

RBC Models

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1 Baseline RBC model

I am considering the Hansen-Style RBC model. Here firms are producing a good using the production function:

$$Y_t := Y_t(Z_t, K_{t-1}, L_t) = Z_t K_{t-1}^\alpha L_t^{1-\alpha}, \quad (1)$$

and households are endowed with a utility function characterised by

$$u_t := u(C_t, L_t) = c_t - \xi \frac{L_t^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}, \quad (2)$$

where $c_t := \log C_t$. Since there are no distortions, I can solve the model in the spirit of Ramsay social planner's problem, i.e., maximising

$$\max_{\{C_t, L_t, K_t\}} \mathbb{E}_t \left[\sum_{k \geq 0} \beta^k u(C_{t+k}, L_{t+k}) \right] \quad (3)$$

subject to

$$C_t \leq Y_t - I_t, \quad (4)$$

where the dynamics of K_t , Y_t and A_t are driven by

$$K_t = I_t + (1 - \delta)K_{t-1} \quad (5)$$

$$Y_t = Z_t K_{t-1}^\alpha L_t^{1-\alpha}, \quad (6)$$

$$\log Z_t = \rho_a \log Z_{t-1} + \varepsilon_t^z, \quad (7)$$

where $\varepsilon_t^z \stackrel{\text{iid}}{\sim} \mathcal{N}(0, \sigma_z^2)$. Note that from (5) I can write I_t in terms time- t and time- $(t-1)$ capital, i.e., $I_t = K_t - (1 - \delta)K_{t-1}$. Hence, I can rewrite (4) as

$$C_t \leq Z_t K_{t-1}^\alpha L_t^{1-\alpha} - (K_t - (1 - \delta)K_{t-1}) \quad (8)$$

On the other hand, notice that

$$\mathbb{E}_t \left[\sum_{k \geq 0} \beta^k u(C_{t+k}, L_{t+k}) \right] = u_t + \beta \mathbb{E}_t [u_{t+1}] + \mathbb{E}_t \left(\sum_{k \geq 2} \beta^k u_{t+k} \right)$$

I can now define the Lagrangian:

$$\mathcal{L}_t := \mathcal{L}_t(C_t, L_t, K_t, \lambda_t) = u_t + \beta \mathbb{E}_t [u_{t+1}] + (\dots) + \lambda_t (A_t K_{t-1}^\alpha L_t^{1-\alpha} - (K_t - (1-\delta)K_{t-1}) - C_t). \quad (9)$$

Now I take the FOC:

$$\partial_{C_t} \mathcal{L}_t = \partial_{C_t} u_t - \lambda_t = 0, \quad (10)$$

$$\partial_{L_t} \mathcal{L}_t = \partial_{L_t} u_t + (1-\alpha)\lambda_t Z_t K_{t-1}^\alpha L_t^{-\alpha} = 0, \quad (11)$$

$$\partial_{K_t} \mathcal{L}_t = \beta \mathbb{E}_t [\partial_{K_t} u_{t+1}] - \lambda_t = 0. \quad (12)$$

Now, subbing (10) into both (11) and (12), I obtain

$$\partial_{L_t} u_t = -(1-\alpha)\partial_{C_t} u_t Z_t K_{t-1}^\alpha L_t^{-\alpha} \quad (13)$$

$$\partial_{C_t} u_t = \beta \mathbb{E}_t [\partial_{K_t} u_{t+1}]. \quad (14)$$

I just need now to compute the partial derivatives:

$$\partial_{C_t} u_t = \frac{1}{C_t}, \quad \partial_{L_t} u_t = -\xi L_t^{\frac{1}{\nu}}, \quad \partial_{K_t} u_{t+1} = \frac{1}{C_t} \left(\alpha Z_{t+1} K_t^{\alpha-1} L_{t+1}^{(1-\alpha)} + (1-\delta) \right), \quad (15)$$

where I have used that the constraint (8) is binding, because $\partial_{C_t} u_t > 0$ by assumption (non-satiation). Notice on the other hand that

$$Z_{t+1} K_t^{\alpha-1} L_{t+1}^{(1-\alpha)} = \frac{Y_{t+1}}{K_t}, \quad Z_t K_{t-1}^\alpha L_t^{-\alpha} = \frac{Y_t}{L_t}.$$

Hence, the two FOC conditions are:

$$\xi L_t^{\frac{1}{\nu}} = (1-\alpha) \frac{Y_t}{L_t C_t} \quad (16)$$

$$\frac{1}{C_t} = \beta \mathbb{E}_t \left[\frac{1}{C_{t+1}} \left(\alpha \frac{Y_{t+1}}{K_t} + (1-\delta) \right) \right] \quad (17)$$

For simplicity, I am going to define two new variables:

$$R_{t+1} := \alpha \frac{Y_{t+1}}{K_t} + (1-\delta) \quad (18)$$

$$W_t := (1-\alpha) \frac{Y_t}{L_t}, \quad (19)$$

which can be interpreted as the marginal value (note that both are defined by means of partial derivatives) of an additional unit of capital at time t inherited in period $t + 1$, and the marginal product of labour (real wage), respectively. Hence, the FOC conditions are now:

$$W_t = \xi C_t L_t^{\frac{1}{\nu}} \quad (20)$$

$$1 = \beta \mathbb{E}_t \left[\frac{C_t}{C_{t+1}} Z_{t+1} \right]. \quad (21)$$

I am now in conditions of defining the equilibrium:

Definition 1 *The equilibrium of the model defined by all the equations (1)-(7) consists of*

1. *Set of prices, R_t, W_t ;*
2. *Allocations, Y_t, C_t, L_t, I_t and K_t ,*
3. *Productivity, Z_t*

all satisfying the FOC conditions (20) and (21), and where the dynamics of Y_t, K_t and Z_t are described by (6), (5), (7), respectively.

In order to solve the model using perturbation methods, I need to find the steady state. My notation for steady states is simply dropping the subscripts. Hence, note that $\mathbb{E}[\log Z_t] = 0$, which means that $\log A = \mathbb{E}[\log Z_t] = 0$ and then $A = 1$. On the other hand, it's also easy to see from the dynamics of K_t , (5), that

$$I = \delta K \quad (22)$$

. Also, from (4),

$$Y = C + I; \quad (23)$$

and from the dynamics of Y_t , (1),

$$Y = K^\alpha L^{1-\alpha}. \quad (24)$$

Putting these three equations together, I get $C + \delta K = K^\alpha L^{1-\alpha}$, which gives us the consumption-to-labour ratio

$$\frac{C}{L} = \left(\frac{K}{L} \right)^\alpha - \delta \frac{K}{L}. \quad (25)$$

Now take (18). Hence,

$$R = \alpha \frac{Y}{K} + (1 - \delta) = \alpha \frac{K^\alpha L^{1-\alpha}}{K} + (1 - \delta) = \alpha \left(\frac{K}{L} \right)^{\alpha-1} + (1 - \delta), \quad (26)$$

and then, from (21),

$$R = \frac{1}{\beta}, \quad (27)$$

Table 1: Parameters of my RBC Model

α	ν	ξ	β	δ	ρ_a	σ_a
0.3	2	4.5	0.99	0.025	0.95	0.01

and then,

$$\frac{1}{\beta} = \alpha \left(\frac{K}{L} \right)^{\alpha-1} + (1 - \delta), \quad (28)$$

from which is easy to identify the capital-to-labour variable:

$$\frac{K}{L} = \left(\frac{1}{\alpha} \left(\frac{1}{\beta} - (1 - \delta) \right) \right)^{\frac{1}{\alpha-1}} = \left[\frac{\alpha}{\frac{1}{\beta} - (1 - \delta)} \right]^{\frac{1}{1-\alpha}}. \quad (29)$$

Then I can express (19) in terms of (29), i.e.,

$$W = (1 - \alpha) \left(\frac{K}{L} \right)^{\alpha}. \quad (30)$$

For the moment, I have identified $Z, R, W, K/L$ and C/L . I am now interested in identifying K and L from which I, Y can be directly computed, and finally C . To do so, let's start from (20) and (19). Then,

$$(1 - \alpha) \left(\frac{K}{L} \right)^{\alpha} = \xi C L^{\frac{1}{\nu}}. \quad (31)$$

Note that here I only have consumption, C , but I can fix it by multiplying and dividing by L , i.e.,

$$(1 - \alpha) \left(\frac{K}{L} \right)^{\alpha} = \xi \frac{C}{L} L^{\frac{1}{\nu}+1}. \quad (32)$$

it's clear then that

$$L = \left[\frac{1 - \alpha}{\xi} \left(\frac{C}{L} \right)^{-1} \left(\frac{K}{L} \right)^{\alpha} \right]^{\frac{\nu}{1+\nu}}. \quad (33)$$

From here, K can be easily computed, because $K = \frac{K}{L}L$ and then I, Y and C . Finally, I am choosing the parameters that can be found in Table 1.

2 RBC-GK model

I will explain the different sectors separately:

2.1 Household Sector

I am considering a representative household who has to solve the following problem:

$$\max_{\{C_t, L_t, B_{t+1}\}} \mathbb{E}_t \left[\sum_{k \geq 0} \beta^{t+k} u_{t+k} \right] \quad (34)$$

subject to his period budget constraint, i.e.,

$$C_t + B_{t+1} = W_t L_t + R_t B_t + \Pi_t. \quad (35)$$

This can be rewritten as

$$v_h(B_t) = \max_{\{C_t, L_t, B_{t+1}\}} \{u_t + \beta \mathbb{E}_t [v_h(B_{t+1})]\} \quad (36)$$

I can write the Lagrangian here as

$$\mathcal{L}_t := \mathcal{L}_t(C_t, L_t, B_{t+1}) = u_t + \beta \mathbb{E}_t [v_h(B_{t+1})] + \lambda_t (W_t L_t + R_t B_t + \Pi_t - C_t - B_{t+1}). \quad (37)$$

where λ_{t+1} is predictable. From here, I just take the first order conditions:

$$\partial_{C_t} \mathcal{L}_t = \partial_{C_t} u_t - \lambda_t = 0 \quad (38)$$

$$\partial_{L_t} \mathcal{L}_t = \partial_{L_t} u_t + \lambda_t W_t = 0 \quad (39)$$

$$\partial_{B_{t+1}} \mathcal{L}_t = \beta \mathbb{E}_t [v'_h(B_{t+1})] - \lambda_t = 0. \quad (40)$$

The two first equations give us that

$$\xi L_t^{\frac{1}{\nu}} = \frac{W_t}{C_t}, \quad (41)$$

where we have use my computations before, i.e., $\partial_{L_t} u_t = -\xi L_t^{\frac{1}{\nu}}$. On the other hand, from (40), we get

$$\lambda_t = \mathbb{E}_t [\lambda_{t+1} R_{t+1}] \quad (42)$$

after making use of the Envelope Theorem:

$$v'(B_{t+1}) = \partial_{B_{t+1}} \mathcal{L}_{t+1} = \lambda_{t+1} R_{t+1}. \quad (43)$$

Finally, since $\lambda_{t+1} = \beta \mathbb{E}_t \left[\frac{1}{C_{t+1}} \right]$ (it can be seen iterating the maximisation and changing the expectation with the maximum under the assumption that u_t is increasing in C_t for all t),

$$1 = \mathbb{E}_t \left[\beta \frac{C_t}{C_{t+1}} R_{t+1} \right] = \mathbb{E}_t [\lambda_{t+1} R_{t+1}], \quad (44)$$

where

$$\Lambda_{t+1} := \beta \frac{C_t}{C_{t+1}}. \quad (45)$$

If we collect them together, we'd have

$$\xi L_t^{\frac{1}{\nu}} = \frac{W_t}{C_t}, \quad (46)$$

$$\mathbb{E}_t [\Lambda_{t+1} R_{t+1}] = 1. \quad (47)$$

2.2 The firm sector

Now consider a firm solving the following problem:

$$J(K_t) := \max_{\{K_{t+1}, L_t, \Pi_t\}} (\Pi_t + \beta \mathbb{E}_t [\Lambda_{t+1} J(K_{t+1})]) \quad (48)$$

subject to

$$Y_t + (1 - \delta)K_t \geq K_{t+1} + W_t L_t + \Pi_t, \quad (49)$$

where

$$Y_t = A_t K_t^\alpha L_t^{1-\alpha} \quad (50)$$

$$\log A_{t+1} = \rho_a \log A_t + \varepsilon_{t+1}^a \quad (51)$$

and. We set again the Lagrangian up:

$$\mathcal{L}_t := \mathcal{L}_t(K_{t+1}, L_t, \Pi_t) = \Pi_t + \beta \mathbb{E}_t [\Lambda_{t+1} J(K_{t+1})] - \lambda_t (K_{t+1} + W_t L_t + \Pi_t - Y_t - (1 - \delta)K_t). \quad (52)$$

The first order conditions are:

$$\partial_{\Pi_t} \mathcal{L}_t = 1 - \lambda_t = 0, \quad (53)$$

$$\partial_{L_t} \mathcal{L}_t = -\lambda_t (W_t - \partial_{L_t} Y_t) = 0, \quad (54)$$

$$\partial_{K_{t+1}} \mathcal{L}_t = \beta \mathbb{E}_t [\Lambda_{t+1} J'(K_{t+1})] - \lambda_t = 0. \quad (55)$$

Also, note that thanks to the Envelope Theorem,

$$J'(K_{t+1}) = \partial_{K_{t+1}} \mathcal{L}_{t+1} = \lambda_{t+1} (\partial_{K_{t+1}} Y_{t+1} + (1 - \delta)). \quad (56)$$

Now, since $\lambda_{t+1} = 1$ exactly by the same reason that $\lambda_t = 1$, and $\partial_{K_{t+1}} Y_{t+1} = \alpha \frac{Y_{t+1}}{K_{t+1}}$, then, the two FOC for this problem are

$$W_t = (1 - \alpha) \frac{Y_t}{L_t}, \quad (57)$$

$$1 = \mathbb{E}_t \left[\Lambda_{t+1} \left(\alpha \frac{Y_{t+1}}{K_{t+1}} + (1 - \delta) \right) \right]. \quad (58)$$

2.3 The banking sector

Each bank is solving the following problem

$$V_b(N_t) = \max_{K_{t+1}} \mathbb{E}_t [\Lambda_{t+1} [(1 - \psi)N_{t+1} + \psi V_b(N_{t+1})]] \quad (59)$$

subject to

$$V_b(N_t) \geq \lambda k_{t+1} \quad (60)$$

where N_t stands for net worth. In this model, bankers use net worth and (new) savings, B_{t+1} , to purchase new capital, i.e., $N_t + B_{t+1} = K_{t+1}$. On the other hand, net worth evolves through retained earnings, i.e.,

$$\begin{aligned} N_{t+1} &= R_{t,t+1}^k K_{t+1} - R_t^h B_{t+1} \\ &= R_{t,t+1}^k K_{t+1} - R_{t+1}^h (K_{t+1} - N_t) \\ &= (R_{t,t+1}^k - R_t^h) K_{t+1} - R_t^h N_t, \end{aligned} \quad (61)$$

where $R_t^h B_{t+1}$ indicates promises to households and $R_{t,t+1}^k K_{t+1}$ indicates the return on assets. We have seen in class that by guessing that is linear, i.e., $V_t(N_t) = A_t N_t$, we can show that

$$A_t = \frac{R_t^h}{1 - \mu_t} \mathbb{E}_t [\Lambda_{t+1} [(1 - \psi) + \psi A_{t+1}]], \quad (62)$$

$$\mu_t = \max \left\{ 1 - \frac{R_t^h N_t}{\lambda K_{t+1}} \mathbb{E}_t [\Lambda_{t+1} [(1 - \psi) + \psi A_t]], 0 \right\}, \quad (63)$$

and then rewrite

$$V(N_t) = \max_{K_{t+1}} \mathbb{E}_t [\hat{\Lambda}_{t+1} [R_{t+1}^k - R_{t+1}^h] K_{t+1}] \quad (64)$$

subject to

$$A_t N_t \geq \lambda K_{t+1}. \quad (65)$$

and where

$$\hat{\Lambda}_{t+1} = \Lambda_{t+1} [(1 - \psi) + \psi A_t]. \quad (66)$$

The Lagrangian of this problem is

$$\mathcal{L}_t = \mathcal{L}_t(K_{t+1}) = \mathbb{E}_t \left[\hat{\Lambda}_{t+1} [R_{t+1}^k - R_{t+1}^h] K_{t+1} \right] - \lambda_t (\lambda K_{t+1} - A_t N_t), \quad (67)$$

and then the first order condition is

$$\partial_{K_{t+1}} \mathcal{L}_t = \mathbb{E}_t \left[\hat{\Lambda}_{t+1} [R_{t+1}^k - R_t^h] \right] - \lambda \lambda_t = 0, \quad (68)$$

i.e.,

$$\mathbb{E}_t \left[\hat{\Lambda}_{t+1} R_{t+1}^k \right] = \mathbb{E}_t \left[\hat{\Lambda}_{t+1} R_t^h \right] + \lambda \mu_t. \quad (69)$$

Now, let's define the aggregate variables in the following way:

$$\begin{aligned} K_{t+1} &= \int_H K_{t+1}(N_{h,t}) dh, \\ B_{t+1} &= \int_H B_{t+1}(N_{h,t}) dh. \end{aligned} \quad (70)$$

If we define the cum interest that bankers need to pay to households as

$$P_t = R_t B_t, \quad (71)$$

and we assume that, in each period, the new bankers start with an aggregate capital of wK_t . Therefore, the aggregate net-worth equals

$$N_t = \psi(R_{t+1}^k K_t - P_t) - wK_t. \quad (72)$$

2.4 The Stationary State

The first thing we should note is that the goods market should clear, i.e.,

$$Y_t = C_t + I_t. \quad (73)$$

On the other hand, if we combine the labour optimality conditions for the household and the firm, i.e., (46) and (57), we get

$$L_t^{\frac{1}{\nu} + \alpha} = \frac{1 - \alpha}{\xi} \frac{A_t K_t^\alpha}{C_t} \implies L_t = \left(\frac{1 - \alpha}{\xi} \frac{A_t K_t^\alpha}{C_t} \right)^{\frac{1}{\frac{1}{\nu} + \alpha}}. \quad (74)$$

Same than before, the return on asset/capital equals the future benefit of having one additional unit of capital, i.e.,

$$R_{t+1}^k = \alpha \frac{Y_{t+1}}{K_t} + (1 - \delta). \quad (75)$$

On the other hand, note that, from (51), and assuming that $A_t = A$ for all t , I have that $A = 1$. Also, I have from (45), and assuming that $\Lambda_t = \Lambda$ for all t , that $\Lambda = \beta$. From (50), I have that the output-to-capital and the labour-to-capital are related by means of

$$\frac{Y}{K} = \left(\frac{L}{K} \right)^{1-\alpha}. \quad (76)$$

If I identify either the output-to-capital or the labour-to-capital, then I will identify W , because, from (57),

$$W = (1 - \alpha) \frac{Y}{L} = (1 - \alpha) \frac{K^\alpha L^{1-\alpha}}{L} = (1 - \alpha) \left(\frac{K}{L} \right)^\alpha. \quad (77)$$

Now, from (58),

$$1 = \beta \left(\alpha \frac{Y}{K} + (1 - \delta) \right) \implies \frac{Y}{K} = \frac{1}{\alpha} \left(\frac{1}{\beta} - (1 - \delta) \right). \quad (78)$$

I have managed then to identify the output-to-capital ratio and, for the reasons that I explained above, also, the labour-to-capital ratio and W . Now, from (49),

$$Y = \delta K + WL + \Pi \implies \frac{\Pi}{K} = \frac{Y}{K} - \delta - W \frac{L}{K}, \quad (79)$$

then $\frac{\Pi}{K}$ is also identified. Now, from (46) and (57),

$$\xi C L^{\frac{1}{\nu}} = (1 - \alpha) \frac{Y}{L}, \quad (80)$$

and therefore

$$L^{\frac{1}{\nu}+1} = \frac{1 - \alpha}{\xi} \frac{Y}{C} \implies L = \left(\frac{1 - \alpha}{\xi} \frac{Y}{C} \right)^{\frac{1}{1+\frac{1}{\nu}}}. \quad (81)$$

We would need then to identify the output-to-consumption ratio. From (62), and using that $R^h = \beta$ and $\Lambda = 1/\beta$,

$$A = \frac{\beta}{1 - \mu} \frac{1}{\beta} [(1 - \psi) + \psi A] \implies A(1 - \mu) = (1 - \psi) + \psi A \implies A = \frac{1 - \psi}{1 - \mu - \psi}, \quad (82)$$

If $\mu = 0$, then $A = 1$. On the other hand, we had

$$A = \lambda \left(\frac{N}{K} \right)^{-1} = \frac{1}{\frac{1}{\lambda} \frac{N}{K}}, \quad (83)$$

so we would need to identify N/K . In that case, from (69),

$$\beta R^k = \beta R^h + \lambda \mu \iff R^k = \frac{1 + \lambda \mu}{\beta}. \quad (84)$$

Then, from (61), and dividing it by K in both sides, we get

$$R^k - \frac{1}{\beta} \frac{B}{K} = \lambda \iff \frac{B}{K} = \beta (R^k - \lambda) = \beta \left(\frac{1}{\beta} - \lambda \right), \quad (85)$$

i.e.,

$$\frac{B}{K} = 1 - \lambda\beta. \quad (86)$$

Now, if we divide in both sides of (72) by K , we get

$$\frac{N}{K} = \psi \left(R^k - R^h \frac{B}{K} \right) - w = \frac{\psi}{\beta} (1 + \lambda\mu - (1 - \lambda\beta)) - w. \quad (87)$$

Hence, we have this system of equations:

$$\begin{cases} \frac{N}{K} = \frac{\psi}{\beta} (1 + \lambda\mu - (1 - \lambda\beta)) - w \\ 1 - \mu - \psi = \frac{1 - \psi}{\lambda} \frac{N}{K} \end{cases} \quad (88)$$

so we have identified also N/K and μ (system of two equations and two unknowns), and then also A . We are about to finish: notice that

$$\frac{C}{K} + \left(1 - \frac{1}{\beta}\right) \frac{B}{K} = W \frac{L}{K} + \frac{\Pi}{K} \implies \frac{C}{K} = W \frac{L}{K} + \frac{\Pi}{K} - \left(1 - \frac{1}{\beta}\right) (1 - \lambda\beta), \quad (89)$$

and then $\frac{C}{K}$ is also identified. From the dynamics of capital, i.e., $K_t = I_t + (1 - \delta)K_{t-1}$, we get $I = \delta K$, then

$$L = \left(\frac{1 - \alpha}{\xi} \left(1 + \delta \frac{K}{C} \right) \right)^{\frac{1}{1 - \frac{1}{\nu}}}. \quad (90)$$

With K in hands, I identify K , and then B , etc.