Modal Analysis of the Gyroscopic Continua: Comparison of Continuous and Discretized Models

Xiao-Dong Yang[1]
Beijing Key Laboratory of Nonlinear Vibrations and Strength of Mechanical Engineering,
College of Mechanical Engineering,
Beijing University of Technology,
Beijing 100124, China
e-mail: jxdyang@163.com

Song Yang
Beijing Key Laboratory of Nonlinear Vibrations and Strength of Mechanical Engineering,
College of Mechanical Engineering,
Beijing University of Technology,
Beijing 100124, China
e-mail: 554634456@qq.com

Ying-Jing Qian
Beijing Key Laboratory of Nonlinear Vibrations and Strength of Mechanical Engineering,
College of Mechanical Engineering,
Beijing University of Technology,
Beijing 100124, China
e-mail: candiceqyj@163.com

Wei Zhang
Beijing Key Laboratory of Nonlinear Vibrations and Strength of Mechanical Engineering,
College of Mechanical Engineering,
Beijing University of Technology,
Beijing 100124, China
e-mail: sandyzhang0@yahoo.com

Roderick V. N. Melnik
M2NeT Laboratory,
The MS2Discovery Interdisciplinary Research Institute,
Wilfrid Laurier University,
75 University Avenue West,
Waterloo, ON N2L 3C5, Canada
e-mail: rmelnik@wlu.ca

The vibrations of gyroscopic continua may induce complex mode functions. The continuous model governed by partial differential equations (PDEs) as well as the discretized model governed by ordinary differential equations (ODEs) are used in the dynamical study of the gyroscopic continua. The invariant manifold method is employed to derive the complex mode functions of the discretized models, which are compared to the mode functions derived from the continuous model. It is found that the complex mode functions constituted by trial functions of the discretized system yield good agreement with that derived by the continuous system. On the other hand, the modal analysis of discretized system demonstrates the phase difference among the general coordinates presented by trial functions, which reveals the physical explanation of the complex modes. [DOI: 10.1115/1.4033752]

Introduction

Mechanical structures that can vibrate about a state of mean rotation are classified as gyroscopic dynamic systems. Wickert [1] used the superposed motions to explain the gyroscopic dynamics that occurs when configurations are measured relative to a noninertial frame. Many examples of gyroscopic systems can be found in the engineering field, including applications of axially moving materials, rotating bodies, and fluid-conveying pipes. The three-body problems in celestial mechanics, especially the orbital dynamics about the Lagrangian libration point, belong to the discretized gyroscopic system.

The presence of the skew-symmetric gyroscopic operator in the governing equations limits analytical results, but enriches the dynamical behaviors dramatically. One significant distinction of the gyroscopic dynamics is the involvement of the complex modes. The vibrations and stability of gyroscopic continua have been studied extensively in the literature. Recent progress on axially moving materials can be found in the review paper by Marynowski and Kapitaniak [2], in which important studies of axially moving strings, beams, and plates have been reported. Some investigations of rotating bodies are found in the papers by Banerjee and Kennedy [3], Banerjee et al. [4], and Rao [5]. Readers who are interested in the dynamics of pipes conveying fluid can refer to the monograph by Païdoussis [6] and the papers by Ibrahim [7], Yu et al. [8], as well as references therein.

The linear governing PDEs of such gyroscopic continua in terms of their displacement w can be written as

$$M\ddot{w} + G\dot{w} + Kw = F(w, \dot{w}, t) \tag{1}$$

where the mass, gyroscopic, and linear stiffness operators are, respectively, denoted by M, G, and K. The distributed time-dependent external and parametric excitations are included in F. For different models of the gyroscopic continua, the gyroscopic and stiffness operators have different forms. For the axially moving material case and the pipes conveying fluid case, the gyroscopic operator is $G=2\Gamma\partial^2/\partial x\partial t$, where Γ denotes the velocity of the axially moving material [9,10] or the fluid in the pipe [11,12]. The gyroscopic operators in the axially moving material give rise to the gyroscopic coupling of different general coordinates for the same transverse direction in the corresponding discretized system. For the rotating beam and the spinning beam case, the gyroscopic operators are in the matrix form $G=[0,2\Omega\partial/\partial x;\ -2\Omega\partial/\partial x, 0]$ (Ω is the spinning angular velocity), due to the fact that the gyroscopic coupling of the rotating bodies is caused by the two-directional coupling instead of general coordinates coupling [13,14]. In the study of string models, the second-order stiffness operator $K=-(1-\Gamma^2)\partial^2/\partial x^2$ has been used [15,16]. Furthermore, if the beam model is involved in the axially moving material, the fourth-order stiffness operator $K=\partial^4/\partial x^4-(1-\Gamma^2)\partial^2/\partial x^2$ is required in the analysis [17,18].

Earlier research focused on the analysis of the natural frequencies, critical speed, and stability of the gyroscopic continua based on the free linear vibrations [9,19–22]. If we have the distributed periodic loading on the material or on the boundaries, the steady-state responses of the nonlinear system become the main focus of the research. In many works, responses for external and parametric resonances have been studied in many literatures by the perturbation methods [23–29] as well as the numerical methods [25,30–33].

Modal analysis is a powerful tool to study the responses, even mode interactions, of the linear nongyroscopic systems [34]. However, when treating the gyroscopic, or general damped systems, the modal analysis becomes complicated because the

[1]Corresponding author.

Contributed by the Applied Mechanics Division of ASME for publication in the Journal of Applied Mechanics. Manuscript received February 17, 2016; final manuscript received May 31, 2016; published online June 16, 2016. Assoc. Editor: Alexander F. Vakakis.

complex modes must be involved [1,35]. By casting the configuration space into state space, Brake and Wickert [36] carried out the modal analysis for the second-order gyroscopic continua by the means of real value vectors instead of complex value vectors.

In recent years, there have been a number of investigations related to the concept of nonlinear normal modes [37–40], which shed a new light on the modal analysis of nonlinear and gyroscopic systems. This new developments are largely due to earlier works by Shaw and Pierre [41,42], who introduced the invariant manifold method to derive the nonlinear normal modes which are not confined to the concept of vibration in-unison modes. Hence, the invariant manifold method has the potential power to study the gyroscopic system.

Two types of modeling approaches are used in this field most frequently: the approach based on the continuous governing PDEs and the approach based on the discretized ODEs. Analysis of PDEs yields complex mode functions, and investigation of discretized ODEs leads to complex relations among trial functions. The complex mode functions derived from the two models have not been systematically investigated. We will fill this gap by applying the invariant manifold method to the complex modal analysis of gyroscopic system.

In this study, taking the axially moving beam as an example, we construct the relationship between the complex modal functions of the continuous PDE model and the modal motions of the discretized ODE model for the gyroscopic continua. Further, we provide in-depth explanation of relations among the trial functions, which are employed as the base for discretization.

Gyroscopic Continua: Axially Moving Beam Models

The axially moving beam is a typical gyroscopic continuum in the engineering field. In the dimensionless form, the linear governing PDEs [10,43,44] can be written as

$$\frac{\partial^2 w}{\partial t^2} + 2\gamma\frac{\partial^2 w}{\partial x \partial t} + \left(\gamma^2 - 1\right)\frac{\partial^2 w}{\partial x^2} + \kappa\frac{\partial^4 w}{\partial x^4} = 0 \tag{2}$$

where w is the dimensionless transverse displacement, t is the dimensionless time, γ denotes the dimensionless velocity, and κ denotes the dimensionless stiffness.

The mode functions and natural frequencies have been derived analytically [28,45,46]. The nth complex mode function is

$$\phi(x) = c_1\left\{e^{i\beta_1 x} - \frac{(\beta_4^2-\beta_1^2)(e^{i\beta_3}-e^{i\beta_1})}{(\beta_4^2-\beta_2^2)(e^{i\beta_3}-e^{i\beta_2})}e^{i\beta_2 x} - \frac{(\beta_4^2-\beta_1^2)(e^{i\beta_2}-e^{i\beta_1})}{\left(\beta_4^2-\beta_3^2\right)\left(e^{i\beta_2}-e^{i\beta_3}\right)}e^{i\beta_3 x} + \left[1 + \frac{(\beta_4^2-\beta_2^2)(e^{i\beta_3}-e^{i\beta_1})}{(\beta_4^2-\beta_2^2)(e^{i\beta_3}-e^{i\beta_2})} + \frac{(\beta_4^2-\beta_1^2)(e^{i\beta_2}-e^{i\beta_1})}{\left(\beta_4^2-\beta_3^2\right)\left(e^{i\beta_2}-e^{i\beta_3}\right)}\right]e^{i\beta_4 x}\right\} \tag{3}$$

where the eigenvalues β_k $(k=1,2,3,4)$ are the nth set of solutions for the following fourth-order algebraic equation:

$$\kappa\beta^4 + (1-\gamma^2)\beta^2 - 2\gamma\beta\omega - \omega^2 = 0 \tag{4}$$

and the following transcendental equation determined by the simple supported boundary conditions:

$$\begin{aligned}&[e^{i(\beta_1+\beta_2)} + e^{i(\beta_3+\beta_4)}](\beta_1^2-\beta_2^2)(\beta_3^2-\beta_4^2)\\ &\quad + [e^{i(\beta_1+\beta_3)} + e^{i(\beta_2+\beta_4)}](\beta_2^2-\beta_4^2)(\beta_3^2-\beta_1^2)\\ &\quad + [e^{i(\beta_3+\beta_2)} + e^{i(\beta_1+\beta_4)}](\beta_1^2-\beta_4^2)(\beta_2^2-\beta_3^2) = 0\end{aligned} \tag{5}$$

In computing the nth set of eigenvalues of β_k $(k=1,2,3,4)$, the corresponding nth natural frequency ω_n can also be derived numerically.

The complex mode functions of the continuous model (3) appear to be complicated and inappropriate to truncate the PDEs in the study of the complex dynamics of the gyroscopic continuum (2). Many researchers [27,34,35] took the sine trial functions to truncate the PDEs for convenience, since such functions satisfy the boundary conditions for the simply supported axially moving beam.

By using the sine trial functions, the solutions to Eq. (2) are assumed as

$$w(x,t) = \sum_{n=1}^{N} q_n \sin(n\pi x) \tag{6}$$

Substituting Eq. (6) into Eq. (2), multiplying both sides of the result by $\sin(k\pi x)$, and integrating the final equation on the length interval [0, 1], we can get the kth equation. Let $k = 1, 2,\ldots, N$, respectively, we obtain a set of ODEs

$$\mathbf{M}\ddot{\mathbf{w}} + \mathbf{G}\dot{\mathbf{w}} + \mathbf{K}\mathbf{w} = \mathbf{0} \tag{7}$$

where the mass matrix $\mathbf{M}$, gyroscopic matrix $\mathbf{G}$, and stiffness matrix $\mathbf{K}$ are all $N \times N$ matrices derived from the Galerkin integration.

Hence, the transverse displacement of the axially moving beam is resolved into the generalized coordinates of the trial functions.

Modal Motions for the Discretized Gyroscopic System

To show the procedure of the invariant manifold method, we use the two-order truncation. Let $N = 2$ in Eq. (7), we obtain

$$\begin{aligned}&\ddot{q}_1 - \frac{16}{3}\gamma\dot{q}_2 + \left(\kappa\pi^2 - \gamma^2 + 1\right)\pi^2 q_1 = 0\\ &\ddot{q}_2 + \frac{16}{3}\gamma\dot{q}_1 + 4\left(4\kappa\pi^2 - \gamma^2 + 1\right)\pi^2 q_2 = 0\end{aligned} \tag{8}$$

The invariant manifold method will be used to study the modal motions of the discretized system (8). During modal motions, we transfer the configuration coordinates into the state vectors

$$[x_1,\ y_1,\ x_2,\ y_2] = [q_1,\ \dot{q}_1,\ q_2,\ \dot{q}_2] \tag{9}$$

Hence, the equations of motion, presented in the first order form, are

$$\begin{aligned}&\dot{x}_1 = y_1, \quad \dot{y}_1 = f_1(x_1,x_2;y_1,y_2) = \frac{16}{3}\gamma\dot{x}_2 - \left(\kappa\pi^2 - \gamma^2 + 1\right)\pi^2 x_1\\ &\dot{x}_2 = y_2, \quad \dot{y}_2 = f_2(x_1,x_2;y_1,y_2) = -\frac{16}{3}\gamma\dot{x}_1 - 4\left(4\kappa\pi^2 - \gamma^2 + 1\right)\pi^2 x_2\end{aligned} \tag{10}$$

One modal motion of the system can be expressed as

$$\begin{aligned}&x_1 = u, \quad y_1 = v\\ &x_2 = a_1 u + a_2 v, \quad y_2 = b_1 u + b_2 v\end{aligned} \tag{11}$$

Since it is a single modal motion of the system, the second coordinate is related linearly to the first coordinate in the invariant state space via the last two equations of Eq. (11).

The invariant manifold based on Eq. (11) constructs a relationship between the first and second oscillators during modal motions. To solve for the coefficients in Eq. (11), we need to consider two conditions. The first condition, may be called compatibility relation, is obtained by equaling the derivative of the third equation to the fourth one in Eq. (11). The second relation, of course, should consider the governing equation (10). Substituting the derivatives of the second and the fourth equation of Eq. (11) back into Eq. (10) yields the second relation. By replacing the variables in all the equations of the two relations by u and v and expanding the final equations into polynomials of u and v, while equating the gathered coefficients of each term to zero, we can determine the unknown coefficients a_1, a_2, b_1, and b_2 based on the following algebraic equations:

$$
\begin{aligned}
b_1 - g a_2 b_1 + k_1 a_2 &= 0 \\
-k_2 a_1 + k_1 b_2 - g b_2 b_1 &= 0 \\
b_2 - a_1 - g a_2 b_2 &= 0 \\
-g - b_1 - g b_2^2 - k_2 a_2 &= 0
\end{aligned}
\tag{12}
$$

Where

$$
g = \frac{16}{3}\gamma,\; k_1 = \left(\kappa\pi^2 - \gamma^2 + 1\right)\pi^2,\; k_2 = 4\left(4\kappa\pi^2 - \gamma^2 + 1\right)\pi^2
\tag{13}
$$

Two sets of real roots are obtained in closed form, which represent the relationship during the two modal motions

$$
\begin{aligned}
&\text{mode 1:} && \text{mode 2:} \\
&a_1 = 0, \quad b_2 = 0 && a_1 = 0, \quad b_2 = 0 \\
&a_2 = \frac{-g^2 + k_2 - k_1 - \sqrt{\left(g^2 + k_1 - k_2\right)^2 + 4k_2 g^2}}{2gk_2} && a_2 = \frac{-g^2 + k_2 - k_1 + \sqrt{\left(g^2 + k_1 - k_2\right)^2 + 4k_2 g^2}}{2gk_2} \\
&b_1 = \frac{a_2 k_1}{a_2 g - 1} && b_1 = b_1 = \frac{a_2 k_1}{a_2 g - 1}
\end{aligned}
\tag{14}
$$

Hence, during the modal motions, the following relationship should be satisfied:

$$
x_2 = a_2 y_1, \quad y_2 = b_1 x_1
\tag{15}
$$

Inspection of Eq. (15) leads to the following features of gyroscopic system. The displacement of one coordinate is proportional to the velocity of the other coordinate. When the first coordinate reaches its maximum value, the second coordinate goes through its equilibrium point with maximum velocity. Discrete gyroscopic systems of two degree-of-freedom (DOF) demonstrate the $\pi/2$ phase difference between the two oscillators, the system behaves like the second oscillator following the first one with a quarter period gap or vice versa.

Now, we return to the final solutions obtained by the discretized system. Substituting the modal motion relations (15) and $q_1 = Ae^{i\omega t}$ into Eq. (6) with $N = 2$ leads to

$$
\begin{aligned}
w(x,t) &= x_1 \sin(\pi x) + a_2 \dot{x}_1 \sin(2\pi x) \\
&= Ae^{i\omega t}\left[\sin(\pi x) + i\omega a_2 \sin(2\pi x)\right]
\end{aligned}
\tag{16}
$$

The analytical complex mode functions by the continuous model are presented in Eq. (3). On the other hand, using Eq. (16), we can extract the complex mode function based on the two-order discretized system as

$$
\phi(x) = A\left[\sin(\pi x) + i\omega a_2 \sin(2\pi x)\right]
\tag{17}
$$

From the above equation, we find that we have the $\pi/2$ phase difference relationship between the two trial functions. The next question appears as to how the complex modal functions behave in the modal motions. It is recognized that a complex mode exhibits two shape functions that vibrate at a single frequency in phase quadrature. In an animated display, the complex modes behave as "galloping nodes": the nodes of the mode functions shift with time [35]. The snapshots for different instances are presented in Fig. 1 to show the modal motions, during which $\gamma = 0.6$. We can locate the leftward traveling node for the first modal motions and the rightward traveling node for the second modal motions. It should be noted that the axially moving direction for the beam is rightward. The leftward and rightward galloping-node modal motions found in the foregoing study are analogous to the backward and forward whirling motions found in the rotor dynamics [47], although the axially moving beam takes the translational movement while the rotors present rotational motions.

Comparison of the Mode Functions by Continuous and Discretized Models

In this section, we compare the complex mode functions derived from the continuous and discretized models.

By using the method of invariant manifold in the N-order truncated ODE (7), we find the relations among the trial functions of $\sin(n\pi x)$ $(n = 1, 2, \ldots, N)$

$$
\begin{matrix}
x_2 = a_2\dot{x}_1, & x_3 = a_3 x_1, & x_4 = a_4\dot{x}_1, & x_5 = a_5 x_1, & \ldots \\
y_2 = b_2 x_1, & y_3 = b_3\dot{x}_1, & y_4 = b_4 x_1, & y_5 = b_5\dot{x}_1, & \ldots
\end{matrix}
\tag{18}
$$

Substituting Eq. (18) into Eq. (6), we can extract the complex mode function based on the N-order discretized system as

$$
\begin{aligned}
\phi(x) = A\{&[\sin(\pi x) + \mathrm{a}_3 \sin(3\pi x) + \cdots + \mathrm{a}_{2k-1}\sin((2k-1)\pi x)] + \\
&i\omega[\sin(2\pi x) + \mathrm{a}_4 \sin(4\pi x) + \cdots + \mathrm{a}_{2k}\sin(2k\pi x)]\}, \\
N = \max&\{2k-1,\, 2k\}
\end{aligned}
\tag{19}
$$

During the modal motions, the displacements of the odd order coordinates are proportional to the displacement of the first order, and the displacements of the even order coordinates are proportional to the velocity of the first order. Hence, we can conclude that there always exists $\pi/2$ phase difference between any of the two adjacent general coordinates of the trial functions for the gyroscopic system presented in the form (7). Based on this rule, the dynamics of the N-DOF system can be represented by just 1DOF system.

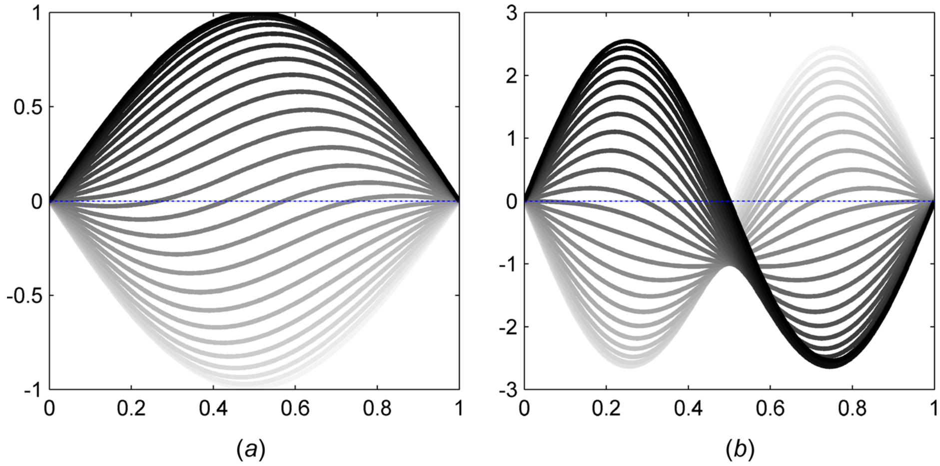

Fig. 1 The snapshots of the axially moving material during modal motions: (*a*) the first modal motions and (*b*) the second modal motions

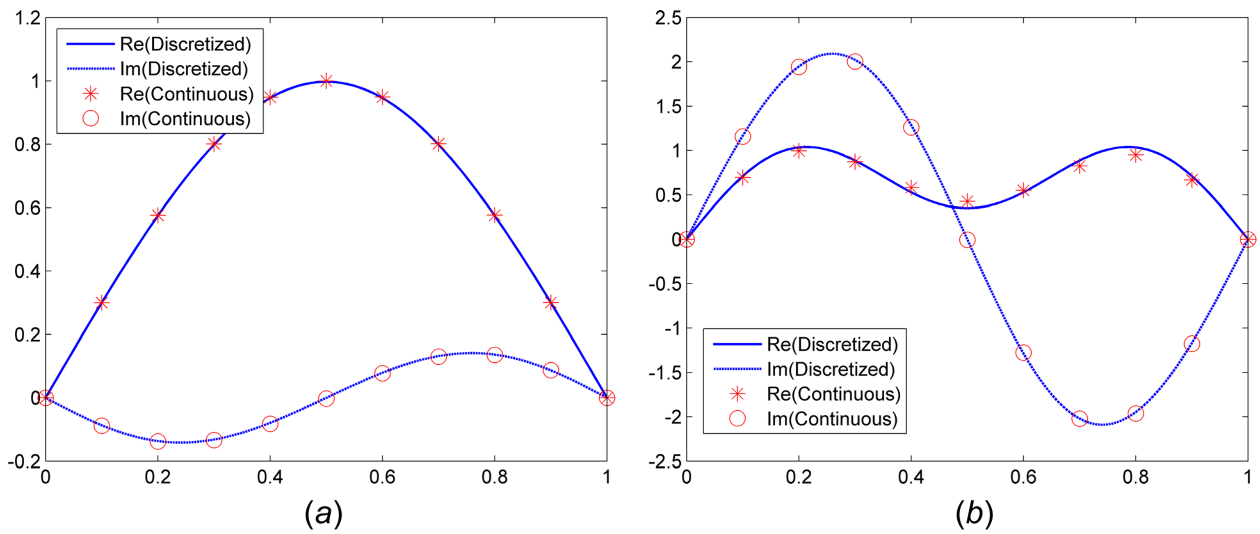

Fig. 2 Comparison of the mode functions by continuous and discretized methods: (*a*) the first mode and (*b*) the second mode

The first and second complex mode functions based on the continuous analysis (3) and fourth-order discretized analysis (19) for $\gamma = 0.6$ are plotted in Figs. 2(*a*) and 2(*b*), respectively. The complex mode functions for the discretized system have good agreement with the numerical results obtained by the continuous system. It should be noted that the good results of mode shapes and frequencies can be guaranteed only if the truncation order is higher than the number of mode we want to study.

Conclusions

In this study, we investigated the complex mode functions of the gyroscopic continua focusing on the axially moving beam as an example. By the invariant manifold method, the relationship among the generalized coordinates has been obtained for the discretized model. The gyroscopic system, such as axially moving material, pipes conveying fluid, rotating shaft, and motions around libration points, can be treated by the invariant manifold method to show the phase-difference and the node-galloping. Main conclusions that have been reached are as follows:

(1) The mode functions of the continuous model of the gyroscopic continua and those obtained with the trial functions based on the discretized model have been systematically compared, demonstrating good agreement.

(2) By studying the discretized system, we found out the relations among the general coordinates (trial functions). There always exists the $\pi/2$ phase difference between any two adjacent general coordinates during modal motions. The even order components of trial functions share the same in-unison motions and are also true for the odd order components, while the even order trial function and odd even order trial function have a phase quadrature gap. Based on this conclusion, the dynamics of the *N*-DOF system can be represented by just 1DOF system.

Acknowledgment

This work was supported in part by the National Natural Science Foundation of China (Project Nos. 11322214, 11172010, 11290152, and 11402007). R.M. was supported by the Natural Sciences and Engineering Research Council (NSERC) of Canada, the Canada Research Chair (CRC) program, and the Bizkaia Talent Grant under the Basque Government through the BERC 2014–2017 program, as well as Spanish Ministry of Economy and Competitiveness MINECO: BCAM Severo Ochoa excellence accreditation SEV-2013-0323.

References

[1] Wickert, J. A., 1996, "Transient Vibration of Gyroscopic Systems With Unsteady Superposed Motion," J. Sound Vib., **195**(5), pp. 797–807.
[2] Marynowski, K., and Kapitaniak, T., 2014, "Dynamics of Axially Moving Continua," Int. J. Mech. Sci., **81**(4), pp. 26–41.
[3] Banerjee, J. R., and Kennedy, D., 2014, "Dynamic Stiffness Method for Inplane Free Vibration of Rotating Beams Including Coriolis Effects," J. Sound Vib., **333**(26), pp. 7299–7312.
[4] Banerjee, J. R., Papkov, S. O., Liu, X., and Kennedy, D., 2015, "Dynamic Stiffness Matrix of a Rectangular Plate for the General Case," J. Sound Vib., **342**, pp. 177–199.
[5] Rao, J. S., 2011, *History of Rotating Machinery Dynamics*, Springer, Dordrecht, The Netherlands.
[6] Païdoussis, M. P., 1998, *Fluid-Structure Interactions: Slender Structures and Axial Flow*, Vol. 1, Elsevier Academic Press, London.
[7] Ibrahim, R. A., 2010, "Overview of Mechanics of Pipes Conveying Fluids—Part I: Fundamental Studies," ASME J. Pressure Vessel Technol., **132**(3), p. 034001.
[8] Yu, D. L., Paidoussis, M. P., Shen, H. J., and Wang, L., 2014, "Dynamic Stability of Periodic Pipes Conveying Fluid," ASME J. Appl. Mech., **81**(1), p. 011008.
[9] Wickert, J. A., and Mote, C. D., 1990, "Classical Vibration Analysis of Axially Moving Continua," ASME J. Appl. Mech., **57**(3), pp. 738–744.
[10] Chen, L. Q., and Yang, X. D., 2005, "Steady-State Response of Axially Moving Viscoelastic Beams With Pulsating Speed: Comparison of Two Nonlinear Models," Int. J. Solids Struct., **42**(1), pp. 37–50.
[11] Chen, L.-Q., Zhang, Y.-L., Zhang, G.-C., and Ding, H., 2014, "Evolution of the Double-Jumping in Pipes Conveying Fluid Flowing at the Supercritical Speed," Int. J. Non-Linear Mech., **58**, pp. 11–21.
[12] Paidoussis, M. P., and Li, G. X., 1993, "Pipes Conveying Fluid: A Model Dynamical Problem," J. Fluids Struct., **7**(7), pp. 823–823.
[13] Hosseini, S. A. A., and Khadem, S. E., 2009, "Free Vibrations Analysis of a Rotating Shaft With Nonlinearities in Curvature and Inertia," Mech. Mach. Theory, **44**(1), pp. 272–288.
[14] Lin, S. C., and Hsiao, K. M., 2001, "Vibration Analysis of a Rotating Timoshenko Beam," J. Sound Vib., **240**(2), pp. 303–322.
[15] Wickert, J. A., and Mote, C. D., 1991, "Traveling Load Response of an Axially Moving String," J. Sound Vib., **149**(2), pp. 267–284.
[16] Chen, L. Q., 2005, "Analysis and Control of Transverse Vibrations of Axially Moving Strings," Appl. Mech. Rev., **58**(2), pp. 91–116.
[17] Ding, H., and Chen, L. Q., 2011, "Natural Frequencies of Nonlinear Vibration of Axially Moving Beams," Nonlinear Dyn., **63**(1–2), pp. 125–134.
[18] Ghayesh, M. H., Kafiabad, H. A., and Reid, T., 2012, "Sub- and Super-Critical Nonlinear Dynamics of a Harmonically Excited Axially Moving Beam," Int. J. Solids Struct., **49**(1), pp. 227–243.
[19] Meirovitch, L., 1974, "A New Method of Solution of the Eigenvalue Problem for Gyroscopic Systems," AIAA J., **12**(10), pp. 1337–1342.
[20] Meirovitch, L., 1975, "A Modal Analysis for the Response of Linear Gyroscopic Systems," ASME J. Appl. Mech., **42**(2), pp. 446–450.
[21] Ulsoy, A. G., Mote, C. D., and Szymani, R., 1978, "Principal Developments in Band Saw Vibration and Stability Research," Holz Roh Werkst., **36**(7), pp. 273–280.
[22] Perkins, N. C., 1990, "Linear Dynamics of a Translating String on an Elastic-Foundation," ASME J. Vib. Acoust., **112**(1), pp. 2–7.
[23] Chen, L. Q., and Zu, J. W., 2008, "Solvability Condition in Multi-Scale Analysis of Gyroscopic Continua," J. Sound Vib., **309**(1–2), pp. 338–342.
[24] Chen, L. Q., Tang, Y. Q., and Lim, C. W., 2010, "Dynamic Stability in Parametric Resonance of Axially Accelerating Viscoelastic Timoshenko Beams," J. Sound Vib., **329**(5), pp. 547–565.
[25] Ding, H., and Chen, L. Q., 2010, "Galerkin Methods for Natural Frequencies of High-Speed Axially Moving Beams," J. Sound Vib., **329**(17), pp. 3484–3494.
[26] Malookani, R. A., and van Horssen, W. T., 2015, "On Resonances and the Applicability of Galerkin's Truncation Method for an Axially Moving String With Time-Varying Velocity," J. Sound Vib., **344**, pp. 1–17.
[27] Ozhan, B. B., 2014, "Vibration and Stability Analysis of Axially Moving Beams With Variable Speed and Axial Force," Int. J. Struct. Stab. Dyn., **14**(6), p. 1450015.
[28] Öz, H. R., Pakdemirli, M., and Boyacı, H., 2001, "Non-Linear Vibrations and Stability of an Axially Moving Beam With Time-Dependent Velocity," Int. J. Non-Linear Mech., **36**(1), pp. 107–115.
[29] Thomsen, J. J., and Dahl, J., 2010, "Analytical Predictions for Vibration Phase Shifts Along Fluid-Conveying Pipes Due to Coriolis Forces and Imperfections," J. Sound Vib., **329**(15), pp. 3065–3081.
[30] Čepon, G., and Boltežar, M., 2007, "Computing the Dynamic Response of an Axially Moving Continuum," J. Sound Vib., **300**(1–2), pp. 316–329.
[31] Ding, H., and Chen, L. Q., 2011, "Approximate and Numerical Analysis of Nonlinear Forced Vibration of Axially Moving Viscoelastic Beams," Acta Mech. Sin., **27**(3), pp. 426–437.
[32] Yang, X. D., Chen, L. Q., and Zu, J. W., 2011, "Vibrations and Stability of an Axially Moving Rectangular Composite Plate," ASME J. Appl. Mech., **78**(1), p. 011018.
[33] Ding, H., Chen, L.-Q., and Yang, S.-P., 2012, "Convergence of Galerkin Truncation for Dynamic Response of Finite Beams on Nonlinear Foundations Under a Moving Load," J. Sound Vib., **331**(10), pp. 2426–2442.
[34] Bishop, R. E. D., and Johnson, D. C., 1979, *The Mechanics of Vibration*, Cambridge University Press, Cambridge, UK.
[35] Lang, G. F., 2012, "Matrix Madness and Complex Confusion. A Review of Complex Modes From Multiple Viewpoints," Sound Vib., **46**(11), pp. 8–12.
[36] Brake, M. R., and Wickert, J. A., 2010, "Modal Analysis of a Continuous Gyroscopic Second-Order System With Nonlinear Constraints," J. Sound Vib., **329**(7), pp. 893–911.
[37] Kerschen, G., Peeters, M., Golinval, J. C., and Vakakis, A. F., 2009, "Nonlinear Normal Modes, Part I: A Useful Framework for the Structural Dynamicist," Mech. Syst. Signal Process., **23**(1), pp. 170–194.
[38] Peeters, M., Viguié, R., Sérandour, G., Kerschen, G., and Golinval, J. C., 2009, "Nonlinear Normal Modes, Part II: Toward a Practical Computation Using Numerical Continuation Techniques," Mech. Syst. Signal Process., **23**(1), pp. 195–216.
[39] Neild, S. A., Champneys, A. R., Wagg, D. J., Hill, T. L., and Cammarano, A., 2015, "The Use of Normal Forms for Analysing Nonlinear Mechanical Vibrations," Philos. Trans. R. Soc. London, Ser. A, **373**(2051), p. 20140404.
[40] Hill, T. L., Cammarano, A., Neild, S. A., and Wagg, D. J., 2015, "Out-of-Unison Resonance in Weakly Nonlinear Coupled Oscillators," Proc. Math. Phys. Eng. Sci./R. Soc., **471**(2173), p. 20140659.
[41] Shaw, S. W., and Pierre, C., 1993, "Normal-Modes for Nonlinear Vibratory-Systems," J. Sound Vib., **164**(1), pp. 85–124.
[42] Shaw, S. W., and Pierre, C., 1994, "Normal-Modes of Vibration for Nonlinear Continuous Systems," J. Sound Vib., **169**(3), pp. 319–347.
[43] Ghayesh, M. H., 2012, "Coupled Longitudinal–Transverse Dynamics of an Axially Accelerating Beam," J. Sound Vib., **331**(23), pp. 5107–5124.
[44] Orloske, K., Leamy, M. J., and Parker, R. G., 2006, "Flexural–Torsional Buckling of Misaligned Axially Moving Beams. I. Three-Dimensional Modeling, Equilibria, and Bifurcations," Int. J. Solids Struct., **43**(14–15), pp. 4297–4322.
[45] Chen, L. Q., and Yang, X. D., 2005, "Stability in Parametric Resonance of Axially Moving Viscoelastic Beams With Time-Dependent Speed," J. Sound Vib., **284**(3–5), pp. 879–891.
[46] Öz, H. R., and Pakdemirli, M., 1999, "Vibrations of an Axially Moving Beam With Time-Dependent Velocity," J. Sound Vib., **227**(2), pp. 239–257.
[47] Shahgholi, M., Khadem, S. E., and Bab, S., 2014, "Free Vibration Analysis of a Nonlinear Slender Rotating Shaft With Simply Support Conditions," Mech. Mach. Theory, **82**, pp. 128–140.