# Advanced Asset Pricing
# Homework 1

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## 1 The model

Let's consider the following model:

$$\begin{cases} \Delta c_{t+1} = \mu + x_t + \varepsilon^c_{t+1} \\ x_t = \rho x_{t-1} + \varepsilon^x_t \\ \Delta d_{t+1} = \lambda x_t + \varepsilon^d_{t+1} \end{cases} \tag{1}$$

where $\varepsilon^i_t \overset{\text{iid}}{\sim} \mathcal{N}(\mu, \sigma^2_i)$. The Utility Function is given by

$$U_t = \left((1-\delta)C^\alpha_t + \delta\ \mathbb{E}_t\left[U^{1-\gamma}_{t+1}\right]^\theta\right)^{\frac{1}{\alpha}}, \tag{2}$$

where

$$\alpha = 1 - \frac{1}{\psi}, \qquad \theta = \frac{\alpha}{1-\gamma}, \tag{3}$$

with $\gamma$ and $\psi$ being the risk aversion and intertemporal elasticity of substitution parameters, respectively. The Stochastic Discount Factor is given by

$$M_{t+1} = \delta^\zeta e^{-\Delta c_{t+1}\frac{\zeta}{\psi}} e^{(\zeta-1)r_{c,t+1}}, \tag{4}$$

where $\zeta = 1/\theta$ (and it's the $\theta$ in Bansal and Yaron (2004)), and

$$R_{c,t+1} = \frac{P_{t+1} + C_{t+1}}{P_t}, \tag{5}$$

i.e., if we set $m_{t+1} = \log M_{t+1}$, then

$$m_{t+1} = \zeta \log \delta - \frac{\zeta}{\psi}\Delta c_{t+1} + (\zeta - 1)r_{c,t+1}. \tag{6}$$

Under No Arbitrage,

$$1 = \mathbb{E}_t\left[e^{m_{t+1} + r_{c,t+1}}\right]. \tag{7}$$

Note that

$$m_{t+1} + r_{c,t+1} = \zeta \log \delta - \frac{\zeta}{\psi}\Delta c_{t+1} + \zeta r_{c,t+1}.$$

On the other hand,

$$
\begin{aligned}
r_{c,t+1} &= \log \frac{P_{t+1}+C_{t+1}}{P_t} = \log \left[\frac{P_{t+1}+C_{t+1}}{C_{t+1}} \frac{C_{t+1}}{C_t} \frac{C_t}{P_t}\right] \\
&= \log \left[\frac{P_{t+1}+C_{t+1}}{C_{t+1}}\right] + \log \frac{C_{t+1}}{C_t} + \log \frac{C_t}{P_t} = \log\left[1+\mathrm{e}^{pc_{t+1}}\right] + \Delta c_{t+1} - pc_t.
\end{aligned}
$$

Now note that if we define $f(x) = \log(1+g(x))$, where $g(x) = \mathrm{e}^x$, its first-order Taylor polynomial around $\mathfrak{pc} = \mathbb{E}[pc_t]$ is given by

$$
h(pc_{t+1}) = \log(1+g(\mathfrak{pc})) + \frac{g'(\mathfrak{pc})}{1+g(\mathfrak{pc})}(pc_{t+1} - \mathfrak{pc}) = \kappa_{0,c} + \kappa_{1,c} pc_{t+1},
$$

where

$$
\kappa_{1,c} = \frac{\mathrm{e}^{\mathfrak{pc}}}{1+\mathrm{e}^{\mathfrak{pc}}}, \qquad \kappa_{0,c} = \log\left(1+\mathrm{e}^{\mathfrak{pc}}\right) - \kappa_{1,c}\mathfrak{pc}. \tag{8}
$$

Therefore, if we conjecture that $pc_t = \mathfrak{pc} + b_c x_t$, then

$$
\begin{aligned}
m_{t+1} + r_{c,t+1} &= \zeta \log \delta - \frac{\zeta}{\psi}\Delta c_{t+1} + \zeta\left(\kappa_{0,c} + \kappa_{1,c} pc_{t+1} + \Delta c_{t+1} - pc_t\right) \\
&= \zeta\left(\left[\log\delta + \kappa_{0,c}\right] + \left[1 - \frac{1}{\psi}\right]\Delta c_{t+1} + \kappa_{1,c} pc_{t+1} - pc_t\right) \\
&= \zeta\left(\left[\log\delta + \kappa_{0,c} + \mathfrak{pc}(\kappa_{1,c}-1)\right] + \alpha\Delta c_{t+1} + b_c(\kappa_{1,c}x_{t+1} - x_t)\right) \\
&= \zeta\left(\left[\log\delta + \kappa_{0,c} + \mathfrak{pc}(\kappa_{1,c}-1)\right] + \alpha[\mu + x_t + \varepsilon^c_{t+1}] + b_c[\kappa_{1,c}(\rho x_t + \varepsilon^x_{t+1}) - x_t]\right) \\
&= \zeta\left(\left[\log\delta + \kappa_{0,c} + \mathfrak{pc}(\kappa_{1,c}-1) + \alpha\mu\right] + \alpha[x_t + \varepsilon^c_{t+1}] + b_c[(\kappa_{1,c}\rho - 1)x_t + \kappa_{1,c}\varepsilon^x_{t+1}]\right) \\
&= \zeta\left(\left[\log\delta + \kappa_{0,c} + \mathfrak{pc}(\kappa_{1,c}-1) + \alpha\mu\right] + [\alpha + b_c(\kappa_{1,c}\rho - 1)]x_t + \alpha\varepsilon^c_{t+1} + \kappa_{1,c}b_c\varepsilon^x_{t+1})\right) \\
&= \zeta\left(\tau + [\alpha + b_c(\kappa_{1,c}\rho - 1)]x_t + \alpha\varepsilon^c_{t+1} + \kappa_{1,c}b_c\varepsilon^x_{t+1})\right).
\end{aligned}
$$

with

$$
\tau = \log\delta + \kappa_{0,c} + \mathfrak{pc}(\kappa_{1,c}-1) + \alpha \tag{9}
$$

Therefore,

$$
\begin{aligned}
1 &= \mathbb{E}_{t-1}\left[\mathrm{e}^{m_{t+1}+r_{c,t+1}}\right] \\
&= \mathrm{e}^{\zeta\tau}\mathrm{e}^{\zeta[\alpha + b_c(\kappa_{1,c}\rho-1)]x_t}\mathbb{E}_t\left[\mathrm{e}^{\zeta\left(\alpha\varepsilon^c_{t+1} + \kappa_{1,c}b_c\varepsilon^x_{t+1})\right)}\right].
\end{aligned}
$$

Therefore, since $x_t$ is still random, and $(1, x_t)$ is an orthogonal basis in $\mathcal{L}^2$, then we need for that equality to be true that

$$
\alpha + b_c(\kappa_{1,c}\rho - 1) = 0,
$$

which is equivalent to

$$
b_c = \frac{\alpha}{1 - \kappa_{1,c}\rho}. \tag{10}
$$

Therefore,

$$
pc_t = \mathfrak{pc} + \frac{\alpha}{1-\kappa_{1,c}\rho}x_t. \tag{11}
$$

Hence,

$$
\begin{aligned}
r_{c,t+1} &= \kappa_{0,c} + \kappa_{1,c}\left[\mathfrak{pc} + b_c x_{t+1}\right] - \mathfrak{pc} - b_c x_t + \mu + x_t + \varepsilon^c_{t+1} \\
&= [\kappa_{0,c} + (\kappa_{1,c}-1)\mathfrak{pc} + \mu] + \kappa_{1,c}b_c[\rho x_t + \varepsilon^x_{t+1}] - b_c x_t + x_t + \varepsilon^c_{t+1} \\
&= \bar{r} + \left[b_c\left(\kappa_{1,c}\rho - 1\right) + 1\right]x_t + \kappa_{1,c}b_c\varepsilon^x_{t+1} + \varepsilon^c_{t+1} \\
&= \bar{r} + [1-\alpha]\,x_t + \kappa_{1,c}b_c\varepsilon^x_{t+1} + \varepsilon^c_{t+1} \\
&= \bar{r} + \frac{1}{\psi}x_t + \frac{\alpha\kappa_{1,c}}{1-\kappa_{1,c}\rho}\varepsilon^x_{t+1} + \varepsilon^c_{t+1},
\end{aligned} \tag{12}
$$

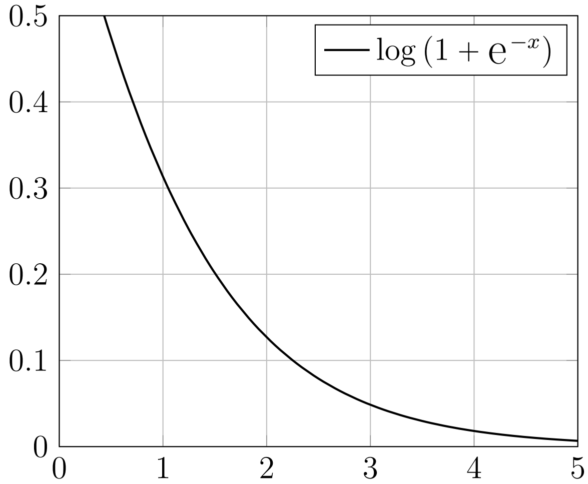


Figure 1: $f(x) = \log\left(1 + \mathrm{e}^{-x}\right)$

with

$$\bar{r} = \kappa_{0,c} + (\kappa_{1,c} - 1)\mathfrak{pc} + \mu = \log\left(1 + \mathrm{e}^{\mathfrak{pc}}\right) - \mathfrak{pc} + \mu. \tag{13}$$

Note that

$$\log\left(1 + \mathrm{e}^{\mathfrak{pc}}\right) - \mathfrak{pc} = \log\left(\frac{1 + \overline{PC}}{\overline{PC}}\right) = \log\left(1 + \mathrm{e}^{-\mathfrak{pc}}\right) \approx 0$$

as long as $\mathfrak{pc}$ is sufficiently high. Hence, just by looking at Figure 1, we can conclude that it is reasonable to assume that

$$\bar{r} = \mu + \log\left(1 + \mathrm{e}^{-\mathfrak{pc}}\right) \approx \mu.$$

Hence,

$$r_{c,t+1} = \mu + \frac{1}{\psi}x_t + \frac{\alpha\kappa_{1,c}}{1 - \kappa_{1,c}\rho}\varepsilon^x_{t+1} + \varepsilon^c_{t+1}, \tag{14}$$

so that $\mathbb{E}[r_{c,t}] = \mu$. Now, let's compute the stochastic discount factor:

$$\begin{aligned}
m_{t+1} &= \zeta \log\delta - \frac{\zeta}{\psi}\Delta c_{t+1} + (\zeta - 1)r_{c,t+1} \\
&= \zeta \log\delta - \frac{\zeta}{\psi}\left(\mu + x_t + \varepsilon^c_{t+1}\right) + \zeta\left(\mu + \frac{1}{\psi}x_t + b_c\kappa_{1,c}\varepsilon^x_{t+1} + \varepsilon^c_{t+1}\right) - r_{c,t+1} \\
&= \zeta\left[\log\delta + \alpha\mu\right] + \zeta b_c\kappa_{1,c}\varepsilon^x_{t+1} + \alpha\zeta\varepsilon^c_{t+1} - r_{c,t+1} \\
&= \left[\zeta\log\delta + (1-\gamma)\mu\right] + b_c\kappa_{1,c}\zeta\varepsilon^x_{t+1} + (1-\gamma)\varepsilon^c_{t+1} - r_{c,t+1} \\
&= \left[\zeta\log\delta + (1-\gamma)\mu - \mu\right] - \frac{1}{\psi}x_t + b_c\kappa_{1,c}(\zeta - 1)\varepsilon^x_{t+1} - \gamma\varepsilon^c_{t+1} \\
&= \left[\zeta\log\delta - \gamma\mu\right] - \frac{1}{\psi}x_t + \frac{\frac{1}{\psi} - \gamma}{1 - \rho\kappa_{1,c}}\kappa_{1,c}\varepsilon^x_{t+1} - \gamma\varepsilon^c_{t+1},
\end{aligned}$$

because $\zeta\alpha = 1 - \gamma$. Hence, the intercept of the stochastic discount factor is

$$\bar{m} = \zeta\log\delta - \gamma\mu. \tag{15}$$

Finally, our stochastic discount factor is given by

$$m_{t+1} = \bar{m} - \frac{1}{\psi}x_t + \frac{\frac{1}{\psi} - \gamma}{1 - \rho\kappa_{1,c}}\kappa_{1,c}\varepsilon^x_{t+1} - \gamma\varepsilon^c_{t+1}. \tag{16}$$

# 2 Useful recursive functions for `dynare++`

## 2.1 The utility-consumption ratio

If we define the consumption ratio by $\Xi_t := \frac{U_t}{C_t}$, then it's clear that

$$\Xi_t = \left[(1-\delta) + \delta \mathbb{E}_t \left[\Xi_t^{1-\gamma} e^{(1-\gamma)\Delta c_{t+1}}\right]^{\theta}\right]^{\frac{1}{\alpha}}.$$

Hence, if we define $\xi_t := \log \Xi_t$, we get the equation that we would include in `dynare++`

$$e^{\alpha \xi_t} = (1-\delta) + \delta \log \mathbb{E}_t \left[e^{(1-\gamma)(\xi_t + \Delta c_{t+1})}\right]^{\theta}. \tag{17}$$

If we define

$$Q_t = \mathbb{E}_t \left[e^{(1-\gamma)(\xi_t + \Delta c_{t+1})}\right],$$

then we can rewrite (17) as

$$\boxed{e^{\alpha \xi_t} = (1-\delta) + \delta e^{\theta q_t},} \tag{18}$$

where $q_t = \log Q_t$. For the initial conditions, just compute the steady state, i.e., $\xi_t = \bar{\xi}$, $\forall t$. In that case,

$$e^{\alpha \bar{\xi}} = (1-\delta) + \delta e^{\alpha(\bar{\xi}+\mu)} \iff (1-\delta e^{\alpha\mu}) e^{\alpha \bar{\xi}} = 1-\delta \iff e^{\alpha \bar{\xi}} = \frac{1-\delta}{1-\delta e^{\alpha\mu}}.$$

and therefore

$$\bar{\xi} = \frac{1}{\alpha}\left(\log(1-\delta) - \log(1-\delta e^{\alpha\mu})\right). \tag{19}$$

## 2.2 The stochastic discount factor

We can express the stochastic discount factor in terms of $\xi_t$:

$$M_t = \delta e^{-\frac{1}{\psi}\Delta c_t} \left( \frac{e^{\xi_t + \Delta c_t}}{\mathbb{E}_{t-1}\left[e^{(1-\gamma)(\xi_t + \Delta c_t)}\right]^{\frac{1}{1-\gamma}}} \right)^{\frac{1}{\psi}-\gamma}, \tag{20}$$

and therefore

$$\begin{aligned} m_t &= \log \delta - \frac{1}{\psi}\Delta c_t + \left(\frac{1}{\psi} - \gamma\right)\left(\xi_t + \Delta c_t - \frac{1}{1-\gamma} \log \mathbb{E}_{t-1}\left[e^{(1-\gamma)(\xi_t + \Delta c_t)}\right]\right) \\ &= \log \delta - \frac{1}{\psi}\Delta c_t + (1-\theta)(1-\gamma)\left(\xi_t + \Delta c_t\right) - (1-\theta) \log \mathbb{E}_{t-1}\left[e^{(1-\gamma)(\xi_t + \Delta c_t)}\right], \end{aligned}$$

because

$$1-\theta = 1 - \frac{1-1/\psi}{1-\gamma} = \frac{1-\gamma-1+1/\psi}{1-\gamma} = \frac{1/\psi - \gamma}{1-\gamma} = \frac{1-\gamma\psi}{\psi(1-\gamma)},$$

which in turns implies that

$$(1-\theta)(1-\gamma) = \frac{1}{\psi} - \gamma.$$

Hence, in `dynare++` we would write

$$\boxed{e^{m_t + \frac{1}{\psi}\Delta c_t} = \delta e^{(1-\theta)(1-\gamma)(\xi + \Delta c_t) - (1-\theta) q_{t-1}},} \tag{21}$$

For the initial conditions, just compute the steady state, i.e., $m_t = \bar{m}$, $\forall t$. In that case,

$$\mathrm{e}^{\bar{m}} = \mathrm{e}^{\log \delta - \frac{1}{\psi}\mu},$$

where the steady state for $\Delta c_t$ is assumed to be its mean, i.e., $\mu$, by stationarity. Hence,

$$\bar{m} = \log \delta - \frac{\mu}{\psi}. \tag{22}$$

## 2.3 The Price-to-Consumption ratio

By No Arbitrage, we know that

$$P_t = \mathbb{E}_t \left[ M_{t+1} \left( P_{t+1} + C_{t+1} \right) \right],$$

which implies that

$$\frac{P_t}{C_t} = \mathbb{E}_t \left[ M_{t+1} \left( \frac{P_{t+1} + C_{t+1}}{C_{t+1}} \right) \frac{C_{t+1}}{C_t} \right],$$

and therefore, in `dynare++`

$$\boxed{\mathrm{e}^{pc_t} = \mathbb{E}_t \left[ \mathrm{e}^{m_{t+1} + \Delta c_{t+1}} \left( 1 + \mathrm{e}^{pc_{t+1}} \right) \right].} \tag{23}$$

For the initial values, let's compute the steady-state, i.e., $pc_t = \bar{pc}$, $\forall t$. In that case,

$$\mathrm{e}^{\bar{pc}} = \mathrm{e}^{\bar{m}+\mu} \left( 1 + \mathrm{e}^{\bar{pc}} \right) \iff \mathrm{e}^{\bar{pc}} \left( 1 - \mathrm{e}^{\bar{m}+\mu} \right) = \mathrm{e}^{\bar{m}+\mu},$$

and therefore

$$\bar{pc} = \log \left( \frac{\mathrm{e}^{\bar{m}+\mu}}{1 - \mathrm{e}^{\bar{m}+\mu}} \right). \tag{24}$$

## 2.4 The Price-to-Dividend ratio

Equivalently for the price-dividend ratio, $pc_t$,

$$\boxed{\mathrm{e}^{pd_t} = \mathbb{E}_t \left[ \mathrm{e}^{m_{t+1} + \Delta d_{t+1}} \left( 1 + \mathrm{e}^{pd_{t+1}} \right) \right].} \tag{25}$$

For the initial values, we just compute the stead-state as we did with the price-to-consumption ratio. To do so,

$$\overline{pd} = \log \left( \frac{\mathrm{e}^{\bar{m}+\mu}}{1 + \mathrm{e}^{\bar{m}+\mu}} \right). \tag{26}$$

## 2.5 The Return on Consumption

This is something that we've already found above. Hence, we just copy the formula:

$$\boxed{\mathrm{e}^{r_{c,t+1} + pc_t - \Delta c_{t+1}} = 1 + \mathrm{e}^{pc_{t+1}}.} \tag{27}$$

For the initial values, we set the steady state to be $\bar{r}_c = r_f = -\bar{m}$.

## 2.6 The Return on Dividend

This is something that we've already found above. Hence, we just copy the formula:

$$\boxed{e^{r_{d,t+1}+pd_t-\Delta d_{t+1}} = 1 + e^{p_d t+1}.} \tag{28}$$

For the initial values, we set the steady state to be $\bar{r}_d = r_f = -\bar{m}$.

# 3 Replicating Bansal and Yaron (2004)

Table 1: This table is supposed to replicate Bansal and Yaron (2004) results. They have been obtained by running the model in `dynare++` using 1200 number of periods of simulations repeated 500 times. The parameters of the model are fixed asin Bansal and Yaron (2004), i.e., $\delta = 0.999$, $\gamma = 10$, $\psi = 1.5$, $\varphi = 2$. The standard deviation of the shocks, $\varepsilon_t^c$, $\varepsilon_t^x$ and $\varepsilon_t^d$ are, respectively, $\sigma_c = 0.0059$, $\sigma_x = 0.0003$, and $\sigma_d = 0.036$. All the shocks are supposed to be uncorrelated. We set $\rho = 0.98$ and $\mu = 0.0015$. All results are given in annualised percentage units.

| | Bootsrapped Mean | Long sample Mean |
|---|---|---|
| Consumption Growth, $\frac{C_{t+1}}{C_t}$ | 1.71) | (1.78) |
| | (2.11) | (2.16) |
| Dividend Growth, $\frac{D_{t+1}}{D_t}$ | 0.90) | (2.07) |
| | (12.84) | (12.44) |
| Return on Consumption, $R_t^C$ | 2.77) | (2.83) |
| | (2.67) | (2.72) |
| Return on Dividends, $R_t^D$ | 3.00) | (4.12) |
| | (14.75) | (14.11) |
| Return on Dividends, $R_t^D$ | 1.66) | (1.67) |
| | (0.41) | (0.33) |

# 4 Additional

## 4.1 Exercise 1

Using the appropriate no-arbitrage condition, derive the equation that allows us to pin-down the coefficient $\kappa_{1,d}$ for the Campbell and Shiller approximation in the Bansal-Yaron (2004) economy. Take $\kappa_{1,c}$ and the intercept of the SDF as known scalars.
**Solution.** Under No Arbitrage,

$$P_t = \mathbb{E}_t \left[ M_{t+1} \left( P_{t+1} + D_{t+1} \right) \right], \tag{29}$$

which is equivalent to

$$\frac{P_t}{D_t} = \mathbb{E}_t\left[M_{t+1}\left(\frac{P_{t+1}+D_{t+1}}{D_{t+1}}\right)\frac{D_{t+1}}{D_t}\right] = \mathbb{E}_t\left[M_{t+1}\left(1+\frac{P_{t+1}}{D_{t+1}}\right)\frac{D_{t+1}}{D_t}\right], \tag{30}$$

i.e.

$$\mathrm{e}^{pd_t} = \mathbb{E}_t\left[\mathrm{e}^{m_{t+1}+\kappa_{0,d}+\kappa_{1,d}pd_{t+1}+\Delta d_{t+1}}\right], \tag{31}$$

because

$$1+\frac{P_{t+1}}{D_{t+1}} = \mathrm{e}^{\log\left(1+\mathrm{e}^{pd_{t+1}}\right)},$$

and

$$\log\left(1+\mathrm{e}^{pd_{t+1}}\right) \approx \log\left(1+\mathrm{e}^{\mathfrak{pd}}\right) + \frac{\mathrm{e}^{\mathfrak{pd}}}{1+\mathrm{e}^{\mathfrak{pd}}}\left(pd_{t+1}-\mathfrak{pd}\right) = \kappa_{0,d}+\kappa_{1,d}pd_{t+1},$$

with

$$\kappa_{1,d} = \frac{\mathrm{e}^{\mathfrak{pd}}}{1+\mathrm{e}^{\mathfrak{pd}}}, \qquad \kappa_{0,d} = \log\left(1+\mathrm{e}^{\mathfrak{pd}}\right) - \kappa_{1,d}\mathfrak{pd}. \tag{32}$$

As we usually do, let's conjecture that $pd_t = \mathfrak{pd} + b_d x_t$. Therefore, equation (31) becomes

$$\mathrm{e}^{\mathfrak{pd}+b_d x_t} = \mathbb{E}_t\left[\mathrm{e}^{m_{t+1}+\kappa_{0,d}+\kappa_{1,d}(\mathfrak{pd}+b_d x_t)+\lambda x_t+\varepsilon^d_{t+1}}\right].$$

Hence, by using our expression for the stochastic discount factor in (16),

$$\begin{aligned}
\mathrm{e}^{\mathfrak{pd}+b_d x_t} &= \mathbb{E}_t\left[\mathrm{e}^{\bar{m}-\frac{1}{\psi}x_t+\frac{\frac{1}{\psi}-\gamma}{1-\rho\kappa_{1,c}}\kappa_{1,c}\varepsilon^x_{t+1}-\gamma\varepsilon^c_{t+1}+\kappa_{0,d}+\kappa_{1,d}(\mathfrak{pd}+b_d x_t)+\lambda x_t+\varepsilon^d_{t+1}}\right] \\
\mathrm{e}^{\mathfrak{pd}-\kappa_{0,d}-\kappa_{1,d}\mathfrak{pd}}\mathrm{e}^{b_c x_t} &= \mathrm{e}^{\bar{m}+\left[\kappa_{1,d}b_d-\frac{1}{\psi}+\lambda\right]x_t}\mathbb{E}_t\left[\mathrm{e}^{\frac{\frac{1}{\psi}-\gamma}{1-\kappa_{1,c}\rho}\kappa_{1,c}\varepsilon^x_{t+1}-\gamma\varepsilon^c_{t+1}+\varepsilon^d_{t+1}}\right].
\end{aligned} \tag{33}$$

Now note that

$$\mathfrak{pd}-\kappa_{0,d}-\kappa_{1,d}\mathfrak{pd} = \mathfrak{pd}-\log(1+\mathrm{e}^{\mathfrak{pd}})+\kappa_{1,d}\mathfrak{pd}-\kappa_{1,d}\mathfrak{pd} = \mathfrak{pd}-\log(1+\mathrm{e}^{\mathfrak{pd}}), \tag{34}$$

and therefore

$$\mathrm{e}^{\mathfrak{pd}-\kappa_{0,d}-\kappa_{1,d}\mathfrak{pd}} = \frac{\mathrm{e}^{\mathfrak{pd}}}{1+\mathrm{e}^{\mathfrak{pd}}} = \kappa_{1,d}.$$

Hence by also assuming that $x_t \approx \mathbb{E}[x_t] = 0$, (33) becomes

$$\kappa_{1,d} = \mathrm{e}^{\bar{m}+\frac{1}{2}\left[\frac{\frac{1}{\psi}-\gamma}{1-\kappa_{1,c}\rho}\right]^2\kappa^2_{1,x}\sigma^2_x+\gamma^2\sigma^2_c+\sigma^2_d}.$$

With our expression for $\bar{m}$ given in (15), we finally obtain[1]:

$$\kappa_{1,d} = \mathrm{e}^{\zeta\log\delta-\gamma\mu+\frac{1}{2}\left[\frac{\frac{1}{\psi}-\gamma}{1-\kappa_{1,c}\rho}\right]^2\kappa^2_{1,c}\sigma^2_x+\frac{\gamma^2}{2}\sigma^2_c+\frac{1}{2}\sigma^2_d}. \tag{35}$$

[1]Remember that $\zeta = \frac{1}{\theta}$ where $\theta = \frac{1-\gamma}{\alpha}$ as in Bansal and Yaron (2004).

## 4.2 Exercise 2

Using the appropriate expression for the stochastic discount factor and the NA-condition for P/C, find the equation that pins down $\kappa_c$.

**Solution.** By No Arbitrage

$$P_t = \mathbb{E}_t[M_{t+1}\left(P_{t+1} + C_{t+1}\right)], \tag{36}$$

which is equivalent to

$$\frac{P_t}{C_t} = \mathbb{E}_t\left[M_{t+1}\frac{P_{t+1} + C_{t+1}}{C_{t+1}}\frac{C_{t+1}}{C_t}\right].$$

and is itself equivalent to

$$\mathrm{e}^{pc_t} = \mathbb{E}_t\left[\mathrm{e}^{m_{t+1}+\log\left(1+\mathrm{e}^{pc_{t+1}}\right)+\Delta c_{t+1}}\right]. \tag{37}$$

Recall that

$$\log\left(1+\mathrm{e}^{pc_{t+1}}\right) \approx \kappa_{0,c} + \kappa_{1,c}pc_{t+1},$$

with $\kappa_{0,c}$ and $\kappa_{1,c}$ as in (8). Also, conjecturing that $pc_t = \mathfrak{pc} + b_c x_t$, with $b_c$ as in (10), we arrive at

$$\begin{aligned}
\mathrm{e}^{\mathfrak{pc}+b_c x_t} &= \mathbb{E}_t\left[\mathrm{e}^{\bar{m}-\frac{1}{\psi}x_t+\frac{\frac{1}{\psi}-\gamma}{1-\rho\kappa_{1,c}}\kappa_{1,c}\varepsilon^x_{t+1}-\gamma\varepsilon^c_{t+1}+\kappa_{0,c}+\kappa_{1,c}(\mathfrak{pc}+b_c x_{t+1})+\mu+x_t+\varepsilon^c_{t+1}}\right], \\
\mathrm{e}^{\mathfrak{pc}-\kappa_{0,c}-\kappa_{1,c}\mathfrak{pc}}\mathrm{e}^{b_c x_t} &= \mathrm{e}^{[\bar{m}+\mu]+[\alpha+\rho\kappa_{1,c}b_c]x_t}\mathbb{E}_{t-1}\left[\mathrm{e}^{\left(b_c+\frac{\frac{1}{\psi}-\gamma}{1-\kappa_{1,c}\rho}\right)\kappa_{1,c}\varepsilon^x_{t+1}+(1-\gamma)\varepsilon^c_{t+1}}\right] \\
\mathrm{e}^{\mathfrak{pc}-\kappa_{0,c}-\kappa_{1,c}\mathfrak{pc}}\mathrm{e}^{b_c x_t} &= \mathrm{e}^{[\bar{m}+\mu]+[\alpha+\rho\kappa_{1,c}b_c]x_t}\mathbb{E}_{t-1}\left[\mathrm{e}^{(1-\gamma)\frac{\kappa_{1,c}}{1-\kappa_{1,c}\rho}\varepsilon^x_{t+1}+(1-\gamma)\varepsilon^c_{t+1}}\right],
\end{aligned}$$

Similarly as before,

$$\mathfrak{pc} - \kappa_{0,c} - \kappa_{1,c}\mathfrak{pc} = \mathfrak{pc} - \log(1+\mathrm{e}^{\mathfrak{pc}}) + \kappa_{1,d}\mathfrak{pc} - \kappa_{1,d}\mathfrak{pc} = \mathfrak{pc} - \log(1+\mathrm{e}^{\mathfrak{pc}}), \tag{38}$$

and therefore

$$\mathrm{e}^{\mathfrak{pc}-\kappa_{0,c}-\kappa_{1,c}\mathfrak{pc}} = \frac{\mathrm{e}^{\mathfrak{pc}}}{1+\mathrm{e}^{\mathfrak{pc}}} = \kappa_{1,c}.$$

Hence, by assumiing also that $x_t \approx \mathbb{E}_t[x_t] = 0$,

$$\kappa_{1,c} = \mathrm{e}^{\bar{m}+\mu}\mathbb{E}_{t-1}\left[\mathrm{e}^{(1-\gamma)\frac{\kappa_{1,c}}{1-\kappa_{1,c}\rho}\varepsilon^x_{t+1}+(1-\gamma)\varepsilon^c_{t+1}}\right]$$

I think it should be

$$\kappa_{1,c} = \mathrm{e}^{\bar{m}+\mu}\mathbb{E}_{t-1}\left[\mathrm{e}^{(1-\gamma)(\varepsilon^x_{t+1}+\varepsilon^c_{t+1})}\right],$$

but I may be missing some minus sign. In my derivations,

$$\kappa_{1,c} = \mathrm{e}^{\bar{m}+\mu+\frac{1}{2}(1-\gamma)^2\left[\frac{\kappa^2_{1,c}}{(1-\kappa_{1,c})^2}\sigma^2_x+\sigma^2_c\right]},$$

but this has no closed form solution... In the case that the $\kappa_{1,c}$-term dissappears, we would have

$$\kappa_{1,c} = \mathrm{e}^{\bar{m}+\mu+\frac{(1-\gamma)^2}{2}\left(\sigma^2_x+\sigma^2_c\right)}.$$