Approximate Models for Nonlinear Control and Hamilton-Jacobi-Bellman Equation in Non-Reflexive Banach Spaces

Melnik, R.V.N.

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Welcome to the European Control Conference 1999

The European Control Conference has grown into an important European forum of control theory and technology with an international format. The strong response (1261 paper submissions for the ECC’99 from 73 countries world wide) reflects the importance the European Control Conference has gained in the field of control and industrial automation. The ECC’99 in Karlsruhe underlines the practical relevance of control by providing Industry Packages (September 1-2, 1999) with emphasis on the interests of the participants from industry under cost/profit-aspects.

Altogether 900 papers (ca. 70 %) have been accepted covering aspects of systems and control ranging from fundamental research to a wide field of engineering applications. The reviews were conducted by the 86 members of the IPC organised within 14 major conference topics resulting in a programme of high quality.

The official opening session on August 31, 1999, will be followed by a plenary talk by K. J. Åström and two semiplenaries by G. Schmidt and I. Mareels. Each of the following conference days will start with two semiplenaries given by (in order): D. Seborg (USA), M. Blanke (DK), M. J. Grimble (U.K.), J. Lewine (F), B. H. Krogh (USA) and S. Campbell (USA). The lecture sessions programme is structured into one morning time slot and two afternoon time slots, offering 14 to 15 parallel sessions each. 84 papers were assigned as poster presentations which are, after a short introduction, on display in the Foyer of the conference centre during preassigned times. Three mini-courses on special topics are integrated into the conference programme. The evening of Sept. 1, 1999, is dedicated to two Round Tables on R&D-topics. There will also be guided tours to major companies around Karlsruhe.

Additionally, tutorial workshops on theory and application oriented topics are offered to be arranged at the conference site prior to the conference on August 30, 1999.

The European Control Conference 1999 will lead all of us into the next century. It provides an excellent opportunity to discuss the role of control and the state of the art at the beginning of this new century within the international control community. I cordially invite you to seize this opportunity and actively participate in the conference. Female researchers are especially encouraged to attend. I am looking forward to welcome you at the European Control Conference 1999 in Karlsruhe.

Paul Frank

Paul M. Frank, General Chairman

Welcome to Karlsruhe

Karlsruhe extends a warm welcome to the visitors of the European Control Conference 1999! The City of Karlsruhe, the scientific community as well as the high-tech industry of Karlsruhe and its Technology Region are happy to host Europe's top conference on Measurement and Control.

Karlsruhe is an industrial and economic center in the middle part of the Upper Rhine Valley, located in the center of Western Europe, close to Heidelberg, Baden-Baden and Strasbourg, to the northern part of the Black Forest, to the Alsace (France) and to the Palatinate region. Frankfurt and Stuttgard are within the range of 1.5 hours by train or car. With the opening of the Federal Supreme Court in 1950 and the Federal Constitutional Court in 1951, the city was called the "Residence of Justice". Petrochemical and electronic industries have settled in Karlsruhe, being complemented, among others, by pharmaceutical and machine tool enterprises.

Karlsruhe provides a fertile soil for research and development in measurement and control. Karlsruhe and its Technology Region show an extraordinary density of researchers due to many well renowned scientific institutions and transfer institutions and agencies. The Technical University of Karlsruhe, with top positions in several rankings, has more than 20,000 students, among them ca. 300 students in control. Ca. 8 chairs within the faculties of electrical and mechanical engineering, of informatics (computer science) and of management sciences are dealing with system and control theory. The Technical College of Karlsruhe, with 4,300 students, adds application oriented research and development. The Research Centre Karlsruhe, three Fraunhofer-Institutes, the Steinbeis Foundation and the Federal and State Research Agencies for Nutrition and Environment Protection: they all need measurement and control for their research and technology developments.

Karlsruhe is a place with high quality of life. The environment of the Rhine valley with its border mountain areas can be reached even by our largely extended public train system. Extensive areas of the Rhine flood plain have been listed as nature conservation areas. The city itself offers a great variety of cultural and culinary entertainment: 76 museums, among them the Baden State Gallery of Fine Arts, 25 theatres, among them the Baden State Theatre, a concert hall and many libraries but also haute cuisine restaurants as well as the "local" cuisines of Baden and the Alsace (France, 30 minutes by car from Karlsruhe) add to the joy of life in Karlsruhe.

It will hardly be possible to explore all those possibilities during the short week of the ECC 99. The local organizing committee and the City of Karlsruhe, however, will provide means for easily exploring these opportunities. Among others, excursions and a conference ticket will be offered to support our guests in accessing the beautiful environment of Karlsruhe including Heidelberg, Baden-Baden and Schwetzingen.

We do hope that your visit to the ECC 99 will be the beginning of a long term friendship with Karlsruhe. Karlsruhe people are open minded people, proud of their city and its possibilities, and they feel members of an European society. They welcome you all at the ECC 99 and many subsequent visits to our city and wish you a successful week in Karlsruhe.

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VDI/VDE-Gesellschaft Mess- und Automatisierungstechnik (GMA)
(VDI/VDE-Society for Measurement and Automatic Control)
Graf-Recke-Straße 84
40239 Duesseldorf
Tel.: +49/211/6214-226
Fax.: +49/211/6214-161
E-Mail: gma@vdi.de

The Organiser

VDI/VDE-Gesellschaft Mess- und Automatisierungstechnik (GMA)

[VDI/VDE-Society for Measurement and Automatic Control (GMA)]
- Portrait -

The technology field of measurement and automatic control enables the industrial user to operate processes and production facilities in the way demanded by presentday criteria of flexibility, productivity, safety and environmental protection. It is one of the fields with the shortest innovation cycles and the highest innovation rates.

As a result of this development the engineer is faced with ever new requirements which he must constantly meet with new methods and expanded knowledge.

The measurement and automatic control technician nowadays relies more than ever before on specialist support, up-to-date information, exchange of information, possibilities for further training and guidelines for his work. All this is provided for him by the GMA (Gesellschaft Mess- und Automatisierungstechnik), the Society for Measurement and Automatic Control.

The GMA is a joint federation of the two engineering associations VDI (Association of German Engineers) and VDE (Association of German Electrical Engineers).

The GMA has about 15.000 personal members and is the German member organisation in IFAC and in IMEKO. About 1000 voluntary specialists from industry and science work in more than 80 technical committees und subcommittees of the GMA.

Main activities of the GMA are
- promotion of the exchange of information between industry, authorities and scientific institutions
- organising congresses, conferences, symposia etc. to promote the flow of information concerning new processes and developments
- preparation of publications, recommendations, guidelines etc. for the improvement of understanding
- scientific preparation for standardisation
- national and international representing the field of measurement and automatic control
- publication and promotion of technical and scientific literature
- engagement in education and post graduate training

With its work the GMA makes a considerable contribution to the further development of measurement and automatic control technology and the dissemination of information and knowledge used nowadays in this field.

Contact: VDI/VDE Gesellschaft Mess- und Automatisierungstechnik (GMA)
P.O. Box 10 11 39
D - 40002 Duesseldorf
Tel.: +49 211 6214 226
Fax.: +49 211 6214 161
Email: gma@vdi.de
Internet: http://www.vdi.de/gma/gma.htm

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In 1990, a number of prominent members of the Systems and Control community from countries of the European Union decided to set up an organisation, named European Union Control Association (EUCA), whose main purposes are to promote initiatives aiming at enhancing scientific exchanges, disseminating information, coordinating research networks and technology transfer in the field of Systems and Control within the Union.

The first of these initiatives consisted in the launch of a series of periodic wide-spectrum conferences, called European Control Conferences (ECC), to be held every second year in countries of the Union.

A second major initiative of the EUCA was the launch of a scientific journal, called European Journal of Control (EJC), whose first issue appeared in September 1995, published on a quarterly basis by Springer Verlag. Its Editor-in-Chief is Professor Ioan Landau.

EUCA consists of up to four members per each country of the European Union, depending on the level of activity in the field of Systems and Control in the particular country. At present, there are 41 members. Each member is appointed for a two-year term and can be re-appointed for at most two additional consecutive terms. The members of EUCA meet at least once a year to oversee the organisation of future ECCs, the status of EJC, to discuss new initiatives of interest to the Systems and Control community in the countries of the European Union, and to deliberate about re-appointments and/or new appointments. In the time interval between annual meetings, matters of interest to EUCA are run by a Governing Board consisting of a President, two vice-Presidents, a Secretary and a Treasurer. EUCA is a non-profit organisation.

EUCA Governing Board

APPROXIMATE MODELS FOR NONLINEAR CONTROL AND HAMILTON-JACOBI-BELLMAN EQUATIONS IN NON-REFLEXIVE BANACH SPACES

R.V.N. Melnik

CSIRO Mathematical and Information Sciences,
Locked Bag 17, North Ryde, NSW 1670, Australia
Fax : +61 7 4631 1775 and e-mail : melnik@usq.edu.au

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Abstract

In many applications of control and games theory we often encounter evolutionary partial differential equations, solutions of which are not smooth enough to satisfy these equations in the classical sense. A classical example is provided by Hamilton-Jacobi-Bellman equations that describe dynamics of value/cost functions which may not be differentiable everywhere. This paper is devoted to the analysis of such situations in both deterministic and stochastic cases.

1 Introduction

Non-smooth dynamic systems appear naturally and frequently in the control field [15]. Typically non-smooth systems fail the Lipschitz continuity requirement that is critical for the definition of classical solutions. Starting from Filippov's works, many fruitful ideas have been developed in this field including the Dini derivative technique, Clarke's generalised gradients, and the viscosity solution theory.

Until recently, the analysis of associated differential equations was predominantly conducted in reflexive Banach spaces. However, non-reflexive Banach spaces, such as L^1, play an increasingly important role in the control context. This has been demonstrated by the introduction of L^1 control theory [14] as well as by an increasing number of engineering applications (see [7, 11] and references therein). The objective of this paper is to contribute to the construction and the analysis of approximate PDE models for the description of non-smooth control systems that work in a dynamic environment.

Three main approaches to the solution of optimal control problems have been reported. These are Pontryagin's maximum principle, Bellman's dynamic programming approach and the Markov control policy approach. Historically, deterministic control problems have been solved with the maximum principle approach whereas stochastic control problems have been solved with the dynamic programming approach. During recent years much effort has been made to clarify the connection between these two approaches in the non-smooth case [5, 1, 16, 17]. Our current deliberation is based on the observation that both these approaches are ultimately connected with Markov control policy techniques [9]. A rigorous justification of this connection is well established only in the case when the value function is a classical solution of a HJB-type PDE [16, 8]. A recent breakthrough in this field was achieved with the viscosity solution theory (see [5, 1] and references in [8, 9]). Nevertheless some important questions remain open and they will be addressed in this paper.

2 Hamilton-Jacobi-Bellman Equation and Nonhomogeneous Conservation Laws

A close connection between Hamilton-Jacobi-Bellman-type equations and optimal control theory is well known [12]. Indeed, in order to derive a partial differential equation which describes the evolution of the optimal value/cost function, we can effectively use Bellman's dynamic programming approach. The resulting equation, considered in the open time-space domain $Q_T = I \times \Omega$ ($\Omega \subseteq B_1$, where B_1 is a given Banach space, $I = (t_0, T)$), has much in common with the nonhomogeneous conservation law (or the conservation law with source)

$$\frac{\partial u}{\partial t} + \frac{\partial F(t,x,u)}{\partial x} = G(t,x,u), \qquad (2.1)$$

where u is the unknown function from the given Banach space B_2, F and G are given functions such that $I \times B_1 \times B_2 \to \mathbb{R}$, $T > t_0$ ($t_0 \geq 0$) is the given number (the possibility of $T = \infty$ is not excluded). One can think of u as the conserved quantity, subject to the appropriate definition of its flux F and the source term G, the initial and boundary conditions.

By setting in (2.1) $F = -uf(t,x,u)$, $G = -u\dfrac{\partial f}{\partial x}$, we obtain the following equation

$$\frac{\partial u}{\partial t} - f(t,x,u)\frac{\partial u}{\partial x} = 0. \tag{2.2}$$

Apart from the fact that this equation (under appropriate assumptions) can be interpreted as the equation for the governing dynamics of a control system, formally it is a special case of a more general partial differential equation, known as the Hamilton-Jacobi-Bellman (HJB) equation. The later equation describes the evolution of the value/cost function of the control system

$$\frac{\partial V}{\partial t} + H(t,x,V,DV) = 0, \tag{2.3}$$

where $V(t,x)$ denotes the value/cost function that is connected with function $u(t,x)$ by the control goal, $DV(t,x)$ denotes the Fréchet derivative with respect to x, and H (often interpreted as the system Hamiltonian) is the given function. Even in this relatively simple case classical global solutions for this equation may not exist and one has to specify in what sense we have to understand the solution of (2.3). Moreover, if the class of generalised solutions is specified, one may expect non-uniqueness of such solutions. A well-known non-uniqueness example provides the functional class $W^{1,\infty}_{\mathrm{loc}}(Q_T) = \{V, DV \in L^\infty_{\mathrm{loc}}(Q_T)\}$ (i.e. V and DV are measurable bounded on each open set, the closure of which is in Q_T) with $\Omega = \mathbb{R}$.

Another difficulty intrinsic to equation (2.3) manifests itself in the 'curse of dimensionality' when attempts to approximate this equation are made. The importance of approximations of Hamilton-Jacobi-Bellman equations for control applications is well known [2]. Along with the classical artificial/vanishing viscosity method and its modifications, one of the main bases for the development of effective numerical procedures for the solution of the HJB equation has been the conservation laws [6, 10]. In the one-dimensional case the analogy between HJB equation (2.3) and the homogeneous conservation law (i.e. equation (2.1) with $G = 0$) is straightforward [6]. This analogy becomes blurred if we recall that in practical applications only an approximation of the function H is available. This fact leads to the conclusion that the key role in the construction of numerical procedures for HJB systems has to be assigned to the non-homogeneous conservation law (2.1), rather than its homogeneous counterpart. The source term G in (2.1) can be interpreted as the perturbation source for the HJB system. Indeed, by applying Bellman's dynamic programming approach in a heuristic manner we can only derive a HJB equation that gives an approximation to the dynamics of the value/cost function. Of course, in some cases we may be able to establish verification theorems that are useful in finding an optimal feedback. However, in the final analysis the quality of this feedback is subject to the quality of approximation of the function H which is associated with the system Hamiltonian. Such an approximation will be denoted by H^δ_ϵ [8]. Similar to the nonhomogeneous conservation laws, the natural space for perturbations in the systems control context is L^1 rather than L^2 [4]. Indeed, L^1 perturbations of control problems is the largest and the most reasonable functional space for perturbations in such problems where control systems work in a dynamic and uncertain environment. This situation arises in a number of important applications of control systems in oceanographic research, aerospace engineering and smart structure technology.

3 Non-smooth Deterministic Control

Three of the most important concepts in the development of control theory have been the Pontryagin maximum principle, the Wiener-Hopf-Kalman H^2 optimal control theory and H^∞ robust control theory [2]. Although based on different ideas for disturbance rejection (a stochastic white noise disturbance model and a deterministic disturbance model respectively), both H^2 and H^∞ theories involve L^2-measures for the controller performance (in order to quantify disturbance rejections in the frequency domain). However, when the control system works in a dynamic uncertain environment, it is important to be able to capture the worst-case peak amplitude response. In such cases L^1 theory becomes more appropriate [14, 2]. Another technique for disturbance rejection is implicitly implemented in the Pontryagin maximum principle. The rigorous logical basis of the Pontryagin maximum principle technique, which leads to a system of ordinary differential equations (rather than to a partial differential equation formally obtainable using the Bellman dynamic programming approach) comes at a cost of certain assumptions which we analyse below.

The application of Pontryagin's maximum principle is quite natural for the solution of the following control problem

$$\begin{aligned} J(t,x;u) \quad &= \quad \int_{t_0}^{T} f_0(\tau, x(\tau), u(\tau, x(\tau)))d\tau + \\ &\qquad \alpha g(T, x(T)) \to \min \end{aligned} \tag{3.1}$$

with the dynamics governed by the equation

$$\frac{dx}{dt} = f(t,x,u) \ \text{ a.e. in } \ [t_0, T]; \tag{3.2}$$

$$x(t_0) = x_0, \ \ u \in U. \tag{3.3}$$

The problem (3.1) – (3.3) is the Bolza problem, where U is the given set of admissible controls, f_0, f are given maps on $I \times B_1 \times B_2$, g is the given map on $I \times B_1$, and α is the given real number.

We introduce Pontryagin-Hamilton's function [13]

$$P(t,x,u,\psi^p,\delta) = -\delta f_0 + f\psi^p, \ \ \delta \geq 0, \tag{3.4}$$

where δ is the scaling factor and ψ^p is the adjoint function. A standard practical recipe that follows from the Pontryagin maximum principle is to consider function $P(t, x, u, \psi^p, \delta)$ as a function of $u(\cdot, \cdot)$ taking all other variables (t, x, ψ^p, δ) as parameters. Then for each fixed set (t, x, ψ^p, δ) we have to solve the following optimisation problem

$$P(t, x, u, \psi^p, \delta) \to \sup, \quad u \in U \tag{3.5}$$

The supremum in (3.5) will be denoted by

$$H_\epsilon^\delta(t, x, \psi^p, \delta) = \sup_{u \in U} P(t, x, u, \psi^p, \delta), \tag{3.6}$$

where the definition of the adjoint function ψ^p is subject to the solution of the linear ordinary differential equation

$$\frac{d\psi^p}{dt} = \delta \frac{\partial f_0}{\partial x} - \frac{\partial f}{\partial x} \psi^p. \tag{3.7}$$

Functions f_0 and f are coupled by control u. In the general case this coupling may be exhibited for all values of $t \in (t_0, T)$. This fact makes it difficult to construct a general model for the adjoint system dynamics. Indeed, one of the main premises for the Pontryagin maximum principle is the Hamiltonian system paradigm

$$\frac{dx}{dt} = \frac{\partial P}{\partial \psi}, \quad \frac{d\psi}{dt} = -\frac{\partial P}{\partial x}, \tag{3.8}$$

where one assumes that function P is a function of t, x and the generalised impulse ψ. However this analogy between the generalised impulse ψ and the adjoint function ψ^p may not be appropriate. Let $P = -\delta f_0 + f\psi$ be the Hamiltonian of the control system described by (3.1) – (3.3). Then both functions f and f_0 become dependent on ψ (due to their coupling via control), which makes the first equality in (3.8) only approximate (assuming small rates of change for f and f_0 with respect to ψ). Further we notice that

$$\frac{d\psi}{dt} = -\frac{\partial P}{\partial x} = \delta \frac{\partial f_0}{\partial x} - \frac{\partial f}{\partial x} \psi - \underline{f \frac{\partial \psi}{\partial x}}. \tag{3.9}$$

If we assume that $f\dfrac{\partial \psi}{\partial x} = 0$, then equations (3.7) and (3.9) become identical, and hence $\psi^p = \psi$. However, in the general case the gradient $\dfrac{\partial \psi}{\partial x}$ may be arbitrarily large. This fact leads to essential difficulties in the application of Pontryagin's maximum principle to a number of control problems.

4 Connection between the Pontryagin Maximum Principle and the Bellman Dynamic Programming Approach

A natural question to ask is how to apply Bellman's dynamic programming approach to (3.1)–(3.3). It is also quite natural to ask how to apply Pontryagin's maximum principle to the solution of stochastic optimal control problems.

One of the key difficulties in answering these questions lies in the fact that a formal derivation of the HJB equation for deterministic control problems requires Taylor's expansion of the value/cost function. This leads to *a priori* excessive assumptions on smoothness of this function. We propose to relax these assumptions by applying Steklov's operator technique.

We recall that the Bellman approach is well suited to stochastic optimal control problems, i.e. for systems whose dynamics are described by the following governing equation

$$\begin{aligned} x(t) &= x(t_0) + \int_{t_0}^{t} f(\tau, x(\tau), u(\tau)) d\tau + \\ & \int_{t_0}^{t} \sigma(s, x(s), u(s, x(s))) d\omega(s) \end{aligned} \tag{4.1}$$

with the requirement

$$E_{t,x}^u [J(t, x; u)] \to \min, \tag{4.2}$$

where functional $J(t, x; u)$ is defined by (3.1), $u(\cdot, \cdot)$ is the control function valued (as above) in subset U, $E_{t,x}^u[J(t, x; u)]$ is the expectation conditioned on $u(t, x)$, $x(\cdot)$ is a $\mathbb{R}$-valued process, f and σ are given functions that serve as drift and diffusion of this process, $\omega(\cdot)$ is a $\mathbb{R}$-valued process which serves as a "driving noise" for the control system. Since values of function u at time t may depend on information about the past states $x(\cdot)$ prior to t, function $u(t, x(\tau))$, $0 \le \tau < t$ is often denoted simply as $u(\cdot)$. We prefer to explicitly indicate the state-dependency.

The result of the application of Bellman's dynamic programming approach to the problem (4.1)–(4.2) is a second order PDE (see [8, 17] and references therein). Formally, one can also write a PDE associated with deterministic optimal control problems such as the Bolza problem (3.1)–(3.3)

$$\frac{\partial V}{\partial t} + H_\epsilon^\delta \left(t, x, \frac{\partial V}{\partial x} \right) = 0, \tag{4.3}$$

where V is the value/cost function defined as

$$V(t, x) = \inf_{u(\cdot, \cdot) \in U(t, x)} J(t, x; u) \tag{4.4}$$

and

$$H_\epsilon^\delta \left(t, x, \frac{\partial V}{\partial x} \right) = \inf_{u \in U} \left(f_0 + f \frac{\partial V}{\partial x} \right). \tag{4.5}$$

The connection between the described approaches is well established only in the classical situation when $V \in C^{1,2}(Q_T)$. In this case

$$\frac{\partial V}{\partial x}(t, x^*(t)) = \psi^p(t), \tag{4.6}$$

where $x^*(\cdot)$ denotes the optimal path of the value/cost function. Of course, it is well known that the assumption on continuous differentiability of the functional (3.1), required for this connection, does not hold in the simplest cases. However, this assumption can be essentially relaxed by using the viscosity solution theory [5, 1] (or equivalent approaches [12]). In this "relaxed" framework (for the non-smooth deterministic case) the connection between the Pontryagin maximum principle and the Bellman dynamic programming approach can be interpreted in the following way (see [16] and references therein)

$$D_x^- V(t, x^*(t)) \subset \{\psi^p\} \subset D_x^+ V(t, x^*(t)); \quad (4.7)$$

$$D_{t,x}^- V(t, x^*(t)) \subset \{H_\epsilon^\delta\} \subset D_{t,x}^+ V(t, x^*(t)), \quad (4.8)$$

where $D_{t,x}^{\pm} V$ denotes superdifferential/subdifferential in the (t, x)-variables and $D_x^{\pm} V$ denotes the superdifferential/subdifferential in the x-variable for each fixed value of $t \in (t_0, T)$. Inclusions (4.7), (4.8) indicate major difficulties with both classical approaches, namely the choice of the equation adequately describing the dynamics of the adjoint process, and infinite dimensionality of the HJB equation. These difficulties become obvious when the control problem is solved numerically. We also note that in the general stochastic case inclusion (4.8) may be violated [17]. In the next sections we develop a unified treatment that allows us to effectively deal with both deterministic and stochastic control.

5 Generalized PDEs of the HJB-type in Control Theory

Non-reflexive Banach spaces, such as L^1, reflect the nature of control problems in the most complete way and allow us to explore further the interplay between optimal, robust and adaptive control paradigms in control theory. In what follows we consider Sobolev spaces $W_1^l(\Omega)$. We recall that a measurable function $f(x)$ belongs to $W_1^l(\Omega)$ (l is a positive integer) if $f \in L^1(\Omega)$, and if Sobolev generalised derivative $f^{(l)}$ of order l exists and $f^{(l)} \in L^1(\Omega)$. The classes W_1^l are converted into Banach spaces by the introduction of the norm

$$\|f\|_{W_1^l} = \|f\|_1 + \left\| f^{(l)} \right\|_1, \quad \|\cdot\|_1 \equiv \|\cdot\|_{L^1}. \quad (5.1)$$

For an elementary time-space region

$$Q_0 = \{(t', x') : \ t \le t' \le t + \Delta t, \ x \le x' \le x + \Delta x \ \} \quad (5.2)$$

we introduce Steklov's averaging operators as follows

$$S^t V(t, x) = \int_t^{t+\theta_1 \Delta t} V(\eta, x) d\eta, \quad (5.3)$$

$$S^x V(t, x) = \int_x^{x+\theta_2 \Delta x} V(t, \mu) d\mu, \quad (5.4)$$

where $0 \le \theta_i \le 1, i = 1, 2$. Choices of $\omega_1 = \Delta x$ and $\omega_2 = \Delta t$ are coupled to initial conditions (t, x) and value/cost function $V(t, x)$. Using the Steklov operator technique we have obtained a Local Optimality Principle that governs the evolution of the value/cost function.

Theorem 5.1 *If $V \in W_1^{1,1}(Q_0)$, then the evolution of the value/cost function is described by the following integro-differential equation*

$$\omega_1^2 \frac{\partial V}{\partial x} + \omega_2^2 \frac{\partial V}{\partial t} + \omega_1 \omega_2 \left[\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \right] +$$
$$\omega_1 \omega_2 \left[\omega_1 \frac{\partial^2 V}{\partial t \partial x} + \omega_2 \frac{\partial^2 V}{\partial x \partial t} \right] +$$
$$\frac{1}{2} \left\{ \int_x^{x+\Delta x} \frac{\partial^2}{\partial \xi^2} (S^x V)(x + \Delta x - \xi) d\xi \mid_{t+\Delta t} + \right.$$
$$\left. \int_t^{t+\Delta t} \frac{\partial^2}{\partial \eta^2} (S^t V)(t + \Delta t - \eta) d\eta \mid_{x+\Delta x} \right\} +$$
$$\omega_2(\omega_1 + \omega_2) f_0 = 0. \quad (5.5)$$

We emphasise that this result is independent of the type of governing equation for the system dynamics and include both cases (3.1)–(3.3) and (4.1)–(4.2).

Our approach to the construction of a general model for the evolution of the value/cost function has much in common with the robust sample-data control theory. It leads to the rapprochement of (worst-case) L^1 theory and statistical approaches in control problems [7]. Indeed, using available statistical information we can construct such functional approximations for ω_1 and ω_2 that allow efficient continuation procedures for the equation (5.5) to the region Q_T. One can view such continuations as supervisory on-line control procedures (see [8] and references therein). The necessity of such supervisory procedures can be explained from the game theoretic point of view where the model uncertainty and the control can be seen as strategies employed by opposing players in a game (control is taken to minimise value/cost function, while the uncertainty opposes it by trying to maximise this function [3]). In this context we recall that both classical approaches (the Pontryagin maximum principle and the Bellman dynamic programming) can be directly applied to controlled systems with a complete model description. This description is unavailable when the systems work in a dynamic environment in the presence of any sort of uncertainty such as parametric type, unmodelled dynamics or external perturbations [3]. Nonreflexive Banach spaces are the most suitable functional spaces for the construction of control models in such situations.

Now we are in a position to derive some important special cases of (5.5).

If $V \in W_1^{2,2}(Q_0)$ then the stochastic control problem

(4.1)–(4.2) is equivalent to the following second order PDE

$$(1+v)\left[\frac{\partial V}{\partial x}+\frac{1}{v}\left(\frac{\partial V}{\partial t}+f_0\right)\right]+$$

$$\omega_1\frac{\partial^2 V}{\partial t\partial x}+\omega_2\frac{\partial^2 V}{\partial x\partial t}=\sigma_1\frac{\partial^2 V}{\partial x^2}+\sigma_2\frac{\partial^2 V}{\partial t^2}, \quad (5.6)$$

where $v=\omega_1/\omega_2$ and σ_1, σ_2 are σ-dependent diffusion functions. The later functions vanish in the case of partial differential equations that deal with non-smooth deterministic controls. More precisely, the following result holds.

Theorem 5.2 *If* $V\in W_1^{1,1}(Q_0)$, $\dfrac{\partial V}{\partial x}\in L^1\left(W_1^1(I),\Omega\right)$, $\dfrac{\partial V}{\partial t}\in L^1\left(I,W_1^1(\Omega)\right)$, *then the equation*

$$(1+v)\left[\frac{\partial V}{\partial x}+\frac{1}{v}\left(\frac{\partial V}{\partial t}+f_0\right)\right]+$$

$$\omega_1\frac{\partial^2 V}{\partial t\partial x}+\omega_2\frac{\partial^2 V}{\partial x\partial t}=0 \quad (5.7)$$

provides $\mathcal{O}(\Delta t+\Delta x)$ *approximation of the Local Optimality Principle.*

The most difficult case for analysis is the deterministic limit of control problems. Assuming $V\in L^1(Q_0)$, this case leads to the equation

$$(1+v)\left[\frac{\partial V}{\partial x}+\frac{1}{v}\left(\frac{\partial V}{\partial t}+f_0\right)\right]=0, \quad (5.8)$$

which (under certain conditions) coincides with the nonhomogeneous conservation law.

6 Local Vector Fields in Solutions of Nonlinear Control Problems

Let $\vec{\psi}=(\psi_1,\psi_2)$, where

$$\psi_1(t,x)=\int_{x_0}^{x}V(t,\mu)d\mu,\quad \psi_2(t,x)=\int_{t_0}^{t}V(\eta,x)d\eta. \quad (6.1)$$

We introduce non-smooth parts of the Hamiltonian using a composition of Steklov's operators as a performance measure in Q_0:

- non-smooth wrt t ($\forall x'\in[x,x+\Delta x]$ and a.e. in $t\in I$)

$$H_1=S^t\otimes S^x\left[\int_t^{t+\Delta t}f_0d\tau\right]+S^t\psi_1 \quad (6.2)$$

- non-smooth wrt x ($\forall t'\in[t,t+\Delta t]$ and a.e. in $x\in\Omega$)

$$H_2=S^x\otimes S^t\left[\int_t^{t+\Delta t}f_0d\tau\right]+S^x\psi_2, \quad (6.3)$$

where $\otimes$ denotes the composition of the corresponding integral operators.

Then system dynamics in the non-smooth case can be described by the following coupled system of canonic equations

$$\begin{cases}\dfrac{\partial H_1}{\partial t}+\kappa_1\mathrm{div}\vec{\psi}=0, & \kappa_1=\dfrac{1}{2}\left(\dfrac{\omega_1}{\omega_2}+\omega_1\right),\\[2mm] \dfrac{\partial H_2}{\partial t}+\kappa_2\mathrm{div}\vec{\psi}=0, & \kappa_2=\dfrac{1}{2}\left(\dfrac{\omega_2}{\omega_1}+\omega_2\right).\end{cases} \quad (6.4)$$

Theorem 6.1 *If* $\omega_1+\omega_2=2$, *then a.e. in* Q_0, H_1 *and* H_2 *are monotone functions in* t *and* x *respectively, simultaneously increasing or decreasing:*

$$\frac{\partial H_1}{\partial t}\frac{\partial H_2}{\partial x}=(\mathrm{div}\vec{\psi})^2. \quad (6.5)$$

For a local optimum point we have

$$\mathrm{div}\vec{\psi}=\mathrm{div}\vec{H}=0,\quad \frac{\partial^2 H_1}{\partial t\partial x}=\frac{\partial^2 H_2}{\partial x\partial t}. \quad (6.6)$$

7 Weak Solutions of Generalized PDEs of the HJB-type

Let $\kappa_{Q_0}(x,t)$ be the characteristic function of set Q_0. Then in the non-smooth deterministic case we define generalised solutions of the PDE associated with the nonlinear control problem (3.1) – (3.3) as follows

Definition 7.1 *A function* $V(t,x)\in W_1^{1,1}(Q_0)$ *is called the generalized solution of the non-smooth deterministic control problem if the integral identity*

$$\int\int_{Q_0}\left\{S^x\otimes S^t\left[(1+v)\left(\frac{\partial V}{\partial x}+\frac{1}{v}\left(\frac{\partial V}{\partial t}+f_0\right)\right)+\right.\right.$$

$$\left.\left.\omega_1\frac{\partial^2 V}{\partial t\partial x}+\omega_2\frac{\partial^2 V}{\partial x\partial t}\right]\right\}\kappa_{Q_0}(x,t)dxdt=0 \quad (7.1)$$

is satisfied.

Theorem 7.1 *Let* $f_0,g,f\in L^1(Q_0)$. *If* $\tilde{F}\equiv f_0(1+1/v)\in L^1_{\mathrm{Lip}}(Q_0)$, *then there exists a unique generalized solution of the non-smooth deterministic control problem. It has mixed derivatives if*

$$\frac{\partial V}{\partial x}\in L^1(W_1^1(I),\Omega),\quad \frac{\partial V}{\partial t}\in L^1(I,W_1^1(\Omega)). \quad (7.2)$$

An analogous result was obtained in the stochastic case, where the generalised solution of (4.1)–(4.2) is defined as

Definition 7.2 *A function* $V(t,x)\in W_1^{1,1}(Q_0)$ *is called the generalized solution of the stochastic control problem if the integral identity*

$$\int\int_{Q_0}\left\{S^x\otimes S^t\left[(1+v)\left(\frac{\partial V}{\partial x}+\frac{1}{v}\left(\frac{\partial V}{\partial t}+f_0\right)\right)+\right.\right.$$

$$\omega_1 \frac{\partial^2 V}{\partial t \partial x} + \omega_2 \frac{\partial^2 V}{\partial x \partial t} -$$

$$\sigma_1 \frac{\partial^2 V}{\partial x^2} - \sigma_2 \frac{\partial^2 V}{\partial t^2} \Bigg] \Bigg\} \kappa_{Q_0}(x,t) dx dt = 0 \tag{7.3}$$

is satisfied.

Theorem 7.2 *Let $f_0, g, f \in L^1(Q_0)$. If $\tilde{F} \equiv f_0(1+1/v) \in L^1_{\mathrm{Lip}}(Q_0)$, then there exists a unique generalized solution of the stochastic control problem. It has mixed and second order derivatives if*

$$V(t,x) \in W_1^{2,2}(Q_0). \tag{7.4}$$

The application of the methodology described above to the analysis of the HJB-type equation in the deterministic case,

$$\frac{\partial V}{\partial x} + \frac{1}{v}\left(\frac{\partial V}{\partial t} + f_0\right) = 0, \tag{7.5}$$

requires the locally relaxed Lipschitz condition:

$$\|S^x \otimes S^t \left(\tilde{F}(x,t,V') - \tilde{F}(x,t,V'')\right)\|_{L^1(Q_0)} \leq$$

$$q \|S^x \otimes S^t (V' - V'')\|_{L^1(Q_0)}. \tag{7.6}$$

The existence and uniqueness theorem for equation (7.5) has been constructively proved using Fejer's sums.

8 Conclusions and Future Directions

In this paper two major approaches for the solution of optimal control problems have been thoroughly analyzed in the framework of generalised solutions. We derived generalized partial differential equations for value/cost functionals in cases that are not covered by standard diffusion processes. Existence and uniqueness theory for these equations has been developed in nonreflexive Banach spaces. The approach proposed in this paper leads to the construction of approximating Markov chains for the derived equations [9]. A deeper investigation of the connection between non-smooth deterministic control problems and stochastic control problems in the class $W_1^{1,1}(Q_0)$ remains a challenging task for future work.

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References

[1] Barron, E. N., Jensen, R., "The Pontryagin maximum principle from dynamic programming and viscosity solutions to first-order partial differential equations", *Transactions of the American Mathematical Society,* **298**, No. 2, 635–641, (1986).

[2] Beard, R. W., McLain, T.W., "Successive Galerkin approximation algorithms for nonlinear optimal and robust control", *Int. J. Control,* **71**, No. 5, 717–743, (1998).

[3] Boltyansky, V.G., Poznyak, A.S., "Robust maximum principle in minimax control, *Int. J. Control,* **72**, No. 4, 305–314, (1999).

[4] Borwein, J. M., Zhu, Q.J., "Variational analysis in nonreflexive spaces and applications to control problems with L^1 perturbations", *Nonlinear Analysis: TAM,* **28**, No. 5, 889–915, (1997).

[5] Crandall, M.G., Lions. P.-L., "Viscosity solutions of Hamilton-Jacobi equations", *Transactions of the American Mathematical Society,* **277**, No. 1, 1–42, (1983).

[6] Crandall, M.G., Lions, P. L., "Two approximations of solutions of Hamiltonian-Jacobi equations", *Mathematics of Computation* **43**, No. 167, 1–19, (1984).

[7] Makila, P.M., "On robust control-oriented identification of discrete and continuous-time systems", *Int. J. Control,* **70**, 319–335, (1998).

[8] Melnik, R.V.N., "On Consistent Regularities of Control and Value Functions", *Numerical Functional Analysis and Optimization,* **18** (3&4), 401 – 426, (1997).

[9] Melnik, R.V.N., "Dynamic System Evolution and Markov Chain Approximation", *Discrete Dynamics in NS, Gordon & Breach,* **2**, 7–39, (1998).

[10] Osher, S., Shu, C-W., "High-order essentially nonoscillatory schemes for Hamilton-Jacobi equations", *SIAM J. Numer. Anal.,* **28**, No. 4, 907–922, (1991).

[11] Partington, J.R., *Interpolation, Identification and Sampling,* Oxford University Press, (1997).

[12] Subbotin, A. I., *Generalised Solutions of First-Order PDEs. The Dynamical Optimization Perspective,* Birkhäuser, (1995).

[13] Vasiliev, F.P., *Numerical Methods for solving extremal problems,* Nauka, (1988).

[14] Vidyasagar, M., "Optimal rejection of persistent bounded disturbances", *IEEE Trans. Automat. Control,* **31**, 527–534, (1986).

[15] Wu, Q. et al "On construction of smooth Lyapunov functions for non-smooth systems", *Int. J. Control,* **69**, 443–457, (1998).

[16] Zhou, X.Y., "Maximum principle, dynamic programming, and their connection in deterministic control", *Journal of Optimization Theory and Applications,* **65**, 363–373, (1990).

[17] Zhou, X.Y., "The connection between the maximum principle and dynamic programming in stochastic control", *Stochastics and Stochastics Report,* **31**, 1–13, (1990).

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