What do algebras form?

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Outline

- **Question:** What do algebras form?

- **Answer 1:** A category in categories (HH^0)
- **Derived Answer 1:** A category in dg cocategories (Hochschild cochains...)

- **Answer 2:** A 2-category with a trace functor (HH_0)
- **Derived Answer 2:** A category in dg cocategories with a trace functor (Hochschild chains...)

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- **Derived Answer 2:** A category in dg cocategories with a trace functor (Hochschild chains...) up to homotopy

Answer 1: Algebras form a **2-category**.

- Objects: algebras $A, B, \ldots$
- 1-Morphisms: bimodules ${}_A M_B$
- 1-Composition: ${}_A M_B \otimes_B {}_B N_C$
- 2-Morphisms: morphisms of bimodules

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$$\{\text{maps of bimodules } {}_f B \to_g B\} \cong Z_A({}_g B_f) \cong HH^0(A, {}_g B_f)$$
$$M \mapsto M(1)$$
$$(M_b : b' \mapsto b \cdot b') \leftarrow b$$

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Can we use Hochschild cohomology or cochains instead of HH^0?

Derived Answer 1: Algebras form a category in dg cocategories.

- Objects: algebras $A, B, \dots$
- Morphisms: a dg cocategory $Bar(Hoch(A, B))$
- Composition:
  $\bullet : Bar(Hoch(A, B)) \otimes Bar(Hoch(B, C)) \to Bar(Hoch(A, C))$
  associative map of dg cocategories

Defining $Bar(Hoch(A, B))$

1. $Hoch(A, B)$ is a dg category with
   - Objects: algebra maps $f : A \to B$
   - Morphisms: $Hoch(A)(f, g) = (C^\bullet(A, {}_fB_g), {}_f\delta_g)$
   - Composition: cup product on cochains

$$\phi \in C^p(A,_f B_g)$$
$$\psi \in C^q(A,_g B_h)$$
$$(\phi \cup \psi)(a_1, \ldots, a_{p+q}) = \pm\phi(a_1, \ldots, a_p)\psi(a_{p+1}, \ldots, a_q)$$

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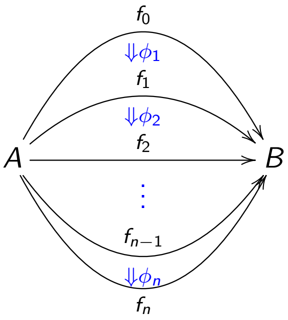
2. $Bar : DGCat \to DGCocat$
   Bar(Hoch(A,B)) has the same objects as Hoch(A,B).



A morphism from f_0 to f_n in Bar(Hoch(A,B))

$$\Delta(\phi_1 \dots \phi_n) = \sum_{0 \le i \le n} \pm \phi_1 \dots \phi_i \otimes \phi_{i+1} \dots \phi_n$$

$$|\phi_1 \dots \phi_n| = \sum_{1 \le i \le n} |\phi_i| - n$$

$$d_{Bar(Hoch(A,B))} = \tilde{d}_{Hoch(A,B)} + d_\cup$$

Derived Answer 1: Algebra form a category in dg cocategories.

- Objects: algebras $A, B, \ldots$
- Morphisms: a dg cocategory $Bar(Hoch(A, B))$
- Composition:
  $\bullet : Bar(Hoch(A, B)) \otimes Bar(Hoch(B, C)) \to Bar(Hoch(A, C))$
  associative map of dg cocategories

$Bar(Hoch(A, C))$ is the cofree dg cocategory on $Hoch(A, C)$:

$$\begin{tikzcd}
Bar(Hoch(A, B)) \otimes Bar(Hoch(B, C)) \arrow[r, "\bullet"] \arrow[dr, dotted, "\bullet"'] & Hoch(A, C) \\
& Bar(Hoch(A, C)) \arrow[u, two heads]
\end{tikzcd}$$

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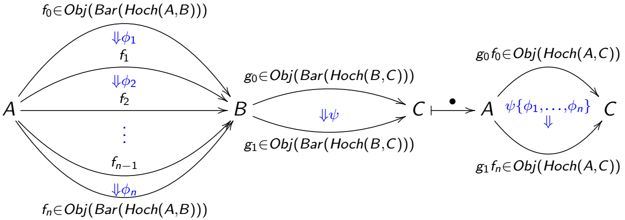

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\end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge]
A \arrow[r, bend left=60, "f_0 \in Obj(Bar(Hoch(A,B)))"] \arrow[r, phantom, bend left=45, "\Downarrow\phi_1"] \arrow[r, bend left=25, "f_1"] \arrow[r, phantom, bend left=15, "\Downarrow\phi_2"] \arrow[r, "f_2"] \arrow[r, phantom, bend right=20, "\vdots"] \arrow[r, bend right=30, "f_{n-1}"'] \arrow[r, phantom, bend right=50, "\Downarrow\phi_n"] \arrow[r, bend right=60, "f_n \in Obj(Bar(Hoch(A,B)))"'] & B \arrow[r, bend left, "g_0 \in Obj(Bar(Hoch(B,C)))"] \arrow[r, phantom, "\Downarrow\psi"] \arrow[r, bend right, "g_1 \in Obj(Bar(Hoch(B,C)))"'] & C \arrow[r, mapsto, "\bullet"] & A \arrow[r, bend left, "g_0 f_0 \in Obj(Hoch(A,C))"] \arrow[r, phantom, "\psi\{\phi_1,\ldots,\phi_n\} \Downarrow"] \arrow[r, bend right, "g_1 f_n \in Obj(Hoch(A,C))"'] & C
\end{tikzcd}$$

$$\psi\{\phi_1,\ldots,\phi_n\}(a_1,\ldots,a_q) = \sum \pm\psi(f_0 a_1,\ldots,f_0 a_{i_1},\phi_1(a_{i_1+1},\ldots), f_1 a_*,\ldots,f_1 a_*,$$
$$\phi_2(a_*,\ldots), f_2 a_*,\ldots,f_n a_q)$$

$Bar(Hoch(A,C))$ is the cofree dg cocategory on $Hoch(A,C)$:

$$\begin{tikzcd} Bar(Hoch(A,B)) \otimes Bar(Hoch(B,C)) \arrow[r, "\bullet"] \arrow[dr, dotted, "\bullet"'] & Hoch(A,C) \\ & Bar(Hoch(A,C)) \arrow[u, two heads] \end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge] A \arrow[r, bend left=60, "f_0 \in Obj(Bar(Hoch(A,B)))"] \arrow[r, bend left=30, "f_1"] \arrow[r, "f_2"] \arrow[r, bend right=30, "f_{n-1}"'] \arrow[r, bend right=60, "f_n \in Obj(Bar(Hoch(A,B)))"'] \arrow[r, phantom, shift left=6, "\Downarrow\phi_1"] \arrow[r, phantom, shift left=2, "\Downarrow\phi_2"] \arrow[r, phantom, shift right=3, "\vdots"] \arrow[r, phantom, shift right=7, "\Downarrow\phi_n"] & B \arrow[r, bend left, "g_0 \in Obj(Bar(Hoch(B,C)))"] \arrow[r, bend right, "g_1 \in Obj(Bar(Hoch(B,C)))"'] \arrow[r, phantom, "\Downarrow\psi"] & C \arrow[r, mapsto, "\bullet"] & A \arrow[r, bend left, "g_0 f_0 \in Obj(Hoch(A,C))"] \arrow[r, bend right, "g_1 f_n \in Obj(Hoch(A,C))"'] \arrow[r, phantom, "\psi\{\phi_1,\dots,\phi_n\} \Downarrow"] & C \end{tikzcd}$$

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Bar(Hoch(A,B)) \otimes Bar(Hoch(B,C)) \arrow[r, "\bullet"] \arrow[dr, dotted, "\bullet"'] & Hoch(A,C) \\
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\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=huge]
A \arrow[r, bend left=70, "f_0\in Obj(Bar(Hoch(A,B)))"] \arrow[r, bend left=35, "f_1"'] \arrow[r, bend left=70, phantom, shift right=3, "\Downarrow\phi_1"] \arrow[r, bend left=35, phantom, shift right=3, "\Downarrow\phi_2"] \arrow[r, "f_2"] \arrow[r, phantom, shift right=4, "\vdots"] \arrow[r, bend right=35, "f_{n-1}"] \arrow[r, bend right=70, "f_n\in Obj(Bar(Hoch(A,B)))"'] \arrow[r, bend right=55, phantom, "\Downarrow\phi_n"] & B \arrow[r, bend left, "g_0\in Obj(Bar(Hoch(B,C)))"] \arrow[r, bend right, "g_1\in Obj(Bar(Hoch(B,C)))"'] \arrow[r, phantom, "\Downarrow\psi"] & C \arrow[r, mapsto, "\bullet"] & A \arrow[r, bend left, "g_0f_0\in Obj(Hoch(A,C))"] \arrow[r, bend right, "g_1f_n\in Obj(Hoch(A,C))"'] \arrow[r, phantom, "\psi\{\phi_1,\ldots,\phi_n\}\Downarrow"] & C
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\begin{tikzcd}
A \arrow[r, bend left, "f_0"] \arrow[r, bend right, "f_n"'] \arrow[r, phantom, "\Downarrow\phi"] & B \arrow[r, bend left, "g_0"] \arrow[r, bend right, "g_1"'] \arrow[r, phantom, "\Downarrow 1\in k"] & C \arrow[r, mapsto, "\bullet"] & A \arrow[r, bend left, "g_0f_0"] \arrow[r, bend right, "g_1f_n"'] \arrow[r, phantom, "\Downarrow\phi"] & C
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Braces are associative. (Getzler-Jones; Voronov-Gerstenhaber, Lyubashenko-Manzyuk; Keller)

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Answer 2: Algebras form a 2-category with a trace functor

Definition

(Kaledin): A <u>trace functor</u> on a 2-category $\mathcal{C}$ is:

- for each $A \in Obj(\mathcal{C})$, a functor $TR_A : \mathcal{C}(A, A) \to k - mod$
- for each pair $A, B \in Obj(\mathcal{C})$, a natural transformation $\tau_!(A, B)$

$$
\begin{tikzcd}
\mathcal{C}(A,B) \otimes \mathcal{C}(B,A) \arrow[rr, "\tau = flip"] \arrow[d, "m"'] & & \mathcal{C}(B,A) \otimes \mathcal{C}(A,B) \arrow[d, "m"'] \\
\mathcal{C}(A,A) \arrow[rr, phantom, "\overset{\Rightarrow}{\tau_!(A,B)}"] \arrow[dr, "TR_A"'] & & \mathcal{C}(B,B) \arrow[dl, "TR_B"] \\
& k - mod &
\end{tikzcd}
$$

- such that $\tau_!(B, A) \circ \tau_!(C, B) \circ \tau_!(A, C) = id$

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$$
\begin{tikzcd}
 & & \mathcal{C}(C,A) \otimes \mathcal{C}(A,C) \otimes \mathcal{C}(B,C) \arrow[drr, "\tau"] \arrow[dd] & & \\
\mathcal{C}(A,B) \otimes \mathcal{C}(B,C) \otimes \mathcal{C}(C,A) \arrow[urr, "\tau"] \arrow[drr] \arrow[r, phantom, "\tau_!(\overset{\Rightarrow}{A},C)"] & \tau_!(\overset{\Leftarrow}{B},A) & & \tau_!(\overset{\Rightarrow}{C},B) & \mathcal{C}(B,C) \otimes \mathcal{C}(C,A) \otimes \mathcal{C}(A,B) \arrow[llll, bend right, "\tau"'] \arrow[dll] \\
 & & k - mod & &
\end{tikzcd}
$$

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  $TR_A : \text{bimodule } {}_AM_A \mapsto M/[A, M] \cong HH_0(A, M)$

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$$\begin{tikzcd}
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\mathcal{C}(A,A) \arrow[dr, "TR_A"'] & \tau_!(\overset{\Rightarrow}{A},B) & \mathcal{C}(B,B) \arrow[dl, "TR_B"] \\
& k - mod &
\end{tikzcd}$$

$$\frac{{}_AM_B \otimes_B {}_BN_A}{[A, M \otimes_B N]} \xrightarrow{\tau_!(A,B)} \frac{{}_BN_A \otimes_A {}_AM_B}{[B, N \otimes_A M]}$$

$$m \otimes n \mapsto n \otimes m$$

such that

$\tau_!(B, A) \circ \tau_!(C, B) \circ \tau_!(A, C) = id$

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\mathcal{C}(A,B) \otimes \mathcal{C}(B,A) \arrow[rr, "\tau = flip"] \arrow[d, "m"'] & & \mathcal{C}(B,A) \otimes \mathcal{C}(A,B) \arrow[d, "m"] \\
\mathcal{C}(A,A) \arrow[dr, "TR_A"'] & \tau_!(\overrightarrow{A,B}) & \mathcal{C}(B,B) \arrow[dl, "TR_B"] \\
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\end{tikzcd}$$

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$$m \otimes n \mapsto n \otimes m$$

such that

$$\tau_!(B, A) \circ \tau_!(C, B) \circ \tau_!(A, C) = id$$

Can we use Hochschild homology or chains instead of HH_0 to extend this to a trace functor on the category in dg cocategories?

Massaging the definition of a trace functor

Definition

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- for each $A \in Obj(\mathcal{C})$, a functor $TR_A : \mathcal{C}(A, A) \to k - mod$

$$\mathcal{C}(A, A)(f, g) \otimes_k TR_A(g) \to TR_A(f)$$

- for each pair $A, B \in Obj(\mathcal{C})$, a natural transformation $\tau_!(A, B)$

$$\begin{tikzcd}
\mathcal{C}(A,B) \otimes \mathcal{C}(B,A) \arrow[rr, "\tau"] \arrow[d, "m"'] & & \mathcal{C}(B,A) \otimes \mathcal{C}(A,B) \arrow[d, "m"'] \\
\mathcal{C}(A,A) \arrow[dr, "TR_A"'] \arrow[rr, phantom, "\Rightarrow \tau_!(A,B)"] & & \mathcal{C}(B,B) \arrow[dl, "TR_B"] \\
& k - mod &
\end{tikzcd}$$

- such that $\tau_!(B, A) \circ \tau_!(C, B) \circ \tau_!(A, C) = id$

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Definition

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- for each $A \in Obj(\mathcal{C})$, a functor $TR_A : \mathcal{C}(A,A) \to k-mod$

$$\mathcal{C}(A,A)(f,g) \otimes_k TR_A(g) \to TR_A(f)$$

- for each pair $A, B \in Obj(\mathcal{C})$, a natural transformation $\tau_!(A,B)$

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\mathcal{C}(A,A) \arrow[dr, "TR_A"'] \arrow[rr, phantom, "\overset{\Rightarrow}{\tau_!(A,B)}"] & & \mathcal{C}(B,B) \arrow[dl, "TR_B"] \\
& k-mod &
\end{tikzcd}$$

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Definition

(Kaledin): A trace functor on a category in k-linear categories $\mathcal{C}$ is:

- for each $A \in Obj(\mathcal{C})$, a left module $T(A)$ over $\mathcal{C}(A, A)$

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\mathcal{C}(A,A) \arrow[dr, "TR_A"'] & \tau_!(\overrightarrow{\overrightarrow{A}},B) & \mathcal{C}(B,B) \arrow[dl, "TR_B"] \\
& k - mod &
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- for each pair $A, B \in Obj(\mathcal{C})$, a map of modules $\tau_!(A,B) : m^*T(A) \to \tau^* m^* T(B)$ over $\mathcal{C}(A,B) \otimes \mathcal{C}(B,A)$

$$\begin{tikzcd}
\mathcal{C}(A,B) \otimes \mathcal{C}(B,A) \arrow[rr, "\tau"] \arrow[d, "m"'] & & \mathcal{C}(B,A) \otimes \mathcal{C}(A,B) \arrow[d, "m"] \\
\mathcal{C}(A,A) \arrow[dr, "TR_A"'] \arrow[rr, phantom, "\overset{\Rightarrow}{\tau_!(A,B)}"] & & \mathcal{C}(B,B) \arrow[dl, "TR_B"] \\
& k - mod &
\end{tikzcd}$$

- such that $\tau_!(B,A) \circ \tau_!(C,B) \circ \tau_!(A,C) = id$

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Definition

(Kaledin): A trace functor on a category in k-linear categories $\mathcal{C}$ is:

- for each $A \in Obj(\mathcal{C})$, a left module $T(A)$ over $\mathcal{C}(A, A)$

$$\mathcal{C}(A, A)(f, g) \otimes_k TR_A(g) \to TR_A(f)$$

- for each pair $A, B \in Obj(\mathcal{C})$, a map of modules $\tau_!(A, B) : m^* T(A) \to \tau^* m^* T(B)$ over $\mathcal{C}(A, B) \otimes \mathcal{C}(B, A)$

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\mathcal{C}(A,A) \arrow[dr, "TR_A"'] & \tau_!(\overrightarrow{A},B) & \mathcal{C}(B,B) \arrow[dl, "TR_B"] \\
& k - mod &
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- such that $\tau^{*2}\tau_!(B, A) \circ \tau^*\tau_!(C, B) \circ \tau_!(A, C) = id$

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\mathcal{C}(A,A) \arrow[dr, "TR_A"'] \arrow[rr, phantom, "\tau_!(\overrightarrow{\overrightarrow{A}},B)"] & & \mathcal{C}(B,B) \arrow[dl, "TR_B"] \\
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Massaging the definition of a trace functor

Definition

(Kaledin): A <u>trace functor</u> on a category in dg cocategories $\mathcal{C}$ is:

- for each $A \in Obj(\mathcal{C})$, a left dg comodule $T(A)$ over $\mathcal{C}(A,A)$

$$\prod_{g\in Obj(\mathcal{C})} \mathcal{C}(A,A)^{\bullet}(f,g) \otimes_k TR_A{}^{\bullet}(g) \leftarrow TR_A{}^{\bullet}(f)$$

- for each pair $A, B \in Obj(\mathcal{C})$, a map of modules $\tau_!(A,B) : m^* T(A) \to \tau^* m^* T(B)$ over $\mathcal{C}(A,B) \otimes \mathcal{C}(B,A)$

$$\begin{tikzcd}
\mathcal{C}(A,B)\otimes\mathcal{C}(B,A) \arrow[rr, "\tau"] \arrow[d, "m"'] & & \mathcal{C}(B,A)\otimes\mathcal{C}(A,B) \arrow[d, "m"] \\
\mathcal{C}(A,A) \arrow[dr, "TR_A"'] \arrow[rr, phantom, "\tau_!(\overset{\rightrightarrows}{A},B)"] & & \mathcal{C}(B,B) \arrow[dl, "TR_B"] \\
& k-mod &
\end{tikzcd}$$

- such that $\tau^{*2}\tau_!(B,A) \circ \tau^*\tau_!(C,B) \circ \tau_!(A,C) = id$

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\mathcal{C}(A,A) \arrow[dr, "TR_A"'] & \tau_!(\overset{\rightrightarrows}{A},B) & \mathcal{C}(B,B) \arrow[dl, "TR_B"] \\
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Unraveling the definition of a trace functor

Definition

Let $\mathcal{C}$ be a category in dg cocategories. Let $\chi(\mathcal{C})$ be the dg category with

- Objects $= \{A_0 \to \ldots \to A_n \to A_0 : A_i \in Obj(\mathcal{C}), n \geq 0\}$
- Morphisms $= \{$linear combinations of compositions of

$$\begin{aligned} \textit{rotations } \tau_n &: \mathcal{A} \mapsto (A_n \to A_0 \to \ldots \to A_n) \\ \textit{coboundaries } \delta_{j,n} &: \mathcal{A} \mapsto (A_0 \to \ldots \to A_j \to A_{j+2\,(mod\ n+1)} \to \ldots \to A_0) \\ \textit{codegeneracies } \sigma_{i,n} &: \mathcal{A} \mapsto (A_0 \to \ldots \to A_i \to A_i \to \ldots \to A_0) \end{aligned}$$

where $\mathcal{A} := (A_0 \to \ldots \to A_n \to A_0)$, subject to the cyclic relations$\}[0]$

Unraveling the definition of a trace functor

Definition

Let $\mathcal{D}$ be the dg category with

- Objects $= \{(\underset{B}{\text{dg cocategory}}, \underset{C}{\text{dg comodule}})\}$
- Morphisms:

$$\mathcal{D}^p\big((B_1, C_1), (B_0, C_0)\big) := \left\{ \begin{matrix} F : B_1 \to B_0 \textit{ dg functor}, \\ F_! : C_1 \to F^* C_0 \textit{ degree-p linear map} \end{matrix} \right\}$$

$$d_{\mathcal{D}}(F, F_!) = \big(F, [d, F_!] = d_{F^*C_0} \circ F_! \pm F_! \circ d_{C_1}\big)$$

Unraveling the definition of a trace functor

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For us, $F^* C_0$ is the categorified version of co-extension of scalars:

$$F^* C_0 = ker(B_1 \otimes C_0 \rightrightarrows B_1 \otimes B_0 \otimes C_0)$$

Unraveling the definition of a trace functor

Let $\mathcal{C}$ be a category in dg cocategories. A trace functor on $\mathcal{C}$ gives a dg functor

$$\chi(\mathcal{C}) \to \mathcal{D}$$

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$$(A_0 \to \ldots \to A_n \to A_0) \mapsto \begin{pmatrix} \mathcal{C}(A_0, A_1) \otimes \ldots \otimes \mathcal{C}(A_n, A_0), \\ m^{*n} T(A_0), \ {\scriptstyle m^n : \mathcal{C}(A_0,A_1) \otimes \ldots \otimes \mathcal{C}(A_n,A_0) \to \mathcal{C}(A_0,A_0)} \end{pmatrix}$$

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$$\delta_{j,n} \mapsto \begin{pmatrix} \ldots \otimes \mathcal{C}(A_j, A_{j+1}) \otimes \mathcal{C}(A_{j+1}, A_{j+2}) \otimes \ldots \xrightarrow{\hat{\delta}_{j,n} = m} \ldots \otimes \mathcal{C}(A_j, A_{j+2}) \otimes \ldots \\ m^{*n} T(A_0) \xrightarrow{\delta_{j,n!} = id} \hat{\delta}^*_{j,n} m^{*n-1} T(A_0) \cong (m^{n-1} \hat{\delta}_{j,n})^* T(A_0) \end{pmatrix}$$

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$$\tau_n \mapsto \begin{pmatrix} \mathcal{C}(A_0, A_1) \otimes \ldots \otimes \mathcal{C}(A_n, A_0) \xrightarrow{\hat{\tau}_n} \mathcal{C}(A_n, A_0) \otimes \ldots \otimes \mathcal{C}(A_{n-1}, A_n) \\ m^{*n} T(A_0) \xrightarrow{\tau_{n!} = m^{*n-1} \tau_!(A_0, A_n)} \hat{\tau}^*_n m^{*n} T(A_n) \\ m^{n-1} : \big(\mathcal{C}(A_0, A_1) \otimes \ldots \otimes \mathcal{C}(A_{n-1}, A_n)\big) \otimes \mathcal{C}(A_n, A_0) \to \mathcal{C}(A_0, A_n) \otimes \mathcal{C}(A_n, A_0) \end{pmatrix}$$

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$\tau_n^{n+1} = id$ is preserved:

- n=2 cocyle relation,
- $n > 2$ pullback of cocycle relation,
- n=1 cocycle relation for $A, B, C = B$ and the fact that $\sigma_{1,1!}$is an identity map

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Functor is DG: $\delta_{j,n!} = id$, $\sigma_{i,n!} = id$, $\tau_{n!} = m^{*n-1} \tau_!$ are maps of DG comodules.

Question: Can we give a dg functor

$$\chi(\mathcal{C}) \to \mathcal{D}$$

$$\text{where } (A_0 \to \ldots \to A_n \to A_0) \mapsto \begin{pmatrix} B(A_0 \to \ldots \to A_n \to A_0) := \\ := Bar(Hoch(A_0, A_1)) \otimes \ldots \otimes Bar(Hoch(A_n, A_0)), \\ C(A_0 \to \ldots \to A_n \to A_0) := m^{*n} T(A_0) \end{pmatrix} ?$$

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Answer: No, but we can give an A_∞-functor.
This will imply that algebras form a category in dg cocategories with a trace functor up to homotopy.

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Rest of this talk:

- Define dg comodules $C(A_0 \to \ldots \to A_0)$ using Hochschild chains
- Describe the A_∞-functor: $\tau_{1!}$, $\tau_{n!}^{n+1} = m^{*n-1} \tau_{1!} \sim id$

Fix algebras $A_0, \ldots, A_n$. Let $\mathcal{A} = (A_0 \to \ldots \to A_n \to A_0)$.
Define a dg comodule $C(\mathcal{A})$ over $B(\mathcal{A})$:

$$C(\mathcal{A})^\bullet(\underbrace{A_0 \stackrel{f_{0,0}}{\to} \ldots \to A_n \stackrel{f_{n,0}}{\to} A_0}_{\in Obj(B(\mathcal{A}))}) :=$$

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$$:= \left\{ \begin{tikzcd} A_0 \arrow[r, bend left=70, "f_{0,0}"] \arrow[r, bend left=30, "f_{0,1}"] \arrow[r, phantom, shift left=4, "\Downarrow\phi_{0,1}"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=30, "f_{0,k_0}"'] \arrow[rrrr, bend right=40, "\alpha", "id_{A_0}"'] & A_1 \arrow[r, dotted, bend left=70] \arrow[r, dotted, bend left=30] \arrow[r, phantom, "\vdots"] \arrow[r, dotted, bend right=30] & \cdots \arrow[r, dotted, bend left=70] \arrow[r, dotted, bend left=30] \arrow[r, phantom, "\vdots"] \arrow[r, dotted, bend right=30] & A_n \arrow[r, bend left=70, "f_{n,0}"] \arrow[r, bend left=30, "f_{n,1}"] \arrow[r, phantom, shift left=4, "\Downarrow\phi_{n,1}"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=30, "f_{n,k_n}"'] & A_0 \end{tikzcd} = \begin{array}{c} (\phi_{0,1}|\ldots|\phi_{0,k_0})\otimes\ldots\otimes(\phi_{n,1}|\ldots|\phi_{n,k_n})\otimes\alpha: \\ s.t.\ \phi_{i,j}\in C^\bullet(A_{i,f_{j-1}}A_{i+1\,f_j}), \\ \alpha\in C_{-\bullet}(A_{0,f_{n,k_n}}\ldots{}_{f_{0,k_0}}A_0) \end{array} \right\}$$

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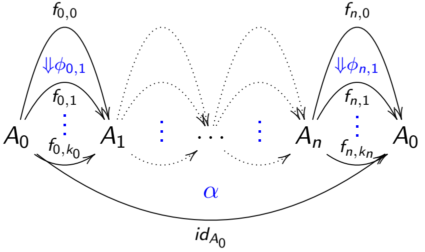
$:= \{$



$$= \begin{array}{c} (\phi_{0,1}|\ldots|\phi_{0,k_0})\otimes\ldots\otimes(\phi_{n,1}|\ldots|\phi_{n,k_n})\otimes\alpha: \\ s.t.\ \phi_{i,j}\in C^{\bullet}(A_{i}, {}_{f_{j-1}}A_{i+1}{}_{f_j}), \\ \alpha\in C_{-\bullet}(A_0, {}_{f_{n,k_n}}\ldots{}_{f_{0,k_0}}A_0) \end{array} \}$$

$$d_{C(A_0\to\ldots\to A_0)} = d_B \otimes id_{C_{-\bullet}} + id_B \otimes b + \tilde{\iota}$$

where $\tilde{\iota}$ is given as follows:

$$
\begin{tikzcd}
C(A_0 \to \ldots \to A_0)^{\bullet}(f_{0,0}, \ldots, f_{n,0}) \arrow[r, "\tilde{\iota}"] \arrow[dr, "\tilde{\iota}", "\text{extend as a coderivation}"'] & C_{-\bullet}(A_0 {}_{f_{n,0}} \ldots {}_{f_{0,0}} A_0) \\
 & C(A_0 \to \ldots \to A_0)^{\bullet}(f_{0,0}, \ldots, f_{n,0}) \arrow[u, twoheadrightarrow]
\end{tikzcd}
$$

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$$\tilde{\iota}\big((\phi_{0,1}|\ldots|\phi_{0,k_0})\otimes\ldots\otimes(\phi_{n,1}|\ldots|\phi_{n,k_n})\otimes\alpha\big) = \iota_{(\phi_{0,1}|\ldots|\phi_{0,k_0})\bullet\ldots\bullet(\phi_{n,1}|\ldots|\phi_{n,k_n})}\alpha$$

$$\iota_\phi(a_0\otimes\ldots a_p) = \pm\phi(a_{d+1},\ldots,a_p)\cdot a_0\otimes a_1\otimes\ldots\otimes a_d \quad \text{where } |\phi| = p-d$$

Defining $\tau_{1!}$

$$C(A_0 \to A_1 \to A_0)^\bullet(f,g) \xrightarrow{\tau_{1!}} C(A_1 \to A_0 \to A_1)^\bullet(g,f)$$

$$\begin{tikzcd}[column sep=huge]
A_0 \arrow[r, bend left=80, "f_{0,0}"] \arrow[r, bend left=30, "f_{0,1}"'] \arrow[r, bend right=30, "f_{0,k_0}"] \arrow[r, phantom, bend left=55, "\Downarrow\phi_{0,1}"] \arrow[r, phantom, "\vdots"] \arrow[rr, bend right=50, "id_{A_0}"'] \arrow[rr, phantom, bend right=35, "\alpha"] & A_1 \arrow[r, bend left=80, "f_{1,0}"] \arrow[r, bend left=30, "f_{1,1}"'] \arrow[r, bend right=30, "f_{1,k_1}"] \arrow[r, phantom, bend left=55, "\Downarrow\phi_{1,1}"] \arrow[r, phantom, "\vdots"] & A_0 & \mapsto \sum & A_1 \arrow[r, bend left=80, "f_{1,0}"] \arrow[r, bend left=30, "f_{1,1}"'] \arrow[r, bend right=30, "f_{1,k_1}"] \arrow[r, phantom, bend left=55, "\Downarrow\phi'_{1,1}"] \arrow[r, phantom, "\vdots"] \arrow[rr, bend right=50, "id_{A_1}"'] \arrow[rr, phantom, bend right=35, "\alpha'"] & A_0 \arrow[r, bend left=80, "f_{0,0}"] \arrow[r, bend left=30, "f_{0,1}"'] \arrow[r, bend right=30, "f_{0,k_0}"] \arrow[r, phantom, bend left=55, "\Downarrow\phi'_{0,1}"] \arrow[r, phantom, "\vdots"] & A_1
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A_0 \arrow[r, bend left=80, "f_{0,0}"] \arrow[r, bend left=40, "f_{0,1}"] \arrow[r, phantom, bend left=60, "\Downarrow\phi_{0,1}"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=20, "f_{0,k_0}"] \arrow[rr, bend right=50, "\alpha", "id_{A_0}"'] & A_1 \arrow[r, bend left=80, "f_{1,0}"] \arrow[r, bend left=40, "f_{1,1}"] \arrow[r, phantom, bend left=60, "\Downarrow\phi_{1,1}"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=20, "f_{1,k_1}"] & A_0
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$$\alpha = a_0 \otimes \ldots \otimes a_n \mapsto \alpha' = f_{0,0}(a_0) \otimes \ldots \otimes f_{0,0}(a_n)$$

Defining $\tau_{1!}$

$$C(A_0 \to A_1 \to A_0)^\bullet(f,g) \xrightarrow{\tau_{1!}} C(A_1 \to A_0 \to A_1)^\bullet(g,f)$$

$$\begin{tikzcd}
A_0 \arrow[r, bend left=60, "f_{0,0}"] \arrow[r, "f_{0,1}"'] \arrow[r, phantom, bend left=30, "\Downarrow\phi"] \arrow[rr, bend right=50, "id_{A_0}"'] \arrow[rr, phantom, bend right=30, "\alpha"] & A_1 \arrow[r, "f_{1,0}"] & A_0 & \overset{?}{\longmapsto} & A_1 \arrow[r, "f_{1,0}"] \arrow[rr, bend right=50, "id_{A_1}"'] \arrow[rr, phantom, bend right=30, "\alpha'"] & A_0 \arrow[r, "f_{0,0}"] & A_1
\end{tikzcd}$$

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\end{tikzcd}$$

$$\overline{\tau_{1!} \circ d}(\phi \otimes \alpha) = \overline{d \circ \tau_{1!}}(\phi \otimes \alpha)$$

Defining $\tau_{1!}$

$$C(A_0 \to A_1 \to A_0)^\bullet(f, g) \xrightarrow{\tau_{1!}} C(A_1 \to A_0 \to A_1)^\bullet(g, f)$$

$$\begin{tikzcd} A_0 \arrow[r, bend left=60, "f_{0,0}"] \arrow[r, phantom, bend left=30, "\Downarrow\phi"] \arrow[r, "f_{0,1}"] \arrow[rr, bend right=50, "id_{A_0}"', "\alpha"] & A_1 \arrow[r, "f_{1,0}"] & A_0 \end{tikzcd} \overset{?}{\mapsto} \begin{tikzcd} A_1 \arrow[r, "f_{1,0}"] \arrow[rr, bend right=50, "id_{A_1}"', "\alpha'"] & A_0 \arrow[r, "f_{0,0}"] & A_1 \end{tikzcd}$$

$$[b, \bar{\tau}_{1!}](\phi \otimes \alpha) \pm \bar{\tau}_{1!}(\delta\phi \otimes \alpha) = [\bar{\tau}_{1!}, \iota_\phi](\alpha)$$

Defining $\tau_{1!}$

$$C(A_0 \to A_1 \to A_0)^{\bullet}(f,g) \xrightarrow{\tau_{1!}} C(A_1 \to A_0 \to A_1)^{\bullet}(g,f)$$

$$\begin{tikzcd} A_0 \arrow[r, bend left=60, "f_{0,0}"] \arrow[r, phantom, shift left=3ex, "\Downarrow\phi"] \arrow[r, "f_{0,1}"] \arrow[rr, bend right, "id_{A_0}"', "\alpha"] & A_1 \arrow[r, "f_{1,0}"] & A_0 \arrow[r, phantom, "\overset{?}{\mapsto}"] & A_1 \arrow[r, "f_{1,0}"] \arrow[rr, bend right, "id_{A_1}"', "\alpha'"] & A_0 \arrow[r, "f_{0,0}"] & A_1 \end{tikzcd}$$

$$[b, \bar{\tau}_{1!}](\phi \otimes \alpha) \pm \bar{\tau}_{1!}(\delta\phi \otimes \alpha) = [\bar{\tau}_{1!}, \iota_\phi](\alpha)$$

$$L_\phi(\alpha) = \sum_{k \geq 1} \pm a_0 \otimes \ldots \otimes \phi(a_k, \ldots) \otimes a_r \otimes \ldots \otimes a_n +$$

$$\sum \pm \phi(a_k, \ldots, a_n, a_0, \ldots) \otimes a_s \otimes \ldots \otimes a_{k-1}$$

$$[b, L_\phi] \pm L_{\delta\phi} = 0$$

Defining $\tau_{1!}$

$$C(A_0 \to A_1 \to A_0)^\bullet(f,g) \xrightarrow{\tau_{1!}} C(A_1 \to A_0 \to A_1)^\bullet(g,f)$$

$$\begin{tikzcd}
A_0 \arrow[r, "f_{0,1}"] \arrow[r, bend left=60, "f_{0,0}"] \arrow[r, phantom, bend left=30, "\Downarrow\phi"] \arrow[rr, bend right, "id_{A_0}"'] \arrow[rr, phantom, bend right=20, "\alpha"] & A_1 \arrow[r, "f_{1,0}"] & A_0 \arrow[r, mapsto, "?"] & A_1 \arrow[r, "f_{1,0}"] \arrow[rr, bend right, "id_{A_1}"'] \arrow[rr, phantom, bend right=20, "\alpha'"] & A_0 \arrow[r, "f_{0,0}"] & A_1
\end{tikzcd}$$

$$[b, \bar{\tau}_{1!}](\phi \otimes \alpha) \pm \bar{\tau}_{1!}(\delta\phi \otimes \alpha) = [\bar{\tau}_{1!}, \iota_\phi](\alpha)$$

$$\bar{\tau}_{1!}(\phi \otimes \alpha) = \sum_{k \geq 1} \pm f_{0,0} a_0 \otimes \ldots \otimes \phi(a_k, \ldots) \otimes f_{0,1} a_r \ldots \otimes f_{0,1} a_n +$$

$$\sum \pm \phi(f_{1,0} f_{0,1} a_k, \ldots, f_{1,0} f_{0,1} a_n, a_0, \ldots) \otimes f_{0,1} a_s \otimes \ldots \otimes f_{0,1} a_{k-1}$$

Defining $\tau_{1!}$

$$\bar{\tau}_{1!}\big((\phi_{0,1}|\ldots|\phi_{0,k_0})\otimes(\phi_{1,1}|\ldots|\phi_{1,k_1})\otimes\alpha\big)=$$

$$=\sum_{\substack{1\le i\le j_1\le\ldots\le j_{2k_1}\le k_0,\\ p}}\pm\phi_{0,1}\Big(\begin{matrix} f_{1,0}f_{0,i}a_p,\ldots,f_{1,0}\phi_{0,i+1}(a_*,\ldots),\\ f_{1,0}f_{0,i+1}a_*,\ldots,f_{1,0}\phi_{0,j_1}(a_*,\ldots),\\ f_{1,0}f_{0,j_1}a_*,\ldots,\phi_{1,1}(f_{0,j_1}a_*,\ldots,\phi_{0,j_1+1}(a_*,\ldots),\ldots),\ldots,\\ \phi_{1,k_1}(f_{0,j_{2k_1-1}}a_*,\ldots,\phi_{0,j_{2k_1-1}+1}(a_*,\ldots),\ldots),\ldots,a_0,\ldots\end{matrix}\Big)\otimes$$

$$\otimes f_{0,1}a_*\otimes\ldots\otimes\phi_{0,2}(a_*,\ldots)\otimes f_{0,2}a_*\otimes\ldots\otimes$$

$$\otimes\phi_{0,i}(a_*,\ldots)\otimes f_{0,i}a_*\otimes\ldots f_{0,i}a_{p-1}+$$

$$\Big(\sum\pm f_{0,0}a_0\otimes\ldots\otimes\phi_{0,1}(a_*,\ldots)\otimes\ldots\otimes\phi_{0,n_0}(a_*,\ldots)\otimes$$

$$\otimes f_{0,n_0}a_*\otimes\ldots\otimes f_{0,n_0}a_n\quad\text{if } k_1=0\Big)$$

First homotopy: $\tau_{1!}^2 \sim id$

$$
\begin{tikzcd}
C(A_0 \to A_1 \to A_0) \arrow[r, "\tau_{1!}"] \arrow[rr, bend right, "id"'] & \hat{\tau}_1^* C(A_1 \to A_0 \to A_1) \arrow[r, "\hat{\tau}_1^* \tau_{1!}"] & \hat{\tau}_1^{*2} C(A_0 \to A_1 \to A_0)
\end{tikzcd}
$$

First homotopy: $\tau_{1!}^2 \sim id$

$$\begin{tikzcd} C(A_0 \to A_1 \to A_0) \arrow[r, "\tau_{1!}"] \arrow[rr, bend right, "id"'] & \hat{\tau}_1^* C(A_1 \to A_0 \to A_1) \arrow[r, "\hat{\tau}_1^* \tau_{1!}"] & \hat{\tau}_1^{*2} C(A_0 \to A_1 \to A_0) \end{tikzcd}$$

$$\begin{tikzcd} A_0 \arrow[r, "f_{0,0}"] \arrow[rr, bend right, "id_{A_0}"'] & A_1 \arrow[r, "f_{1,0}"] \arrow[d, phantom, "\alpha"] & A_0 \\ & {} & \end{tikzcd}$$

$$\alpha = a_0 \otimes \ldots \otimes a_n \overset{\tau_{1!}}{\mapsto} f_{0,0}a_0 \otimes \ldots \otimes f_{0,0}a_n \overset{\hat{\tau}_{1!}^*\tau_{1!}}{\mapsto} f_{1,0}f_{0,0}a_0 \otimes \ldots \otimes f_{1,0}f_{0,0}a_n$$

First homotopy: $\tau_{1!}^2 \sim id$

$$
\begin{tikzcd}
C(A_0 \to A_1 \to A_0) \arrow[r, "\tau_{1!}"] \arrow[rr, bend right, "id"'] & \hat{\tau}_1^* C(A_1 \to A_0 \to A_1) \arrow[r, "\hat{\tau}_1^* \tau_{1!}"] & \hat{\tau}_1^{*2} C(A_0 \to A_1 \to A_0)
\end{tikzcd}
$$

$$
\begin{tikzcd}
A_0 \arrow[r, "f_{0,0}"] \arrow[rr, bend right, "id_{A_0}"'] & A_1 \arrow[r, "f_{1,0}"] \arrow[d, phantom, "\alpha"] & A_0 \\
 & {} &
\end{tikzcd}
$$

$$\alpha = a_0 \otimes \ldots \otimes a_n \overset{\tau_{1!}}{\mapsto} f_{0,0}a_0 \otimes \ldots \otimes f_{0,0}a_n \overset{\hat{\tau}_{1!}^* \tau_{1!}}{\mapsto} f_{1,0}f_{0,0}a_0 \otimes \ldots \otimes f_{1,0}f_{0,0}a_n$$

$$f_{1,0}f_{0,0}\alpha - \alpha = [b, B](\alpha)$$

$$B(a_0 \otimes \ldots \otimes a_n) = \sum_{0 \leq i \leq n} \pm 1 \otimes f_{1,0}f_{0,0}a_i \otimes \ldots \otimes f_{1,0}f_{0,0}a_n \otimes a_0 \otimes \ldots \otimes a_{i-1}$$

First homotopy: $\tau_{1!}^2 \sim id$

$$\begin{tikzcd} C(A_0 \to A_1 \to A_0) \arrow[r, "\tau_{1!}"] \arrow[rr, bend right, "id"'] & \hat{\tau}_1^* C(A_1 \to A_0 \to A_1) \arrow[r, "\hat{\tau}_1^* \tau_{1!}"] & \hat{\tau}_1^{*2} C(A_0 \to A_1 \to A_0) \end{tikzcd}$$

$$B\big((\phi_{0,1}|\ldots|\phi_{0,k_0}) \otimes (\phi_{1,1}|\ldots|\phi_{1,k_1}) \otimes \alpha\big) =$$

$$= \sum_{\substack{0 \le j_1 \le \ldots \le j_{2k_1} \le k_0 \\ p}} \pm 1 \otimes f_{1,0} f_{0,0} a_p \otimes \ldots \otimes f_{1,0}\phi_{0,1}(a_*, \ldots) \otimes$$

$$\otimes f_{1,0} f_{0,1} a_* \otimes \ldots \otimes f_{1,0}\phi_{0,j_1}(a_*, \ldots) \otimes$$

$$\otimes f_{1,0} f_{0,j_1} a_* \otimes \ldots \otimes \phi_{1,1}(f_{0,j_1} a_*, \ldots, \phi_{0,j_1+1}(a_*, \ldots), \ldots) \otimes$$

$$\otimes \ldots \otimes \phi_{1,k_1}(f_{0,j_{2k_1-1}} a_*, \ldots, \phi_{0,j_{2k_1-1}+1}(a_*, \ldots), \ldots) \otimes \ldots \otimes$$

$$\otimes a_0 \otimes \ldots \otimes a_{p-1}$$

In the language of A_∞-functors

$$\chi(\mathcal{C}) \to \mathcal{D}$$

$$\mathcal{A}{:=}(A_0{\to}A_1{\to}A_0) \mapsto \begin{pmatrix} B(\mathcal{A}), \\ C(\mathcal{A}) \end{pmatrix}$$

$$\tau_1 \mapsto \begin{pmatrix} \hat{\tau}_1 : B(\mathcal{A}) \to B(\tau_1\mathcal{A}) \\ \tau_{1!} : C(\mathcal{A}) \to \hat{\tau}_1^* C(\tau_1\mathcal{A}) \end{pmatrix}$$

$$(\tau_1, \tau_1) \mapsto \begin{pmatrix} id : B(\mathcal{A}) \to B(\mathcal{A}) \\ B : C(\mathcal{A}) \to C(\mathcal{A}) \end{pmatrix}$$

$$(\tau_1, \tau_1, \tau_1) \mapsto \begin{pmatrix} \hat{\tau}_1 : B(\mathcal{A}) \to B(\tau_1\mathcal{A}) \\ 0 : C(\mathcal{A}) \to \hat{\tau}_1^* C(\tau_1\mathcal{A}) \end{pmatrix}$$

$$\vdots$$

The A_∞ relations mean:

- $\tau_{1!}$ is a map of complexes
- $B^2 = 0$
- The following diagram commutes:

$$\begin{tikzcd} C(\mathcal{A}) \arrow[r, "\tau_{1!}"] \arrow[d, "B"] & \hat{\tau}_1^* C(\tau_1 \mathcal{A}) \arrow[d, "\hat{\tau}_1^* B"] \\ C(\mathcal{A}) \arrow[r, "\tau_{1!}"'] & \hat{\tau}_1^* C(\tau_1 \mathcal{A}) \end{tikzcd}$$

For higher $n > 1$, we want to find a homotopy between "$\tau_{n!}^{n+1}$" and id.

For higher $n > 1$, we want to find a homotopy between "$\tau_{n!}^{n+1}$" and id. However, it is sufficient to find a homotopy between $\hat{\tau}_n^{*2}\delta_{0,n!} \circ \hat{\tau}_n^{*}\tau_{n!} \circ \tau_{n!}$ and $\hat{\delta}^*_{n-1,n}\tau_{n-1!} \circ \delta_{n-1,n!}$.

$$
\begin{tikzcd}
{[n]} \arrow[r, "\tau_n"] \arrow[d, "\delta_{n-1,n}"] & {[n]} \arrow[r, "\tau_n"] & {[n]} \arrow[dl, "\delta_{0,n}"] \\
{[n-1]} \arrow[r, "\tau_{n-1}"'] & {[n-1]} &
\end{tikzcd}
$$

For higher $n > 1$, we want to find a homotopy between "$\tau_{n!}^{n+1}$" and id. However, it is sufficient to find a homotopy between $\hat{\tau}_n^{*2}\delta_{0,n!} \circ \hat{\tau}_n^{*}\tau_{n!} \circ \tau_{n!}$ and $\hat{\delta}_{n-1,n}^{*}\tau_{n-1!} \circ \delta_{n-1,n!}$.

$$\begin{tikzcd}
{[n]} \arrow[r, "\tau_n"] \arrow[d, "\delta_{n-1,n}"] & {[n]} \arrow[r, "\tau_n"] & {[n]} \arrow[dl, "\delta_{0,n}"] \\
{[n-1]} \arrow[r, "\tau_{n-1}"'] & {[n-1]} &
\end{tikzcd}$$

Strategy: Find such a homotopy, $\mathcal{B}$, for $n = 2$, and use $\hat{\delta}_0^{*n-2}\mathcal{B}$ for $n > 2$.

$$
\begin{aligned}
\chi(\mathcal{C}) &\to \mathcal{D}\\
\mathcal{A} &\mapsto (B(\mathcal{A}), C(\mathcal{A}))\\
\mu = \tau_{n-1} \circ \delta_{n-1,n} = \delta_{0,n} \circ \tau_n^2 &\mapsto \begin{pmatrix} \hat{\tau}_{n-1} \circ \hat{\delta}_{n-1,n} = \hat{\delta}_{0,n} \circ \hat{\tau}_n^2 \\ \hat{\tau}_n^{*2} \delta_{0,n!} \circ \hat{\tau}_n^{*} \tau_{n!} \circ \tau_{n!} \end{pmatrix}\\
(\delta_{0,n}, \tau_n^2) &\mapsto \begin{pmatrix} \hat{\delta}_{0,n} \circ \hat{\tau}_n^2 \\ 0 \end{pmatrix}\\
(\tau_{n-1}, \delta_{n-1,n}) &\mapsto \begin{pmatrix} \hat{\tau}_{n-1} \circ \hat{\delta}_{n-1,n} \\ \mathcal{B} \end{pmatrix}\\
(\tau_{n-1}, \delta_{n-1,n}, \lambda) &\mapsto \begin{pmatrix} \hat{\tau}_{n-1} \circ \hat{\delta}_{n-1,n} \circ \hat{\lambda} \\ 0 \end{pmatrix}\\
&\vdots
\end{aligned}
$$

For $n > 1$, the A_∞ relations mean:

- $\tau_{n!}$ is a map of complexes
- $\mathcal{B}^2 = 0$
- We have a pair of homotopic maps:

$$
\begin{tikzcd}
C(A_0 \to \ldots \to A_n \to A_0) \arrow[d, bend right, "{\substack{\hat{\tau}_n^{*2}\tau_{n!}\circ\hat{\tau}_n^{*}\tau_{n!}\circ\tau_{n!} \\ \text{“apply } \tau_{n!} \text{ 3 times”}}}"'] \arrow[d, bend left, "{\substack{(\delta_{n-2,\widehat{n-1}}\delta_{n-1,n})^*\tau_{n-2!} \\ \text{“brace together the last 3 algebras,} \\ \text{then apply } \tau_{n-2!} \text{ once”}}}"] \\
C(A_{n-2} \to A_{n-1} \to A_n \to A_0 \to \ldots \to A_{n-2}).
\end{tikzcd}
$$

For $n > 1$, the A_∞ relations mean:

- $\tau_{n!}$ is a map of complexes
- $\mathcal{B}^2 = 0$
- They are homotopic via two homotopies:

$$
\begin{tikzcd}[column sep=large]
\begin{array}{c}(\delta_{n-2,\widehat{n-1\delta}_{n-1,n}})^*\tau_{n-2!} \\ \text{“brace together } A_{n-2}, A_{n-1}, A_n, \\ \text{then apply } \tau_{n-2!}\text{”}\end{array} \arrow[r, "\cong"] \arrow[d, "\cong"'] & \hat{\delta}^*_{n-1,n}(\hat{\delta}^*_{n-2,n-1}\tau_{n-2!}) \arrow[r, "\hat{\delta}^*_{n-1,n}\mathcal{B}_{n-1}"] & \hat{\delta}^*_{n-1,n}(\hat{\tau}^*_{n-1}\tau_{n-1!} \circ \tau_{n-1!}) \arrow[d, "\cong"] \\
(\delta_{n-2,\widehat{n-1\delta}_{n-2,n}})^*\tau_{n-2!} \arrow[d, "\hat{\delta}^*_{n-2,n}\mathcal{B}_{n-1}"'] & & \begin{array}{c}\hat{\tau}_n^{*2}\tau_{n!} \circ \hat{\delta}^*_{n-1,n}\tau_{n-1!} \\ \text{“brace together } A_{n-1}, A_n \\ \text{and apply } \tau_{n-1!}, \\ \text{then apply } \tau_{n!}\text{”}\end{array} \arrow[d, "\tau_n^{*2}\tau_{n!}\circ\mathcal{B}_n"] \\
\hat{\delta}^*_{n-2,n}(\hat{\tau}^*_{n-1}\tau_{n-1!} \circ \tau_{n-1!}) \arrow[r, "\cong"'] & \begin{array}{c}\hat{\tau}^*_n(\hat{\delta}^*_{n-1,n}\tau_{n-1!}) \circ \tau_{n!} \\ \text{“apply } \tau_{n!}, \\ \text{then brace together } A_{n-1}, A_{n-2} \\ \text{and apply } \tau_{n-1!}\text{”}\end{array} \arrow[r, "\hat{\tau}^*_n\mathcal{B}_n\circ\tau_{n!}"] & \begin{array}{c}\hat{\tau}_n^{*2}\tau_{n!} \circ \hat{\tau}^*_n\tau_{n!} \circ \tau_{n!} \\ \text{“apply } \tau_{n!} \text{ three times”}\end{array}
\end{tikzcd}
$$

Summary: We have a given an A_∞-functor $\chi(\mathcal{C}) \to \mathcal{D}$, which implies that algebras form a category in dg cocategories with a trace up to homotopy.

To get a category in dg categories with a trace up to homotopy, apply (categorified) Cobar(-).

Thank you!