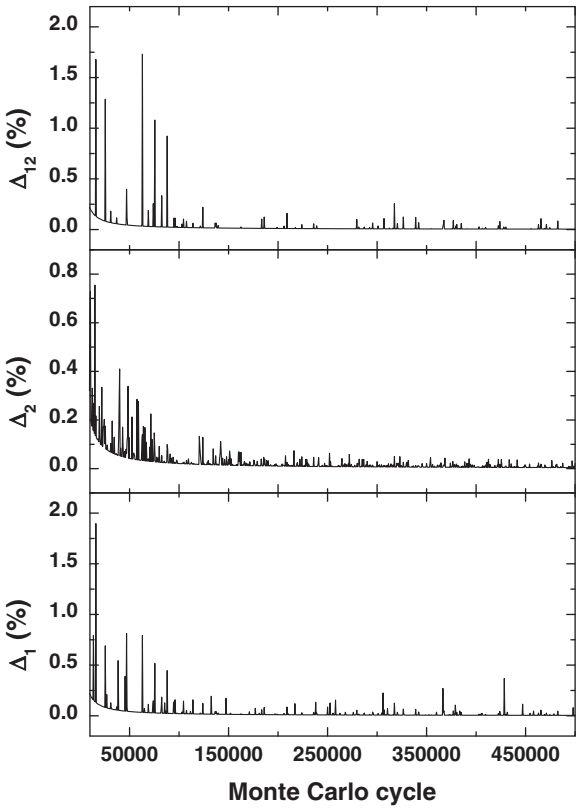

Figure 4. The calculated $\Delta_i (i=1,2)$ and Δ_{12} with $\rho=0.4$ for different quality processes.

If historical default rates are given, we can estimate Z_i as follows:

$$Z_i = \arg\min_{Z_i} \sum_t \left(\frac{P_i(Z_i,t)}{t} - \frac{\widetilde{A}_i(t)}{t} \right)^2 \tag{12}$$

where $P_i(Z_i,t)$ are the theoretical default probabilities (as determined by Equation (11)) and $\widetilde{A}_i(t)$ are the historical default rates. For the A-rated firm considered here, the optimized Z_i value was evaluated in [7] as 8.06. By substituting the optimized Z_i-value into Equation (11), we get the theoretical cumulative default rates without jumps, given in Figure 5 by the line of circles.

Now, let us consider the UNIF method, briefly described in Section 3 (see also further details in [20]). First, the developed Monte Carlo simulation allows us to obtain the estimated density $\widehat{f}_i(t)$ by using kernel estimator method. We get also the default rate $P_i(t)$ for firm i.

Then we minimize the difference between our model and the historical default data to obtain the optimized parameters in our model:

$$\text{argmin} \left(\sum_i \sqrt{\sum_{t_j} \left(\frac{P_i(t_j) - \widetilde{A}_i(t_j)}{t_j} \right)^2} \right) \tag{13}$$

DOI: 10.1002/asmb

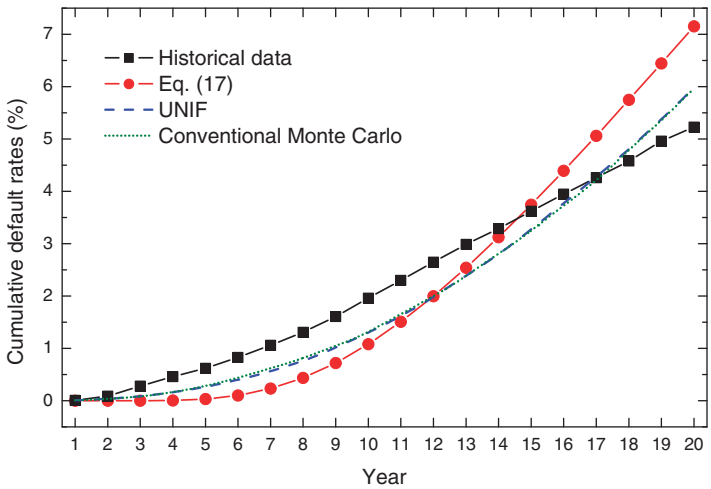

Figure 5. Historical, theoretical, and simulated cumulative default rates. The theoretical value was calculated by using Equation (11). All the simulations were performed with Monte Carlo runs $N = 100000$; for conventional Monte Carlo method, the discretization size of time period has been taken $\Delta = 0.005$.

As in [20], for convenience, we reduce the number of optimizing parameters by:

- Setting $X_i(0) = 2$ and $\ln(\kappa) = 0$.
- Setting the growth rate γ of debt value equal to the growth rate μ of the firm's value [7], so that the default of firm is non sensitive to μ (we have taken $\mu = -0.001$ in our computation, next reported).
- Supposing that the interjumps $(T_j - T_{j-1})$ are distributed according to an exponential distribution with mean value equal to 1.
- The arrival rate for jumps satisfies the Poisson distribution with intensity parameter λ, while the jump-size has a normal distribution $Z_t \sim N(\mu_Z, \sigma_Z)$.

As a result, we only need to optimize $\sigma, \lambda, \mu_Z, \sigma_Z$ for this firm, This is done by minimizing the differences between our simulated default rates and the given historical data. The minimization was performed by using quasi-Newton procedure implemented as a Scilab program.

The optimized parameters for the A-rated firm are $\sigma = 0.09000984, \lambda = 0.10001559, \mu_Z = -0.20003641$, and $\sigma_Z = 0.50000485$. Then, by using these optimized parameters, we carried out a final simulation with Monte Carlo runs $N = 100000$. The simulated cumulative default rates by using the UNIF method are shown in Figure 5 by the dash line. For comparison, we have carried out the CMC simulation with the same optimized parameters. The resulting simulated default rates are displayed by dotted line in Figure 5. All the simulations reported here were carried out on a 2.4 GHz AMD Opteron(tm) Processor. The optimal bandwidth and CPU time are given in Table II.

From Figure 5, we can conclude that our simulations give similar results to that theoretically predicted by Equation (11), and exceed them for short time horizon. The UNIF method gives exactly the same default curve as the CMC method, but the former outperforms the latter

DOI: 10.1002/asmb

Table II. The optimal bandwidth h_{opt}, and CPU time per Monte Carlo run of the simulations.

	Optimal bandwidth	CPU time per Monte Carlo run
Conventional Monte Carlo	0.891077	0.119668
UNIF	0.655522	0.000621

All the simulations were performed with Monte Carlo runs $N = 100000$; the discretization size of time horizon for the conventional Monte Carlo method was $\Delta = 0.005$.

substantially in terms of computational time. The UNIF methodology is much faster compared with the conventional method and is extremely useful in practical applications.

4.3. Default correlations: comparison with closed-form solutions

Our final example concerns the default correlation of two A-rated firms (A,A). In Table III we provide the information on the default correlation of firms (A,A) for 1-, 2-, 5- and 10-year. The values in the second column were calculated using the closed-form solution derived in [7].

In order to implement the UNIF method, we use assumptions, similar to the ones before, in order to reduce the number of optimizing parameters:

1. Setting $X(0) = 2$ and $\ln(\kappa) = 0$ for all firms.
2. Setting $\gamma = \mu = -0.001$ for all firms.
3. Since we are considering two same rated firms (A,A), we choose σ as:

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \tag{14}$$

where $\sigma\sigma^{\top} = H_0$ such that

$$\sigma\sigma^{\top} = H_0 = \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \tag{15}$$

and

$$\begin{aligned} \sigma_1^2 &= \sigma_{11}^2 + \sigma_{12}^2 \\ \sigma_2^2 &= \sigma_{21}^2 + \sigma_{22}^2 \\ \rho_{12} &= \frac{\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22}}{\sigma_1\sigma_2} \end{aligned} \tag{16}$$

In (16), ρ_{12} reflects the correlation of diffusion parts of the state vectors of the two firms.

4. The arrival rate for jumps satisfies the Poisson distribution with intensity parameter λ for all firms, and we use the parameters optimized from single A-rated firm, i.e. $\lambda = 0.10001559$ for all the firms.
5. As before, we generate the same interjump times $(T_j - T_{j-1})$ according to an exponential distribution with mean value equal to 1 for all firms. Furthermore, the jump-size has a normal

DOI: 10.1002/asmb

Table III. Theoretical and simulated default correlations (%) of firms (A,A). The simulations were performed with Monte Carlo runs $N = 100000$.

Year	Reference [7]	UNIF
1	0.00	0.00
2	0.02	2.47
5	1.65	6.58
10	7.75	9.28

distribution $Z_t \sim N(\mu_{Z_i}, \sigma_{Z_i})$, and we use the parameters optimized from a single A-rated firm, i.e. $\mu_{Z_i} = -0.20003641$ and $\sigma_{Z_i} = 0.50000485$ for all the firms.

As a result, there are only four parameters left to optimize: $\sigma_{11}, \sigma_{12}, \sigma_{21}$, and σ_{22}. The optimization was carried out by using the quasi-Newton procedure implemented as a Scilab program. The resulting optimized parameters are $\sigma_{11} = 0.06963755, \sigma_{12} = 0.02993134, \sigma_{21} = 0.03387809$, and $\sigma_{22} = 0.06691001$. We can easily get $\sigma_1 = 0.0757976, \sigma_2 = 0.0749978$, and $\rho_{12} = 0.7673104$. The parameter ρ_{12} represents the correlation between diffusion parts of the state vectors of two firms.

The simulated default correlations are displayed in the third column of Table III. Observe that, the UNIF method gives a little larger default correlation compared with the theoretical predicted by Equation (11). This is mainly because our optimized ρ_{12} is larger than 0.4 used in [7], and we have used the same interjump times $(T_j - T_{j-1})$ for all the firms. Nevertheless, the UNIF method gives the correct default correlation trend, as the default correlation becomes larger with increasing time.

5. CONCLUSION

We analyzed the FPT problem in the context of multivariate and correlated JDP by developing further the fast Monte Carlo-type numerical method—the UNIF method—in the multivariate case. We demonstrated the efficiency of the developed methodology on the analysis of the default rates and default correlations for several different, but correlated processes, in which we incorporated jumps to reflect external shocks or other unpredicted events. Furthermore, we provided an application example of simulating default correlations based on the available historic data and compared the results of our simulations with available theoretical results. Finally, we note that the developed methodology provides an efficient tool for further practical applications, including credit analysis, barrier option pricing, macroeconomic dynamics, and the evaluation of risk in other areas of business and industry.

ACKNOWLEDGEMENTS

This work, conducted in the M^{2}NeT Laboratory (http://www.m2netlab.wlu.ca), was supported by the NSERC and CRC program. It was made possible by the facilities of the Canadian Shared Hierarchical Academic Research Computing Network (SHARCNET). We would like to thank the anonymous referees for their helpful suggestions.

DOI: 10.1002/asmb

REFERENCES

1. Tuckwell HC, Wan FYM. First-passage time of Markov processes to moving barriers. *Journal of Applied Probability* 1984; **21**:695–709.
2. Weiss GH. First passage time problems in chemical physics. *Advances in Chemical Physics* 1967; **13**:1–18.
3. Deo CS *et al*. First passage time Markov chain analysis of rare events for kinetic Monte-Carlo: double kink nucleation during dislocation glide. *Modelling and Simulation in Material Science and Engineering* 2002; **10**:581–596.
4. De Falco D, Tamascelli D. Dynamical kickback and noncommuting impurities in a spin chain. *International Journal of Quantum Information* 2008; **6**(Suppl. 1):807–813.
5. Li DX. On default correlation: a copula approach. *Journal of Fixed Income* 2000; **9**:43–54.
6. Chen RR, Sopranzetti BJ. The valuation of default-triggered credit derivatives. *Journal of Financial and Quantitative Analysis* 2003; **38**:359–382.
7. Zhou C. An analysis of default correlation and multiple defaults. *Review of Financial Studies* 2001; **14**:555–576.
8. Hull J, White A. Valuing credit default swaps II: modeling default correlations. *Journal of Derivatives* 2001; **8**:12–22.
9. Jarrow RA, Lando D, Yu F. Default risk and diversification: theory and empirical implications. *Mathematical Finance* 2005; **15**(1):1–26.
10. Das SR, Duffie D, Kapadia N, Saita L. Common failings: how corporate defaults are correlated. *Journal of Finance* 2007; **62**(1):93–117.
11. Giesecke K. Correlated default with incomplete information. *Journal of Banking and Finance* 2004; **28**(7): 1521–1545.
12. Giesecke K. Default and information. *Journal of Economic Dynamics and Control* 2006; **30**(11):2281–2303.
13. Zhou C. The term structure of credit spreads with jump risk. *Journal of Banking and Finance* 2001; **25**:2015–2040.
14. Kou SG, Wang H. First passage times of a jump diffusion process. *Advances in Applied Probability* 2003; **35**:504–531.
15. Yu J. Closed-form likelihood approximation and estimation of jump-diffusions with an application to the realignment risk of the Chinese Yuan. *Journal of Econometrics* 2007; **141**:1245–1280.
16. Das SR, Uppal R. Systemic risk and international portfolio choice. *The Journal of Finance* 2004; **LIX**(6): 2809–2834.
17. Kole E, Koedijk K, Verbeek M. Portfolio implications of systemic crises. *Journal of Banking and Finance* 2006; **30**:2347–2369.
18. Denuit M, Frostig E. Comparison of dependence in factor models with application to credit risk portfolios. *Probability in the Engineering and Information Sciences* 2008; **22**:151–160.
19. Pesaran MH *et al*. Macroeconomic dynamics and credit risk: a global perspective. *Journal of Money, Credit, and Banking* 2006; **38**(5):1211–1261.
20. Zhang D, Melnik RVN. Solving stochastic differential equations with jump-diffusion efficiently: applications to FPT problems in credit risk. *Dynamics of Continuous, Discrete and Impulsive Systems. Series A: Mathematical Analysis* 2007; **14**(S2):128–133.
21. Luciano E, Schoutens W. A multivariate jump-driven financial asset model. *Quantitative Finance* 2006; **6**(5): 385–402.
22. Kaupuzs J, Melnik RVN, Rimsans J. Monte Carlo test of the Goldstone mode singularity in 3D XY model. *European Physical Journal B* 2007; **55**(4):363–370.
23. Kaupuzs J, Melnik RVN, Rimsans J. Advanced Monte Carlo study of the Goldstone mode singularity in the 3D XY model. *Communications in Computational Physics* 2008; **4**(1):124–134.
24. Melnik RVN, Uhlherr A, Hodgkin J, de Hoog F. Distance geometry algorithms in molecular modelling of polymer and composite systems. *Computers and Mathematics with Applications* 2003; **45**(1–3):515–534.
25. Melnik RVN. Markov chain network training and conservation law approximations: linking microscopic and macroscopic models for evolution. *Applied Mathematics and Computation* 2008; **199**(1):315–333.
26. Zhang D, Melnik RVN. Monte-Carlo simulations of the FPT for multivariate transformed Brownian motions with jumps. Preprint, 2008.
27. Wei XL, Melnik RVN, Moreno-Hagelsieb G. *Modelling Dynamics of Genetic Networks as a Multiscale Process*. Computational Science—ICCS 2005, PT 3, Lecture Notes in Computer Science, vol. 3516. Springer: Berlin, 2005; 134–138.
28. Melnik RVN, Povitsky A. Wave phenomena in physics and engineering: new models, algorithms, and applications. *Mathematics and Computers in Simulation* 2004; **65**(4–5):299–302.

DOI: 10.1002/asmb

29. Melnik RVN, Roberts A. Computational models for multi-scale coupled dynamic problems. *Future Generation Computer Systems* 2004; **20**(3):453–464.
30. Mahapatra DR, Melnik RVN. Three-dimensional mathematical models of phase transformation kinetics in shape memory alloys. *Dynamics of Continuous, Discrete and Impulsive Systems. Series B—Applications and Algorithms* 2005; **2**(Sp. Iss. SI):557–562.
31. Wang LX, Melnik RVN. Two-dimensional analysis of shape memory alloys under small loadings. *International Journal for Multiscale Computational Engineering* 2006; **4**(2):291–304.
32. Melnik RVN, Povitsky A. A special issue on modelling coupled and transport phenomena in nanotechnology. *Journal of Computational and Theoretical Nanoscience* 2006; **3**(4):10. DOI: 10.1166/jctn.2006.001.
33. Yang XD, Melnik RVN. Effect of internal viscosity on Brownian dynamics of DNA molecules in shear flow. *Computational Biology and Chemistry* 2007; **31**(2):110–114.
34. Mahapatra DR, Melnik RVN. Modelling and analysis of collagen piezoelectricity in human cornea. *Dynamics of Continuous, Discrete and Impulsive Systems. Series A—Mathematical Analysis* 2006; **13**(Suppl.):377–384.
35. Mahapatra DR, Melnik RVN. Finite element modelling and simulation of phase transformations in shape memory alloy thin films. *International Journal for Multiscale Computational Engineering* 2007; **5**(1):65–71.
36. Melnik RVN. Deterministic and stochastic dynamics with hyperbolic HJB-type equations. *Dynamics of Continuous, Discrete and Impulsive Systems. Series A—Mathematical Analysis* 2003; **10**(1–3):317–330.
37. Melnik RVN. On consistent regularities of control and value functions. *Numerical Functional Analysis and Optimization* 1997; **18**(3–4):401–426.
38. Kloeden PE, Platen E, Schurz H. *Numerical Solution of SDE Through Computer Experiments* (3rd revised edn). Springer: Germany, 2003.
39. Favella L, Reineri MT, Ricciardi LM, Sacerdote L. First passage time problems and some related computational methods. *Cybernetics and Systems* 1982; **13**:95–128.
40. Giraudo MT, Sacerdote L. Simulation methods in neuronal modelling. *BioSystems* 1998; **48**:77–83.
41. Atiya AF, Metwally SAK. Efficient estimation of first passage time density function for jump-diffusion processes. *SIAM Journal on Scientific Computing* 2005; **26**:1760–1775.
42. Metwally S, Atiya A. Using Brownian bridge for fast simulation of jump-diffusion processes and barrier options. *The Journal of Derivatives* 2002; **10**:43–54.
43. Melnik RVN, Roberts AJ. Thermomechanical behaviour of thermoelectric SMA actuators. *Journal de Physique IV* 2001; **11**(PR8):515–520.
44. Melnik KN, Melnik RVN. Optimal cubature formulae and recovery of fast-oscillating functions from an interpolational class. *BIT* 2001; **41**(4):748–775.
45. Melnik RVN, Jenkins DR. On computational control of flow in airblast atomisers for pulmonary drug delivery. *International Journal of Pharmaceutics* 2002; **239**(1–2):23–35.
46. Melnik RVN, Roberts AJ. Computational models for materials with shape memory: towards a systematic description of coupled phenomena. *Proceedings of Computational Science—ICCS 2002*. Lecture Notes in Computer Science, vol. 2330(Part 2). Springer: Berlin, 2002; 490–499.
47. Huai Y, Melnik RVN, Thogersen PB. Computational analysis of temperature rise phenomena in electric induction motors. *Applied Thermal Engineering* 2003; **23**(7):779–795.
48. Melnik RVN. Modelling coupled dynamics: piezoelectric elements under changing temperature conditions. *International Communications in Heat and Mass Transfer* 2003; **30**(1):83–92.
49. Willatzen M, Melnik RVN, Galeriu C, Voon LCLY. Finite element analysis of nanowire superlattice structures. *Proceedings of Computational Science and its Applications—ICCSA 2003*. Lecture Notes in Computer Science, vol. 2668(Part 2). Springer: Berlin, 2003; 755–763.
50. Melnik RVN, Rimshans J. Numerical analysis of fast charge transport in optically sensitive semiconductors. *Dynamics of Continuous, Discrete and Impulsive Dynamical Systems. Series B—Applications and Algorithms* 2003; (Suppl.):102–107.
51. Zotsenko KN, Melnik RVN. Optimal minimax algorithm for integrating fast oscillatory functions in two dimensions. *Engineering Computations* 2004; **21**(7–8):834–847.
52. Melnik RVN, Strunin DV, Roberts AJ. Nonlinear analysis of rubber-based polymeric materials with thermal relaxation models. *Numerical Heat Transfer. Part A—Applications* 2005; **47**(6):549–569.
53. Wang LX, Melnik R. Dynamics of shape memory alloys patches with mechanically induced transformations. *Discrete and Continuous Dynamical Systems* 2006; **15**(4):1237–1252.
54. Kamath H, Willatzen M, Melnik RVN. Vibration of piezoelectric elements surrounded by fluid media. *Ultrasonics* 2006; **44**(1):64–72.

55. Mahapatra DR, Melnik RVN. Numerical simulation of phase transformations in shape memory alloy thin films. *Proceedings of Computational Science—ICCS 2006*, vol. 3992(Part 2), 2006; 114–121.
56. Wu Z, Melnik RVN, Borup F. Model-based analysis and simulation of airflow control systems of ventilation units in building environments. *Building and Environment* 2007; **42**(1):203–217.
57. Mahapatra DR, Melnik RVN. Finite element approach to modelling evolution of 3D shape memory materials. *Mathematics and Computers in Simulation* 2007; **76**(1–3):141–148.
58. Wang LX, Melnik RVN. Thermo-mechanical wave propagations in shape memory alloy rod with phase transformations. *Mechanics of Advanced Materials and Structures* 2007; **14**(8):665–676.
59. Wang LX, Melnik RVN. Modifying macroscale variant combinations in a two-dimensional structure using mechanical loadings during thermally induced transformation. *Materials Science and Engineering A* 2008; **481**(Sp. Iss.):190–193.
60. Radulovic N, Willatzen M, Melnik RVN, Voon LCLY. Influence of the metal contact size on the electron dynamics and transport inside the semiconductor heterostructure nanowire. *Journal of Computational and Theoretical Nanoscience* 2006; **3**(4):551–559.
61. Angelov C, Melnik RVN, Buur J. The synergistic integration of mathematics, software engineering, and usercentred design: exploring new trends in education. *Future Generation Computer Systems* 2003; **19**(8):1299–1307.
62. Cont R, Tankov P. *Financial Modelling with Jump Processes*. Chapman & Hall, CRC: London, Boca Raton, FL, 2003.
63. Crescenzo AD, Nardo ED, Ricciardi LM. Simulation of first-passage times for alternating Brownian motions. *Methodology and Computing in Applied Probability* 2005; **7**:161–181.
64. Ngwira B, Gerrard R. Stochastic pension fund control in the presence of Poisson jumps. *Insurance*: *Mathematics and Economics* 2007; **40**:283–292.
65. Wang J, Zhang W-J, Liang J-R *et al.* Fractional nonlinear diffusion equation and first passage time. *Physica A* 2008; **387**(1):764–772.
66. Pinaud D. Quantifying search effort of moving animals at several spatial scales using first-passage time analysis: effect of the structure of environment and tracking systems. *Journal of Applied Ecology* 2008; **45**(1):91–99.
67. Boulougouris GS, Theodorou DN. Dynamical integration of a Markovian web: a first passage time approach. *Journal of Chemical Physics* 2007; **127**(8):084903.
68. Song JH, Kiureghian AD. Joint first-passage probability and reliability of systems under stochastic excitation. *Journal of Engineering Mechanics* 2006; **132**(1):65–77.
69. Kolomeisky AB, Fisher ME. Molecular motors: a theorist's perspective. *Annual Review of Physical Chemistry* 2007; **58**:675–695.
70. Duffie D, Pan J, Singleton K. Transform analysis and option pricing for affine jump-diffusions. *Econometrica* 2000; **68**:1343–1376.
71. Silverman BW. *Density Estimation for Statistics and Data Analysis*. Chapman & Hall: London, 1986.

DOI: 10.1002/asmb