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Substitution method: A technique to study dynamics of both non-gyroscopic and gyroscopic systems

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ABSTRACT

Although the substitution idea has been used occasionally in some engineering structures to study vibration characteristics, it is still short of systematic study in the vibrational engineering field. We summarize the techniques of the substitution method in this paper and apply such skill further to continuous and nonlinear systems, by which the dimension of the system can be reduced. By illustrated examples of frequency analysis on Timoshenko beam and composite sandwich structure, the substitution method has verified to be valid and efficient. The gyroscopic systems, represented by both rotating structures and orbits around libration points have also been presented to show the power of substitution method. Illustrated by the current examples, it is concluded that the substitution method has wide potential applications via studying the functional relations among all the degree-of-freedoms (DOFs) of non-gyroscopic or gyroscopic systems.

1. Introduction

Free vibration models of linear discrete and continuous vibratory mechanical systems have been widely used in studying vibrating engineering structures [1–5]. In discrete systems, the spatial variation of the deflection from the equilibrium position is entirely characterized by a finite number of different amplitudes. In continuous systems, the amplitude of deflection is defined by a continuous function of the spatial coordinates. Mathematically, the difference of the two types is that, in discrete systems, vibrations are expressed by ordinary differential equations (ODEs), while in continuous systems, vibrations are expressed by partial differential equations (PDEs), which are much more difficult to solve. It is noted that in engineering applications, continuous systems are often truncated to finite dimension discrete systems by using the finite element method or assumed mode truncation methods.

The purpose of free vibration analysis or modal analysis is to find the characteristic of the structures, which include the natural frequencies and the corresponding mode shapes. Traditional technique widely used in treating free vibration of linear multiple-degree-of-freedom (MDOF) discrete and continuous systems is the standard eigenvalue procedure [6]. The normal

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mode usually implies a linear functional relationship among all the DOFs during the synchronous modal motions. Rosenberg [7] expanded the linear normal mode concept to nonlinear case, by which linear or nonlinear functional relations can be determined to show the modal motions of nonlinear systems. Furthermore, some scientists [8–11] have focused on the study of nonlinear normal modes by approximate techniques, such as perturbation methods.

In general, for MDOF discrete systems, the number of degree of freedoms only indicates the number of ordinary differential equations and it does not bring complex calculation process compared with the single-DOF system. While for complex continuous systems, the increasing degree of freedoms often means not only more equations but more complicated modeling and calculation processes, which makes it easier to yield false results. It should be noted that the term DOF defined in continuous system in this study is different as defined in the discrete systems. The continuous systems are usually referred as infinite DOF systems. In this study, the dimension of continuous systems can be viewed as motion directions. A beam undergoes both torsional and bending vibrations can be called a 2-DOF continuous system.

For the linear discrete case, the substitution method is mathematically equivalent to the standard eigenvalue method with the difference that the standard eigenvalue method yields the natural frequencies first and the substitution method yields the normal modes first. The substitution idea to differential equations is similar to Gaussian elimination method treating algebraic equations. On the other hand, Lagrange multiplier method applies 'combined' but not 'elimination' idea in treating the constrained extreme value problems.

For continuous systems, the frequencies and mode shapes are usually obtained by numerical methods [12,13]. Inspired by the linear functional relations among the DOFs of a system, the substitution method is presented in this paper to simplify the modeling and analysis processes of both discrete and continuous systems.

In this study, we expand the applications of substitution method to the analysis of continuous systems, which is verified to be efficient to reduce the dimension of continuous systems. The concept of substitution is adopted to study the vibration of Timoshenko beam and composite structure. Further, the application of substitution method to gyroscopic systems is discussed.

2. Vibration of MDOF discrete systems

Free vibration of n-DOF systems is governed by

$$\mathbf{M}\ddot{\mathbf{q}}(t)+\mathbf{K}\mathbf{q}(t)=0, \tag{1}$$

where $\mathbf{M}$ and $\mathbf{K}$ are respectively mass and stiffness matrixes. It is assumed that all the DOFs $\mathbf{q}=\begin{bmatrix} q_1 & q_2 & \dots & q_n \end{bmatrix}^T$ are undergoing synchronous motion during modal vibrations: all the DOFs reach maxima and go through the equilibriums at the same instance. Hence, the solution to Eq. (1) can be written in the form of $\mathbf{q}=\mathbf{A}e^{\mathrm{i}\omega t}$, where $\mathbf{A}$ is the vector of amplitudes, ω is the natural frequency. By substituting the assumed solution back into Eq. (1), the following standard eigenvalue problem can be obtained

$$\left[\mathbf{K}-\omega^2\mathbf{M}\right]\mathbf{A}=\mathbf{0}. \tag{2}$$

The natural frequencies can then be determined by the characteristic equation of nontrivial solution condition $\left|\mathbf{K}-\omega^2\mathbf{M}\right|=\mathbf{0}$. The corresponding eigenvector $\mathbf{A}$ is referred as normal mode, or mode shape, or principal mode of vibration. It should be noted that positive definite and symmetric matrixes $\mathbf{M}$ and $\mathbf{K}$ guarantee real values of both natural frequencies and eigenvectors.

For the linear system, the functional relationship inspires another analytical technique, substitution method, to solve for the frequencies and normal modes. The functional relations among the DOFs are assumed first, and by substituting the assumed relation back into the governing equations, the dimension of the MDOF system will be reduced and further the frequencies can be obtained by such technique.

As an example of the substitution method, we consider a 2-DOF system as presented in Fig. 1, the governing equation of which is

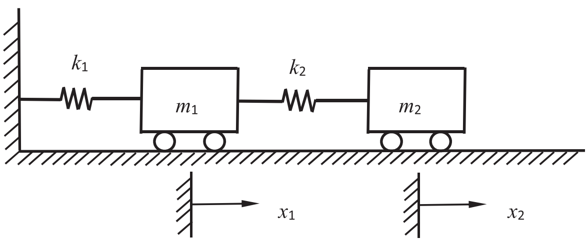

Fig. 1. Diagram of a typical 2-DOF system.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix}+\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{3}$$

where x_1, x_2 are the coordinate of the two oscillators, respectively, and m_j, k_j $(j=1{,}2)$ are the mass and elastic stiffness, respectively.

Based on the in-unison vibration concept of modal motions, the linear functional relation between the two oscillators during anyone of the modal motions is assumed as

$$x_2=\lambda x_1, \tag{4}$$

where λ is a coefficient being determined. Substituting the relation (4) back into Eq. (3) yields

$$\begin{aligned} &\ddot{x}_1+\left(\frac{k_1+k_2}{m_1}-\frac{k_2}{m_1}\lambda\right)x_1=0, \\ &\ddot{x}_1+\frac{\lambda-1}{\lambda}\frac{k_2}{m_2}x_1=0. \end{aligned} \tag{5}$$

By comparing the two equations of Eq. (5), the coefficient λ can be calculated as

$$\lambda=\frac{\frac{k_1+k_2}{m_1}-\frac{k_2}{m_2}\pm\sqrt{\left(\frac{k_1+k_2}{m_1}-\frac{k_2}{m_2}\right)^2+4\frac{k_2^2}{m_1m_2}}}{2\frac{k_2}{m_1}}. \tag{6}$$

The modal motions of the two DOFs are now denoted by the first oscillator of either equation in Eq. (5), while the second one will 'follow' the first one by the rule of functional relation (4). Usually either oscillator, which is the first oscillator in this study, can be taken as master coordinate and the remainder DOFs as slave coordinates in the nonlinear normal modes analysis [14,15]. By substituting the relation between the master coordinate and the slave coordinates, the MDOF system can reduced to single-DOF system. The natural frequencies can be obtained straightforward as $\omega^2=\frac{k_1+k_2}{m_1}-\frac{k_2}{m_1}\lambda$, or $\omega^2=\frac{\lambda-1}{\lambda}\frac{k_2}{m_2}$ in Eq. (5).

It should be noted that the coefficient λ is real for the non-gyroscopic vibrating systems and is complex for gyroscopic vibrating systems.

3. Vibration of Timoshenko beam

Compared to the Euler-Bernoulli beam model, the Timoshenko beam model takes into account shear deformation and rotational bending effects, making it suitable for describing the behavior of thick beams. Due to the fact that the normal deformation and shear deformation are not related, one more degree-of-freedom, rotating angle, is supplemented.

As presented in Fig. 2, the transverse normal does not remain perpendicular to the mid-surface after deformation. The total rotating angle ϕ of transverse normal can be assumed as superposition of the mid-surface angle $\partial w/\partial x$ and the angle caused by shear deformation θ. Hence the strains and stresses can be expressed as

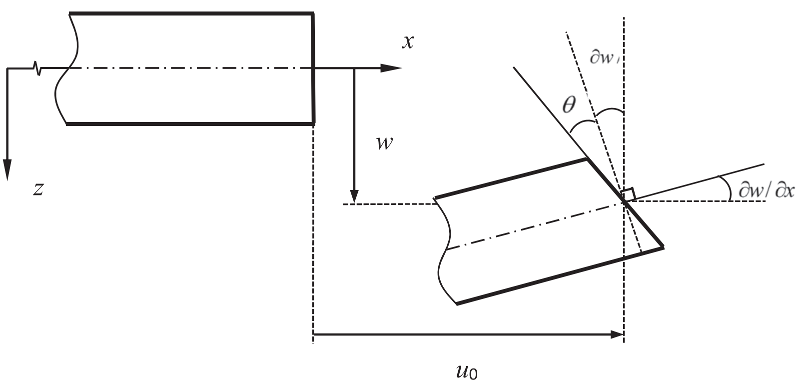

Fig. 2. Deformation diagram of the Timoshenko beam.

$$\varepsilon_x = -z\left(\frac{\partial\theta}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right), \gamma_{xz} = -\theta, \tag{7}$$

$$\sigma_x = -Ez\left(\frac{\partial\theta}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right), \tau_{xz} = -G_{xz}\theta. \tag{8}$$

The equations of motion of the Timoshenko beam will be derived by Hamilton's principle

$$\delta\int_{t_0}^{t_1}(T-U)dt = 0, \tag{9}$$

where T and U are respectively the kinetic and strain energies of the structure,

$$T = \frac{1}{2}\int_V \rho\left(\dot{u}_1^2 + \dot{w}_1^2\right)dV, \;\; U = \frac{1}{2}\int_V (\tau_{xz}\gamma_{xz} + \sigma_x\varepsilon_x)dV. \tag{10}$$

3.1. Traditional method

Traditional method deals with the transverse deflection w and the rotating angle θ in the meantime. The solutions are assumed to cast into mode vectors and generalized coordinates vectors,

$$w(x,t) = \sum_{n=1}^{N}\gamma_n(x)^T p_n(t) = \boldsymbol{\gamma}^T(x)\mathbf{p}(t), \tag{11}$$

$$\theta(x,t) = \sum_{n=1}^{N}\chi_n(x)^T g_n(t) = \boldsymbol{\chi}^T(x)\mathbf{g}(t), \tag{12}$$

in which $\mathbf{p}(t) = [p_1, ..., p_N]^T$ and $\mathbf{g}(t) = [g_1, ..., g_N]^T$ are the generalized coordinates of the beam system, and $\boldsymbol{\gamma}(x) = [\gamma_1, ..., \gamma_N]^T$ and $\boldsymbol{\chi}(x) = [\chi, ..., \chi_N]^T$ are the assumed modes which meet the geometric boundary conditions. For simply supported Timoshenko beams, the first N-term assumed modes are expressed as

$$\gamma_n(x) = \sin\frac{n\pi}{L}x, \;\; \chi_n(x) = \cos\frac{n\pi}{L}x, (n = 1, 2, 3...N) \tag{13}$$

Substituting Eqs. (7), (8) and (10)–(13) into Eq. (9) and performing the variation operation in terms of w and θ, the discretized ordinary differential equations of motion for the Timoshenko beam with length L, width b, thickness h, elastic modulus E, shear modulus G_{xz}, and density ρ will be obtained as follows

$$\mathbf{M}\ddot{\mathbf{X}}(t) + \mathbf{K}\mathbf{X}(t) = 0, \tag{14}$$

where $\mathbf{M}$ and $\mathbf{K}$ are the $2N \times 2N$ structural mass matrix and stiffness matrix, expressions of which are given in Appendix A. Symbol $\mathbf{X}(t) = [p(t)^T, g(t)^T]^T$ is the generalized coordinate vectors of the beam. The angular frequencies of the Timoshenko beam ω_n and corresponding modes can be obtained by eigenproblem solutions to such $2N$-term discretized system. Hence the shear deformation is considered, it is no doubt that two frequency spectra are detected contrary to one frequency spectrum found for Euler-Bernoulli beam model, since the N-term truncation results to $2N$ equations. The physical explanations of the supplemented frequencies and modes due the additional DOF are still a research focus in the fundamental mechanic investigations [16,17]. In the following subsection, it will be found that N-term of substitution method leads to N equations and one frequency spectrum similar to Euler-Bernoulli beam.

3.2. Substitution method

During one modal motion, the transverse deflection w and the shear angle θ behave as vibration-in-unison, hence it is reasonable to assume that there exists a relation between the two variables. The transverse deflection w is assumed as the master coordinate and rotating angle θ as the slave coordinate. According to the simply supported condition, the functional relation of the two DOFs $\theta = \lambda\frac{\partial w}{\partial x}$ can be used and then the displacements will be given as

$$u = -z(1+\lambda)\frac{\partial w}{\partial x}, w = w. \tag{15}$$

It is clear that undetermined coefficient λ is the key pivot of the substitution method.
Further, both the strains and stresses can be obtained as

$$\varepsilon_x = -z(1+\lambda)\frac{\partial^2 w}{\partial x^2}, \gamma_{xz} = -\lambda\frac{\partial w}{\partial x}, \tag{16}$$

$$\sigma_x = -Ez(1+\lambda)\frac{\partial^2 w}{\partial x^2}, \tau_{xz} = -G_{xz}\lambda\frac{\partial w}{\partial x}. \tag{17}$$

The equation of motion of the Timoshenko beam can be derived by Hamilton's principle as

$$\frac{Eh^3}{12}(1+\lambda)^2\frac{\partial^4 w}{\partial x^4} - G_{xz}h\lambda^2\frac{\partial^2 w}{\partial x^2} = \frac{\rho h^3}{12}(1+\lambda)^2\frac{\partial^2 \ddot{w}}{\partial x^2} - \rho h\ddot{w}. \tag{18}$$

For simply supported beams, the transverse deflection function can be expressed as

$$w(x,t) = \sum_{n=1}^{N} a_n \sin\frac{n\pi x}{L}e^{i\omega_n t}, \tag{19}$$

where a_n and ω_n are the amplitude and angular frequency of the transverse displacement, respectively. It should be noted that after truncation the eigenproblem will be N-term discretization, contrary to the traditional $2N$-term discretized system Eq. (14).

On the other hand, according to the force balance equation in the transverse direction, one has

$$\frac{\partial Q_{xz}}{\partial x} = b\rho\frac{\partial^2 w}{\partial t^2}, \tag{20}$$

in which

$$Q_{xz} = bh\tau_{xz} = -G_{xz}bh\lambda\frac{\partial w}{\partial x}. \tag{21}$$

Substituting Eqs. (19) and (21) into Eq. (20), the key pivot λ is then expressed as function of natural frequency

$$\lambda = -\left(\frac{L}{n\pi}\right)^2\frac{\rho\omega_n^2}{G_{xz}}. \tag{22}$$

Substituting Eqs. (19) and (22) back into Eq. (18), the angular frequencies of the simply supported Timoshenko beam ω_n can be calculated.

In order to validate the effectiveness of the substitution method, the comparison with published literature has been provided in Table 1. Reference [18] has employed discrete singular convolution method to find the numerical solution of

Table 1
Comparison of the natural frequencies via different methods.

	ω_i ($i = 1,2,3,4,5$)	ω_1	ω_2	ω_3	ω_4	ω_5
$h/L = 0.01$	Ref. [18]	3.1413	6.2810	9.4196	12.5494	15.6749
	Traditional method	3.1407	6.2813	9.4185	12.5517	15.6792
	Present method	3.1414	6.2818	9.4185	12.5507	15.6818
$h/L = 0.02$	Ref. [18]	3.1405	6.2747	9.3963	12.4995	15.5785
	Traditional method	3.1414	6.2758	9.3999	12.5077	15.5944
	Present method	3.1403	6.2759	9.4024	12.5088	15.5959
$h/L = 0.1$	Ref. [18]	3.1157	6.2314	9.2553	12.1813	14.9927
	Traditional method	3.1189	6.1133	8.9043	11.4664	13.8083
	Present method	3.1188	6.1141	8.9049	11.4669	13.8072
$h/L = 0.2$	Ref. [18]	3.0453	5.6716	7.8395	9.6571	11.2220
	Traditional method	3.0566	5.7332	7.9767	9.8774	11.5245
	Present method	3.0576	5.7334	7.9767	9.8763	11.5245

Timoshenko beam, the data of which are used as benchmark values in the current study. It is found that the present substitution method yields satisfactory results although the discretization order is only one.

By the substitution method, the displacement and rotation coupled two-dimensional equations are reduced to one-dimensional case, which makes the analysis more concise. In the next section, more complicated example is introduced to illustrate the application of substitution idea.

4. Vibration of composite sandwich structures

The substitution idea presents an alternative option in studying free vibration of composite structures, such as sandwich beams as shown in Fig. 3. The assumptions are made as follows: (1) For the thin face sheets, only bending stiffness due to elastic modulus E is considered, and the shear modulus is neglected; (2) For the core layer, only shear modulus G_{xz} is considered and the core is too weak to provide considerable bending stiffness to the sandwich beam. Therefore, the face sheets will rotate by an angle $\partial w/\partial x$ around y axis.

4.1. Traditional method

Traditional method considers both deflection w and rotation angle θ at the same time. The displacements in x-direction for different layers of the sandwich beam are given as follows:

$$\begin{aligned} u_t &= -\frac{h_c}{2}\theta - \left(z - \frac{h_c}{2}\right)\frac{\partial w}{\partial x}, \\ u_c &= -z\theta, \\ u_b &= \frac{h_c}{2}\theta - \left(z + \frac{h_c}{2}\right)\frac{\partial w}{\partial x}. \end{aligned} \tag{23}$$

where the subscripts t, b and c represent the top face sheet, bottom face sheet and the core, respectively.

The strains and stresses in different layers of the composite sandwich beam are

$$\varepsilon_t = -\frac{h_c}{2}\frac{\partial\theta}{\partial x} - \left(z - \frac{h_c}{2}\right)\frac{\partial w}{\partial x}, \gamma_c = -z\frac{\partial\theta}{\partial x}, \varepsilon_b = \frac{h_c}{2}\frac{\partial\theta}{\partial x} - \left(z + \frac{h_c}{2}\right)\frac{\partial w}{\partial x}, \tag{24}$$

$$\sigma_t = -E\left[\frac{h_c}{2}\frac{\partial\theta}{\partial x} + \left(z - \frac{h_c}{2}\right)\frac{\partial w}{\partial x}\right], \gamma_c = -G_{xz}z\frac{\partial\theta}{\partial x}, \varepsilon_b = E\left[\frac{h_c}{2}\frac{\partial\theta}{\partial x} - \left(z + \frac{h_c}{2}\right)\frac{\partial w}{\partial x}\right], \tag{25}$$

where ε and γ represent respectively the normal and shear strains, and σ and τ denote respectively the normal and shear stresses.

According to Hamilton's principle, together with assumed solutions in the form of Eqs. (11)–(13), we obtain the standard linear ordinary differential equation.

$$\mathbf{M}\ddot{\mathbf{X}}(t) + \mathbf{K}\mathbf{X}(t) = 0, \tag{26}$$

where $\mathbf{M}$ and $\mathbf{K}$ are respectively the $2N \times 2N$ structural mass matrix and stiffness matrix, which are listed in Appendix B. Therefore, the natural frequencies of the composite sandwich beam can be obtained by such eigenproblem of the $2N$-term discretized system.

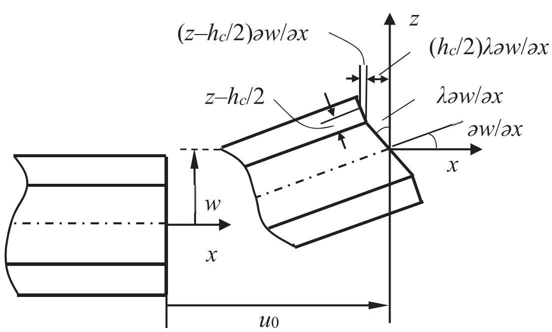

Fig. 3. Deformation diagram of a composite sandwich beam.

4.2. Substitution method

Based on the substitution idea and taking w as the master coordinate, it is assumed that the core layer will rotate by an angle $\lambda \partial w/\partial x$ around y axis, where λ is the ratio to be determined. Hence, we prescribe a functional relation between the deformations of face sheet and core layer.

By such relation, the displacements in x-direction for different layers of the composite sandwich beam are given as

$$u_t = -\left[\frac{h_c}{2}(\lambda-1)+z\right]\frac{\partial w}{\partial x}, u_c = -\lambda z\frac{\partial w}{\partial x}, u_b = \left[\frac{h_c}{2}(\lambda-1)-z\right]\frac{\partial w}{\partial x}, \tag{27}$$

where the subscripts t, c, and b represent the top face sheet, the core layer and the bottom face sheet, respectively. The strains and stresses in the different layers of the composite sandwich beam can be obtained as follows:

$$\varepsilon_t = -\left[\frac{h_c}{2}(\lambda-1)+z\right]\frac{\partial^2 w}{\partial x^2}, \gamma_c = (1-\lambda)\frac{\partial w}{\partial x}, \varepsilon_b = \left[\frac{h_c}{2}(\lambda-1)-z\right]\frac{\partial^2 w}{\partial x^2}, \tag{28}$$

$$\sigma_t = -E\left[\frac{h_c}{2}(\lambda-1)+z\right]\frac{\partial^2 w}{\partial x^2}, \tau_c = G_{xz}(1-\lambda)\frac{\partial w}{\partial x}, \sigma_b = E\left[\frac{h_c}{2}(\lambda-1)-z\right]\frac{\partial^2 w}{\partial x^2}. \tag{29}$$

The governing equations of the composite sandwich beam with equal face sheets can be established by Hamilton's principle as

$$\begin{aligned} &E\left[\frac{h_c^2 h_f}{2}(\lambda-1)^2+\frac{h_c h^2-h_c^3}{4}(\lambda-1)+\frac{h^3-h_c^3}{12}\right]\frac{\partial^4 w}{\partial x^4}-G_{xz}h_c(\lambda-1)^2\frac{\partial^2 w}{\partial x^2} \\ &= -\left(2\rho_f h_f+\rho_c h_c \ddot{w}\right)+\left\{\rho_f\left[\frac{h_c^2 h_f}{2}(\lambda-1)^2+\frac{h_c h^2-h_c^3}{4}(\lambda-1)+\frac{h^3-h_c^3}{12}\right]+\frac{\rho_c h_c^3}{12}\lambda^2\right\}\frac{\partial^2 \ddot{w}}{\partial x^2}. \end{aligned} \tag{30}$$

where h_f, h_c and h denote the thickness of the face sheets, the core layer and the overall beam, respectively.

By considering the transverse force balance, the coefficient λ is then obtained as

$$\lambda = 1-\frac{L}{n\pi}\frac{\left(2\rho_f h_f+\rho_c h_c\right)\omega_n^2}{h_c G_{xz}} \tag{31}$$

For the simply supported boundary conditions, the natural frequencies can be obtained by substituting Eq. (31) and the assumed solution Eq. (19) into Eq. (30). By the substitution method, the DOFs of the system have been reduced.

The efficiency of substitution method used in vibration analysis of the composite sandwich beams is verified by comparing with the theoretical results of Ref. [19] for the given parameter values in Table 2. The first five natural frequencies are calculated which are shown in Table 3. The relative small errors demonstrate that the substitution method is capable of analyzing vibration characteristics after introducing a suitable relation.

5. Applications to the gyroscopic systems

Synchronicity is the feature of the modal motions of general vibrating structures. Hence, the amplitudes of all the DOFs are related functionally. The idea of the substitution method is just based on such functional relations among all the DOFs. However, there exists another type of vibrating structures, gyroscopic systems, which behave not as vibration-in-unison.

For example, a mass point in a rotating frame vibrates in an out-of-unison manner with 90° phase difference: as one coordinate of the point reaches maxima and the other coordinate goes through the zero position with maximum velocity, observed from the rotating frame of reference. The substitution method is still applicable for such gyroscopic structures.

As presented in Fig. 4, the mass point m is attached to a rotating structure with velocity Ω by perpendicular arranged springs with constant k. The xy reference frame is fixed on the rotating structure. The governing equations can be obtained by the Newton's second law as

Table 2
Parameter values.

E (Gpa)	G_{xz}	ρ_f (kg/m^3)	ρ_c (kg/m^3)	h_1, h_3 (mm)	h_2 (mm)	L (mm)
210	0.022	7900	60	2	30	300

Table 3
Comparison of natural frequencies of a composite sandwich beam (Hz).

Mode	1	2	3	4	5
Substitution method	251	537	876	1285	1776
Ref. [19].	245	521	856	1266	1762

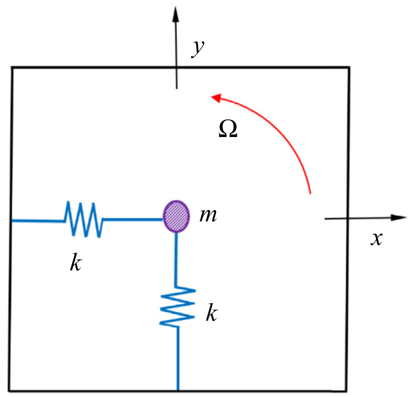

Fig. 4. The vibration of a mass in a rotating frame.

$$\begin{aligned} &m\ddot{x} - 2\Omega\dot{y} + \left(k - \Omega^2\right)x = 0, \\ &m\ddot{y} - 2\Omega\dot{x} + \left(k - \Omega^2\right)y = 0. \end{aligned} \tag{32}$$

If we substitute the relation $y = \lambda x$ into Eq. (32) and compare the two equations, we find that $\lambda = \pm \mathrm{i}$, which means the forward and backward circular motion of the mass point. The pure imaginary ratio of λ denotes the out-of-unison mode motion, during which the two degrees have a 90° phase difference. Further applying the relation $y = \pm \mathrm{i} \cdot x$ and $x = Ce^{\mathrm{i}\omega t}$ back to the first equation of Eq. (32), the natural frequencies can be obtained as

$$\omega_{1,2} = \mp\frac{\Omega}{m} + \sqrt{\left(\frac{\Omega}{m}\right)^2 + \frac{k - \Omega^2}{m}}. \tag{33}$$

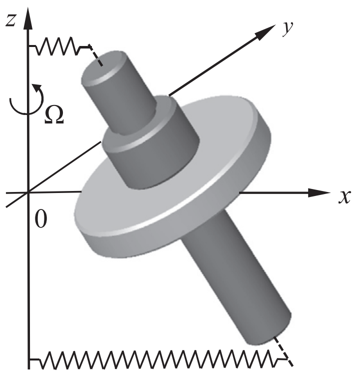

Fig. 5. Diagram of a rotor with four degrees of freedom.

Without rotation, the two natural frequencies are equal due to the symmetry of the two directions. With the introduction of the rotating velocity coupling the two directions, two different frequencies appear, which describe the forward and backward circular motions instead of the linear motions.

Further, we study the free vibration of 4-DOF rigid rotor with rotating velocity Ω as presented in Fig. 5 [20], which can be described by the following ordinary differential equations

$$\begin{aligned}
&m\ddot{X}+D_{11}X+D_{12}\varphi_y=0,\\
&m\ddot{Y}+D_{11}Y-D_{12}\varphi_x=0,\\
&J_t\ddot{\varphi}_y-J_p\dot{\varphi}_x+D_{12}X+D_{22}\varphi_y=0,\\
&J_t\ddot{\varphi}_x+J_p\dot{\varphi}_y-D_{12}Y+D_{22}\varphi_x=0,
\end{aligned} \tag{34}$$

where X and Y are respectively the two transverse displacements of the rotor, ϕ_x and ϕ_y are angles with respect to x and y axis respectively, J_p and J_t are respectively polar moment of inertia about the rotation axis and transversal moment of inertia about any axis in the rotation plane, m is the mass and D is the stiffness due to the elastic boundaries.

The linear functional relations between the four freedoms during anyone of the modal motions are assumed as

$$Y=d_1X,\ \ \phi_x=d_2\phi_y \tag{35}$$

where d_1 and d_2 are coefficients being determined. Substituting the relations (35) back into Eq. (34) yields

$$\begin{aligned}
&m\ddot{X}+D_{11}X+D_{12}\varphi_y=0,\\
&md_1\ddot{X}+D_{11}d_1X-D_{12}d_2\varphi_y=0\\
&J_t\ddot{\varphi}_y-\Omega J_pd_2\dot{\varphi}_y+D_{12}X+D_{22}\varphi_y=0,\\
&J_td_2\ddot{\varphi}_y+\Omega J_p\dot{\varphi}_y-D_{12}d_1X+D_{22}d_2\varphi_y=0,
\end{aligned} \tag{36}$$

from which d_1 and d_2 can both be calculated as $d_1=\pm\mathrm{i},\ \ d_2=\mp\mathrm{i}$, and then $Y=\pm\mathrm{i}X, \phi_x=\mp\mathrm{i}\phi_y$. Thus, 4-DOF system can be reduced to 2-DOF one as

$$\begin{aligned}
&m\ddot{X}+D_{11}X+D_{12}\varphi_y=0,\\
&J_t\ddot{\varphi}_y-\Omega J_pd_2\dot{\varphi}_y+D_{12}X+D_{22}\varphi_y=0.
\end{aligned} \tag{37}$$

Similarly, it is further assumed that $\phi_y=d_3X$, where d_3 is the coefficient to be determined, and the solution of X can be given as $X=\varsigma e^{\mathrm{i}\omega t}$. Eq. (37) can be deduced as

$$\begin{aligned}
&-m\omega^2+D_{11}+D_{12}d_3=0,\\
&-J_td_3\omega^2-\mathrm{i}\Omega J_pd_2d_3\omega+D_{12}+D_{22}d_3=0.
\end{aligned} \tag{38}$$

From the first equation of Eq. (38), d_3 is expressed as

$$d_3=\frac{m\omega^2-D_{11}}{D_{12}}, \tag{39}$$

By substituting Eq. (39) into the second equation of Eq. (38), the natural frequencies can be obtained by solving the quartic equation.

The ratios d_1 and d_2 are pure imaginary, which implies that the 90° phase difference between the two directions, deformations X and Y, angles ϕ_x and ϕ_y. On the other hand, the real value of ratio d_3 means the in-unison relation between the deformation and angle in the same direction. For the linear discrete cases including non-gyroscopic and gyroscopic systems, the substitution method is equivalent to the eigenproblem in the complex domain.

Lastly, we study another gyroscopic nonlinear system: periodic orbits around libration points. For example, the equations governing $L4$ planar periodic orbits [21,22] measured on a rotating frame are

$$\begin{aligned}
&\ddot{\xi}-g\dot{\eta}+k_1\xi=\sum_{n=2}^{\infty}\sum_{i+j+k=n,i,j,k\in N}\alpha_{ijk}\xi^i\eta^j\zeta^k,\\
&\ddot{\eta}+g\dot{\xi}+k_2\eta=\sum_{n=2}^{\infty}\sum_{i+j+k=n,i,j,k\in N}\beta_{ijk}\xi^i\eta^j\zeta^k,
\end{aligned} \tag{40}$$

where ξ, η are distances of space craft to the $L4$ point in three directions, k_1, k_2 are the linear gravity forces and the right-hand sides are the nonlinear gravity forces.

The ratio between the distances of the two directions is now complex, neither pure real nor pure imaginary. To simplify cumbersome mathematical manipulations, we consider the relations of both distances and velocities of the two directions during periodic orbit motion, which makes the ratios real by the cost of doubled undetermined coefficients. The nonlinear relations between the two directions are assumed as

$$\begin{aligned}\eta &= F_1(u,v)\\ &= a_1\xi + a_2\dot{\xi} + a_3\xi\dot{\xi} + a_4\xi^2 + a_5\dot{\xi}^2 + a_6\xi^2\dot{\xi} + a_7\xi\dot{\xi}^2 + a_8\xi^3 + a_9\dot{\xi}^3 + \ldots\\ \dot{\eta} &= F_2(u,v)\\ &= b_1\xi + b_2\dot{\xi} + b_3\xi\dot{\xi} + b_4\xi^2 + b_5\dot{\xi}^2 + b_6\xi^2\dot{\xi} + b_7\xi\dot{\xi}^2 + b_8\xi^3 + b_9\dot{\xi}^3 + \ldots\end{aligned} \tag{41}$$

where up to 3 order polynomials are used. Readers may try higher orders for more accurate results on the cost of more mathematical manipulations. By utilizing the polynomial expansions and their derivatives, associated with the governing equations, we can determine the nonlinear relations between the two directions. By substituting Eq. (41) back into any equation of (40), the 1-DOF equation in the following form can be obtained

$$\ddot{u} + S_1 u = S_2 u^2 + S_3\dot{u}^2 + S_4 u\dot{u}^2 + S_5 u^3 \tag{42}$$

where the coefficient can be found in the reference [21].

The substitution method has been verified effective in the computation of periodic orbits by reducing the nonlinear 2-DOF gyroscopic system into nonlinear 1-DOF system, which then can be solved straightforwardly by traditional analytical or numerical methods [22–24].

For the gyroscopic continua, such as axially moving structures, the substitution method can still be used. By choosing appropriate base functions, the vibrations of gyroscopic continua can be expressed on the generalized coordinates. Yang et al. [25] have found the functional relations among the base functions when studying the axially moving beams, involving imaginary ratios. The introduction of the imaginary ratios among the base functions triggers non-standing wave modal motions, equivalent to the non-unison modes found in the discrete gyroscopic systems. After the governing equations of gyroscopic continua has been casted in the generalized coordinates, the substitution method then can be used straightforwardly.

6. Conclusions

In this study, the applications of the substitution method has been investigated and summarized by some simple examples. The main idea of the substitution method is to assume relations among all the DOFs of a system, and further the system dimension can be reduced and at the same time the relations can be determined. For the linear non-gyroscopic discrete system, the substitution method is equivalent to the standard eigenproblem method. The relations among the DOFs describe exactly the modes of the system.

By studying the vibrations of Timoshenko beam and sandwich structure, it is found that the substitution method has potential to be used in the continuous system. On the other hand, the application of imaginary or complex coefficients makes possible to expand the method to gyroscopic systems. Some examples are given to show the power of such method in the study of gyroscopic systems.

In this study, the DOFs of the systems are usually less. For the systems with more DOFs, the substitution method still works. However, the coefficients to be determined in the linear relations among the DOFs will increase, which makes the procedure more cumbersome. As for the nonlinear case, the polynomial relations in the state space can be used to describe the relations, which makes the determination of the coefficient complicated if higher order accuracy is demanded.

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Appendix A

The mass matrix and stiffness matrix in Eq. (14) are given as follows:

$$
\begin{aligned}
&\mathbf{M} = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{12} & \mathbf{M}_{22} \end{bmatrix}, \mathbf{K} = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{12} & \mathbf{K}_{22} \end{bmatrix}, \\
&\mathbf{M}_{11} = b\rho \int_0^L \left(\frac{h^3}{12} \frac{\partial \boldsymbol{\eta}}{\partial x} \frac{\partial \boldsymbol{\eta}^T}{\partial x} + h\boldsymbol{\eta}\boldsymbol{\eta}^T \right) \mathrm{d}x, \mathbf{M}_{12} = b\rho \int_0^L \frac{h^3}{12} \frac{\partial \boldsymbol{\eta}}{\partial x} \boldsymbol{\chi}^T \mathrm{d}x, \\
&\mathbf{M}_{22} = b\rho \int_0^L \frac{h^3}{12} \boldsymbol{\chi}\boldsymbol{\chi}^T \mathrm{d}x, \\
&\mathbf{K}_{11} = \frac{Ebh^3}{12} \int_0^L \frac{\partial^2 \boldsymbol{\eta}}{\partial x^2} \frac{\partial^2 \boldsymbol{\eta}^T}{\partial x^2} \mathrm{d}x, \ \mathbf{K}_{12} = \frac{Ebh^3}{12} \int_0^L \frac{\partial^2 \boldsymbol{\eta}}{\partial x^2} \frac{\partial \boldsymbol{\chi}^T}{\partial x} \mathrm{d}x, \\
&\mathbf{K}_{22} = \frac{Ebh^3}{12} \int_0^L \frac{\partial \boldsymbol{\chi}}{\partial x} \frac{\partial \boldsymbol{\chi}^T}{\partial x} \mathrm{d}x + G_{xz}bh \int_0^L \boldsymbol{\chi}\boldsymbol{\chi}^T \mathrm{d}x.
\end{aligned}
\tag{A1}
$$

Appendix B

The mass matrix and stiffness matrix in Eq. (26) are given as follows:

$$
\begin{aligned}
&\mathbf{M} = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{12} & \mathbf{M}_{22} \end{bmatrix}, \mathbf{K} = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{12} & \mathbf{K}_{22} \end{bmatrix}, \\
&\mathbf{M}_{11} = \frac{2}{3} b\rho_f h_f^3 \int_0^L \frac{\partial \boldsymbol{\eta}}{\partial x} \frac{\partial \boldsymbol{\eta}^T}{\partial x} \mathrm{d}x + \left(2b\rho_f h_f + b\rho_c h_c \right) \int_0^L \boldsymbol{\eta}\boldsymbol{\eta}^T \mathrm{d}x, \\
&\mathbf{M}_{12} = \frac{1}{2} b\rho_f h_c h_f^2 \int_0^L \frac{\partial \boldsymbol{\eta}}{\partial x} \boldsymbol{\chi}^T \mathrm{d}x, \\
&\mathbf{M}_{22} = \left(\frac{1}{2} b\rho_f h_f h_c^2 + \frac{1}{12} b\rho_c h_c^3 \right) \int_0^L \boldsymbol{\chi}\boldsymbol{\chi}^T \mathrm{d}x, \\
&\mathbf{K}_{11} = \frac{2}{3} E\rho_f h_f^3 \int_0^L \frac{\partial^2 \boldsymbol{\eta}}{\partial x^2} \frac{\partial^2 \boldsymbol{\eta}^T}{\partial x^2} \mathrm{d}x + G_{xz}bh_c \int_0^L \frac{\partial \boldsymbol{\eta}}{\partial x} \frac{\partial \boldsymbol{\eta}^T}{\partial x} \mathrm{d}x, \\
&\mathbf{K}_{12} = \frac{1}{2} Ebh_f^2 h_c \int_0^L \frac{\partial^2 \boldsymbol{\eta}}{\partial x^2} \frac{\partial \boldsymbol{\chi}^T}{\partial x} \mathrm{d}x - G_{xz}bh_c \int_0^L \frac{\partial \boldsymbol{\eta}}{\partial x} \boldsymbol{\chi}^T \mathrm{d}x, \\
&\mathbf{K}_{22} = \frac{1}{2} Ebh_f h_c^2 \int_0^L \frac{\partial \boldsymbol{\chi}}{\partial x} \frac{\partial \boldsymbol{\chi}^T}{\partial x} \mathrm{d}x + G_{xz}bh_c \int_0^L \boldsymbol{\chi}\boldsymbol{\chi}^T \mathrm{d}x.
\end{aligned}
\tag{B1}
$$

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