Replication - Advanced Econometrics 2

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This model by *** formalises the idea that the reference cycle is best measured by looking at co-movements across several aggregate time series rather than just a single variable, like GDP or GNP. Importantly, we are here talking about one single but unobserved variable, "the state of the economy", denoted here by C_t, which can be proxied, according to ***, by the experimental CEI.

An important assumption here is that each roughly coincident series has a component that is attributable to the single unobserved variable and a component which is unique (idosyncratic component), with all of the idiosyncratic omponents of each observed variable being uncorrelated. Those idiosyncratic components are also assumed to be uncorrelated with C_t at all leads and lags. Our goal here is thus to study an "unobserved single index" or "'dynamic factor" model.

1 The Coincident Index: The M'odel

The transition equation for the state is given by

$$\begin{bmatrix} C_t^* \\ u_t^* \\ C_{t-1} \end{bmatrix} = \begin{bmatrix} \Phi^* & 0 & 0 \\ 0 & D^* & 0 \\ Z_c & 0 & 1 \end{bmatrix} \begin{bmatrix} C_{t-1}^* \\ u_{t-1}^* \\ C_{t-1} \end{bmatrix} + \begin{bmatrix} Z_c & 0 \\ 0 & Z_u \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \eta_t \\ \varepsilon_t \end{bmatrix} \tag{1}$$

where

$$
\begin{aligned}
C_t^* &= \begin{bmatrix} \Delta C_t & \Delta C_{t-1} & \cdots \Delta C_{t-p+1} \end{bmatrix}' \\
u_t^* &= \begin{bmatrix} u_t' & u_{t-1}' & \cdots & u_{t-k+1}' \end{bmatrix}' \\
\Phi^* &= \begin{bmatrix} \phi_1 & \cdots & \phi_{p-1} & \phi_p \\ & I_{p-1} & & 0 \end{bmatrix} \\
D^* &= \begin{bmatrix} D_1 & \cdots & D_{k-1} & D_k \\ & I_{n(k-1)} & & 0 \end{bmatrix}
\end{aligned}
\tag{2}
$$

and

$$
\begin{aligned}
Z_c &= \begin{bmatrix} 1 & 0_{1\times(p-1)} \end{bmatrix} \\
Z_u &= \begin{bmatrix} I_n & 0_{n\times n(k-1)} \end{bmatrix}
\end{aligned}
\tag{3}
$$

and where I_n denotes the $n \times n$ identity matrix, $0_{n\times k}$ denotes a $n \times k$ matrix of zeros, and $D_i = \text{diag}(d_{1,i}, \ldots, d_{n,i})$, where $d_j(L) = 1 - \sum_{i=1}^{k} d_{j,i} L^i$.

The measurement equation is:

$$
Y_t = \beta \mathbf{1} + \begin{bmatrix} \gamma Z_c & Z_u & 0 \end{bmatrix} \begin{bmatrix} C_t^* \\ u_t^* \\ C_{t-1} \end{bmatrix}, \tag{4}
$$

where $\mathbf{1}$ is a vector of ones in $\mathbb{R}^n$, with n being the number of time series variables used to estimate the index; and $\gamma \in \mathbb{R}^n$ containes each specific loading with respect to C_t*. Note then that $\gamma Z_c \in \mathbb{R}^{n\times(p-1)}$. The system can be rewritten more compactly in the standard form

$$
\alpha_t = T\alpha_{t-1} + R\zeta_t \tag{5}
$$

$$
Y_t = \beta \mathbf{1} + Z\alpha_t + \xi_t \tag{6}
$$

where

$$
\begin{aligned}
\alpha_t &= \begin{bmatrix} (C_t^*)' & (u_t^*)' & C_{t-1}' \end{bmatrix}' \\
\zeta_t &= \begin{bmatrix} \eta_t & \varepsilon_t' \end{bmatrix}'
\end{aligned}
\tag{7}
$$

and where T_t, R and Z respectively denote the transition matrix, the selection matrix, and the selection matrix. The covariance matrix of ζ_t is $\mathbb{E}\zeta_t\zeta_t' = \Sigma$. We also assume that $\sigma^2(\eta_t) = 1$ and that Σ is diagonal. The Kalman filter prediction equations are:

$$\_t|t-1 = T_t \alpha_{t-1|t-1} \tag{8}$$

$$P_{t|t-1} = T_t P_{t-1|t-1} T_t' + R\Sigma R' \tag{9}$$

The forecast of Y_t at time $t-1$ is

$$Y_{t|t-1} = \beta \mathbf{1} + Z\alpha_{t|t-1},$$

and the forecast error is

$$\nu_t = Y_t - \beta \mathbf{1} - Z\alpha_{t|t-1}.$$

The updating equations of the filter are:

$$\alpha_{t|t} = \alpha_{t|t-1} + P_{t|t-1} Z' F_t^{-1} \nu_t \tag{10}$$

$$P_{t|t} = P_{t|t-1} - P_{t|t-1} Z' F_t^{-1} Z P_{t|t-1} \tag{11}$$

where $F_t = \mathbb{E}[\nu_t \nu_t'] = Z P_{t|t-1} Z' + H$. In this specification, and following Stock and Watson (1989), we take T_t to be constant and $H = 0$.

2 The Coincident Index: Empirical Results

Following Stock and Watson (1989), we take data on Industrial Production (IP), total personal income less transfer payments (DPI), total manufacturing and trade sales (TS), and average workweek hours (AW).Note that my data is different from those used by Stock and Watson (1989). I am taking data from 2014-12-15 to 2022-08-31.

2.1 Preliminary analysis

The first step in specifying the model is to test for whether the series are integrated and, if they are, whether they are cointegrated. As in Stock and Watson (1989), for each of the the coincident indicators, Dickey and Fuller (1979) test for a unit root (against the alternative that the series are stationary, perhaps around a linear time trend) was unable to reject (at the 10% level) the hypothesis that the series are integrated. P-values are reported in Table 1.

Table 1: P-values of the test Dickey and Fuller (1979) for a unit root applied to the four series used in the index estimation. We fail to reject in every case at thte 10%.

	IP	DPI	TS	AW
p value	0.08	0.33	0.98	0.47

The subsequent aplication of the Engle and Granger (1987) test of the nul hypothesis that the four series are not cointegrated against the alternative of cointegration failed to reject at the 10% significance level. Thus these tests provided no evidence against the hypothesis that each series is integrated but they are not cointegrated. I therefore estimated the model using for the first diference of the logarithm of each of the coincident series, standardized to have zero mean and unit variance.

Table 2: P-values of the (Engle & Granger, 1987) for cointegration to the four series used in the index estimation. We fail to reject in every case at the 10% level.

	IP	DPI	TS	AW
IP	-	0.321725	0.174665	0.175054
DPI	0.814755	-	0.091015	0.538742
TS	0.971659	0.139333	-	0.731533
AW	0.210531	0.149117	0.026043	-

2.2 Maximum Likelihood Estimation

The parameters of the single-index model have been estimated using IP, DPI, TS and AW over the periods 2006:12-2022:08. As in Stock and Watson (1989), a second order autoregressive specification has been adopted for ΔC_t, so that $p = 2$. Also, errors u_t are modelled as an $AR(2)$, i.e., $k = 2$. The loglikelihood for this model is 274.819. The maximum likelihood estimates of the parameters of the single-index model are presented in Tables hemaximumlikelihodestimatesoftheparametersofthesingle-index model are presented in Table 3

Table 3: The estimation period is 2006:12-2022:08. The parameters were estimated by Gaussian maximum likelihod as described in the text. The parameters are $\gamma = (\gamma_1, \ldots, \gamma_4)$, $D(L) = \mathrm{diag}\,(d_1(L), \ldots, d_4(L))$, where $d_i(L) = 1 - d_{i1}L - d_{12}L^2$ and $\Sigma = \mathrm{diag}\left(1, \sigma_1^2, \ldots, \sigma_4^2\right)$. Maximum likelihood is $\mathcal{L} = 274.819$ at the 42688th iteration.

	IP	DPI	TS	AW
γ_i	0.592400	-0.160700	0.585400	0.320700
d_{1i}	-0.136700	-0.277700	0.298100	-0.857200
d_{2i}	-0.010300	-0.796800	-0.288700	-0.060500
σ_i	0.399400	0.611400	0.013000	0.831200
	ϕ_1		ϕ_1	
	0.994600		0.639000	

Figure 1: Evolution of the maximum likelihood, $\mathcal{L}$, during the iteration process. X-axis is plotted in log-scale.

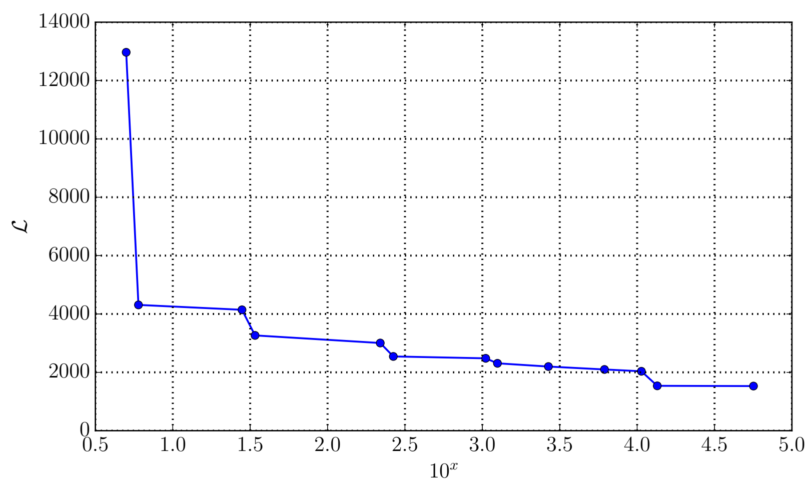

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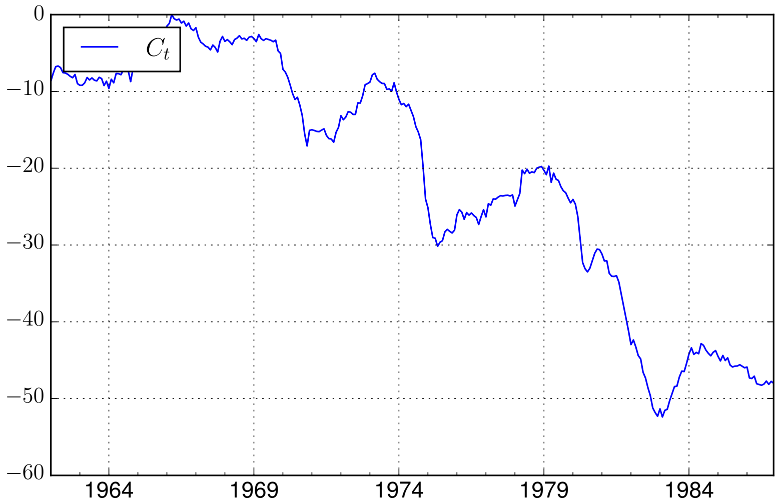
Figure 2: $C_{t|t} = \begin{bmatrix} Z_c & 0 & 1 \end{bmatrix} \alpha_{t|t}$.



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