

Replication - Advanced Econometrics 2

Rubén Fernández-Fuertes

January 31, 2023

This model was proposed by Stock and Watson (1988) formalises the idea that the reference cycle is best measured by looking at co-movements across several aggregate time series rather than just a single variable, like GDP or GNP. Importantly, we are here talking about one single but unobserved variable, “the state of the economy”, denoted here by C_t , which can be proxied, according to (Stock & Watson, 1988, 1989), by the experimental CEI.

An important assumption here is that each roughly coincident series has a component that is attributable to the single unobserved variable and a component which is unique (idiosyncratic component), with all of the idiosyncratic components of each observed variable being uncorrelated. Those idiosyncratic components are also assumed to be uncorrelated with C_t at all leads and lags. Our goal here is thus to study an “unobserved single index” or “dynamic factor” model. The methodology would be using the Kalman Filter.

1 The Coincident Index: The Model

The model for the Coincident Economic Indicators (Stock & Watson, 1989) is

$$\begin{aligned}\Delta Y_t &= \beta + \gamma(L)\Delta C_t + u_t \\ D(L)u_t &= \varepsilon_t \\ \phi(L)\Delta C_t &= \delta + \eta_t,\end{aligned}\tag{1}$$

where L denotes the lag operator, and $\phi(L)$, $\gamma(L)$ and $D(L)$ are respectively scalar, vector, and matrix lag polynomials. Following the spirit of Stock and Watson (1988), we take $\gamma(L) = \gamma \in \mathbb{R}^n$, where n is the number of variables used in Y_t , and Y_t is the vector of logarithms. For simplicity, and again in the spirit of Stock and Watson (1988), we take the degree of ϕ to be $p = 2$ and the degree of $D(L)$ to be $k = 2$. Note that the latter is a matrix of lag polynomials. For simplicity, this matrix is assumed to be diagonal, i.e., $D(L) = \text{diag}(d_1(L), \dots, d_k(L))$. Now, let's write our equations in the state space in order to view it as a Kalman filter.

The transition equation for the state is given by

$$\begin{bmatrix} C_t^* \\ u_t^* \\ C_{t-1} \end{bmatrix} = \begin{bmatrix} \Phi^* & 0 & 0 \\ 0 & D^* & 0 \\ Z_c & 0 & 1 \end{bmatrix} \begin{bmatrix} C_{t-1}^* \\ u_{t-1}^* \\ C_{t-1} \end{bmatrix} + \begin{bmatrix} Z_c & 0 \\ 0 & Z_u \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \eta_t \\ \varepsilon_t \end{bmatrix} \quad (2)$$

where

$$\begin{aligned} C_t^* &= [\Delta C_t \quad \Delta C_{t-1} \quad \dots \quad \Delta C_{t-p+1}]' \\ u_t^* &= [u_t' \quad u_{t-1}' \quad \dots \quad u_{t-k+1}']' \\ \Phi^* &= \begin{bmatrix} \phi_1 & \dots & \phi_{p-1} & \phi_p \\ & I_{p-1} & & 0 \end{bmatrix} \\ D^* &= \begin{bmatrix} D_1 & \dots & D_{k-1} & D_k \\ & I_{n(k-1)} & & 0 \end{bmatrix} \end{aligned} \quad (3)$$

and

$$\begin{aligned} Z_c &= \begin{bmatrix} 1 & 0_{1 \times (p-1)} \end{bmatrix} \\ Z_u &= \begin{bmatrix} I_n & 0_{n \times n(k-1)} \end{bmatrix} \end{aligned} \quad (4)$$

and where I_n denotes the $n \times n$ identity matrix, $0_{n \times k}$ denotes a $n \times k$ matrix of zeros, and $D_i = \text{diag}(d_{1,i}, \dots, d_{n,i})$, where $d_j(L) = 1 - \sum_{i=1}^k d_{j,i} L^i$. Note that we are augmenting our vector to be able to recover C_t as $C_t = [Z_c \ 0 \ 1] \alpha_t$.

The measurement equation is:

$$Y_t = \beta \mathbf{1} + \begin{bmatrix} \gamma Z_c & Z_u & 0 \end{bmatrix} \begin{bmatrix} C_t^* \\ u_t^* \\ C_{t-1} \end{bmatrix}, \quad (5)$$

where $\mathbf{1}$ is a vector of ones in \mathbb{R}^n , with n being the number of time series variables used to estimate the index; and $\gamma \in \mathbb{R}^n$ contains each specific loading with respect to C_t^* . Note then that $\gamma Z_c \in \mathbb{R}^{n \times (p-1)}$. The system can be rewritten more compactly in the standard form

$$\alpha_t = T \alpha_{t-1} + R \zeta_t \quad (6)$$

$$Y_t = \beta \mathbf{1} + Z \alpha_t + \xi_t \quad (7)$$

where

$$\begin{aligned} \alpha_t &= \begin{bmatrix} (C_t^*)^T & (u_t^*)^T & C_{t-1}^T \end{bmatrix}^T \\ \zeta_t &= \begin{bmatrix} \eta_t & \varepsilon_t^T \end{bmatrix}^T \end{aligned} \quad (8)$$

and where T_t , R and Z respectively denote the transition matrix, the selection matrix, and the selection matrix. The covariance matrix of ζ_t is $\mathbb{E} \zeta_t \zeta_t' = \Sigma$. We also assume that $\sigma^2(\eta_t) = 1$ and that Σ is diagonal. The Kalman filter prediction equations are:

$$\alpha_{t|t-1} = T_t \alpha_{t-1|t-1} \quad (9)$$

$$P_{t|t-1} = T_t P_{t-1|t-1} T_t' + R \Sigma R' \quad (10)$$

with $\mathbf{1} \in \mathbb{R}^n$ being a vector of ones. The forecast of Y_t at time $t-1$ is thus

$$Y_{t|t-1} = \beta \mathbf{1} + Z \alpha_{t|t-1},$$

and the forecast error is

$$\nu_t = Y_t - \beta \mathbf{1} - Z\alpha_{t|t-1}.$$

The updating equations of the filter are:

$$\alpha_{t|t} = \alpha_{t|t-1} + P_{t|t-1}Z'F_t^{-1}\nu_t \quad (11)$$

$$P_{t|t} = P_{t|t-1} - P_{t|t-1}Z'F_t^{-1}ZP_{t|t-1} \quad (12)$$

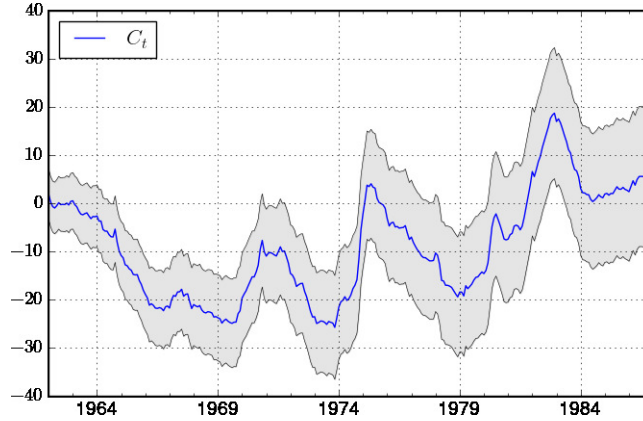
where $F_t = \mathbb{E}[\nu_t\nu_t'] = ZP_{t|t-1}Z' + H$. In this specification, and following Stock and Watson (1989), we take T_t to be constant, i.e., $T_t = T$.

Note that γ can be taken more generally as $\gamma(L)$, as it is done, for instance, in Stock and Watson (1988). However, for the sake of simplicity now, we are just considering $\gamma(L) = \gamma$ to be just a vector in \mathbb{R}^n .

2 The Coincident Index: Empirical Results from 1959:01-1989:01

Following Stock and Watson (1988, 1989), we take data on Industrial Production (IP), total personal income less transfer payments (GMYXP), total manufacturing and trade sales (MT82), and employee-hours in non-agricultural establishments (LPMHU). Note that in their first paper (1988) they use employees on non agricultural payrolls. The reason of this change is to following Moore's (1988) recommendation. Because of over time and part-timework, employee-hours measures more directly fluctuations in labour input than does the number of employees. Hence, our results are going to be different from those of Stock and Watson (1988) for that reason. Data has been downloaded from Stock's webpage.

Figure 1: $C_{t|t}$ together with a 5% confidence interval (assuming gaussianity).



2.1 Preliminary analysis

The first step in specifying the model is to test for whether the series are integrated and, if they are, whether they are cointegrated. As in Stock and Watson (1988, 1989), for each of the coincident indicators, Dickey and Fuller (1979) test for a unit root (against the alternative that the series are stationary, perhaps around a linear time trend) was unable to reject (at the 10% level) the hypothesis that the series are integrated. P-values are reported in Table 1.

Table 1: P-values of the test Dickey and Fuller (1979) for a unit root applied to the four series used in the index estimation. We fail to reject in every case at the 10%.

	IP	GMYP8	MT82	LPMHU
p value	0.97	0.99	0.98	0.99

The subsequent application of the Engle and Granger (1987) test of the null hypothesis that the four series are not cointegrated against the alternative of cointegration failed to reject at the 10% significance level. Thus these tests provided no evidence against the hypothesis that each series is integrated but they are not cointegrated. I therefore estimated the model using for the first difference of the logarithm of each of the coincident

series, standardized to have zero mean and unit variance.

Table 2: P-values of the (Engle & Granger, 1987) for cointegration to the four series used in the index estimation. We fail to reject in every case at the 10% level.

	IP	GMYP8	MT82	LPMHU
IP	-	0.013209	0.230167	0.000807
GMYP8	0.01397	-	0.020279	0.128943
MT82	0.248144	0.020003	-	0.121991
LPMHU	0.051041	0.134312	0.130508	-

2.2 Maximum Likelihood Estimation

The parameters of the single-index model have been estimated using IP, DPI, TS and AW over the periods 1959:2-1983:12. As in Stock and Watson (1989), a second order autoregressive specification has been adopted for ΔC_t , so that $p = 2$. Also, errors u_t are modelled as an $AR(2)$, i.e., $k = 2$. I have followed Stock and Watson (1988) and considered γ to be constant, i.e., $\gamma = \gamma_0 \in \mathbb{R}^4$. The loglikelihood for this model is 424.529. The maximum likelihood estimates of the parameters of the single-index model are presented in Table 3. I have followed Gupta and Mehra (1974) for maximising the log likelihood. I'm aware that there's another alternative proposed by Shumway and Stoffer (1982) using the EM algorithm, but I have preferred the former to used `scipy` packages in `Python`. The basic idea is that the negative log likelihood is minimised by the Newton-Raphson Method, iterating the process until the log likelihood stabilises. This method may have problems depending on the initial conditions. Standard errors are computed as the square root of the diagonal of the (approximated) Hessian matrix of \mathcal{L} (Shumway & Stoffer, 2019), which can be easily recovered with an easy command.

The CEI, $C_{t|t}$, is plotted in Figure 1. Notice that

$$C_{t|t} = [Z_c \ 0 \ 1] \hat{\alpha}_{t|t}.$$

Table 3: The estimation period is 1959:2-1983:12. The parameters were estimated by Gaussian maximum likelihood as described in the text. The parameters are $\gamma = (\gamma_1, \dots, \gamma_4)$, $D(L) = \text{diag}(d_1(L), \dots, d_4(L))$, where $d_i(L) = 1 - d_{i1}L - d_{i2}L^2$ and $\Sigma = \text{diag}(1, \sigma_1^2, \dots, \sigma_4^2)$. Maximum likelihood is $\mathcal{L} = 424.529$.

	IP	GMVX	LPMH	LPMHU
γ_i	-0.7219 (0.033)	-0.4976 (0.0283)	-0.4625 (0.0283)	-0.5001 (0.0274)
d_{1i}	-0.0204 (0.0858)	-0.1163 (0.1191)	-0.4273 (0.0769)	-0.4453 (0.0489)
d_{2i}	-0.1492 (0.1071)	0.1513 (0.0515)	-0.2311 (0.0996)	-0.1206 (0.0675)
σ_i	0.4681 (0.0362)	0.78 (0.033)	0.7425 (0.0309)	0.7132 (0.0308)

I have also included a 5% confidence interval band measured by

$$C_{t|t} \pm 1.96\sqrt{\hat{P}_{t|t}},$$

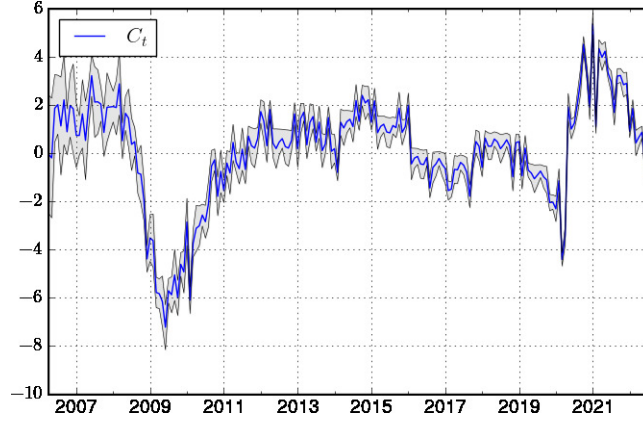
where

$$\hat{P}_{t|t} = [Z_c \ 0 \ 1] \hat{\alpha}_{t|t} [Z_c \ 0 \ 1]^T$$

3 The Coincident Index: Empirical Results from 2007:01-2022:12

I want to find the Coincident Economic Index with more recent data. Then, I take US data from Refinitiv on Industrial Production (USIPMANG), total disposable personal incomes (USGPDYD), total machinery manufacturing (USPPM), and weekly average work hours in nonfarm payrolls (USWRKW). I am taking data from 2007:01-2012:12.

Figure 2: $C_{t|t}$ together with a 5% confidence interval (assuming gaussianity) using data from 2007:01 to 2022:12.



3.1 Preliminary analysis

I repeat the same analysis: I check whether the series are integrated and, if they are, whether they are cointegrated. For each of the the coincident indicators, Dickey and Fuller (1979) test for a unit root (against the alternative that the series are stationary, perhaps around a linear time trend) was unable to reject (at the 10% level) the hypothesis that the series are integrated. P-values are reported in Table 4.

Table 4: P-values of the test Dickey and Fuller (1979) for a unit root applied to the four series used in the index estimation. We fail to reject in every case at thte 10%.

	USIPMANG	USGPYD	USPPM	USWRKW
p value	0.08	0.91	0.98	0.47

The subsequent aplication of the Engle and Granger (1987) test of the null hypothesis that the four series are not cointegrated against the alternative of cointegration failed to reject at the 10% significance level. Thus these tests provided no evidence against the hypothesis that each series is integrated but they are not cointegrated. I therefore estimated the model using for the first difference of the logarithm of each of the coincident series, standardized to have zero mean and unit variance.

Table 5: P-values of the (Engle & Granger, 1987) for cointegration to the four series used in the index estimation. We fail to reject in every case at the 10% level.

	USIPMANG	USGPYD	USPPM	USWRKW
USIPMANG	-	0.131099	0.195242	0.038813
USGPYD	0.961745	-	0.792259	0.764748
USPPM	0.992393	0.118382	-	0.940171
USWRKW	0.696816	0.452826	0.08429	-

3.2 Maximum Likelihood Estimation

The parameters of the single-index model have been estimated using USIPMANG, USGPYD, USPPM, USWRKW over the periods 2007:01-2022:12. I repeat the second order autoregressive specification has been adopted for ΔC_t , so that $p = 2$. Also, errors u_t are modelled as an $AR(2)$, i.e., $k = 2$; and γ is assumed to be constant, i.e., $\gamma = \gamma_0 \in \mathbb{R}^4$. The loglikelihood for this model is 233.490. The maximum likelihood estimates of the parameters of the single-index model are presented in Table 6. The procedure is the same than before.

Table 6: The estimation period is 1959:2-1983:12. The parameters were estimated by Gaussian maximum likelihood as described in the text. The parameters are $\gamma = (\gamma_1, \dots, \gamma_4)$, $D(L) = \text{diag}(d_1(L), \dots, d_4(L))$, where $d_i(L) = 1 - d_{i1}L - d_{i2}L^2$ and $\Sigma = \text{diag}(1, \sigma_1^2, \dots, \sigma_4^2)$. Maximum likelihood is $\mathcal{L} = 233.491$.

	USIPMANG	USGPYD	USPPM	USWRKW
γ_i	0.2289 (0.0499)	0.133 (0.0611)	-0.0059 (0.0382)	0.8573 (0.0503)
d_{1i}	0.2115 (0.0613)	-0.5554 (0.0794)	0.4363 (0.057)	-0.2912 (0.0055)
d_{2i}	-0.219 (0.069)	-0.1022 (0.0699)	0.3789 (0.0408)	-0.9998 (0.011)
σ_i	-0.9257 (0.0436)	-0.8277 (0.0546)	0.6663 (0.0342)	-0.0 (0.0134)

References

- Dickey, D. A., & Fuller, W. A. (1979). Distribution of the estimators for autoregressive time series with a unit root. *Journal of the American statistical association*, 74(366a), 427–431.
- Engle, R. F., & Granger, C. W. (1987). Co-integration and error correction: representation, estimation, and testing. *Econometrica: journal of the Econometric Society*, 251–276.
- Gupta, N., & Mehra, R. (1974). Computational aspects of maximum likelihood estimation and reduction in sensitivity function calculations. *IEEE transactions on automatic control*, 19(6), 774–783.
- Shumway, R. H., & Stoffer, D. S. (1982). An approach to time series smoothing and forecasting using the em algorithm. *Journal of time series analysis*, 3(4), 253–264.
- Shumway, R. H., & Stoffer, D. S. (2019). *Time series: a data analysis approach using r*. Chapman and Hall/CRC.
- Stock, J. H., & Watson, M. W. (1988). *A probability model of the coincident economic indicators*. National Bureau of Economic Research Cambridge, Mass., USA.
- Stock, J. H., & Watson, M. W. (1989). New indexes of coincident and leading economic indicators. *NBER macroeconomics annual*, 4, 351–394.