

Biophysical and Stochastic Modelling of Single Neurons and Neuronal Populations

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This project uses theoretical electrophysiology to build and simulate models of intracellular voltage dynamics. Several models will be explored with their derivations from basic neuronal and synaptic electrophysiology by applying numerous modelling techniques to differential equations. Statistical methods will also be discussed as a need to account for the stochastic nature for voltage dynamics of neuronal populations. These models will then be used to develop a working understanding of the firing rates of populations of disconnected neurons. Ultimately this project provides the analysis to show that the Fokker-Planck equation is sufficient [3, 7] to calculate firing rates for populations of neurons with fluctuating synaptic input for both linear and non-linear integrate-and-fire models [18].

I. Introduction

The goal of this project is to develop reliable descriptions of neuronal voltage dynamics from non-deterministic mathematical models. The properties of these models will be analysed analytically and supported with computer simulations that approximate the solutions that will be discussed. The models in use have been used in numerous publications and have been studied extensively [7, 8, 9]. These studies provide

validation for the mathematical techniques and models used in this project and will be discussed further below.

The importance of this analysis lies in its ability to explain the underlying interactions of the nervous system due to the models being built to accurately replicate the behaviour of natural processes. Understanding how spike mechanisms work may help define their relationships in complex neuronal networks which provide functionality for numerous processes. This would

further understanding on how these complex systems behave.

II. Theory

A. Neuron Models

Neurons are the primary component of the nervous system and consist of excitable cells which communicate with other cells via specialised connection pathways called synapses. The mechanisms by which these interactions take place can be developed using basic neuronal and synaptic electrophysiology which form the basis of understanding for this project.

The neuron itself consists of three main functional components: the *soma* is the cell body, the *dendrite* receives synaptic inputs from neighbouring neurons and the *axon* generates action potentials which propagate through axon transmitters which then pass the signal across a synapse. Synaptic inputs from nearby neurons may cause the generation and propagation of action potentials within the axon which causes the neuron to spike if the inputs exceed some threshold value.

One of the earliest models of this process was developed in 1907 by Louis Lapicque [4] who proposed a simple capacitor circuit which explained excitations in nerve structures using data obtained from frog nerve stimulation. This was a de-

terministic model used to calculate spike rates where a membrane capacitor would discharge upon exceeding a threshold value resulting in the generation of an action potential. Lapicque argued that neuronal membranes could be modelled as a leaky capacitor in such a way that the data he presented correlated to Weiss' fundamental law of electrostimulation [5] which used a constant voltage decay term. This model did not define the mechanism by which spike generation occurred and the necessary inputs for such an occurrence. Rather it suggested that the membrane time constant which caused the leaky nature of the model was linked with the excitability of a neuron. In the 1960s there were multiple studies which analysed the model proposed by Lapicque from which the modern day integrate-and-fire (IF) [4] model emerged.

The basic IF model requires the membrane capacitance of the neuron to exceed a threshold value before action potential is generated after which the neuron exhibits a refractory state. This model has some inherent problems such as inability to account for trans-membrane ionic currents and firing frequency behaviours. This has subsequently led to the development of leaky, quadratic and exponential IF models which provide more physiological detail while still being analytically solvable.

The IF model is most commonly used for describing the behaviour of neuronal

systems due to its ability to be analysed mathematically while retaining sufficient complexity. Such a model uses synaptic inputs that are assumed to be stochastic and can be described by a Poisson process. It is important to require stochastic inputs rather than having a deterministic model like Lapicque since it is difficult to accurately determine input currents for neuronal systems [6]. An extensive analysis of stochastic IF models was conducted by Burkitt [3], most relevant for this project are the discussions on Gaussian approximations, Ornstein-Uhlenbeck processes and the Fokker-Planck equation which provide solutions for spiking patterns of the neuron and will be discussed in detail.

These advanced theoretical models have been developed to determine the probability of neurons firing. A study [1] (Fourcaud-Trocmé et al) analytically calculated the firing response of a neuron exposed to fluctuating noisy inputs such as Gaussian white noise using a non-linear IF model. This study developed the exponential IF model for neuron dynamics.

A following experimental study to match time-series data of a neuron under fluctuating inputs to existing mathematical models was undertaken in 2008 (Badel et al) [2]. The study found that under stochastic inputs, the exponential integrate and fire model proposed by Fourcaud

proved to be sufficiently reliable at modelling the time-series data.

There are other notable models such as the Hodgkin-Huxley (H-H) model [10] which describes the generation of action potentials where voltages are determined by the concentrations of charged particles present in membrane channels. This laid out how concentrations of sodium and potassium ions, in conjunction with a leakage current made of chloride ions, could describe the ionic currents in neuronal membranes. The primary focus of this project will be on the basic IF model mentioned and how it can be made more complex to provide detailed descriptions of spiking behaviour for both singular neurons and disconnected populations. This is in part due to the large number of parameters involved in models such as the H-H model, which make them increasingly difficult to analyse mathematically.

B. Synaptic Drive

Synaptic drive, or potential, is the potential difference across postsynaptic membrane that is caused by the action potential across a neuronal synapse [6]. A typical neuron will have a resting potential of -60 mV and a threshold voltage which must be passed in order for an action potential to be fired. Multiple synaptic potentials are required to exceed a threshold value, due to their small

individual magnitudes, so action potentials can only be triggered by a sum of different pulses.

Synaptic potentials are produced as one of two forms - excitatory postsynaptic potentials (EPSPs) and inhibitory postsynaptic potentials (IPSPs). They are referred to as *excitatory* and *inhibitory* depending on whether they increase or decrease the likelihood of postsynaptic action occurring. It is important to note that the vast majority of neurons receive inputs from both excitatory and inhibitory synapses so it is important to understand the distinction between them and precisely how they affect the mechanisms involved. Generally speaking the way in which excitatory and inhibitory postsynaptic pulses behave is the same and is often only affected by the concentration of permeable ions in its corresponding channel and can be distinguished by the reversal potential of the neuron.

Neuronal membranes have channels which allows for the flow of ions from inside the cell to outside and vice versa, this inherently makes the neuron potential leaky. This allows trans-membrane ionic currents I_{ion} to flow which can be represented as [12]

$$C \frac{dV_m}{dt} + I_{ion} = I \quad (1)$$

where V_m is the membrane voltage.

It is possible to consider time dependent synaptic conductance and current in

terms of the number of open and closed post-synaptic channels in a neuron. [13]

The state of a channel x is said to have two states $x = 1$ when open or $x = 0$ when closed. Upon release of neurotransmitters at the synapse all channels are open and decay at a closing rate given by β . Assuming that the total number of open channels X can be expressed as $X \simeq N\langle x \rangle$ where

$$\frac{d\langle x \rangle}{dt} = -\beta\langle x \rangle. \quad (2)$$

Substituting $\tau_s = \frac{1}{\beta}$ into the solution allows the time-dependant synaptic conductance to be expressed in the form

$$g_s(t) = \bar{g}_s e^{\frac{t}{\tau_s}}. \quad (3)$$

The voltage equation for a neuron taking into account its leaky membrane can be written as

$$C \frac{dV}{dt} = g_i(E_i - V) + I \quad (4)$$

where i corresponds to each synaptic input such that

$$C \frac{dV}{dt} = g_L(E_L - V) + g_e(E_e - V) + g_i(E_i - V) \quad (5)$$

as expected with multiple EPSP and IPSP inputs. By substituting the time-dependant synaptic conductance equation

of the form

$$g_s(t)(E_s - V) = I \quad (6)$$

B. Eq (3) can be written as

$$\frac{dV}{dt} = \frac{g_l(E_L - V) + g_s(t)(E_s - V)}{C} \quad (7)$$

where E_L is the leaky reversal potential, E_s is the reversal potential due to an excitatory or inhibitory neuron and g_L is the leaky conductance.

This equation can be solved for an explicit function for $V(t)$ which, although difficult to work with, can be compared with a much simpler Euler approximation. This model can be made more realistic by considering $g_s(t)$ as a probabilistic function whereby spike events are distributed as

$$\frac{dx}{dt} = -\frac{x}{\tau_m} + a\delta(t - t_1). \quad (8)$$

This uses independent and identically distributed values to evaluate the change in conductance at spike times and models the decay in the absence of a spike.

C. Stochastic Model

The problem has been simplified thus far by considering only direct inputs above the threshold value as triggers. In reality, the neuronal potential inputs occur at high frequency with significant fluctuations; this is referred to as a *noisy* input. The aim

is to develop a method to tackle stochastic synaptic inputs using the conductance equations that have been formalised above.

C..1 Fast Synaptic Model

Using the same idea as the detailed conductance input model governed by

$$\frac{dx}{dt} = -\frac{x}{\tau} + a\delta(t - t_1) \quad (9)$$

consider applying these events as rapid spike triggers for the conductance equation with a fixed rate R_s and an exponential decay constant γ_s . This defines a noisy conductance equation [14] with fast synaptic inputs and can be written as

$$\tau_s \frac{dg_s}{dt} = -g_s + \gamma_s \tau_s \sum_{t_{sk}} \delta(t - t_{sk}). \quad (10)$$

It is possible to compute the solution to this stochastic differential equation however the summation of delta functions proves to be difficult to solve analytically. This is in part due to the nature of the integral

$$\int_0^T \sum \delta(t' - t_1) dt' \simeq RT + \sqrt{RT} \phi_k \quad (11)$$

where ϕ_k is a unit Gaussian number with mean $\langle \phi \rangle = 0$ and variance $\langle \phi^2 \rangle = 1$. The aim is then to produce an approximation which holds the same statistical values for mean and variance while being expressed as a sum of Gaussian random variables which are much simpler to compute analytically.

Statistically, the summation of dirac functions can be represented by a Poisson process where the distribution of spikes N has mean and variance $\langle N \rangle = \text{Var}(N) = R\Delta$. The rationale for approximating using Gaussian numbers can be seen by considering the following probability distribution for N

$$P(N) = \frac{e^{-\lambda} \lambda^N}{N!} \quad (12)$$

where $\lambda = \langle N \rangle$. It is clear to see that as $R\Delta \gg 1$ the distribution is representative of a Gaussian.

C..2 Gaussian White Noise

The stochastic dirac function can be discretised over a time steps of length Δ such that

$$X(t_j) = \frac{1}{\Delta} \int_{t_j}^{t_j+\Delta} \sum_{t_{sk}} \delta(t' - t_{sk}) dt'. \quad (13)$$

This function is sufficiently smooth over time scales less than τ_s provided $R\Delta \gg 1$ as before. Using our expression for the expected value of $\langle N \rangle$ the Poisson value of Poisson distribution can be expressed using a Gaussian variable

$$X(t_j) \simeq \frac{N}{\Delta} = R_s + \sqrt{R_s} \frac{\phi_j}{\sqrt{\Delta}} \quad (14)$$

where ϕ_j is a unit Gaussian random number with mean $\langle \phi_j \rangle = 0$ and unit variance $\langle \phi_j^2 \rangle$. Now it is possible to define a function

$\xi(t)$ under the limit $\Delta \rightarrow 0$ such that

$$X(t) = \sum_{t_s} \delta(t - t_{sk}) \simeq R_s + \sqrt{R_s} \xi(t) \quad (15)$$

where

$$\xi(t_j) = \lim_{\Delta \rightarrow 0} \frac{\phi_j}{\sqrt{\Delta}}. \quad (16)$$

It is now important to show that it behaves under integration over finite time periods T and has the same statistical values as the Poisson case. By considering the integral over a time $T = N\Delta$ and the definition of $\xi(t)$, it is possible to write

$$\int_0^T \xi(t') dt' \simeq \sum_{k=1}^{\frac{T}{\Delta}} \Delta \frac{\phi_k}{\sqrt{\Delta}}. \quad (17)$$

The following summation properties can be used

$$\langle \sum_i \phi_i \rangle = 0 \quad (18)$$

$$\langle [\sum_i \phi_i]^2 \rangle = \sum_i \langle \phi_i^2 \rangle \quad (19)$$

to express the variability as a Gaussian number.

$$\sum_{k=1}^N \Delta \frac{\phi_k}{\sqrt{\Delta}} = \sqrt{N\Delta} \phi = \sqrt{T} \phi. \quad (20)$$

Here it is important to note that although the function $\xi(t)$ was not well defined due to the arbitrary choice of Δ , its integral exists independent of this choice and is thus behaved and well defined under integration.

The statistical properties of $\xi(t)$ can also be determined in order to show that

it agrees with the Poisson model. The aim is to show that these two expressions have the same mean and variance

$$\sum_{t_s} \delta(t - t_1) \simeq R_s + \sqrt{R_s} \xi(t). \quad (21)$$

From Poisson statistics, it can be shown that for the Poisson model, both mean and variance are equal to a value RT . What is left is to show the same holds for our Gaussian approximation. The mean can be computed by considering the integral over finite time.

$$\int_0^T R + \sqrt{R} \xi(t') dt' = RT + \sqrt{R} \int_0^T dt' \langle \xi(t') \rangle \quad (22)$$

where $\langle \xi(t) \rangle = 0$ by definition of $\xi(t)$. Thus it is clear they have the same mean. And now the variance can be computed as

$$R \int_0^T dt' \int_0^T dt'' \langle \xi(t') \xi(t'') \rangle \quad (23)$$

where

$$\langle \xi(t) \xi(t') \rangle = \frac{\langle \phi_j \phi_k \rangle}{\Delta} = \frac{\delta_{jk}}{\Delta} \rightarrow \delta(t - t_1) \quad (24)$$

thus

$$R \int_0^T dt' \int_0^T dt'' \delta(t - t_1) = RT \quad (25)$$

concluding the statistical comparison.

Substituting the Gaussian approximation of the Poisson process C.2 Eq (15), into the fast-synaptic conductance equation

tion

$$\tau \dot{v} = -v + a\tau \sum \delta(t - t_{sk}) \quad (26)$$

results in a stochastic differential equation of the form

$$\frac{dV}{dt} = \frac{Ra\tau - V}{\tau} + a\sqrt{R}\xi(t) \quad (27)$$

which can be re-parameterised in terms of mean and standard deviation given as the Ornstein–Uhlenbeck process

$$\tau \dot{v} = \mu - v + \sigma \sqrt{R\tau} \xi(t) \quad (28)$$

whereby $\langle v \rangle = aR\tau$ and $\langle \sigma^2 \rangle = \frac{\tau a^2 R}{2}$

D. Populations of Neurons

Now that a working understanding of singular neuron dynamics has been established, it is useful to consider the effects of a fluctuating synaptic input on a population of disconnected neurons in order to determine their distribution and firing rates [15].

Given the probability density $P(V, t)$ of finding a neuron with voltage V at time t , consider the flux of voltage trajectories $J(V)$ across a voltage boundary. Clearly the flux across a voltage range $V \rightarrow V + \delta v$ over time δt is given by $(J(V) - J(V + \delta v))\delta t$. This flux must correspond with a change in the probability $P(V, t)$ of finding a neuron in said voltage range. Thus by

taking the continuum limit

$$\frac{\partial P}{\partial t} + \frac{\partial J}{\partial V} = 0. \quad (29)$$

This result can also be derived from the Taylor expansion of excitatory and inhibitory contributions to the flux in the form

$$\Delta \frac{dP}{dt} = \Delta P(V - a_e)R_e + \Delta P(V - a_i)R_i - \Delta P(V)[R_e - R_i]. \quad (30)$$

The flux $J(V)$ is comprised of a deterministic drift component which is proportional to the time derivative of average voltage velocity and a fluctuating component proportional to the density gradient due to the noisy nature of the input. These quantities can be expressed as

$$J_{drift} = \frac{dV}{dt}P = \frac{(\mu - V)}{\tau}P \quad (31)$$

$$J_{fluc} = -\frac{\alpha^2}{\tau} \frac{dP}{dV}. \quad (32)$$

Since total flux $J(V)$ is given by $J = J_{drift} + J_{fluc}$ and $\langle V \rangle^2 - \mu^2 = \sigma_V^2 = \alpha^2$, it follows that

$$\tau J = (\mu - V)P - \sigma_V^2 \frac{dP}{dV}. \quad (33)$$

The following Fokker-Planck equation is derived from the substitution of D. Eq(29) and D. Eq(33)

$$\tau \frac{\partial P}{\partial t} = \sigma^2 \frac{\partial^2 P}{\partial V^2} + \frac{\partial}{\partial V}((V - \mu)P). \quad (34)$$

D..1 Firing Rate of LIF Populations

A number of observations can be made directly from the behaviour of a neuron. By considering a fixed spike threshold value V_{th} and reset value V_{re} note that a neuron is instantaneously reset to a voltage V_{re} upon reaching the threshold value. This implies that $P(V_{th}) = 0$ and that the firing rate $r = J(V_{th})$. Consider now the steady state approximation whereby $\frac{\partial P}{\partial t} = 0$. Due to the observations made above, it is possible to evaluate D. Eq(33) at V_{th} such that the firing rate

$$r = -\frac{\sigma_V^2}{\tau} \frac{\partial P}{\partial V} \Big|_{V_{th}}. \quad (35)$$

Note also that since neuron voltage is reset to V_{re} the flux around around the reset voltage obeys

$$J_{re-} + r = J_{re+}. \quad (36)$$

It can be assumed that the flux below the reset voltage is zero since neuronal voltage will tend towards the stable fixed point at $V = \mu$. Thus we can use a heavy side function to express the flux as a function of the firing rate in D..1 Eq(35)

$$J = r\theta(V - V_{re}) \text{ for } V \leq V_{th}. \quad (37)$$

By making the following substitutions

$$x = \frac{V - \mu}{\sigma_V} \quad (38)$$

$$P(V) dv = p(x) dx \quad (39)$$

D. Eq(34) can be written in the form

$$-\tau r \theta(x - x_{re}) = xp + \frac{\partial p}{\partial x} = e^{-x^2/2} \frac{\partial}{\partial x} (e^{x^2/2} p). \quad (40)$$

Integrating from x to x_{th} yields an expression of the form

$$p(x) = \tau r e^{-x^2/2} \int_x^{x_{th}} dy e^{y^2/2} \theta(y - x_{re}). \quad (41)$$

By using the normalisation condition

$$\int_{-\infty}^{x_{th}} dx p(x) = 1 \quad (42)$$

D..1 Eq(41) can be written in the form

$$\tau r \int_{x_{re}}^{x_{th}} dy \int_{-\infty}^y dx e^{y^2/2} e^{-x^2/2} = 1. \quad (43)$$

Now by making the substitution $x = y - z$, the firing rate can be simplified as

$$r = \frac{1}{\tau} Z \quad (44)$$

where

$$Z = \int_0^\infty \frac{dz}{z} (e^{x_{th}z} - e^{x_{re}z}) e^{-z^2/2}. \quad (45)$$

It should be noted that this integral is strictly positive for all $t \geq 0$ and can be seen by applying L'Hopital's rule on the limit of small t .

D..2 Exponential-IF (EIF) Model

The LIF model that has been used thus far can be improved by adding in an exponential term that accounts for instantaneous activation of sodium currents. This produces voltage dynamics that more accurately depict the spiking behaviour of a neuron. The exponential IF model is given by

$$\tau_L \frac{dV}{dt} = \mu_L - V + \Delta_T e^{(V-V_T)/\Delta_T} + \sigma_v \sqrt{2\tau} \xi(t) \quad (46)$$

where the noise term is taken from the Ornstein-Uhlenbeck process. To simplify the analysis, it is possible to first consider the noise free solution for the firing rate.

Note that above an sufficiently large voltage V_T , the voltage diverges in finite time and resets such that $V_{th} = \infty$. As a result consider the effective threshold as V_T with a spike initiation voltage range Δ_T .

By considering the phase plane of the EIF model, it can be shown that the voltage undergoes a continuous phase transition. This means that there is a critical point $\mu^* = V_T - \Delta_T$ where the fixed points merge. For potentials lower than the critical value, the function exhibits two fixed points, one stable and the other unstable, allowing the excitability of the neuron. If the resting potential is sufficiently larger than the critical value, the voltage diverges and is reset instantly resulting in a model

that spontaneously oscillates.

Above the critical point the firing rate $r = 1/T$ where

$$\int_{V_{re}}^{V_{th}} \frac{dV}{F(V)} = \frac{T}{\tau_L} \quad (47)$$

and $F(V) = \tau_L \frac{dV}{dt}$. This integral must be calculated numerically and it should be noted that a discontinuity is expected at the critical value where the phase transition occurs.

D..3 Firing Rate of EIF Populations

The Fokker-Planck equation D. Eq(34) can now be adapted for a population of EIF neurons with a noisy input.

$$\tau r \theta(V - V_{re}) = P(\mu - V + \Delta_T e^{\frac{(V-V_T)}{\Delta T}}) - \sigma_v^2 \frac{dP}{dV} \quad (48)$$

which can be solved as a first order differential equation that is linear in P. The probability density function $P(V)$ can then be expressed in the form

$$P(V) = \frac{r\tau}{\sigma_v^2} e^{-h(v)} \int_V^\infty du \theta(u - V_{re}) e^{h(u)} \quad (49)$$

where

$$h(v) = \frac{1}{\sigma_v^2} \left(\frac{v^2}{2} - \mu v - \Delta_T^2 e^{\frac{v-V_T}{\Delta T}} \right). \quad (50)$$

Note that the heavy side step function in D..3 Eq(49) makes it sufficient to compute P(V) for only $V > V_{re}$. The same normal-

isation condition as the LIF model holds

$$\int_{-\infty}^{V_{th}} dV P(V) = 1 \quad (51)$$

which allows the firing rate r , to be expressed as

$$\tau \int_{-\infty}^{V_{th}} dv I(v) = \frac{\sigma_v^2}{r} \quad (52)$$

with

$$I(v) = e^{-h(v)} \int_V^\infty du \theta(u - V_{re}) e^{h(u)}. \quad (53)$$

III. Results

A. Post-Synaptic Voltage and Conductance

Synaptic conductance in the fast synapse approximation is given by

$$\frac{dx}{dt} = -\frac{x}{\tau_m} + a\delta(t - t_{sk}) \quad (54)$$

where $\delta(t - t_{sk})$ simulates independent and identically distributed events corresponding to the opening of synaptic channels. This is computed as a Poisson process with a fixed rate which is set at $R = \frac{5}{1000} \text{ ms}^{-1}$

By considering the synaptic conductance in the form $g_s(t)(E_s - V) = I$ and using the values already computed, the voltage equation can be evaluated in the form

$$\frac{dV}{dt} = \frac{g_L(E_L - V) + g_s(t)(E_s - V)}{C} \quad (55)$$

where the reversal potential E_s depends on

the excitatory or inhibitory nature of the neuron; here it suffices to have $E_S = 0$. The values of both conductance and voltage at each time step are shown in Figure 1.

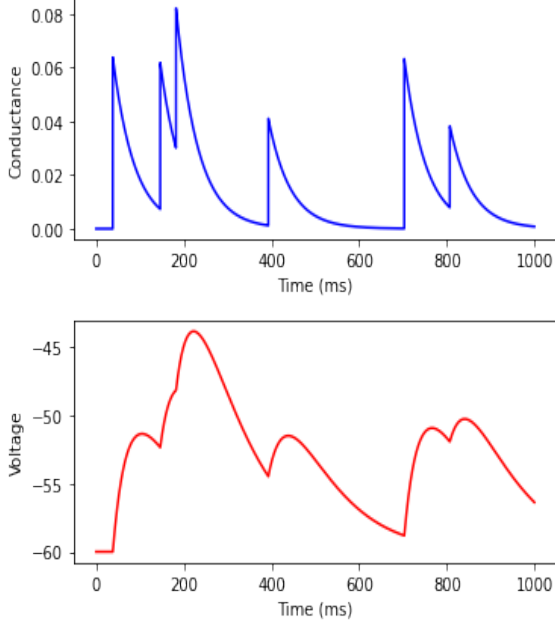


Figure 1: Graph of conductance variation due to the opening and closing of synaptic channels (top) and the resulting effect on neuronal voltage computed from conductance values (bottom). Parameters $\tau = 50\text{ms}$, $g_L = 0.1\text{nS}$, $E_L = -60\text{mV}$, $E_S = 0$.

B. Stochastic Fast Synapse Model

Approach the modelling of neuronal voltage by first considering the theoretical Poisson model given by

$$\frac{dV}{dt} = -\frac{V}{\tau_m} + a \sum_{t_{sk}} \delta(t - t_{sk}) \quad (56)$$

which produces a voltage distribution as shown at the top of Figure 2.

The Poisson distributed events are then discretised over an arbitrary interval and using a Gaussian approximation express

the stochastic term as a fixed average with variability given by a Gaussian number

$$E[\sum_{t_{sk}} \delta(t - t_{sk})] = R\Delta + \phi_k \sqrt{R\Delta} \quad (57)$$

in order to solve the stochastic voltage equation C.2 Eq (27) using an Euler method of the form

$$V(t + \Delta) = V_k + \frac{\Delta}{\tau} (aR\tau - V_k) + a\sqrt{R\Delta}\phi_k \quad (58)$$

which produces the values shown in red in Figure 2.

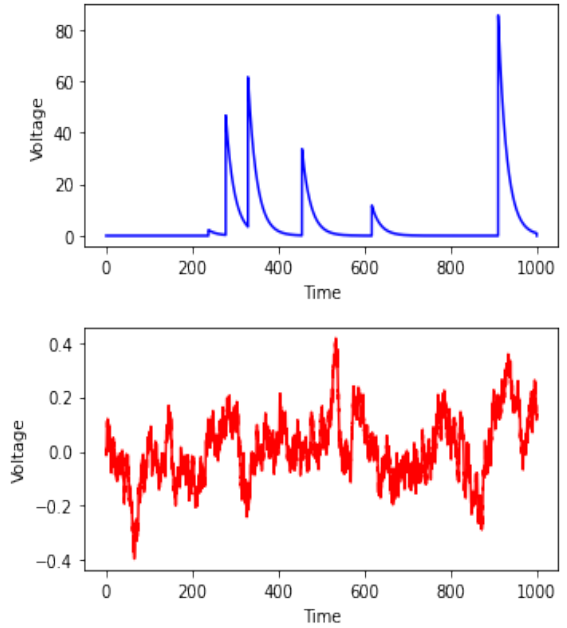


Figure 2: Comparison of voltage described by the previous theoretical Poisson model (top) and the Gaussian approximation (bottom) with fixed mean and variance $\mu = 10\text{mV}$ and $\sigma_V^2 = 5\text{mV}^2$

The degree to which the approximation is statistically valid in comparison to the theoretical model was calculated using voltage variance reparameterisation using the Ornstein-Uhlenbeck process as mentioned

in C..2 Eq (28).

C. Stochastic IF Neuron

The voltage equation is adapted and written in terms of voltage variance and mean voltage with stochastic Gaussian input

$$\tau_0 \frac{dV}{dt} = E_0 - V + \sigma_V \sqrt{t\tau_0} \xi(t). \quad (59)$$

The aim is to make the spike dynamic of the neuron more realistic by considering the refractory state of a neuron. This is done by putting a physical limit of the voltage at a threshold value V_{th} which causes an instantaneous reset to a refractory value V_{re} and can be seen in Figure 3.

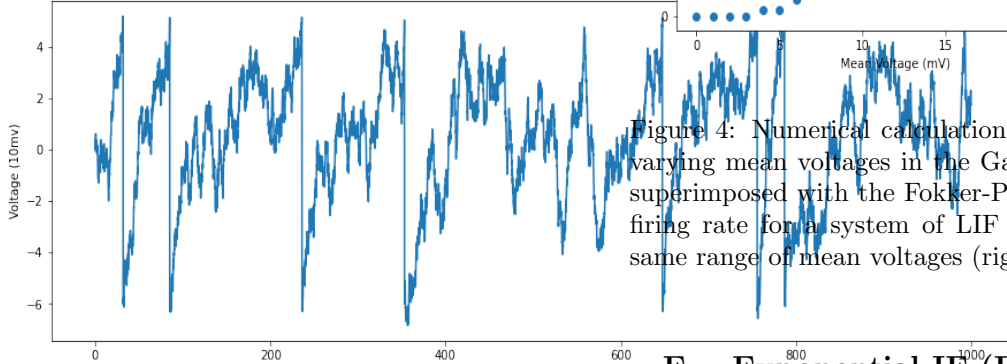


Figure 3: Synaptic voltage with stochastic Gaussian input account for refractory states with reset voltage $V_{re} = -60$ mV. Simulated with mean voltage $\mu = 20$ mV and an expected rate of spikes at $R = \frac{2}{1000} \text{ms}^{-1}$ over a time period of 10 seconds.

D. Populations of LIF Neurons

The dynamics of a system of disconnected neurons is described by the Fokker-Planck equation D. Eq(34) and can be used to ex-

press the firing rate D..1 Eq(44) in the form

$$r = \frac{1}{\tau} \int_0^\infty \frac{dz}{z} (e^{x_{th}z} - e^{x_{re}z}) e^{-z^2/2}. \quad (60)$$

The integral can be solved for small values of t by considering the expansion of $(e^{x_{th}z} - e^{x_{re}z})$ due to L'Hopital's rule. By considering a range of mean voltages it is possible to determine the expected firing rate and compare with numerically calculated rates from the Gaussian case as shown in Figure 4.

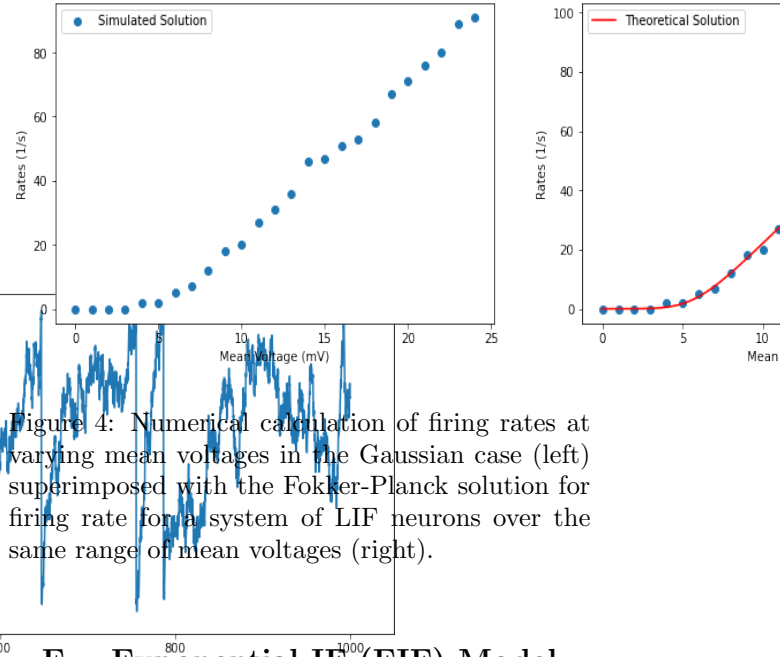


Figure 4: Numerical calculation of firing rates at varying mean voltages in the Gaussian case (left) superimposed with the Fokker-Planck solution for firing rate for a system of LIF neurons over the same range of mean voltages (right).

E. Exponential IF (EIF) Model

Using the exponential variant of the leaky stochastic model described in D..2 Eq(46) it is possible to produce a more realistic spike dynamic as in Figure 5. Adding in a fluctuating Gaussian input, the voltage equation is expressed as

$$\tau_L \frac{dV}{dt} = \mu_L - V + \Delta_T e^{(V-V_T)/\Delta_T} + \sigma_v \sqrt{2\tau} \xi(t). \quad (61)$$

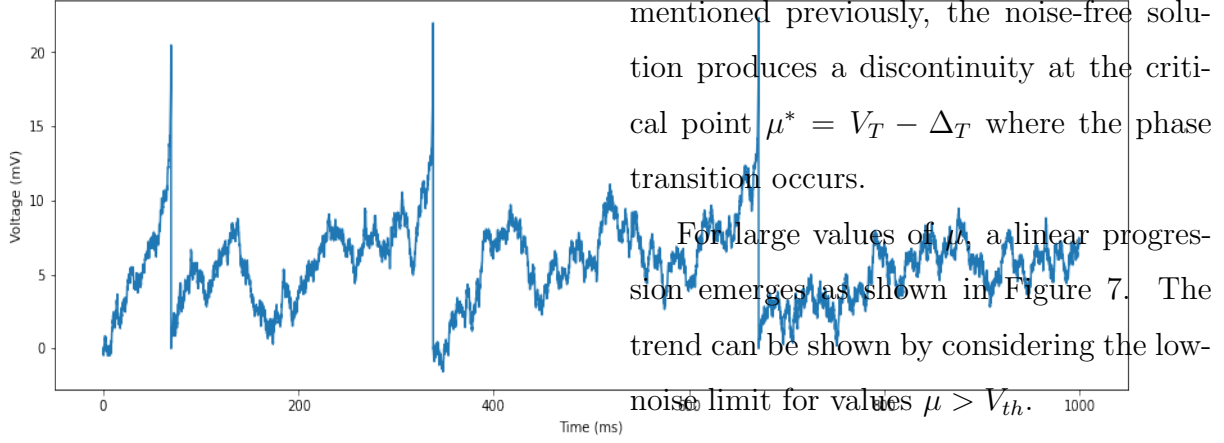


Figure 5: Voltage dynamics of the EIF model over a run time of 1000ms with $V_{re} = 0\text{mV}$. Arbitrary threshold $V_{th} = 20\text{mV}$ and membrane potential $V_T = 6\text{mV}$ at an expected mean voltage of $\mu = 5\text{mV}$.

F. Populations of EIF Neurons

As before, the firing rate can be calculated both from the voltage simulation and the theoretical form. Figure 6 shows the scatter of firing rates of the EIF model over a range of voltages. The theoretical noise-free expression for the firing rate can be numerically calculated above a critical value D..2 Eq(47) as

$$\int_{V_{re}}^{V_{th}} \frac{dV}{F(V)} = \frac{T}{\tau_L} \quad (62)$$

with $r = 1/T$.

The Fokker-Planck equation can also be solved to consider a population of EIF neurons D..3 Eq(48) with a noisy input whereby the firing rate can be expressed as

$$r = \frac{\sigma_V^2}{\tau \int_{-\infty}^{V_{th}} dv I(v)} \quad (63)$$

where $I(V)$ is given in D..3 Eq(53). As

mentioned previously, the noise-free solution produces a discontinuity at the critical point $\mu^* = V_T - \Delta_T$ where the phase transition occurs.

For large values of μ , a linear progression emerges as shown in Figure 7. The trend can be shown by considering the low-noise limit for values $\mu > V_{th}$.

$$Z \simeq \int_0^\infty \frac{dz}{z} (e^{zx_{th}} - e^{zx_{re}}) = \int_{x_{re}}^{x_{th}} dx \int_0^\infty dz e^{xz} = \log\left(\frac{x_{re}}{x_{th}}\right) \quad (64)$$

This allows the firing rate for this limit to be expressed as

$$r \simeq \frac{1}{\tau \log\left(\frac{\mu - V_{re}}{\mu - v_{th}}\right)} \quad (65)$$

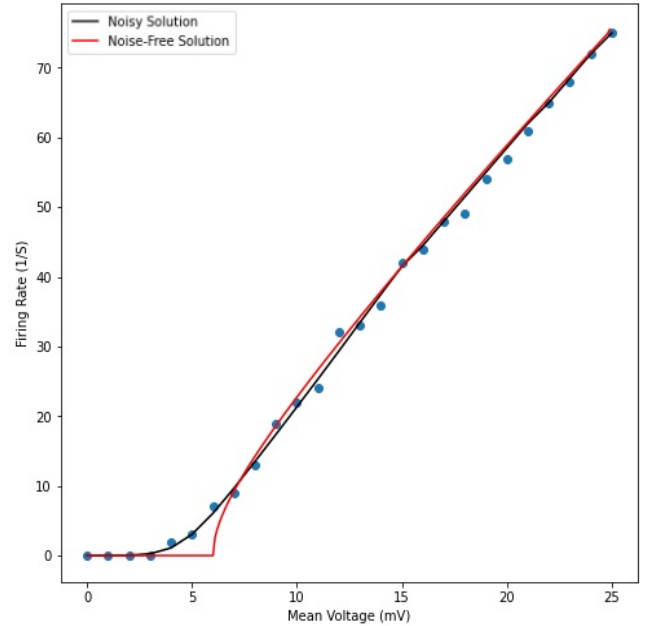


Figure 6: Scatter showing firing rates of EIF neurons calculated from simulated voltage dynamics at different mean voltages. Red curve showing the noise-free solution to the Fokker-Planck equation for EIF neurons with an observable discontinuity at the critical point. Black curve shows the noisy solution to the Fokker-Planck equation. Both curves tend towards a constant value as expected from the low-noise limit for mean voltages $\mu > V_{th}$.

G. Spike Probability Distributions

D.3 Eq(49) Allows the numerical calculation of the probability density function for different values of μ

$$P(V) = \frac{r\tau}{\sigma_v^2} e^{-h(v)} \int_V^\infty du \theta(u - V_{re}) e^{h(u)} \quad (66)$$

which can then be normalised and fitted over a histogram of the voltage distribution to provide a representation of finding a neuron with a specific voltage V for any given time.

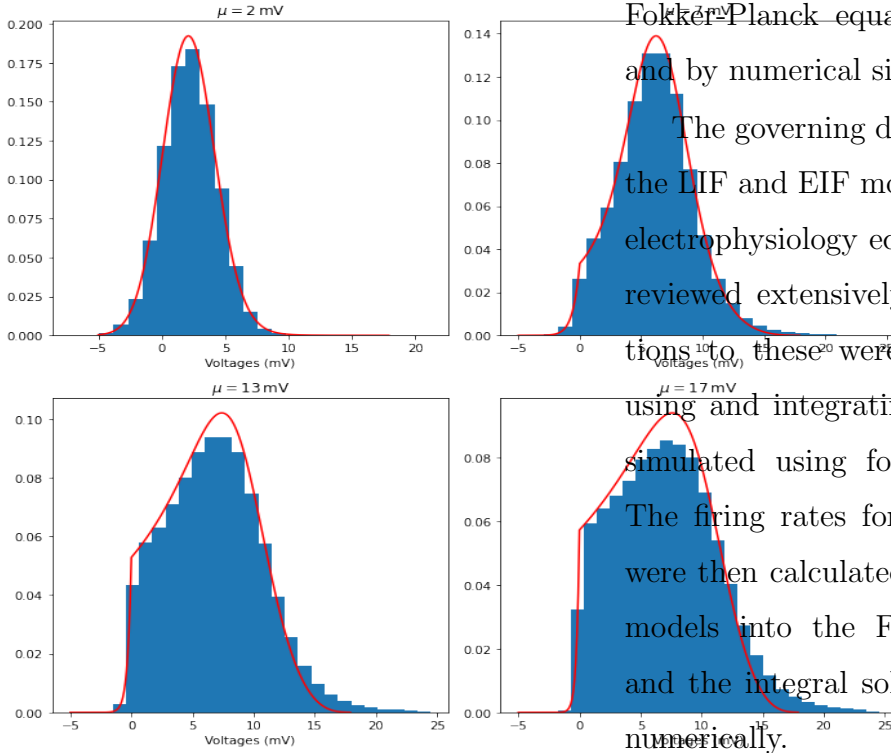


Figure 7: Probability distributions for mean voltages $\mu = 2, 7, 13, 17$ mV in the steady state, fitted over the respective voltage distributions and normalised with discontinuities occurring at $V_{re} = 0$ mV as expected. Voltage data from simulating and EIF neuron over an extended period of 10s.

Due to the heavy side step function in the expression for the probability density

Eq(66), a discontinuity is expected at values $V = V_{re}$ and can clearly be seen for sufficiently large μ .

IV. Conclusions

This project showed the derivations and simulations of non-linear and exponential integrate-and-fire (IF) models for neuronal voltage dynamics. It presents an accurate method for calculating the firing rate of populations of disconnected neurons with the aforementioned models using the Fokker-Planck equation both analytically and by numerical simulation.

The governing differential equations for the LIF and EIF models were derived from electrophysiology equations and have been reviewed extensively [2, 3, 7]. The solutions to these were explored analytically using integrating factor methods and simulated using forward Euler methods. The firing rates for neuronal populations were then calculated by substituting these models into the Fokker-Planck equation and the integral solutions were calculated numerically.

The models allow the numerical calculation of expected firing rates which were then compared to the theoretical solution given by the Fokker-Planck equations. The fact that these solutions were comparable via two different approaches shows that the Fokker-Planck equation is reliable at pro-

ducing exact solutions for the firing rates of neurons for both linear and non-linear models.

Additionally, the probability distributions for the EIF model was shown to correspond with the theoretical description derived from the Fokker-Planck equation and was shown to be consistent across a range of mean voltages.

The limitations for this approach lie in the assumptions made on how a population of neurons behave. For the purposes of this project, populations were considered disconnected meaning the voltage dynamics of a singular neuron would not impact that of any other neuron. The logical progression is to therefore consider what happens when network feedback occurs in a sufficiently large population of neurons.

Consider a population of LIF neurons with fluctuating synaptic input as discussed earlier. If the size of the population is sufficiently large the expected feedback from the system can be described as $A \propto Nr$ where the feedback A is proportional to the size of the population N and the firing rate of the neurons r .

Now the mean voltage of a neuron can be considered as the sum of external voltage input and the network feedback $\tilde{\mu} = \mu_{ext} + Ar$ which then allows the firing rate to be calculated as $r = \frac{\tilde{\mu} - \mu_{ext}}{A}$. Again this behaviour can be calculated over a range of mean voltages to determine the average

firing rates of a population.

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