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Coupled stochastic systems of Skorokhod type: Well-posedness of a mathematical model and its applications

Thi Kim Thoa Thieu[1] 🄳 | Adrian Muntean[2] | Roderick Melnik[1,3]

[1]MS2Discovery Interdisciplinary Research Institute, Wilfrid Laurier University, Waterloo, Ontario, Canada

[2]Department of Mathematics and Computer Science and Center for Societal Risk Research (CSR), Karlstad University, Karlstad, Sweden

[3]BCAM—Basque Center for Applied Mathematics, Bilbao, Spain

Correspondence
Thi Kim Thoa Thieu, MS2Discovery Interdisciplinary Research Institute, Wilfrid Laurier University, 75 University Ave W, Waterloo, ON N2L 3C5, Canada. Email: tthieu@wlu.ca

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Population dynamics with complex biological interactions, accounting for uncertainty quantification, are critical for many application areas. However, due to the complexity of biological systems, the mathematical formulation of the corresponding problems faces the challenge that the corresponding stochastic processes should, in most cases, be considered in bounded domains. We propose a model based on a coupled system of reflecting Skorokhod-type stochastic differential equations with jump-like exit from a boundary. The setting describes the population dynamics of active and passive populations. As main working techniques, we use compactness methods and Skorokhod's representation of solutions to SDEs posed in bounded domains to prove the well-posedness of the system. This functional setting is a new point of view in the field of modeling and simulation of population dynamics. We provide the details of the model, as well as representative numerical examples, and discuss the applications of a Wilson–Cowan-type system, modeling the dynamics of two interacting populations of excitatory and inhibitory neurons. Furthermore, the presence of random input current, reflecting factors together with Poisson jumps, increases firing activity in neuronal systems.

KEYWORDS

compound Poisson process, excitatory neurons, finite activity jumps, inhibitory neurons, population dynamics, reflecting boundary condition, Skorokhod equations, stochastic differential equations, well-posedness, Wilson–Cowan equations

MSC CLASSIFICATION

60H10, 60H35, 92B05

1 | INTRODUCTION

In recent years, the modeling of population dynamics arising from biological systems offers many challenging questions to some of the most advanced areas of science and technology. In order to model reliably population dynamics, accounting for the complexity in the interactions among populations, considering for uncertainty quantification is critical for many application areas. One of the main challenges is that the corresponding stochastic processes should, in most cases, be considered in bounded domains. Before studying such stochastic models in detail, the question of well-posedness has to be addressed. There are several results for stochastic differential equations (SDEs) with reflecting boundary conditions,[1–5] one of them being the seminal contribution of Skorokhod,[6] where the author provided the existence and uniqueness to one-dimensional stochastic equations for diffusion processes in a bounded region. A direct approach to the solution of the

reflecting boundary conditions and reductions to the case of nonsmooth domains are reported in Lions and Sznitman.[7] Extending results by Tanaka, Saisho[8] proved the existence and uniqueness of solutions to the Skorokhod equation posed in a bounded domain in $\mathbb{R}^d$ where a reflecting boundary condition was applied. Dupuis and Ishii[9] studied the strong existence and uniqueness of the SDEs with reflecting boundary conditions for domains that might have corners. In addition, the existence, uniqueness, and stability of solutions of multidimensional SDE's with reflecting boundary conditions were provided in Słomiński,[10] where the author obtained results on the existence and uniqueness of strong and weak solutions to the SDE for any driving semimartingale and in a more general domain. The models of SDEs in a bounded domain have been known for a long time and yet, only a few relevant results are available in the context of population dynamics for the problems posed in confined domains. As far as we are aware, one of the first questions in this setting was posed in the modeling and simulation study[11] while considering the evacuation dynamics of a mixed active–passive pedestrian populations in a complex geometry in the presence of a fire as well as of a slowly spreading smoke curtain. From a stochastic processes perspective, various lattice gas models for active–passive pedestrian dynamics have been recently explored in Cirillo et al[12] and Colangeli et al.[13] An extended lattice gas model to simulate pedestrian counter flow under the open boundary conditions by considering the human subconscious behavior and different maximum velocities has been reported in Kuang et al,.[14] Cirillo et al[15] have considered a simple exclusion process to describe the pedestrian counter flow through the lattice. See also Thieu et al[16] for a result on the weak solvability of a deterministic system of parabolic partial differential equations describing the interplay of a mixture of flows for active–passive populations of pedestrians.

In general, the purely diffusive Brownian motion with random fluctuations of continuous sample paths in population dynamics is often regarded as noise in a dynamic system. However, the diffusive fluctuations can be large and abrupt events that appear at random times throughout the analyzed dynamic process. Therefore, in such cases, the above description of these diffusive fluctuations would be incomplete to reflect real population dynamics, and the models based on jump-diffusion stochastic processes would provide more accurate descriptions for the underlying population dynamics.[17–19] After constructing new mathematical models, the rigorous proofs of their well-posedness would be the next important step in their justification. Many models have been proposed in the field of population dynamics describing various scenarios. However, when it comes to the models of population dynamics described by more realistic SDEs, their well-posedness is often lacking in the literature. Specifically, motivated by previous studies,[7,8,20–22] in this paper, we are interested in a coupled system of reflecting Skorokhod-type SDEs with jumps that can provide a more realistic modeling framework for the dynamics of active and passive populations. Furthermore, previous results on the reflecting Skorokhod-type SDEs with jump-like exits from a boundary did not address such an important issue as the dependence on parameters with the vigor it deserves. In this paper, we prove the well-posedness of a coupled system of reflecting Skorokhod-type SDEs with jump-like exit from a boundary for active–passive population dynamics by showing the existence, uniqueness and the dependence on the parameters of its solutions. From the modeling perspective, our approach is novel, opening new routes for investigation of population dynamics, including the computability of solutions and identification of model parameters. Taking the inspirations from the applications of population dynamics and neuroscience,[23,24] we provide details of an application of our active–passive population dynamics model in a Wilson–Cowan-type system describing the dynamics of two interacting populations of excitatory and inhibitory neurons. This study will bridge the gap between practical applications of this vital class of mathematical models and their rigorous mathematical analysis.

2 | MATHEMATICAL MODEL: COUPLED STOCHASTIC PROCESSES IN BOUNDED DOMAIN

We start from considering the dynamics of active–passive population dynamics. Each population is considered in a one-dimensional domain, then the whole system will be embedded in a two-dimensional domain, which we refer to as Ω. Let $D \subset \mathbb{R}$ satisfies the assumption (A_2) in Section 2.3. We denote $S = (0, T)$ for some $T \in \mathbb{R}_+$. We refer to $\bar{D} \times S$ as D_T, note that $\bar{D}$ denotes the closure of D.

2.1 | Active particle population

Our main focus in the remainder of this section is to find an explicit formula for a solution of the reflection problem, which is similar to the Skorokhod-like problem but involves the possibility of a jump-like exit from zero. For $t \in S$, $f_A : D_T \to \mathbb{R}$, and $g_A : D_T \times D_T \to \mathbb{R}^2$, let X_A denote the active population at time t. We assume that the dynamics of active population

is governed by the following model (see, e.g., Pilipenko[20] and Przybyłlowicz et al.[21]):

$$
\begin{cases}
dX_A(t) = f_A(X_A(t), X_P(t))dt + g_A(X_A(t), X_P(t))dW_A(t) + d\phi_A(t) + \int_{\mathbb{R}} \rho_A(X_A(s), y)\nu(dy, ds), \\
X_A(0) = X_{A,0},
\end{cases}
\tag{1}
$$

where ρ_A is a measurable function such that $\rho_A : \mathbb{R} \to \mathbb{R}$, $W_A = (W_A(t))_{t \in [0,T]}$ is a two-dimensional standard Brownian motion, while ν is a Poisson random measure with finite jump intensity, associated with a scalar compound Poisson process (clarified below by relationship 7).

2.2 | Passive particle population

The case of passive particle populations is treated in a way similar described above. For $t \in S$, $f_P : D_T \to \mathbb{R}$, and $g_P : D_T \times D_T \to \mathbb{R}^2$, let X_P denote the passive population inside the domain $\bar{D}$. The dynamics of the passive population is described here by a system of SDEs as follows (see, e.g., Pilipenko[20] and Przybyłlowicz et al.[21]):

$$
\begin{cases}
dX_P(t) = f_P(X_A(t), X_P(t))dt + g_P(X_A(t), X_P(t))dW_P(t) + d\phi_P(t) + \int_{\mathbb{R}} \rho_P(X_P(s), y)\nu(dy, ds), \\
X_P(0) = X_{P,0},
\end{cases}
\tag{2}
$$

where ρ_P is a measurable function $\rho_P : \mathbb{R} \to \mathbb{R}$, $W_P = (W_P(t))_{t \in [0,T]}$ is a two-dimensional standard Brownian motion, while ν is a Poisson random measure with finite jump intensity, associated with a scalar compound Poisson process (clarified below by relationship 7).

The proposed dynamics (1)–(2) are the general structures of our active–passive population dynamics. We will discuss further detailed model descriptions as well as the applications of the population dynamics in Section 4.

2.3 | The Skorokhod equation

Now, having representations for active and passive populations, we would like to consider a system of stochastic Skorokhod-type equations and analyze their properties. We consider the following equation (see, e.g., Pilipenko[20] and Przybyłlowicz et al.[21]):

$$
dX(t) = df(X_A(t), X_P(t))dt + dg(X_A(t), X_P(t))dW(t) + d\Phi(t) + \int_{\mathbb{R}} \rho(X(s), y)\nu(dy, ds).
\tag{3}
$$

Let $\{W(t), t \geq 0\}$ be a Wiener process and let $\{J(t), t \geq 0\}$ be a nondecreasing Lévy process independent of $\{W(t), t \geq 0\}$ with finite Lévy measure μ. The jump measure ν is a Poisson random measure with finite jump intensity, associated with a compound Poisson process that can be represented by the following form:

$$
J(t) = \sum_{n=1}^{N(t)} \xi_k,
\tag{4}
$$

where $N(t)$, $t \geq 0$ is a Poisson process with intensity $\alpha = \mu((0, \infty))$ and $\{\xi_n, n \geq 1\}$ are independent identically distributed random variables independent of $N(t)$, $t \geq 0$ such that $P(\xi_n \leq x) = \mu((0, x])/\alpha$.

2.3.1 | Assumptions

We rely on the following assumptions:

(A$_1$) The functions $f : D_T \times D_T \to \mathbb{R} \times \mathbb{R}$, and $g : D_T \times D_T \to \mathbb{R}^2 \times \mathbb{R}^2$ satisfy the global Lipschitz conditions.

(A$_2$) ∂D is $C^{2,\alpha}$ with $\alpha \in (0, 1)$.

(A$_3$) There exists a constant $C_\rho \in (0, \infty)$ such that the jump coefficient $\rho : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the following inequality for all $x \in \mathbb{R}$ (see, e.g., Przybyłllowicz et al.[21]):

$$\int_{\mathbb{R}} \|\rho(x, y)\|^2 \psi(dy) \leq C_\rho(1 + \|x\|^2), \tag{5}$$

and for all $x, z \in \mathbb{R}$,

$$\int_{\mathbb{R}} \|\rho(x, y) - \rho(z, y)\|^2 \psi(dy) \leq C_\rho \|x - z\|^2, \tag{6}$$

where ψ is the distribution of ξ. Moreover, ρ is a bounded function.

It is worth mentioning that assumptions (A$_1$) and (A$_2$) correspond to the modeling of the situation in Section 4, while (A$_3$)–(A$_6$) are of technical nature, corresponding to the type of solution we are searching for; clarifications in this direction are given in the next sections.

Let us denote $\rho(x, y) = y \cdot \rho(x)$. Then, the jump process can be considered as a compound Poisson process, that is, for all $t \in [0, T]$, we have (see, e.g., Przybyłllowicz et al.[21]):

$$\int_0^t \int_{\mathbb{R}} \rho(X(s), y)\nu(dy, ds) = \sum_{k=1}^{N_t} \xi_k \cdot \rho(X(\tau_k)) = \sum_{0 < s \leq t} \rho(X(s))\Delta J(s) = \int_0^t \rho(X(s))dJ(s). \tag{7}$$

2.3.2 | Concept of solution

Take $x \in \partial D$ arbitrarily fixed. We define the set $\mathcal{N}_x$ of inward normal unit vectors at $x \in \partial D$ by

$$\begin{aligned} \mathcal{N}_x &= \cup_{r>0} \mathcal{N}_{x,r}, \\ \mathcal{N}_{x,r} &= \left\{ \mathbf{n} \in \mathbb{R}^2 : |\mathbf{n}| = 1, B(x - r\mathbf{n}, r) \cap D = \varnothing \right\}, \end{aligned} \tag{8}$$

where $B(z, r) = \{y \in \mathbb{R}^2 : |y - z| < r\}, z \in \mathbb{R}^2, r > 0$. Mind that, in general, it can happen that $\mathcal{N}_x = \varnothing$. In this case, the uniform exterior sphere condition is not satisfied (see, for instance, the examples provided in fig. 5 in Cholaquidis et al[25] and in page 4 in Choulli[26]).

We complement our list of assumptions (A$_1$)–(A$_3$) with three specific conditions on the geometry of the domain D:

(A$_4$) (Uniform exterior sphere condition). There exists a constant $r_0 > 0$ such that

$$\mathcal{N}_x = \mathcal{N}_{x,r_0} \neq \varnothing \text{ for any } x \in \partial D.$$

(A$_5$) There exist constants $\delta > 0$ and $\delta' \in [1, \infty)$ with the following property: for any $x \in \partial D$, there exists a unit vector $\mathbf{l}_x$ such that

$$\langle \mathbf{l}_x, \mathbf{n} \rangle \geq 1/\delta' \text{ for any } \mathbf{n} \in \bigcup_{y \in B(x,\delta) \cap \partial D} \mathcal{N}_y,$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^2$.

(A$_6$) There exist $\delta'' > 0$ and $\nu > 0$ such that for each $x_0 \in \partial D$ we can find a function $f \in C^2(\mathbb{R}^2)$ satisfying

$$\langle y - x, \mathbf{n} \rangle + \frac{1}{\nu} \langle \nabla f(x), \mathbf{n} \rangle |y - x|^2 \geq 0, \tag{9}$$

for any $x \in B(x_0, \delta'') \cap \partial D, y \in B(x_0, \delta'') \cap \partial \bar{D}$ and $\mathbf{n} \in \mathcal{N}_x$.

The following relation is called the Skorokhod equation: Find $(\xi, \phi) \in C(\bar{S}, \mathbb{R}^2) \times C(\bar{S})$ such that

$$\xi(t) = w(t) + \phi(t), \tag{10}$$

where $w \in C(\bar{S}, \mathbb{R}^2)$ is given so that $w(0) \in \bar{D}$. The solution of (10) is a pair (ξ, ϕ), which satisfies the following two conditions:

(a) $\xi \in C(\bar{S}, \bar{D})$;
(b) $\phi \in C(\bar{S})$ with bounded variation on each finite time interval satisfying $\phi(0) = 0$ and

$$\phi(t) = \int_0^t \mathbf{n}(y)d|\phi|_y,$$

$$|\phi|_t = \int_0^t \mathbb{1}_{\partial D}(\xi(y))d|\phi|_y,$$

$$\tag{11}$$

where

$$\mathbf{n}(y) \in \mathcal{N}_{\xi(y)} \text{ if } \xi(y) \in \partial D,$$

$$|\phi|_t = \text{ total variation of } \phi \text{ on } [0, t]$$

$$= \sup_{\mathcal{T} \in \mathcal{G}([0,t])} \sum_{k=1}^{n_\mathcal{T}} |\phi(t_k) - \phi(t_k - 1)|.$$

$$\tag{12}$$

In (12), we denote by $\mathcal{G}([0, t])$ the family of all partitions of the interval $[0, t]$ and take a partition $\mathcal{T} = \{0 = t_0 < t_1 < \ldots < t_{n_\mathcal{T}} = t\} \in \mathcal{G}([0, t])$. The supremum in (12) is taken over all partitions of type $0 = t_0 < t_1 < \ldots < t_{n_\mathcal{T}} = t$.

Conditions (a) and (b) guarantee that ξ is a *reflecting process* on $\bar{D}$.
It is easily seen from the definition that

$$\xi_1(t) = w_1(t), \ldots, w_{d-1}(t) = \xi_{d-1},$$

and

$$\xi_d(t)) = w_d(t) + \phi(t), \int_0^t \mathbb{1}_{\xi_d(y) \notin \partial D} d|\phi|_y.$$

We define a multidimensional Skorokhod's map $\Gamma : C(\bar{S}) \to C(\bar{S})$ such that

$$\Gamma w(t) = \Gamma(w_1, \ldots, w_d)(t) = (\Gamma w_1(t), \ldots, \Gamma w_{d-1}(t), \Gamma w_d(t)). \tag{13}$$

Hence, the pair (ξ_d, ϕ) is the exact solution of the one-dimensional Skorokhod problem ξ_d. Therefore, it holds

$$\phi(t) = -\min_{y \in [0,t]} \{w_d(y), 0\}, \quad \xi_d(t) = w_d(t) - \min_{y \in [0,t]} \{w_d(y), 0\} = \Gamma w_d(t). \tag{14}$$

The multidimensional Skorokhod's map Γ satisfies the Lipschitz condition in a space of continuous functions.

Theorem 1. *Assume conditions* (A$_4$) *and* (A$_5$). *Then, for any* $w \in C(\bar{S}, \mathbb{R}^2)$ *with* $w(0) \in \bar{D}$, *there exists a unique solution* $\xi(t, w)$ *of Equation (10) such that* $\xi(t, w)$ *is continuous in* (t, w).

For the proof of this theorem, we refer the reader to Theorem 4.1 in Saisho.[8]
To come closer to the model equations for active–passive population dynamics described above in Sections 2.1–2.3, we introduce the mappings

$$f : D_T \times D_T \to \mathbb{R}^2, \ g : D_T \times D_T \to \mathbb{R}^{2 \times 2}$$

and consider the Skorokhod-like system on the probability space $(\Omega, \mathcal{F}, P)$

$$dX(t) = f(X(t))dt + g(X(t))dW(t) + \int_{\mathbb{R}} \rho(X(s), y)\nu(dy, ds) + d\Phi(t), \tag{15}$$

with

$$X(0) = X_0 \in \bar{D}, \tag{16}$$

where the inital value X_0 is assumed to be an $\mathcal{F}_0$-measurable random variable and $B(t)$ is a 2×2-dimensional $\mathcal{F}_t$-Brownian motion with $B(0) = 0$. Here, $\{\mathcal{F}_t\}$ is a filtration such that $\mathcal{F}_0$ contains all P-negligible sets and $\mathcal{F}_t = \cap_{\epsilon > 0} \mathcal{F}_{t+\epsilon}$, while $\Phi_t(t)$ is defined in the Definition 1 below. Further, properties of the structure of (15)–(16) are listed in Section 3. Similarly to the deterministic case, we can now define the following concept of solutions to (15).

Definition 1. A pair $(X(t), \Phi(t))$ is called solution to (15)–(16) if the following conditions hold:

i. $X(t)$ is a $\bar{D}$-valued $\mathcal{F}_t$-adapted continuous process;
ii. $\Phi(t)$ is an $\mathbb{R}^2$-valued $\mathcal{F}_t$-adapted continuous process with bounded variation on each finite time interval such that $\Phi(0) = 0$ with

$$\Phi(t) = \int_0^t \mathbf{n}(y) d|\Phi|_y,$$

$$\tag{17}$$

$$|\Phi|(t) = \int_0^t \mathbb{1}_{\partial D}(X(y)) d|\Phi|_y.$$

iii. $\mathbf{n}(s) \in \mathcal{N}_{X(s)} \in \partial D$.

Note that the Definition 1 ensures that $X(t)$ entering (15)–(16) is a reflecting process on $\bar{D}$.

3 | WELL-POSEDNESS OF SKOROKHOD-TYPE SYSTEM

In this section, we establish the well-posedness of the Skorokhod-type system by showing the existence, uniqueness, and stability of solutions to the problem (15)–(16) in the sense of Definition 1.

We use the compactness method together with the continuity result of the deterministic case stated in Theorem 1 for proving the existence of solutions to (15)–(16). We follow the arguments by Da Prato and Zabczyk[27], Section 8.3 and a result of Flandoli and Gatarek[28] for martingale solutions. The starting point of this argument is based on considering a sequence $\{X_t^n\}$ of solutions of the following system of Skorokhod-type SDEs:

$$\begin{cases} dX^n(t) & = f(X^n(h^n(t)))dt + g(X^n(h^n(t)))dW(t) + \int_{\mathbb{R}} \rho(X^n(s), y)\nu(dy, ds) + d\Phi^n(t), \\ X^n(0) & = X_0 \in \bar{D}, \end{cases} \tag{18}$$

where $X_0^n \in \bar{D}$ is given.

For convenience, we recast the solution to the system (1) and (2) in terms of the vector $X_t^n(t)$, $n \in \mathbb{N}$, such that

$$X^n(t) := (X_A^n(t), X_P^n(t)), f(X^n(t)) := (f_A(X_A^n(t), X_P^n(t)), f_P(X_A^n(t), X_P^n(t))),$$
$$g(X^n(t)) := (g_A(X_A^n(t), X_P^n(t)), g_P(X_A^n(t), X_P^n(t))), \Phi^n(t) := (\phi_A^n(t), \phi_P^n(t)),$$
$$J(t) := (J_A(t), J_P(t)), X^n(0) := (X_{A,0}, X_{P,0}), X(t) := (X_A(t), X_P(t)),$$
$$f(X(t)) := (f_A(X_A(t), X_P(t)), f_P(X_A(t), X_P(t))),$$
$$g(X(t)) := (g_A(X_A(t), X_P(t)), g_P(X_A(t), X_P(t))),$$
$$\Phi(t) := (\phi_A(t), \phi_P(t)), J(t) := (J_A(t), J_P(t)), W(t) := (W_A(t), W_P(t)).$$

Let us introduce the following step functions:

$$h^n(0) = 0, \tag{19}$$

$$h^n(t) = (k - 1)2^{-n}, \tag{20}$$

$$(k - 1)2^{-n} < t \le k2^{-n}, \ k = 1, 2, \dots, n \text{ and } n \ge 1. \tag{21}$$

Using Theorem 1, we have a unique solution of (18). Furthermore, each value of $X_t^n(t)$ is obtained within $0 \leq t \leq k2^{-n}$ and $X_t^n(t)$ is attained for $k2^{-n} < t \leq (k+1)2^{-n}$ that is uniquely determined as the solution of the following Skorokhod equation:

$$X^n(t) = X^n(k2^{-n}) + f(X^n(k2^{-n}))(t - k2^{-n}) + g(X^n(k2^{-n}))\{W(t) - W(k2^{-n})\} + \int_{\mathbb{R}} \rho(X^n(k2^{-n}), y)\nu(dy, ds) + \Phi^n(t). \quad (22)$$

Let us denote

$$Y^n(t) := X_0 + \int_0^t f(X^n(h^n(y)))dy + \int_0^t g(X^n(h^n(y)))dW(y) + \int_0^t \int_{\mathbb{R}} \rho(X^n(k2^{-n}), y)\nu(ds, dy). \quad (23)$$

Then, $X^n(t) = (\Gamma Y_t^n)(t)$, we also have

$$Y^n(t) := X_0 + \int_0^t f((\Gamma Y^n)(h^n(y)))dy + \int_0^t g((\Gamma Y^n)(h^n(y)))dW(y) + \int_0^t \int_{\mathbb{R}} \rho(X^n(h^n(s)), y)\nu(ds, dy). \quad (24)$$

We define the family of laws

$$\left\{ \mathcal{P}(Y^n); 0 \leq t \leq T, n \geq 1 \right\}. \quad (25)$$

Accordingly, (25) is a family of probability distributions of Y^n. Let $\mathcal{P}^n$ be the laws of Y^n.

3.1 | Statement of the main theoretical results of the paper

The main theoretical results of this paper are stated in Theorems 2–4 below. In this section, the focus lies on ensuring the well-posedness of Skorokhod solutions with jump-like exit from a boundary to our population dynamics problem.

Theorem 2 (Existence). *Assume that* (A_1)–(A_3) *hold. There exists at least a weak solution to the Skorokhod-type system (15)–(16) in the sense of Definition 1.*

Theorem 3 (Uniqueness). *Assume that* (A_1)–(A_6) *hold. There is a unique strong solution to (15)–(16).*

Theorem 4 (Dependence on parameters). *Assume that* (A_1)–(A_3) *hold and*

$$\lim_{k \to \infty} E(|X_0^k - X_0|^2) = 0. \quad (26)$$

Suppose that $X_t^n(t) \in C(\bar{S}; \bar{D} \times \bar{D})$ *solves*

$$\begin{cases} dX^n(t) = f(X^n(t))dt + g(X^n(t))dW(t) + \int_{\mathbb{R}} \rho(X^n(s), y)\nu(ds, dy) + d\Phi^n(t), \\ X^n(0) = X_0^n \in \bar{D}, \end{cases} \quad (27)$$

where $X_0^n \in \bar{D}$ *is given and* $\Phi^n(t)$ *is defined as a sequence of* $\Phi(t)$ *in Definition 1. Then,*

$$\lim_{n \to \infty} E(\max_{0 \leq t \leq T} |X^n(t) - X(t)|^2) = 0, \quad (28)$$

where $X_t \in C(\bar{S}; \bar{D} \times \bar{D})$ *is the unique solution of the following problem:*

$$\begin{cases} dX(t) = f(X(t))dt + g(X(t))dW(t) + \int_{\mathbb{R}} \rho(X(s), y)\nu(ds, dy) + d\Phi(t), \\ X(0) = X_0 \in \bar{D}. \end{cases} \quad (29)$$

These statements are proven in Sections 3.1.1–3.1.3.

3.1.1 | Proof of the existence

Let us start with handling the tightness of the laws $\{\mathcal{P}^n\}$ through the following Lemma.

Lemma 1. *Assume that* (A_1)–(A_3) *hold. Then, the family* $\{\mathcal{P}^n\}$ *given by (25) is tight in* $C(\bar{S}, \mathbb{R}^2 \times \mathbb{R}^2)$.

Proof. Let us introduce the following relative compact set in $C(\bar{S}, \mathbb{R}^2 \times \mathbb{R}^2)$

$$K_{M_1 M_2} = \left\{ f \in C(\bar{S}; \mathbb{R}^2 \times \mathbb{R}^2) : \|f\|_{L^\infty(S;\mathbb{R}^2 \times \mathbb{R}^2)} \le M_1, [f]_{C^\alpha(\bar{S};\mathbb{R}^2 \times \mathbb{R}^2)} \le M_2 \right\}. \tag{30}$$

Now, we will show that for a given $\epsilon > 0$, there are $M_1, M_2 > 0$ such that

$$P(Y^n(\cdot) \in K_{M_1 M_2}) \le \epsilon, \text{ for all } n \in \mathbb{N}. \tag{31}$$

This means that

$$P(\|Y^n(t)\|_{L^\infty(S;\mathbb{R}^2 \times \mathbb{R}^2)} > M_1 \text{ or } [Y^n(\cdot)]_{C^\alpha(\bar{S};\mathbb{R}^2 \times \mathbb{R}^2)} > M_2) \le \epsilon. \tag{32}$$

A sufficient condition for this to happen is

$$P(\|Y^n(t)\|_{L^\infty(S;\mathbb{R}^2 \times \mathbb{R}^2)} > M_1) < \frac{\epsilon}{2} \text{ and } P([Y^n(\cdot)]_{C^\alpha(\bar{S};\mathbb{R}^2 \times \mathbb{R}^2)} > M_2) < \frac{\epsilon}{2}, \tag{33}$$

where $Y(\cdot)$ denotes either $Y(t)$ or $Y(r)$.

We consider first $P(\|Y^n(\cdot)\|_{L^\infty(S,\mathbb{R}^2 \times \mathbb{R}^2)} > M_1) < \frac{\epsilon}{2}$. Using Markov's inequality (see, e.g., Jacod and Protter[29], Corollary 5.1), we get

$$P(\|Y^n(t)\|_{L^\infty(S;\mathbb{R}^2 \times \mathbb{R}^2)} > M_1) \le \frac{1}{M_1} E[\sup_{t \in \bar{S}} |Y^n(t)|], \tag{34}$$

but

$$\sup_{t \in \bar{S}} |Y^n(t)| = \sup_{t \in \bar{S}} \left\{ \left| X_{A,0} + \int_0^t f_A((\Gamma Y^n)(h^n(y)))dy + \int_0^t g_A((\Gamma Y^n)(h^n(y)))dW_A(y) \right|, \right.$$
$$\left. \left| X_{P,0} + \int_0^t f_P((\Gamma Y^n)(h^n(y)))dy + \int_0^t g_P((\Gamma Y^n)(h^n(y)))dW_P(y) \right| \right\}. \tag{35}$$

We estimate

$$\sup_{t \in \bar{S}} |Y^n(t)| \le \sup_{t \in \bar{S}} \left\{ \left| X_{A,0} \right| + \left| \int_0^t f_A((\Gamma Y^n)(h^n(y)))dy \right| + \left| \int_0^t g_A((\Gamma Y^n)(h^n(y)))dW(y) \right|, \right.$$
$$\left. \left| X_{P,0} \right| + \left| \int_0^t f_P((\Gamma Y^n)(h^n(y)))dy \right| + \left| \int_0^t g_P((\Gamma Y^n)(h^n(y)))dW(y) \right| \right\}. \tag{36}$$

Since F_1, F_2 are bounded, then we have

$$\int_0^T f_A((\Gamma Y^n)(h^n(y)))dy \le C \text{ and } \int_0^T f_P((\Gamma Y^n)(h^n(y)))dy \le C. \tag{37}$$

Taking the expectation on (36), we are led to

$$E\left[\sup_{t \in \bar{S}} |Y^n(t)|\right] \le C + E\left[\sup_{t \in \bar{S}} \left| \int_0^t g((\Gamma Y^n)(h^n(y)))dW(y) \right|\right]. \tag{38}$$

On the other hand, the Burlkholder–Davis–Gundy's inequality[*] implies

$$E\left[\sup_{t\in\bar{S}}\left|\int_0^t g((\Gamma Y^n)(h^n(y)))dB(y)\right|\right] \leq K_{1/2}E\left[\int_0^t |g((\Gamma Y^n)(h^n(y)))|^2 dy\right]^{1/2}$$
$$\leq K_{1/2}t^{1/2}, \tag{39}$$

for $K_{1/2} > 0$.

Then, we have the following estimate:

$$E\left[\sup_{t\in\bar{S}}|Y^n(t)|\right] \leq C + E\left[\int_0^t |g((\Gamma Y^n)(h^n(y)))|^2 dy\right]^{1/2} \leq C. \tag{40}$$

Hence, for $\varepsilon > 0$, we can choose $M_1 > 0$ such that $P(\|Y^n\|_{L^\infty(S;\mathbb{R}^2\times\mathbb{R}^2)} > M_1) < \frac{\varepsilon}{2}$.

In the sequel, we consider the second inequality $P([Y^n(\cdot)]_{C^\alpha(\bar{S};\mathbb{R}^2\times\mathbb{R}^2)} > M_2) < \frac{\varepsilon}{2}$, this reads

$$P([Y^n(\cdot)]_{C^\alpha(\bar{S};\mathbb{R}^2\times\mathbb{R}^2)} > M_2) = P\left(\sup_{t\neq r;t,r\in\bar{S}} \frac{|Y^n(t)-Y^n(r)|}{|t-r|^\alpha} > M_2\right) \leq \frac{\varepsilon}{2}. \tag{41}$$

Let us introduce another class of compact sets now in the Sobolev space $W^{\alpha,p}(0,T;\mathbb{R}^2\times\mathbb{R}^2)$ (which for suitable exponents $\alpha p - \gamma > 1$ lies in $C^\gamma(\bar{S};\mathbb{R}^2\times\mathbb{R}^2)$). Additionally, we recall the relatively compact sets $K'_{M_1 M_2}$, defined as in Appendix A1, such that

$$K'_{M_1 M_2} = \left\{ f \in C(\bar{S};\mathbb{R}^2\times\mathbb{R}^2) : \|f\|_{L^\infty(S;\mathbb{R}^2\times\mathbb{R}^2)} \leq M_1, [f]_{W^{\alpha,p}(S;\mathbb{R}^2\times\mathbb{R}^2)} \leq M_2 \right\}, \tag{42}$$

where $\alpha p > 1$ (see, e.g., Colangeli et al[13] and Flandoli and Gatarek[28]). Having this in mind, we wish to prove that there exits $\alpha \in (0,1)$ and $p > 1$ with $\alpha p > 1$ together with the property: given $\varepsilon > 0$, there is $M_2 > 0$ such that

$$P([Y^n(\cdot)]_{W^{\alpha,p}(S;\mathbb{R}^2\times\mathbb{R}^2)} > M_2) < \frac{\varepsilon}{2} \text{ for every } n \in \mathbb{N}. \tag{43}$$

Using Markov's inequality, we obtain

$$P([Y^n(\cdot)]_{W^{\alpha,p}(S;\mathbb{R}^2\times\mathbb{R}^2)} > M_2) \leq \frac{1}{M_2}E\left[\int_0^T\int_0^T \frac{|Y^n(t)-Y^n(r)|^p}{|t-r|^{1+\alpha p}}dtdr\right] = \frac{C}{M_2}\int_0^T\int_0^T \frac{E[|Y^n(t)-Y^n(r)|^p]}{|t-r|^{1+\alpha p}}dtdr. \tag{44}$$

For $t > r$, we have

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[*]See, for example, Karatzars and Shreve,[30][Theorem 3.28] (The Burlkholder–Davis–Gundy's inequality). Let $M \in \mathcal{M}^{c,\text{loc}}$ and $M_t^* := \max_{0\leq s\leq t}|M_s|$. For every $m > 0$, there exists universal positive constants k_m, K_m (depending only on m), such that the inequalities

$$k_m E(\langle M\rangle_T^m \leq E[(M_T^*)^{2m}] \leq K_m E(\langle M\rangle_T^m)$$

hold for every stopping time T. Note that $\mathcal{M}^{c,\text{loc}}$ denotes the space of continuous local martingales and $\langle X\rangle$ represents the quadratic variance process of $X \in \mathcal{M}^{c,\text{loc}}$.

$$Y^n(t) - Y^n(r) = \begin{bmatrix} \int_r^t f_A((\Gamma Y_A^n)(h^n(y)), (\Gamma Y_P^n)(h^n(y)))dy \\ \int_r^t f_P((\Gamma Y_A^n)(h^n(y)), (\Gamma Y_P^n)(h^n(y)))dy \end{bmatrix}$$

$$+ \begin{bmatrix} \int_r^t g_P((\Gamma Y_A^n)(h^n(y)), (\Gamma Y_P^n)(h^n(y)))dW_A(y) \\ \int_r^t g_P((\Gamma Y_A^n)(h^n(y)), (\Gamma Y_P^n)(h^n(y)))dW_P(y) \end{bmatrix} + \begin{bmatrix} \int_r^t \rho_A(y)dJ_A(y) \\ \int_r^t \rho_P(y)dJ_P(y) \end{bmatrix}. \tag{45}$$

Let us introduce some further notation. For a vector $u = (u_1, u_2)$, we set $|u| := |u_1| + |u_2|$. At this moment, we consider the following expression:

$$|Y^n(t) - Y_r^n(r)| = \left| \int_r^t f_A((\Gamma Y_A^n)(h^n(y)), (\Gamma Y_P^n)(h^n(y)))dy + \int_r^t g_A((\Gamma Y_A^n)(h^n(y)), (\Gamma Y_P^n)(h^n(y)))dW_A(y) \right|$$

$$+ \left| \int_r^t f_P((\Gamma Y_A^n)(h^n(y)), (\Gamma Y_P^n)(h^n(y)))dy + \int_r^t g_P((\Gamma Y_A^n)(h^n(y)), (\Gamma Y_P^n)(h^n(y)))dW_P(y) \right| \tag{46}$$

$$+ \left| \int_r^t \rho_A((\Gamma Y_A^n)(h^n(y)))dJ_A(y) + \int_r^t \rho_P((\Gamma Y_P^n)(h^n(y)))dJ_P(y) \right|.$$

Taking the modulus up to the power $p > 1$ and applying Minkowski inequality, we have

$$|Y_t^n(t) - Y_r^n(r)|^p = \left(\left| \int_r^t f_A((\Gamma Y_A^n)(h^n(y)), (\Gamma Y_P^n)(h^n(y)))dy + \int_r^t g_A((\Gamma Y_A^n)(h^n(y)), (\Gamma Y_P^n)(h^n(y)))dW_A(y) \right| \right.$$

$$+ \left| \int_r^t f_P((\Gamma Y_A^n)(h^n(y)), (\Gamma Y_P^n)(h^n(y)))dy + \int_r^t g_P((\Gamma Y_A^n)(h^n(y)), (\Gamma Y_P^n)(h^n(y)))dW_P(y) \right|$$

$$+ \left. \left| \int_r^t \rho_A((\Gamma Y_A^n)(h^n(y)))dJ_A(y) + \int_r^t \rho_P((\Gamma Y_P^n)(h^n(y)))dJ_P(y) \right| \right)^p$$

$$\leq C \left(\left| \int_r^t f_A((\Gamma Y_A^n)(h^n(y)), (\Gamma Y_P^n)(h^n(y)))dy + \int_r^t g_A((\Gamma Y_A^n)(h^n(y)), (\Gamma Y_P^n)(h^n(y)))dW_A(y) \right|^p \right.$$

$$+ \left| \int_r^t f_P((\Gamma Y_A^n)(h^n(y)), (\Gamma Y_P^n)(h^n(y)))dy + \int_r^t g_P((\Gamma Y_A^n)(h^n(y)), (\Gamma Y_P^n)(h^n(y)))dW_P(y) \right|^p \tag{47}$$

$$+ \left. \left| \int_r^t \rho_A((\Gamma Y_A^n)(h^n(y)))dJ_A(y) + \int_r^t \rho_P((\Gamma Y_P^n)(h^n(y)))dJ_P(y) \right|^p \right)$$

$$\leq C \left(\int_r^t |f_A((\Gamma Y_A^n)(h^n(y)), (\Gamma Y_P^n)(h^n(y)))|^p dy + \int_r^t |g_A((\Gamma Y_A^n)(h^n(y)), (\Gamma Y_P^n)(h^n(y)))|^p dW_A(y) \right.$$

$$+ \int_r^t |f_P((\Gamma Y_A^n)(h^n(y)), (\Gamma Y_P^n)(h^n(y)))|^p dy + \int_r^t |g_P((\Gamma Y_A^n)(h^n(y)), (\Gamma Y_P^n)(h^n(y)))|^p dW_P(y)$$

$$+ \left. \int_r^t |\rho_A((\Gamma Y_A^n)(h^n(y)))|^p dJ_A(y) + \int_r^t |\rho_P((\Gamma Y_P^n)(h^n(y)))|^p dJ_P(y) \right).$$

Taking the expectation on (47), we obtain the following estimate:

$$E[|Y^n(t) - Y^n(r)|^p] \le C(t - r)^p + CE\left[\left|\int_r^t \sigma((\Gamma Y^n)(h^n(y)))dW(y)\right|^p\right]. \tag{48}$$

Applying the Burkholder–Davis–Gundy's inequality to the second term of the right-hand side of (48), we obtain

$$E\left[\left|\int_r^t \sigma((\Gamma Y^n)(h^n(y)))dW(y)\right|^p\right] \le CE\left[\left(\int_r^t dy\right)^{p/2}\right] \le C(t - r)^{p/2}. \tag{49}$$

On the other hand, if $\alpha < \frac{1}{2}$, then

$$\int_0^T \int_0^T \frac{1}{|t - r|^{1 + (\alpha - \frac{1}{2})p}} dt dr < \infty. \tag{50}$$

Consequently, we can pick $\alpha < \frac{1}{2}$. Taking now $p > 2$ together with the constraint $\alpha p > 1$, we can find $M_2 > 0$ such that

$$P([Y^n(t)]_{W^{\alpha,p}(S;\mathbb{R}^2 \times \mathbb{R}^2)} > M_2) < \frac{\varepsilon}{2}. \tag{51}$$

This argument completes the proof of the lemma. □

From Lemma 1, we have obtained that the sequence $\{\mathcal{P}^n\}$ is tight in $C(\bar{S}; \mathbb{R}^2 \times \mathbb{R}^2)$. Applying the Prokhorov's theorem, there are subsequences $\{\mathcal{P}^{n_k}\}$ which converge weakly to some $\mathcal{P}(Y(t))$ as $n \to \infty$. For simplicity of the notation, we denote these subsequences by $\{\mathcal{P}^n\}$. This means that we have $\{\mathcal{P}^n\}$ converging weakly to some probability measure $\mathcal{P}$ on Borel sets in $C(\bar{S}; \mathbb{R}^2 \times \mathbb{R}^2)$.

Since we have that $\mathcal{P}^n(Y^n(t))$ converges weakly to $\mathcal{P}(Y(t))$ as $n \to \infty$, by using the "Skorokhod Representation Theorem," there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ with the filtration $\{\tilde{\mathcal{F}}_t\}$ and $\tilde{Y}^n(t)$, $\tilde{Y}(t)$ belonging to $C(\bar{S}; \mathbb{R}^2 \times \mathbb{R}^2)$ with $n \in \mathbb{N}$, such that $\mathcal{P}(\tilde{Y}) = \mathcal{P}(Y)$, $\mathcal{P}(\tilde{Y}^n(t)) = \mathcal{P}(Y^n(t))$, and $\tilde{Y}^n(t) \to \tilde{Y}(t)$ as $n \to \infty$, $\tilde{P}$–a.s. Moreover, let $(\tilde{X}^n(t), \tilde{\Phi}^n(t))$ and $(\tilde{X}(t), \tilde{\Phi}(t))$ be the solutions of the following Skorokhod equations:

$$\begin{aligned}
\tilde{X}^n(t) &= \tilde{Y}^n(t) + \tilde{\Phi}^n(t), \\
\tilde{X}(t) &= \tilde{Y}(t) + \tilde{\Phi}(t),
\end{aligned} \tag{52}$$

respectively. Then, the continuity result in Theorem 1 implies that the sequence $(\tilde{X}^n(t), \tilde{\Phi}^n(t))$ converges to $(\tilde{X}(t), \tilde{\Phi}(t)) \in C(\bar{S}; \bar{D} \times \bar{D}) \times C(\bar{S})$ uniformly in $t \in \bar{S}$, $\tilde{P}$–a.s as $n \to \infty$. Hence, we still need to prove that $\tilde{Y}^n(t)$ converges to $\tilde{Y}(t)$ in some sense, where we denote

$$\tilde{Y}^n(t) := \tilde{X}_0 + \int_0^t f(\tilde{X}^n(h_n(y))dy + \int_0^t g(\tilde{X}^n(h_n(y))d\tilde{W}(y) + \int_0^t \rho(X^n(y))dJ(y), \tag{53}$$

and

$$\tilde{Y}(t) := \tilde{X}_0 + \int_0^t f(\tilde{X}^n(y))dy + \int_0^t g(\tilde{X}^n(y))d\tilde{W}(y) + \int_0^t \rho(X(y))dJ(y). \tag{54}$$

To complete the proof of the existence of solutions to the problem (15)–(16) in the sense of Definition 1, we consider the following lemma.

Lemma 2. *The pair $(\tilde{X}(t), \tilde{\Phi}(t)) \in C(\bar{S}; \bar{D} \times \bar{D}) \times C(\bar{S})$ cf. (52) is a solution of the Skorokhod-type system*

$$\tilde{X}(t) = \tilde{Y}(t) + \tilde{\Phi}(t), \tag{55}$$

where

$$\tilde{Y}(t) := \tilde{X}_0 + \int_0^t f(\tilde{X}^n(y))dy + \int_0^t g(\tilde{X}^n(y))d\tilde{W}(y), \tag{56}$$

and $\tilde{X}_0 \in \bar{D}$.

Proof. We consider the term $\int_0^t g(\tilde{X}^n(h_n(y)))d\tilde{W}(y)$ with the step process $g(\tilde{X}^n(h_n(y)))$. Approximating this stochastic integral by Riemann–Stieltjes sums (see, e.g., Evans[31]), it yields

$$\int_0^t g(\tilde{X}^n(h_n(y)))d\tilde{W}(y) = \sum_{k=0}^{n-1} g(\tilde{X}^n(h_n(t)))(W(t_{k+1}^n) - W(t_k^n)). \tag{57}$$

By taking the limit $n \to \infty$ in (57), this gives

$$\lim_{n\to\infty} \int_0^t g(\tilde{X}^n(h_n(y)))d\tilde{W}(y) = \lim_{n\to\infty} \sum_{k=0}^{n-1} g(\tilde{X}^n(h_n(t)))(W(t_{k+1}^n) - W(t_k^n))$$

$$= \sum_{k=0}^{n-1} g(\tilde{X}(t))(W(t_{k+1}) - W(t_k)) = \int_0^t g(\tilde{X}(y))d\tilde{W}(y). \tag{58}$$

By using the fact that $(\tilde{X}^n(t), \tilde{\Phi}^n(t))$ converges to $(\tilde{X}(t), \tilde{\Phi}(t)) \in C(\bar{S}; \bar{D} \times \bar{D}) \times C(\bar{S})$ uniformly in $t \in [0, T]$ $\tilde{P}$-a.s as $n \to \infty$ together with (58), we obtain that

$$\tilde{X}^n(t) = \tilde{X}_0 + \int_0^t f(\tilde{X}^n(h^n(y)))dy + \int_0^t g(\tilde{X}^n(h^n(y)))d\tilde{W}(y) + \int_0^t \rho(X^n(h^n(y)))dJ(y) + \tilde{\Phi}^n(t) = \tilde{Y}^n(t) + \tilde{\Phi}^n(t) \tag{59}$$

converges to

$$\tilde{X}(t) = \tilde{X}_0 + \int_0^t f(\tilde{X}(y))dy + \int_0^t g(\tilde{X}(y))d\tilde{W}(y) + \int_0^t \rho(X(y))dJ(y) + \tilde{\Phi}(t) = \tilde{Y}(t) + \tilde{\Phi}(t), \quad \tilde{P} - \text{a.s as } n \to \infty. \tag{60}$$

□

3.1.2 | Proof of the uniqueness

We take $X(t), X'(t) \in C(\bar{S}; \bar{D} \times \bar{D})$ as two solutions to (15)–(16) with the same initial values $X(0) = X'(0)$.

Moreover, suppose that the supports of b and σ are included in the same ball $B(x_0, \delta)$ for some $x_0 \in \partial D$. Let us recall the assumption (A_6), where D satisfies the following condition: There exists a positive number η such that for each $x_0 \in \partial D$ we can find $h \in C^2(\mathbb{R}^2)$ satisfying

$$\langle y - x, \mathbf{n}\rangle + \frac{1}{\eta}\langle \nabla h(x), \mathbf{n}\rangle |y - x|^2 \geq 0,$$

for any $x \in B(x_0, \delta') \cap \partial D$, $y \in B(x_0, \delta'') \cap \partial \bar{D}$ and $\mathbf{n} \in \mathcal{N}_x$. Using the proof idea of Lemma 5.3 in Saisho,[8] we consider $f(x) = \langle \mathbf{l}, x - x_0\rangle$. Then, we have

$$\langle X(s) - X'(s), d\Phi(s) - d\Phi'(s)\rangle - \frac{1}{\eta}|X(s) - X'(s)|^2\langle \mathbf{l}, d\Phi(s) - d\Phi'(s)\rangle$$

$$= -(\langle X(s) - X'(s), d\Phi(s)\rangle + \frac{1}{\eta}|X(s) - X'(s)|^2\langle \mathbf{l}, d\Phi(s)\rangle) \tag{61}$$

$$-(\langle X(s) - X'(s), d\Phi'(s)\rangle + \frac{1}{\eta}|X(s) - X'(s)|^2\langle \mathbf{l}, d\Phi'(s)\rangle).$$

Moreover, using the assumption (A_6), we have the following estimates:

$$\langle X(s) - X'(s), d\Phi(s)\rangle + \frac{1}{\eta}|X(s) - X'(s)|^2\langle \mathbf{l}, d\Phi(s)\rangle \geq 0, \tag{62}$$

and

$$\langle X(s) - X'(s), d\Phi'(s)\rangle + \frac{1}{\eta}|X(s) - X'(s)|^2\langle \mathbf{l}, d\Phi'(s)\rangle \geq 0. \tag{63}$$

Combining (61)–(63), we obtain the following estimate:

$$\langle X(s) - X'(s), d\Phi(s) - d\Phi'(s)\rangle - \frac{1}{\eta}|X(s) - X'(s)|^2\langle \mathbf{l}, d\Phi(s) - d\Phi'(s)\rangle \leq 0, \tag{64}$$

where $\mathbf{l}$ is the unit vector appearing in condition (A_5).

Let $\mathcal{O}$ be a bounded smooth open set in R^2, there exists $\psi \in C^2(R^2)$ such that

$$\exists \eta > 0, \forall x \in \mathcal{O}, \frac{\partial \psi}{\partial \mathbf{n}} \leq \eta \text{ (see, e.g., Lions and Sznitman[7]).}$$

We consider the following quantity:

$$\Psi(t) = |X(t) - X'(t)|^2 \exp\left\{-\frac{1}{\eta}(\psi(X(t)) - \psi'(X(t)))\right\}. \tag{65}$$

Using similar ideas as in Lions and Sznitman[7] (see Theorem 3.1 and Proposition 4.1) together with applying Ito's formula, we have the following estimate:

$$|X(t) - X'(t)|^2 \exp\left\{-\frac{1}{\eta}(\psi(X(t)) - \psi'(X(t)))\right\} \leq 2\left(\exp\left\{-\frac{1}{\eta}(\psi(X(y)) - \psi'(X(y)))\right\}\int_0^t (b(X(y)) - b(X'(y)))dy\right.$$

$$+ \exp\left\{-\frac{1}{\eta}(\psi(X(y)) - \psi'(X(y)))\right\}\int_0^t (\sigma(X_y(y)) - \sigma(X(y)))dW(y)\Bigg)^2 + \exp\left\{-\frac{1}{\eta}(\psi(X(y)) - \psi'(X(y)))\right\}$$

$$\times \int_0^t \left(2\langle X(y) - X'(y), \mathbf{l}\rangle - \frac{1}{\eta}|X(y) - X'(y)|^2\right)d\psi(y) + \exp\left\{-\frac{1}{\eta}(\psi(X(y)) - \psi'(X(y)))\right\}$$

$$\times \int_0^t \left(2\langle X(y) - X'(y), \mathbf{l}\rangle - \frac{1}{\eta}|X(y) - X'(y)|^2\right)d\psi'(y) + 2\int_0^t |b(X(y)) - b(X'(y))|^2 \exp\left\{-\frac{2}{\eta}(\psi(X(y)) - \psi'(X(y)))\right\}dy$$

$$+ 2\int_0^t |\sigma(X(y)) - \sigma(X(y))|^2 \exp\left\{-\frac{2}{\eta}(\psi(X(y)) - \psi'(X(y)))\right\}dy + \int_0^t \left(2\langle X(y) - X'(y), \mathbf{l}\rangle - \frac{1}{\eta}|X(y) - X'(y)|^2\right)$$

$$\times \exp\left\{-\frac{1}{\eta}(\psi(X(y)) - \psi'(X(y)))\right\}d\psi(y) + \int_0^t \left(2\langle X(y) - X'(y), \mathbf{l}\rangle - \frac{1}{\eta}|X(y) - X'(y)|^2\right)$$

$$\times \exp\left\{-\frac{1}{\eta}(\psi(X(y)) - \psi'(X(y)))\right\}d\psi'(y). \tag{66}$$

On the other hand, taking the expectation on both sides of (66) and using the Lipschitz condidion to the first term of the right-hand side together with (61), we are led to

$$E\left(|X(t) - X'(t)|^2 \exp\left\{-\frac{1}{\eta}(\psi(X(t)) - \psi'(X(t)))\right\}\right) \le C \int_0^t E\left(|X(y) - X'(y)|^2 \exp\left\{-\frac{2}{\eta}(\psi(X(y)) - \psi'(X(y)))\right\}\right) dy.$$

(67)

This also implies that

$$E[|X(t) - X'(t)|^2] \le C \int_0^t E[|X(y) - X'(y)|^2] dy.$$

(68)

Hence, $X(t) = X'(t)$ for all $t \in [0, T]$. Then, the pathwise uniqueness of solutions to (15)–(16) holds true. Moreover, the pathwise uniqueness implies the uniqueness of strong solutions (see in Ikeda and Watanabe,[32] Theorem IV-1.1). On the other hand, combining the Lemma 2 and the result provided in Yamada and Watanabe,[33] the system of reflected SDEs (15)–(16) admits a unique strong solution $(X(t), \Phi(t)) \in C(\bar{S}; \bar{D} \times \bar{D}) \times C(\bar{S})$.

3.1.3 | Proof of the dependence on the parameters

Proof. Let us recall our system of SDEs from (18), Section 3:

$$\begin{cases} dX^n(t) = f(X^n(t))dt + g(X_t^n(t))dW(t) + \int_{\mathbb{R}} \rho(X^n(t), y)\nu(dy, dt) + d\Phi^n(t), \\ X^n(0) = X_0^n \in \bar{D} \text{ for } n \ge 1. \end{cases}$$

(69)

For its solution, we have

$$X_t^n(t) = X_0^n + \int_0^t f(X_z^n(z))dz + \int_0^t g(X^n(z))dB(z) + \Phi^n(t).$$

(70)

Let us consider the following equation:

$$X^n(t) - X_t(t) = X_0^n - X_0 + \int_0^t f(X^n(y))dy - \int_0^t f(X(y))dy + \int_0^t g(X^n(y))dW(y) - \int_0^t g(X(y))dW(y)$$

$$+ \int_0^t \rho(X^n(y))dJ(y) + \int_0^t \rho(X(y))dJ(y) + \Phi^n(t) - \Phi(t).$$

(71)

Since $(a + b + c + d + e)^2 \le 5a^2 + 5b^2 + 5c^2 + 5d^2$ for any $a, b, c, d, e \in \mathbb{R}$, we have the following estimate:

$$|X^n(t) - X(t)|^2 \le 5|X_0^n - X_0|^2 + 5\left|\int_0^t f(X^n(y))dy - \int_0^t f(X(y))dy\right|^2 + 5\left|\int_0^t g(X^n(y))dW(y) - \int_0^t g(X(y))dW(y)\right|^2$$

$$+ 5\left|\int_0^t \rho(X^n(y))dJ(y) - \int_0^t \rho(X(y))dJ(y)\right|^2 + 5|\Phi^n(t) - \Phi(t)|^2.$$

(72)

Taking the expectation on both sides of (72), we have

$$
\begin{aligned}
E(|X^n(t) - X(t)|^2) \leq{} & 5E(|X_0^n - X_0|^2) + 5E\left(\left|\int_0^t f(X^n(y))dy - \int_0^t f(X(y))dy\right|^2\right) \\
& + 5E\left(\left|\int_0^t g(X^n(y))dW(y) - \int_0^t \sigma(X(y))dW(y)\right|^2\right) \\
& + 5E\left(\left|\int_0^t \rho(X^n(y))dJ(y) - \int_0^t \rho(X(y))dJ(y)\right|^2\right) + 5E\left(|\Phi^n(t) - \Phi(t)|^2\right).
\end{aligned}
\tag{73}
$$

To begin with, we consider the second and the third terms of the right-hand side of (73). Using Cauchy–Schwarz inequality together with the assumption that f, g are Lipschitz functions, we are led to

$$
\begin{aligned}
E\left(\left|\int_0^t f(X^n(y))dy - \int_0^t f(X(y))dy\right|^2\right) &\leq CE\left(\int_0^t |f(X^n(y)) - f(X(y))|^2 dy\right) \\
&\leq C\int_0^t E(|X^n(y) - X(y)|^2)dy,
\end{aligned}
\tag{74}
$$

and

$$
\begin{aligned}
E\left(\left|\int_0^t g(X^n(y))dW(y) - \int_0^t g(X(y))dW(y)\right|^2\right) &= E\left(\int_0^t |g(X^n(y)) - g(X(y))|^2 dy\right) \\
&\leq C\int_0^t E(|X^n(y) - X(y)|^2)dy.
\end{aligned}
\tag{75}
$$

Moreover, using (14), it yields

$$
\begin{aligned}
|\Phi^n(t) - \Phi(t)| \leq{} & 2|X_0^n - X_0| + 2\left|\int_0^t f(X^n(y))dy - \int_0^t f(X(y))dy\right| \\
& + 2\left|\int_0^t g(X^n(y))dW(y) - \int_0^t g(X(y))dB(y)\right| + 2\left|\int_0^t \rho(X^n(y))dJ(y) - \int_0^t \rho(X(y))dJ(y)\right|.
\end{aligned}
\tag{76}
$$

Since $(a + b + c + d)^2 \leq 4a^2 + 4b^2 + 4c^2 + 4d^2$ for all $a, b, c, d \in \mathbb{R}$, we have

$$
\begin{aligned}
|\Phi^n(t) - \Phi(t)|^2 \leq{} & 8|X_0^n - X_0|^2 + 8\left|\int_0^t f(X^n(y)) - \int_0^t f(X(y))dy\right|^2 + 8\left|\int_0^t g(X^n(y))dW(y) - \int_0^t g(X(y))dW(y)\right|^2 \\
& + 8\left|\int_0^t \rho(X^n(y))dJ(y) - \int_0^t \rho(X(y))dJ(y)\right|^2.
\end{aligned}
\tag{77}
$$

Taking the expectation on both sides of (77), we are led to

$$
E(|\Phi_t^n(t) - \Phi_t(t)|^2) \le 8E(|X_0^n - X_0|^2) + 8E\left(\left|\int_0^t f(X^n(y)) - \int_0^t f(X(y))dy\right|^2\right)
$$
$$
+ 8E\left(\left|\int_0^t g(X^n(y))dW(y) - \int_0^t g(X(y))dW(y)\right|^2\right) \tag{78}
$$
$$
+ 8E\left(\left|\int_0^t \rho(X^n(y))dJ(y) - \int_0^t \rho(X(y))dJ(y)\right|^2\right).
$$

By applying Cauchy–Schwarz's inequality to the second and third terms of the right-hand side of (78), we have the following estimate:

$$
E(|\Phi^n(t) - \Phi(t)|^2) \le 8E(|X_0^n - X_0|^2) + CE\left(\int_0^t |f(X^n(y))dy - f(X(y))|^2 dy\right)
$$
$$
+ 8E\left(\int_0^t |g(X^n(y))dW(y) - g(X(y))|^2 dy\right) + 8E\left(\int_0^t |\rho(X^n(y)) - \rho(X(y))|^2 dJ(y)\right). \tag{79}
$$

Using again the assumption that b, σ are Lipschitz functions, we get next the following estimate:

$$
E(|\Phi^n(t) - \Phi(t)|^2) \le 8E(|X_0^n - X_0|^2) + C\int_0^t E(|X^n(y) - X(y)|^2)dy
$$
$$
+ C\int_0^t E(|X^n(y) - X(y)|^2)dy + C\int_0^t E(|X^n(y) - X(y)|^2 dJ(y)). \tag{80}
$$

Then, by using (74), (75), and (80), the inequality (70) becomes

$$
E(|X^n(t) - X(t)|^2) \le CE(|X_0^n - X_0|^2) + C\int_0^t E(|X^n(y) - X(y)|^2)dy + C\int_0^t E(|X^n(y) - X(y)|^2 dJ(y)), \tag{81}
$$

for $0 \le t \le T$. Applying Gronwall's inequality to (81) yields

$$
E(|X^n(t) - X(t)|^2) \le CE(|X_0^n - X_0|^2). \tag{82}
$$

Moreover, we have that

$$
\max_{0 \le t \le T} |X^n(t) - X(t)|^2 \le 5|X_0^n - X_0|^2 + C\int_0^T |X^n(t) - X(t)|^2 dt + 5\max_{0 \le t \le T}\left|\int_0^T g(X^n(t)) - g(X(t))dB(t)\right|^2
$$
$$
+ C\max_{0 \le t \le T}\int_0^T |X^n(y) - X(y)|^2 dJ(y) + 5\max_{0 \le t \le T}|\Phi_t^n(t) - \Phi_t(t)|^2. \tag{83}
$$

After taking the expectation on both sides of (83), we apply the martingale inequality to the third term on the right-hand side of the resulting inequality, which reads

$$
E\left(\max_{0\leq t\leq T}|X_t^n(t)-X_t(t)|^2\right) \leq 5E(|X_0^n-X_0|^2) + C\int_0^T E(|X_t^n(t)-X_t(t)|^2)dt
$$

$$
+ C\int_0^T E(|X_t^n(t)-X_t(t)|^2)dt + 5E\left(\max_{0\leq t\leq T}|\Phi_t^n(t)-\Phi_t(t)|^2\right) + C\int_0^t E(|X^n(y)-X(y)|^2 dJ(y)) \tag{84}
$$

$$
\leq CE(|X_0^n-X_0|^2) + C\int_0^T E(|X_t^n(t)-X_t(t)|^2)dt + C\int_0^t E(|X^n(y)-X(y)|^2 dJ(y)).
$$

Finally, using (80) and (82), we obtain the desired estimate:

$$
E\left(\max_{0\leq t\leq T}|X^n(t)-X(t)|^2\right) \leq CE(|X_0^n-X_0|^2). \tag{85}
$$

By using the fact that $\lim_{n\to\infty}E(|X_0^n-X_0|^2)=0$, we obtain the following estimate:

$$
\lim_{n\to\infty} E\left(\max_{0\leq t\leq T}|X^n(t)-X(t)|^2\right) = 0. \tag{86}
$$

$\square$

# 4 | APPLICATIONS OF COUPLED STOCHASTIC PROCESSES IN BOUNDED DOMAINS

In this section, we provide an application of our coupled stochastic processes in the bounded domain for active–passive population dynamics (1)–(2) proposed in Section 2. In particular, we discuss the application of our active–passive population dynamics by considering a Wilson–Cowan-type system describing the dynamics of two interacting populations of excitatory and inhibitory neurons in brain cell systems.

In general, in the study of biological systems, the descriptions of individual cells may be appropriate for primitive systems. However, to model reliably living systems with complex biological interactions, a large number of cells needs to be accounted for. For instance, the human brain consists of approximately 10^{11} neurons and is connected to 10^4 other neurons.[34,35] To better understand the resulting neural activity requires appropriate models that can track the average firing rate across many areas of a neuronal network. Therefore, from a large population of densely coupled neurons, Wilson and Cowan[23,24] have derived an effective system for the proportion of cells in a population that are active per unit time. In this section, we consider an application of our active–passive population dynamics in the model of synaptically coupled excitatory and inhibitory neurons in the neocortex. In particular, we study a system of stochastic Wilson–Cowan-type equations with reflection and possible jump-like exit from a boundary that allows us to model the dynamics of two interacting populations of excitatory and inhibitory neurons (see, e.g., Figure 1). Let us recall the following Wilson–Cowan system, considering a two-dimensional dynamic case (see, e.g., Wilson and Cowan[23])

$$
\begin{cases}
\tau_E \frac{dr_E}{dt} = -r_E(t) + (1-\delta_E r_E(t))F_E(w_{EE}r_E(t) - w_{EI}r_I(t) + I_E^{\text{ext}}, \theta_E, a_E), \\
\tau_I \frac{dr_I}{dt} = -r_I(t) + (1-\delta_I r_I(t))F_I(w_{IE}r_E(t) - w_{II}r_I(t) + I_I^{\text{ext}}, \theta_I, a_I),
\end{cases} \tag{87}
$$

where $r_E(t)$ and $r_I(t)$ are the proportions of excitatory and inhibitory cells firing per unit time at the instant t, respectively. Here, w_{EE} and w_{II} represent the strengths of connection between excitatory and inhibitory cells, respectively, while w_{EI} describes the strength of connection from excitatory cells to inhibitory cells and w_{IE} denotes the strength of connection from inhibitory cells to excitatory cells. Moreover, τ_E and τ_I denote the refractory periods of excitatory and inhibitory cells after a trigger, respectively, while r_E and r_I are the absolute refractory periods, θ_E and θ_I are the threshold of the excitatory and inhibitory populations. We also assume that $r_E=0$ and $r_I=0$ correspond to a low-activity resting states of excitatory

FIGURE 1 Sketch of networks of interacting excitatory and inhibitory populations [Color figure can be viewed at wileyonlinelibrary.com]
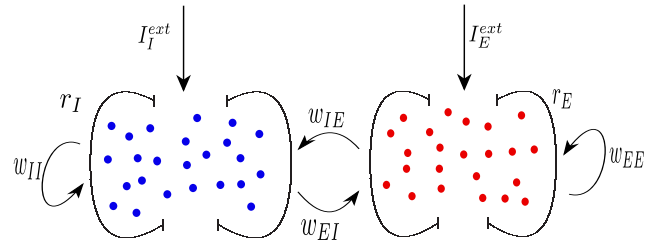
and inhibitory cells. In (87), functions F_E and F_I represent the nonlinearities typically chosen to be sigmoidal defined as (see, e.g., Wilson and Cowan[23])

$$F(x, \theta, a) := \frac{1}{1 + \exp[-a(x - \theta)]} - \frac{1}{1 + \exp(a\theta)}. \tag{88}$$

In general, an excitatory transmitter generates an electrical signal called an action potential in the receiving neuron, while an inhibitory transmitter prevents such electrical signals (see, e.g., Chan et al.[36]). Hence, we assume that an excitatory population can be considered an active population, while an inhibitory population can be seen as a passive population. Neuron dynamics are intensively computed and often deal with many challenges from severe accuracy degradation if the input data is corrupted with noise. Furthermore, the noise is normally assumed as purely diffusive noise, namely, as random fluctuations with continuous sample paths. However, such a description is incomplete due to the fact that the diffusive fluctuations are large and abrupt events appear at random times throughout the time series.[18,19,37] Additionally, in some cases, we can observe also negative firing rates, while the firing rates are normally positive.[38] We know that a negative firing rate is not physiologically plausible and the dynamical model can be interpreted to capture activity relative to a baseline resting state rather than total activity.[39] In this work, we observe that under some cases of random input current, the firing rates of excitatory and inhibitory cells go below the low-activity resting-state level. Thus, we can set a reflecting boundary to our population dynamics to avoid such negative firing rate cases. To get closer to the real scenarios in biological systems, jump-diffusion stochastic processes provide a more appropriate framework to model these data. In particular, using the descriptions provided in the previous sections, we focus on investigating the system of stochastic Wilson-Cowan-type equations with reflection and possible jump-like exit from a boundary, which reads

$$\begin{cases} dr_E(t) &= \frac{1}{\tau_E}(-r_E(t) + (1 - \delta_E r_E(t))\tilde{F}_E(w_{EE}r_E(t) - w_{EI}r_I(t) + I_E^{\text{ext}}, \theta_E, a_E))dt \\ &\quad + \frac{\sigma_E^{\text{ext}}}{\tau_E}(1 - \delta_E r_E(t))dW_E(t) + d\phi_E(t) + \int_{\mathbb{R}} \rho_E(r_E(t))dJ_E(t), \\ dr_I(t) &= \frac{1}{\tau_I}(-r_I(t) + (1 - \delta_I r_I(t))\tilde{F}_I(w_{IE}r_E(t) - w_{II}r_I + I_I^{\text{ext}}, \theta_I, a_I))dt \\ &\quad + \frac{\sigma_I^{\text{ext}}}{\tau_I}(1 - \delta_I r_I(t))dW_I(t) + d\phi_I(t) + \int_{\mathbb{R}} \rho_I(r_I(t))dJ_I(t), \end{cases} \tag{89}$$

In (89), we assume that $F_E(w_{EE}r_E - w_{EI}r_I + I_E^{\text{ext}}, \theta_E, a_E) = \tilde{F}_E(w_{EE}r_E - w_{EI}r_I + I_E^{\text{ext}}, \theta_E, a_E)) + \sigma_E^{\text{ext}}dW_E(t)$ and $F_I = (w_{IE}r_E - w_{II}r_I + I_I^{\text{ext}}, \theta_I, a_I)) + \sigma_I^{\text{ext}}dW_I(t)$. Note that $f_A(X_A(t), X_P(t))$ and $f_P(X_A(t), X_P(t))$ (in 1) become $\frac{1}{\tau_E}(-r_E + (1 - \delta_E r_E)\tilde{F}_E(w_{EE}r_E - w_{EI}r_I + I_E^{\text{ext}}, \theta_E, a_E))$ and $\frac{1}{\tau_I}(-r_I + (1 - \delta_I r_I)\tilde{F}_I(w_{IE}r_E - w_{II}r_I + I_I^{\text{ext}}, \theta_I, a_I))$, respectively. Similarly, $g_A(X_A(t), X_P(t))$ and $g_P(X_A(t), X_P(t))$ (in 1) become $\frac{\sigma_E^{\text{ext}}}{\tau_E}(1 - \delta_E r_E)$ and $\frac{\sigma_I^{\text{ext}}}{\tau_I}(1 - \delta_I r_I)$, while $\rho_A, \rho_P, \phi_A, \phi_P$ can be considered as $\rho_E, \rho_I, \phi_E, \phi_I$, respectively. Applying Theorems 2–4, the system (89) is well-posedness. As we have mentioned in Section 1, after constructing a new mathematical model, the theoretical proof of its well-posedness would be an important step in it justification. In what follows, after obtaining the well-posedness of the system (89) as the next step, we will provide the simulation results of the system's dynamics.

In particular, the simulations presented in this section have been carried out by using by a discrete-time integration based on the Euler method implemented in Python. In this section, for $t \geq 0$, we consider the case of $r_E(t), r_I(t) \geq 0$ with $L(0) = 0$ and

$$\phi_E(t) = \int_0^t \mathbb{1}_{r_E(s)=0}d\phi_E, \quad \text{and} \quad \phi_I(t) = \int_0^t \mathbb{1}_{r_I(s)=0}d\phi_I. \tag{90}$$

This condition implies that the process can increase only when r_E and r_I hit 0 (see, e.g., Pilipenko[20]). In other words, this is the reflecting boundary condition at 0 in one-dimensional for each population in the domain $D = [0, \infty)$. Moreover, we use a set of compound Poission process as in (4) to describe the jump-like exit at the boundary.

In the simulations, we choose the parameter set as follows: $r_E = r_I = 0.2$, $\tau_E = 1$ (ms), $\tau_I = 2$ (ms), $\theta_E = 2.8$, $\theta_I = 4$, $a_E = 1.2$, $a_I = 1$, $w_{EE} = 12$, $w_{EI} = 4$, $w_{IE} = 13$, $w_{II} = 11$, $dt = 0.1$ (ms). These parameters have also been used in Wilson and Cowan.[23]
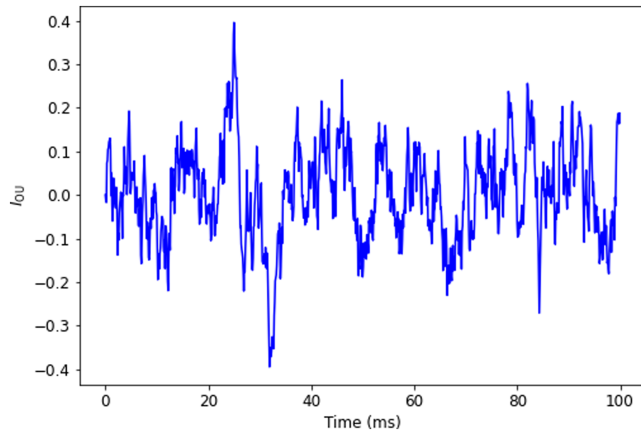
FIGURE 2 Ornstein–Uhlenbeck input current profile satisfies the Ornstein–Uhlenbeck process (91). Parameters: $\mu = 0$, $\gamma = 1$, and $\sigma = 0.1$ [Color figure can be viewed at wileyonlinelibrary.com]
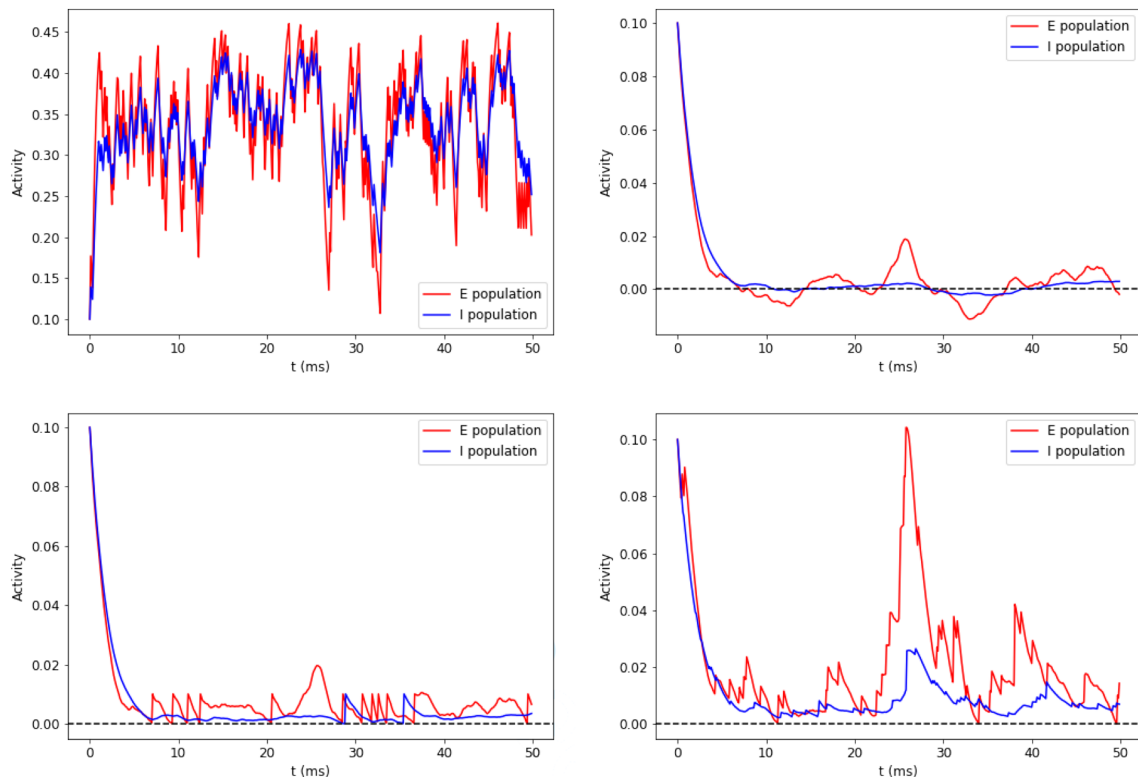
FIGURE 3 The population trajectories of excitatory (red color) and inhibitory populations (blue color) of the system (89). Top left: Gaussian white noise input current. Top right: Ornstein–Uhlenbeck input current (see, e.g., Figure 2). Bottom left: Ornstein–Uhlenbeck input current with reflecting boundary condition at 0. Bottom right: Ornstein–Uhlenbeck input current with reflecting boundary condition at 0 and with jumps. Parameters: $\mu = 0$, $\gamma = 1$, and $\sigma = 0.1$ [Color figure can be viewed at wileyonlinelibrary.com]

Let $\{V(t) : t \geq 0\}$ be an Ornstein–Uhlenbeck process (see, e.g., Figure 2) defined on $[r, \infty)$ with drift $(\mu - V(t)/\gamma)$ and constant diffusion parameter σ. Then, the process $\{V(t) : t \geq 0\}$ satisfies the following SDE:

$$\begin{cases} dV(t) = (\mu - \frac{1}{\gamma}V(t))dt + \sigma dW(t), \\ V(0) \in [r, \infty), \end{cases} \tag{91}$$

where $W(t)$ denotes Gaussian white noise.

The main representative numerical results of our analysis are shown in Figure 3, where we have plotted the population trajectories of excitatory and inhibitory populations.

In particular, in the top left panel of Figure 3, we have plotted the population trajectories of excitatory and inhibitory populations under only Gaussian white noise input current. We see that there are fluctuations in the time evolution of the proportions of both excitatory and inhibitory cells firing. However, in the top right panel of Figure 3, when we replace the Gaussian white noise current with the Ornstein–Uhlenbeck input current presented in Figure 2, we observe that the firing activity of excitatory and inhibitory cells fluctuates to values less than zero. Therefore, we add the reflecting factor to our system with the Ornstein–Uhlenbeck input current, and we see that the firing activity increases to values larger than 0 (the resting states of excitatory and inhibitory cells) in the bottom left panel of Figure 3. Moreover, in the presence of Poisson jumps and the reflecting factor in our system with Ornstein–Uhlenbeck input current, the firing activities of excitatory and inhibitory cells increase dramatically, as seen in the bottom right panel of Figure 3, compared to the case presented in the bottom left panel of the same figure. Specifically, the firing rate of excitatory cells increases to 0.1 at $t = 27$ (ms) and the firing rate of the inhibitory cells increases to 0.02 at the same time. However, the firing rate of inhibitory cells is less than the firing rate of excitatory cells in the presence of jumps in the system. The numerical results show that the presence of the reflecting boundary condition together with the jump-like exits from the boundary could help to avoid the negative firing rate in the case of random input current with large random fluctuations and the appearance of abrupt events at random times throughout the time series.

Additionally, we have observed that in the presence of Ornstein–Uhlenbeck input current, reflecting factors together with the Poisson jumps increase the firing activity of excitatory and inhibitory populations. This effect may lead to an improvement in the response of neurons to each stimulus in neuronal systems.

5 | CONCLUSIONS

We have proposed a model based on a coupled system of reflecting Skorokhod-type SDEs with jumps. We have analyzed the well-posedness of such systems. In particular, using compactness methods and Skorokhod's representation of solutions to SDEs with the jump-like exit from a boundary, we have shown the existence and uniqueness of the solutions of these systems. Additionally, the structure of the Skorokhod problem allowed us to prove also the solution dependence on the parameters of our system. On the other hand, the mathematical setting of our system of SDEs has demonstrated a new point of view useful for the field of modeling and simulations of population dynamics. We provided details of the models along with representative numerical examples and discussed the applications of our population dynamics in applications to neuronal dynamics. In particular, we have considered a system of stochastic Wilson–Cowan-type equations with reflection and possible jump-like exit from a boundary. Our numerical results have shown that the presence of Ornstein–Uhlenbeck input current, reflecting factors together with the Poisson jumps, strongly affects the firing activities of excitatory and inhibitory populations in a neuronal system.

CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

AUTHOR CONTRIBUTIONS

In this work, TKTT contributed with major, essential parts to all proofs and mathematical derivations in the manuscript. Discussions and feedback were provided continuously by AM and RM. TKTT drafted the manuscript and produced all figures and was responsible for the submission process to publication.

ORCID

Thi Kim Thoa Thieu https://orcid.org/0000-0002-3873-2245

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APPENDIX A: TECHNICAL PRELIMINARIES FOR SECTION 3

In the proof of existence in Section 3, we used compactness arguments. Here, we provide necessary details for such arguments to hold. We recall the classical Ascoli–Arzelà theorem[40]:

A family of functions $U \subset C(\bar{S}; \mathbb{R}^d)$ is relatively compact (with respect to the uniform topology) if

i. for every $t \in \bar{S}$, the set $\{h(t); h \in U\}$ is bounded;

ii. for every $\varepsilon > 0$ and $t, s \in \bar{S}$, there is $\bar{\delta} > 0$ such that

$$|h(t) - h(s)| \leq \varepsilon, \tag{A1}$$

whenever $|t - s| \leq \bar{\delta}$ for all $h \subset U$.

For a function $h : \bar{S} \to \mathbb{R}^d$, we introduce the definition of Hölder seminorms as

$$[h]_{C^\alpha(\bar{S}; \mathbb{R}^d)} = \sup_{t \neq s; t, s \in \bar{S}} \frac{|h(t) - h(s)|}{|t - s|^\alpha}, \tag{A2}$$

for $\alpha \in (0, 1)$ and the supremum norm as

$$\|h\|_{L^\infty(S; \mathbb{R}^d)} = \operatorname*{ess\,sup}_{t \in \bar{S}} |h(t)|. \tag{A3}$$

We refer to Adams and Fournier[41] and Gilbarg and Trudinger[42] for more details on the used function spaces.

In fact, the simple sufficient conditions for (i) and (ii) are

i′. there is $M_1 > 0$ such that $\|h\|_{L^\infty(S; \mathbb{R}^d)} \leq M_1$ for all $h \in U$;

ii′. for some $\alpha \in (0, 1)$, there is an $M_2 > 0$ such that $[h]_{C^\alpha(\bar{S}; \mathbb{R}^d)} \leq M_2$ for all $h \in U$.

Hence, we have the sets

$$K_{M_1 M_2} := U = \left\{ h \in C(\bar{S}; \mathbb{R}^d); \|h\|_{L^\infty(S; \mathbb{R}^d)} \le M_1, [h]_{C^\alpha(\bar{S}; \mathbb{R}^d)} \le M_2 \right\} \tag{A4}$$

are relatively compact in $C(\bar{S}; \mathbb{R}^d)$.

For $\alpha \in (0, 1)$, $T > 0$ and $p > 1$, the space $W^{\alpha,p}(S; \mathbb{R}^d)$ is defined as the set of all $h \in L^p(S; \mathbb{R}^d)$ such that

$$[h]_{W^{\alpha,p}(S; \mathbb{R}^d)} := \int_0^T \int_0^T \frac{|h(t) - h(s)|^p}{|t - s|^{1 + \alpha p}} \, dt \, ds < \infty.$$

This space is endowed with the norm

$$\|h\|_{W^{\alpha,p}(S; \mathbb{R}^d)} = \|h\|_{L^p(S; \mathbb{R}^d)} + [h]_{W^{\alpha,p}(S; \mathbb{R}^d)}.$$

Moreover, we have the following embedding:

$$W^{\alpha,p}(S; \mathbb{R}^d) \subset C^\gamma(\bar{S}; \mathbb{R}^d) \text{ for } \alpha p - \gamma > 1,$$

and $[h]_{C^\gamma(\bar{S}; \mathbb{R}^d)} \le C_{\gamma,\alpha,p} \|h\|_{W^{\alpha,p}(S; \mathbb{R}^d)}$ (see, e.g., in Theorem 6, Chapter 5 in Evans[43]). Relying on the Ascoli–Arzelà theorem, we have the following situation:

ii''. for some $\alpha \in (0, 1)$ and $p > 1$ with $\alpha p > 1$, there is $M_2 > 0$ such that $[h]_{W^{\alpha,p}(S; \mathbb{R}^d)} \le M_2$ for all $h \in U' \subset C(\bar{S}; \mathbb{R}^d)$.

If (i') and (ii'') hold, then the set

$$K'_{M_1 M_2} := U' = \left\{ h \in C(\bar{S}; \mathbb{R}^d); \|h\|_{L^\infty(S; \mathbb{R}^d)} \le M_1, [h]_{W^{\alpha,p}(S; \mathbb{R}^d)} \le M_2 \right\} \tag{A5}$$

is relatively compact in $C(\bar{S}; \mathbb{R}^d)$, if $\alpha p > 1$ (see, e.g., Colangeli et al[13] and Flandoli and Gatarek[28]). In the main part of the manuscript, this result was formulated in formula (42).