RBC Models

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1 Baseline RBC model

I am considering the Hansen-Style RBC model. Here firms are producing a good using the production function:

$$Y_t := Y_t(A_t, K_{t-1}, L_t) = A_t K_{t-1}^{\alpha} L_t^{1-\alpha}, \tag{1}$$

and households are endowed with a utility function characterised by

$$u_t := u\left(C_t, L_t\right) = c_t - \xi \frac{L_t^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}, \tag{2}$$

where $c_t := \log C_t$. Since there are no distortions, I can solve the model in the spirit of Ramsay social planner's problem, i.e., maximising

$$\max_{\{C_t, L_t, K_t\}} \mathbb{E}_t \left[\sum_{k \geq 0} \beta^k u\left(C_{t+k}, L_{t+k}\right) \right] \tag{3}$$

subject to

$$C_t \leq Y_t - I_t, \tag{4}$$

where the dynamics of K_t, Y_t and A_t are driven by

$$K_t = I_t + (1-\delta) K_{t-1} \tag{5}$$

$$Y_t = A_t K_{t-1}^{\alpha} L_t^{1-\alpha}, \tag{6}$$

$$\log A_t = \rho_a \log A_{t-1} + \varepsilon_t^a, \tag{7}$$

where $\varepsilon_t^a \overset{\text{iid}}{\sim} \mathcal{N}\left(0, \sigma_a^2\right)$. Note that from (5) I can write I_t in terms time-t and time-$(t-1)$ capital, i.e., $I_t = K_t - (1-\delta)K_{t-1}$. Hence, I can rewrite (4) as

$$C_t \leq A_t K_{t-1}^{\alpha} L_t^{1-\alpha} - (K_t - (1-\delta) K_{t-1}) \tag{8}$$

On the other hand, notice that

$$\mathbb{E}_t\left[\sum_{k\geq 0}\beta^k u\left(C_{t+k}, L_{t+k}\right)\right] = u_t + \beta\mathbb{E}_t\left[u_{t+1}\right] + \mathbb{E}_t\left(\sum_{k\geq 2}\beta^k u_{t+k}\right)$$

I can now define the Lagrangian:

$$\mathcal{L}_t := \mathcal{L}_t\left(C_t, L_t, K_t, \lambda_t\right) = u_t + \beta\mathbb{E}_t\left[u_{t+1}\right] + (\cdots) + \lambda_t\left(A_t K_{t-1}^{\alpha} L_t^{1-\alpha} - (K_t - (1-\delta)K_{t-1}) - C_t\right). \tag{9}$$

Now I take the FOC:

$$\partial_{C_t}\mathcal{L}_t = \partial_{C_t}u_t - \lambda_t = 0, \tag{10}$$

$$\partial_{L_t}\mathcal{L}_t = \partial_{L_t}u_t + (1-\alpha)\lambda_t A_t K_{t-1}^{\alpha} L_t^{-\alpha} = 0, \tag{11}$$

$$\partial_{K_t}\mathcal{L}_t = \beta\mathbb{E}_t\left[\partial_{K_t}u_{t+1}\right] - \lambda_t = 0. \tag{12}$$

Now, subbing (10) into both (11) and (12), I obtain

$$\partial_{L_t}u_t = -(1-\alpha)\partial_{C_t}u_t A_t K_{t-1}^{\alpha} L_t^{-\alpha} \tag{13}$$

$$\partial_{C_t}u_t = \beta\mathbb{E}_t\left[\partial_{K_t}u_{t+1}\right]. \tag{14}$$

I just need now to compute the partial derivatives:

$$\partial_{C_t}u_t = \frac{1}{C_t}, \qquad \partial_{L_t}u_t = -\xi L_t^{\frac{1}{\nu}}, \qquad \partial_{K_t}u_{t+1} = \frac{1}{C_t}\left(\alpha A_{t+1}K_t^{\alpha-1}L_{t+1}^{(1-\alpha)} + (1-\delta)\right), \tag{15}$$

where I have used that the constraint (8) is binding, because $\partial_{C_t}u_t > 0$ by assumption (non-satiation). Notice on the other hand that

$$A_{t+1}K_t^{\alpha-1}L_{t+1}^{(1-\alpha)} = \frac{Y_{t+1}}{K_t}, \qquad A_t K_{t-1}^{\alpha}L_t^{-\alpha} = \frac{Y_t}{L_t}.$$

Hence, the two FOC conditions are:

$$\xi L_t^{\frac{1}{\nu}} = (1-\alpha)\frac{Y_t}{L_t C_t} \tag{16}$$

$$\frac{1}{C_t} = \beta\mathbb{E}_t\left[\frac{1}{C_{t+1}}\left(\alpha\frac{Y_{t+1}}{K_t} + (1-\delta)\right)\right] \tag{17}$$

For simplicity, I am going to define two new variables:

$$Z_{t+1} := \alpha\frac{Y_{t+1}}{K_t} + (1-\delta) \tag{18}$$

$$W_t := (1-\alpha)\frac{Y_t}{L_t}, \tag{19}$$

which can be interepreted as the marginal value (note that both are defined by means of partial derivatives) of an additional unit of capital at time t inherited in period $t+1$, and the marginal product of labmy (real wage), respectively. Hence, the FOC conditions are now:

$$W_t = \xi C_t L_t^{\frac{1}{\nu}} \tag{20}$$

$$1 = \beta \mathbb{E}_t \left[\frac{C_t}{C_{t+1}} Z_{t+1} \right]. \tag{21}$$

I am now in conditions of defining the equilibrium:

Definition 1 *The equilibrium of the model defined by all the equations (1)-(7) consists of*

1. Set of prices, Z_t, W_t*;*

2. Allocations, Y_t*,* C_t*,* L_t*,* I_t *and* K_t*,*

3. Productivity, A_t

all satisfying the FOC conditions (20) and (21), and where the dynamics of Y_t*,* K_t *and* A_t *are described by (6), (5), (7), respectively.*

In order to solve the model using perturbation methods, I need to find the steaty state. My notation for steady states is simply dropping the subscripts. Hence, note that $\mathbb{E}[\log A_t] = 0$, which means that $\log A = \mathbb{E}[\log A_t] = 0$ and then $A = 1$. On the other hand, it's also easy to see from the dynamics of K_t, (5), that

$$I = \delta K \tag{22}$$

. Also, from (4),

$$Y = C + I; \tag{23}$$

and from the dyanmics of Y_t, (1),

$$.Y = K^{\alpha} L^{1-\alpha}. \tag{24}$$

Putting these three equations together, I get $C + \delta K = K^{\alpha} L^{1-\alpha}$, which gives us the consumption-to-labmy ratio

$$\frac{C}{L} = \left(\frac{K}{L} \right)^{\alpha} - \delta \frac{K}{L}. \tag{25}$$

Now take (18). Hence,

$$Z = \alpha \frac{Y}{K} + (1-\delta) = \alpha \frac{K^{\alpha} L^{1-\alpha}}{K} + (1-\delta) = \alpha \left(\frac{K}{L} \right)^{\alpha - 1} + (1-\delta), \tag{26}$$

and then, from (21), $\frac{1}{\beta} = Z$ and then,

$$\frac{1}{\beta} = \alpha \left(\frac{K}{L} \right)^{\alpha - 1} + (1-\delta), \tag{27}$$

Table 1: Parameters of my RBC Model

α	ν	ξ	β	δ	ρ_a	σ_a
0.3	2	4.5	0.99	0.025	0.95	0.01

from which is easy to identify the capital-to-labmy variable:

$$\frac{K}{L} = \left(\frac{1}{\alpha}\left(\frac{1}{\beta} - (1-\delta)\right)\right)^{\frac{1}{\alpha-1}} = \left[\frac{\alpha}{\frac{1}{\beta} - (1-\delta)}\right]^{\frac{1}{1-\alpha}}. \tag{28}$$

Then I can express (19) in terms of (28), i.e.,

$$W = (1-\alpha)\left(\frac{K}{L}\right)^{\alpha}. \tag{29}$$

For the moment, I have identified $A, Z, W, K/L$ and C/L. I am now interested in idenfitying K and L from which I, Y can be directly computed, and finally C. To do so, let's start from (20) and (19). Then,

$$(1-\alpha)\left(\frac{K}{L}\right)^{\alpha} = \xi C L^{\frac{1}{\nu}}. \tag{30}$$

Note that here I only have consumption, C, but I can fix it by multiplying and dividing by L, i.e.,

$$(1-\alpha)\left(\frac{K}{L}\right)^{\alpha} = \xi \frac{C}{L} L^{\frac{1}{\nu}+1}. \tag{31}$$

it's clear then that

$$L = \left[\frac{1-\alpha}{\xi}\left(\frac{C}{L}\right)^{-1}\left(\frac{K}{L}\right)^{\alpha}\right]^{\frac{\nu}{1+\nu}}. \tag{32}$$

From here, K can be easily computed, because $K = \frac{K}{L}L$ and then I, Y and C. Finally, I am choosing the parameters that can be found in Table 1.

2 RBC-GK model

I will explain the different sectors separately:

2.1 Household Sector

I am considering a representative household who has to solve the following problem:

$$\max_{\{C_t, L_t, B_{t+1}\}} \mathbb{E}_t\left[\sum_{k\geq 0}\beta^{t+k}u_{t+k}\right] \tag{33}$$

subject to his period budget constraint, i.e.,

$$C_t + B_{t+1} = W_t L_t + R_t B_t + \Pi_t. \tag{34}$$

This can be rewritten as

$$v_h(B_t) = \max_{\{C_t, L_t, B_{t+1}\}} \left\{u_t + \beta \mathbb{E}_t \left[v_h(B_{t+1})\right]\right\} \tag{35}$$

I can write the Lagrangian here as

$$\mathcal{L}_t := \mathcal{L}_t \left(C_t, L_t, B_{t+1}\right) = u_t + \beta \mathbb{E}_t \left[v_h(B_{t+1})\right] + \lambda_t \left(W_t L_t + R_t B_t + \Pi_t - C_t - B_{t+1}\right). \tag{36}$$

where λ_{t+1} is predictable. From here, I just take the first order conditions:

$$\partial_{C_t} \mathcal{L}_t = \partial_{C_t} u_t - \lambda_t = 0 \tag{37}$$

$$\partial_{L_t} \mathcal{L}_t = \partial_{L_t} u_t + \lambda_t W_t = 0 \tag{38}$$

$$\partial_{B_{t+1}} \mathcal{L}_t = \beta \mathbb{E}_t \left[v_h'(B_{t+1})\right] - \lambda_t = 0. \tag{39}$$

The two first equations give us that

$$\xi L_t^{\frac{1}{\nu}} = \frac{W_t}{C_t}, \tag{40}$$

where we have use my computations before, i.e., $\partial_{L_t} u_t = -\xi L_t^{\frac{1}{\nu}}$. On the other hand, from (39), we get

$$\lambda_t = \mathbb{E}_t \left[\lambda_{t+1} R_{t+1}\right] \tag{41}$$

after making use of the Envelope Theorem:

$$v'\left(B_{t+1}\right) = \partial_{B_{t+1}} \mathcal{L}_{t+1} = \lambda_{t+1} R_{t+1}. \tag{42}$$

Finally, since $\lambda_{t+1} = \beta \mathbb{E}_t \left[\frac{1}{C_{t+1}}\right]$ (it can be seen iterating the maximisation and changing the expectation with the maximum under the assumption that u_t is increasing in C_t for all t),

$$1 = \mathbb{E}_t \left[\beta \frac{C_t}{C_{t+1}} R_{t+1}\right] = \mathbb{E}_t \left[\Lambda_{t+1} R_{t+1}\right], \tag{43}$$

where

$$\Lambda_{t+1} := \beta \frac{C_t}{C_{t+1}}. \tag{44}$$

If we collect them together, we'd have

$$\xi L_t^{\frac{1}{\nu}} = \frac{W_t}{C_t}, \tag{45}$$

$$\mathbb{E}_t \left[\Lambda_{t+1} R_{t+1} \right] = 1. \tag{46}$$

2.2 The firm sector

Now consider a firm solving the following problem:

$$J(K_t) := \max_{\{K_{t+1}, L_t, \Pi_t\}} \left(\Pi_t + \beta \mathbb{E}_t \left[\Lambda_{t+1} J(K_{t+1}) \right] \right) \tag{47}$$

subject to

$$Y_t + (1-\delta) K_t \geq K_{t+1} + W_t L_t + \Pi_t, \tag{48}$$

where

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha} \tag{49}$$

$$\log A_{t+1} = \rho_a \log A_t + \varepsilon_{t+1}^a \tag{50}$$

and. We set again the Lagrangian up:

$$\mathcal{L}_t := \mathcal{L}_t \left(K_{t+1}, L_t, \Pi_t \right) = \Pi_t + \beta \mathbb{E}_t \left[\Lambda_{t+1} J(K_{t+1}) \right] - \lambda_t \left(K_{t+1} + W_t L_t + \Pi_t - Y_t - (1-\delta) K_t \right). \tag{51}$$

The first order conditions are:

$$\partial_{\Pi_t} \mathcal{L}_t = 1 - \lambda_t = 0, \tag{52}$$

$$\partial_{L_t} \mathcal{L}_t = -\lambda_t \left(W_t - \partial_{L_t} Y_t \right) = 0, \tag{53}$$

$$\partial_{K_{t+1}} \mathcal{L}_t = \beta \mathbb{E}_t \left[\Lambda_{t+1} J'(K_{t+1}) \right] - \lambda_t = 0. \tag{54}$$

Also, note that thanks to the Envelope Theorem,

$$J'(K_{t+1}) = \partial_{K_{t+1}} \mathcal{L}_{t+1} = \lambda_{t+1} \left(\partial_{K_{t+1}} Y_{t+1} + (1-\delta) \right). \tag{55}$$

Now, since $\lambda_{t+1} = 1$ exactly by the same reason that $\lambda_t = 1$, and $\partial_{K_{t+1}} Y_{t+1} = \alpha \frac{Y_{t+1}}{K_{t+1}}$, then, the two FOC for this problem are

$$W_t = (1-\alpha)\frac{Y_t}{L_t}, \tag{56}$$

$$1 = \mathbb{E}_t\left[\Lambda_{t+1}\left(\alpha\frac{Y_{t+1}}{K_{t+1}} + (1-\delta)\right)\right]. \tag{57}$$

2.3 The banking sector

Each bank is solving the following problem

$$V_b(N_t) = \max_{K_{t+1}} \mathbb{E}_t\left[\Lambda_{t+1}\left[(1-\psi)N_{t+1} + \psi V_b(N_{t+1})\right]\right] \tag{58}$$

subject to

$$V_b(N_t) \geq \lambda k_{t+1} \tag{59}$$

where N_t stands for net worth. In this model, bankers use net worth and (new) savings, B_{t+1}, to purchase new capital, i.e., $N_t + B_{t+1} = K_{t+1}$. On the other hand, net worth evolves through retained earnings, i.e.,

$$\begin{aligned} N_{t+1} &= R^k_{t,t+1}K_{t+1} - R^h_t B_{t+1} \\ &= R^k_{t,t+1}K_{t+1} - R^h_{t+1}\left(K_{t+1} - N_t\right) \\ &= \left(R^k_{t,t+1} - R^h_t\right)K_{t+1} - R^h_t N_t, \end{aligned} \tag{60}$$

where $R^h_t B_{t+1}$ indicates promises to households and $R^k_{t,t+1}K_{t+1}$ indicates the return on assets. We have seen in class that by guessing that is linear, i.e., $V_t(N_t) = A_t N_t$, we can show that

$$A_t = \frac{R^h_t}{1-\mu_t}\mathbb{E}_t\left[\Lambda_{t+1}\left[(1-\psi) + \psi A_{t+1}\right]\right], \tag{61}$$

$$\mu_t = \max\left\{1 - \frac{R^h_t N_t}{\lambda K_{t+1}}\mathbb{E}_t\left[\Lambda_{t+1}\left[(1-\psi) + \psi A_t\right]\right], 0\right\}, \tag{62}$$

and then rewrite

$$V\left(N_t\right) = \max_{K_{t+1}} \mathbb{E}_t\left[\hat{\Lambda}_{t+1}\left[R^k_{t+1} - R^h_{t+1}\right]K_{t+1}\right] \tag{63}$$

subject to

$$A_t N_t \geq \lambda K_{t+1}. \tag{64}$$

and where

$$\hat{\Lambda}_{t+1} = \Lambda_{t+1}\left[(1-\psi) + \psi A_t\right]. \tag{65}$$

The Lagrangian of this problem is

$$\mathcal{L}_t = \mathcal{L}_t\left(K_{t+1}\right) = \mathbb{E}_t\left[\hat{\Lambda}_{t+1}\left[R^k_{t+1} - R^h_{t+1}\right]K_{t+1}\right] - \lambda_t\left(\lambda K_{t+1} - A_t N_t\right), \tag{66}$$

and then the first order condition is

$$\partial_{K_{t+1}}\mathcal{L}_t = \mathbb{E}_t\left[\hat{\Lambda}_{t+1}\left[R^k_{t+1} - R^h_t\right]\right] - \lambda\lambda_t = 0, \tag{67}$$

i.e.,

$$\mathbb{E}_t\left[\hat{\Lambda}_{t+1}R^k_{t+1}\right] = \mathbb{E}_t\left[\hat{\Lambda}_{t+1}R^h_t\right] + \lambda\mu_t. \tag{68}$$

Now, let's define the aggregate variables in the following way:

$$\begin{aligned} K_{t+1} &= \int_H K_{t+1}\left(N_{h,t}\right)\mathrm{d}h, \\ B_{t+1} &= \int_H B_{t+1}\left(N_{h,t}\right)\mathrm{d}h. \end{aligned} \tag{69}$$

If we define the cum interest that bankers need to pay to households as

$$P_t = R_t B_t, \tag{70}$$

and we assume that, in each period, the new bankers start with an aggregate capital of wK_t. Therefore, the aggregate net-worth equals

$$N_t = \psi\left(R^k_{t+1}K_t - P_t\right) - wK_t. \tag{71}$$

2.4 The Stationary State

The first thing we should note is that the goods market should clear, i.e.,

$$Y_t = C_t + I_t. \tag{72}$$

On the other hand, if we combine the labour optimality conditions for the household and the firm, i.e., (45) and (56), we get

$$L_t^{\frac{1}{\nu}+\alpha} = \frac{1-\alpha}{\xi}\frac{A_t K_t^\alpha}{C_t} \implies L_t = \left(\frac{1-\alpha}{\xi}\frac{A_t K_t^\alpha}{C_t}\right)^{\frac{1}{\frac{1}{\nu}+\alpha}}. \tag{73}$$

Same than before, the return on asset/capital equals the future benefit of having one additional unit of capital, i.e,

$$R^k_{t+1} = \alpha\frac{Y_{t+1}}{K_t} + (1-\delta). \tag{74}$$

On the other hand, note that, from (50), and assuming that $A_t = A$ for all t, I have that $A = 1$. Also, I have from (44), and assuming that $\Lambda_t = \Lambda$ for all t, that $\Lambda = \beta$. From (49), I have that the output-to-capital and the labour-to-capital are related by means of

$$\frac{Y}{K} = \left(\frac{L}{K}\right)^{1-\alpha}. \tag{75}$$

If I idenfity either the output-to-capital or the labour-to-capital, then I will identify W, because, from (56),

$$W = (1-\alpha)\frac{Y}{L} = (1-\alpha)\frac{K^\alpha L^{1-\alpha}}{L} = (1-\alpha)\left(\frac{K}{L}\right)^\alpha. \tag{76}$$

Now, from (57),

$$1 = \beta\left(\alpha\frac{Y}{K} + (1-\delta)\right) \implies \frac{Y}{K} = \frac{1}{\alpha}\left(\frac{1}{\beta} - (1-\delta)\right). \tag{77}$$

I have managed then to identify the output-to-capital ratio and, for the reasons that I explained above, also, the labour-to-capital ratio and W. Now, from (48),

$$Y = \delta K + WL + \Pi \implies \frac{\Pi}{K} = \frac{Y}{K} - \delta - W\frac{L}{K}, \tag{78}$$

then $\frac{\Pi}{K}$ is also identified. Now, from (45) and (56),

$$\xi C L^{\frac{1}{\nu}} = (1-\alpha)\frac{Y}{L}, \tag{79}$$

and therefore

$$L^{\frac{1}{\nu}+1} = \frac{1-\alpha}{\xi}\frac{Y}{C} \implies L = \left(\frac{1-\alpha}{\xi}\frac{Y}{C}\right)^{\frac{1}{1+\frac{1}{\nu}}}. \tag{80}$$

We would need then to identify the output-to-consumption ratio. From (61), and using that $R^h = \beta$ and $\Lambda = 1/\beta$,

$$A = \frac{\beta}{1-\mu}\frac{1}{\beta}\left[(1-\psi) + \psi A\right] \implies A(1-\mu) = (1-\psi) + \psi A \implies A = \frac{1-\psi}{1-\mu-\psi}, \tag{81}$$

If $\mu = 0$, then $A = 1$. On the other hand, we had

$$A = \lambda\left(\frac{N}{K}\right)^{-1} = \frac{1}{\frac{1}{\lambda}\frac{N}{K}}, \tag{82}$$

so we would need to identify N/K. In that case, from (68),

$$\beta R^k = \beta R^h + \lambda\mu \iff R^k = \frac{1+\lambda\mu}{\beta}. \tag{83}$$

Then, from (60), and dividing it by K in both sides, we get

$$R^k - \frac{1}{\beta}\frac{B}{K} = \lambda \iff \frac{B}{K} = \beta\left(R^k - \lambda\right) = \beta\left(\frac{1}{\beta} - \lambda\right), \tag{84}$$

i.e.,

$$\frac{B}{K} = 1 - \lambda\beta. \tag{85}$$

Now, if we divide in both sides of (71) ny K, we get

$$\frac{N}{K} = \psi\left(R^k - R^h\frac{B}{K}\right) - w = \frac{\psi}{\beta}(1 + \lambda\mu - (1 - \lambda\beta)) - w. \tag{86}$$

Hence, we have this system of equations:

$$\begin{cases} \frac{N}{K} = \frac{\psi}{\beta}(1 + \lambda\mu - (1 - \lambda\beta)) - w \\ 1 - \mu - \psi = \frac{1-\psi}{\lambda}\frac{N}{K} \end{cases} \tag{87}$$

so we have identified also N/K and μ (system of two equations and two unknowns), ant then also A. We are about to finish: notice that

$$\frac{C}{K} + \left(1 - \frac{1}{\beta}\right)\frac{B}{K} = W\frac{L}{K} + \frac{\Pi}{K} \implies \frac{C}{K} = W\frac{L}{K} + \frac{\Pi}{K} - \left(1 - \frac{1}{\beta}\right)(1 - \lambda\beta)\,, \tag{88}$$

and then $\frac{C}{K}$ is also identified. From the dynamics of capital, i.e., $K_t = I_t + (1-\delta)K_{t-1}$, we get $I = \delta K$, then

$$L = \left(\frac{1-\alpha}{\xi}\left(1 + \delta\frac{K}{C}\right)\right)^{\frac{1}{1-\frac{1}{\nu}}}. \tag{89}$$

With K in hands, I idenfity K, and then B, etc.