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# Optimal-by-accuracy and optimal-by-order cubature formulae in interpolational classes

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**Abstract**

In this paper the problem of optimal integration for fast oscillatory functions of two variables is solved constructively in the case where a priori information is limited. The connection of this problem with the problem of optimal recovery of a function from interpolational classes is analysed using properties of majorants and minorants for these functional classes. © 2002 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

In the solution of many classes of problems such as statistical processing of experimental data, boundary problems for PDEs, signal processing, modelling systems of automative regulation and image recognition we often have to compute integrals of the form

$$I^2(f) = \int_0^1 \int_0^1 f(x_1,x_2)\varphi_1(x_1)\varphi_2(x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2, \tag{1.1}$$

where $\varphi_1(x_1)$, $\varphi_2(x_2)$ are known integrable functions and $f(x_1,x_2)$ belongs to a given functional class $F_N$. An important special case of this problem is the computation of

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integrals

$$I_2^2(f)=\int_0^1\int_0^1 f(x_1,x_2)\sin(\omega_1x_1)\sin(\omega_2x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2, \tag{1.2}$$

$$I_3^2(f)=\int_0^1\int_0^1 f(x_1,x_2)\cos(\omega_1x_1)\cos(\omega_2x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2, \tag{1.3}$$

where $\omega_1,\omega_2$ are real numbers with $|\omega_i|\geqslant 2\pi,\ i=1,2$.

Integrands in (1.2), (1.3) are typical examples of rapidly oscillatory functions that occur in various applications, often in the context of the Fourier or Fourier–Bessel integral transforms [21,14]. Integrating fast oscillatory functions is beset with difficulties even in the one-dimensional case (see, for example, [1,7,9,11,15,16,28]). Indeed, assume that we have to integrate the product $f(x)\exp(-\mathrm{i}\omega x)$ on an interval $(a,b)$, where $\omega(b-a)\gg 1$. Since $\Re e(f(x)\exp(-\mathrm{i}\omega x))$ and $\Im m(f(x)\exp(-\mathrm{i}\omega x))$ have approximately $\omega(b-a)/\pi$ zeros on the interval $(a,b)$, even if $f(x)$ is a smooth function, we have to choose a polynomial of degree $n\gg\omega(b-a)/\pi$ in order to achieve an adequate level of approximation. It is well known that the use of such a high degree polynomial may lead to instability [12], a difficulty which is exacerbated in the two-dimensional case [13,20,6]. Moreover, since a priori information about the integrand is typically given inaccurately in the majority of practical problems, optimisation issues in numerical integration of fast oscillatory functions become of primary importance.

In this paper we construct optimal-by-accuracy and optimal-by-order cubature formulae for computing integrals (1.1)–(1.3) in interpolational classes $C^2_{2,\,L_1,\,L_2,\,N}$ and $C^2_{2,\,L,\,L,\,N}$. These classes are defined as follows:

- $C^2_{2,\,L_1,\,L_2,\,N}$ is the class of functions defined in the domain $\pi_2, \pi_2=\{\mathbf{x}=(x_1,x_2)\colon 0\leqslant x_i\leqslant 1,\ i=1,2\}$, satisfying the Lipschitz condition with constant $L_1$ and $L_2$ in each variable,

$$|f(\bar{x}_1,x_2)-f(\bar{\bar{x}}_1,x_2)|\leqslant L_1|\bar{x}_1-\bar{\bar{x}}_1|,\quad |f(x_1,\bar{x}_2)-f(x_1,\bar{\bar{x}}_2)|\leqslant L_2|\bar{x}_2-\bar{\bar{x}}_2|, \tag{1.4}$$

  and taking fixed values $f(x_1)=f_1,\ldots,f(x_N)=f_N$ at fixed nodes $x_1,\ldots,x_N$, respectively;
- $C^2_{2,L,L,N}$ is the class of functions defined in the domain $\pi_2$, $\pi_2=\{\mathbf{x}=(x_1,x_2)\colon 0\leqslant x_i\leqslant 1,\ i=1,2\}$, satisfying the Lipschitz condition with constant $L$ in both variables,

$$|f(\bar{x}_1,x_2)-f(\bar{\bar{x}}_1,x_2)|\leqslant L|\bar{x}_1-\bar{\bar{x}}_1|,\quad |f(x_1,\bar{x}_2)-f(x_1,\bar{\bar{x}}_2)|\leqslant L|\bar{x}_2-\bar{\bar{x}}_2|, \tag{1.5}$$

  and taking fixed values $f(x_1)=f_1,\ldots,f(x_N)=f_N$ at fixed nodes $x_1,\ldots,x_N$, respectively.

In what follows, it is assumed that these functional classes are nonempty.

In order to obtain optimal-by-accuracy and optimal-by-order solutions of problems (1.1)–(1.3) we use the method of limit functions [24,22,16,18]. The method consists of the definition of upper, $I^+(F_N)$, and lower, $I^-(F_N)$, limits of set of possible values of integral (1.1) (and, hence, (1.2), (1.3) as a special case) on functions from class $F_N$ by the following formula:

$$I^+(F_N)=\sup_{f\in F_N} I^2(f),\quad I^-(F_N)=\inf_{f\in F_N} I^2(f) \tag{1.6}$$

and the determination of the value

$$I^*(F_N) = \frac{I^+(F_N) + I^-(F_N)}{2}, \tag{1.7}$$

taken as the optimal-by-accuracy value of the integral $I^2(f)$. In this case $I^*(F_N)$ is the Chebyshev centre of undefinability domain of values $I^2(f)$ on class $F_N$ (see, for example, [19, p. 171]). The Chebyshev radius coincides with $\delta(F_N)$, defined as follows:

$$\delta(F_N) = \tfrac{1}{2}(I^+(F_N) - I^-(F_N)). \tag{1.8}$$

In a special case, where $\varphi_1(x_1) = \varphi_2(x_2) = 1$, we come to the problem of computing the optimal-by-accuracy value $I_1^*(F_N)$ for integrals

$$I_1^2(f) = \iint_{\pi_2} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \tag{1.9}$$

with $f \in F_N$ and $X = (x_1, x_2)$.

It is known (see, for example, [3,22] and references therein), that the problem of optimal-by-accuracy integration on class $F_N$ is closely connected with the problem of optimal-by-accuracy recovery of $f(X) \in F_N$ at point $X = (x_1, x_2) \in \pi_2$.

**Definition 1.1.** Let $F_N$ be a class of functions defined in a domain $D$. Then a function $A^+_{F_N}(X)$ $(A^-_{F_N}(X))$ is called a majorant (minorant) of the class $F_N$, if the conditions

(a) $A^+_{F_N}(X) \geqslant f(X)(A^-_{F_N}(X) \leqslant f(X))$ for all $f \in F_N$, $X = (x_1, x_2) \in D$ and
(b) $A^+_{F_N}(X) \in F_N(A^-_{F_N} \in F_N)$

are satisfied.

The value of

$$f^*(X) = \tfrac{1}{2}(A^+_{F_N}(X) + A^-_{F_N}(X)) \tag{1.10}$$

(with $A^+_{F_N}(X), A^-_{F_N}(X)$ majorant and minorant of class $F_N$, respectively) is taken as the optimal-by-accuracy recovery of $f(X)$ at $X \in \pi_2$. Further, in this paper, we assume that $F_N = C^2_{2,\,L_1,\,L_2,\,N}$ or $F_N = C^2_{2,L,L,N}$. The error $\bar{\delta}(F_N, X)$ of the recovery of function $f(X) \in F_N$ at point $X$ has the form

$$\bar{\delta}(F_N, X) = \frac{A^+_{F_N}(X) - A^-_{F_N}(X)}{2}. \tag{1.11}$$

Then, the optimal-by-accuracy cubature formulae for computing (1.9) is [24,22]

$$I_1^*(F_N) = \iint_{\pi_2} f^*(X)\,\mathrm{d}X \tag{1.12}$$

with the Chebyshev radius, $\bar{\bar{\delta}}(F_N)$, of the domain of undefinability of integral (1.9) in the form

$$\bar{\bar{\delta}}(F_N) = \iint_{\pi_2} \bar{\bar{\delta}}(F_N, X)\,\mathrm{d}X. \tag{1.13}$$

For a constructive solution of problems (1.10)–(1.11) and (1.12)–(1.13), as well as for the construction of efficient cubature formulae for computing integrals (1.1)–(1.3) we have to consider properties of majorants and minorants of the functional classes that are investigated.

## 2. Properties of majorants and minorants of interpolational classes $C^2_{\bar{2},\,L_1,\,L_2,\,N}$ and $C^2_{2,L,L,N}$

From Definition 1.1 it follows that if there exists such a function $g_1(X)$ (or $g_2(X)$) with $g_1(X) = \max_{f\in F_N} f(X)$ (or $g_2(X) = \min_{f\in F_N} f(X)$) from class $F_N$, then it coincides with the majorant (or minorant) of the class of functions that is investigated. We also note that

$$A^+_{F_N}(X) = \sup_{f\in F_N} f(X) = \min_{v=1,\dots,N}(f_v + L\|X - X_v\|), \tag{2.1}$$

$$A^-_{F_N}(X) = \inf_{f\in F_N} f(X) = \max_{v=1,\dots,N}(f_v - L\|X - X_v\|), \tag{2.2}$$

where $X = (x_1, x_2)$, $D = \pi_2$. In the case $F_N = C^2_{\bar{2},\,L_1,\,L_2,\,N}$ we define

$$\|X\| = \|X\|_{\bar{2}} = |x_1| + \frac{L_2}{L_1}|x_2| \tag{2.3}$$

and for the case $F_N = C^2_{2,L,L,N}$

$$\|X\| = \|X\|_2 = |x_1| + |x_2|. \tag{2.4}$$

On the example of class $F_N = C^2_{\bar{2},\,L_1,\,L_2,\,N}$ we will show that functions $A^+_{F_N}(X)$ and $A^-_{F_N}(X)$ defined by (2.1) and (2.2) indeed satisfy Definition 1.1.

First, let us show that function $A^+_{C^2_{\bar{2},\,L_1,\,L_2,\,N}}(X)$ satisfies condition (a). For any $f(X) \in C^2_{\bar{2},\,L_1,\,L_2,\,N}$ we have

$$\begin{aligned} f(X) - A^+_{C^2_{\bar{2},\,L_1,\,L_2,\,N}}(X) &= f(X) - \min_{v=1,\dots,N}(f_v + L_1\|X - X_v\|_{\bar{2}}) \\ &= f(X) - f_{v_0} - L_1\|X - X_{v_0}\|_{\bar{2}} \leqslant 0. \end{aligned} \tag{2.5}$$

The last inequality in (2.5) follows from the fact that inequalities (1.4) and the relationship

$$|f(\bar{X}) - f(\bar{\bar{X}})| \leqslant L_1\|\bar{X} - \bar{\bar{X}}\|_{\bar{2}}, \quad \bar{X} = (x_1, x_2), \quad \bar{\bar{X}} = (\bar{\bar{x}}_1, \quad \bar{\bar{x}}_2) \tag{2.6}$$

are equivalent. Indeed,

$$\begin{aligned} |f(\bar{X}) - f(\bar{\bar{X}})| &= |f(\bar{x}_1, \bar{x}_2) - f(\bar{\bar{x}}_1, \bar{\bar{x}}_2)| = |f(\bar{x}_1, \bar{x}_2) - f(\bar{\bar{x}}_1, \bar{x}_2) + f(\bar{\bar{x}}_1, \bar{x}_2) - f(\bar{\bar{x}}_1, \bar{\bar{x}}_2)| \\ &\leqslant L_1|\bar{x}_1 - \bar{\bar{x}}_1| + L_2|\bar{x}_2 - \bar{\bar{x}}_2| = L_1\|\bar{X} - \bar{\bar{X}}\|_{\bar{2}}. \end{aligned} \tag{2.7}$$

In other words, from the definition of class $C_{\bar{2},L_1,L_2,N}$ and relationship (2.1) inequality (2.6) follows immediately. The inverse statement is also true. Let $\bar{X}=(\bar{x}_1,\bar{x}_2)$, $\tilde{X}=(\bar{x}_1,\bar{\bar{x}}_2)$, $\bar{\bar{X}}=(\bar{\bar{x}}_1,\bar{\bar{x}}_2)$. Then

$$|f(\bar{x}_1,\bar{x}_2)-f(\bar{x}_1,\bar{\bar{x}}_2)| \leqslant L_1\|\bar{X}-\tilde{X}\|_{\bar{2}} = L_1\frac{L_2}{L_1}|\bar{x}_2-\bar{\bar{x}}_2| = L_2|\bar{x}_2-\bar{\bar{x}}_2| \tag{2.8}$$

and

$$|f(\bar{x}_1,\bar{\bar{x}}_2)-f(\bar{\bar{x}}_1,\bar{\bar{x}}_2)| \leqslant L_1\|\tilde{X}-\bar{\bar{X}}\|_{\bar{2}} = L_1|\bar{x}_1-\bar{\bar{x}}_1|. \tag{2.9}$$

Now let us show that function $A^{+}_{C^2_{\bar{2},L_1,L_2,N}}(X)$ belongs to class $C^2_{\bar{2},L_1,L_2,N}$, i.e., satisfies (1.4) or (2.6) and $A^{+}_{C^2_{\bar{2},L_1,L_2,N}}(X_v)=f_v,\ v=1,\dots,N$. Indeed,

$$\begin{aligned} A^{+}_{C^2_{\bar{2},L_1,L_2,N}}(\bar{X})-A^{+}_{C^2_{\bar{2},L_1,L_2,N}}(\bar{\bar{X}}) &= f_{v_0}+L_1\|X_{v_0}-\bar{X}\|_{\bar{2}}-f_{v_1}-L_1\|X_{v_1}-\bar{\bar{X}}\|_{\bar{2}}\\ &\leqslant f_{v_1}+L_1\|X_{v_1}-\bar{X}\|_{\bar{2}}-f_{v_1}-L_1\|X_{v_1}-\bar{\bar{X}}\|_{\bar{2}}\\ &= L_1\|X_{v_1}-\bar{X}\|_{\bar{2}}-L_1\|X_{v_1}-\bar{\bar{X}}\|_{\bar{2}}\\ &\leqslant L_1\|\bar{X}-\bar{\bar{X}}\|_{\bar{2}}. \end{aligned} \tag{2.10}$$

The second last inequality in (2.10) follows from

$$A^{+}_{C^2_{\bar{2},L_1,L_2,N}}(\bar{X}) = \min_{v=1,\dots,N}(f_v+L_1\|X_v-\bar{X}\|_{\bar{2}}). \tag{2.11}$$

Therefore,

$$A^{+}_{C^2_{\bar{2},L_1,L_2,N}}(\bar{X}) \leqslant f_{v_1}+L_1\|X_{v_1}-\bar{X}\|_{\bar{2}}. \tag{2.12}$$

The fact that $A^{+}_{C^2_{\bar{2},L_1,L_2,N}}(X_v)=f_v,\ v=1,\dots,N$ follows from nonemptiness of class $C^2_{\bar{2},L_1,L_2,N}$ and relationship (1.4). Hence, we have shown that function

$$A^{+}_{C^2_{\bar{2},L_1,L_2,N}}(X) = \min_{v=1,\dots,N}(f_v+L_1\|X-X_v\|_{\bar{2}}) \tag{2.13}$$

is a majorant of class $C^2_{\bar{2},L_1,L_2,N}$. In a similar way, it can be shown that

$$A^{-}_{C^2_{\bar{2},L_1,L_2,N}}(X) = \max_{v=1,\dots,N}(f_v-L_1\|X-X_v\|_{\bar{2}}). \tag{2.14}$$

Let us define more precisely our set of nodes $X_1,X_2,\dots,X_N$.

- For class $C^2_{2,L,L,N}$ we denote this set as $\Delta_1=\{X_s\}_{s=1,\dots,N}$, $N=(m+1)^2$. We assume that $\Delta_1$ has the following structure:

$$X_s=(x_{1,i};x_{2,j}),\quad s=(i-1)(m+1)+j,$$

$$x_{1,i}=(i-1)\frac{1}{m},\quad x_{2,j}=(j-1)\frac{1}{m},\quad i,j=1,m+1. \tag{2.15}$$

The grid $\Delta_1$ splits the domain $\pi_2$ into $m^2$ equal squares $K_p$, $p=1,\dots,m^2$ with sides $1/m$. We will call such squares elementary.

- For class $C^2_{2,\,L_1,L_2,N}$ we denote the set of nodes $X_1,X_2,\dots,X_N$ by $\Delta_2$ (see Fig. 4), where

$$\Delta_2=\{X_s\}_{s=1,\dots,N},\quad N=(m+1)(m_1+1),\quad X_s=(x_{1,i};x_{2,j})$$

with the nodes defined as

$$\text{(a)}\quad x_{1,i}=(i-1)\frac{1}{m},\ i=1,\dots,m+1,\quad x_{2,j}=(j-1)\frac{L_2}{mL_1},\ j=1,\dots,m_1, \tag{2.16}$$

and

$$\text{(b)}\quad x_{1,i}=(i-1)\frac{1}{m},\ i=1,\dots,m+1,\quad x_{2,m_1+1}=1-\frac{m_1L_2}{mL_1}.$$

It should be noted that points $(x_{1,i},x_{2,m_1+2})$ (denoted in Fig. 4 by "$\circ$") where $x_{2,m_1+2}=1$ are not included in $\Delta_2$. In (2.16) the index $s$ denotes the number of the node computed by the formula (see Fig. 5)

$$s=(i-1)(m_1+1)+j,\quad i=1,\dots,m+1,\ j=1,\dots,m_1+1,\ m_1=\left[m\frac{L_1}{L_2}\right].$$

In this case, grid $\Delta_2$ splits square $\pi_2$ into $mm_1$ equal rectangles $\bar{K}_p$, $p=1,\dots,mm_1$ with sides $1/m$, $L_2/(mL_1)$, and $m$ equal rectangles $\tilde{K}_p$, $p=1,\dots,m$ with sides $1/m$, $(1-m_1L_2/(mL_1))$. Such rectangles will also be referred to as elementary.

Further, we introduce the following notation:

1. $h=h_1=1/m$, $h_2=L_2/(mL_1)$, $\bar{h}_2=1-m_1L_2/(mL_1)$;
2. $\sigma(K_p)=\{s_i\}_{i=1,\dots,4}$, where $s_1=(i_1-1)(m+1)+j_1$, $s_2=(i_1-1)(m+1)+j_1+1$, $s_3=i_1(m+1)+j_1+1$, $s_4=i_1(m+1)+j_1$ are numbers of nodes $X_{s_1},X_{s_2},X_{s_3},X_{s_4}$, which correspond to vertices of the elementary square $K_p$, $p=1,\dots,m^2$, $p=(i_1-1)m+j_1$, $i_1=1,\dots,m$, $j_1=1,\dots,m$;
3. $\sigma(\bar{K}_p)=\{s_i\}_{i=1,\dots,4}$, where $s_1,s_2,s_3,s_4$ are defined as before, i.e., they give the numbers of nodes $X_{s_1},X_{s_2},X_{s_3},X_{s_4}$, which correspond to vertices of the elementary rectangle $\bar{K}_p$, $p=(i_1-1)m+j_1$, $i_1=1,\dots,m$, $j_1=1,\dots,m_1$;
4. $\sigma(\tilde{K}_p)=\{s_i\}_{i=1,4}$, where $s_1=(p-1)(m_1+1)+m_1+1$, $s_4=p(m_1+1)+m_1+1$ are numbers of nodes $X_{s_1}$, $X_{s_4}$ that correspond to *two-out-of-four vertices* of the elementary rectangle $\tilde{K}_p$, $p=1,\dots,m$;
5. finally, let $f_{s_i}=f(X_{s_i})$, $i=1,2,3,4$.

In other words, in all elementary rectangles we consider the values $f_{s_i}$ in all nodes of the grid $\Delta_2$, except $(x_{1,i},x_{2,m_1+1})$, $i=1,\dots,m+1$.

**Theorem 2.1.** *Let $F_N=C^2_{2,\,L_1,L_2,N}$ or $C^2_{2,L,L,N}$. Then in elementary region $K\in\pi_2$, the majorant and minorant of class $F_N$ has the form*

$$A^+_{F_N}(X)=\min_{s\in\sigma(K)}(f_s+L\|X-X_s\|),\quad A^-_{F_N}(X)=\max_{s\in\sigma(K)}(f_s-L\|X-X_s\|),\quad X\in K, \tag{2.17}$$

*where for $F_N = C^2_{2,\,L_1,\,L_2,\,N}$ elementary region $K$ is either $\bar{K}_p$, $p=(i_1-1)m_1+j_1$, $i_1=1,\dots,m$, $j_1=1,\dots,m_1$ or $\tilde{K}_p$, $p=1,\dots,m$; and for $F_N = C^2_{2,\,L,\,L,\,N}$ elementary region $K$ coincides with $K_p$, $p=1,\dots,m^2$.*

**Proof.** We will prove the theorem for the majorant $A^+_{F_N}(X)$. For the minorant $A^-_{F_N}(X)$ the proof is analogous.

Let $F_N = C^2_{\bar{2},L_1,L_2,N}$, $K=\bar{K}_{\tilde{p}}$, $\tilde{p}=(\tilde{i}-1)m_1+\tilde{j}$, where $\tilde{i}$, $\tilde{j}$ are certain fixed values of indices $i_1$ and $j_1$. In this case $X_{s_1}=(x_{1,\tilde{i}};x_{2,\tilde{j}})$, $X_{s_2}=(x_{1,\tilde{i}};x_{2,\tilde{j}+1})$, $X_{s_3}=(x_{1,\tilde{i}+1};x_{2,\tilde{j}+1})$, $X_{s_4}=(x_{1,\tilde{i}+1};x_{2,\tilde{j}})$. The rest of the nodes of the grid $\Delta_2$ (except $(x_{1,i},x_{2,m_1+1})$, $i=1,\dots,m+1$) is grouped in the following way (see Fig. 6):

The four groups of nodes are given by

$$X_{v_1}=\begin{cases}(x_{1,\tilde{i}-k_1};x_{2,\tilde{j}-k_2});\ k_1=1,\dots,\tilde{i}-1,\ k_2=1,\dots,\tilde{j}-1,\\ (x_{1,\tilde{i}};x_{2,\tilde{j}-k_2}),\ k_2=1,\dots,\tilde{j}-1,\\ (x_{1,\tilde{i}-k};x_{2,\tilde{j}}),\ k_1=1,\dots,\tilde{i}-1,\end{cases}$$

$$X_{v_2}=\begin{cases}(x_{1,\tilde{i}-k_1};x_{2,\tilde{j}+k_2}),\ k_1=1,\dots,\tilde{i}-1,\ k_2=2,\dots,m_1-\tilde{j},\\ (x_{1,\tilde{i}};x_{2,\tilde{j}+k_2});\ k_2=2,\dots,m_1-\tilde{j},\\ (x_{1,\tilde{i}-k_1};x_{2,\tilde{j}+1});\ k_1=1,\dots,\tilde{i}-1,\end{cases}$$

$$X_{v_3}=\begin{cases}(x_{1,\tilde{i}+k_1};x_{2,\tilde{j}+k_2});\ k_1=2,\dots,m+1-\tilde{i},\ k_2=2,\dots,m_1-\tilde{j},\\ (x_{1,\tilde{i}+1};x_{2,\tilde{j}+k_2});\ k_2=2,\dots,m_1-\tilde{j},\\ (x_{1,\tilde{i}+k_1};x_{2,\tilde{j}+1});\ k_1=2,\dots,m+1-\tilde{i},\end{cases}$$

$$X_{v_4}=\begin{cases}(x_{1,\tilde{i}+k_1};x_{2,\tilde{j}-k_2});\ k_1=2,\dots,m+1-\tilde{i},\ k_2=1,\dots,\tilde{j}-1,\\ (x_{1,\tilde{i}+1};x_{2,\tilde{j}-k_2});\ k_2=1,\dots,\tilde{j}-1,\\ (x_{1,\tilde{i}+k_1};x_{2,\tilde{j}});\ k_1=2,\dots,m+1-\tilde{i}.\end{cases}$$

We introduce the functions $g_v(X)$ defined for each node of the grid $\Delta_2$ excluding $(x_{1,i},x_{2,m_1+1})$, $i=1,\dots,m+1$ as follows:

$$g_v(X)=f_v+L_1\|X-X_v\|_{\bar{2}}, \tag{2.18}$$

where $v=(i-1)(m_1+1)+j$, $i=1,\dots,m+1$, $j=1,\dots,m_1$. In other words, $v\in\sigma(\Delta_2)$, where $\sigma(\Delta_2)$ denotes the set of numbers of all nodes of the grid $\Delta_2$, except $(x_{1,i},x_{2,m_1}+1)$, $i=1,\dots,m+1$.

Let further $g_{v_l}(X)$, $l=1,2,3,4$ be the functions of the form (2.18) defined for the group of nodes $X_{v_l}$, $l=1,2,3,4$, respectively. Then, it is easy to show that functions defined by (2.18) have the following property:

$$g_{v_l}(X)\geqslant g_{s_l}(X)\quad \forall X\in\bar{K}_{\tilde{p}},\ \ l=1,2,3,4, \tag{2.19}$$

where $g_{s_l}(X)$ are the functions of the form (2.18) defined for the nodes $X_{s_l}$, $l=1,2,3,4$, respectively.

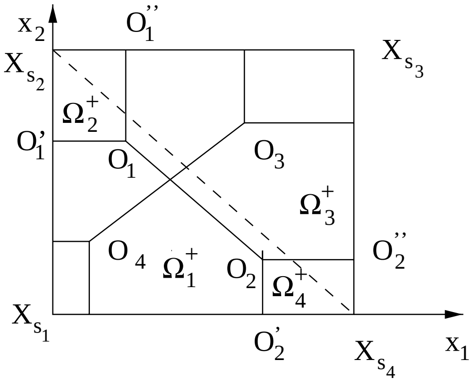


Fig. 1. Splitting of elementary square $K_p$.

Indeed, for the points of $X_{v_1}$-type given in the form $(x_{1,\tilde{i}-k_1}; x_{2,\tilde{j}-k_2})$, $k_1=1,\ldots,\tilde{i}-1$, $k_2=1,\ldots,\tilde{j}-1$, we have

$$\begin{aligned} g_{v_1}(X) - g_{s_1}(X) = f_{v_1} + L_1\|X - X_{v_1}\|_{\tilde{2}} - f_{s_1} - L_1\|X - X_{s_1}\|_{\tilde{2}} \geqslant - L_1k_1h_1 - L_2k_2h_2 \\ + L_1(x_1 - (\tilde{i} - k_1)h_1) + L_2(x_2 - (\tilde{j} - k_2)h_2) - L_1(x_1 - \tilde{i}h_1) \\ - L_2(x_2 - \tilde{j}h_2) = 0. \end{aligned} \tag{2.20}$$

With slight modifications this formula can be easily obtained for other points of $X_{v_1}$-type. In a way similar to (2.20) it can be shown that $g_{v_l}(X) - g_{s_l}(X) \geqslant 0$ $\forall X \in \bar{K}_{\tilde{p}}$ and for $l = 2,3,4$. This means that for all functions $g_v(X), v \in \sigma(\Delta_2) \setminus \sigma(\bar{K}_{\tilde{p}})$ the inequality

$$g_v(X) \geqslant \min_{s\in\sigma(\bar{K}_{\tilde{p}})} g_s(X) \tag{2.21}$$

holds $\forall X \in \bar{K}_{\tilde{p}}$. Analogously, we reason for $K = \tilde{K}_p$, $p = 1,\ldots,m$. In this case we have to consider nodes $X_{v_1}$ and $X_{v_4}$.

Therefore, we have proved that for an arbitrary function $g_v(X)$, $v \in \sigma(\Delta_2) \setminus \sigma(K)$ there exists function $g_s(X)$, $s \in \sigma(K)$ such that $g_v(X) \geqslant g_s(X)$, which confirms the statement of Theorem 2.1. The case $F_N = C^2_{2,L,L,N}$ can be considered similarly. □

Theorem 2.1 has several important consequences. Let us consider the class $C^2_{2,L,L,N}$. We place the origin of the plane $(x_1,x_2)$ in the left lower vertex of elementary square $K_p$, i.e., at the node $X_{s_1}$. Then we split elementary square $K_p$ into parts $\Omega_1^+$, $\Omega_2^+$, $\Omega_3^+$ and $\Omega_4^+$ as shown in Fig. 1. The equations of five lines that split $K_p$ into $\Omega_l^+$, $l = 1,2,3,4$ have the following forms:

$$g_{s_1}(X) = g_{s_3}(X), \quad f_{s_1} + L\|X - X_{s_1}\|_2 = f_{s_3} + L\|X - X_{s_3}\|_2,$$

$$L(x_1 + x_2) - L(h - x_1 + h - x_2) = f_{s_3} - f_{s_1}, \quad x_2 = -x_1 + \frac{f_{s_3} - f_{s_1}}{2L} + h \tag{2.22}$$

for the line through $O_1$, $O_2$;

$$g_{s_2}(X) = g_{s_3}(X), \quad x_1 = \frac{f_{s_3} - f_{s_2}}{2L} + \frac{h}{2} \tag{2.23}$$

for $O_1$, $O_1''$;

$$g_{s_1}(X) = g_{s_2}(X), \quad x_2 = \frac{f_{s_2} - f_{s_1}}{2L} + \frac{h}{2} \tag{2.24}$$

for $O_1$, $O_1'$;

$$g_{s_1}(X) = g_{s_4}(X), \quad x_1 = \frac{f_{s_4} - f_{s_1}}{2L} + \frac{h}{2} \tag{2.25}$$

for $O_2'$, $O_2$;

$$g_{s_4}(X) = g_{s_3}(X), \quad x_2 = \frac{f_{s_3} - f_{s_4}}{2L} + \frac{h}{2} \tag{2.26}$$

for $O_2$, $O_2''$. Note that in Fig. 1 we present the case where $f_{s_2} + f_{s_4} \geqslant f_{s_3} + f_{s_1}$ and $f_{s_1} > f_{s_3}$.

**Corollary 2.1.** *The majorant of the class $C^2_{2,L,L,N}$ for $X \in K_p$ has the form*

$$A^+_{C^2_{2,L,L,N}}(X) = \bar{g}_{s_l}(X), \quad X \in \Omega_l^+, \ l = 1,2,3,4, \tag{2.27}$$

*where* $\bar{g}_{s_l}(X) = f_{s_l} + L\|X - X_{s_l}\|_2$, $\bigcup_{l=1}^4 \Omega_l^+ = K_p$, $p = 1,\ldots,m^2$.

**Proof.** We limit ourselves to the case presented in Fig. 1 where

(a) $f_{s_2} + f_{s_4} \geqslant f_{s_3} + f_{s_1}$, $f_{s_1} > f_{s_3}$.

Three other cases,

(b) $f_{s_2} + f_{s_4} \geqslant f_{s_3} + f_{s_1}$, $f_{s_1} \leqslant f_{s_3}$,
(c) $f_{s_1} + f_{s_3} > f_{s_2} + f_{s_4}$, $f_{s_2} > f_{s_4}$,
(d) $f_{s_1} + f_{s_3} > f_{s_2} + f_{s_4}$, $f_{s_2} \leqslant f_{s_4}$

are considered analogously.

Let $X \in \Omega_1^+$. It is easy to show that

$$\bar{g}_{s_1}(X) \leqslant \bar{g}_{s_i}(X), \quad i = 2,3,4 \ \forall X \in \Omega_1^+. \tag{2.28}$$

Indeed, let us prove this inequality, for example, for $i = 3$. We have

$$\begin{aligned} f_{s_1} + L\|X - X_{s_1}\|_2 - f_{s_3} - L\|X - X_{s_3}\|_2 &= f_{s_1} - f_{s_3} + L(x_1 + x_2) - L(h - x_1 + h - x_2) \\ &= f_{s_1} - f_{s_3} + 2L(x_1 + x_2 - h) \\ &\leqslant f_{s_1} - f_{s_3} + (f_{s_3} - f_{s_1} + 2Lh) - 2Lh = 0. \end{aligned} \tag{2.29}$$

The last inequality in (2.29) follows from the fact that in domain $\Omega_1^+$ we have

$$x_2 \leqslant -x_1 + \frac{f_{s_3} - f_{s_1}}{2L} + h. \tag{2.30}$$

An analogous statement can also be formulated for the majorant of class $C^2_{\bar{2},\,L_1,L_2,N}$ when $X \in \bar{K}_p$. In this case (which is similar to that in Fig. 1) we have

$$X_{s_1} = (0;0), \quad X_{s_2} = (0;h_2), \quad X_{s_3} = (h_1;h_2), \quad X_{s_4} = (h_1;0). \tag{2.31}$$

Elementary rectangle $\bar{K}_p$ is split into sub-regions $\bar{\Omega}_1^+$, $\bar{\Omega}_2^+$, $\bar{\Omega}_3^+$, $\bar{\Omega}_4^+$ by the following five lines:

$$x_2 = -\frac{L_1}{L_2}x_1 + \frac{f_{s_3} - f_{s_1}}{2L_2} + \frac{L_1 h_1}{2L_2} + \frac{h_2}{2} \tag{2.32}$$

for the line through $O_1$, $O_2$;

$$x_1 = \frac{f_{s_3} - f_{s_2}}{2L_1} + \frac{h_1}{2} \tag{2.33}$$

for $O_1$, $O_1''$;

$$x_2 = \frac{f_{s_2} - f_{s_1}}{2L_2} + \frac{h_2}{2} \tag{2.34}$$

for $O_1'$, $O_1$;

$$x_1 = \frac{f_{s_4} - f_{s_1}}{2L_1} + \frac{h_1}{2} \tag{2.35}$$

for $O_2'$, $O_2$;

$$x_2 = \frac{f_{s_3} - f_{s_4}}{2L_2} + \frac{h_2}{2} \tag{2.36}$$

for $O_2$, $O_2''$. □

**Corollary 2.2.** *The majorant of the class* $C^2_{\bar{2},\,L_1,L_2,N}$, $X \in \bar{K}_p$ *has the form*

$$A^+_{C^2_{\bar{2},L_1,L_2,N}}(X) = \bar{g}_{s_l}(X), \quad X \in \bar{\Omega}_l^+, \; l = 1,2,3,4, \tag{2.37}$$

*where* $\bar{g}_{s_l}(X) = f_{s_l} + L_1\|X - X_{s_l}\|_{\bar{2}}$, $\bigcup_{l=1}^4 \bar{\Omega}_l^+ = \bar{K}_p$, $p = 1,\ldots,mm_1$.

**Proof.** The proof is analogous to the proof of Corollary 2.1. □

Now, let us consider elementary rectangle $\tilde{K}_p$ (see Fig. 2). As before, we place the origin in the lower left vertex of $\tilde{K}_p$. The line $(O_1, O_2)$, whose equation is

$$x_1 = \frac{f_{s_2} - f_{s_1}}{2L_1} + \frac{h_1}{2} \tag{2.38}$$

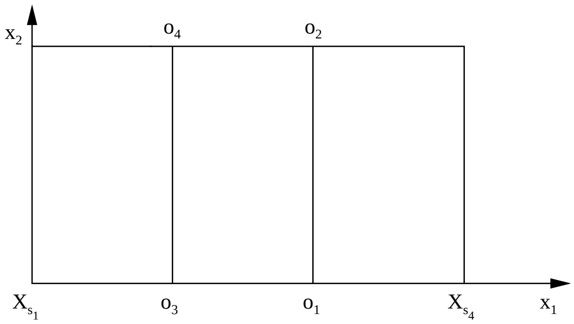


Fig. 2. Splitting of elementary rectangle $\tilde{K}_p$.

splits elementary rectangle $\tilde{K}_p$ into sub-regions $\tilde{\Omega}_1^+$ and $\tilde{\Omega}_2^+$. Therefore, the statement analogous to Corollary 2.2 also holds.

**Corollary 2.3.** *The majorant of the class $C^2_{\bar{2},\,L_1,L_2,N}$ for $X \in \tilde{K}_p$ has the form*

$$A^+_{C^2_{\bar{2},\,L_1,L_2,N}}(X) = \tilde{g}_{s_l}(X), \quad X \in \tilde{\Omega}_l^+,\ l = 1,2, \tag{2.39}$$

*where* $\tilde{g}_{s_l}(X) = f_{s_l} + L_1\|X - X_{s_l}\|_{\bar{2}}$, $\tilde{\Omega}_1^+ \cup \tilde{\Omega}_2^+ = \tilde{K}_p$, $p = 1,\dots,m$.

Therefore, Corollaries 2.1–2.3 allow us to single out regions of linearity of the majorant $A^+_{F_N}(X)$ and the minorant $A^-_{F_N}(X)$ for classes $C^2_{\bar{2},\,L_1,L_2,N}$ and $C^2_{2,L,L,N}$, represent them in $\pi_2$ and, finally, solve constructively the problem of optimal-by-accuracy recovery of function $f(X)$ from these classes at point $X \in \pi_2$.

## 3. On optimal integration of function products in classes $C^2_{\bar{2},\,L_1,L_2,N}$ and $C^2_{2,L,L,N}$

In this section we deal with the general integral (1.1) assuming that $f(x_1,x_2) \in C^2_{\bar{2},\,L_1,L_2,N}$ or $f(x_1,x_2) \in C^2_{2,L,L,N}$ and that $\varphi_1(x_1)$, $\varphi_2(x_2)$ are given integrable functions. We introduce the following notation:

$$\bar{I}^*_p = \frac{1}{2}\iint_{K_p} (A^+_{C^2_{2,L,L,N}}(X) + A^-_{C^2_{2,L,L,N}}(X))\varphi_1(x_1)\varphi_2(x_2)\,\mathrm{d}X, \quad p = 1,\dots,m^2,$$

$$\bar{\bar{I}}^*_p = \frac{1}{2}\iint_{\bar{K}_p} (A^+_{C^2_{\bar{2},\,L_1,L_2,N}}(X) + A^-_{C^2_{\bar{2},\,L_1,L_2,N}}(X))\varphi_1(x_1)\varphi_2(x_2)\,\mathrm{d}X, \quad p = 1,\dots,mm_1,$$

$$\tilde{I}^*_p = \frac{1}{2}\iint_{\tilde{K}_p} (A^+_{C^2_{\bar{2},\,L_1,L_2,N}}(X) + A^-_{C^2_{\bar{2},\,L_1,L_2,N}}(X))\varphi_1(x_1)\varphi_2(x_2)\,\mathrm{d}X, \quad p = 1,\dots,m. \tag{3.1}$$

We recall that in Section 2 we obtained explicit forms of functions $A^+_{F_N}(X)$, $A^-_{F_N}(X)$ for $F_N = C^2_{2,L,L,N}$ and $F_N = C^2_{\bar{2},\,L_1,L_2,N}$. Each of the domains $K_p, \bar{K}_p, \tilde{K}_p$ was split into sub-regions in which $A^+_{F_N}(X)$

and $A^-_{F_N}(X)$ were linear functions. Below we show how the results obtained in Section 2 can be applied to the constructive solution of the problem of computing integrals in (3.1).

**Theorem 3.1.** *If functions* $\varphi_1(x_1)$ *and* $\varphi_2(x_2)$ *do not change sign for* $x_1, x_2 \in [0,1]$, *then optimal-by-accuracy cubature formulae for computing integrals* (1.1) *have the form*

$$\bar{I}^* = \sum_{p=1}^{m^2} \bar{I}^*_p \quad \textit{when } F_N = C^2_{2,L,L,N} \tag{3.2}$$

*and the form*

$$\bar{\bar{I}}^* = \sum_{p=1}^{mm_1} \bar{I}^*_p + \sum_{l=1}^{m} \tilde{I}^*_l \quad \textit{when } F_N = C^2_{2,\,L_1,\,L_2,\,N}. \tag{3.3}$$

*The Chebyshev radius of the undefinability domain of integral values is defined by the formula*

$$\delta(F_N) = \frac{1}{2} \iint_{\pi_2} (A^+_{F_N}(X) - A^-_{F_N}(X)) |\varphi_1(x_1)\varphi_2(x_2)| \, \mathrm{d}X. \tag{3.4}$$

**Proof.** If $\varphi_1(x_1)\varphi_2(x_2) > 0$, then $\forall X \in \pi_2$, $X = (x_1, x_2)$ we have

$$I^{\pm}(F_N) = \iint_{\pi_2} A^{\pm}_{F_N}(X) \varphi_1(x_1)\varphi_2(x_2) \, \mathrm{d}X. \tag{3.5}$$

Similarly, if $\varphi_1(x_1)\varphi_2(x_2) < 0$, then

$$I^{\pm}(F_N) = \iint_{\pi_2} A^{\mp}_{F_N}(X) \varphi_1(x_1)\varphi_2(x_2) \, \mathrm{d}X. \tag{3.6}$$

The statement of the theorem follows from relationships (3.5) and (3.6) by taking into account (3.1), (1.6) and (1.7). □

**Remark 3.1.** Let the domain $\pi_2$ be split into sub-regions $\Phi_q$, $q = 1, \dots, Q$ where the product of functions $\varphi_1(x_1)$ and $\varphi_2(x_2)$ preserves the sign. Then in each sub-region $\Phi_q$ cubature formulae (3.2), (3.3) are optimal-by-accuracy. However, in all domain $\pi_2$ Theorem 3.1 does not hold in general.

When Theorem 3.1 fails, the majorant of the class $F_N$ is different from $A^+_{F_N}(X)$. Let, for example, $F_N = C^2_{2,L,L,N}$. We consider two neighbouring elementary squares $K_p$ and $K_{p+1}$ in which the sign of the product $\varphi_1(x_1)\varphi_2(x_2)$ changes from "+" to "−".

In the transition from the region $K_p$ to region $K_{p+1}$ the function

$$\gamma^+(X) = \begin{cases} A^+_{F_N}(X), & X \in K_p, \\ A^-_{F_N}(X), & X \in K_{p+1} \end{cases} \tag{3.7}$$

has a discontinuity, hence the Lipschitz condition is violated and $\gamma^+(X) \notin F_N$.

Let us choose $\gamma^+(X)$ in the following form:

$$\gamma^+(X) = \begin{cases} \min(A^+_{F_N}(X), l(X)), & X \in K_p, \\ \max(A^-_{F_N}(X), l(X)), & X \in K_{p+1}, \end{cases} \tag{3.8}$$

where function $l(X)$ performs "sewing" $A^+_{F_N}(X)$ and $A^-_{F_N}(X)$ in the transition from $K_p$ to $K_{p+1}$.

In this case $\gamma^+(X)$ satisfies Definition 1.1 for the majorant of class $F_N$ and the following relationship:

$$\iint_{K_p \cup K_{p+1}} \gamma^+(X)\varphi_1(x_1)\varphi_2(x_2)\,\mathrm{d}X = \sup_{f \in F_N} \iint_{K_p} f(X)\varphi_1(x_1)\varphi_2(x_2)\,\mathrm{d}X + \inf_{f \in F_N} \iint_{K_{p+1}} f(X)\varphi_1(x_1)\varphi_2(x_2)\,\mathrm{d}X. \tag{3.9}$$

takes place. Then

$$\begin{aligned} I^+(F_N) &= \sup_{f \in F_N} \iint_{\pi_2} f(X)\varphi_1(x_1)\varphi_2(x_2)\,\mathrm{d}X = \iint_{\pi_2} \gamma^+(X)\varphi_1(x_1) \times \varphi_2(x_2)\,\mathrm{d}X \\ &= \sum_{p=1}^{m^2} \iint_{K_p} \gamma^+(X)\varphi_1(x_1)\varphi_2(x_2)\,\mathrm{d}X = \sum_{p=1}^{m^2} I_p^+. \end{aligned} \tag{3.10}$$

The choice of function $l(X)$ in (3.8) is determined by the condition

$$I_p^+ + I_{p+1}^+ = \sup_{f \in F_N} \iint_{K_p \cup K_{p+1}} f(X)\varphi_1(x_1)\varphi_2(x_2)\,\mathrm{d}X. \tag{3.11}$$

We also note that the need of "sewing" $A^+_{F_N}(X)$ and $A^-_{F_N}(X)$ directly follows from the fact that $\gamma^+(X) \in F_N$.

We define the function $l(X)$ in the following form

$$l(X) = -Lx_1 + B_1(x_2)x_2 + B_2(x_2), \tag{3.12}$$

where $B_1(x_2), B_2(x_2)$ are certain piecewise constant functions, values of which are defined by relationship (3.8).

In the general case the problem of finding $B_1(x_2), B_2(x_2)$ in (3.12) is fairly difficult. Even in a simple case when zeros of functions $\varphi_1(x_1), \varphi_2(x_2)$ coincide with grid nodes, then the problem of construction of function $\gamma^+(X)$ is too difficult for this approach to be used in practice. Thus the need arises for a simpler close-to-optimal method.

**Corollary 3.1.** *Let functions $\varphi_1(x_1), \varphi_2(x_2)$ change sign when $x_1, x_2 \in [0, 1]$. Then the error of formulae* (3.2), (3.3) *will not be more than twice the optimal error.*

**Proof.** In Section 2 we constructed the majorants $A^+_{F_N}(X)$ and the minorants $A^-_{F_N}(X)$ in the cases when $F_N = C^2_{2,\,L_1,L_2,N}$ and $F_N = C^2_{2,L,L,N}$. We also recall that the function

$$f^*(X) = \tfrac{1}{2}(A^+_{F_N}(X) + A^-_{F_N}(X)) \tag{3.13}$$

is the optimal-by-accuracy approximation of function $f(X) \in F_N$.

For $F_N = C^2_{2,L,L,N}$ we denote

$$\hat{I}^*(f^*) = \iint_{\pi_2} f^*(X)\varphi_1(x_1)\varphi_2(x_2)\,\mathrm{d}X, \quad X = (x_1, x_2). \tag{3.14}$$

In contrast to $\gamma^+(X)$ in the form (3.7), function $f^*(X) \in C^2_{2,L,L,N}$ is continuous as the sum of two continuous functions. By the same token, it satisfies the Lipschitz condition and passes through points $f_v$, $v = 1,\ldots,N$. Hence, $\hat{I}^*(f^*) \in [I^-(F_N), I^+(F_N)]$, where $I^+(F_N)$ and $I^-(F_N)$ are limits of the values of integral (1.1) when $F_N = C^2_{2,L,L,N}$ (taking into account changes of the sign of functions $\varphi_1(x_1)$, $\varphi_2(x_2)$). Moreover, the optimal-by-accuracy value of (1.1) for this class is

$$I^*(F_N) = \tfrac{1}{2}(I^+(F_N) + I^-(F_N)) \tag{3.15}$$

with the error determined as

$$\delta(F_N) = \tfrac{1}{2}(I^+(F_N) - I^-(F_N)). \tag{3.16}$$

It is easy to see that when functions $\varphi_1(x_1), \varphi_2(x_2)$ change sign for $x_1, x_2 \in [0,1]$, $\hat{I}^*(f^*) \neq I^*(F_N)$ and the inequality

$$|\hat{I}^*(f^*) - I^*(F_N)| \leqslant \delta(F_N) \tag{3.17}$$

holds. Taking into account that

$$|I^2(f) - I^*(F_N)| \leqslant \delta(F_N), \tag{3.18}$$

and using the triangle inequality, from (3.17) and (3.18) we have

$$|\hat{I}^*(f^*) - I^2(f)| \leqslant 2\delta(F_N). \tag{3.19}$$

Inequality (3.19) leads to the statement of the theorem. The case $F_N = C^2_{2,\,L_1,L_2,N}$ is considered analogously. $\square$

It is worthwhile noting that relationships (3.8)–(3.18) hold not only for the function $f^*(X)$ but for any recovered function $\tilde{f}$ from the class $F_N$. We also note that a spline-based approach proposed earlier in [17] can also be used for the construction of efficient cubature formulae for computing the integral $I^2(f)$ in classes $C^2_{2,\,L_1,L_2,N}$ and $C^2_{2,L,L,N}$. Indeed, let $C^2_{1,L,N\times M}$ be the class of functions such that

- these functions are defined on $\pi_2$ by their values in the nodes $(x_{1,i}; x_{2,j})$, $(i=1,\ldots,N,\ j=1,\ldots,M)$ of an arbitrary grid on $\pi_2$;
- they satisfy the condition $\sup_{x_1,x_2}\max(|f'_{x_1}|, |f'_{x_2}|) \leqslant L$.

Then, it can be shown, for example, that the class $C^2_{1,L,N\times M}$ is close to classes $C^2_{\tilde{2},\,L_1,L_2,N}$ and $C^2_{2,L,L,N}$ and functions $\tilde{S}_1(x_1,x_2)$ and $\tilde{S}_2(x_1,x_2)$ constructed using a linear-spline approximation (see (3.1) in [17]) belong to classes $C^2_{\tilde{2},\,L_1,L_2,N}$ and $C^2_{2,L,L,N}$, respectively. Moreover, in the general case, cubature formulae

$$Q_1(\tilde{S}_1)=\int_0^1\int_0^1 \tilde{S}_1(x_1,x_2)\varphi_1(x_1)\varphi_2(x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2, \tag{3.20}$$

$$Q_2(\tilde{S}_2)=\int_0^1\int_0^1 \tilde{S}_2(x_1,x_2)\varphi_1(x_1)\varphi_2(x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2, \tag{3.21}$$

have the same accuracy properties as formulae (3.2) and (3.3). However, it is reasonable to apply formulae (3.20) and (3.21) only in the case when we know not Lipschitz constants themselves but only their estimates. Although the method of integrand approximation by a linear spline proposed in [17] (see also [2,3,16–18,24–27]) allows us to construct optimal-by-order cubature formulae without knowledge of Lipschitz constants, that method is unable to constructively compute error estimates for such formulae.

If a priori information about the problem is known exactly, then the approach proposed in this section has a number of advantages. First of all we admit that if zeros of functions $\varphi_1(x_1)$, $\varphi_2(x_2)$ are located relatively sparsely with respect to grid nodes (the weak oscillations case [18]), then in regions with constant sign of functions $\varphi_1(x_1),\varphi_2(x_2)$ formulae (3.2) and (3.3) will be optimal-by-accuracy. Then, in this case $v(F_N,\hat{I}^*,f)=|I^2(f)-I^*(F_N)|$. Moreover, the proposed approach allows us to simultaneously construct an estimate of optimal error $v(F_N,\hat{I}^*,f)$ (using (3.18) and (3.4)):

$$v(F_N,\hat{I}^*,f)\leqslant\frac{1}{2}\iint_{\pi_2}(A^+_{F_N}(X)-A^-_{F_N}(X))|\varphi_1(x_1)\varphi_2(x_2)|\,\mathrm{d}X. \tag{3.22}$$

Therefore, in the case of strong oscillations of functions $\varphi_1(x_1),\varphi_2(x_2)$ under exact a priori information, the application of cubature formulae (3.2) and (3.3) is more favourable.

## 4. The choice of grids in the class $C^2_{2,L,L,N}$

The passage from a functional class $F$ to an interpolational class $F_N$ is usually due to the desire to maximise the usage of a priori available information about the problem. However, in working with interpolational classes it is important to realise that, in practice, we often have to deal with functions with fairly complicated structures. Hence, for computing functional characteristics (such as function values) we may need an expensive physical or computational experiment. Such situations occur in automative design problems, signal and image processing and many other applications [4,5,10,8]. This leads to a dilemma. Indeed, on the one hand, we have to in the most complete way obtain a priori information about the problem. On the other hand, we have to decrease the number of expensive function evaluations.

In the construction of optimal-by-accuracy and optimal-by-order cubature formulae for computing integrals $I_i^2(f)$, $i=1,2,3$ in the class $C^2_{2,L,L,N}$, the resolution of this dilemma requires the

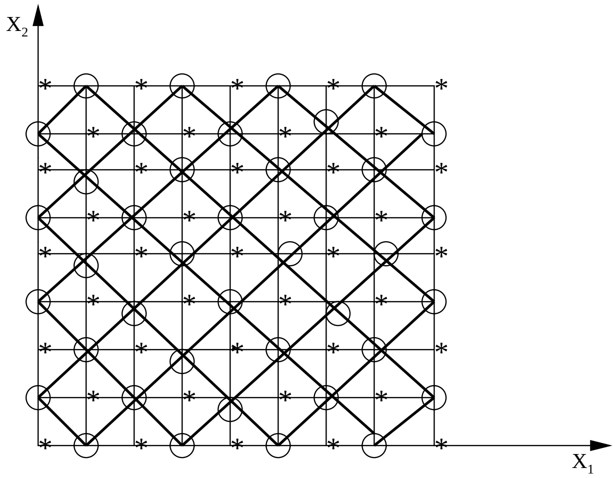


Fig. 3. Splitting $\pi_2$ by the grid $\gamma$.

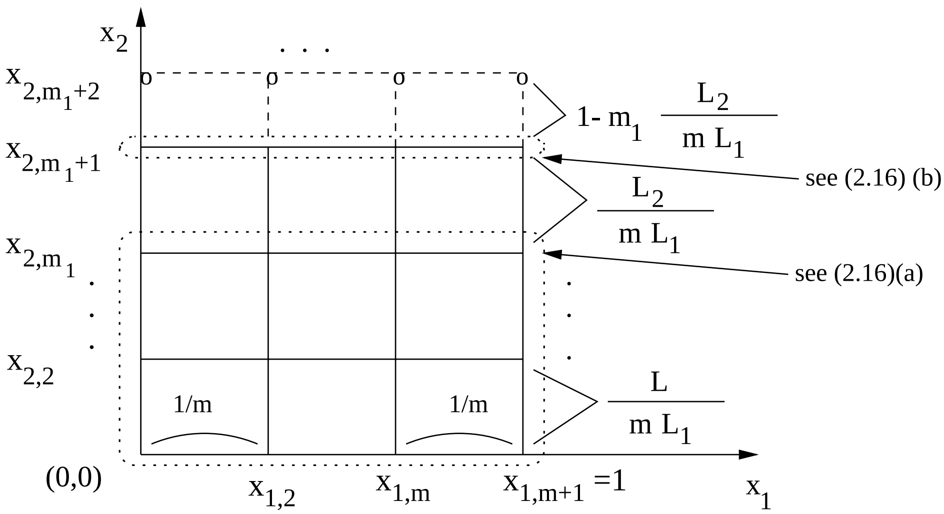


Fig. 4. Construction of the grid $\Delta_2$.

consideration of optimal (in a certain sense specified bellow) grids in $\pi_2$ that would allow us to compute function values only at nodes of such a grid.

Let $F_N = C^2_{2,L,L,N}$. First, we consider the grid $\gamma$ which splits $\pi_2$ into $4n^2$ equal elementary squares $K_p$ with side $1/(2n)$, $p=1,\dots,4n^2$ (see Fig. 3). Then, we split $\gamma$ into two subsets: $\gamma_1 \cup \gamma_2 = \gamma$, $\gamma_1 \cap \gamma_2 = \emptyset$. Let us assume that the grid $\gamma_1 \subset \gamma$ consists of nodes $X_{v_1} = ((i-1)1/2n;(j-1)1/2n)$, $v_1 = (i-1)(2n+1)+j$, $i=\{1,3,\dots,2n+1\}$, $j=\{2,4,\dots,2n\}$ and nodes $X_{v_2} = ((i-1)1/2n,(j-1)1/2n)$, $v_2 = (i-1)(2n+1)+j+1$, $i=\{2,4,\dots,2n\}$, $j=\{1,3,\dots,2n+1\}$. Similarly, let the grid $\gamma_2 \subset \gamma$ consist of the nodes $X_{\mu_1} = ((i-1)1/2n,(j-1)1/2n)$, $\mu_1 = (i-1)(2n+1)+j$, $i,j=\{1,3,\dots,2n+1\}$ and nodes $X_{\mu_2} = ((i-1)1/2n,(j-1)1/2n)$, $\mu_2 = (i-1)(2n+1)+j$, $i,j=\{2,4,\dots,2n\}$. The grid

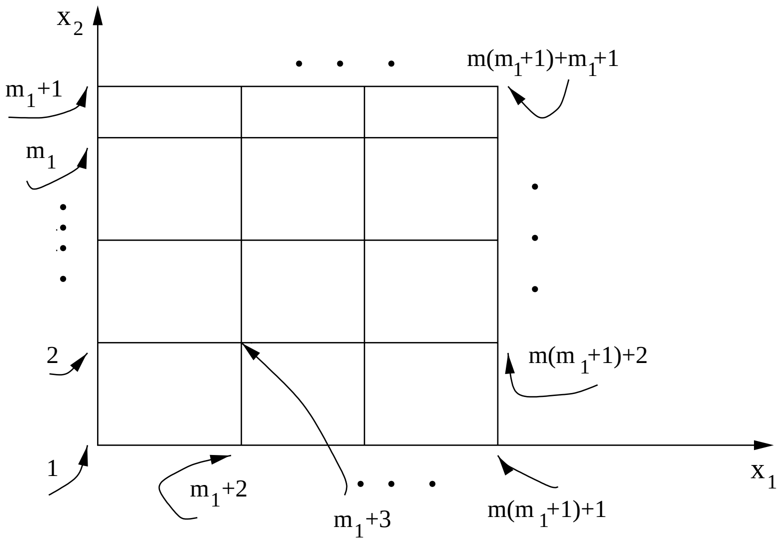


Fig. 5. Numbering nodes of the grid $\Delta_2$.

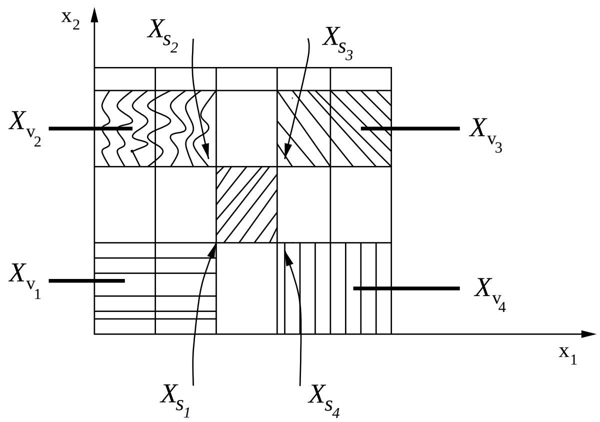


Fig. 6. Groups of nodes of the grid $\Delta_2$.

$\gamma_1$ splits the domain $\pi_2$ into certain elementary regions of rhombic forms and their parts. The nodes of $\gamma_2$ are centres of these rhombuses. In Fig. 3 the nodes of $\gamma_1$ are highlighted with circles, and the nodes of $\gamma_2$ are highlighted with stars. Further denote

- the set of nodes from $\gamma_1$ that lie on the sides of $\pi_2$ by $\tilde{\gamma}_1$;
- the set of nodes from $\gamma_2$ that lie on the sides of $\pi_2$ by $\tilde{\gamma}_2$;
- the set of nodes from $\tilde{\gamma}_2$ that consists of the vertices of $\pi_2$ by $\gamma_2^*$.

It is easy to see that $\tilde{\gamma}_2$ consists completely of nodes of the form $X_{\mu_1}$.

We assume that function $f(X)$ may be given by its values not in all nodes of the grid $\gamma$, but only at nodes of the grid $\gamma_1 \subset \gamma$. Therefore, we allow the situation when we are given function values not in $N=(2n+1)^2$ nodes, but only at $\bar{N}=2n(n+1)$ nodes, i.e., only at two vertices of the square $K_p$, $p=1,\dots,4n^2$. Hence, instead of the class $C^2_{2,L,L,N}$ it is more reasonable to consider the class $C^2_{2,L,L,\bar{N}}$, which is defined as follows. It is the class of such functions that are defined in the domain $\pi_2=\{X=(x_1,x_2)\colon 0 \leqslant x_i \leqslant 1,\ i=1,2\}$, that satisfy the Lipschitz condition with constant $L$ (in each variable) and that take fixed values $f_1,\dots,f_{\bar{N}}$ at nodes $X_1,\dots,X_{\bar{N}}$ of grid $\gamma_1$, respectively.

Let us consider a certain node $X_{s_0}=(\bar{i}\bar{h},\bar{j}\bar{h})$, $X_{s_0}\in\gamma_2\setminus\tilde{\gamma}_2$, $\bar{h}=1/(2n)$. From the set of all nodes we single out a subset of nodes that is defined as follows: $X_{s_1}=((\bar{i}-1)\bar{h},\bar{j}\bar{h})$, $X_{s_2}=(\bar{i}\bar{h},(\bar{j}+1)\bar{h})$, $X_{s_3}=((\bar{i}+1)\bar{h},\bar{j}h)$, $X_{s_4}=(\bar{i}\bar{h},(\bar{j}-1)\bar{h})$. Let $\sigma(\{X_{s_l}\}_{l=1,2,3,4})=\{s_l\}_{l=1,2,3,4}$, $f(X_{s_l})=f_{s_l}$, $l=1,2,3,4$. Then the following result holds.

**Theorem 4.1.** *Let $F_N=C^2_{2,\,L,\,L,\,\bar{N}}$. Then $\forall X_{s_0}\in\gamma_2\setminus\tilde{\gamma}_2$ we have*

$$A^+_{C^2_{2,\,L,\,L,\,\bar{N}}}(X_{s_0})=\min_{s\in\sigma(\{X_{s_l}\}_{l=1,\,2,\,3,\,4})} f_s+L\bar{h},\quad A^-_{C^2_{2,\,L,\,L,\,\bar{N}}}(X_{s_0})=\max_{s\in\sigma(\{X_{s_l}\}_{l=1,\,2,\,3,\,4})} f_s-L\bar{h}. \tag{4.1}$$

**Proof.** First, we introduce the functions defined $\forall X_v\in\gamma_1\setminus\{X_{s_l}\}_{l=1,2,3,4}$ as

$$g_{s_l}(X)=f_{s_l}+L\|X-X_{s_l}\|_2,\ l=1,2,3,4,\quad g_v(X)=f_v+L\|X-X_v\|_2. \tag{4.2}$$

Let us show that

$$g_v(X_{s_0})-g_{s_l}(X_{s_0})\geqslant 0\quad \forall v\in\sigma(\gamma_1\setminus\{X_{s_l}\}_{l=1,2,3,4}),\ \forall s_l,\ l=1,2,3,4. \tag{4.3}$$

As an example, we consider the node $X_{\tilde{v}}=((\bar{i}-1-k_1)\bar{h},(\bar{j}+k_2)\bar{h})$, $X_{\tilde{v}}\in\gamma_1\setminus\{X_{s_l}\}_{l=1,2,3,4}$. We have

$$\begin{aligned} g_{\tilde{v}}(X_{s_0})-g_{s_1}(X_{s_0}) &= f_{\tilde{v}}+L\|X_{s_0}-X_{\tilde{v}}\|_2-f_{s_1}-L\|X_{s_0}-X_{s_1}\|_2 \\ &\geqslant -L(k_1+k_2)\bar{h}+L(k_1+k_2+1)\bar{h}-L\bar{h}=0. \end{aligned} \tag{4.4}$$

Similarly, it can be proved that inequalities analogous to (4.3) also hold for $s_l$, $l=2,3,4$. From relationship (2.1) and inequalities (4.3) it follows that

$$A^+_{C^2_{2,L,L,\bar{N}}}(X_{s_0})=\min_{i=1,\dots,N}(f_s+L\|X_{s_0}-X_{s_i}\|_2)=\min_{s\in\sigma(\{X_{s_l}\}_{l=1,\,2,\,3,\,4})} f_s+L\bar{h}. \tag{4.5}$$

Therefore, we have shown that the value of the majorant of the class $C^2_{2,L,L,\bar{N}}$ at any point $X_{s_0}\in\gamma_2\setminus\tilde{\gamma}_2$ is determined by its value at four closest to $X_{s_0}$ nodes of the grid $\gamma_1$ (i.e., by the nodes of the form $X_{s_l}$ for which $\|X_{s_0}-X_{s_l}\|_2=1/(2n)$, $l=1,2,3,4$). For the minorant $A^-_{C^2_{2,L,L,\bar{N}}}$ the proof is analogous. □

**Corollary 4.1.** *Let $X_{s_0}\in\tilde{\gamma}_2\setminus\gamma_2^*$. Then the following relationships hold*:

$$A^+_{C^2_{2,L,L,\bar{N}}}(X_{s_0})=\min_{l=1,\,2,\,3} f_{s_l}+L\bar{h},\quad A^-_{C^2_{2,\,L,\,L,\,\bar{N}}}(X_{s_0})=\max_{l=1,\,2,\,3} f_{s_l}-L\bar{h} \tag{4.6}$$

*where $s_l$ is the number of the node $X_{s_l}$ of the grid $\gamma_1$ such that $\|X_{s_0}-X_{s_l}\|_2=1/(2n)$, $l=1,2,3$.*

**Corollary 4.2.** *Let $X_{s_0} \in \gamma_2^*$. Then the following relationships hold*:

$$A^+_{C^2_{2,\,L,\,L,\,\bar N}}(X_{s_0}) = \min(f_{s_1}, f_{s_2}) + L\bar h, \quad A^-_{C^2_{2,\,L,\,L,\,\bar N}}(X_{s_0}) = \max(f_{s_1}, f_{s_2}) - L\bar h \tag{4.7}$$

*where $s_1$, $s_2$ are the numbers of the nodes $X_{s_1}$, $X_{s_2}$ of the grid $\tilde\gamma_1$ such that $\|X_{s_0} - X_{s_l}\|_2 = 1/(2n)$, $l = 1, 2$.*

Proofs of the Corollaries 4.1 and 4.2 are analogous to the proof of Theorem 4.1.

As we mentioned before, the difficulties in the realisation of approach (1.6)–(1.8) lie in the need for constructive computation of quantities $I_i^+(F_N), I_i^-(F_N)$, $i = 1, 2, 3$. In Sections 2 and 3 we constructively solved this problem under the assumption that functions of the given class are known at the nodes of the grid $\gamma$. Hence, for computing $I_i^+(C^2_{2,L,L,\bar N})$ we extend the definition of $f(X)$ as $f(X) = A^+_{C^2_{2,L,L,\bar N}}(X)$, $X \in \gamma_2$, and for computing $I_i^-(C^2_{2,L,L,\bar N})$ we extend the definition of $f(X)$ as $f(X) = A^-_{C^2_{2,L,L,\bar N}}(X)$, $X \in \gamma_2$, where in both cases $i = 1, 2, 3$.

**Lemma 4.1.** *For the majorant of the class $C^2_{2,L,L,\bar N}$ the following relationship holds*:

$$A^+_{C^2_{2,L,L,\bar N}}(X) = \min_{\mu=1,\dots,N}(f_\mu + L\|X - X_\mu\|_2) \tag{4.8}$$

*with $f_\mu = f(X_\mu)$ for $X_\mu \in \gamma_1$ and with $f_\mu = A^+_{C^2_{2,L,L,\bar N}}(X_\mu)$ for $X_\mu \in \gamma_2$.*

**Proof.** Let us consider the function $B^+(X) = \min_{\mu=1,\dots,N}(f_\mu + L\|X - X_\mu\|_2)$ with $f_\mu = f(X_\mu)$ for $X_\mu \in \gamma_1$ and with $f_\mu = A^+_{C^2_{2,L,L,\bar N}}(X_\mu)$ for $X_\mu \in \gamma_2$.

According to the definition of the majorant we have

$$A^+_{C^2_{2,L,L,\bar N}}(X) = \min_{v=1,\dots,\bar N}(f_v + L\|X - X_v\|_2). \tag{4.9}$$

Let us now show that functions $A^+_{C^2_{2,L,L,\bar N}}(X)$ and $B^+(X)$ coincide. Indeed,

$$\begin{aligned} A^+_{C^2_{2,L,L,\bar N}}(X) - B^+(X) &= \min_{v=1,\dots,\bar N}(f_v + L\|X - X_v\|_2) - \min_{\mu=1,\dots,N}(f_\mu + L\|X - X_\mu\|_2) \\ &= f_{v_0} + L\|X - X_{v_0}\|_2 - f_{\mu_0} - L\|X - X_{\mu_0}\|_2 \\ &\geqslant f_{v_0} + L\|X - X_{v_0}\|_2 - f_{v_0} - L\|X - X_{v_0}\|_2 = 0. \end{aligned} \tag{4.10}$$

The inequality in (4.10) follows from the fact that $\gamma_1 \subset \gamma$ and $B^+(X) \leqslant f_{v_0} + L\|X - X_{v_0}\|_2$.

If $X_{\mu_0} \in \gamma_1$, then

$$\begin{aligned} &f_{v_0} + L\|X - X_{v_0}\|_2 - f_{\mu_0} - L\|X - X_{\mu_0}\|_2 \\ &\quad\leqslant f_{\mu_0} + L\|X - X_{\mu_0}\|_2 - f_{\mu_0} - L\|X - X_{\mu_0}\|_2 = 0. \end{aligned} \tag{4.11}$$

On the other hand, if $X_{\mu_0} \in \gamma_2$, then from Theorem 4.1 and its corollaries it follows that

$$\begin{aligned} &f_{v_0} + L\|X - X_{v_0}\|_2 - f_{\mu_0} - L\|X - X_{\mu_0}\|_2 \\ &\quad= f_{v_0} + L\|X - X_{v_0}\|_2 - f_{v_1} - L\bar h - L\|X - X_{\mu_0}\| \end{aligned}$$

$$\leqslant f_{v_1} + L\|X - X_{v_1}\|_2 - f_{v_1} - L(\bar{h} + |x_1 - x_{1,\mu_0}| + |x_2 - x_{2,\mu_0}|)$$

$$\leqslant L(\bar{h} + |x_1 - x_{1,\mu_0}| + |x_2 - x_{2,\mu_0}|) - L(\bar{h} + |x_1 - x_{1,\mu_0}| + |x_2 - x_{2,\mu_0}|) = 0. \tag{4.12}$$

Inequalities (4.10)–(4.12) lead to the statement of the lemma. □

Analogously to the above proof it can be shown that

$$A^{-}_{C^2_{2,L,L,\bar{N}}}(X) = \max_{\mu=1,\dots,N}(f_\mu - L\|X - X_\mu\|_2) \tag{4.13}$$

with $f_\mu = f(X_\mu)$ for $X_\mu \in \gamma_1$ and with $f_\mu = A^{-}_{C^2_{2,L,L,\bar{N}}}(X_\mu)$ for $X_\mu \in \gamma_2$.

The results obtained in Sections 2–4 allow us to efficiently solve problems (1.10)–(1.13) as well as construct cubature formulae for computing (1.1)–(1.3) with a substantial reduction of required a priori information. Indeed, we propose to use the information about values of function $f(X)$ only at those nodes of the grid $\gamma_1$ which are centres of such balls with radius $1/(2n)$ that cover $\pi_2$ in an optimal way.

The grids $\gamma$ and $\gamma_1$ split the domain $\pi_2$ into elementary regions $K_p$ and $K'_d$, respectively. Therewith, diameters of $K_p$ and $K'_d$ coincide in the given norm. From this fact it follows that the accuracy estimates for the recovery of $f^*(X)$ in classes $C^2_{2,L,L,N}$ and $C^2_{2,L,L,\bar{N}}$ coincide in spite of the fact that the number of nodes of the grid $\gamma$ is two times larger than the number of nodes of the grid $\gamma_1$. Finally, we note that the use of the optimal-by-accuracy recovery of function $f^*(X) \in C^2_{2,L,L,N}$ in formulae (3.2) and (3.3) justifies the application of the proposed grids $\gamma_1$ for these cubature formulae. Advantages of the proposed approach can be easily observed when $\varphi_1(x_1) = 1$, $\varphi_2(x_2) = 1$.

We conclude this section with the following result.

**Theorem 4.2.** *The following estimate*

$$\bar{\bar{\delta}}(C^2_{2,L,L,N}) \leqslant \bar{\bar{\delta}}(C^2_{2,L,L,\bar{N}}) \tag{4.14}$$

*holds.*

**Proof.** It is easy to see that

$$A^{+}_{C^2_{2,L,L,\bar{N}}}(X) \geqslant A^{+}_{C^2_{2,L,L,N}}(X), \quad A^{-}_{C^2_{2,L,L,\bar{N}}}(X) \leqslant A^{-}_{C^2_{2,L,L,N}}(X), \quad X \in \pi_2. \tag{4.15}$$

In addition, when $f(X) = \text{const}$, $X \in \gamma$ we have

$$\delta(C^2_{2,L,L,N}) = \bar{\bar{\delta}}(C^2_{2,L,L,N}) = \bar{\bar{\delta}}(C^2_{2,L,L,\bar{N}}). \tag{4.16}$$

From (4.15) and (4.16) the statement of the theorem follows immediately. □

## 5. Optimal-by-accuracy cubature formulae for functions from classes $C^2_{\bar{2},\,L_1,L_2,N}$ and $C^2_{2,L,L,N}$

Let $F_N = C^2_{\bar{2},\,L_1,L_2,N}$. Using Corollary 2.2 and relationships (2.32)–(2.36), it is easy to show that the splitting of $\bar{K}_p$ into regions $\bar{\Omega}^{+}_l$, $l = 1,2,3,4$, $p = 1,\dots,mm_1$ is determined by points $O_1(\bar{\bar{x}}_{1,i}, \bar{\bar{x}}_{2,j})$,

$O_2=(\bar{x}_{1,i},\bar{x}_{2,j})$ (the situation is similar to that shown in Fig. 1) with

$$\bar{x}_{1,i}=x_{1,i}+\frac{h_1}{2}+\frac{f_{i+1,j}-f_{i,j}}{2L_1}\delta_1+\frac{f_{i+1,j+1}-f_{i,j+1}}{2L_1}\delta_2,$$
$$\bar{\bar{x}}_{1,i}=x_{1,i}+\frac{h_1}{2}+\frac{f_{i+1,j+1}-f_{i,j+1}}{2L_1}\delta_1+\frac{f_{i+1,j}-f_{i,j}}{2L_1}\delta_2, \tag{5.1}$$

$$\bar{x}_{2,j}=x_{2,j}+\frac{h_2}{2}+\frac{f_{i+1,j+1}-f_{i+1,j}}{2L_2},\quad \bar{\bar{x}}_{2,j}=x_{2,j}+\frac{h_2}{2}+\frac{f_{i,j+1}-f_{i,j}}{2L_2}, \tag{5.2}$$

$$\delta_1=\tfrac{1}{2}(1-\operatorname{sign}(f_{i,j}+f_{i+1,j+1}-f_{i,j+1}-f_{i+1,j})),$$
$$\delta_2=\tfrac{1}{2}(1+\operatorname{sign}(f_{i,j}+f_{i+1,j+1}-f_{i,j+1}-f_{i+1,j})) \tag{5.3}$$

and $i=1,\dots,m,\ j=1,\dots,m_1$. In an analogous way we obtain that the splitting of $\bar{K}_p$ into regions $\bar{\Omega}_l^-,\ l=1,2,3,4,\ p=1,\dots,mm_1$ is determined by points $\mathrm{O}_3=(\tilde{x}_{1,i},\tilde{x}_{2,j})$, $\mathrm{O}_4=(\tilde{\tilde{x}}_{1,i},\tilde{\tilde{x}}_{2,i})$ (similar to Fig. 1) with

$$\tilde{x}_{1,i}=x_{1,i}+\frac{h_1}{2}-\frac{f_{i+1,j+1}-f_{i,j+1}}{2L_1}\delta_1-\frac{f_{i+1,j}-f_{i,j}}{2L_1}\delta_2,$$
$$\tilde{\tilde{x}}_{1,i}=x_{1,i}+\frac{h_1}{2}-\frac{f_{i+1,j}-f_{i,j}}{2L_1}\delta_1-\frac{f_{i+1,j+1}-f_{i,j+1}}{2L_1}\delta_2, \tag{5.4}$$

$$\tilde{x}_{2,j}=x_{2,j}+\frac{h_2}{2}-\frac{f_{i+1,j+1}-f_{i+1,j}}{2L_2},\quad \tilde{\tilde{x}}_{2,j}=x_{2,j}+\frac{h_2}{2}-\frac{f_{i,j+1}-f_{i,j}}{2L_2}, \tag{5.5}$$

and $\delta_1$, $\delta_2$ defined by relationships (5.3), $i=1,\dots,m,\ j=1,\dots,m_1$. Using Corollary 2.3 and relationship (2.38) we split $\tilde{K}_p$ into $\tilde{\Omega}_l^+,\ l=1,2$ by points $O_1=(\hat{x}_{1,i},x_{2,m_1+1})$, $O_2=(\hat{x}_{1,i},1)$, and similarly, we split $\tilde{K}_p$ into $\tilde{\Omega}_l^-,\ l=1,2$ by points $O_3=(\dot{x}_{1,i},x_{2,m_1+1})$, $O_4=(\dot{x}_{1,i},1)$ (see Fig. 1), where

$$\hat{x}_{1,i}=x_{1,i}+\frac{h_1}{2}+\frac{f_{i+1,m_1+1}-f_{i,m_1+1}}{2L_1},\quad \dot{x}_{1,i}=x_{1,i}+\frac{h_1}{2}-\frac{f_{i+1,m_1+1}-f_{i,m_1+1}}{2L_1} \tag{5.6}$$

and $i=1,\dots,m,\ p=1,\dots,m$.

Now let $F_N=C^2_{2,L,L,N}$. Using Corollary 2.1 and relationships (2.22)–(2.26), it is easy to show that the splitting of $K_p$ into $\Omega_l^+,\ l=1,\dots,4$ is determined by points $O_1=(\bar{\bar{x}}_{1,i},\bar{\bar{x}}_{2,j})$, $O_2=(\bar{x}_{1,i},\bar{x}_{2,j})$ and the splitting $K_p$ into $\Omega_l^-,\ l=1,2,3,4$ is determined by points $O_3=(\tilde{x}_{1,i},\tilde{x}_{2,j})$, $O_4=(\tilde{\tilde{x}}_{1,i},\tilde{\tilde{x}}_{2,j})$ (see Fig. 1), where $\bar{x}_{1,i}$, $\bar{\bar{x}}_{1,i}$, $\tilde{x}_{1,i}$, $\tilde{\tilde{x}}_{1,i}$ $(i=1,\dots,m)$ and $\bar{x}_{2,j}$, $\bar{\bar{x}}_{2,j}$, $\tilde{x}_{2,j}$, $\tilde{\tilde{x}}_{2,j}$ $(j=1,\dots,m,\ p=1,\dots,m^2)$ are computed by formulae (5.1)–(5.5), respectively, for $L_1=L_2=L$ and $h_1=h_2=h$.

We start by considering the problem of computing optimal-by-accuracy values of integral $I_1^2(f)$. Let us introduce the following notation:

$$U_p^*=\frac{1}{2}(U_p^++U_p^-)=\frac{1}{2}\left(\iint_{K_p}A^+_{C^2_{2,L,L,N}}(X)\,\mathrm{d}X+\iint_{K_p}A^-_{C^2_{2,L,L,N}}(X)\,\mathrm{d}X\right), \tag{5.7}$$

where $p = 1,\dots,m^2$;

$$\bar{U}_p^* = \frac{1}{2}(\bar{U}_p^+ + \bar{U}_p^-) = \frac{1}{2}\left(\iint_{\bar{K}_p} A^+_{C^2_{\bar{2},\,L_1,L_2,N}}(X)\,\mathrm{d}X + \iint_{\bar{K}_p} A^-_{C^2_{\bar{2},\,L_1,L_2,N}}(X)\,\mathrm{d}X\right), \tag{5.8}$$

where $p = 1,\dots,mm_1$ and

$$\tilde{U}_p^* = \frac{1}{2}(\tilde{U}_p^+ + \tilde{U}_p^-) = \frac{1}{2}\left(\iint_{\tilde{K}_p} A^+_{C^2_{\bar{2},\,L_1,L_2,N}}(X)\,\mathrm{d}X + \iint_{\tilde{K}_p} A^-_{C^2_{\bar{2},\,L_1,L_2,N}}(X)\,\mathrm{d}X\right), \tag{5.9}$$

where $p = 1,\dots,m$. From Theorem 3.1 it follows that the optimal-by-accuracy cubature formula for computing integral $I_1^2(f)$ has the form

$$U^* = \sum_{p=1}^{m^2} U_p^* \quad \text{when } F_N = C^2_{2,L,L,N} \tag{5.10}$$

and the form

$$\bar{U}^* = \sum_{p=1}^{mm_1} \bar{U}_p^* + \sum_{p=1}^{m} \tilde{U}_p^* \quad \text{when } F_N = C^2_{\bar{2},\,L_1,L_2,N}. \tag{5.11}$$

Moreover,

$$\bar{\bar{\delta}}(C^2_{2,L,L,N}) = \frac{1}{2}\sum_{p=1}^{m^2}(U_p^+ - U_p^-),$$

$$\bar{\bar{\delta}}(C^2_{\bar{2},\,L_1,L_2,N}) = \frac{1}{2}\left(\sum_{p=1}^{mm_1}(\bar{U}_p^+ - \bar{U}_p^-) + \sum_{p=1}^{m}(\tilde{U}_p^+ - \tilde{U}_p^-)\right). \tag{5.12}$$

Let

$$\bar{\kappa}_{1,i} = x_{1,i} + \frac{h_1}{2} + \delta\left(\frac{f_{i+1,j} - f_{i,j}}{2L_1}\mu_1 + \frac{f_{i+1,j+1} - f_{i,j+1}}{2L_1}\mu_2\right),$$

$$\bar{\bar{\kappa}}_{1,i} = x_{1,i} + \frac{h_1}{2} + \delta\left(\frac{f_{i+1,j+1} - f_{i,j+1}}{2L_1}\mu_1 + \frac{f_{i+1,j} - f_{i,j}}{2L_1}\mu_2\right), \tag{5.13}$$

$$\bar{\kappa}_{2,j} = x_{2,j} + \frac{h_2}{2} + \delta\frac{f_{i+1,j+1} - f_{i+1,j}}{2L_2},$$

$$\bar{\bar{\kappa}}_{2,j} = x_{2,j} + \frac{h_2}{2} + \delta\frac{f_{i,j+1} - f_{i,j}}{2L_2}, \tag{5.14}$$

$$\tilde{\kappa}_{1,i} = x_{1,i} + \frac{h_1}{2} + \delta\frac{f_{i+1,m_1+1} - f_{i,m_1+1}}{2L_1}, \quad i = 1,\dots,m,\ j = 1,\dots,m_1, \tag{5.15}$$

where

$$\mu_1 = \tfrac{1}{2}((1-\delta)\delta_2 + (1+\delta)\delta_1), \quad \mu_2 = \tfrac{1}{2}((1-\delta)\delta_1 + (1+\delta)\delta_2), \tag{5.16}$$

$\delta \in \{-1, 1\}$, and $\delta_1$, $\delta_2$ defined by (5.3). Then the equation of the line that passes points $(\bar{\kappa}_{1,i}, \bar{\kappa}_{2,j})$, $(\bar{\bar{\kappa}}_{1,i}, \bar{\bar{\kappa}}_{2,j})$ has the form

$$\frac{x_1 - \bar{\kappa}_{1,i}}{\bar{\bar{\kappa}}_{1,i} - \bar{\kappa}_{1,i}} = \frac{x_2 - \bar{\kappa}_{2,j}}{\bar{\bar{\kappa}}_{2,j} - \bar{\kappa}_{2,j}}, \quad \text{or} \quad (x_2 - \bar{\kappa}_{2,j})(\bar{\bar{\kappa}}_{1,i} - \bar{\kappa}_{1,i}) = (x_1 - \bar{\kappa}_{1,i})(\bar{\bar{\kappa}}_{2,j} - \bar{\kappa}_{2,j}), \tag{5.17}$$

that immediately leads to

$$x_2 = \bar{\kappa}_{2,j} + (x_1 - \bar{\kappa}_{1,i})(\bar{\bar{\kappa}}_{2,j} - \bar{\kappa}_{2,j})/(\bar{\bar{\kappa}}_{1,i} - \bar{\kappa}_{1,i}). \tag{5.18}$$

Since

$$\bar{\bar{\kappa}}_{1,i} - \bar{\kappa}_{1,i} = \delta(\mu_1 - \mu_2)\frac{f_{i+1,j+1} - f_{i,j+1} - f_{i+1,j} + f_{i,j}}{2L_1}, \tag{5.19}$$

and

$$\bar{\bar{\kappa}}_{2,j} - \bar{\kappa}_{2,j} = \frac{\delta}{2L_2}(f_{i,j+1} + f_{i+1,j} - f_{i,j} - f_{i+1,j+1}), \tag{5.20}$$

we have

$$(\bar{\bar{\kappa}}_{2,j} - \bar{\kappa}_{2,j})/(\bar{\bar{\kappa}}_{1,i} - \bar{\kappa}_{1,i}) = -\frac{L_1}{L_2}\mu_1 + \frac{L_1}{L_2}\mu_2. \tag{5.21}$$

From (5.21) we get

$$x_2 = \bar{\kappa}_{2,j} + \frac{L_1}{L_2}(\mu_1 - \mu_2)(\bar{\kappa}_{1,i} - x_1), \quad i = 1, \ldots, m, \ j = 1, \ldots, m_1. \tag{5.22}$$

It is easy to see that Eq. (5.22) is the equation of the lines $(O_1, O_2)$ and $(O_3, O_4)$ with $\delta = 1$ and $\delta = -1$, respectively (similar to the situation shown in Fig. 1). By using Corollaries 2.2, 2.3 and by taking into account relationships (5.13)–(5.22) we get

$$\begin{aligned}
\bar{U}_p^{\pm} = & \int_{x_{1,i}}^{\bar{\bar{\kappa}}_{1,i}} \int_{x_{2,j}}^{\bar{\bar{\kappa}}_{2,j}} (f_{i,j} + \delta(L_1(x_1 - x_{1,i}) + L_2(x_2 - x_{2,j})))\,\mathrm{d}x_2\,\mathrm{d}x_1 \\
& + \int_{x_{1,i}}^{\bar{\bar{\kappa}}_{1,i}} \int_{\bar{\kappa}_{2,j}}^{x_{2,j+1}} (f_{i,j+1} + \delta(L_1(x_1 - x_{1,i}) + L_2(x_{2,j+1} - x_2)))\,\mathrm{d}x_2\,\mathrm{d}x_1 \\
& + \int_{\bar{\kappa}_{1,i}}^{x_{1,i+1}} \int_{\bar{\kappa}_{2,j}}^{x_{2,j+1}} (f_{i+1,j+1} + \delta(L_1(x_{1,i+1} - x_1) + L_2(x_{2,j+1} - x_2)))\,\mathrm{d}x_2\,\mathrm{d}x_1 \\
& + \int_{\bar{\kappa}_{1,i}}^{x_{1,i+1}} \int_{x_{2,j}}^{\bar{\kappa}_{2,j}} (f_{i+1,j} + \delta(L_1(x_{1,i+1} - x_1) + L_2(x_2 - x_{2,j})))\,\mathrm{d}x_2\,\mathrm{d}x_1
\end{aligned}$$

$$+\int_{\bar{\bar{\kappa}}_{1,i}}^{\bar{\kappa}_{1,i}}\int_{x_{2,j}}^{\bar{\kappa}_{2,j}+(L_1/L_2)(\mu_1-\mu_2)(\bar{\kappa}_{1,i}-x_1)}(\mu_1 f_{i,j}+\mu_2 f_{i+1,j}+\delta(L_1((\mu_1-\mu_2)x_1-\mu_1 x_{1,i}+\mu_2 x_{1,i+1})$$

$$+L_2(x_2-x_{2,j})))\,\mathrm{d}x_2\,\mathrm{d}x_1+\int_{\bar{\bar{\kappa}}_{1,i}}^{\bar{\kappa}_{1,i}}\int_{\bar{\kappa}_{2,j}+(L_1/L_2)(\mu_1-\mu_2)(\bar{\kappa}_{1,i}-x_1)}^{x_{2,j+1}}(\mu_1 f_{i+1,j+1}+\mu_2 f_{i,j+1}$$

$$+\delta(L_1((\mu_2-\mu_1)x_1-\mu_2 x_{1,i}+\mu_1 x_{1,i+1})+L_2(x_{2,j+1}-x_2)))\,\mathrm{d}x_2\,\mathrm{d}x_1, \tag{5.23}$$

where $p=1,\dots,mm_1,\ i=1,\dots,m,\ j=1,\dots,m_1$. Setting $\delta=1$ and then $\delta=-1$ in (5.23), we get expressions for $\bar{U}_p^+$ and for $\bar{U}_p^-$, respectively ($p=1,\dots,mm_1$). By computing the sum $\bar{S}_p^{\pm}$ for the first four integrals in (5.23) and the sum $\bar{\bar{S}}_p^{\pm}$ ($p=1,\dots,mm_1$) of the last two integrals from (5.23) we obtain that

$$\bar{U}_p^{\pm}=\bar{S}_p^{\pm}+\bar{\bar{S}}_p^{\pm}=(\bar{\bar{\kappa}}_{1,i}-x_{1,i})(x_{2,j+1}f_{i,j+1}-x_{2,j}f_{i,j}+\delta h_2 L_1\bar{\bar{\kappa}}_{1,i})+(x_{1,i+1}-\bar{\kappa}_{1,i})$$
$$\times(x_{2,j+1}f_{i+1,j+1}-x_{2,j}f_{i+1,j}+\delta h_2 L_1\bar{\kappa}_{1,i})+(\bar{\kappa}_{1,i}-\bar{\bar{\kappa}}_{1,i})\big((\bar{\kappa}_{2,j}-x_{2,j})(\mu_1(f_{i,j}-\delta L_1 x_{1,i})$$
$$+\mu_2(f_{i+1,j}+\delta L_1 x_{1,i+1}))+(x_{2,j+1}-\bar{\kappa}_{2,j})(\mu_1(f_{i+1,j+1}+\delta L_1 x_{1,i+1})+\mu_2(f_{i,j+1}-\delta L_1 x_{1,i}))$$
$$+2\delta L_1(\mu_1-\mu_2)\bar{\kappa}_{1,i}\bar{\kappa}_{2,j}+\frac{\delta L_2}{2}(x_{2,j}^2+x_{2,j+1}^2+\bar{\kappa}_{2,j}^2)$$
$$-\frac{\delta L_1^2}{L_2}\left(\frac{1}{3}(\bar{\kappa}_{1,i}^2+\bar{\kappa}_{1,i}\bar{\bar{\kappa}}_{1,i})-\bar{\bar{\kappa}}_{1,i}^2\right)\bigg), \tag{5.24}$$

where $p=1,\dots,mm_1$. In the same way we can compute $\tilde{U}_p^{\pm},\ p=1,\dots,m$:

$$\tilde{U}_p^{\pm}=\int_{x_{1,i}}^{\tilde{\kappa}_{1,i}}\int_{x_{2,m_1+1}}^{1}(f_{i,m_1+1}+\delta(L_1(x_1-x_{1,i})+L_2(x_2-x_{2,m_1+1})))\,\mathrm{d}x_2\,\mathrm{d}x_1$$
$$+\int_{\tilde{\kappa}_{1,i}}^{x_{1,i+1}}\int_{x_{2,m_1+1}}^{1}(f_{i+1,m_1+1}+\delta(L_1(x_{1,i+1}-x_1)+L_2(x_2-x_{2,m_1+1})))\,\mathrm{d}x_2\,\mathrm{d}x_1$$
$$=(f_{i,m_1+1}-\delta(L_1x_{1,i}+L_2x_{2,m_1+1}))(\tilde{\kappa}_{1,i}-x_{1,i})(1-x_{2,m_1+1})+(f_{i+1,m_1+1}$$
$$+\delta(L_1x_{1,i+1}+L_2x_{2,m_1+1}))(x_{1,i+1}-\tilde{\kappa}_{1,i})(1-x_{2,m_1+1})+\delta\bigg(\frac{L_1}{2}((\tilde{\kappa}_{1,i}^2-x_{1,i}^2)(1-x_{2,m_1+1})$$
$$-(x_{1,i+1}^2-\tilde{\kappa}_{1,i}^2)(1-x_{2,m_1+1}))+\frac{L_2}{2}((1-x_{2,m_1+1}^2)(\tilde{\kappa}_{1,i}-x_{1,i})+(1-x_{2,m_1+1}^2)$$
$$(x_{1,i+1}-\tilde{\kappa}_{1,i})))=(1-x_{2,m_1+1})(f_{i,m_1+1}(\tilde{\kappa}_{1,i}-x_{1,i})+f_{i+1,m_1+1}(x_{1,i+1}-\tilde{\kappa}_{1,i})$$
$$+\delta\left(L_1\left(\frac{1}{2}(x_{1,i}^2+x_{1,i+1}^2)+\tilde{\kappa}_{1,i}(\tilde{\kappa}_{1,i}-x_{1,i}-x_{1,i+1})\right)+\frac{1}{2}L_2h_1\times(1-x_{2,m_1+1})\right)\bigg). \tag{5.25}$$

If we let $L_1 = L_2 = L$, $h_1 = h_2 = h$ in (5.24) we get an expression for $U_p^{\pm}$, $p = 1,\dots,m^2$. Therefore, we have proved the following theorem.

**Theorem 5.1.** *Optimal-by-accuracy cubature formulae for computing the integral $I_1^2(f)$ in the classes $C^2_{\bar{2},L_1,L_2,N}$ and $C^2_{2,L,L,N}$ have the forms* (5.11) *and* (5.10), *respectively. The values of $\bar{U}_p^{\pm}$ $(p = 1,\dots,mm_1)$ and $\tilde{U}_p^{\pm}$ $(p = 1,\dots,m)$ in* (5.10), (5.11) *are computed by formulae* (5.24) *and* (5.25), *respectively, and the value of $U_p^{\pm}$ $(p = 1,\dots,m^2)$ is computed by formula* (5.24) *for $L_1 = L_2 = L$, $h_1 = h_2 = h$. Error estimates of cubature formulae* (5.10) *and* (5.11) *are determined from relationships* (5.12).

## 6. Optimal-by-order cubature formulae for computing integrals with fast oscillatory functions in classes $C_{\bar{2},\,L_1,L_2,N}$ and $C^2_{2,L,L,N}$

The approach described in Section 5 can be applied to the computation of integrals $I_2^2(f)$ and $I_3^2(f)$ in functional classes $C^2_{\bar{2},\,L_1,L_2,N}$ and $C^2_{2,L,L,N}$. In such cases cubature formulae can be derived in explicit forms. In this section, we consider a special case of the integral $I^2(f)$ when $\varphi_1(x_1) = \sin(\omega_1 x_1)$, $\varphi_2(x_2) = \sin(\omega_2 x_2)$, $|\omega_i| \geqslant 2\pi$, $i = 1,2$.

Let $F_N = C^2_{\bar{2},\,L_1,L_2,N}$. As before, the splitting of the region $\bar{K}_p$ into sub-regions $\bar{\Omega}_l^{+}$, $l = 1,2,3,4$, $p = 1,\dots,mm_1$ is determined by points $O_1(\bar{\bar{\kappa}}_{1,i},\bar{\bar{\kappa}}_{2,j})$, $O_2(\bar{\kappa}_{1,i},\bar{\kappa}_{2,j})$ (similar to Fig. 1) with $\bar{\kappa}_{1,i},\bar{\bar{\kappa}}_{1,i},\bar{\kappa}_{2,j}$, $\bar{\bar{\kappa}}_{2,j}$ $(i = 1,\dots,m,\ j = 1,\dots,m_1)$ computed by formulae (5.1)–(5.3). Analogously, the splitting of $\bar{K}_p$ into sub-region $\bar{\Omega}_l^{-}$, $l = 1,2,3,4$, $p = 1,\dots,mm_1$ is determined by points $O_3 = (\tilde{\kappa}_{1,i},\tilde{\kappa}_{2,j})$, $O_4 = (\tilde{\tilde{\kappa}}_{1,i},\tilde{\tilde{\kappa}}_{2,j})$ (similar to Fig. 1) with $\tilde{\kappa}_{1,i},\tilde{\tilde{\kappa}}_{1,i},\tilde{\kappa}_{2,j},\tilde{\tilde{\kappa}}_{2,j}$ $(i = 1,\dots,m,\ j = 1,\dots,m_1)$ computed by formulae (5.4) and (5.5). The splitting of the region $\tilde{K}_p$ into subregions $\tilde{\Omega}_l^{+}$, $l = 1,2$ is performed by the points $O_1(\hat{x}_{1,i},x_{2,m_1+1})$, $O_2 = (\hat{x}_{1,i},1)$. Finally, the splitting of the region $\tilde{K}_p$ into sub-regions $\tilde{\Omega}_l^{-}$, $l = 1,2$ is performed by the points $O_3 = (\dot{x}_{1,i},x_{2,m_1+1})$, $O_4 = (\dot{x}_{1,i},1)$ (see Fig. 2) with $\hat{x}_{1,i}$, $\dot{x}_{1,i}$ computed by formula (5.6). Let

$$\bar{T}_p^{\pm} = \iint_{\bar{K}_p} A^{\pm}_{C^2_{\bar{2},\,L_1,L_2,N}}(X)\sin(\omega_1 x_1)\sin(\omega_2 x_2)\,\mathrm{d}X, \quad p = 1,\dots,mm_1,$$

$$\tilde{T}_p^{\pm} = \iint_{\tilde{K}_p} A^{\pm}_{C^2_{\bar{2},\,L_1,L_2,N}}(X)\sin(\omega_1 x_1)\sin(\omega_2 x_2)\,\mathrm{d}X, \quad p = 1,\dots,m. \tag{6.1}$$

Taking into account Corollary 3.1 we conclude that the optimal-by-order cubature formula with constant not exceeding 2 (see also [23,16]) for computing integral $I_2^2(f)$ in the class $C^2_{\bar{2},\,L_1,L_2,N}$ has the form

$$\bar{T}^{*} = \frac{1}{2}\left(\sum_{p=1}^{mm_1}(\bar{T}_p^{+} + \bar{T}_p^{-}) + \sum_{p=1}^{m}(\tilde{T}_p^{+} + \tilde{T}_p^{-})\right), \tag{6.2}$$

therewith

$$v(C^2_{2,\,L_1,L_2,N}, \bar{T}^*, f) \leqslant \frac{1}{2}\left(\sum_{p=1}^{mm_1}(\max(\bar{T}^+_p, \bar{T}^-_p) - \min(\bar{T}^+_p, \bar{T}^-_p)\right)$$

$$+\sum_{p=1}^{m}(\max(\tilde{T}^+_p, \tilde{T}^-_p) - \min(\tilde{T}^+_p, \tilde{T}^-_p)). \tag{6.3}$$

By using Corollaries 2.2, 2.3 and by taking into account relationships (5.13)–(5.22) we obtain that $\bar{T}^{\pm}_p$, $p = 1,\dots,mm_1$ can be determined as follows:

$$\bar{T}^{\pm}_p = \int_{x_{1,i}}^{\bar{\bar{\kappa}}_{1,i}}\int_{x_{2,j}}^{\bar{\bar{\kappa}}_{2,j}}(f_{i,j} + \delta(L_1(x_1 - x_{1,i}) + L_2(x_2 - x_{2,j})))\sin(\omega_1 x_1)\sin(\omega_2 x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1$$

$$+\int_{x_{1,i}}^{\bar{\bar{\kappa}}_{1,i}}\int_{\bar{\bar{\kappa}}_{2,j}}^{x_{2,j+1}}(f_{i,j+1} + \delta(L_1(x_1 - x_{1,i}) + L_2(x_{2,j+1} - x_2)))\sin(\omega_1 x_1)\sin(\omega_2 x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1$$

$$+\int_{\bar{\kappa}_{1,i}}^{x_{1,i+1}}\int_{\bar{\kappa}_{2,j}}^{x_{2,j+1}}(f_{i+1,j+1} + \delta(L_1(x_{1,i+1}-x_1) + L_2(x_{2,j+1}-x_2)))\sin(\omega_1 x_1)\sin(\omega_2 x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1$$

$$+\int_{\bar{\kappa}_{1,i}}^{x_{1,i+1}}\int_{x_{2,j}}^{\bar{\kappa}_{2,j}}(f_{i+1,j} + \delta(L_1(x_{1,i+1} - x_1) + L_2(x_2 - x_{2,j})))\sin(\omega_1 x_1)\sin(\omega_2 x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1$$

$$+\int_{\bar{\bar{\kappa}}_{1,i}}^{\bar{\kappa}_{1,i}}\int_{\bar{\kappa}_{2,j}}^{\bar{\kappa}_{2,j}+(L_1/L_2)(\mu_1-\mu_2)(\bar{\kappa}_{1,i}-x_1)}(\mu_1 f_{1,j} + \mu_2 f_{i+1,j} + \delta(L_1((\mu_1-\mu_2)x_1 - \mu_1 x_{1,i} + \mu_2 x_{1,i+1})$$

$$+L_2(x_2 - x_{2,j})))\sin(\omega_1 x_1)\sin(\omega_2 x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1 + \int_{\bar{\bar{\kappa}}_{1,i}}^{\bar{\kappa}_{1,i}}\int_{\bar{\kappa}_{2,j}+(L_1/L_2)(\mu_1-\mu_2)(\bar{\kappa}_{1,i}-x_1)}^{x_{2,j+1}}$$

$$(\mu_1 f_{i+1,j+1} + \mu_2 f_{i,j+1} + \delta(L_1((\mu_2 - \mu_1)x_1 - \mu_2 x_{1,i} + \mu_1 x_{1,i+1})$$

$$+L_2(x_2 - x_{2,j})))\sin(\omega_1 x_1)\sin(\omega_2 x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1. \tag{6.4}$$

Setting $\delta = 1$ in (6.4) we get the expression for $\bar{T}^+_p$. Similarly, setting $\delta = -1$ in (6.4) gives the expression for $\bar{T}^-_p$, $p = 1,\dots,mm_1$. The sum $\bar{S}^{\pm}_p$ of the first four integrals in (6.4) and the sum $\bar{\bar{S}}^{\pm}_p$ ($p = 1,\dots,mm_1$) of the last two integrals in (6.4) can be computed explicitly. By combining the results of these computations, we get that

$$\bar{T}^{\pm}_p = \sum_{k=1}^{4}\bar{W}^{\pm}_k, \quad p = 1,\dots,mm_1, \tag{6.5}$$

where

$$\bar{W}^{\pm}_1 = \frac{1}{\omega_1\omega_2}((\cos\omega_1\bar{\bar{\kappa}}_{1,i} - \cos\omega_1 x_{1,i})(f_{i,j}(\cos\omega_2\bar{\bar{\kappa}}_{2,j} - \cos\omega_2 x_{2,j})$$

$$+ f_{i,j+1}(\cos\omega_2 x_{2,j+1} - \cos\omega_2\bar{\bar{\kappa}}_{2,j})) + (\cos\omega_1 x_{1,i+1} - \cos\omega_1\bar{\kappa}_{1,i})(f_{i+1,j}(\cos\omega_2\bar{\kappa}_{2,j}$$

$$-\cos\omega_2 x_{2,j}) + f_{i+1,j+1}(\cos\omega_2 x_{2,j+1} - \cos\omega_2\bar{\kappa}_{2,j})) + (\cos\omega_1\bar{\bar{\kappa}}_{1,i} - \cos\omega_1\bar{\kappa}_{1,i})$$
$$\times((\mu_1 f_{i,j} + \mu_2 f_{i+1,j})\cos\omega_2 x_{2,j} - (\mu_1 f_{i+1,j+1} + \mu_2 f_{i,j+1})\cos\omega_2 x_{2,j+1})); \tag{6.6}$$

$$\bar{W}_2^{\pm} = \frac{\delta L_1}{\omega_1\omega_2}((\cos\omega_2 x_{2,j} - \cos\omega_2 x_{2,j+1})(x_{1,i+1}\cos\omega_1\bar{\kappa}_{1,i} + x_{1,i}\cos\omega_1\bar{\bar{\kappa}}_{1,i}$$
$$-\frac{1}{\omega_1}(\sin\omega_1 x_{1,i} + \sin\omega_1 x_{1,i+1})) + \left(\frac{\sin\omega_1\bar{\kappa}_{1,i}}{\omega_1} - \bar{\kappa}_{1,i}\cos\omega_1\bar{\kappa}_{1,i}\right)((1+(\mu_1-\mu_2))$$
$$\cos\omega_2 x_{2,j} - (1-(\mu_1-\mu_2))\cos\omega_2 x_{2,j+1}) + \left(\frac{\sin\omega_1\bar{\bar{\kappa}}_{1,i}}{\omega_1} - \bar{\bar{\kappa}}_{1,i}\cos\omega_1\bar{\bar{\kappa}}_{1,i}\right)$$
$$\times((1-(\mu_1-\mu_2))\cos\omega_2 x_{2,j} - (1+(\mu_1-\mu_2))\cos\omega_2 x_{2,j+1}), \tag{6.7}$$

$$\bar{W}_3^{\pm} = \frac{\delta L_2}{\omega_1\omega_2}((\cos\omega_1 x_{1,i} - \cos\omega_1\bar{\bar{\kappa}}_{1,i})((x_{2,j} + x_{2,j+1} - 2\bar{\bar{\kappa}}_{2,j})\cos\omega_2\bar{\bar{\kappa}}_{2,j}$$
$$+\frac{2}{\omega_2}\sin\omega_2\bar{\bar{\kappa}}_{2,j}) + (\cos\omega_1\bar{\kappa}_{1,i} - \cos\omega_1 x_{1,i+1})((x_{2,j} + x_{2,j+1} - 2\bar{\kappa}_{2,j})\cos\omega_2\bar{\kappa}_{2,j}$$
$$+\frac{2}{\omega_2}\sin\omega_2\bar{\kappa}_{2,j}) + \frac{1}{\omega_2}(\cos\omega_1 x_{1,i+1} - \cos\omega_1 x_{1,i})(\sin\omega_2 x_{2,j} + \sin\omega_2 x_{2,j+1})), \tag{6.8}$$

$$\bar{W}_4^{\pm} = \frac{1}{\omega_2}\frac{((\cos(\omega_2\bar{\kappa}_{2,j} - \omega_1\bar{\kappa}_{1,i}) - \cos(\omega_2\bar{\bar{\kappa}}_{2,j} - \omega_1\bar{\bar{\kappa}}_{1,i})}{2(\omega_1 + \omega_2\frac{L_1}{L_2}(\mu_1-\mu_2))}$$
$$+\frac{\cos(\omega_2\bar{\kappa}_{2,j} + \omega_1\bar{\kappa}_{1,i}) - \cos(\omega_2\bar{\bar{\kappa}}_{2,j} + \omega_1\bar{\bar{\kappa}}_{1,i})}{2(\omega_1 - \omega_2\frac{L_1}{L_2}(\mu_1-\mu_2))}(\mu_1(f_{i,j} - f_{i+1,j+1})$$
$$+\mu_2(f_{i+1,j} - f_{i,j+1}) + \delta(L_1(\mu_1-\mu_2)(2\bar{\kappa}_{1,i} - x_{1,i+1} - x_{1,i})$$
$$+L_2(2\bar{\kappa}_{2,j} - x_{2,j} - x_{2,j+1}))) - \frac{\delta L_2}{\omega_2}\frac{(\sin(\omega_2\bar{\kappa}_{2,j} - \omega_1\bar{\kappa}_{1,i}) - \sin(\omega_2\bar{\bar{\kappa}}_{2,j} - \omega_1\bar{\bar{\kappa}}_{1,i})}{\omega_1 + \omega_2\frac{L_1}{L_2}(\mu_1-\mu_2)}$$
$$+\frac{\sin(\omega_2\bar{\kappa}_{2,j} - \omega_1\bar{\kappa}_{1,i}) - \sin(\omega_2\bar{\bar{\kappa}}_{2,j} - \omega_1\bar{\bar{\kappa}}_{1,i})}{\omega_1 - \omega_2\frac{L_1}{L_2}(\mu_1-\mu_2)}. \tag{6.9}$$

In a similar way, we compute $\tilde{T}_p^{\pm}$, $p = 1,\dots,m$:

$$\tilde{T}_p^{\pm} = \int_{x_{1,i}}^{\tilde{\kappa}_{1,i}}\int_{x_{2,m_1+1}}^{1}(f_{i,m_1+1} + \delta(L_1(x_1 - x_{1,i}) + L_2(x_2 - x_{2,m_1+1})))\sin\omega_1 x_1\sin\omega_2 x_2\,\mathrm{d}x_2\,\mathrm{d}x_1$$
$$+\int_{\tilde{\kappa}_{1,i}}^{x_{1,i+1}}\int_{x_{2,m_1+1}}^{1}(f_{i+1,m_1+1} + \delta(L_1(x_{1,i+1} - x_1) + L_2(x_2 - x_{2,m_1+1})))$$

$$\sin\omega_1x_1\sin\omega_2x_2\,\mathrm{d}x_2\,\mathrm{d}x_1=\frac{1}{\omega_1\omega_2}((f_{i,m_1+1}-\delta(L_1x_{1,i}+L_2x_{2,m_1+1}))(\cos\omega_1\tilde{\kappa}_{1,i}-\cos\omega_1x_{1,i})$$
$$\times(\cos\omega_2-\cos\omega_2x_{2,m_1+1})+(f_{i+1,m_1+1}+\delta(L_1x_{1,i+1}-L_2x_{2,m_1+1}))(\cos\omega_1x_{1,i+1}-\cos\omega_1x_{1,i})$$
$$\times(\cos\omega_2-\cos\omega_2x_{2,m_1+1}))+\delta\biggl(\frac{L_1}{\omega_1\omega_2}(\cos\omega_2x_{2,m_1+1}-\cos\omega_2)\biggl(\frac{1}{\omega_1}(\sin\omega_1\tilde{\kappa}_{1,i}-\sin\omega_1x_{1,i})$$
$$-\tilde{\kappa}_{1,i}\cos\omega_1\tilde{\kappa}_{1,i}+x_{1,i}\cos\omega_1x_{1,i}-\frac{1}{\omega_1}(\sin\omega_1x_{1,i+1}-\sin\omega_1\tilde{\kappa}_{1,i})+x_{1,i+1}\cos\omega_1x_{1,i+1}$$
$$-\tilde{\kappa}_{1,i}\cos\omega_1\tilde{\kappa}_{1,i}\bigr)+\frac{L_2}{\omega_1\omega_2}\left(\frac{1}{\omega_2}(\sin\omega_2-\sin\omega_2x_{2,m_1+1})-\cos\omega_2+x_{2,m_1+1}\cos\omega_2x_{2,m_1+1}\right)$$
$$\times\,(\cos\omega_1x_{1,i}-\cos\omega_1\tilde{\kappa}_{1,i}+\cos\omega_1\tilde{\kappa}_{1,i}-\cos\omega_1x_{1,i+1})\bigr)=\frac{1}{\omega_1\omega_2}\bigl((\cos\omega_2-\cos\omega_2x_{2,m_1+1})$$
$$\times\,\bigl(f_{i,m_1+1}(\cos\omega_1\tilde{\kappa}_{1,i}-\cos\omega_1x_{1,i})+f_{i+1,m_1+1}(\cos\omega_1x_{1,i+1}-\cos\omega_1\tilde{\kappa}_{1,i})+\delta L_1$$
$$\times\left((2\tilde{\kappa}_{1,i}-x_{1,i}-x_{1,i+1})\cos\omega_1\tilde{\kappa}_{1,i}+\frac{1}{\omega_1}(\sin\omega_1x_{1,i}+\sin\omega_1x_{1,i+1}-2\sin\omega_1\tilde{\kappa}_{1,i})\right)\Bigr)+\delta L_2$$
$$\times\,(\cos\omega_1x_{1,i}-\cos\omega_1x_{1,i+1})((x_{2,m_1+1}-1)\cos\omega_2+\frac{1}{\omega_2}(\sin\omega_2-\sin\omega_2x_{2,m_1+1}))\Bigr). \quad (6.10)$$

By setting $\delta=1$ in (6.10) we get the explicit expression for $\tilde{T}^+_p$, and by setting $\delta=-1$ we obtain the explicit expression for $\tilde{T}^-_p$, $p=1,\dots,m$. Therefore, we have proved the following theorem.

**Theorem 6.1.** *Optimal-by-order (with constant not exceeding 2) cubature formulae for computing the integral $I_2^2(f)$ in the class $C^2_{2,\,L_1,L_2,N}$ have the form (6.2) with $\bar{T}^{\pm}_p$ $(p=1,\dots,mm_1)$ and $\tilde{T}^{\pm}_p$ $(p=1,\dots,m)$ computed by formulae (6.5)–(6.9) and (6.10), respectively. The error estimate of cubature formulae (6.2) is determined by relationship (6.3).*

Now let $F_N=C^2_{2,L,L,N}$. As it was mentioned in Section 5, the splitting of $K_p$ into regions $\Omega^+_l$, $l=1,2,3,4$ is determined by points $O_1(\bar{\bar{x}}_{1,i},\bar{\bar{x}}_{2,j})$, $O_2=(\bar{x}_{1,i},\bar{x}_{2,j})$, and the splitting of $K_p$ into regions $\Omega^-_l$, $l=1,2,3,4$ is determined by points $O_3=(\tilde{x}_{1,i},\tilde{x}_{2,j})$, $O_4=(\tilde{\tilde{x}}_{1,i},\tilde{\tilde{x}}_{2,j})$ (see Fig. 1), where $\bar{x}_{1,i},\bar{\bar{x}}_{1,i},\tilde{x}_{1,i},\tilde{\tilde{x}}_{1,i}$ $(i=1,\dots,m)$ and $\bar{x}_{2,j},\bar{\bar{x}}_{2,j},\tilde{x}_{2,j},\tilde{\tilde{x}}_{2,j}$ $(j=1,\dots,m,\ p=1,\dots,m^2)$ are computed by formulae (5.1)–(5.5), respectively, for $L_1=L_2=L$ and $h_1=h_2=h$. Let

$$T^{\pm}_p=\iint_{K_p}A^{\pm}_{C^2_{2,L,L,N}}(X)\sin\omega_1x_1\sin\omega_2x_2\,\mathrm{d}X,\quad p=1,\dots,m^2. \quad (6.11)$$

By taking into account Corollary 3.1, we obtain that optimal-by-order (with constant not exceeding 2) cubature formulae for computing integrals $I_2^2(f)$ in class $C^2_{2,L,L,N}$ have the form

$$T^* = \frac{1}{2}\sum_{p=1}^{m^2}(T_p^+ + T_p^-), \tag{6.12}$$

therewith

$$v(C^2_{2,L,L,N}, T^*, f) \leqslant \frac{1}{2}\sum_{p=1}^{m^2}(\max(T_p^+, T_p^-) - \min(T_p^+, T_p^-)). \tag{6.13}$$

By setting $L_1 = L_2 = L$ and $h_1 = h_2 = h$ in (6.5) we obtain the expression for $T_p^{\pm}$, $p = 1, \dots, m^2$. This leads to the following result.

**Theorem 6.2.** *Optimal-by-order* (*with constant not exceeding* 2) *cubature formulae for computing the integral $I_2^2(f)$ in the class $C^2_{2,L,L,N}$ have the form* (6.12) *with $T_p^{\pm}$ ($p = 1, \dots, m^2$) computed by formulae* (6.5)–(6.9) *for $L_1 = L_2 = L$ and $h_1 = h_2 = h$. The error estimate of cubature formulae* (6.12) *is determined by relationship* (6.13).

Explicit forms of optimal-by-order cubature formulae for computing the integral $I_3^2(f)$ in the classes $C_{\bar{2},\, L_1, L_2, N}$ and $C^2_{2,\, L,\, L,\, N}$ can be derived analogously.

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## References

[1] A. Alaylioglu, Numerical evaluation of finite Fourier integrals, J. Comput. Appl. Math. 9 (1983) 305–313.
[2] A.I. Berezovskii, V.V. Ivanov, On optimal-by-accuracy uniform spline approximation, Soviet Math. (Iz. VUZ) 10 (1977) 14–24.
[3] A.I. Berezovskii, N.E. Nechiporenko, Optimal accuracy approximation of functions and their derivatives, J. Soviet Math. 54 (1991) 799–812.
[4] R.E. Blahut, Fast Algorithms for Digital Signal Processing, Addison-Wesley, Reading, MA, 1987.
[5] S.C. Chan, K.L. Ho, A new two-dimensional fast cosine transform, IEEE Trans. Signal Process. 39 (2) (1991) 481–485.
[6] R. Cools, Constructing Cubature Formulae: the Science Behind the Art, Acta Numerica, Cambridge University Press, Cambridge, 1997, pp. 1–54.
[7] P. Davis, P. Rabinowitz, Methods of Numerical Integration, Academic Press, New York, 1984.
[8] B. Drachman, J. Ross, Approximation of certain functions given by integrals with highly oscillatory integrands, IEEE Trans. Antennas and Propagation 42 (9) (1994) 1355–1356.
[9] B. Einarson, Numerical calculation of Fourier integrals with cubic splines, BIT 8 (3) (1968) 279–286.
[10] O.K. Ersoy, Fourier-Related Transforms, Fast Algorithms, and Applications, Prentice-Hall, Englewood Cliffs, NJ, 1997.
[11] L.N. Filon, On a quadrature formula for trigonometric integrals, Proc. Roy. Soc. Edinburgh 49 (1928) 38–47.

[12] Q. Haider, L.C. Liu, Fourier and Bessel transformations of highly oscillatory functions, J. Phys. A 25 (1992) 6755–6760.
[13] H.H. Hopkins, Numerical evaluation of a class of double integrals of oscillatory functions, IMA J. Numer. Anal. 9 (1989) 61–80.
[14] T. Iwaniec, A. Lutoborski, Asymptotic expansions of multiple integrals of rapidly oscillating functions, Math. Comput. 50 (1988) 215–228.
[15] D. Levin, Fast integration of rapidly oscillatory functions, J. Comput. Appl. Math. 67 (1996) 95–101.
[16] K.N. Melnik, R.V.N. Melnik, On computational aspects of certain optimal digital signal processing algorithms, in: J. Noye, M. Teubner, A. Gill (Eds.), Proceedings of Computational Technique and Applications Conference: CTAC97, World Scientific, Singapore, 1998, pp. 433–440.
[17] K.N. Melnik, R.V.N. Melnik, A note on optimal-by-order cubature formulae for fast oscillatory functions in Lipschitz classes, Technical Report SC-MC-9811, Department of Mathematics and Computing, University of Southern Queensland, 1998.
[18] K.N. Melnik, R.V.N. Melnik, Optimal-by-order quadrature formulae for fast oscillatory functions with inaccurately given a priori information, J. Comput. Appl. Math. 110 (1999) 45–72.
[19] V.A. Morozov, Regularizations Methods for Ill-Posed Problems, CRC Press, Boca Raton, FL, 1993.
[20] I.P. Mysovskih, Interpolatorische Kubaturformulen, Institut fur Geometrie und Praktische Matematik der RWTH Aachen, Bericht, Vol. 74, 1992.
[21] R. Piessens, M. Branders, A survey on numerical methods for computation of Bessel functions integrals, Rend. Sem. Mat. Univ. Politec. Torino 43 (1985) 249–265.
[22] A. Sucharev, Minimax Models in the Theory of Numerical Methods, Kluwer Academic Publishers, Dordrecht, 1992.
[23] J.F. Traub, H. Wozniakowski, A General Theory of Optimal Algorithms, Academic Press, New York, 1980.
[24] V.K. Zadiraka, N.T. Abatov, Optimally exact algorithms for solutions of a certain numerical integration problem, Ukrainian Math. J. 43 (1991) 43–54.
[25] V.K. Zadiraka, S.Z. Kasenov, Optimal-by-accuracy quadrature formulae for computing Fourier transform of finite functions from $C_{L,N}$, Ukrainian Math. J. 38 (2) (1986) 233–237.
[26] V.A. Zheludev, Periodic splines and the fast Fourier transform, Comput. Math. Math. Phys. 32 (2) (1992) 149.
[27] V.A. Zheludev, Processing of periodic signals using spline-wavelets, Radioelectron. Commun. Syst. 38 (3) (1995) 1.
[28] Ya.M. Zhileikin, A.B. Kukarkin, A fast Fourier–Bessel transformation algorithm, Comput. Math. Math. Phys. 35 (7) (1995) 901.