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Complex systems in finance: Monte Carlo evaluation of first passage time density functions

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Abstract

Many examples of complex systems are provided by applications in finance and economics areas. Some of intrinsic features of such systems lie with the fact that their parts are interacting in a non-trivial dynamic manner and they can be subject to stochastic forces and jumps. The mathematical models for such systems are often based on stochastic differential equations and efficient computational tools are required to solve them. Here, on an example from the credit risk analysis of multiple correlated firms, we develop a fast Monte-Carlo type procedure for the analysis of complex systems such as those occurring in the financial market. Our procedure is developed by combining the fast Monte-Carlo method for one-dimensional jump-diffusion processes and the generation of correlated multidimensional variates. As we demonstrate on the evaluation of first passage time density functions in credit risk analysis, this allows us to analyze efficiently multivariate and correlated jump-diffusion processes.

Keywords: Monte Carlo simulations, credit risk, dynamic interactions, complex systems, stochastic differential equations, multidimensional, jump-diffusion processes.

1. Introduction

Complex systems usually consist of parts that interact in a dynamic non-trivial manner [14, 15]. Such systems are found frequently in financial applications and economics. For example, in the market economy, individual companies are inevitably linked together via dynamically changing economic conditions. Therefore, the default events of companies are often correlated, especially in the same industry. Probably, Zhou [20] and Hull et al [7] were the first to incorporate default correlation into the Black-Cox first passage structural model, but they have not included the jumps. The standard Brownian motion model for market behavior falls short of explaining empirical observations of market returns and their underlying derivative prices Zhou [21], Kou et al [9]. Models describing jump-diffusion processes (JDPs) represent an important alternative to the standard Brownian motion model [2] and provide a convenient framework for investigating many complex phenomena. As soon as jumps are incorporated in the model for most practical cases we have to resort to numerical procedures where Monte Carlo methods remain a primary candidate for applications.

The conventional Monte Carlo methods are very straightforward to implement. We discretize the time period into N intervals with N being large enough in order to avoid discretization bias [8]. The main drawback of this procedure

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is that we need to evaluate the processes at each discretized time which is very time-consuming. Many researchers have contributed to the field of enhancement of the efficiency of Monte Carlo simulations. Among others, Kuchler et al [10] discussed the solution of SDEs in the framework of weak discrete time approximations and Liberati et al [12] considered the strong approximation where the SDE is driven by a high intensity Poisson process. Atiya and Metwally [2, 16] have developed a fast Monte Carlo-type numerical methods to solve the FPT problem. In our recent contributions, we reported an extension of this fast Monte-Carlo-type method in the context of multiple non-correlated jump-diffusion processes [18, 19]. In this contribution, we generalize our previous fast Monte-Carlo method (for non-correlated jump-diffusion cases) to multivariate (and correlated) jump-diffusion processes. The developed technique provides an efficient tool for a number of applications, including credit risk and option pricing [11, 1]. We demonstrate the applicability of this technique to the evaluation of first passage time density functions in the default analysis.

2. Models for joint default in the financial market

If the firm (i) assets value $V_i(t)$ falls below a threshold level $D_{V_i}(t)$, the firm defaults. Note that finding the threshold level is already a challenge in applications (one of the reasons for that is because firms often rearrange their liability structure when they have credit problems). Here we assume that $D_{V_i}(t) = \kappa_i \exp(\gamma_i t)$ as proposed by Zhou [20], where γ_i is the growth rate of the firm's liabilities, while κ_i is responsible for capturing the liability structure of the firm (often defined as the firm's short-term liability plus 50% of the firm's long-term liability). We set $X_i(t) = \ln[V_i(t)]$, in which case the threshold of $X_i(t)$ is $D_i(t) = \gamma_i t + \ln(\kappa_i)$, and in what follows our main interest is in the dynamics of process $X_i(t)$.

The default correlation measures the strength of the default relationship between different firms and for two firms i and j with respective probabilities P_i and P_j it is defined as

$$\rho_{ij} = \frac{P_{ij} - P_i P_j}{\sqrt{P_i(1-P_i)P_j(1-P_j)}}, \tag{1}$$

where P_{ij} is the probability of joint default. The default correlation defined by (1) plays a key role in the joint default with important implications in the field of credit analysis and other applications.

The first passage time model to describe default correlations of two firms under the "bivariate diffusion process" can be described as follows Zhou [20]:

$$\begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} dt + \Omega \begin{bmatrix} dz_1 \\ dz_2 \end{bmatrix}, \tag{2}$$

where μ_1 and μ_2 are constant drift terms, z_1 and z_2 are two independent standard Brownian motions, and Ω is a constant 2×2 matrix such that

$$\Omega \cdot \Omega' = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}.$$

The coefficient ρ reflects the correlation between the movements in the asset values of the two firms. If we assume that $\mu_i = \gamma_i$ $(i = 1, 2)$, then the probability that firm i defaults at time t can be easily calculated as:

$$P_i(t) = 2 \cdot \Phi\left(-\frac{X_i(0) - \ln(\kappa_i)}{\sigma_i \sqrt{t}}\right) = 2 \cdot \Phi\left(-\frac{Z_i}{\sqrt{t}}\right), \tag{3}$$

where

$$Z_i \equiv \frac{X_i(0) - \ln(\kappa_i)}{\sigma_i}$$

is the standardized distance of firm i to its default point and $\Phi(\cdot)$ denotes the cumulative probability distribution function for a standard normal variable.

However, this and other classical models do not include jumps in the processes, while it is well-known that jumps are a major factor in credit risk analysis and other applications. Multivariate jump-diffusion processes can provide a convenient way to describe multivariate and correlated processes with jumps.

2.1. Multivariate jump-diffusion processes

On a complete probability space (Ω, F, P) with information filtration (F_t) we assume that $\vec{X}_t = \ln(\vec{V}_t)$ is a Markov process in some state space $D \subset \mathbb{R}^n$ such that it solves the stochastic differential equation [4]:

$$d\vec{X}_t = \vec{\mu}(\vec{X}_t)dt + \vec{\sigma}(\vec{X}_t)d\vec{W}_t + d\vec{Z}_t, \tag{4}$$

where $\vec{W}$ is an (F_t)-standard Brownian motion in $\mathbb{R}^n$; $\vec{\mu} : D \to \mathbb{R}^n$, $\vec{\sigma} : D \to \mathbb{R}^{n\times n}$, and $\vec{Z}$ is a pure jump process whose jumps have a fixed probability distribution $\vec{\nu}$ on $\mathbb{R}^n$ such that they arrive with intensity $\{\vec{\lambda}(\vec{X}_t) : t \geq 0\}$, for some $\vec{\lambda} : D \to [0, \infty)$. Under these conditions, the above model is reduced to an affine model and the theory of affine processes as a class of time-homogeneous Markov processes arising often in the context of applications in finance, including credit risk modeling, has been developed in [5, 6]. What we are interested in our contribution is multivariate transformed Brownian motions with jumps and in the estimation of the default rate of a firm during a given time period. This problem is reduced to a first passage time problem. The difficulties arise from the fact that the multiple processes as well as their first passage times are indeed correlated, so the simulation must reflect the correlations of first passage times. We propose a solution to circumvent these difficulties by combining the fast Monte-Carlo method of one-dimensional jump-diffusion processes and the generation of correlated multidimensional variables, generalizing previous results on non-correlated jump-diffusion situations.

Although for jump-diffusion processes, the closed form solutions are usually unavailable, between each two jumps the process is a Brownian bridge for a univariate jump-diffusion process. In Atiya et al [2] the one-dimensional first passage time distribution has been deduced for time period $[0, T]$. In order to evaluate multiple processes, we obtain multi-dimensional formulas and reduce them to computable forms.

Therefore, we consider N_{firm} firms $\vec{X}_t = [X_1, X_2, ..., X_{N_{\text{firm}}}]^T$, each X_i describes the process of individual firm i. We may expect that each process X_i satisfies the following SDE:

$$\begin{aligned} dX_i &= \mu_i dt + \sum_j \sigma_{ij} dW_j + dZ_i \\ &= \mu_i dt + \sigma_i d\tilde{W}_i + dZ_i, \end{aligned} \tag{5}$$

where $\tilde{W}_i$ is also a standard Brownian motion and σ_i is:

$$\sigma_i = \sqrt{\sum_j \sigma_{ij}^2}.$$

We assume that in the interval $[0, T]$, the total number of jumps for firm i is M_i. Let the jump instants be $T_1, T_2, \cdots, T_{M_i}$. Let $T_0 = 0$ and $T_{M_i+1} = T$. The quantities τ_j equal interjump times, which are $T_j - T_{j-1}$. Following the notation of Atiya et al [2], let $X_i(T_j^-)$ be the process value immediately before the jth jump, and $X_i(T_j^+)$ be the process value immediately after the jth jump. The jump-size is $X_i(T_j^+) - X_i(T_j^-)$, and we can use such jump-sizes to generate $X_i(T_j^+)$ sequentially.

Let $A_i(t)$ be the event consisting of process X_i crossing the threshold level $D_i(t)$ for the first time in the interval $[t, t + dt]$, then the conditional interjump first passage density is defined as [2]:

$$g_{ij}(t) = P(A_i(t) \in dt | X_i(T_{j-1}^+), X_i(T_j^-)). \tag{6}$$

For firm i, after generating a series of first passage times s_i, we use a kernel density estimator with Gaussian kernel to estimate the first passage time density (FPTD) f. Such kernels allow quite straightforward kernel combinations, and therefore present a natural choice for a number of generalizations of the procedure described here. The kernel density estimator is based on centering a kernel function of a bandwidth as follows:

$$\widehat{f} = \frac{1}{N} \sum_{i=1}^{N} K(h, t - s_i), \tag{7}$$

where

$$K(h, t - s_i) = \frac{1}{\sqrt{\pi/2}h} \exp\left(-\frac{(t - s_i)^2}{h^2/2}\right).$$

The optimal bandwidth in the kernel function K can be calculated by using standard procedures (at least in the one-dimensional case) and we do not focus on this here. However, we note that the kernel estimator for the multivariate case involves the evaluation of the joint conditional interjump first passage time density and the methodology for such an evaluation is quite involved compared to the one-dimensional case.

3. The methodology of solution

First, let us recall the conventional Monte-Carlo procedure in application to the analysis of the evolution of firm X_i within the time period $[0, T]$. We divide the time period into n small intervals $[0, t_1]$, $[t_1, t_2]$, $\cdots$, $[t_{n-1}, T]$. In each Monte Carlo run, we need to calculate the value of X_i at each discretized time t. As usual, in order to exclude discretization bias, the number n must be large. This procedure exhibits substantial computational difficulties when applied to jump-diffusion processes. Indeed, for a typical jump-diffusion process, let T_{j-1} and T_j be any successive jump instants, as described above. Then, in the conventional Monte Carlo method, although there is no jump occurring in the interval $[T_{j-1}, T_j]$, yet we need to evaluate X_i at each discretized time t in $[T_{j-1}, T_j]$. This very time-consuming procedure results in a serious shortcoming of the conventional Monte-Carlo methodology.

To remedy the situation, two modifications of the conventional procedure were recently proposed [2, 16] that allow a potential speed-up of the conventional methodology of up to 10-30 times. One of the modifications, the uniform sampling method, involves samplings using the uniform distribution. The other is the inverse Gaussian density sampling method. Both methodologies were developed for the univariate case.

The major improvement of the UNIF method is based on the fact that it only evaluates X_i at generated jump times, while between each two jumps the process is a Brownian bridge. Hence, we just consider the probability of X_i crossing the threshold in (T_{j-1}, T_j) instead of evaluating X_i at each discretized time t. More precisely, in the UNIF method, we assume that the values of $X_i(T_{j-1}^+)$ and $X_i(T_j^-)$ are known as two end points of the Brownian bridge, the probability that firm i defaults in (T_{j-1}, T_j) is $1 - P_{ij}$ which can easily be computed. Then we generate a variable s_i from a distribution uniform in the interval $[T_{j-1}, T_{j-1} + \frac{T_j - T_{j-1}}{1-P_{ij}}]$. If the generated point s_i falls in the interjump interval $[T_{j-1}, T_j]$, then we have successfully generated the first passage time s_i and can neglect the other intervals and perform another Monte Carlo run. On the other hand, if the generated point s_i falls outside the interval $[T_{j-1}, T_j]$ (which happens with probability P_{ij}), then that point is "rejected". This means that no boundary crossing has occurred in the interval, and we proceed to the next interval and repeat the whole process again.

In what follows, we focus on the further development of the UNIF method and extend it to multivariate and correlated jump-diffusion processes. In order to implement the UNIF method for our multivariate model, we need to consider several points:

1. We assume that the arrived jumps follow the Poisson process. The intensity λ of the Poisson process and the distribution of $(T_j - T_{j-1})$ are the same for each firm. This assumption may not always be fulfilled as one may argue that the intensity λ could be different for different firms which implies that different firms may endure different jump rates. However, in the real market economy once a firm (let us call it "A") encounters sudden economic hazard, its correlated firms may also endure the same hazard. Furthermore, it is common that other firms may help firm "A" to pull out, which may result in a simultaneous jump for them. Therefore, as a first approximation, it is reasonable to employ the simultaneous jumps processes for all the different firms.
2. As for the jump-size, we generate it by a given distribution which can be different for different firms to reflect specifics of the jump process for each firm. In the current contribution, we exemplify our description by considering an exponential distribution (mean value μ_T) for $(T_j - T_{j-1})$ and a normal distribution (mean value μ_J and standard deviation σ_J) for the jump-size. We can use any other distribution when appropriate.
3. An array `IsDefault` (whose size is the number of firms denoted by N_{firm}) is used to indicate whether firm i has defaulted in this Monte Carlo run. If the firm defaults, then we set `IsDefault`$(i) = 1$, and will not evaluate it during this Monte Carlo run.
4. Most importantly, as we have mentioned before, the default events of firm i are inevitably correlated with other firms, for example firm $i+1$. The default correlation of firms i and $i+1$ can easily be calculated. Hence, firm i's first passage time s_i is indeed correlated with s_{i+1} – the first passage time of firm $i+1$. We must generate several correlated s_i in each interval $[T_{j-1}, T_{j-1} + \frac{T_j - T_{j-1}}{1-P_{ij}}]$ which is the key point for multivariate correlated processes.

Finally, we note that the default happening at time s_i also means that time s_i is exactly the first passage time for firm i. Therefore, the correlation of s_i and s_{i+1} is the same as the default correlation of firms i and $i+1$:

$$\rho(s_i, s_{i+1}) = \rho_{i,i+1}(t) = \frac{P_i(t) + P_{i+1}(t) - P_i(t)P_{i+1}(t) - P_{i\cup i+1}(t)}{\sqrt{P_i(t)[1-P_i(t)]P_{i+1}(t)[1-P_{i+1}(t)]}}, \tag{8}$$

where in practice t can be chosen as the midpoint of the interval.

Next, we will give a brief description of the sum-of-uniforms method which is used to generate correlated uniform random variables, followed by the description of the multivariate and correlated UNIF method and the model calibration.

3.1. Sum-of-uniforms method

In the above sections, we have reduced the solution of the original problem to a series of one-dimensional jump-diffusion processes. The first passage time distribution in an interval $[T_{j-1}, T_j]$ (between two successive jumps) was obtained. Here, we will describe how to generate several correlated s_i in $[T_{j-1}, T_{j-1} + \frac{T_j - T_{j-1}}{1-P_{ij}}]$ whose correlations can be described by Eq. (8).

Let us introduce a new variable $b_{ij} = \frac{T_j - T_{j-1}}{1-P_{ij}}$, then we have $s_i = b_{ij}Y_i + T_{j-1}$, where Y_i is uniformly distributed in $[0, 1]$. Moreover, the correlation of Y_i and Y_{i+1} equals $\rho(s_i, s_{i+1})$. Now we can generate the correlated uniform random variables $Y_1, Y_2, \cdots$ by using the sum-of-uniforms (SOU) method [3, 17] in the following steps:

1. Generate Y_1 from numbers uniformly distributed in $[0, 1]$.
2. For $i = 2, 3, \cdots$, generate $W_i \sim U(0, c_{i-1,i})$, where $U(0, c_{i-1,i})$ denotes a uniform random number over range $(0, c_{i-1,i})$. Chen [3] has obtained the relationship of parameter $c_{i-1,i}$ and the correlation $\rho(s_{i-1}, s_i)$ (abbreviated as $\rho_{i-1,i}$) as follows:

$$\rho_{i-1,i} = \begin{cases} \dfrac{1}{c_{i-1,i}} - \dfrac{0.3}{c_{i-1,i}^2}, & 0 \le \rho_{i-1,i} \le 0.7, \quad c_{i-1,i} \ge 1, \\ 1 - 0.5c_{i-1,i}^2 + 0.2c_{i-1,i}^3, & \rho_{i-1,i} \ge 0.7, \quad c_{i-1,i} < 1, \\ -\dfrac{1}{c_{i-1,i}} + \dfrac{0.3}{c_{i-1,i}^2}, & -0.7 \le \rho_{i-1,i} \le 0, \quad c_{i-1,i} \ge 1, \\ -1 + 0.5c_{i-1,i}^2 - 0.2c_{i-1,i}^3, & \rho_{i-1,i} \le -0.7, \quad c_{i-1,i} < 1. \end{cases}$$

If Y_{i-1} and Y_i are positively correlated, then let

$$Z_i = Y_{i-1} + W_i.$$

If Y_{i-1} and Y_i are negatively correlated, then let

$$Z_i = 1 - Y_{i-1} + W_i.$$

Let $Y_i = F(Z_i)$, where for $c_{i-1,i} \ge 1$,

$$F(Z) = \begin{cases} Z^2/(2c_{i-1,i}), & 0 \le Z \le 1, \\ (2Z-1)/(2c_{i-1,i}), & 1 \le Z \le c_{i-1,i}, \\ 1 - (1 + c_{i-1,i} - Z)^2/(2c_{i-1,i}), & c_{i-1,i} \le Z \le 1 + c_{i-1,i}, \end{cases}$$

and for $0 < c_{i-1,i} \le 1$,

$$F(Z) = \begin{cases} Z^2/(2c_{i-1,i}), & 0 \le Z \le c_{i-1,i}, \\ (2Z - c_{i-1,i})/2, & c_{i-1,i} \le Z \le 1, \\ 1 - (1 + c_{i-1,i} - Z)^2/(2c_{i-1,i}), & 1 \le Z \le 1 + c_{i-1,i}. \end{cases}$$

By carrying out the above two steps, we can generate correlated uniform random variables Y_1, Y_2,..., leading to the relationship $s_i = b_{ij}Y_i + T_{j-1}$ whose correlations automatically satisfy Eq. (8). Note also that $\rho(s_i, s_{i+1})$ should be computed before generating the correlated uniform random variables Y_1, Y_2,..., and hence, in a practical implementation, we should approximate time t where $\rho(s_i, s_{i+1})$ is computed as discussed. One such possible choice has already been mentioned above and, from a practical point of view, it will work well as long as $\rho(s_i, s_{i+1})$ is a slowly varying function in $[T_{j-1}, T_j]$.

3.2. Uniform sampling method

In this subsection, we will describe our algorithm for multivariate jump-diffusion processes, which is an extension of the one-dimensional case developed earlier by other authors (e.g. Atiya et al [2], Metwally et al [16]).

Consider N_{firm} firms in the given time period $[0, T]$. First, we generate the jump instant T_j by generating interjump times $(T_j - T_{j-1})$ and set all the $\texttt{IsDefault}(i) = 0$ $(i = 1, 2, \cdots, N_{\text{firm}})$ to indicate that no firm has defaulted at the beginning.

Note that for each process X_i we can make the following observations:

1. If no jump occurs, as described by Eq. (5), the interjump size $(X_i(T_j^-) - X_i(T_{j-1}^+))$ follows a normal distribution of mean $\mu_i(T_j - T_{j-1})$ and standard deviation $\sigma_i\sqrt{T_j - T_{j-1}}$. We get

$$\begin{aligned} X_i(T_j^-) &\sim X_i(T_{j-1}^+) + \mu_i(T_j - T_{j-1}) + \sigma_i\sqrt{T_j - T_{j-1}}\,\tilde{W}_i \\ &\sim X_i(T_{j-1}^+) + \mu_i(T_j - T_{j-1}) + \sum_{k=1}^{N_{\text{firm}}} \sigma_{ik}\sqrt{T_j - T_{j-1}}\,W_i, \end{aligned}$$

   where the initial state is $X_i(0) = X_i(T_0^+)$.
2. If a jump occurs, we simulate the jump-size by a normal distribution or another distribution when appropriate, and compute the postjump value:

$$X_i(T_j^+) = X_i(T_j^-) + Z_i(T_j).$$

This completes the procedure for generating beforejump and postjump values $X_i(T_j^-)$ and $X_i(T_j^+)$, respectively. As before, $j = 1, \cdots, M$ where M is the total number of jumps for all the firms. We compute P_{ij}. To recur the first passage time density (FPTD) $f_i(t)$, we have to consider three possible cases that may occur for each non-default firm i:

1. **First passage happens inside the interval.** We know that if $X_i(T_{j-1}^+) > D_i(T_{j-1})$ and $X_i(T_j^-) < D_i(T_j)$, then the first passage happened in the time interval $[T_{j-1}, T_j]$. To evaluate when the first passage happened, we introduce a new variable b_{ij} as $b_{ij} = \frac{T_j - T_{j-1}}{1 - P_{ij}}$. We generate several correlated uniform numbers Y_i by using the SOU method as described in Section 3.1, then compute $s_i = b_{ij}Y_i + T_{j-1}$. If s_i belongs to interval $[T_{j-1}, T_j]$, then the first passage time occurred in this interval. We set $\texttt{IsDefault}(i) = 1$ to indicate that firm i has defaulted and compute the conditional boundary crossing density $g_{ij}(s_i)$. To get the density for the entire interval $[0, T]$, we use $\widehat{f_{i,n}}(t) = \left(\frac{T_j - T_{j-1}}{1 - P_{ij}}\right) g_{ij}(s_i) * K(h_{opt}, t - s_i)$, where n is the iteration number of the Monte Carlo cycle.
2. **First passage does not happen in this interval.** If s_i does not belong to interval $[T_{j-1}, T_j]$, then the first passage time has not yet occurred in this interval.
3. **First passage happens at the right boundary of the interval.** If $X_i(T_j^+) < D_i(T_j)$ and $X_i(T_j^-) > D_i(T_j)$, then T_{I_i} is the first passage time and $I_i = j$, we evaluate the density function using kernel function $\widehat{f_{i,n}}(t) = K(h_{opt}, t - T_{I_i})$, and set $\texttt{IsDefault}(i) = 1$.

Next, we increase j and examine the next interval and analyze the above three cases for each non-default firm again. After running N times the Monte Carlo cycle, we get the FPTD of firm i as $\widehat{f_i}(t) = \frac{1}{N}\sum_{n=1}^{N}\widehat{f_{i,n}}(t)$.

3.3. Model calibration and concluding remarks on the developed methodology

We need to calibrate the developed model that is to numerically choose or optimize the parameters, and in the specific case of the credit analysis these parameters include drift, volatility and jumps to fit the most liquid market data.

As already mentioned, after Monte Carlo simulation we obtain the estimated density $\hat{f}_i(t)$ by using the kernel estimator method. Then we minimize the difference between our model and historical default data $\widetilde{A}_i(t)$ to obtain the optimized parameters in the model (such as σ_{ij}, arrival intensity λ in Eq. (5)):

$$\text{argmin}\left(\sum_i \sqrt{\sum_{t_j}\left(\frac{P_i(t_j) - \widetilde{A}_i(t_j)}{t_j}\right)^2}\right). \tag{9}$$

Note that in practice, the generated by using SOU method s_i are not obtained according to the conditional boundary crossing density $g_{ij}(s_i)$. Instead, in order to obtain an appropriate density estimate, the right hand side summation in Eq. (7) can be viewed as a finite sample estimate in a way proposed by Atiya et al [2]. For the multidimensional density estimate, we need to evaluate the joint conditional boundary crossing density. This problem can be divided into several one-dimensional density estimation subproblems if the processes are non-correlated [18]. As for the general case of multivariate correlated processes, the joint density is not available analytically and numerical approximations are necessary for different classes of special cases. Before moving to specific examples, demonstrating numerical efficiency of the developed methodology, we remark that the developed method belongs to the class of methodologies based on Brownian bridge simulations or more generally large deviations methodologies. In several special cases, recent theoretical results on estimating barrier crossing probabilities of the associated Brownian bridges are available in the literature (with upper and lower limits). With a few exceptions, most such results concern one-dimensional Brownian bridges only [16, 13] for further details on these issues.

4. Applications and discussion

Our first task is to describe the first passage time density functions (based on which we can determine default rates of the corresonding firms). Since there is no option value that can be used, we will employ Eq.(9) to optimize the parameters in our model. For convenience, we reduce the number of optimizing parameters by:

1. Setting $X_i(0) = 2$ and $\ln(\kappa_i) = 0$.
2. Setting the growth rate γ_i of debt value equivalent to the growth rate μ_i of the firm's value [20], so the default of firm is non-sensitive to μ_i. In our computations, we set $\mu_i = -0.001$.
3. The interjump times $(T_j - T_{j-1})$ satisfy an exponential distribution with mean value equal to 1.
4. The arrival rate for jumps satisfies the Poisson distribution with intensity parameter λ, where the jump size is a normal distribution $Z_i \sim N(\mu_{Z_i}, \sigma_{Z_i})$.

As a result, we only need to optimize σ_i, λ, μ_{Z_i}, σ_{Z_i} for each firm. This is done by minimizing the differences between our simulated default rates and historical data. Moreover, as mentioned above, we will use the same arrival rate λ and distribution of $(T_j - T_{j-1})$ for differently rated firms, so we first optimize four parameters for, e.g., the A-rated firm, and then set the parameter λ of other three firms the same as A's.

The minimization was performed by using the quasi-Newton procedure implemented as a Scilab program. The optimized parameters for each differently rated firm were found by using the UNIF method. In each step of the optimization we choose the Monte Carlo runs $N = 50,000$.

By using the optimized parameters, we carried out the final simulation with Monte Carlo runs $N = 500,000$. The estimated first passage time density function of these four firms are shown in Fig. 1 (top). The simulated cumulative default rates (line) together with historical data (squares) are given in Fig. 1 (bottom). The theoretical data denoted as circles in Fig. 1 (bottom) were computed by using Eq. (3) where the Z_i were evaluated in [20] as 8.06, 6.46, 3.73 and 2.10 for A-, Baa-, Ba- and B-rated firms, respectively.

From Fig. 1, we can conclude that the density functions of A- and Baa-rated firms still have the trend to increase, which means the default rates of A- and Baa-rated firms may increase little faster in future. As for Ba- and B-rated firms, their density functions have decreased, so their default rates may increase very slowly or be kept at a constant level. Mathematically speaking, the cumulative default rates of A- and Baa-rated firms are convex function, while the cumulative default rates of Ba- and B-rated firms are concave.

5. Conclusion

In this contribution, we develop a fast Monte-Carlo type procedure for the analysis of complex systems such as those occurring in the financial market. As an example, we have analyzed the credit risk problems of multiple correlated firms in a structural model framework, where we incorporated jumps to reflect the external shocks or other unpredicted events. Our new procedure was developed by combining the fast Monte-Carlo method for one-dimensional jump-diffusion processes and the generation of correlated multidimensional variates. This allows us to analyze multivariate and correlated jump-diffusion processes. The developed approach generalizes further previously

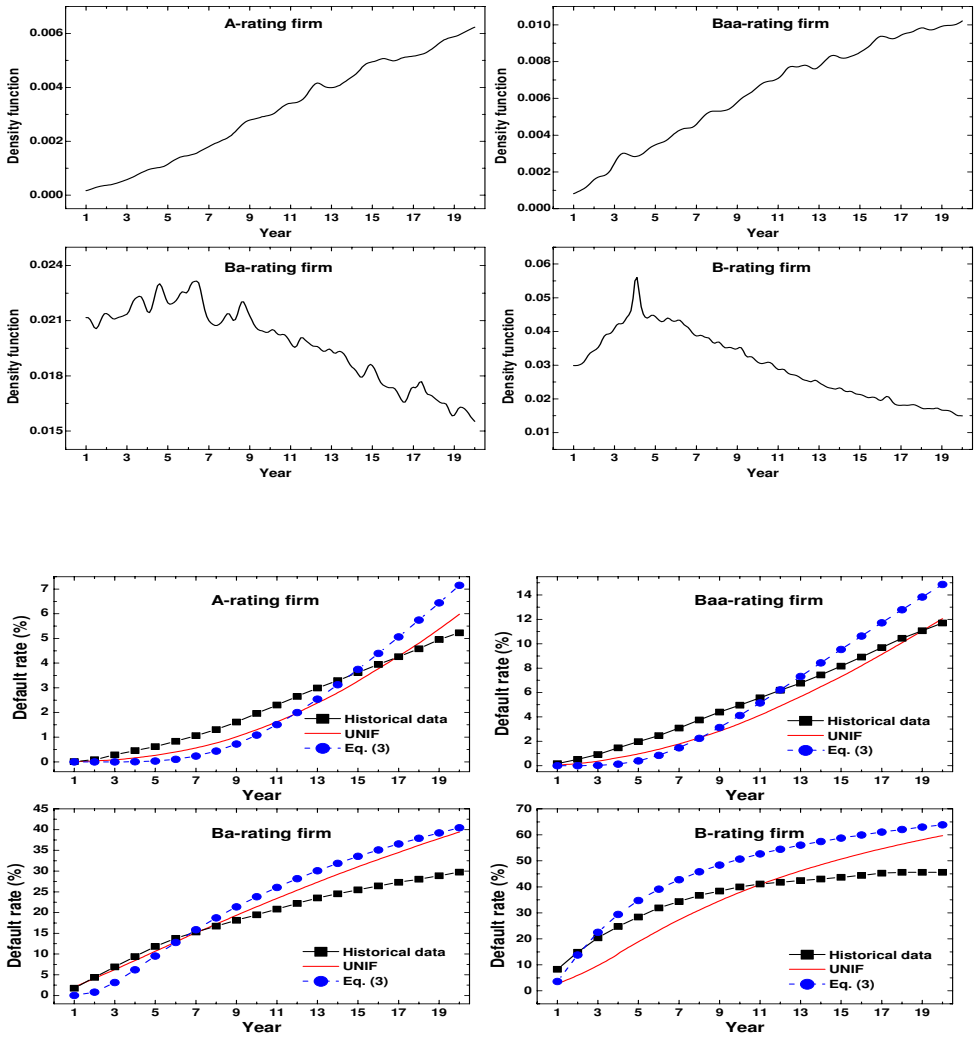

Figure 1: Estimated density function for differently rated firms with all the simulations were performed with Monte Carlo runs $N = 500,000$ (top). Historical (squares), theoretical (circles) and simulated (line) cumulative default rates for differently rated firms with all the simulations were performed with Monte Carlo runs $N = 500,000$ (bottom).

discussed non-correlated jump-diffusion cases for multivariate and correlated jump-diffusion processes. Finally, we have applied the developed technique to analyze the default events of multiple correlated firms.

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