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Optimal minimax algorithm for integrating fast oscillatory functions in two dimensions

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Abstract *In this paper, we give a complete description of efficient formulae for the numerical integration of fast oscillating functions of two variables. The focus is on the case encountered frequently in many engineering applications where an accurate value of the Lipschitz constant is not available. Using spline approximations, we demonstrate the main idea of our approach on the example of piecewise bilinear interpolation, and propose optimal-by-order (with a constant not exceeding two) cubature formulae that are applicable for a wide range of oscillatory patterns. This property makes the formulae indispensable in many engineering applications dealing with signal processing and image recognition. Illustrative results of numerical experiments are presented.*

1. Introduction

In many areas of electrical, electronic, civil, and geomechanical engineering, in particular those pertinent to signal processing and image recognition, we come to the problem of computing integrals with rapidly oscillating functions. This problem arises frequently in the context of computing integral transforms such as Fourier of Fourier-Bessel in applications of the technology dealing with transmission and processing of signals by various means, as well as in many other areas of engineering applications (Alaylioglu, 1983; Brachman and Moore, 1999; Ersoy, 1997; Haider and Liu, 1992; Zhileikin and Kukarkin, 1995). An effective solution of this problem is paramount not only for technological innovations related to signal processing, but also in such diverse areas as image recognition, the solution of boundary value problems, modelling optical and automated control systems, nuclear physics, crystallography, scattering theory, and diffraction problems (Blahut, 1987; Chan and Ho, 1991; Drachman and Ross, 1994; Escobar *et al.*, 1997; Hopkins, 1989; Zheludev, 1995).

While there are important applied problems that require dealing with infinite domains (Brachman and Moore, 1999; Davies, 1989; Fettis and Pexton, 1982), a large proportion of problems in signal and image processing, solution of boundary value problems and in other areas of applications have to deal with finite domains of integration (Johnston and Elliott, 2000; Pieper, 1999) and references therein), and on a number of occasions this requires to go to dimensions higher than one. This paper complements a series of papers (Melnik and Melnik, 1998, 1999, 2001). In Melnik and Melnik (1999) we derived rigorously *optimal-by-order quadrature formulae* for integration of fast oscillatory functions in interpolational classes. The purpose of this paper is to consider algorithmic issues of the problem most important in engineering

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applications, and to show how our previous results are extended to the multivariate case.

In what follows we describe the basics of an efficient algorithmic procedure for the construction of optimal (in the specified sense) formulae for numerical integration of fast oscillatory functions of two variables. In order to explain the difficulties arising in integrating rapidly oscillating functions, we start from the one-dimensional case. Consider the product $f(x)\exp(-i\omega x)$ on an interval (a, b), where $\omega(b-a) \gg 1$ and $f(x)$ is a smooth function. Both functions, $\mathcal{R}e(f(x)\exp(-i\omega x))$ and $\mathcal{I}m(f(x)\exp(-i\omega x))$, have on the interval (a, b) approximately $\omega(b-a)/\pi$ zeros. Therefore, if we would like to approximate such functions by polynomials we need polynomials of degree $n \gg \omega(b-a)/\pi$ (indeed, a polynomial of degree n has no more than n zeros). This is not only impractical in many applications but also may lead to instability of computation (Haider and Liu, 1992). In many practical situations it is more reasonable to treat the oscillating factor $\exp(-\omega ix)$ (the same can be said about $\sin(\omega x)$ or $\cos(\omega x)$) as a weight function. The idea of taking an oscillating factor into account in quadrature formula coefficients has been extensively developed since Filon's work (see, for example, Alaylioglu (1983), Filon (1928), Levin (1996) and references therein). Since information about the integrand often comes from measurements or experiments it is important to be able to optimise the use of information available by using optimal in some sense formulae for numerical integration. In the one-dimensional case optimal formulae for numerical integration of fast oscillatory functions when *a priori* information about the integrand is given inaccurately were recently constructed in Melnik and Melnik (1998, 1999).

In the two-dimensional case the difficulties in numerical integration of fast oscillatory functions essentially increase. Different ways to overcome such difficulties were discussed by many authors (see, for example, Hopkins (1989), Mysovskih (1992) and Zhileikin and Kukarkin (1995). A recent survey on the construction of cubature formulae to approximate multivariate integrals can be found in Cools (1997). However, results on optimal cubature formulae when *a priori* information is given inaccurately are still lacking in the literature. Consider the problem of computing integrals

$$I^n(f) = \int_{a_1}^{b_1} \ldots \int_{a_n}^{b_n} f(x_1,\ldots,x_n)\varphi_1(x_1)\ldots\varphi_n(x_n)\,\mathrm{d}x_1\ldots\mathrm{d}x_n, \tag{1.1}$$

where $\varphi_k(x_k)$, $k = 1,\ldots,n$ are known integrable functions and $f(x_1, x_2, \ldots, x_n)$ belongs to some predefined functional class F. From a mathematical point of view the only *a priori* information available about function f is the knowledge that $f \in F$ (Sukharev, 1992). On frequent occasions, this limited information becomes a starting point for formal mathematical analysis and numerical approximations to equation (1.1). If, however, information about function f comes from measurements or experiments it is quite natural to narrow the given class F by assuming only that function f is given by a fixed table of its values $f(X_1) = f_1, \ldots, f(X_K) = f_K$ in K fixed points $\{X_i\}_{i=1}^{K}$ from its domain of definition Ω. Apart from the fact that such a consideration put us closer to realistic situations, it allows us to improve the quality of formulae for numerical integration by the maximal use of available information about the function (Melnik and Melnik, 1999). A new narrowed class, denoted here as F_K, is an interpolational class playing a key role in the solution of many practical problems of signal processing.

In what follows, we explain the basic idea of our approach that leads to the notion of optimality-by-order for cubature formulae. We consider the integral

$$I^2(f) = \int_a^b \int_c^d f(x_1, x_2)\varphi_1(x_1)\varphi_2(x_2)\, \mathrm{d}x_1\, \mathrm{d}x_2, \tag{1.2}$$

where $f \in F_K$. In many applications a quite natural class to consider is $F_K \equiv C^2_{1,L,N\times M}$, where superindex 2 indicates that the functions of this class are functions of two variables. These (continuous) functions have piecewise continuous first derivatives (indicated by subindex 1) bounded by constant L. More precisely, interpolational class $C^2_{1,L,N\times M}$ is defined as a class of bivariate functions defined on $\Omega = [a,b] \times [c,d]$, given by their fixed values f_{ij} at $K = N \times M$ nodes $(x_{1,i}; x_{2,j})$ of an arbitrary grid on Ω with $i = 1,\ldots,N$, $j = 1,\ldots,M$, and such that

$$\sup_{(x_1,x_2)\in\Omega} \max(|f'_{x_1}|, |f'_{x_2}|) \leq L. \tag{1.3}$$

All results in this paper are formulated for this interpolational class. Finally, it is important to note that numerical approximations to equation (1.2) derived in the next sections do not rely on any *a priori* estimates of the Lipschitz constant L and all results obtained in this paper are valid for any arbitrary constant L.

The paper is organized as follows. A minimax definition of optimality for problems of numerical integration is given in Section 2. In Section 3 we derive optimal-by-order cubature formulae using spline-approximations. In Section 4 we consider algorithmic aspects of computing estimates for Fourier transforms of two-variable functions. Numerical results are also presented in this section, and finally, in Section 5, Conclusions and future directions are discussed.

2. Minimax optimality in numerical integration for signal processing

We aim at the construction of cubature formulae for numerical integration of equation (1.2) that give the best result for the worst function in the class. This minimax concept of optimality in the theory of numerical algorithms (sometimes called the pure strategy of decision making) goes back to Chebyshev's work and is the most natural when all *a priori* information about the problem is contained in the fact that $f \in F_K$. Since we believe that this concept is best suited to most problems of signal processing and image recognition leading to equation (1.2), other concepts of optimality (i.e. the sequential strategy of decision making, concepts of optimality based on probabilistic approaches) are not discussed here. For further insight refer Heinrich *et al.* (2003), Hickernell (2003), Sukharev (1992) and Traub and Wozniakowski (1980).

If we denote by $r(F_K, A, f)$ the result of application of algorithm A to function f, then the error of integration of this function by A is

$$v(F_K, A, f) = |I^2(f) - r(F_K, A, f)|. \tag{2.1}$$

As the worst function in class F_K we take a function that provides

$$\sup_{f\in F_k} v(F_K, A, f)$$

for the given algorithm A on the set $\mathscr{M}$ of all cubature formulae that use information consisting of the definition of class F_K. We consider the following minimax characteristic of the algorithm quality

$$\delta(F_K) = \inf_{A \in \mathscr{M}} \sup_{f \in F_K} v(F_K, A, f). \tag{2.2}$$

Now we are in a position to give the formal definition of optimal algorithms for numerical integration of equation (1.2).

Definition 2.1. A cubature formula on which $\delta(F_K)$ is achieved (provided it exists) is called optimal-by-accuracy for the given class F_K. If for a cubature formula A^0

$$v(F_N, A^0, f) \leq \delta(F_N) + \eta, \quad \eta \geq 0, \tag{2.3}$$

then A^0 is called an optimal cubature formula on the class F_K with accuracy up to η. If $\eta = o(\delta(F_K))$ or $\eta = O(\delta(F_K))$, then A^0 is called an asymptotically optimal or optimal-by-order cubature formula, respectively.

In a quite general setting, the construction of numerical integration algorithms for equation (1.2) can be thought as the solution of a problem $\mathscr{P}(I, S)$ with input dataset $I \in \mathscr{I}$ and output dataset $S \in \mathscr{S}$ where $\mathscr{I}$ and $\mathscr{S}$ are certain metric spaces with metrics ρ_1 and ρ_2, respectively. Whenever instead of I an approximate input dataset $\tilde{I}$ is given (which is typical in the majority of applications), we have to deal with an uncertainty domain $\mathscr{D}_1$ induced by the approximate nature of input data. It is important to underline that the uncertainty domain $\mathscr{D}_1 \subset \mathscr{I}$ always gives rise to an uncertainty domain of output dataset $\mathscr{D}_2 \subset \mathscr{S}$. This set $\mathscr{D}_2$ becomes the most complete characteristic of the problem solution. In principle any element of the uncertainty domain $\mathscr{D}_2$ can be considered as the solution of the problem (1.2). Given *a priori* inaccurate information, as an optimal solution of the problem, it is reasonable to choose a point for which the maximum distance along $\mathscr{D}_2$ is the minimal among all points in $\mathscr{S}$. If $\mathscr{D}_2$ is a bounded set in the metric space ($\mathscr{S}$, ρ_2), then an element $x_0 \in \mathscr{S}$ for which

$$\sup_{y \in \mathscr{D}_2} \rho_2(x_0, y) = \inf_{x \in \mathscr{S}} \sup_{y \in \mathscr{D}_2} \rho_2(x, y) \tag{2.4}$$

is known to be the Chebyshev center of $\mathscr{D}_2$ and the quantity (2.4) is the Chebyshev radius of this set. If $\mathscr{D}_2$ is unbounded further *a priori* information on the location of the solution set in $\mathscr{S}$ has to be sought (Melnik and Melnik, 1999).

For the interpolational class $F_K = C^2_{1,L,N\times M}$, considered in this paper, the quantity (see also (Melnik and Melnik, 1999; Sukharev, 1992))

$$I^*(F_K) = -\frac{1}{2}(I^+(F_K) + I^-(F_K)) \tag{2.5}$$

is taken as optimal-by-accuracy value of integral $I^2(f)$, where

$$I^+(F_K) = \sup_{f \in F_K} I^2(f), \quad I^-(F_K) = \inf_{f \in F_K} I^2(f) \tag{2.6}$$

are, respectively, the upper and lower limits of the set of possible values of integrals (1.2) in the domain of integration on the functions of class F_K. Then

$$\delta(F_K) = \frac{1}{2}(I^+(F_K) - I^-(F_K)). \tag{2.7}$$

In this case $I^*(F_K)$ is the Chebyshev center of uncertainty domain of values $I^2(f)$ on class F_K. The Chebyshev radius of this domain coincides with $\delta(F_K)$.

3. Cubature formulae for fast oscillating functions via spline-based approximations

Compared to the one-dimensional case (Melnik and Melnik, 1999), constructing efficient numerical approximations for integrating fast oscillating functions of two variables is a much more involved procedure (Hopkins, 1989). In this section, we derive simple and efficient cubature formulae that do not require the explicit value of the Lipschitz constant L from equation (1.3).

The close connection of spline approximations with the problems of numerical integration can be traced back to classical works of Kolmogorov, Nikolskii and others (see, for example, Ahlberg *et al.*, 1967; Korneichuk *et al.*, 1996; Nikolskii, 1964; Nurnberger, 1989). In the context of numerical integration of fast oscillatory functions, one of the first works on applications of spline-based approximations was due to Einarson (1968). It was also shown that spline-based approximations in numerical integration can lead to optimal-by-accuracy and optimal-by-order quadrature formulae which were obtained for some important functional classes (Berezovskii and Ivanov, 1977; Melnik and Melnik, 1998, 1999; Zadiraka and Abatov, 1991; Zadiraka and Kasenov, 1986). However, the results in this direction are typically limited to the one-dimensional case. In spite of their potential in applications to many important problems of geomechanical, civil, electrical, electronic engineering pertinent to signal processing and image recognition as well as to other areas, explicit forms of such optimal-by-order formulae for numerical integration of fast oscillating functions in the two-dimensional case have not been reported in the literature.

In what follows we assume that the accuracy of the definition of function $f(x_1, x_2) \in F_K$ in an $N \times M$ grid of nodes is known (for example, as a result of measurements). In the case where $F_K = C^2_{1,L,N\times M}$ it is reasonable to apply integration formulae obtained by the residual minimization method (Morozov, 1984; TiKhonov *et al.*, 1995, 1996) where the value of L is not involved. Let us replace the integrand $f(x_1, x_2) \in C^2_{1,L,N\times M}$ in (1.2) by the bilinear spline $S(x_1, x_2)$ of the following form

$$S(x_1, x_2) = (1-u)((1-t)f_{i,j} + tf_{i+1,j}) + u((1-t)f_{i,j+1} + tf_{i+1,j+1}), \tag{3.1}$$

where

$$(x_1, x_2) \in \Omega_{i,j}, \quad \Omega_{i,j} = [x_{1,i}, x_{1,i+1}] \times [x_{2,j}, x_{2,j+1}] \subset \Omega, \tag{3.2}$$

and

$$t = \frac{x_1 - x_{1,i}}{x_{1,i+1} - x_{1,i}}, \quad u = \frac{x_2 - x_{2,j}}{x_{2,j+1} - x_{2,j}}, \quad i = 1, \ldots, N-1, \quad j = 1, \ldots, M-1. \tag{3.3}$$

From Berezovskii and Nechiporenko (1991) it follows that $S(x_1, x_2) \in C^2_{1,L,N\times M}$. Therefore, the cubature formula

$$\tilde{R}(S) = \int_a^b \int_c^d S(x_1, x_2)\varphi_1(x_1)\varphi_2(x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 \tag{3.4}$$

is optimal-by-order with a constant not exceeding 2 (Melnik and Melnik, 1998, 1999; Traub and Wozniakowski, 1980). This result is independent of the mutual arrangement of nodes of an arbitrary of nodes of an arbitrary grid on Ω and points where functions $\varphi_1(x_1)$, $\varphi_2(x_2)$ change their signs in Ω.

Important special cases of integrals (1.2) include those of the following form

$$\begin{Bmatrix} I_{\mathrm{s}}^2(f) \\ I_{\mathrm{c}}^2(f) \end{Bmatrix} = \int_a^b \int_c^d f(x_1, x_2) \begin{Bmatrix} \sin(\omega x_1)\sin(\omega x_2) \\ \cos(\omega x_1)\cos(\omega x_2) \end{Bmatrix} \mathrm{d}x_1\,\mathrm{d}x_2. \tag{3.5}$$

For the sine and cosine integrals in equation (3.5), the cubature formula (3.4), constructed using a bilinear spline (3.1)-(3.3), has the form

$$\begin{aligned}
\begin{Bmatrix} \tilde{R}_2(\omega, S) \\ \tilde{R}_3(\omega, S) \end{Bmatrix} &= \sum_{i=1}^{N-1}\sum_{j=1}^{M-1} \frac{(f_{i,j} + f_{i+1,j+1} - f_{i,j+1} - f_{i+1,j})}{\omega^4 (x_{1,i+1} - x_{1,i})(x_{2,j+1} - x_{2,j})} \left(\begin{Bmatrix} \sin(\omega x_{1,i+1}) \\ \cos(\omega x_{2,j+1}) \end{Bmatrix} \right. \\
&\quad \left. - \begin{Bmatrix} \sin(\omega x_{1,i}) \\ \cos(\omega x_{2,j}) \end{Bmatrix} \right) \times \left(\begin{Bmatrix} \sin(\omega x_{2,j+1}) \\ \cos(\omega x_{2,j+1}) \end{Bmatrix} - \begin{Bmatrix} \sin(\omega x_{2,j}) \\ \cos(\omega x_{2,j}) \end{Bmatrix} \right) \\
&\quad + \sum_{i=1}^{N-1} \frac{\left(\begin{Bmatrix} \sin(\omega x_{1,i+1}) \\ \cos(\omega x_{1,i+1}) \end{Bmatrix} - \begin{Bmatrix} \sin(\omega x_{1,i}) \\ \cos(\omega x_{1,i}) \end{Bmatrix} \right)}{\omega^3 (x_{1,i+1} - x_{1,i})} \\
&\quad \times \left((f_{i+1,1} - f_{i,1}) \begin{Bmatrix} \cos(\omega c) \\ -\sin(\omega c) \end{Bmatrix} - (f_{i+1,M} - f_{i,M}) \begin{Bmatrix} \cos(\omega d) \\ -\sin(\omega d) \end{Bmatrix} \right) \\
&\quad + \sum_{j=1}^{M-1} \frac{\left(\begin{Bmatrix} \sin(\omega x_{2,j+1}) \\ \cos(\omega x_{2,j+1}) \end{Bmatrix} - \begin{Bmatrix} \sin(\omega x_{2,j}) \\ \cos(\omega x_{2,j}) \end{Bmatrix} \right)}{\omega^2 (x_{2,j+1} - x_{2,j})} \\
&\quad \times \left((f_{1,j+1} - f_{1,j}) \begin{Bmatrix} \cos(\omega a) \\ -\sin(\omega a) \end{Bmatrix} - (f_{N,j+1} - f_{N,j}) \begin{Bmatrix} \cos(\omega b) \\ -\sin(\omega b) \end{Bmatrix} \right) \\
&\quad + \frac{1}{\omega^2} \left(f_{1,1} \begin{Bmatrix} \cos(\omega a) \\ \sin(\omega a) \end{Bmatrix} \begin{Bmatrix} \cos(\omega c) \\ \sin(\omega c) \end{Bmatrix} + f_{N,M} \begin{Bmatrix} \cos(\omega b) \\ \sin(\omega b) \end{Bmatrix} \begin{Bmatrix} \cos(\omega d) \\ \sin(\omega d) \end{Bmatrix} \right. \\
&\quad \left. - f_{N,1} \begin{Bmatrix} \cos(\omega b) \\ \sin(\omega b) \end{Bmatrix} \begin{Bmatrix} \cos(\omega c) \\ \sin(\omega c) \end{Bmatrix} - f_{1,M} \begin{Bmatrix} \cos(\omega a) \\ \sin(\omega a) \end{Bmatrix} \begin{Bmatrix} \cos(\omega d) \\ \sin(\omega d) \end{Bmatrix} \right).
\end{aligned} \tag{3.6}$$

The formula (3.6) holds for any arbitrary nonuniform grid constructed on Ω. In the special case of a uniform grid with steps h_i in the x_i-direction ($i = 1, 2$), cubature formula (3.6) takes the form

$$\begin{Bmatrix}\tilde{\tilde{R}}_2(\omega,S)\\ \tilde{\tilde{R}}_3(\omega,S)\end{Bmatrix}=\frac{4\sin(\omega h_1/2)\sin(\omega h_2/2)}{\omega^4 h_1 h_2}\sum_{i=1}^{N-1}\sum_{j=1}^{M-1}(f_{i,j}+f_{i+1,j+1}-f_{i+1,j}-f_{i,j+1})$$
$$\times\begin{Bmatrix}\cos\left(\omega\left(i+\frac{1}{2}\right)h_1\right)\\ \sin\left(\omega\left(i+\frac{1}{2}\right)h_1\right)\end{Bmatrix}\begin{Bmatrix}\cos\left(\omega\left(j+\frac{1}{2}\right)h_2\right)\\ \sin\left(\omega\left(j+\frac{1}{2}\right)h_2\right)\end{Bmatrix}+\frac{2\sin(\omega h_1/2)}{\omega^3 h_1}$$
$$\times\sum_{i=1}^{N-1}\begin{Bmatrix}\cos\left(\omega\left(i+\frac{1}{2}\right)h_1\right)\\ \sin\left(\omega\left(i+\frac{1}{2}\right)h_1\right)\end{Bmatrix}\times\left((f_{i+1,1}-f_{i,1})\begin{Bmatrix}\cos(\omega c)\\ \sin(\omega c)\end{Bmatrix}\right.$$
$$\left.-(f_{i+1,M}-f_{i,M})\begin{Bmatrix}\cos(\omega d)\\ \sin(\omega d)\end{Bmatrix}\right)+\frac{2\sin(\omega h_2/2)}{\omega^3 h_2}\sum_{j=1}^{M-1}\begin{Bmatrix}\cos\left(\omega\left(j+\frac{1}{2}\right)h_2\right)\\ \sin\left(\omega\left(j+\frac{1}{2}\right)h_2\right)\end{Bmatrix}$$
$$\times\left((f_{1,j+1}-f_{1,j})\begin{Bmatrix}\cos(\omega a)\\ \sin(\omega a)\end{Bmatrix}-(f_{N,j+1}-f_{N,j})\begin{Bmatrix}\cos(\omega b)\\ \sin(\omega b)\end{Bmatrix}\right)$$
$$+\frac{1}{\omega^2}\left(f_{1,1}\begin{Bmatrix}\cos(\omega a)\\ \sin(\omega a)\end{Bmatrix}\begin{Bmatrix}\cos(\omega c)\\ \sin(\omega c)\end{Bmatrix}+f_{N,M}\begin{Bmatrix}\cos(\omega b)\\ \sin(\omega b)\end{Bmatrix}\begin{Bmatrix}\cos(\omega d)\\ \sin(\omega d)\end{Bmatrix}\right)$$
$$-f_{N,1}\begin{Bmatrix}\cos(\omega b)\\ \sin(\omega b)\end{Bmatrix}\begin{Bmatrix}\cos(\omega c)\\ \sin(\omega c)\end{Bmatrix}\ -f_{1,M}\begin{Bmatrix}\cos(\omega a)\\ \sin(\omega a)\end{Bmatrix}\begin{Bmatrix}\cos(\omega d)\\ \sin(\omega d)\end{Bmatrix}. \tag{3.7}$$

We note that formulae (3.6) and (3.7) contain all available *a priori* information about the problem. As an example, in the next section we use these formulae for computing approximations to Fourier transforms.

4. Computing approximations to Fourier transforms in the two-dimensional case

In this section, we consider functions that have bounded first derivatives and that given by their exact values in $K = M \times N$ nodes of a uniform grid. The need in such a consideration arises by a number of applications in signal processing and image recognition, where it is important to compute, different values $\tilde{I}_c(\omega_{k_1}, \omega_{k_2})$, $\tilde{I}_s(\omega_{k_1}, \omega_{k_2})$, approximating sin- and cos-Fourier transformations:

$$I_s(\omega_{k_1}, \omega_{k_2}) = \int_a^b \int_c^d f(x,y) \sin(\omega_{k_1} x) \sin(\omega_{k_2} y)\, dx\, dy, \tag{4.1}$$

$$I_c(\omega_{k_1}, \omega_{k_2}) = \int_a^b \int_c^d f(x,y) \cos(\omega_{k_1} x) \cos(\omega_{k_2} y)\, dx\, dy, \tag{4.2}$$

where $\omega_{k_1} = 2\pi k_1/(b-a)$, $\omega_{k_2} = 2\pi k_2/(d-c)$, $k_p = 1,\ldots,M_p - 1$, $M_p = 2^{m_p} + 1$ $(M_1 \equiv N, M_2 \equiv M)$, and $m_p \geq 3$ $(p = 1,2)$ are integer numbers. As in Section 3, we assume that function $f(x,y)$ is finite in the domain $\Omega = [a,b] \times [c,d]$ and that $f(x_i, y_j) = f_{ij}$ where f_{ij} are given real numbers, (x_i, y_j) are nodes of the uniform grid $\Delta = \Delta_x \times \Delta_y$, $i = 1,\ldots,M_1$, and $j = 1,\ldots,M_2$ with

$$\Delta_x = \{a = x_1 < x_2 < \ldots < x_{M_1} = b\}, \quad \Delta_y = \{c = y_1 < y_2 < \ldots < y_{M_2} = d\}, \tag{4.3}$$

The algorithm for computing K approximate values of (4.1) and (4.2) is based on the results of equations in Section 3. First, note that, given the input data M_1, M_2, a, b, c, d, $\{x_i\}$, $\{y_i\}$, $\{f_{ij}\}$, $i = 1,\ldots,M_1$ and $j = 1,\ldots,M_2$, all auxiliary constants are found in a straightforward manner. These include frequencies $\omega_{k_1} = 2\pi k_1/(b-a)$ and $\omega_{k_2} = 2\pi k_2/(d-c)$, where $k_p = 1,\ldots,M_i - 1$, $p = 1,2$, steps $h_1 = (b-a)/(M_1 - 1)$, $h_2 = (d-c)/(M_2 - 1)$, and the auxiliary values $\{\sin(\omega_{k_1} x_i)\}$, $\{\cos(\omega_{k_1} x_i)\}$ and $\{\sin(\omega_{k_2} y_j)\}$, $\{\cos(\omega_{k_2} y_j)\}$. Having that in mind, we are in a position to execute the loop with respect to the values of $k_p = 1,\ldots,M_i - 1$, and $p = 1,2$ as follows.

Algorithm 4.1. Compute values $\bar{S}_1(\omega_{k_1}, \omega_{k_2})$ by the formula

$$\begin{aligned}\bar{S}_1(\omega_{k_1}, \omega_{k_2}) = {} & \frac{1}{\omega_{k_1}^2 \omega_{k_2} h_1} \sum_{i=1}^{M_1 - 1} (\sin(\omega_{k_1} x_{i+1}) - \sin(\omega_{k_1} x_i)) \\ & \times ((f_{i+1,1} - f_{i,1}) \cos(\omega_{k_2}, c) - (f_{i+1,M_2} - f_{i,M_2}) \cos(\omega_{k_2}, d));\end{aligned} \tag{4.4}$$

Compute values $\bar{S}_2(\omega_{k_1}, \omega_{k_2})$ by the formula

$$\begin{aligned}\bar{S}_2(\omega_{k_1}, \omega_{k_2}) = {} & \frac{1}{\omega_{k_1} \omega_{k_2}^2 h_2} \sum_{j=1}^{M_2 - 1} (\sin(\omega_{k_2} y_{j+1}) - \sin(\omega_{k_2} y_j)) \\ & \times ((f_{1,j+1} - f_{1,j}) \cos(\omega_{k_1} a) - (f_{M_1 j+1} - f_{M_1 j}) \cos(\omega_{k_1}, b));\end{aligned} \tag{4.5}$$

Compute values $\bar{S}_3(\omega_{k_1}, \omega_{k_2})$ by the formula

$$\begin{aligned}\bar{S}_3(\omega_{k_1}, \omega_{k_2}) = {} & \frac{1}{\omega_{k_1} \omega_{k_2}^2 h_1 h_2} \sum_{i=1}^{M_1 - 1} \sum_{j=1}^{M_2 - 1} (f_{ij} + f_{i+1,j+1} - f_{i,j+1} - f_{i+1,j}) \\ & \times (\sin(\omega_{k_1} x_{i+1}) - \sin(\omega_{k_1} x_i))(\sin(\omega_{k_2} y_{j+1}) - \sin(\omega_{k_2} y_j));\end{aligned} \tag{4.6}$$

Compute estimates $\bar{R}(\omega_{k_1}, \omega_{k_2})$ for the sin Fourier transform by the formula

$$\bar{R}(\omega_{k_1},\omega_{k_2}) = \bar{S}_3(\omega_{k_1},\omega_{k_2}) + \bar{S}_1(\omega_{k_1},\omega_{k_2}) + \bar{S}_2(\omega_{k_1},\omega_{k_2})$$

$$+\frac{1}{\omega_{k_1}\omega_{k_2}}(f_{1,1}\cos(\omega_{k_1}a)\cos(\omega_{k_2}c) + (f_{M_1,M_2}\cos(\omega_{k_1}b)\cos(\omega_{k_2}d) \tag{4.7}$$

$$-(f_{M_1,1}\cos(\omega_{k_1}b)\cos(\omega_{k_2}c) - (f_{1,M_2}\cos(\omega_{k_1}a)\cos(\omega_{k_2}d));$$

Compute values $\bar{\bar{S}}_1(\omega_{k_1},\omega_{k_2})$ by the formula

$$\bar{\bar{S}}_1(\omega_{k_1},\omega_{k_2}) = \frac{1}{\omega_{k_1}^2\omega_{k_2}h_1}\sum_{i=1}^{M_1-1}(\cos(\omega_{k_1}x_{i+1}) - \cos(\omega_{k_1}x_i))$$
$$\times((f_{i,1} - f_{i+1,1})\sin(\omega_{k_2}c) - (f_{i,M_2} - f_{i+1,M_2})\sin(\omega_{k_2}d)); \tag{4.8}$$

Compute values $\bar{\bar{S}}_2(\omega_{k_1},\omega_{k_2})$ by the formula

$$\bar{\bar{S}}_2(\omega_{k_1},\omega_{k_2}) = \frac{1}{\omega_{k_1}\omega_{k_2}^2h_2}\sum_{j=1}^{M_2-1}(\cos(\omega_{k_2}y_{j+1}) - \cos(\omega_{k_2}y_j))$$
$$\times((f_{1,j} - f_{1,j+1})\sin(\omega_{k_1}a) - (f_{M_1,j} - f_{M_1,j+1})\sin(\omega_{k_1}b)); \tag{4.9}$$

Compute values $\bar{\bar{S}}_3(\omega_{k_1},\omega_{k_2})$ by the formula

$$\bar{\bar{S}}_3(\omega_{k_1},\omega_{k_2}) = \frac{1}{\omega_{k_1}\omega_{k_2}^2h_1h_2}\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1}(f_{ij} + f_{i+1,j+1} - f_{i,j+1} - f_{i+1,j})$$
$$\times(\cos(\omega_{k_1}x_{i+1}) - \cos(\omega_{k_1}x_i))(\cos(\omega_{k_2}y_{j+1}) - \cos(\omega_{k_2}y_j)); \tag{4.10}$$

Compute estimates $\bar{\bar{R}}(\omega_{k_1},\omega_{k_2})$ for the sin Fourier transform by the formula

$$\bar{\bar{R}}(\omega_{k_1},\omega_{k_2}) = \bar{\bar{S}}_3(\omega_{k_1},\omega_{k_2}) + \bar{\bar{S}}_1(\omega_{k_1},\omega_{k_2}) + \bar{\bar{S}}_2(\omega_{k_1},\omega_{k_2})$$

$$+\frac{1}{\omega_{k_1},\omega_{k_2}}(f_{1,1}\sin(\omega_{k_1}a)\sin(\omega_{k_2}c) + f_{M_1,M_2}\sin(\omega_{k_1}b)\sin(\omega_{k_2}d) \tag{4.11}$$

$$-f_{M_1,1}\sin(\omega_{k_1}b)\sin(\omega_{k_2}c) - f_{1,M_2}\sin(\omega_{k_1}a)\sin(\omega_{k_2}d));$$

Output data $\{\omega_{k_1}\}$, $\{\omega_{k_2}\}$, $\bar{R}(\omega_{k_1},\omega_{k_2})$, $\bar{\bar{R}}(\omega_{k_1},\omega_{k_2})$.

We note that for computing $\bar{S}_1(\omega_{k_1},\omega_{k_2})$, $\bar{S}_2(\omega_{k_1},\omega_{k_2})$, $\bar{\bar{S}}_1(\omega_{k_1},\omega_{k_2})$, $\bar{\bar{S}}_2(\omega_{k_1},\omega_{k_2})$, we used the fast Fourier transform (FFT) algorithm in its discrete version (see, for example, Blahut, 1987; Chan and Ho, 1991; Ersoy, 1997). The results obtained were used for computing values of $\bar{S}_3(\omega_{k_1},\omega_{k_2})$, $\bar{\bar{S}}_3(\omega_{k_1},\omega_{k_2})$, $k_p = 1,\ldots,M_p - 1$, and $p = 1, 2$.

For the given problem the hereditary error is zero. By assuming that the calculations are performed in a floating-point regime with round-off the results of arithmetic operations using the standard rule up to τ binary digits in normalised mantissae of numbers, we can derive the round-off errors for all main arithmetic operations participating in computing $\bar{R}(\omega_{k_1}, \omega_{k_2})$. For example, of fl(·) is the result of computing $x_1 \pm x_2$ on a computer with the above characteristics, we have that $\mathrm{fl}(x_1 \pm x_2) = (x_1 \pm x_2)(1 + \varepsilon_1)$ where $|\varepsilon_1| \leq 2^{-\tau}$. Furthermore, we have (Zadiraka and Abatov, 1991; Zadiraka and Melnikova, 1993) that

$$\mathrm{fl}\left(\prod_{i=1}^{n} x_i\right) = \prod_{i=1}^{n} x_i(1+\varepsilon_1) \quad \text{and} \quad \mathrm{fl}\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} x_i(1+\xi_i)$$

where the bounds for constants ξ_i, $i = 1, \ldots, n$ and ε_1 can be given as follows: $(1 - 2^{-\tau})^{n-1} \leq 1 + \varepsilon_1 \leq (1 + 2^{-\tau})^{n-1}$ with the same bounds for ξ_1, and

$$(1-2^{-\tau})^{n+1-i} \leq 1 + \xi_i \leq (1+2^{-\tau})^{n+1-i}, \quad i = 2, \ldots, n.$$

If we assume that $(n + 2 - i)2^{-\tau} \leq 0.1$ we can conclude that

$$(1 \pm 2^{-\tau})^{n+2-i} \leq 1 \pm 1.06(n+2-i)2^{-\tau}.$$

Applying these estimates to equation (4.7), we derive the following *a priori* estimate of round-off error for computing $\bar{R}(\omega_{k_1}, \omega_{k_2})$

$$\begin{aligned}
\varepsilon \leq \frac{2^{\tau}}{\omega_{k_1}^2 \omega_{k_2}^2 h_1 h_2} & ((14 + 1.06(M_1 - 1))\omega_{k_2} h_2 \max_{1 \leq i \leq M_1 - 1} \left|(\sin(\omega_{k_1} x_{i+1}) - \sin(\omega_{k_1} x_i))\right. \\
& \times ((f_{i+1,1} - f_{i,1})\cos(\omega_{k_2} c) - (f_{i+1,M_2} - f_{i,M_2})\cos(\omega_{k_2} d))\Big| \\
& + (14 + 1.06(M_2 - 1)\omega_{k_1} h_1 \max_{1 \leq i \leq M_2 - 1} |(\sin(\omega_{k_2} y_{j+1}) - \sin(\omega_{k_2} y_j))((f_{1,j+1} - f_{1,j}). \\
& \times \cos(\omega_{k_1} a) - (f_{M_1,j+1} - f_{M_1,j})\cos(\omega_{k_1} b))|(17 + 1.06(M_2 - 1) \\
& + 1.06(M_1 - 1) \max_{1 \leq i \leq M_1 - 1, 1 \leq j \leq M_2 - 1} |(f_{i,j} + f_{i+1,j+1} - f_{i,j+1} - f_{i+1,j})(\sin(\omega_{k_1} x_{i+1}) \\
& - \sin(\omega_{k_1} x_i))(\sin(\omega_{k_2} y_{j+1}) - \sin(\omega_{k_2} y_j))| + 19\omega_{k_1}\omega_{k_2} h_1 h_2 |f_{1,1}\cos(\omega_{k_1} a)\cos(\omega_{k_2} c) \\
& + f_{M_1,M_2}\cos(\omega_{k_1} b)\cos(\omega_{k_2} d) - f_{M_1,1}\cos(\omega_{k_1} b)\cos(\omega_{k_2} c) \\
& - f_{1,M_2}\cos(\omega_{k_1} a)\cos(\omega_{k_2} d)|).
\end{aligned} \tag{4.12}$$

The estimate of the round-off error in computing $\bar{\bar{R}}(\omega_{k_1}, \omega_{k_2})$ can be obtained analogously.

Algorithm 4.1 was applied to computing estimates for the sin- and cos- Fourier transforms on a unit square

$$\int_0^1 \int_0^1 f(x,y)\sin(\omega_k x)\sin(\omega_k y)\mathrm{d}x\mathrm{d}y \text{ and } \int_0^1 \int_0^1 f(x,y)\cos(\omega_k x)\cos(\omega_k y)\mathrm{d}x\mathrm{d}y, \tag{4.13}$$

in a wide range of frequencies, ω_k. The results present in the tables that follow were rounded-off in the third digit after the dot (from the originally kept seven digits) and only four significant digits are shown.

Example 1. Let $f(x,y) = 5x + 2y$, $M_1 = 2^5 + 1$, $M_2 = 2^5 + 1$. We keep the same notation as in Melnik and Melnik (1999), that is we denote by RS and RC the estimates of sin- and cos- Fourier transforms, and by ST and CT the values of $I_2(\omega_k)$ and $I_3(\omega_k)$ obtained analytically. The results of computations, given in Table I, confirm that the cubature formulae (3.7) are exact on linear functions, taking into account the round-off error.

The next two examples demonstrate the efficiency of formulae (3.7) for nonlinear functions.

Example 2. Let $f(x,y) = 7x^3 + 5y^2$, $M_1 = 2^5 + 1$, $M_2 = 2^5 + 1$. Computational results are given in Table II.

Example 3. Let $f(x,y) = \exp(x) + \exp(y)$, $M_1 = 2^7 + 1$, $M_2 = 2^7 + 1$. The results of computations for this example are given in Table III.

Example 4. Finally, our last example demonstrates the efficiency of the algorithm in the case where $M_1 \neq M_2$. We compute estimates of sin- and cos- Fourier transforms

$$I_{\mathrm{s}}(\omega_{k_1}, \omega_{k_2}) = \int_0^1 \int_0^1 (\exp(x) + \exp(y)) \sin(\omega_{k_1} x) \sin(\omega_{k_2} y)\, \mathrm{d}x\, \mathrm{d}y, \tag{4.14}$$

$$I_{\mathrm{c}}(\omega_{k_1}, \omega_{k_2}) = \int_0^1 \int_0^1 (\exp(x) + \exp(y)) \cos(\omega_{k_1} x) \cos(\omega_{k_2} y)\, \mathrm{d}x\, \mathrm{d}y \tag{4.15}$$

for frequencies $\omega_{k_1} \geq 2\pi$, $\omega_{k_2} \geq 2\pi$ and $M_1 = 2^5 + 1$, $M_2 = 2^4 + 1$ (Table IV).

Computations for different values of M_1 and M_2 in a wide range of frequencies ω_{k_1}, ω_{k_2} showed that with increasing frequencies, the accuracy of computation increases for cubature formula (3.7). For the same set of frequencies $(\omega_{k_1}, \omega_{k_2})$,

Table I.
$f(x,y) = 5x + 2y$

Frequency	RS	RC	|ST − RS|	|CT − RC|
7.069×10^{0}	-2.491×10^{-2}	6.594×10^{-2}	0.000	0.000
1.592×10^{2}	2.095×10^{-4}	2.050×10^{-4}	0.000	0.000
5.160×10^{2}	-5.400×10^{-6}	1.310×10^{-5}	0.000	0.000
4.742×10^{3}	2.000×10^{-7}	2.000×10^{-7}	0.000	0.000

Table II.
$f(x,y) = 7x^3 + 5y^2$

Frequency	RS	RC	|ST − RS|	|CT − RC|
7.069×10^{0}	-2.893×10^{-2}	1.541×10^{-1}	1.070×10^{-5}	4.040×10^{-5}
1.592×10^{2}	3.655×10^{-4}	3.520×10^{-4}	4.000×10^{-7}	2.000×10^{-7}
5.160×10^{2}	-9.300×10^{-6}	2.270×10^{-5}	0.000	0.000
4.742×10^{3}	4.000×10^{-7}	4.000×10^{-7}	0.000	0.000

Table III.
$f(x,y) = \exp(x) + \exp(y)$

Frequency	RS	RC	|ST − RS|	|CT − RC|
7.069×10^{0}	-7.473×10^{-3}	5.696×10^{-2}	0.000	5.000×10^{-7}
1.592×10^{2}	2.811×10^{-4}	1.599×10^{-4}	0.000	0.000
5.160×10^{2}	-2.000×10^{-6}	1.020×10^{-5}	0.000	0.000
4.742×10^{3}	3.000×10^{-7}	2.000×10^{-7}	0.000	0.000

Frequency, ω_{k_1}	Frequency, ω_{k_2}	RS	RC	$\|ST-RS\|$	$\|CT-RC\|$
7.069×10^0	6.807×10^0	-8.656×10^{-3}	4.344×10^{-2}	2.400×10^{-6}	9.000×10^{-6}
7.069×10^0	2.283×10^2	-1.626×10^{-4}	2.107×10^{-3}	1.000×10^{-7}	0.000
7.069×10^0	4.093×10^3	-8.000×10^{-6}	9.610×10^{-5}	0.000	0.000
1.592×10^2	6.807×10^0	-1.271×10^{-3}	2.299×10^{-3}	4.000×10^{-7}	5.000×10^{-7}
1.592×10^2	2.283×10^2	1.958×10^{-4}	1.116×10^{-4}	0.000	0.000
1.592×10^2	4.093×10^3	1.300×10^{-5}	5.100×10^{-6}	0.000	0.000
5.160×10^2	6.807×10^0	-1.293×10^{-4}	5.810×10^{-4}	0.000	1.000×10^{-7}
5.160×10^2	2.283×10^2	-5.800×10^{-6}	2.820×10^{-5}	0.000	0.000
5.160×10^2	4.093×10^3	-3.000×10^{-7}	1.300×10^{-6}	0.000	0.000
4.742×10^3	6.807×10^0	-4.270×10^{-5}	-7.740×10^{-5}	0.000	0.000
4.742×10^3	2.283×10^2	6.600×10^{-6}	-3.800×10^{-6}	0.000	0.000
4.742×10^3	4.093×10^3	4.000×10^{-7}	-2.000×10^{-7}	0.000	0.000

Table IV. $f(x,y) = \exp(x) + \exp(y)$

a variation of steps h_1 and h_2 does not substantially influence the accuracy of computations. Hence, in this example, the accuracy of cubature formula (3.7) is practically independent of the mutual arrangement of grid nodes and zeros of oscillating functions ($\sin(\omega_{k_1}x)$ ($\sin(\omega_{k_2}y)$ and ($\cos(\omega_{k_1}x)$ ($\cos(\omega_{k_2}y)$, respectively). Finally, we note that asymptotically, for large K, the number of arithmetic operations for our algorithm using the FFT constitutes 1 percent of the number of arithmetical operations of the method that does not use FFT.

5. Conclusions

The design and implementation of optimal algorithms (and algorithms close to optimal with respect to their computational characteristics) is comparable in importance with that of new computer hardware. In this paper, we constructed and tested cubature formulae for numerical integration of fast oscillatory functions from interpolational classes without invoking any *a priori* estimates of the Lipschitz constant. Our cubature formulae are optimal-by-order (with a constant not exceeding two) and can be applied for practically arbitrary oscillation patterns of integrand. We proposed an efficient algorithm for computing approximations to Fourier transformations in the two-dimensional case. A simple computational implementation of this algorithm is an important feature in the design of efficient software application packages that are potentially useful in a number of areas of signal processing and image recognition. Finally, we note that our results can be generalised to functional classes given by quasi-metrics.

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