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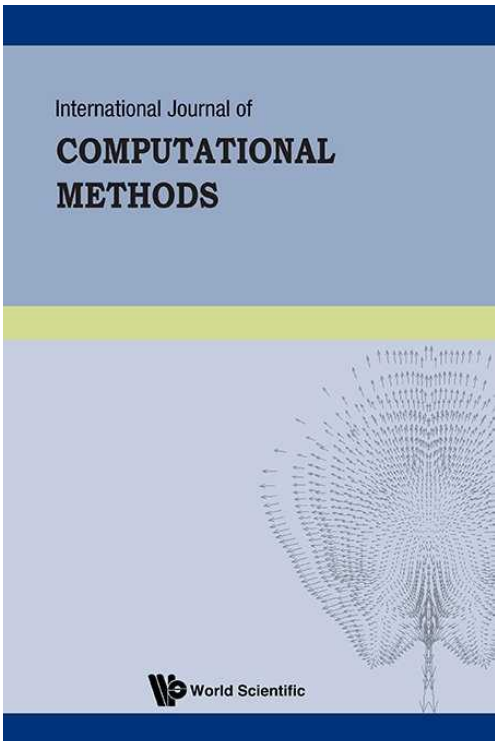

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Computational Aspects of Monte-Carlo Simulations of the First Passage Time for Multivariate Transformed Brownian Motions with Jumps

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Abstract

Many problems in science, engineering, and finance require the information on the first passage time (FPT) of a stochastic process. Mathematically, such problems are often reduced to the evaluation of the probability density of the time for such a process to cross a certain level, a boundary, or to enter a certain region. While in other areas of applications the FPT problem can often be solved analytically, in finance we usually have to resort to the application of numerical procedures, in particular when we deal with jump-diffusion stochastic processes (JDP). In this paper, we propose a Monte-Carlo-based methodology for the solution of the first passage time problem in the context of multivariate (and correlated) jump-diffusion processes. The developed technique provides an efficient tool for a number of applications, including credit risk and option pricing. We demonstrate its applicability to the analysis of default rates and default correlations of several different, but correlated firms via a set of empirical data.

Keywords: First passage time; Monte Carlo simulation; Multivariate jump-diffusion processes; Credit risk; Option Pricing; Brownian bridge simulations; Large deviations methodologies, Complex systems.

1 Introduction

First Passage Time (FPT) problems appear frequently in many areas of applications, including sciences (physics, biology, chemistry), engineering, socio-economic modelling. While the methodology developed here for solving such problems can be applied in these and other areas too, in this paper we focus on FPT problems in finance.

Credit risk can be defined as the possibility of a loss occurring due to the financial failure to meet contractual debt obligations. This is one of the measures of the likelihood that a party will default on a financial agreement. There exist two classes of credit risk models: structural models and reduced form models [53, 28]. Structural models can be traced back to the influential works by Black, Scholes and Merton [3, 36], while reduced form models seem to originate from contribution [21]. As pointed out in [14], it is possible to connect these two

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model approaches via the model's information assumptions. In particular, it is known (e.g., [14]) that if we assume for a structural model that investors do not have complete information about the market dynamics leading to the firm's default, we can derive from such a model a cumulative rate of default which is consistent with a reduced form model (see [14, 20] and references therein). The major focus in the current contribution is given to structural credit-risk models. Before moving to such models, we note that the systems under consideration here are complex [46]. The interest to different aspects of the analysis of complex systems has been increasing steadily over the recent years in scientific, engineering and biomedical applications, as well as in finance and business, corporate governance and control, social sciences and economics [33, 48, 37, 46, 22]. In Section 4 we focus on one of such examples taken from the area of credit risk. However, the tools and methods we discussed in Sections 1-3 can also be useful in other areas of applications mentioned above. Finally, we remark that the systems considered here belong to a special class of complex systems where the presence of human factors is becoming increasingly important [34]. The development of advanced mathematical models where such factors are accounted for are at the beginning of their development and the interested reader can consult [34] and references therein on further details.

In structural credit-risk models, a default occurs when a company cannot meet its financial obligations, or in other words, when the firm's value falls below a certain threshold. The major problem in credit risk analysis is connected with our ability to answer the questions on (a) when a default occurs within a given time period and (b) what is the default rate during such a time period. This problem can be reduced to a first passage time problem, that can be formulated mathematically as a certain stochastic differential equation (SDE). It concerns the estimation of the probability density of the time for a random process to cross a specified threshold level. Therefore, it is natural that the FPT problem occurs also frequently in other areas of applications, including many branches of science and engineering [47, 39, 5, 43, 25].

An important phenomenon that we account for in our discussion here lies with the fact that, in the market economy, individual companies are inevitably linked together via dynamically changing economic conditions. Therefore, the default events of companies are often correlated, especially in the same industry. The authors of [52] and [16] were the first to incorporate default correlation into the Black-Cox first passage structural model, but they have not included the jumps. As pointed out in [53] and [26], the standard Brownian motion model for market behavior falls short of explaining empirical observations of market returns and their underlying derivative prices. In the meantime, jump-diffusion processes (JDPs) have established themselves as a sound alternative to the standard Brownian motion model [2]. Multivariate jump-diffusion models provide a convenient framework for investigating default correlations with jumps and become more readily accepted in the financial world as an efficient modeling tool.

However, as soon as jumps are incorporated in the model, except for very basic applications where analytical solutions are available, for most practical cases we have to resort to numerical procedures. Examples of known analytical solutions include problems where the jump sizes are doubly exponential or exponentially distributed [26] as well as the jumps can have only nonnegative values (assuming that the crossing boundary is below the process starting value) [4]. For other situations, Monte Carlo methods remain a primary candidate for applications.

The conventional Monte Carlo methods are very straightforward to implement. We discretize the time period into N intervals with N being large enough in order to avoid discretization bias [23]. The main drawback of this procedure is that we need to evaluate the processes at each discretized time which is very time-consuming. Many researchers have contributed to the field of enhancement of the efficiency of Monte Carlo simulations. Among others, the authors of [27] discussed the solution of SDEs in the framework of weak discrete time approximations and in [6] the authors considered the strong approximation where the SDE is driven by a high intensity Poisson process. Atiya and Metwally [2, 35] have recently developed a fast Monte Carlo-type numerical methods to solve the FPT problem. In our recent contribution, we reported an extension of this fast Monte-Carlo-type method in the context of multiple non-correlated jump-diffusion processes [50].

In this contribution, we generalize our previous fast Monte-Carlo method (for non-correlated jump-diffusion cases) to multivariate (and correlated) jump-diffusion processes. The developed technique provides an efficient tool for a number of applications, including credit risk and option pricing. We demonstrate the applicability of this technique to the analysis of default rates and default correlations of several different correlated firms via a set of empirical data.

The paper is organized as follows. Section 2 provides details of our model in the context of multivariate jump-diffusion processes. The algorithms and the calibration of the model are presented in Section 3. Section 4 demonstrates how the model works via analyzing the credit risk of multi-correlated firms. Conclusions are given in Section 5.

2 Model description

As mentioned in the introduction, when we deal with jump-diffusion stochastic processes, we usually have to resort to the application of numerical procedures. Although Monte Carlo procedures provide a natural tool in such case, the one-dimensional fast Monte-Carlo method cannot be directly generalized to the multivariate and correlated jump-diffusion case (e.g. [50, 51]). The difficulties arise from the fact that the multiple processes as well as their first passage times are indeed correlated, so the simulation must reflect the correlations of first passage times. In this contribution, we propose a solution to circumvent these difficulties by combining the fast Monte-Carlo method of one-dimensional jump-diffusion processes and the generation of correlated multidimensional variates. This approach generalizes our previous results for the non-correlated jump-diffusion case to multivariate and correlated jump-diffusion processes.

In this section, first, we present a probabilistic description of default events and default correlations. Next, we describe the multivariate jump-diffusion processes and provide details on the first passage time distribution under the one-dimensional Brownian bridge (the sum-of-uniforms method which is used to generate correlated multidimensional variates will be described in Section 3.1). Finally, we apply kernel estimation in the context of our problem that can be used to represent the first passage time density function.

2.1 Default and default correlation

In a structural model, a firm i defaults when it can not meet its financial obligations, or in other words, when the firm assets value $V_i(t)$ falls below a threshold level $D_{V_i}(t)$. Generally speaking, finding the threshold level $D_{V_i}(t)$ is one of the challenges in using the structural methodology in credit risk modeling, since in reality firms often rearrange their liability structure when they have credit problems. In this contribution, we use an exponential form defining the threshold level $D_{V_i}(t) = \kappa_i \exp(\gamma_i t)$ as proposed in [52], where γ_i can be interpreted as the growth rate of the firm's liabilities. Coefficient κ_i captures the liability structure of the firm and is usually defined as the firm's short-term liability plus 50% of the firm's long-term liability. If we set $X_i(t) = \ln[V_i(t)]$, then the threshold of $X_i(t)$ is $D_i(t) = \gamma_i t + \ln(\kappa_i)$. Our main interest is in the process $X_i(t)$.

Prior to moving further, we define a default correlation that measures the strength of the default relationship between different firms. We follow [46] in this. Take two firms i and j as an example, whose probabilities of default are P_i and P_j, respectively. Then the default correlation can be defined as

$$\rho_{ij} = \frac{P_{ij}^{\mathrm{jd}} - P_i P_j}{\sqrt{P_i(1-P_i)P_j(1-P_j)}}, \tag{1}$$

where P_{ij}^{jd} is the probability of joint default.

From Eq. (1) we have $P_{ij}^{\mathrm{jd}} = P_i P_j + \rho_{ij}\sqrt{P_i(1-P_i)P_j(1-P_j)}$. Let us assume that $P_i = P_j = 5\%$. If these two firms are independent, i.e., the default correlation equals zero, then the probability of joint default is $P_{ij}^{\mathrm{jd}} = 0.25\%$. If the two firms are positively correlated, for example, $\rho_{ij} = 0.4$, then the probability that both firms default becomes $P_{ij}^{\mathrm{jd}} = 2.15\%$ which is almost 10 times higher than in the former case. Thus, the default correlation ρ_{ij} plays a key role in the joint default with important implications in the field of credit analysis. The authors of [52] and [16] were the first to incorporate default correlation into the Black-Cox first passage structural model.

The author of [52] has proposed a first passage time model to describe default correlations of two firms under the "bivariate diffusion process":

$$\begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} dt + \Omega \begin{bmatrix} dz_1 \\ dz_2 \end{bmatrix}, \tag{2}$$

where μ_1 and μ_2 are constant drift terms, z_1 and z_2 are two independent standard Brownian motions, and Ω is a constant 2×2 matrix such that

$$\Omega \cdot \Omega' = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}.$$

The coefficient ρ reflects the correlation between the movements in the asset values of the two firms. If we assume that $\mu_i = \gamma_i$ $(i = 1, 2)$, then the probability that firm i defaults at time t can be easily calculated as:

$$P_i(t) = 2 \cdot \Phi\left(-\frac{X_i(0) - \ln(\kappa_i)}{\sigma_i\sqrt{t}}\right) = 2 \cdot \Phi\left(-\frac{Z_i}{\sqrt{t}}\right), \tag{3}$$

where

$$Z_i \equiv \frac{X_i(0) - \ln(\kappa_i)}{\sigma_i}$$

is the standardized distance of firm i to its default point and $\Phi(\cdot)$ denotes the cumulative probability distribution function for a standard normal variable.

For the simplified model $\mu_i = \gamma_i$, the probability that at least one firm defaults by time t can be written as [52]:

$$\begin{aligned} P_{i\cup j}(t) &= 1 - \frac{2r_0}{\sqrt{2\pi t}} \cdot \mathrm{e}^{-\frac{r_0^2}{4t}} \cdot \sum_{n=1,3,\cdots} \frac{1}{n} \sin\left(\frac{n\pi\theta_0}{\alpha}\right) \\ &\quad \cdot \left[I_{\frac{1}{2}\left(\frac{n\pi}{\alpha}+1\right)}\left(\frac{r_0^2}{4t}\right) + I_{\frac{1}{2}\left(\frac{n\pi}{\alpha}-1\right)}\left(\frac{r_0^2}{4t}\right) \right], \end{aligned} \tag{4}$$

where $I_\nu(z)$ is the modified Bessel function I with order ν and

$$\begin{aligned} \alpha &= \begin{cases} \tan^{-1}\left(-\frac{\sqrt{1-\rho^2}}{\rho}\right), & \text{if } \rho < 0, \\ \pi + \tan^{-1}\left(-\frac{\sqrt{1-\rho^2}}{\rho}\right), & \text{otherwise,} \end{cases} \\ \theta_0 &= \begin{cases} \tan^{-1}\left(\frac{Z_2\sqrt{1-\rho^2}}{Z_1-\rho Z_2}\right), & \text{if } (\cdot) > 0, \\ \pi + \tan^{-1}\left(\frac{Z_2\sqrt{1-\rho^2}}{Z_1-\rho Z_2}\right), & \text{otherwise,} \end{cases} \\ r_0 &= Z_2/\sin(\theta_0). \end{aligned}$$

The author of [52] also obtained analytical solutions of the general model in which $\mu_i \neq \gamma_i$. These generalized results involve the double integral of Bessel functions which is more difficult to implement, but they affect the default correlation relatively little. Hence, we will only consider the simplified model with $\mu_i = \gamma_i$ in the following. Then, the default correlation of these two firms is

$$\rho_{ij}(t) = \frac{P_i(t) + P_j(t) - P_i(t)P_j(t) - P_{i\cup j}(t)}{\sqrt{P_i(t)[1 - P_i(t)]P_j(t)[1 - P_j(t)]}}. \tag{5}$$

However, none of the above known models includes jumps in the processes. At the same time, it is well-known that jumps are a major factor in credit risk analysis. With jumps included in such analysis, a firm can default instantaneously because of a sudden drop in its value which is impossible under a diffusion process. The author of [53] has shown the importance of jump risk in the credit risk analysis of an obligor. He implemented a simulation method to show the effect of jump risk in the credit spread of defaultable bonds. He showed that the misspecification of stochastic processes governing the dynamics of firm value, i.e., falsely specifying a jump-diffusion process as a continuous Brownian motion process, can substantially understate the credit spreads of corporate bonds. Similarly, it is reasonable to include jumps into the multiple processes, which may represent reliable dynamics and help analyze the credit risk in a realistic manner. Multivariate jump-diffusion processes can provide a convenient way to describe multivariate and correlated processes with jumps.

2.2 Multivariate jump-diffusion processes

Let us consider a complete probability space (Ω, F, P) with information filtration (F_t). Suppose that $\boldsymbol{X}_t = \ln(\boldsymbol{V}_t)$ is a Markov process in some state space $D \subset \mathbb{R}^n$, solving the stochastic differential equation [11]:

$$d\boldsymbol{X}_t = \boldsymbol{\mu}(\boldsymbol{X}_t)dt + \boldsymbol{\sigma}(\boldsymbol{X}_t)d\boldsymbol{W}_t + d\boldsymbol{Z}_t, \tag{6}$$

where $\boldsymbol{W}$ is an (F_t)-standard Brownian motion in $\mathbb{R}^n$; $\boldsymbol{\mu} : D \to \mathbb{R}^n$, $\boldsymbol{\sigma} : D \to \mathbb{R}^{n\times n}$, and $\boldsymbol{Z}$ is a pure jump process whose jumps have a fixed probability distribution $\boldsymbol{\nu}$ on $\mathbb{R}^n$ such that they arrive with intensity $\{\boldsymbol{\lambda}(\boldsymbol{X}_t) : t \geq 0\}$, for some $\boldsymbol{\lambda} : D \to [0, \infty)$. Under these conditions, the above model is reduced to an affine model if [1, 11]:

$$\begin{aligned}
&\boldsymbol{\mu}(\boldsymbol{X}_t, t) = \boldsymbol{K}_0 + \boldsymbol{K}_1\boldsymbol{X}_t \\
&(\boldsymbol{\sigma}(\boldsymbol{X}_t, t)\boldsymbol{\sigma}(\boldsymbol{X}_t, t)^\top)_{ij} = (\boldsymbol{H}_0)_{ij} + (\boldsymbol{H}_1)_{ij}\boldsymbol{X}_j \\
&\boldsymbol{\lambda}(\boldsymbol{X}_t) = \boldsymbol{l}_0 + \boldsymbol{l}_1 \cdot \boldsymbol{X}_t,
\end{aligned} \tag{7}$$

where $\boldsymbol{K} = (\boldsymbol{K}_0, \boldsymbol{K}_1) \in \mathbb{R}^n \times \mathbb{R}^{n\times n}$, $\boldsymbol{H} = (\boldsymbol{H}_0, \boldsymbol{H}_1) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n\times n}$, $\boldsymbol{l} = (\boldsymbol{l}_0, \boldsymbol{l}_1) \in \mathbb{R}^n \times \mathbb{R}^{n\times n}$. The theory of affine processes as a class of time-homogeneous Markov processes arising often in the context of applications in finance, including credit risk modeling, has been developed in [12, 13].

Notice that the coefficients $\boldsymbol{\mu}$, $\boldsymbol{\sigma}$, and $\boldsymbol{\lambda}$ in Eq. (6) depend on the state vector $\boldsymbol{X}$ as described by Eq. (7). Such SDEs usually need to be solved numerically, for example by using the Euler-Maruyama scheme [18, 19]. As a first step, we simplify Eq. (6) and (7) as follows:

1. $\boldsymbol{K}_1 = \boldsymbol{0}$, $\boldsymbol{H}_1 = \boldsymbol{0}$ and $\boldsymbol{l}_1 = \boldsymbol{0}$ in Eq. (7) which means that the drift term, the diffusion process (Brownian motion) and the arrival intensity are independent of the state vector $\boldsymbol{X}_t$;

2. The distribution of the jump-size $\boldsymbol{Z}_t$ is also independent of $\boldsymbol{X}_t$.

In this scenario, we can rewrite Eq. (6) as

$$d\boldsymbol{X}_t = \boldsymbol{\mu} dt + \boldsymbol{\sigma} d\boldsymbol{W}_t + d\boldsymbol{Z}_t, \tag{8}$$

where

$$\boldsymbol{\mu} = \boldsymbol{K}_0,\ \boldsymbol{\sigma}\boldsymbol{\sigma}^\top = \boldsymbol{H}_0,\ \boldsymbol{\lambda} = \boldsymbol{l}_0.$$

In other words, what we are interested in our contribution is multivariate transformed Brownian motions with jumps as described by Eq. (8). One of the major problems in credit risk analysis is to estimate the default rate of a firm during a given time period. This problem is reduced to a first passage time problem. From Eq. (8) we can obtain computable multi-dimensional formulas of FPT distributions.

2.3 First passage time distribution of Brownian bridge

Although for jump-diffusion processes, the closed form solutions are usually unavailable, yet between each two jumps the process is, generally speaking, a Brownian bridge for a univariate jump-diffusion process. The authors of [2] have deduced the one-dimensional first passage time distribution for time period $[0, T]$. In order to evaluate multiple processes, we obtain multi-dimensional formulas from Eq. (8) and reduce them to computable forms.

Here we follow [45] and we recall the basic idea for completeness. Let us consider N_{firm} firms $\boldsymbol{X}_t = [X_1, X_2, ..., X_{N_{\text{firm}}}]^T$, each X_i describes the process of individual firm i. From Eq. (8), we may expect that each process X_i satisfies the following SDE:

$$\begin{aligned} dX_i &= \mu_i dt + \sum_j \sigma_{ij} dW_j + dZ_i \\ &= \mu_i dt + \sigma_i d\tilde{W}_i + dZ_i, \end{aligned} \tag{9}$$

where $\tilde{W}_i$ is also a standard Brownian motion and σ_i is:

$$\sigma_i = \sqrt{\sum_j \sigma_{ij}^2}.$$

We assume that in the interval $[0, T]$, the total number of jumps for firm i is M_i. Let the jump instants be $T_1, T_2, \cdots, T_{M_i}$. Let $T_0 = 0$ and $T_{M_i+1} = T$. The quantities τ_j equal interjump times, which are $T_j - T_{j-1}$. Following the notation of [2], let $X_i(T_j^-)$ be the process value immediately before the jth jump, and $X_i(T_j^+)$ be the process value immediately after the jth jump. The jump-size is $X_i(T_j^+) - X_i(T_j^-)$, and we can use such jump-sizes to generate $X_i(T_j^+)$ sequentially.

Let $A_i(t)$ be the event consisting of process X_i crossing the threshold level $D_i(t)$ for the first time in the interval $[t, t+dt]$, then the conditional interjump first passage density is defined as [2]:

$$g_{ij}(t) = P(A_i(t) | X_i(T_{j-1}^+), X_i(T_j^-)). \tag{10}$$

If we only consider one interval $[T_{j-1}, T_j]$, we can obtain

$$\begin{aligned} g_{ij}(t) &= \frac{X_i(T_{j-1}^+) - D_i(t)}{2 y_i \pi \sigma_i^2} (t - T_{j-1})^{-\frac{3}{2}} (T_j - t)^{-\frac{1}{2}} \\ &\quad * \exp\left(- \frac{[X_i(T_j^-) - D_i(t) - \mu_i (T_j - t)]^2}{2 (T_j - t) \sigma_i^2} \right) \\ &\quad * \exp\left(- \frac{[X_i(T_{j-1}^+) - D_i(t) + \mu_i (t - T_{j-1})]^2}{2 (t - T_{j-1}) \sigma_i^2} \right), \end{aligned} \tag{11}$$

where

$$y_i = \frac{1}{\sigma_i\sqrt{2\pi\tau_j}} \exp\left(-\frac{[X_i(T_{j-1}^+) - X_i(T_j^-) + \mu_i\tau_j]^2}{2\tau_j\sigma_i^2}\right).$$

After getting a result in one interval, we combine the results to obtain the density for the whole interval $[0, T]$. It is clear that the process X_i may be viewed as a Brownian bridge $B(s)$ with $B(T_{j-1}^+) = X_i(T_{j-1}^+)$ and $B(T_j^-) = X_i(T_j^-)$ in the interval $[T_{j-1}, T_j]$, i.e. between each of the two successive jumps. Then the probability that the minimum of $B(s)$ is always above the boundary level is

$$P_{ij} = \begin{cases} 1 - \exp\left(-\frac{2[X_i(T_{j-1}^+)-D_i(t)][X_i(T_j^-)-D_i(t)]}{\tau_j\sigma_i^2}\right), & \text{if } X_i(T_j^-) > D_i(t), \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

The event "$B(s)$ is below the threshold level" means that the default happens or already happened, and its probability is $1 - P_{ij}$. If $s = s_i$ with s_i being the first passage time, let $L(s_i) \equiv L_i$ denote the index of the interjump period in which the time s_i falls in $[T_{L_i-1}, T_{L_i}]$. Also, let I_i represent the index of the first jump, which happened in the simulated jump instant,

$$\begin{aligned} I_i \;\; = \;\; & \min(j : X_i(T_k^-) > D_i(t); k = 1, \ldots, j, \text{ and} \\ & X_i(T_k^+) > D_i(t); k = 1, \ldots, j-1, \text{ and} \\ & X_i(T_j^+) \le D_i(t)). \end{aligned} \tag{13}$$

If no such I_i exists, then we set $I_i = 0$. By combining Eq. (11), (12) and (13), we get the probability of X_i crossing the boundary level in the whole interval $[0, T]$ as

$$\begin{aligned} & P(A_i(s_i) \in ds | X_i(T_{j-1}^+), X_i(T_j^-), j = 1, \ldots, M_i + 1) \\ = & \begin{cases} g_{iL_i}(s_i) \prod_{k=1}^{L_i-1} P_{ik} & \text{if } L_i < I_i \text{ or } I_i = 0, \\ g_{iL_i}(s_i) \prod_{k=1}^{L_i-1} P_{ik} + \prod_{k=1}^{L_i} P_{ik}\delta(s_i - T_{I_i}) & \text{if } L_i = I_i, \\ 0 & \text{if } L_i > I_i, \end{cases} \end{aligned} \tag{14}$$

where δ is the Dirac's delta function.

2.4 The kernel estimator

For firm i, after generating a series of first passage times s_i, we use a kernel density estimator with Gaussian kernel to estimate the first passage time density (FPTD) f. The kernel density estimator is based on centering a kernel function of a bandwidth as follows:

$$\widehat{f} = \frac{1}{N}\sum_{i=1}^{N} K(h, t - s_i), \tag{15}$$

where

$$K(h, t - s_i) = \frac{1}{\sqrt{\pi/2}h} \exp\left(-\frac{(t - s_i)^2}{h^2/2}\right).$$

The optimal bandwidth in the kernel function K can be calculated as [42]:

$$h_{opt} = \left(2N\sqrt{\pi}\int_{-\infty}^{\infty} (f_t'')^2 dt\right)^{-0.2}, \tag{16}$$

where N is the number of generated points and f_t is the true density. Here we use the approximation for the distribution as a gamma distribution, proposed in [2]:

$$f_t = \frac{\alpha^\beta}{\Gamma(\beta)} t^{\beta-1} \exp(-\alpha t). \tag{17}$$

So the integral in Eq. (16) becomes:

$$\int_0^\infty (f_t'')^2 dt = \sum_{i=1}^{5} \frac{W_i \alpha^i \Gamma(2\beta - i)}{2^{(2\beta-i)}(\Gamma(\beta))^2}, \tag{18}$$

where

$$W_1 = A^2, \;\; W_2 = 2AB, \;\; W_3 = B^2 + 2AC, \;\; W_4 = 2BC, \;\; W_5 = C^2,$$

and

$$A = \alpha^2, \;\; B = -2\alpha(\beta - 1), \;\; C = (\beta - 1)(\beta - 2).$$

From Eq. (18), it follows that in order to get a nonzero bandwidth, we have to have constraint β to be at least equal to 3. Also, we note that the estimates of both α and β can be obtained by the method of moments in the same way as described [2].

The kernel estimator for the multivariate case involves the evaluation of the joint conditional interjump first passage time density, as discussed in Section 3. The methodology for such an evaluation is quite involved compared to the one-dimensional case and we will focus on these details elsewhere. In what follows we highlight the main steps of the procedure.

3 The methodology of solution

First, let us recall the conventional Monte-Carlo procedure in application to the analysis of the evolution of firm X_i within the time period $[0, T]$. We divide the time period into n small intervals $[0, t_1]$, $[t_1, t_2]$, $\cdots$, $[t_{n-1}, T]$ as displayed in Fig. 1(a). In each Monte Carlo run, we need to calculate the value of X_i at each discretized time t. As usual, in order to exclude discretization bias, the number n must be large. This procedure exhibits substantial computational difficulties when applied to jump-diffusion processes. Indeed, for a typical jump-diffusion process, as shown in Fig. 1(a), let T_{j-1} and T_j be any successive jump instants, as described above. Then, in the conventional Monte Carlo method, although there is no jump occurring in the interval $[T_{j-1}, T_j]$, yet we need to evaluate X_i at each discretized time t in $[T_{j-1}, T_j]$. This very time-consuming procedure results in a serious shortcoming of the conventional Monte-Carlo methodology.

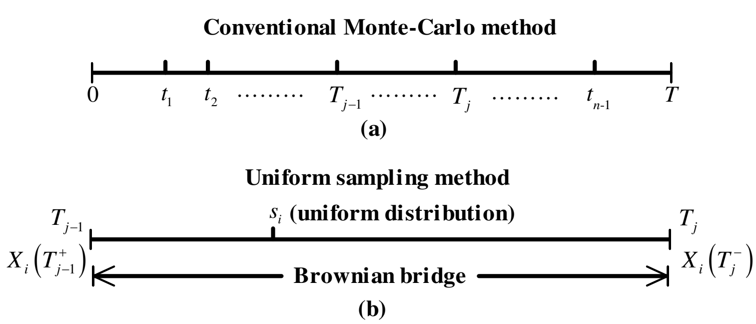

Figure 1: Schematic diagram of (a) conventional Monte Carlo and (b) uniform sampling (UNIF) method.

To remedy the situation, two modifications of the conventional procedure were recently proposed in [2, 35]. They allow a potential speed-up of the conventional methodology of up to 10-30 times. One of the modifications, the uniform sampling method, involves samplings using the uniform distribution. The other is the inverse Gaussian density sampling method. Both methodologies were developed for the univariate case.

The major improvement of the UNIF method is based on the fact that it only evaluates X_i at generated jump times, while between each two jumps the process is a Brownian bridge (see Fig. 1(b)). Hence, we just consider the probability of X_i crossing the threshold in (T_{j-1}, T_j) instead of evaluating X_i at each discretized time t. More precisely, in the UNIF method, we assume that the values of $X_i(T_{j-1}^{+})$ and $X_i(T_j^{-})$ are known as two end points of the Brownian bridge, the probability that firm i defaults in (T_{j-1}, T_j) is $1 - P_{ij}$ which can be computed according to Eq. (12). Then we generate a variable s_i from a distribution uniform in the interval $[T_{j-1}, T_{j-1} + \frac{T_j - T_{j-1}}{1-P_{ij}}]$. If the generated point s_i falls in the interjump interval $[T_{j-1}, T_j]$, then we have successfully generated the first passage time s_i and can neglect the other intervals and perform another Monte Carlo run. On the other hand, if the generated point s_i falls outside the interval $[T_{j-1}, T_j]$ (which happens with probability P_{ij}), then that point is "rejected". This means that no boundary crossing has occurred in the interval, and we proceed to the next interval and repeat the whole process again.

We focus next on the further development of the UNIF method and extend it to multivariate and correlated jump-diffusion processes. This follows [45] and we remind here the main steps. In order to implement the UNIF method for our multivariate model, as described in Eq. (8), we need to consider several points:

1. We assume that the arrived jumps follow the Poisson process. The intensity λ of the Poisson process and the distribution of $(T_j - T_{j-1})$ are the same for each firm. This assumption may not always be fulfilled as one may argue that the intensity λ could be different for different firms which implies that different firms may endure different jump rates. However, in the real market economy once a firm (let us call it "A") encounters sudden economic hazard, its correlated firms may also endure the same hazard. Furthermore, it is common that other firms may help firm "A" to pull out, which may result in a simultaneous jump for them. Therefore, as a first approximation, it is reasonable to employ the simultaneous jumps processes for all the different firms.

2. As for the jump-size, we generate it by a given distribution which can be different for different firms to reflect specifics of the jump process for each firm. In the current contribution, we exemplify our description by considering an exponential distribution (mean value μ_T) for $(T_j - T_{j-1})$ and a normal distribution (mean value μ_J and standard deviation σ_J) for the jump-size. We can use any other distribution when appropriate.

3. An array `IsDefault` (whose size is the number of firms denoted by N_{firm}) is used to indicate whether firm i has defaulted in this Monte Carlo run. If the firm defaults, then we set `IsDefault`$(i) = 1$, and will not evaluate it during this Monte Carlo run.

4. Most importantly, as we have mentioned before, the default events of firm i are inevitably correlated with other firms, for example firm $i+1$. The default correlation of firms i and $i+1$ is described by Eq. (5). Hence, firm i's first passage time s_i is indeed correlated with s_{i+1} – the first passage time of firm $i+1$. We must generate several correlated s_i in each interval $[T_{j-1}, T_{j-1} + \frac{T_j - T_{j-1}}{1-P_{ij}}]$ which is the key point for multivariate correlated processes.

   Finally, we note that the default happening at time s_i also means that time s_i is exactly the first passage time for firm i. Therefore, the correlation of s_i and s_{i+1} is the same as the default correlation of firms i and $i+1$:

$$\rho(s_i, s_{i+1}) = \rho_{i,i+1}(t) = \frac{P_i(t) + P_{i+1}(t) - P_i(t)P_{i+1}(t) - P_{i \cup i+1}(t)}{\sqrt{P_i(t)[1 - P_i(t)]P_{i+1}(t)[1 - P_{i+1}(t)]}}, \tag{19}$$

where in practice t can be chosen as the midpoint of the interval.

Next, we will give a brief description of the sum-of-uniforms method which is used to generate correlated uniform random variables, followed by the description of the multivariate and correlated UNIF method and the model calibration.

3.1 Sum-of-uniforms method

The main result until now is that we have reduced the solution of the original problem to a series of one-dimensional jump-diffusion processes as described by Eq. (9). We follow [45] recalling that the first passage time distribution in an interval $[T_{j-1}, T_j]$ (between two successive jumps) was obtained in section 2.3. Here, we will describe how to generate several correlated s_i in $[T_{j-1}, T_{j-1} + \frac{T_j - T_{j-1}}{1-P_{ij}}]$ whose correlations can be described by Eq. (19).

Let us introduce a new variable $b_{ij} = \frac{T_j - T_{j-1}}{1-P_{ij}}$, then we have $s_i = b_{ij} Y_i + T_{j-1}$, where Y_i is uniformly distributed in $[0, 1]$. Moreover, the correlation of Y_i and Y_{i+1} equals $\rho(s_i, s_{i+1})$. Now we can generate the correlated uniform random variables $Y_1, Y_2, \cdots$ by using the sum-of-uniforms (SOU) method [7, 49] in the following steps:

1. Generate Y_1 from numbers uniformly distributed in $[0, 1]$.

2. For $i = 2, 3, \cdots$, generate $W_i \sim U(0, c_{i-1,i})$, where $U(0, c_{i-1,i})$ denotes a uniform random number over range $(0, c_{i-1,i})$. The author of [7] has obtained the relationship of parameter $c_{i-1,i}$ and the correlation $\rho(s_{i-1}, s_i)$ (abbreviated as $\rho_{i-1,i}$) as follows:

$$\rho_{i-1,i} = \begin{cases} \dfrac{1}{c_{i-1,i}} - \dfrac{0.3}{c_{i-1,i}^2}, & 0 \le \rho_{i-1,i} \le 0.7, \quad c_{i-1,i} \ge 1, \\ 1 - 0.5c_{i-1,i}^2 + 0.2c_{i-1,i}^3, & \rho_{i-1,i} \ge 0.7, \quad c_{i-1,i} < 1, \\ -\dfrac{1}{c_{i-1,i}} + \dfrac{0.3}{c_{i-1,i}^2}, & -0.7 \le \rho_{i-1,i} \le 0, \quad c_{i-1,i} \ge 1, \\ -1 + 0.5c_{i-1,i}^2 - 0.2c_{i-1,i}^3, & \rho_{i-1,i} \le -0.7, \quad c_{i-1,i} < 1. \end{cases}$$

If Y_{i-1} and Y_i are positively correlated, then let

$$Z_i = Y_{i-1} + W_i.$$

If Y_{i-1} and Y_i are negatively correlated, then let

$$Z_i = 1 - Y_{i-1} + W_i.$$

Let $Y_i = F(Z_i)$, where for $c_{i-1,i} \ge 1$,

$$F(Z) = \begin{cases} Z^2/(2c_{i-1,i}), & 0 \le Z \le 1, \\ (2Z-1)/(2c_{i-1,i}), & 1 \le Z \le c_{i-1,i}, \\ 1 - (1 + c_{i-1,i} - Z)^2/(2c_{i-1,i}), & c_{i-1,i} \le Z \le 1 + c_{i-1,i}, \end{cases}$$

and for $0 < c_{i-1,i} \le 1$,

$$F(Z) = \begin{cases} Z^2/(2c_{i-1,i}), & 0 \le Z \le c_{i-1,i}, \\ (2Z - c_{i-1,i})/2, & c_{i-1,i} \le Z \le 1, \\ 1 - (1 + c_{i-1,i} - Z)^2/(2c_{i-1,i}), & 1 \le Z \le 1 + c_{i-1,i}. \end{cases}$$

By carrying out the above two steps, we can generate correlated uniform random variables Y_1, Y_2,..., leading to the relationship $s_i = b_{ij} Y_i + T_{j-1}$ whose correlations automatically satisfy Eq. (19). Note also that $\rho(s_i, s_{i+1})$ should be computed before generating the correlated uniform random variables Y_1, Y_2,..., and hence, in a practical implementation, we should approximate time t where $\rho(s_i, s_{i+1})$ is computed according to Eq. (3), (4), and (19). One such possible choice has already been mentioned above and, from a practical point of view, it will work well as long as $\rho(s_i, s_{i+1})$ is a slowly varying function in $[T_{j-1}, T_j]$.

3.2 Uniform sampling method

In this subsection, we will describe our algorithm for multivariate jump-diffusion processes, which is an extension of the one-dimensional case developed earlier by other authors (e.g. [2, 35]).

Consider N_{firm} firms in the given time period $[0, T]$. First, we generate the jump instant T_j by generating interjump times $(T_j - T_{j-1})$ and set all the $\texttt{IsDefault}(i) = 0$ $(i = 1, 2, \cdots, N_{\text{firm}})$ to indicate that no firm has defaulted at the beginning.

From Fig. 1(b) and Eq. (9), we can conclude that for each process X_i we can make the following observations:

1. If no jump occurs, as described by Eq. (9), the interjump size $(X_i(T_j^-) - X_i(T_{j-1}^+))$ follows a normal distribution of mean $\mu_i(T_j - T_{j-1})$ and standard deviation $\sigma_i \sqrt{T_j - T_{j-1}}$. We get

$$\begin{aligned} X_i(T_j^-) &\sim X_i(T_{j-1}^+) + \mu_i(T_j - T_{j-1}) + \sigma_i \sqrt{T_j - T_{j-1}}\, \tilde{W}_i \\ &\sim X_i(T_{j-1}^+) + \mu_i(T_j - T_{j-1}) + \sum_{k=1}^{N_{\text{firm}}} \sigma_{ik} \sqrt{T_j - T_{j-1}}\, W_i, \end{aligned}$$

   where the initial state is $X_i(0) = X_i(T_0^+)$.

2. If a jump occurs, we simulate the jump-size by a normal distribution or another distribution when appropriate, and compute the postjump value:

$$X_i(T_j^+) = X_i(T_j^-) + Z_i(T_j).$$

This completes the procedure for generating beforejump and postjump values $X_i(T_j^-)$ and $X_i(T_j^+)$, respectively. As before, $j = 1, \cdots, M$, where M is the total number of jumps for all the firms. We compute P_{ij} according to Eq. (12). To recur the first passage time density (FPTD) $f_i(t)$, we have to consider three possible cases that may occur for each non-default firm i. In particular, following [45], we may have:

1. **First passage happens inside the interval.** We know that if $X_i(T_{j-1}^+) > D_i(T_{j-1})$ and $X_i(T_j^-) < D_i(T_j)$, then the first passage happened in the time interval $[T_{j-1}, T_j]$. To evaluate when the first passage happened, we introduce a new variable b_{ij} as $b_{ij} = \frac{T_j - T_{j-1}}{1 - P_{ij}}$. We generate several correlated uniform numbers Y_i by using the SOU method as described in Section 3.1, then compute $s_i = b_{ij} Y_i + T_{j-1}$. If s_i belongs to interval $[T_{j-1}, T_j]$, then the first passage time occurred in this interval. We set $\texttt{IsDefault}(i) = 1$ to indicate that firm i has defaulted and compute the conditional boundary crossing density $g_{ij}(s_i)$ according to Eq. (11). To get the density for the entire interval $[0, T]$, we use $\widehat{f}_{i,n}(t) = \left(\frac{T_j - T_{j-1}}{1 - P_{ij}}\right) g_{ij}(s_i) * K(h_{opt}, t - s_i)$, where n is the iteration number of the Monte Carlo cycle.

2. **First passage does not happen in this interval.** If s_i does not belong to interval $[T_{j-1}, T_j]$, then the first passage time has not yet occurred in this interval.

3. **First passage happens at the right boundary of the interval.** If $X_i(T_j^+) < D_i(T_j)$ and $X_i(T_j^-) > D_i(T_j)$ (see Eq. (13)), then T_{I_i} is the first passage time and $I_i = j$, we evaluate the density function using kernel function $\widehat{f}_{i,n}(t) = K(h_{opt}, t - T_{I_i})$, and set $\texttt{IsDefault}(i) = 1$.

Next, we increase j and examine the next interval and analyze the above three cases for each non-default firm again. After running N times the Monte Carlo cycle, we get the FPTD of firm i as $\widehat{f}_i(t) = \frac{1}{N}\sum_{n=1}^{N} \widehat{f}_{i,n}(t)$.

3.3 Model calibration and concluding remarks on the developed methodology

In order to provide a reasonable credit analysis, we need to calibrate the developed model or, in other words, to numerically choose or optimize the parameters, such as drift, volatility and jumps to fit the most liquid market data. In what follows, we will use the historical default data to optimize the parameters in the model based on the least-square methodology.

As already mentioned in Sections 2.4 and 3.2, after Monte Carlo simulation we obtain the estimated density $\hat{f}_i(t)$ by using the kernel estimator method. The cumulative default rates for firm i in our model is defined as:

$$P_i(t) = \int_0^t \widehat{f}_i(\tau) d\tau. \tag{20}$$

Then we minimize the difference between our model and historical default data $\widetilde{A}_i(t)$ to obtain the optimized parameters in the model (such as σ_{ij}, arrival intensity λ in Eq. (9)):

$$\operatorname{argmin} \left(\sum_i \sqrt{\sum_{t_j} \left(\frac{P_i(t_j) - \widetilde{A}_i(t_j)}{t_j} \right)^2} \right). \tag{21}$$

Note that in practice, the generated by using SOU method s_i are not obtained according to the conditional boundary crossing density $g_{ij}(s_i)$ as described by Eq. (11). Instead, in order to obtain an appropriate density estimate, the right hand side summation in Eq. (15) can be viewed as a finite sample estimate in a way proposed in [2]. For the multidimensional density estimate, we need to evaluate the joint conditional boundary crossing density. This problem can be divided into several one-dimensional density estimation subproblems if the processes are non-correlated [50]. As for the general case of multivariate correlated processes, the joint density is not available analytically and numerical approximations are necessary for different classes of special cases. In its generality, this task is highly non-trivial and the underlying difficulty consists of the fact that the positions of the generated s_i can be biased if their conditional distribution in the j-th interjump time interval (g_{ij}) is not uniform. However, even in this case several possibilities exist, including various modified sampling methodologies (e.g., [44]) as well as the ranked simulated sampling [40]. Perhaps, the simplest way to avoid the mentioned difficulty at the approximation level in the multivariate case would be to use transformations from a correlated multivariate normal (with arbitrary correlations) to uniform for each dimension and then adjust the correlation appropriately [31].

Our specific examples in the next section are given for multiple correlated firms via set of historical default data. However, the multivariate situation still requires a more detailed analysis of CPU times. Despite being applicable in the multivariate case, many methodologies are usually compared for the one-dimensional case (e.g., [30]). In that case we already mentioned earlier in this section that the applied here Monte Carlo procedure provides a speed-up with factor 10-30 compared to the conventional Monte Carlo methodology [2].

Before moving to specific examples, demonstrating numerical efficiency of the developed methodology, we remark that the developed method belongs to the class of methodologies based on Brownian bridge simulations or more generally large deviations methodologies. In several special cases, recent theoretical results on estimating barrier crossing probabilities of the associated Brownian bridges are available in the literature (with upper and lower limits). With a few exceptions, most such results concern one-dimensional Brownian bridges only. The interested reader should consult [35, 41, 24, 38, 29] for further details on these issues.

4 Applications and discussion

In this section, we demonstrate the developed model at work for analyzing the default events of multiple correlated firms via a set of historical default data.

4.1 Density function and default rate

First, for completeness, let us consider a set of historical default data of differently rated firms as presented in [52]. Our first task is to describe the first passage time density functions and default rates of these firms. This result was reported briefly in [46] and we provide here complete details.

Since there is no option value that can be used, we will employ Eq.(21) to optimize the parameters in our model. For convenience, we reduce the number of optimizing parameters by:

1. Setting $X_i(0) = 2$ and $\ln(\kappa_i) = 0$.

2. Setting the growth rate γ_i of debt value equivalent to the growth rate μ_i of the firm's value [52], so the default of firm is non-sensitive to μ_i. In our computations, we set $\mu_i = -0.001$.

3. The interjump times $(T_j - T_{j-1})$ satisfy an exponential distribution with mean value equal to 1.

4. The arrival rate for jumps satisfies the Poisson distribution with intensity parameter λ, where the jump size is a normal distribution $Z_i \sim N(\mu_{Z_i}, \sigma_{Z_i})$.

As a result, we only need to optimize σ_i, λ, μ_{Z_i}, σ_{Z_i} for each firm. This is done by minimizing the differences between our simulated default rates and historical data. Moreover, as mentioned above, we will use the same arrival rate λ and distribution of $(T_j - T_{j-1})$ for differently rated firms, so we first optimize four parameters for, e.g., the A-rated firm, and then set the parameter λ of other three firms the same as A's.

The minimization was performed by using the quasi-Newton procedure implemented as a Scilab program. The optimized parameters for each firm are described in Table 1. In each step of the optimization we choose the Monte Carlo runs $N = 50,000$.

Table 1: Optimized parameters for differently rated firms by using the UNIF method.

	σ_i	λ	μ_{Z_i}	σ_{Z_i}
A	0.0900	0.1000	-0.2000	0.5000
Baa	0.0894	0.1000	-0.2960	0.6039
Ba	0.1587	0.1000	-0.5515	1.6412
B	0.4500	0.1000	-0.8000	1.5000

By using these optimized parameters, we carried out the final simulation with Monte Carlo runs $N = 500,000$. The estimated first passage time density function of these four firms are shown in Fig. 2. The simulated cumulative default rates (line) together with historical data (squares) are given in Fig. 3. The theoretical data denoted as circles in Fig. 3 were computed by using Eq. (3) where the Z_i were evaluated in

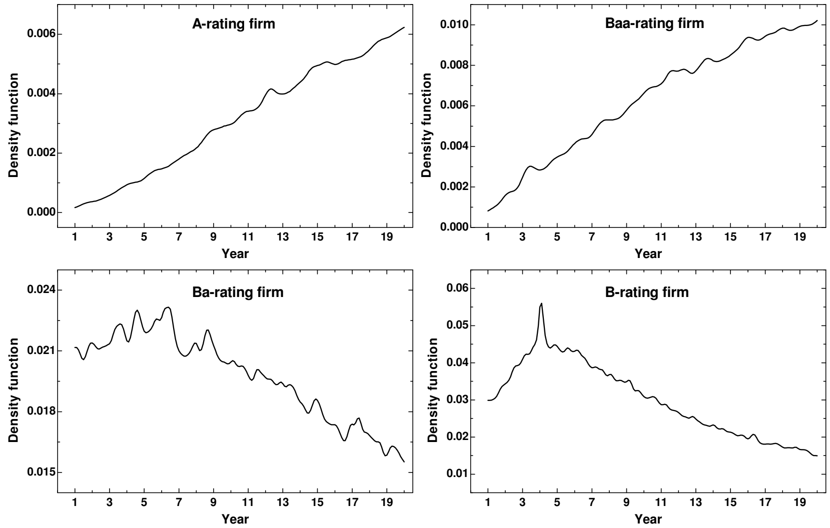

Figure 2: Estimated density function for differently rated firms. All the simulations were performed with Monte Carlo runs $N = 500,000$.

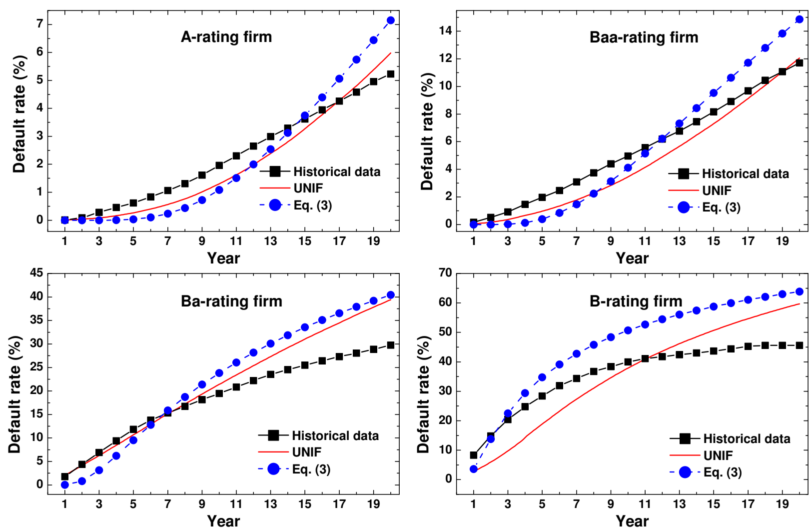

Figure 3: Historical (squares), theoretical (circles) and simulated (line) cumulative default rates for differently rated firms. All the simulations were performed with Monte Carlo runs $N = 500,000$.

Table 2: The optimal bandwidth h_{opt}, parameters α, β for the true density estimate of differently rated firms. All the simulations were performed with Monte Carlo runs $N = 500,000$.

	α	β	Optimal bandwidth
A	0.206699	3	0.655522
Baa	0.219790	3	0.537277
Ba	0.252318	3	0.382729
B	0.327753	3	0.264402

[52] as 8.06, 6.46, 3.73 and 2.10 for A-, Baa-, Ba- and B-rated firms, respectively. In Table 2, we give the optimal bandwidth and parameters α, β for the true density estimate.

Based on these results, we conclude that:

1. Simulations give similar or better results to the analytical results predicted by Eq. (3).

2. A- and Baa-rated firms have a smaller Brownian motion part. Their parameters σ_i are much smaller than those of Ba- and B-rated firms.

3. The optimized parameters σ_i of A- and Baa-rated firms are similar, but the jump parts $(\mu_{Z_i}, \sigma_{Z_i})$ are different, which explains their different cumulative default rates and density functions. Indeed, Baa-rated firm may encounter more severe economic hazard (large jump-size) than A-rated firm.

4. As for Ba- and B-rated firms, except for the large σ, both of them have large μ_{Z_i} and especially large σ_{Z_i}, which indicate that the loss due to sudden economic hazard may fluctuate a lot for these firms. Hence, the large σ_i, μ_{Z_i} and σ_{Z_i} account for their high default rates and low credit qualities.

5. From Fig. 2, we can conclude that the density functions of A- and Baa-rated firms still have the trend to increase, which means the default rates of A- and Baa-rated firms may increase little faster in future. As for Ba- and B-rated firms, their density functions have decreased, so their default rates may increase very slowly or be kept at a constant level. Mathematically speaking, the cumulative default rates of A- and Baa-rated firms are convex function, while the cumulative default rates of Ba- and B-rated firms are concave.

4.2 Correlated default

Our final example concerns the default correlation of two firms. If we do not include jumps in the model, the default correlation can be calculated by using Eq. (3), (4) and (5). In Tables 3-6 we present comparisons of our results with those based on closed form solutions provided in [52] with $\rho = 0.4$. In what follows we follow closely [45] but with more complete information on default data.

Table 3: One year default correlations (%). All the simulations are performed with Monte Carlo runs $N = 500,000$

	UNIF				[52]			
	A	Baa	Ba	B	A	Baa	Ba	B
A	-0.01				0.00			
Baa	-0.02	3.69			0.00	0.00		
Ba	2.37	4.95	19.75		0.00	0.01	1.32	
B	2.80	6.63	22.57	26.40	0.00	0.00	2.47	12.46

Next, let us consider the default correlations under the multivariate jump-diffusion processes. We use the following conditions in our multivariate UNIF method:

1. Setting $X_i(0) = 2$ and $\ln(\kappa_i) = 0$ for all firms.

2. Setting $\gamma_i = \mu_i$ and $\mu_i = -0.001$ for all firms.

Table 4: Two year default correlations (%). All the simulations are performed with the Monte Carlo runs $N = 500,000$

	UNIF				[52]			
	A	Baa	Ba	B	A	Baa	Ba	B
A	2.35				0.02			
Baa	2.32	4.25			0.05	0.25		
Ba	4.17	7.17	20.28		0.05	0.63	6.96	
B	4.73	8.23	23.99	29.00	0.02	0.41	9.24	19.61

Table 5: Five year default correlations (%). All the simulations are performed with the Monte Carlo runs $N = 500,000$

	UNIF				[52]			
	A	Baa	Ba	B	A	Baa	Ba	B
A	6.45				1.65			
Baa	6.71	9.24			2.60	5.01		
Ba	7.29	10.88	22.91		2.74	7.20	17.56	
B	6.77	10.93	22.97	27.93	1.88	5.67	18.43	24.01

Table 6: Ten year default correlations (%). All the simulations are performed with the Monte Carlo runs $N = 500,000$

	UNIF				[52]			
	A	Baa	Ba	B	A	Baa	Ba	B
A	8.79				7.75			
Baa	10.51	13.80			9.63	13.12		
Ba	9.87	14.23	22.50		9.48	14.98	22.51	
B	8.50	12.54	20.49	24.98	7.21	12.28	21.80	24.37

3. Since we are considering two correlated firms, we choose $\boldsymbol{\sigma}$ as,

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}, \tag{22}$$

where $\boldsymbol{\sigma}\boldsymbol{\sigma}^\top = \boldsymbol{H}_0$ such that,

$$\boldsymbol{\sigma}\boldsymbol{\sigma}^\top = \boldsymbol{H}_0 = \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix},$$

and

$$\begin{cases} \sigma_1^2 = \sigma_{11}^2 + \sigma_{12}^2, \\ \sigma_2^2 = \sigma_{21}^2 + \sigma_{22}^2, \\ \rho_{12} = \dfrac{\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22}}{\sigma_1\sigma_2}. \end{cases} \tag{23}$$

In Eq. (23), ρ_{12} reflects the correlation of the diffusion parts of the processes of the two firms. In order to compare with the standard Brownian motion and to evaluate the default correlations between different firms, we set all the $\rho_{12} = 0.4$ as in [52]. Furthermore, we use the optimized σ_1 and σ_2 in Table 1 for firm 1 and 2, respectively. Assuming $\sigma_{12} = 0$, we get,

$$\begin{cases} \sigma_{11} = \sigma_1, \\ \sigma_{12} = 0, \\ \sigma_{21} = \rho_{12}\sigma_2, \\ \sigma_{22} = \sqrt{1-\rho_{12}^2}\sigma_2. \end{cases}$$

4. The arrival rate for jumps satisfies the Poisson distribution with intensity parameter $\lambda = 0.1$ for all firms. The jump size is a normal distribution $Z_i \sim N(\mu_{Z_i}, \sigma_{Z_i})$, where μ_{Z_i} and σ_{Z_i} can be different for different firms to reflect specifics of the jump process for each firm. We adopt the optimized parameters given in Table 1.

5. As before, we generate the same interjump times $(T_j - T_{j-1})$ that satisfy an exponential distribution with mean value equal to 1 for each of the two firms.

We carry out the UNIF method to evaluate the default correlations via the following formula:

$$\rho_{12}(t) = \frac{1}{N}\sum_{n=1}^{N} \frac{P_{12,n}(t) - P_{1,n}(t)P_{2,n}(t)}{\sqrt{P_{1,n}(t)(1-P_{1,n}(t))P_{2,n}(t)(1-P_{2,n}(t))}}, \tag{24}$$

where $P_{12,n}(t)$ is the probability of joint default for firms 1 and 2 in each Monte Carlo cycle, $P_{1,n}(t)$ and $P_{2,n}(t)$ are the cumulative default rates of firm 1 and 2, respectively, in each Monte Carlo cycle.

The simulated default correlations for one-, two-, five- and ten-years are given in Tables 3-6. All the simulations were performed with the Monte Carlo runs $N = 500,000$. Comparing simulated default correlations with the theoretical data for standard Brownian motions, we can conclude as in [45] that similarly to conclusions of [52], the default correlations of same rated firms are usually large compared to differently rated firms. Furthermore, the default correlations tend to increase over long time and may converge to a stable value.

Note also that in our simulations, the one year default correlations of (A,A) and (A,Baa) are negative. This is because they seldom default jointly during one year. Note, however, that the default correlations of other firms are positive and usually larger than in the results presented in [52].

Next observation is that for two and five years, the default correlations of different firms increase. This can be explained by the fact that their individual first passage time density functions increase during these time periods, hence the probability of joint default increases.

Finally, as for ten year default correlations, our simulated results are almost identical to the theoretical data for standard Brownian motions. The differences are that the default correlations of (Ba,Ba), (Ba,B) and (B,B) decrease from the fifth year to tenth year in our simulations. The reason is that the first passage time density functions of Ba- and B-rated firms begin to decrease from the fifth year, hence the probability of joint default may increase slowly.

These two examples demonstrated the efficiency of the developed methodology while applied to the analysis of multivariate jump-diffusion processes. The key to success is an efficient combination of the fast Monte Carlo method for one-dimensional jump-diffusion processes, implemented via the uniform sampling method, and the generation of multi-dimensional variates based on the sum-of-uniforms methodology. The proposed methodology can be developed further along the following lines. Firstly, as we pointed out in Section 3.3, the evaluation of the joint density for multivariate correlated processes is a non-trivial task and a systematic comparison of different techniques represents an open task. Secondly, although in Section 3 we assumed that the distribution of the default interval is the same for all firms and the arrived jumps follow the Poisson process with the same intensity λ, this can be relaxed. In particular, both jump rates (intensities) and the jump times can be different for different firms. However, the effectiveness of the developed methodology is likely to decrease. We also note that after the original version of this paper was submitted several other applications of the fast Monte-Carlo procedure [2] have appeared in the literature. This procedure can be combined with the Markov Chain approximation approach. The latter approach is a powerful tool in solving many problems involving complex dynamics [32]. The authors of [15] have applied the mentioned combination to the solution corporate securities pricing which allowed them to develop an efficient finite dimensional approximative filter for the asset value which is important for problems with incomplete information typical for finance, among other areas.

5 Conclusions

In this contribution, we have analyzed the credit risk problems of multiple correlated firms in a structural model framework, where we incorporated jumps to reflect the external shocks or other unpredicted events. By combining the fast Monte-Carlo method for one-dimensional jump-diffusion processes and the generation of correlated multidimensional variates, we have developed a fast Monte-Carlo type procedure for the analysis of multivariate and correlated jump-diffusion processes. The developed approach generalizes previously discussed non-correlated jump-diffusion cases for multivariate and correlated jump-diffusion processes. Finally, we have applied the developed technique to analyze the default events of multiple correlated firms via a set of historical default data. The developed methodology provides an efficient computational technique that is applicable in other areas of credit risk and for the pricing of options. In the latter case, several problems may arise, including an efficient approximation of the joint probability of the first passage time and the terminal value of the underlying process for barrier type options. Comparisons with other methodologies developed for the multivariate case of option pricing problems (e.g., [17, 8]) would also represent an important task.

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