

# Advanced Asset Pricing

## Homework 1

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### 1 The model

Let's consider the following model:

$$\begin{cases} \Delta c_{t+1} = \mu + x_t + \varepsilon_{t+1}^c \\ x_t = \rho x_{t-1} + \varepsilon_t^x \\ \Delta d_{t+1} = \lambda x_t + \varepsilon_{t+1}^d \end{cases} \quad (1)$$

where  $\varepsilon_t^i \stackrel{\text{iid}}{\sim} \mathcal{N}(\mu, \sigma_i^2)$ . The Utility Function is given by

$$U_t = \left( (1 - \delta) C_t^\alpha + \delta \mathbb{E}_t [U_{t+1}^{1-\gamma}]^\theta \right)^{\frac{1}{\alpha}}, \quad (2)$$

where

$$\alpha = 1 - \frac{1}{\psi}, \quad \theta = \frac{\alpha}{1 - \gamma}, \quad (3)$$

with  $\gamma$  and  $\psi$  being the risk aversion and intertemporal elasticity of substitution parameters, respectively. The Stochastic Discount Factor is given by

$$M_{t+1} = \delta^\zeta e^{-\Delta c_{t+1} \frac{\zeta}{\psi}} e^{(\zeta-1)r_{c,t+1}}, \quad (4)$$

where  $\zeta = 1/\theta$  (and it's the  $\theta$  in Bansal and Yaron (2004)), and

$$R_{c,t+1} = \frac{P_{t+1} + C_{t+1}}{P_t}, \quad (5)$$

i.e., if we set  $m_{t+1} = \log M_{t+1}$ , then

$$m_{t+1} = \zeta \log \delta - \frac{\zeta}{\psi} \Delta c_{t+1} + (\zeta - 1)r_{c,t+1}. \quad (6)$$

Under No Arbitrage,

$$1 = \mathbb{E}_t [e^{m_{t+1} + r_{c,t+1}}]. \quad (7)$$

Note that

$$m_{t+1} + r_{c,t+1} = \zeta \log \delta - \frac{\zeta}{\psi} \Delta c_{t+1} + \zeta r_{c,t+1}.$$

On the other hand,

$$\begin{aligned} r_{c,t+1} &= \log \frac{P_{t+1} + C_{t+1}}{P_t} = \log \left[ \frac{P_{t+1} + C_{t+1}}{C_{t+1}} \frac{C_{t+1}}{C_t} \frac{C_t}{P_t} \right] \\ &= \log \left[ \frac{P_{t+1} + C_{t+1}}{C_{t+1}} \right] + \log \frac{C_{t+1}}{C_t} + \log \frac{C_t}{P_t} = \log [1 + e^{pc_{t+1}}] + \Delta c_{t+1} - pc_t. \end{aligned}$$

Now note that if we define  $f(x) = \log(1 + g(x))$ , where  $g(x) = e^x$ , its first-order Taylor polynomial around  $\mathbf{pc} = \mathbb{E}[pc_t]$  is given by

$$h(pc_{t+1}) = \log(1 + g(\mathbf{pc})) + \frac{g'(\mathbf{pc})}{1 + g(\mathbf{pc})} (pc_{t+1} - \mathbf{pc}) = \kappa_{0,c} + \kappa_{1,c} pc_{t+1},$$

where

$$\kappa_{1,c} = \frac{e^{\mathbf{pc}}}{1 + e^{\mathbf{pc}}}, \quad \kappa_{0,c} = \log(1 + e^{\mathbf{pc}}) - \kappa_{1,c} \mathbf{pc}. \quad (8)$$

Therefore, if we conjecture that  $pc_t = \mathbf{pc} + b_c x_t$ , then

$$\begin{aligned} m_{t+1} + r_{c,t+1} &= \zeta \log \delta - \frac{\zeta}{\psi} \Delta c_{t+1} + \zeta (\kappa_{0,c} + \kappa_{1,c} pc_{t+1} + \Delta c_{t+1} - pc_t) \\ &= \zeta \left( [\log \delta + \kappa_{0,c}] + \left[ 1 - \frac{1}{\psi} \right] \Delta c_{t+1} + \kappa_{1,c} pc_{t+1} - pc_t \right) \\ &= \zeta ([\log \delta + \kappa_{0,c} + \mathbf{pc}(\kappa_{1,c} - 1)] + \alpha \Delta c_{t+1} + b_c(\kappa_{1,c} x_{t+1} - x_t)) \\ &= \zeta ([\log \delta + \kappa_{0,c} + \mathbf{pc}(\kappa_{1,c} - 1)] + \alpha[\mu + x_t + \varepsilon_{t+1}^c] + b_c[\kappa_{1,c}(\rho x_t + \varepsilon_{t+1}^x) - x_t]) \\ &= \zeta ([\log \delta + \kappa_{0,c} + \mathbf{pc}(\kappa_{1,c} - 1) + \alpha\mu] + \alpha[x_t + \varepsilon_{t+1}^c] + b_c[(\kappa_{1,c}\rho - 1)x_t + \kappa_{1,c}\varepsilon_{t+1}^x]) \\ &= \zeta ([\log \delta + \kappa_{0,c} + \mathbf{pc}(\kappa_{1,c} - 1) + \alpha\mu] + [\alpha + b_c(\kappa_{1,c}\rho - 1)]x_t + \alpha\varepsilon_{t+1}^c + \kappa_{1,c}b_c\varepsilon_{t+1}^x) \\ &= \zeta (\tau + [\alpha + b_c(\kappa_{1,c}\rho - 1)]x_t + \alpha\varepsilon_{t+1}^c + \kappa_{1,c}b_c\varepsilon_{t+1}^x). \end{aligned}$$

with

$$\tau = \log \delta + \kappa_{0,c} + \mathbf{pc}(\kappa_{1,c} - 1) + \alpha \quad (9)$$

Therefore,

$$\begin{aligned} 1 &= \mathbb{E}_{t-1} [e^{m_{t+1} + r_{c,t+1}}] \\ &= e^{\zeta\tau} e^{\zeta[\alpha + b_c(\kappa_{1,c}\rho - 1)]x_t} \mathbb{E}_t [e^{\zeta(\alpha\varepsilon_{t+1}^c + \kappa_{1,c}b_c\varepsilon_{t+1}^x)}]. \end{aligned}$$

Therefore, since  $x_t$  is still random, and  $(1, x_t)$  is an orthogonal basis in  $\mathcal{L}^2$ , then we need for that equality to be true that

$$\alpha + b_c(\kappa_{1,c}\rho - 1) = 0,$$

which is equivalent to

$$b_c = \frac{\alpha}{1 - \kappa_{1,c}\rho}. \quad (10)$$

Therefore,

$$pc_t = \mathbf{pc} + \frac{\alpha}{1 - \kappa_{1,c}\rho} x_t. \quad (11)$$

Hence,

$$\begin{aligned} r_{c,t+1} &= \kappa_{0,c} + \kappa_{1,c} [\mathbf{pc} + b_c x_{t+1}] - \mathbf{pc} - b_c x_t + \mu + x_t + \varepsilon_{t+1}^c \\ &= [\kappa_{0,c} + (\kappa_{1,c} - 1)\mathbf{pc} + \mu] + \kappa_{1,c} b_c [\rho x_t + \varepsilon_{t+1}^x] - b_c x_t + x_t + \varepsilon_{t+1}^c \\ &= \bar{r} + [b_c(\kappa_{1,c}\rho - 1) + 1] x_t + \kappa_{1,c} b_c \varepsilon_{t+1}^x + \varepsilon_{t+1}^c \\ &= \bar{r} + [1 - \alpha] x_t + \kappa_{1,c} b_c \varepsilon_{t+1}^x + \varepsilon_{t+1}^c \\ &= \bar{r} + \frac{1}{\psi} x_t + \frac{\alpha \kappa_{1,c}}{1 - \kappa_{1,c}\rho} \varepsilon_{t+1}^x + \varepsilon_{t+1}^c, \end{aligned} \quad (12)$$

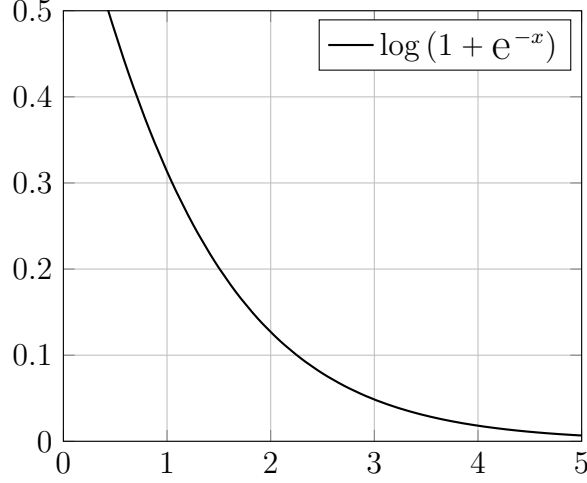


Figure 1:  $f(x) = \log(1 + e^{-x})$

with

$$\bar{r} = \kappa_{0,c} + (\kappa_{1,c} - 1)\mathbf{p}\mathbf{c} + \mu = \log(1 + e^{\mathbf{p}\mathbf{c}}) - \mathbf{p}\mathbf{c} + \mu. \quad (13)$$

Note that

$$\log(1 + e^{\mathbf{p}\mathbf{c}}) - \mathbf{p}\mathbf{c} = \log\left(\frac{1 + \overline{PC}}{\overline{PC}}\right) = \log(1 + e^{-\mathbf{p}\mathbf{c}}) \approx 0$$

as long as  $\mathbf{p}\mathbf{c}$  is sufficiently high. Hence, just by looking at Figure 1, we can conclude that it is reasonable to assume that

$$\bar{r} = \mu + \log(1 + e^{-\mathbf{p}\mathbf{c}}) \approx \mu.$$

Hence,

$$r_{c,t+1} = \mu + \frac{1}{\psi}x_t + \frac{\alpha\kappa_{1,c}}{1 - \kappa_{1,c}\rho}\varepsilon_{t+1}^x + \varepsilon_{t+1}^c, \quad (14)$$

so that  $\mathbb{E}[r_{c,t}] = \mu$ . Now, let's compute the stochastic discount factor:

$$\begin{aligned} m_{t+1} &= \zeta \log \delta - \frac{\zeta}{\psi} \Delta c_{t+1} + (\zeta - 1)r_{c,t+1} \\ &= \zeta \log \delta - \frac{\zeta}{\psi} (\mu + x_t + \varepsilon_{t+1}^c) + \zeta \left( \mu + \frac{1}{\psi}x_t + b_c \kappa_{1,c} \varepsilon_{t+1}^x + \varepsilon_{t+1}^c \right) - r_{c,t+1} \\ &= \zeta [\log \delta + \alpha \mu] + \zeta b_c \kappa_{1,c} \varepsilon_{t+1}^x + \alpha \zeta \varepsilon_{t+1}^c - r_{c,t+1} \\ &= [\zeta \log \delta + (1 - \gamma)\mu] + b_c \kappa_{1,c} \zeta \varepsilon_{t+1}^x + (1 - \gamma) \varepsilon_{t+1}^c - r_{c,t+1} \\ &= [\zeta \log \delta + (1 - \gamma)\mu - \mu] - \frac{1}{\psi}x_t + b_c \kappa_{1,c} (\zeta - 1) \varepsilon_{t+1}^x - \gamma \varepsilon_{t+1}^c \\ &= [\zeta \log \delta - \gamma \mu] - \frac{1}{\psi}x_t + \frac{\frac{1}{\psi} - \gamma}{1 - \rho \kappa_{1,c}} \kappa_{1,c} \varepsilon_{t+1}^x - \gamma \varepsilon_{t+1}^c, \end{aligned}$$

because  $\zeta \alpha = 1 - \gamma$ . Hence, the intercept of the stochastic discount factor is

$$\bar{m} = \zeta \log \delta - \gamma \mu. \quad (15)$$

Finally, our stochastic discount factor is given by

$$m_{t+1} = \bar{m} - \frac{1}{\psi}x_t + \frac{\frac{1}{\psi} - \gamma}{1 - \rho \kappa_{1,c}} \kappa_{1,c} \varepsilon_{t+1}^x - \gamma \varepsilon_{t+1}^c. \quad (16)$$

## 2 Useful recursive functions for dynare++

### 2.1 The utility-consumption ratio

If we define the consumption ratio by  $\Xi_t := \frac{U_t}{C_t}$ , then it's clear that

$$\Xi_t = \left[ (1 - \delta) + \delta \mathbb{E}_t \left[ \Xi_t^{1-\gamma} e^{(1-\gamma)\Delta c_{t+1}} \right]^\theta \right]^{\frac{1}{\alpha}}.$$

Hence, if we define  $\xi_t := \log \Xi_t$ , we get the equation that we would include in **dynare++**

$$e^{\alpha \xi_t} = (1 - \delta) + \delta \log \mathbb{E}_t \left[ e^{(1-\gamma)(\xi_t + \Delta c_{t+1})} \right]^\theta. \quad (17)$$

If we define

$$Q_t = \mathbb{E}_t \left[ e^{(1-\gamma)(\xi_t + \Delta c_{t+1})} \right],$$

then we can rewrite (17) as

$$\boxed{e^{\alpha \xi_t} = (1 - \delta) + \delta e^{\theta q_t}}, \quad (18)$$

where  $q_t = \log Q_t$ . For the initial conditions, just compute the steady state, i.e.,  $\xi_t = \bar{\xi}$ ,  $\forall t$ . In that case,

$$e^{\alpha \bar{\xi}} = (1 - \delta) + \delta e^{\alpha(\bar{\xi} + \mu)} \iff (1 - \delta e^{\alpha \mu}) e^{\alpha \bar{\xi}} = 1 - \delta \iff e^{\alpha \bar{\xi}} = \frac{1 - \delta}{1 - \delta e^{\alpha \mu}}.$$

and therefore

$$\bar{\xi} = \frac{1}{\alpha} (\log(1 - \delta) - \log(1 - \delta e^{\alpha \mu})). \quad (19)$$

### 2.2 The stochastic discount factor

We can express the stochastic discount factor in terms of  $\xi_t$ :

$$M_t = \delta e^{-\frac{1}{\psi} \Delta c_t} \left( \frac{e^{\xi_t + \Delta c_t}}{\mathbb{E}_{t-1} \left[ e^{(1-\gamma)(\xi_t + \Delta c_t)} \right]^{\frac{1}{1-\gamma}}} \right)^{\frac{1}{\psi} - \gamma}, \quad (20)$$

and therefore

$$\begin{aligned} m_t &= \log \delta - \frac{1}{\psi} \Delta c_t + \left( \frac{1}{\psi} - \gamma \right) \left( \xi_t + \Delta c_t - \frac{1}{1-\gamma} \log \mathbb{E}_{t-1} \left[ e^{(1-\gamma)(\xi_t + \Delta c_t)} \right] \right) \\ &= \log \delta - \frac{1}{\psi} \Delta c_t + (1 - \theta)(1 - \gamma) (\xi_t + \Delta c_t) - (1 - \theta) \log \mathbb{E}_{t-1} \left[ e^{(1-\gamma)(\xi_t + \Delta c_t)} \right], \end{aligned}$$

because

$$1 - \theta = 1 - \frac{1 - 1/\psi}{1 - \gamma} = \frac{1 - \gamma - 1 + 1/\psi}{1 - \gamma} = \frac{1/\psi - \gamma}{1 - \gamma} = \frac{1 - \gamma\psi}{\psi(1 - \gamma)},$$

which in turns implies that

$$(1 - \theta)(1 - \gamma) = \frac{1}{\psi} - \gamma.$$

Hence, in **dynare++** we would write

$$\boxed{e^{m_t + \frac{1}{\psi} \Delta c_t} = \delta e^{(1-\theta)(1-\gamma)(\xi_t + \Delta c_t) - (1-\theta)q_{t-1}}}, \quad (21)$$

For the initial conditions, just compute the steady state, i.e.,  $m_t = \bar{m}$ ,  $\forall t$ . In that case,

$$e^{\bar{m}} = e^{\log \delta - \frac{1}{\psi} \mu},$$

where the steady state for  $\Delta c_t$  is assumed to be its mean, i.e.,  $\mu$ , by stationarity. Hence,

$$\bar{m} = \log \delta - \frac{\mu}{\psi}. \quad (22)$$

## 2.3 The Price-to-Consumption ratio

By No Arbitrage, we know that

$$P_t = \mathbb{E}_t [M_{t+1} (P_{t+1} + C_{t+1})],$$

which implies that

$$\frac{P_t}{C_t} = \mathbb{E}_t \left[ M_{t+1} \left( \frac{P_{t+1} + C_{t+1}}{C_{t+1}} \right) \frac{C_{t+1}}{C_t} \right],$$

and therefore, in `dynare++`

$$\boxed{e^{pc_t} = \mathbb{E}_t [e^{m_{t+1} + \Delta c_{t+1}} (1 + e^{pc_{t+1}})]}. \quad (23)$$

For the initial values, let's compute the steady-state, i.e.,  $pc_t = \bar{pc}$ ,  $\forall t$ . In that case,

$$e^{\bar{pc}} = e^{\bar{m} + \mu} (1 + e^{\bar{pc}}) \iff e^{\bar{pc}} (1 - e^{\bar{m} + \mu}) = e^{\bar{m} + \mu},$$

and therefore

$$\bar{pc} = \log \left( \frac{e^{\bar{m} + \mu}}{1 - e^{\bar{m} + \mu}} \right). \quad (24)$$

## 2.4 The Price-to-Dividend ratio

Equivalently for the price-dividend ratio,  $pc_t$ ,

$$\boxed{e^{pd_t} = \mathbb{E}_t [e^{m_{t+1} + \Delta d_{t+1}} (1 + e^{pd_{t+1}})]}. \quad (25)$$

For the initial values, we just compute the steady-state as we did with the price-to-consumption ratio. To do so,

$$\bar{pd} = \log \left( \frac{e^{\bar{m} + \mu}}{1 - e^{\bar{m} + \mu}} \right). \quad (26)$$

## 2.5 The Return on Consumption

This is something that we've already found above. Hence, we just copy the formula:

$$\boxed{e^{r_{c,t+1} + pc_t - \Delta c_{t+1}} = 1 + e^{pc_{t+1}}}. \quad (27)$$

For the initial values, we set the steady state to be  $\bar{r}_c = r_f = -\bar{m}$ .

## 2.6 The Return on Dividend

This is something that we've already found above. Hence, we just copy the formula:

$$\boxed{e^{r_{d,t+1} + pd_t - \Delta d_{t+1}} = 1 + e^{pd_{t+1}}.} \quad (28)$$

For the initial values, we set the steady state to be  $\bar{r}_d = r_f = -\bar{m}$ .

## 3 Replicating Bansal and Yaron (2004)

Table 1: This table is supposed to replicate Bansal and Yaron (2004) results. They have been obtained by running the model in **dynare++** using 1200 number of periods of simulations repeated 500 times. The parameters of the model are fixed as in Bansal and Yaron (2004), i.e.,  $\delta = 0.999$ ,  $\gamma = 10$ ,  $\psi = 1.5$ ,  $\varphi = 2$ . The standard deviation of the shocks,  $\varepsilon_t^c$ ,  $\varepsilon_t^x$  and  $\varepsilon_t^d$  are, respectively,  $\sigma_c = 0.0059$ ,  $\sigma_x = 0.0003$ , and  $\sigma_d = 0.036$ . All the shocks are supposed to be uncorrelated. Finally, we set  $\rho = 0.98$  and  $\mu = 0.0015$ .

	(1)	(2)	(3)	(4)
Variable	Mean SS	Std SS	Mean LS	Std LS
$\Delta c_{t+1}$	1.71	2.11	1.78	2.16
$\Delta d_{t+1}$	0.90	12.84	2.07	12.44
$r_c$	2.77	2.67	2.83	2.72
$r_m$	3.00	14.75	4.12	14.11
$r_m$	1.66	0.41	1.67	0.33

## 4 Additional

### 4.1 Exercise 1

Using the appropriate no-arbitrage condition, derive the equation that allows us to pin-down the coefficient  $\kappa_{1,d}$  for the Campbell and Shiller approximation in the Bansal-Yaron (2004) economy. Take  $\kappa_{1,c}$  and the intercept of the SDF as known scalars.

**Solution.** Under No Arbitrage,

$$P_t = \mathbb{E}_t [M_{t+1} (P_{t+1} + D_{t+1})], \quad (29)$$

which is equivalent to

$$\frac{P_t}{D_t} = \mathbb{E}_t \left[ M_{t+1} \left( \frac{P_{t+1} + D_{t+1}}{D_{t+1}} \right) \frac{D_{t+1}}{D_t} \right] = \mathbb{E}_t \left[ M_{t+1} \left( 1 + \frac{P_{t+1}}{D_{t+1}} \right) \frac{D_{t+1}}{D_t} \right], \quad (30)$$

i.e.

$$e^{pd_t} = \mathbb{E}_t [e^{m_{t+1} + \kappa_{0,d} + \kappa_{1,d} pd_{t+1} + \Delta d_{t+1}}], \quad (31)$$

because

$$1 + \frac{P_{t+1}}{D_{t+1}} = e^{\log(1+e^{pd_{t+1}})},$$

and

$$\log(1 + e^{pd_{t+1}}) \approx \log(1 + e^{p\bar{d}}) + \frac{e^{p\bar{d}}}{1 + e^{p\bar{d}}} (pd_{t+1} - p\bar{d}) = \kappa_{0,d} + \kappa_{1,d}pd_{t+1},$$

with

$$\kappa_{1,d} = \frac{e^{p\bar{d}}}{1 + e^{p\bar{d}}}, \quad \kappa_{0,d} = \log(1 + e^{p\bar{d}}) - \kappa_{1,d}p\bar{d}. \quad (32)$$

As we usually do, let's conjecture that  $pd_t = p\bar{d} + b_dx_t$ . Therefore, equation (31) becomes

$$e^{p\bar{d} + b_dx_t} = \mathbb{E}_t \left[ e^{m_{t+1} + \kappa_{0,d} + \kappa_{1,d}(p\bar{d} + b_dx_t) + \lambda x_t + \varepsilon_{t+1}^d} \right].$$

Hence, by using our expression for the stochastic discount factor in (16),

$$\begin{aligned} e^{p\bar{d} + b_dx_t} &= \mathbb{E}_t \left[ e^{\bar{m} - \frac{1}{\bar{\psi}}x_t + \frac{\frac{1}{\bar{\psi}} - \gamma}{1 - \rho\kappa_{1,c}} \kappa_{1,c}\varepsilon_{t+1}^x - \gamma\varepsilon_{t+1}^c + \kappa_{0,d} + \kappa_{1,d}(p\bar{d} + b_dx_t) + \lambda x_t + \varepsilon_{t+1}^d} \right] \\ e^{p\bar{d} - \kappa_{0,d} - \kappa_{1,d}p\bar{d}} e^{b_dx_t} &= e^{\bar{m} + [\kappa_{1,d}b_d - \frac{1}{\bar{\psi}} + \lambda]x_t} \mathbb{E}_t \left[ e^{\frac{\frac{1}{\bar{\psi}} - \gamma}{1 - \kappa_{1,c}\rho} \kappa_{1,c}\varepsilon_{t+1}^x - \gamma\varepsilon_{t+1}^c + \varepsilon_{t+1}^d} \right]. \end{aligned} \quad (33)$$

Now note that

$$p\bar{d} - \kappa_{0,d} - \kappa_{1,d}p\bar{d} = p\bar{d} - \log(1 + e^{p\bar{d}}) + \kappa_{1,d}p\bar{d} - \kappa_{1,d}p\bar{d} = p\bar{d} - \log(1 + e^{p\bar{d}}), \quad (34)$$

and therefore

$$e^{p\bar{d} - \kappa_{0,d} - \kappa_{1,d}p\bar{d}} = \frac{e^{p\bar{d}}}{1 + e^{p\bar{d}}} = \kappa_{1,d}.$$

Hence by also assuming that  $x_t \approx \mathbb{E}[x_t] = 0$ , (33) becomes

$$\kappa_{1,d} = e^{\bar{m} + \frac{1}{2} \left[ \frac{\frac{1}{\bar{\psi}} - \gamma}{1 - \kappa_{1,c}\rho} \right]^2 \kappa_{1,c}^2 \sigma_x^2 + \gamma^2 \sigma_c^2 + \sigma_d^2}.$$

With our expression for  $\bar{m}$  given in (15), we finally obtain<sup>1</sup>:

$$\kappa_{1,d} = e^{\zeta \log \delta - \gamma\mu + \frac{1}{2} \left[ \frac{\frac{1}{\bar{\psi}} - \gamma}{1 - \kappa_{1,c}\rho} \right]^2 \kappa_{1,c}^2 \sigma_x^2 + \frac{\gamma^2}{2} \sigma_c^2 + \frac{1}{2} \sigma_d^2}. \quad (35)$$

## 4.2 Exercise 2

Using the appropriate expression for the stochastic discount factor and the NA-condition for P/C, find the equation that pins down  $\kappa_c$ .

**Solution.** By No Arbitrage

$$P_t = \mathbb{E}_t[M_{t+1}(P_{t+1} + C_{t+1})], \quad (36)$$

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<sup>1</sup>Remember that  $\zeta = \frac{1}{\theta}$  where  $\theta = \frac{1-\gamma}{\alpha}$  as in Bansal and Yaron (2004).

which is equivalent to

$$\frac{P_t}{C_t} = \mathbb{E}_t \left[ M_{t+1} \frac{P_{t+1} + C_{t+1}}{C_{t+1}} \frac{C_{t+1}}{C_t} \right].$$

and is itself equivalent to

$$e^{pc_t} = \mathbb{E}_t \left[ e^{m_{t+1} + \log(1 + e^{pc_{t+1}}) + \Delta c_{t+1}} \right]. \quad (37)$$

Recall that

$$\log(1 + e^{pc_{t+1}}) \approx \kappa_{0,c} + \kappa_{1,c} pc_{t+1},$$

with  $\kappa_{0,c}$  and  $\kappa_{1,c}$  as in (8). Also, conjecturing that  $pc_t = \mathbf{p}\mathbf{c} + b_c x_t$ , with  $b_c$  as in (10), we arrive at

$$\begin{aligned} e^{\mathbf{p}\mathbf{c} + b_c x_t} &= \mathbb{E}_t \left[ e^{\bar{m} - \frac{1}{\Psi} x_t + \frac{\frac{1}{\Psi} - \gamma}{1 - \rho \kappa_{1,c}} \kappa_{1,c} \varepsilon_{t+1}^x - \gamma \varepsilon_{t+1}^c + \kappa_{0,c} + \kappa_{1,c} (\mathbf{p}\mathbf{c} + b_c x_{t+1}) + \mu + x_t + \varepsilon_{t+1}^c} \right], \\ e^{\mathbf{p}\mathbf{c} - \kappa_{0,c} - \kappa_{1,c} \mathbf{p}\mathbf{c}} e^{b_c x_t} &= e^{[\bar{m} + \mu] + [\alpha + \rho \kappa_{1,c} b_c] x_t} \mathbb{E}_{t-1} \left[ e^{\left( b_c + \frac{\frac{1}{\Psi} - \gamma}{1 - \kappa_{1,c} \rho} \right) \kappa_{1,c} \varepsilon_{t+1}^x + (1 - \gamma) \varepsilon_{t+1}^c} \right] \\ e^{\mathbf{p}\mathbf{c} - \kappa_{0,c} - \kappa_{1,c} \mathbf{p}\mathbf{c}} e^{b_c x_t} &= e^{[\bar{m} + \mu] + [\alpha + \rho \kappa_{1,c} b_c] x_t} \mathbb{E}_{t-1} \left[ e^{(1 - \gamma) \frac{\kappa_{1,c}}{1 - \kappa_{1,c} \rho} \varepsilon_{t+1}^x + (1 - \gamma) \varepsilon_{t+1}^c} \right], \end{aligned}$$

Similarly as before,

$$\mathbf{p}\mathbf{c} - \kappa_{0,c} - \kappa_{1,c} \mathbf{p}\mathbf{c} = \mathbf{p}\mathbf{c} - \log(1 + e^{\mathbf{p}\mathbf{c}}) + \kappa_{1,d} \mathbf{p}\mathbf{c} - \kappa_{1,d} \mathbf{p}\mathbf{c} = \mathbf{p}\mathbf{c} - \log(1 + e^{\mathbf{p}\mathbf{c}}), \quad (38)$$

and therefore

$$e^{\mathbf{p}\mathbf{c} - \kappa_{0,c} - \kappa_{1,c} \mathbf{p}\mathbf{c}} = \frac{e^{\mathbf{p}\mathbf{c}}}{1 + e^{\mathbf{p}\mathbf{c}}} = \kappa_{1,c}.$$

Hence, by assumiing also that  $x_t \approx \mathbb{E}_t[x_t] = 0$ ,

$$\kappa_{1,c} = e^{\bar{m} + \mu} \mathbb{E}_{t-1} \left[ e^{(1 - \gamma) \frac{\kappa_{1,c}}{1 - \kappa_{1,c} \rho} \varepsilon_{t+1}^x + (1 - \gamma) \varepsilon_{t+1}^c} \right]$$

I think it should be

$$\kappa_{1,c} = e^{\bar{m} + \mu} \mathbb{E}_{t-1} \left[ e^{(1 - \gamma)(\varepsilon_{t+1}^x + \varepsilon_{t+1}^c)} \right],$$

but I may be missing some minus sign. In my derivations,

$$\kappa_{1,c} = e^{\bar{m} + \mu + \frac{1}{2}(1 - \gamma)^2 \left[ \frac{\kappa_{1,c}^2}{(1 - \kappa_{1,c})^2} \sigma_x^2 + \sigma_c^2 \right]},$$

but this has no closed form solution... In the case that the  $\kappa_{1,c}$ -term dissappears, we would have

$$\kappa_{1,c} = e^{\bar{m} + \mu + \frac{(1 - \gamma)^2}{2} (\sigma_x^2 + \sigma_c^2)}.$$