

# Title

## Subtitle

Rebecca Wei

Northwestern University

Date/Event

# Background

## Theorem

(Hochschild-Kostant-Rosenberg, '62) Let  $A$  be a regular, commutative algebra over a field  $k$  of characteristic 0. Then,

$$\begin{aligned}(C_{\bullet}(A, A), b) &\xrightarrow{\sim} \Omega_{A/k}^{\bullet} \\ (C^{\bullet}(A, A), \delta) &\xrightarrow{\sim} \wedge^{\bullet} T_A = \wedge^{\bullet}(\operatorname{Der}_k(A, A)).\end{aligned}$$

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## Theorem

(Kontsevich, '97) Let  $A = C^{\infty}(M)$  for  $M$  a smooth real manifold. Then, there is an  $L_{\infty}$  map

$$(C^{\bullet+1}(A, A), \delta, [\cdot, \cdot]_{\operatorname{Ger}}) \xrightarrow{\sim} (\wedge^{\bullet+1} T_A, d = 0, [\cdot, \cdot]_{SN}).$$

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(Tamarkin, '98) Dependent on the choice of a Drinfeld associator, there is a  $\operatorname{Ger}_{\infty}$  map

$$(C^{\bullet+1}(A, A), \delta, [\cdot, \cdot]_{\operatorname{Ger}}, \cup, \dots) \xrightarrow{\sim} (\wedge^{\bullet} T_A, d = 0, [\cdot, \cdot]_{SN}, \wedge).$$

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(Dolgushev-Tamarkin-Tsygan, '08) There is a  $\operatorname{Calc}_{\infty}$  map

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Can we use this calculus structure to create a cyclic object?

# A cyclic object in dg cocategories

Fix an algebra,  $A$ .

Define a dg category,  $Hoch(A)$ :

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$$\begin{aligned} {}_f \delta_g(\phi)(a_1 \otimes \dots \otimes a_n) = & \epsilon_\phi \left( \textcolor{red}{f}(a_1) \cdot \phi(a_2, \dots, a_n) + \right. \\ & + \sum_{1 \leq i \leq n-1} (-1)^i \phi(a_1, \dots, a_i a_{i+1}, \dots, a_n) + \\ & \left. + (-1)^n \phi(a_1, \dots, a_{n-1}) \cdot \textcolor{red}{g}(a_n) \right) \\ \epsilon_\phi = & (-1)^{|\phi|+1} \end{aligned}$$

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Composition: cup product on cochains

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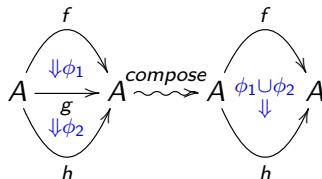
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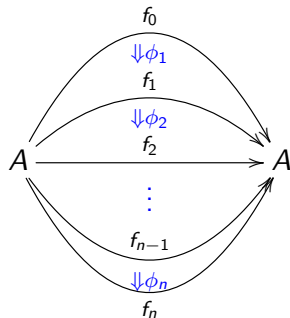
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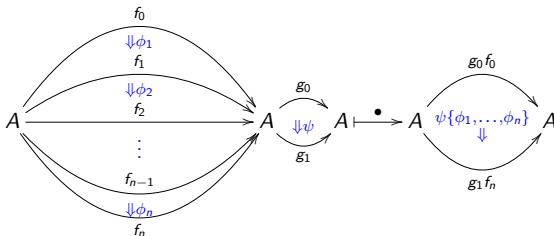
A morphism from  $f_0$  to  $f_n$  in  $Bar(Hoch(A))$

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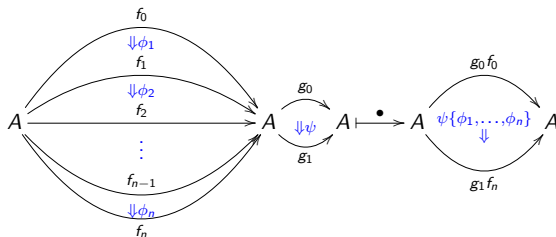


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$$\psi\{\phi_1, \dots, \phi_n\}(a_1, \dots, a_q) = \sum_{1 \leq i_1 \leq \dots \leq i_n \leq q} \pm \psi(a_1, \dots, a_{i_1}, \phi_1(a_{i_1+1}, \dots), \dots, \phi_n(a_{i_n+1}, \dots), \dots)$$

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Fix an algebra,  $A$ .

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In this context, braces,  $\bullet$ , give multilinear maps:

$$\begin{array}{ccc} Bar(Hoch(A)) \otimes Bar(Hoch(A)) & \xrightarrow{\bullet} & Hoch(A) \\ & \searrow \bullet & \uparrow \\ & & Bar(Hoch(A)) \end{array}$$

Then,  $(Bar(Hoch(A)), \bullet)$  is an algebra in  $DGCocats$ .



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But we have more...

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Fix algebras,  $A_0, A_1, \dots, A_n$ .

We will define a dg cocategory  $B(A_0 \rightarrow A_1 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$   
where  $B(A_0 \rightarrow A_0) := \text{Bar}(\text{Hoch}(A_0))$  for  $n=0$ .

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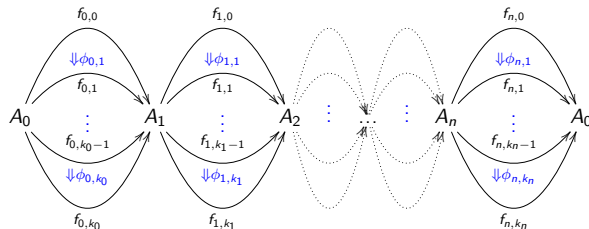
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Objects:  $\{A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \dots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_0\}$

A morphism from  $(A_0 \xrightarrow{f_{0,0}} \dots \xrightarrow{f_{n,0}} A_0)$  to  $(A_0 \xrightarrow{f_{0,k_0}} \dots \xrightarrow{f_{n,k_n}} A_0)$ :



$$\phi_{i,j} \in C^\bullet(A_{i,f_{i,j-1}} A_{i+1} f_{i,j})$$

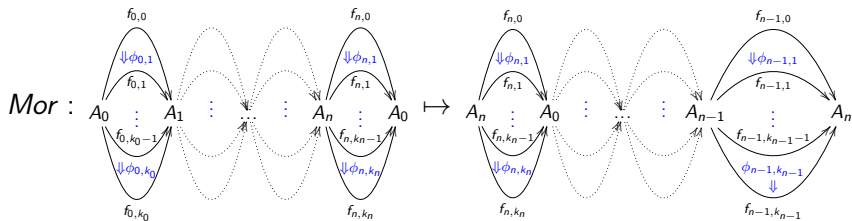
# A cyclic object in dg cocategories

## Example

We have a dg functor

$$\hat{\tau}_n : B(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0) \rightarrow B(A_n \rightarrow A_0 \rightarrow \dots \rightarrow A_n)$$

$$Obj : (f_0, \dots, f_n) \mapsto (f_n, f_0, \dots, f_{n-1})$$



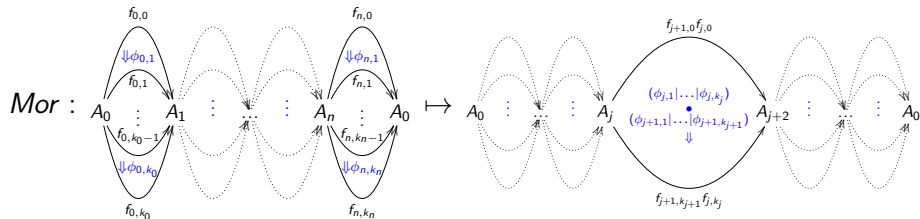
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For  $n \geq 1, 0 \leq j < n$ , we have a dg functor

$$\hat{\delta}_{j,n} : B(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0) \rightarrow B(A_0 \rightarrow \dots \rightarrow A_j \rightarrow A_{j+2} \xrightarrow{(mod\ n+1)} \dots \rightarrow A_0)$$

$$Obj : (f_0, \dots, f_n) \mapsto (f_0, \dots, f_{j+1} f_j, \dots, f_n)$$



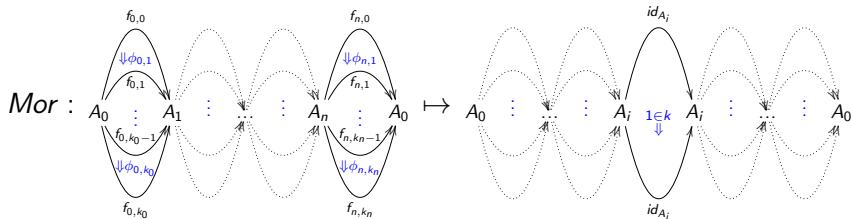
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For  $n \geq 0, 0 \leq i \leq n$ , we have a dg functor

$$\hat{\sigma}_{i,n} : B(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0) \rightarrow B(A_0 \rightarrow \dots \rightarrow A_i \rightarrow A_i \rightarrow \dots \rightarrow A_0)$$

$$Obj : (f_0, \dots, f_n) \mapsto (f_0, \dots, f_{i-1}, id_{A_i}, f_i, \dots, f_n)$$



# A sheafy-cyclic object in DGCocat

## Definition

Let  $\chi$  be the category with objects  $\{A_0 \rightarrow A_1 \rightarrow \dots \rightarrow A_n \rightarrow A_0\}$  and morphisms compositions of

$$\text{rotations } \tau_n : \mathcal{A} \mapsto (A_n \rightarrow A_0 \rightarrow \dots \rightarrow A_n)$$

$$\text{coboundaries } \delta_{j,n} : \mathcal{A} \mapsto (A_0 \rightarrow \dots \rightarrow A_j \rightarrow A_{j+2 \pmod{n+1}} \rightarrow \dots \rightarrow A_0)$$

$$\text{codegeneracies } \sigma_{i,n} : \mathcal{A} \mapsto (A_0 \rightarrow \dots \rightarrow A_i \rightarrow A_i \rightarrow \dots \rightarrow A_0)$$

where  $\mathcal{A} := (A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$ , subject to the cyclic relations.



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## Proposition

We have a functor  $\chi \rightarrow DGCocat$

$$\text{Objects} : (A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0) \mapsto B(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$$

$$\text{Generating morphisms} : \lambda \mapsto \hat{\lambda}$$

# Motivating Question

Each dg cocategory  $B(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$  has a dg comodule  $C(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$  (which is a categorified version of the bar complex of Hochschild chains as a module over Hochschild cochains via contraction).

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- Define dg categories  $\chi_\infty$  and  $\mathcal{D}_\infty$
- Define dg comodules  $C(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$
- Define the  $A_\infty$ -functor  $\mathcal{F}$



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$\chi_\infty$ :

Objects: same objects as  $\chi = \{A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0\}$

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$$\mathcal{D}_\infty^p((B_1, C_1), (B_0, C_0)) := \left\{ \begin{array}{l} F : B_1 \rightarrow B_0 \text{ dg functor,} \\ F_! : C_1 \rightarrow F^* C_0 \text{ degree-}p \text{ linear map} \end{array} \right\}$$
$$d_{\mathcal{D}_\infty}(F, F_!) = (F, [d, F_!] = d_{F^* C_0} \circ F_! \pm F_! \circ d_{C_1})$$

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$F^* C_0$  is the categorified version of co-extension of scalars:

$$F^* C_0 = \ker(B_1 \otimes C_0 \rightrightarrows B_1 \otimes B_0 \otimes C_0)$$

# Dg comodules over dg cocategories

## Definition

A **dg comodule**  $C$  over a dg cocategory  $B$  consists of the following data:

- for each object  $f \in B$ , a complex  $C^\bullet(f)$ , and
- maps of complexes

$$\Delta_C(f) : C^\bullet(f) \rightarrow \prod_{g \in \text{Obj}(B)} B^\bullet(f, g) \otimes C^\bullet(g).$$

such that the following diagrams for coassociativity and counitality commute:

$$\begin{array}{ccc}
 C^\bullet(f) & \xrightarrow{\Delta_C} & \prod_{g \in \text{Obj}(B)} B^\bullet(f, g) \otimes C^\bullet(g) \\
 \Delta_C \downarrow & & \downarrow id_B \otimes \Delta_C \\
 \prod_{g \in \text{Obj}(B)} B^\bullet(f, g) \otimes C^\bullet(g) & \xrightarrow{\Delta_B \otimes id_C} & \prod_{g, g' \in \text{Obj}(B)} B^\bullet(f, g) \otimes B^\bullet(g, g') \otimes C^\bullet(g')
 \end{array}
 \qquad
 \begin{array}{ccc}
 C^\bullet(f) & \xrightarrow{\Delta_C} & \prod_{g \in \text{Obj}(B)} B^\bullet(f, g) \otimes C^\bullet(g) \\
 \searrow id & & \downarrow \epsilon_B \otimes id_C \\
 & & C^\bullet(f)
 \end{array}$$

# Dg comodules over dg cocategories

Fix algebras  $A_0, \dots, A_n$ .

Define a dg comodule over  $B(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$ :

$$C(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)^\bullet(f_{0,0}, \dots, f_{n,0}) :=$$

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$$:= \left\{ \begin{array}{c} \begin{array}{c} f_{0,0} \\ \downarrow \phi_{0,1} \\ f_{0,1} \\ \vdots \\ f_{0,k_0} \end{array} \begin{array}{c} A_0 \\ \vdots \\ A_1 \end{array} \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \begin{array}{c} f_{n,0} \\ \downarrow \phi_{n,1} \\ f_{n,1} \\ \vdots \\ f_{n,k_n} \end{array} \begin{array}{c} A_n \\ \vdots \\ A_0 \end{array} \\ \alpha \end{array} \right\} = \left\{ \begin{array}{l} (\phi_{0,1} | \dots | \phi_{0,k_0}) \otimes \dots \otimes (\phi_{n,1} | \dots | \phi_{n,k_n}) \otimes \alpha: \\ \text{s.t. } \phi_{i,j} \in C^\bullet(A_i, f_{j-1} A_{i+1} f_j), \\ \alpha \in C_{-\bullet}(A_0, f_{n,k_n} \dots f_{0,k_0} A_0) \end{array} \right\}$$

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$$:= \{ A_0 \xrightarrow{f_{0,0}} A_1 \xrightarrow{f_{0,1}} \dots \xrightarrow{f_{n,1}} A_n \xrightarrow{f_{n,0}} A_0 \mid \begin{array}{l} \text{dotted arrows } A_i \xrightarrow{\dots} A_{i+1} \\ \text{dotted arrows } A_{i+1} \xrightarrow{\dots} A_i \end{array} \} =$$

$$d_{C(A_0 \rightarrow \dots \rightarrow A_0)} = d_B \otimes id_{C_{-\bullet}} + id_B \otimes b + \tilde{t}$$



# Dg comodules over dg cocategories

where  $\tilde{\iota}$  is given as follows:

$$\begin{array}{ccc}
 C(A_0 \rightarrow \dots \rightarrow A_0)^\bullet(f_{0,0}, \dots, f_{n,0}) & \xrightarrow{\tilde{\iota}} & C_{-\bullet}(A_0, f_{n,0} \dots f_{0,0} A_0) \\
 & \searrow \tilde{\iota} & \uparrow \\
 & \text{extend as a} & \\
 & \text{coderivation} & \\
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 \end{array}$$

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 & \searrow \tilde{\iota} \text{ extend as a coderivation} & \uparrow \\
 & C(A_0 \rightarrow \dots \rightarrow A_0)^\bullet(f_{0,0}, \dots, f_{n,0}) &
 \end{array}$$

$$\begin{aligned}
 \tilde{\iota}((\phi_{0,1} | \dots | \phi_{0,k_0}) \otimes \dots \otimes (\phi_{n,1} | \dots | \phi_{n,k_n}) \otimes \alpha) &= \iota_{(\phi_{0,1} | \dots | \phi_{0,k_0})} \bullet \dots \bullet (\phi_{n,1} | \dots | \phi_{n,k_n}) \alpha \\
 \iota_\phi(a_0 \otimes \dots \otimes a_p) &= \pm \phi(a_{d+1}, \dots, a_n) \cdot a_0 \otimes a_1 \otimes \dots \otimes a_d \quad \text{where } |\phi| = n - d
 \end{aligned}$$

An  $A_\infty$ -functor  $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

$$\mathcal{A} := (A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$$

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$$\mathcal{A} \mapsto (B(\mathcal{A}), C(\mathcal{A}))$$

# An $A_\infty$ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

$$\mathcal{A} := (A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$$

$$\chi_\infty \rightarrow \mathcal{D}_\infty$$

$$\mathcal{A} \mapsto (B(\mathcal{A}), C(\mathcal{A}))$$

Generating Morphisms:  $\sigma_{i,n} \mapsto \begin{pmatrix} B(\mathcal{A}) \xrightarrow{\hat{\sigma}_{i,n}} B(\sigma_{i,n}\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{id} \hat{\sigma}_{i,n}^* C(\sigma_{i,n}\mathcal{A}) \end{pmatrix}$

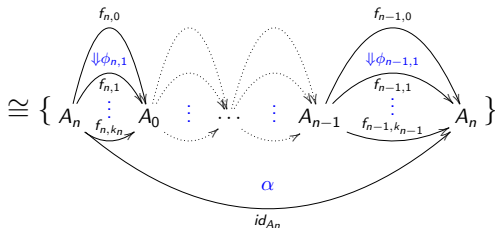
$$\delta_{j,n} \mapsto \begin{pmatrix} B(\mathcal{A}) \xrightarrow{\hat{\delta}_{j,n}} B(\delta_{j,n}\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{id} \hat{\delta}_{j,n}^* C(\delta_{j,n}\mathcal{A}) \end{pmatrix} \quad \tau_n \mapsto \begin{pmatrix} B(\mathcal{A}) \xrightarrow{\hat{\tau}_n} B(\tau_n\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{\tau_n!} \hat{\tau}_n^* C(\tau_n\mathcal{A}) \end{pmatrix}$$

An  $A_\infty$ -functor  $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

$$\hat{\tau}_n^* C(\tau_n \mathcal{A})^\bullet(\underbrace{f_0, \dots, f_n}_{\in \text{Obj}(B(\mathcal{A}))}) \cong C(\tau_n \mathcal{A})^\bullet(f_n, f_0, \dots, f_{n-1}) \text{ as complexes.}$$

An  $A_\infty$ -functor  $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

$$\hat{\tau}_n^* C(\tau_n \mathcal{A})^\bullet(\underbrace{f_0, \dots, f_n}_{\in \text{Obj}(B(\mathcal{A}))}) \cong C(\tau_n \mathcal{A})^\bullet(f_n, f_0, \dots, f_{n-1}) \text{ as complexes.}$$



## An $A_\infty$ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

$$\hat{\tau}_n^* C(\tau_n \mathcal{A})^\bullet(\underbrace{f_0, \dots, f_n}_{\in \text{Obj}(B(\mathcal{A}))}) \cong C(\tau_n \mathcal{A})^\bullet(f_n, f_0, \dots, f_{n-1}) \text{ as complexes.}$$

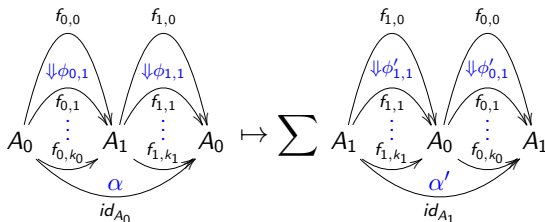
$$n = 1 : \quad C(A_0 \rightarrow A_1 \rightarrow A_0)^\bullet(f, g) \xrightarrow{\tau_{1!}} C(A_1 \rightarrow A_0 \rightarrow A_1)^\bullet(g, f)$$



# An $A_\infty$ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

$$\hat{\tau}_n^* C(\tau_n \mathcal{A})^\bullet(\underbrace{f_0, \dots, f_n}_{\in \text{Obj}(B(\mathcal{A}))}) \cong C(\tau_n \mathcal{A})^\bullet(f_n, f_0, \dots, f_{n-1}) \text{ as complexes.}$$

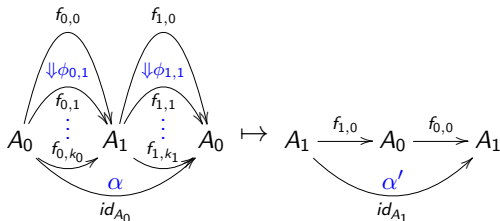
$$n = 1 : \quad C(A_0 \rightarrow A_1 \rightarrow A_0)^\bullet(f, g) \xrightarrow{\tau_{1!}} C(A_1 \rightarrow A_0 \rightarrow A_1)^\bullet(g, f)$$



# An $A_\infty$ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

$$\hat{\tau}_n^* C(\tau_n \mathcal{A})^\bullet(\underbrace{f_0, \dots, f_n}_{\in \text{Obj}(B(\mathcal{A}))}) \cong C(\tau_n \mathcal{A})^\bullet(f_n, f_0, \dots, f_{n-1}) \text{ as complexes.}$$

$$n = 1 : \quad C(A_0 \rightarrow A_1 \rightarrow A_0)^\bullet(f, g) \xrightarrow{\tau_{1!}} C(A_1 \rightarrow A_0 \rightarrow A_1)^\bullet(g, f)$$

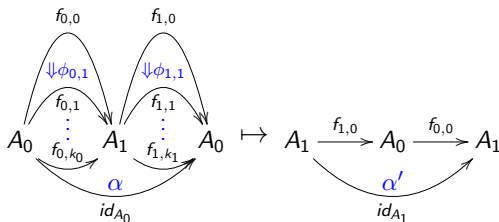


# An $A_\infty$ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

**Strategy:** Give these maps to cogenerators to define  $\tau_{1!}$ , then let

$$\begin{aligned} \tau_{n!} : C(\mathcal{A}) &\cong \hat{\delta}_0^{*n-1} C(A_0 \rightarrow A_n \rightarrow A_0) \xrightarrow{\hat{\delta}_0^{*n-1} \tau_{1!}} \hat{\delta}_0^{*n-1} \hat{\tau}_1^* C(A_n \rightarrow A_0 \rightarrow A_n) \cong \\ &\cong (\widehat{\tau_1 \delta_0^{n-1}})^* C(A_n \rightarrow A_0 \rightarrow A_n) \cong (\widehat{\delta_0^{n-1} \tau_n})^* C(A_n \rightarrow A_0 \rightarrow A_n) \cong \\ &\cong \hat{\tau}_n^* \hat{\delta}_0^{*n-1} C(A_n \rightarrow A_0 \rightarrow A_n) \cong \hat{\tau}_n^* C(\tau_n \mathcal{A}). \end{aligned}$$

$$n = 1 : \quad C(A_0 \rightarrow A_1 \rightarrow A_0)^\bullet(f, g) \xrightarrow{\tau_{1!}} C(A_1 \rightarrow A_0 \rightarrow A_1)^\bullet(g, f)$$



An  $A_\infty$ -functor  $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

**Defining**  $\tau_1!$

# An $A_\infty$ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

**Defining**  $\tau_1!$

$$\begin{array}{ccc} A_0 \xrightarrow{f_{0,0}} A_1 \xrightarrow{f_{1,0}} A_0 & \xrightarrow{?} & A_1 \xrightarrow{f_{1,0}} A_0 \xrightarrow{f_{0,0}} A_1 \\ \searrow \alpha \nearrow & & \searrow \alpha' \nearrow \\ & id_{A_0} & id_{A_1} \end{array}$$

# An $A_\infty$ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

**Defining**  $\tau_1!$

$$\begin{array}{c} A_0 \xrightarrow{f_{0,0}} A_1 \xrightarrow{f_{1,0}} A_0 \xrightarrow{?} A_1 \xrightarrow{f_{1,0}} A_0 \xrightarrow{f_{0,0}} A_1 \\ \quad \quad \quad \underbrace{\hspace{1.5cm}}_{\substack{\alpha \\ id_{A_0}}} \hspace{1.5cm} \underbrace{\hspace{1.5cm}}_{\substack{\alpha' \\ id_{A_1}}} \end{array}$$

$$\alpha = a_0 \otimes \dots \otimes a_n \mapsto \alpha' = f_{0,0}(a_0) \otimes \dots \otimes f_{0,0}(a_n)$$

# An $A_\infty$ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

**Defining  $\tau_1!$**

The diagram illustrates a relationship between two  $A_\infty$ -functors. On the left, a sequence of objects  $A_0 \rightarrow A_1 \rightarrow A_0$  is shown. The first arrow is labeled  $f_{0,1}$ , and the second is  $f_{1,0}$ . A curved arrow from  $A_0$  to  $A_1$  is labeled  $f_{0,0}$  above and  $\alpha$  below. A curved arrow from  $A_1$  to  $A_0$  is labeled  $id_{A_0}$  below. A blue arrow labeled  $\phi$  points from the  $f_{0,0}$  arrow to the  $f_{0,1}$  arrow. On the right, a similar sequence  $A_1 \rightarrow A_0 \rightarrow A_1$  is shown, with arrows labeled  $f_{1,0}$  and  $f_{0,0}$ . A curved arrow from  $A_1$  to  $A_0$  is labeled  $\alpha'$  below, and a curved arrow from  $A_0$  to  $A_1$  is labeled  $id_{A_1}$  below. A central arrow labeled  $\vdash ?$  points from the left diagram to the right diagram.

# An $A_\infty$ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

**Defining  $\tau_{1!}$**

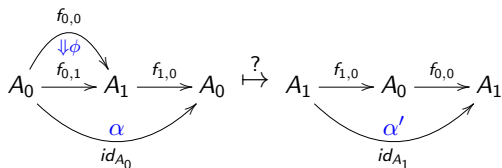
$$\begin{array}{c}
 \begin{array}{ccccc}
 & f_{0,0} & & & \\
 & \downarrow \phi & & & \\
 A_0 & \xrightarrow{f_{0,1}} & A_1 & \xrightarrow{f_{1,0}} & A_0 \\
 & \searrow \alpha & \nearrow id_{A_0} & & \\
 & & & & 
 \end{array}
 \quad \xrightarrow{?} \quad
 \begin{array}{ccccc}
 & f_{1,0} & & f_{0,0} & \\
 & & & & \\
 A_1 & \xrightarrow{f_{1,0}} & A_0 & \xrightarrow{f_{0,0}} & A_1 \\
 & \searrow \alpha' & \nearrow id_{A_1} & & 
 \end{array}
 \end{array}$$

$$\overline{\tau_{1!} \circ d(\phi \otimes \alpha)} = \overline{d \circ \tau_{1!}(\phi \otimes \alpha)}$$



# An $A_\infty$ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

**Defining**  $\tau_{1!}$



$$[b, \bar{\tau}_{1!}](\phi \otimes \alpha) \pm \bar{\tau}_{1!}(\delta\phi \otimes \alpha) = [\bar{\tau}_{1!}, \iota_\phi](\alpha)$$

# An $A_\infty$ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

**Defining**  $\tau_{1!}$

$$\begin{array}{c}
 \begin{array}{ccccc}
 & \overset{f_{0,0}}{\curvearrowright} & & & \\
 & \Downarrow \phi & & & \\
 A_0 & \xrightarrow{f_{0,1}} & A_1 & \xrightarrow{f_{1,0}} & A_0 \\
 & \searrow \alpha & \nearrow id_{A_0} & & \\
 & & & & 
 \end{array}
 \mapsto
 \begin{array}{ccccc}
 & \overset{f_{1,0}}{\curvearrowright} & & & \\
 & \searrow \alpha' & \nearrow id_{A_1} & & \\
 A_1 & \xrightarrow{f_{1,0}} & A_0 & \xrightarrow{f_{0,0}} & A_1
 \end{array}
 \end{array}$$

$$[b, \bar{\tau}_{1!}](\phi \otimes \alpha) \pm \bar{\tau}_{1!}(\delta\phi \otimes \alpha) = [\bar{\tau}_{1!}, \iota_\phi](\alpha)$$

$$\begin{aligned}
 L_\phi(\alpha) = & \sum_{k \geq 1} \pm a_0 \otimes \dots \otimes \phi(a_k, \dots) \otimes a_r \otimes \dots \otimes a_n + \\
 & \sum \pm \phi(a_k, \dots, a_n, a_0, \dots) \otimes a_s \otimes \dots \otimes a_{k-1} \\
 [b, L_\phi] \pm L_{\delta\phi} = & 0
 \end{aligned}$$

# An $A_\infty$ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

**Defining**  $\tau_{1!}$

$$\begin{array}{c}
 \begin{array}{ccccc}
 & \xrightarrow{f_{0,0}} & & & \\
 A_0 & \xrightarrow{f_{0,1}} & A_1 & \xrightarrow{f_{1,0}} & A_0 \\
 & \searrow \alpha & & & \\
 & id_{A_0} & & & 
 \end{array}
 \xrightarrow{?}
 \begin{array}{ccccc}
 & \xrightarrow{f_{1,0}} & & & \\
 A_1 & \xrightarrow{f_{1,0}} & A_0 & \xrightarrow{f_{0,0}} & A_1 \\
 & \searrow \alpha' & & & \\
 & id_{A_1} & & & 
 \end{array}
 \end{array}$$

$$[b, \bar{\tau}_{1!}](\phi \otimes \alpha) \pm \bar{\tau}_{1!}(\delta\phi \otimes \alpha) = [\bar{\tau}_{1!}, \iota_\phi](\alpha)$$

$$\begin{aligned}
 \bar{\tau}_{1!}(\phi \otimes \alpha) = & \sum_{k \geq 1} \pm f_{0,0} a_0 \otimes \dots \otimes \phi(a_k, \dots) \otimes f_{0,1} a_r \dots \otimes f_{0,1} a_n + \\
 & \sum \pm \phi(f_{1,0} f_{0,1} a_k, \dots, f_{1,0} f_{0,1} a_n, a_0, \dots) \otimes f_{0,1} a_s \otimes \dots \otimes f_{0,1} a_{k-1} \\
 [b, \bar{\tau}_{1!}(\phi, -)] \pm \bar{\tau}_{1!}(\delta\phi, -) = & [\bar{\tau}_{1!}, \iota_\phi]
 \end{aligned}$$

# An $A_\infty$ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

**Defining**  $\tau_1!$

$$\begin{aligned}
 & \bar{\tau}_1! \left( (\phi_{0,1} | \dots | \phi_{0,k_0}) \otimes (\phi_{1,1} | \dots | \phi_{1,k_1}) \otimes \alpha \right) = \\
 = & \sum_{\substack{1 \leq i \leq n_0 \\ i \leq j_1 \leq \dots \leq j_{2k_1} \leq k_0 \\ p}} \pm \phi_{0,1} \left( \begin{array}{c} f_{1,0} f_{0,i} a_p, \dots, f_{1,0} \phi_{0,i+1}(a_*, \dots), \\ f_{1,0} f_{0,i+1} a_*, \dots, f_{1,0} \phi_{0,j_1}(a_*, \dots), \\ f_{1,0} f_{0,j_1} a_*, \dots, \phi_{1,1}(f_{0,j_1} a_*, \dots, \phi_{0,j_1+1}(a_*, \dots), \dots), \dots, \\ \phi_{1,k_1}(f_{0,j_{2k_1-1}} a_*, \dots, \phi_{0,j_{2k_1-1}+1}(a_*, \dots), \dots), \dots, a_0, \dots \end{array} \right) \otimes \\
 & \otimes f_{0,1} a_* \otimes \dots \otimes \phi_{0,2}(a_*, \dots) \otimes f_{0,2} a_* \otimes \dots \otimes \\
 & \otimes \phi_{0,i}(a_*, \dots) \otimes f_{0,i} a_* \otimes \dots \otimes f_{0,i} a_{p-1} + \\
 & \left( \sum \pm f_{0,0} a_0 \otimes \dots \otimes \phi_{0,1}(a_*, \dots) \otimes \dots \otimes \phi_{0,n_0}(a_*, \dots) \otimes \right. \\
 & \quad \left. \otimes f_{0,n_0} a_* \otimes \dots \otimes f_{0,n_0} a_n \quad \text{if } k_1 = 0 \right)
 \end{aligned}$$

# An $A_\infty$ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

$$\mathcal{A} := (A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$$

$$\chi_\infty \rightarrow \mathcal{D}_\infty$$

$$\mathcal{A} \mapsto (B(\mathcal{A}), C(\mathcal{A}))$$

Generating Morphisms:  $\sigma_{i,n} \mapsto \begin{pmatrix} B(\mathcal{A}) \xrightarrow{\hat{\sigma}_{i,n}} B(\sigma_{i,n}\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{id} \hat{\sigma}_{i,n}^* C(\sigma_{i,n}\mathcal{A}) \end{pmatrix}$

$$\delta_{j,n} \mapsto \begin{pmatrix} B(\mathcal{A}) \xrightarrow{\hat{\delta}_{j,n}} B(\delta_{j,n}\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{id} \hat{\delta}_{j,n}^* C(\delta_{j,n}\mathcal{A}) \end{pmatrix} \quad \tau_n \mapsto \begin{pmatrix} B(\mathcal{A}) \xrightarrow{\hat{\tau}_n} B(\tau_n\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{\tau_n! = d_0^{*n-1} \tau_1!} \hat{\tau}_n^* C(\tau_n\mathcal{A}) \end{pmatrix}$$

# An $A_\infty$ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

**n=1:**

$$C(A_0 \rightarrow A_1 \rightarrow A_0) \xrightarrow{\tau_1!} \hat{\tau}_1^* C(\tau_1(A_0 \rightarrow A_1 \rightarrow A_0)) \xrightarrow{\hat{\tau}_1^* \tau_1!} \hat{\tau}_1^{*2} C(\tau_1^2(A_0 \rightarrow A_1 \rightarrow A_0))$$

$\text{-----} \xrightarrow{\quad id \quad} \text{-----}$

An  $A_\infty$ -functor  $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

**n=1:**

$$C(A_0 \rightarrow A_1 \rightarrow A_0) \xrightarrow{\tau_1!} \hat{\tau}_1^* C(\tau_1(A_0 \rightarrow A_1 \rightarrow A_0)) \xrightarrow{\hat{\tau}_1^* \tau_1!} \hat{\tau}_1^{*2} C(\tau_1^2(A_0 \rightarrow A_1 \rightarrow A_0))$$

$\searrow \quad \nearrow$   
 $id$

$$A_0 \xrightarrow{f_{0,0}} A_1 \xrightarrow{f_{1,0}} A_0$$

$\alpha$   
 $id_{A_0}$

$$\alpha = a_0 \otimes \dots \otimes a_n \xrightarrow{\tau_{1!}} f_{0,0} a_0 \otimes \dots \otimes f_{0,0} a_n \xrightarrow{\hat{\tau}_{1!}^* \tau_{1!}} f_{1,0} f_{0,0} a_0 \otimes \dots \otimes f_{1,0} f_{0,0} a_n$$

An  $A_\infty$ -functor  $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

**n=1:**

$$C(A_0 \rightarrow A_1 \rightarrow A_0) \xrightarrow{\tau_{1!}} \hat{\tau}_1^* C(\tau_1(A_0 \rightarrow A_1 \rightarrow A_0)) \xrightarrow{\hat{\tau}_1^* \tau_{1!}} \hat{\tau}_1^{*2} C(\tau_1^2(A_0 \rightarrow A_1 \rightarrow A_0))$$

$\searrow id \nearrow$

$$A_0 \xrightarrow{f_{0,0}} A_1 \xrightarrow{f_{1,0}} A_0$$

$\alpha$   
 $id_{A_0}$

$$\alpha = a_0 \otimes \dots \otimes a_n \xrightarrow{\tau_{1!}} f_{0,0}a_0 \otimes \dots \otimes f_{0,0}a_n \xrightarrow{\hat{\tau}_{1!}^* \tau_{1!}} f_{1,0}f_{0,0}a_0 \otimes \dots \otimes f_{1,0}f_{0,0}a_n$$

$$f_{1,0}f_{0,0}\alpha - \alpha = [b, B](\alpha)$$

$$B(a_0 \otimes \dots \otimes a_n) = \sum_{0 \leq i \leq n} \pm 1 \otimes f_{1,0} f_{0,0} a_i \otimes \dots \otimes f_{1,0} f_{0,0} a_n \otimes a_0 \otimes \dots \otimes a_{i-1}$$



# An $A_\infty$ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

**n=1:**

$$C(A_0 \rightarrow A_1 \rightarrow A_0) \xrightarrow{\tau_{1!}} \hat{\tau}_1^* C(\tau_1(A_0 \rightarrow A_1 \rightarrow A_0)) \xrightarrow{\hat{\tau}_1^* \tau_{1!}} \hat{\tau}_1^{*2} C(\tau_1^2(A_0 \rightarrow A_1 \rightarrow A_0))$$

$\xrightarrow{id}$

$$\begin{array}{c}
 \begin{array}{ccccc}
 & f_{0,0} & & & \\
 & \downarrow \phi & & & \\
 A_0 & \xrightarrow{f_{0,1}} & A_1 & \xrightarrow{f_{1,0}} & A_0 \\
 & \uparrow \alpha & & & \\
 & id_{A_0} & & & 
 \end{array}
 \end{array} = \phi \otimes \alpha$$

$$(\overline{\hat{\tau}^* \tau_{1!} \circ \tau_{1!} - id})(\phi \otimes \alpha) = "f_{1,0} \circ L_\phi(\alpha)"$$

An  $A_\infty$ -functor  $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

**n=1:**

$$C(A_0 \rightarrow A_1 \rightarrow A_0) \xrightarrow{\tau_1!} \hat{\tau}_1^* C(\tau_1(A_0 \rightarrow A_1 \rightarrow A_0)) \xrightarrow{\hat{\tau}_1^* \tau_1!} \hat{\tau}_1^{*2} C(\tau_1^2(A_0 \rightarrow A_1 \rightarrow A_0))$$

$\text{---} id \text{---}$

$$\begin{aligned} & \overline{(\hat{\tau}^* \tau_{1!} \circ \tau_{1!} - id)}(\phi \otimes \alpha) = "f_{1,0} \circ L_\phi(\alpha)" \\ &= \sum_{k \geq 1} \pm f_{1,0} f_{0,0} a_0 \otimes \dots \otimes f_{1,0} \phi(a_k, \dots) \otimes f_{1,0} f_{0,1} a_r \dots \otimes f_{1,0} f_{0,1} a_n + \\ & \quad \sum \pm f_{1,0} \phi(f_{1,0} f_{0,1} a_k, \dots, f_{1,0} f_{0,1} a_n, a_0, \dots) \otimes f_{1,0} f_{0,1} a_s \otimes \dots \otimes f_{1,0} f_{0,1} a_{k-1} \\ & \stackrel{?}{=} [\iota_\phi, B](\alpha) \pm [b, B](\phi \otimes \alpha) \pm B(\delta \phi \otimes \alpha) \end{aligned}$$

An  $A_\infty$ -functor  $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

**n=1:**

$$C(A_0 \rightarrow A_1 \rightarrow A_0) \xrightarrow{\tau_{1!}} \hat{\tau}_1^* C(\tau_1(A_0 \rightarrow A_1 \rightarrow A_0)) \xrightarrow{\hat{\tau}_1^* \tau_{1!}} \hat{\tau}_1^{*2} C(\tau_1^2(A_0 \rightarrow A_1 \rightarrow A_0))$$

$\searrow \quad \nearrow$   
 $id$

$$\begin{aligned} & \overline{(\hat{\tau}^* \tau_{1!} \circ \tau_{1!} - id)}(\phi \otimes \alpha) = "f_{1,0} \circ L_\phi(\alpha)" \\ &= \sum_{k \geq 1} \pm f_{1,0} f_{0,0} a_0 \otimes \dots \otimes f_{1,0} \phi(a_k, \dots) \otimes f_{1,0} f_{0,1} a_r \dots \otimes f_{1,0} f_{0,1} a_n + \\ & \quad \sum \pm f_{1,0} \phi(f_{1,0} f_{0,1} a_k, \dots, f_{1,0} f_{0,1} a_n, a_0, \dots) \otimes f_{1,0} f_{0,1} a_s \otimes \dots \otimes f_{1,0} f_{0,1} a_{k-1} \\ & \stackrel{?}{=} [\iota_\phi, B](\alpha) \pm [b, B](\phi \otimes \alpha) \pm B(\delta \phi \otimes \alpha) \end{aligned}$$

$$B(\phi \otimes \alpha) = \sum_{0 \leq i \leq n} \pm 1 \otimes f_{1,0} f_{0,0} a_i \otimes \dots \otimes f_{1,0} \phi(a_*, \dots) \otimes f_{1,0} f_{0,1} a_* \otimes \dots \otimes f_{1,0} f_{0,1} a_n \otimes a_0 \otimes \dots \otimes a_{i-1}$$

# An $A_\infty$ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

**n=1:**

$$\begin{aligned}
 & B((\phi_{0,1}|\dots|\phi_{0,k_0}) \otimes (\phi_{1,1}|\dots|\phi_{1,k_1}) \otimes \alpha) = \\
 = & \sum_{\substack{0 \leq j_1 \leq \dots \leq j_{2k_1} \leq k_0 \\ p}} \pm 1 \otimes f_{1,0} f_{0,0} a_p \otimes \dots \otimes f_{1,0} \phi_{0,1}(a_*, \dots) \otimes \\
 & \otimes f_{1,0} f_{0,1} a_* \otimes \dots \otimes f_{1,0} \phi_{0,j_1}(a_*, \dots) \otimes \\
 & \otimes f_{1,0} f_{0,j_1} a_* \otimes \dots \otimes \phi_{1,1}(f_{0,j_1} a_*, \dots, \phi_{0,j_1+1}(a_*, \dots), \dots) \otimes \\
 & \otimes \dots \otimes \phi_{1,k_1}(f_{0,j_{2k_1-1}} a_*, \dots, \phi_{0,j_{2k_1-1}+1}(a_*, \dots), \dots) \otimes \dots \otimes \\
 & \otimes a_0 \otimes \dots \otimes a_{p-1}
 \end{aligned}$$

# An $A_\infty$ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

$$\chi_\infty \rightarrow \mathcal{D}_\infty$$

$$\mathcal{A} \mapsto (B(\mathcal{A}), C(\mathcal{A}))$$

$$\tau_1 \mapsto \left( \begin{array}{l} \hat{\tau}_1 : B(\mathcal{A}) \rightarrow B(\tau_1 \mathcal{A}) \\ \tau_{1!} : C(\mathcal{A}) \rightarrow \hat{\tau}_1^* C(\tau_1 \mathcal{A}) \end{array} \right)$$

$$(\tau_1, \tau_1) \mapsto \left( \begin{array}{l} id : B(\mathcal{A}) \rightarrow B(\mathcal{A}) \\ B : C(\mathcal{A}) \rightarrow C(\mathcal{A}) \end{array} \right)$$

$$(\tau_1, \tau_1, \tau_1) \mapsto \left( \begin{array}{l} \hat{\tau}_1 : B(\mathcal{A}) \rightarrow B(\tau_1 \mathcal{A}) \\ 0 : C(\mathcal{A}) \rightarrow \hat{\tau}_1^* C(\tau_1 \mathcal{A}) \end{array} \right)$$

$$\vdots$$

# An $A_\infty$ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

The  $A_\infty$  relations mean:

- $\tau_1!$  is a map of complexes
- $B^2 = 0$
- The following diagram commutes:

$$\begin{array}{ccc} C(\mathcal{A}) & \xrightarrow{\tau_1!} & \hat{\tau}_1^* C(\tau_1 \mathcal{A}) \\ \downarrow B & & \downarrow \hat{\tau}_1^* B \\ C(\mathcal{A}) & \xrightarrow{\tau_1!} & \hat{\tau}_1^* C(\tau_1 \mathcal{A}) \end{array}$$

## An $A_\infty$ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

For higher  $n > 1$ , we want to find a homotopy between “ $\tau_n^{n+1}$ ” and  $id$ .

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$$\begin{array}{ccccc} [n] & \xrightarrow{\tau_n} & [n] & \xrightarrow{\tau_n} & [n] \\ \downarrow \delta_{n-1,n} & & & \swarrow \delta_{0,n} & \\ [n-1] & \xrightarrow{\tau_{n-1}} & [n-1] & & \end{array}$$



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**Strategy:** Find such a homotopy,  $\mathcal{B}$ , for  $n = 2$ , and use  $\hat{\delta}_0^{*n-2} \mathcal{B}$  for  $n > 2$ .

# An $A_\infty$ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

$$\chi_\infty \rightarrow \mathcal{D}_\infty$$

$$\mathcal{A} \mapsto (B(\mathcal{A}), C(\mathcal{A}))$$

$$\mu = \tau_{n-1} \circ \delta_{n-1,n} = \delta_{0,n} \circ \tau_n^2 \mapsto \begin{pmatrix} \hat{\tau}_{n-1} \circ \hat{\delta}_{n-1,n} = \hat{\delta}_{0,n} \circ \hat{\tau}_n^2 \\ \hat{\tau}_n^{*2} \delta_{0,n!} \circ \hat{\tau}_n^* \tau_{n!} \circ \tau_{n!} \end{pmatrix}$$

$$(\delta_{0,n}, \tau_n^2) \mapsto \begin{pmatrix} \hat{\delta}_{0,n} \circ \hat{\tau}_n^2 \\ 0 \end{pmatrix}$$

$$(\tau_{n-1}, \delta_{n-1,n}) \mapsto \begin{pmatrix} \hat{\tau}_{n-1} \circ \hat{\delta}_{n-1,n} \\ \mathcal{B} \end{pmatrix}$$

$$(\tau_{n-1}, \delta_{n-1,n}, \lambda) \mapsto \begin{pmatrix} \hat{\tau}_{n-1} \circ \hat{\delta}_{n-1,n} \circ \hat{\lambda} \\ 0 \end{pmatrix}$$

$$\vdots$$

# An $A_\infty$ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

For  $n > 1$ , the  $A_\infty$  relations mean:

- $\tau_{n!}$  is a map of complexes
- $\mathcal{B}^2 = 0$
- We have a pair of homotopic maps:

$$\begin{array}{ccc}
 C(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0) & & \\
 \hat{\tau}_n^{*2} \tau_{n!} \circ \hat{\tau}_n^* \tau_{n!} \circ \tau_{n!} & \begin{array}{c} \curvearrowright \\ \downarrow \quad \downarrow \end{array} & (\delta_{n-2, n-1} \widehat{\delta_{n-1, n}})^* \tau_{n-2!} \\
 \text{"apply } \tau_{n!} \text{ 3 times"} & & \text{"brace together the last 3 algebras,} \\
 & & \text{then apply } \tau_{n-2!} \text{ once"} \\
 C(A_{n-2} \rightarrow A_{n-1} \rightarrow A_n \rightarrow A_0 \rightarrow \dots \rightarrow A_{n-2}). & & 
 \end{array}$$

# An $A_\infty$ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

For  $n > 1$ , the  $A_\infty$  relations mean:

- $\tau_{n!}$  is a map of complexes
- $\mathcal{B}^2 = 0$
- They are homotopic via two homotopies:

$$\begin{array}{ccc}
 (\widehat{\delta_{n-2,n-1} \delta_{n-1,n}})^* \tau_{n-2!} & \xrightarrow{\cong} & \hat{\delta}_{n-1,n}^* (\hat{\delta}_{n-2,n-1}^* \tau_{n-2!}) \xrightarrow{\hat{\delta}_{n-1,n}^* \mathcal{B}_{n-1}} \hat{\delta}_{n-1,n}^* (\hat{\tau}_{n-1}^* \tau_{n-1!} \circ \tau_{n-1!}) \\
 \text{"brace together } A_{n-2}, A_{n-1}, A_n, \text{ then apply } \tau_{n-2!}" & & \\
 \downarrow \cong & & \downarrow \cong \\
 (\widehat{\delta_{n-2,n-1} \delta_{n-2,n}})^* \tau_{n-2!} & & \hat{\tau}_n^{*2} \tau_{n!} \circ \hat{\delta}_{n-1,n}^* \tau_{n-1!} \\
 \downarrow \hat{\delta}_{n-2,n}^* \mathcal{B}_{n-1} & & \text{"brace together } A_{n-1}, A_n \text{ and apply } \tau_{n-1!}, \text{ then apply } \tau_{n!}" \\
 \downarrow & & \downarrow \tau_n^{*2} \tau_{n!} \circ \mathcal{B}_n \\
 \hat{\delta}_{n-2,n}^* (\hat{\tau}_{n-1}^* \tau_{n-1!} \circ \tau_{n-1!}) & \xrightarrow{\cong} & \hat{\tau}_n^* (\hat{\delta}_{n-1,n}^* \tau_{n-1!}) \circ \tau_{n!} \xrightarrow{\hat{\tau}_n^* \mathcal{B}_n \circ \tau_{n!}} \hat{\tau}_n^{*2} \tau_{n!} \circ \hat{\tau}_n^* \tau_{n!} \circ \tau_{n!} \\
 & \text{"apply } \tau_{n!}, \text{ then brace together } A_{n-1}, A_{n-2} \text{ and apply } \tau_{n-1!}" & \text{"apply } \tau_{n!} \text{ three times"}
 \end{array}$$

# Summary:

We have a given a “homotopically sheafy-cyclic object in dg cocategories with a dg comodule”, i.e., an  $A_\infty$ -functor  $\chi_\infty \rightarrow \mathcal{D}_\infty$ .

Time and interest permitting: “A homotopically sheafy-cyclic object in dg categories with a dg module”

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- Rectify  $\mathcal{F}$  to a dg functor  $U(\chi_\infty) \rightarrow \mathcal{D}_\infty$

$$\mathrm{Hom}_{A_\infty\text{-Cat}}(\chi_\infty, \mathcal{D}_\infty) = \mathrm{Hom}_{\mathrm{DGCat}}(U(\chi_\infty), \mathcal{D}_\infty)$$

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- Rectify  $\mathcal{F}$  to a dg functor  $U(\chi_\infty) \rightarrow \mathcal{D}_\infty$
- Give a dg functor  $Adj : \mathcal{D}_\infty \rightarrow \mathcal{D}_1$

$$\mathcal{D}_1^\bullet((B_1, C_1), (B_0, C_0)) = \left\{ \begin{array}{l} F : B_1 \rightarrow B_0 \text{ dg functor} \\ F_! : F_\# C_1 \rightarrow C_0 \text{ linear map} \\ \text{“co-restriction of scalars”} \end{array} \right\}$$

$$Hom_{B_1}(C_1, F^* C_0) = Hom_{B_0}(F_\# C_1, C_0)$$

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- Rectify  $\mathcal{F}$  to a dg functor  $U(\chi_\infty) \rightarrow \mathcal{D}_\infty$
- Give a dg functor  $Adj : \mathcal{D}_\infty \rightarrow \mathcal{D}_1$
- Give a dg functor  $Cobar : \mathcal{D}_1 \rightarrow \mathcal{E}$

$$Obj(\mathcal{E}) = \{(\text{dg category}, \text{dg module})\}$$

$$\mathcal{E}^\bullet((A_1, M_1), (A_0, M_0)) = \left\{ \begin{array}{l} F : A_1 \rightarrow A_0 \text{ dg functor} \\ F_! : M_1 \rightarrow F^* M_0 \text{ linear map} \\ \text{“restriction of scalars”} \end{array} \right\}$$



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- Rectify  $\mathcal{F}$  to a dg functor  $U(\chi_\infty) \rightarrow \mathcal{D}_\infty$
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$$U(\chi_\infty) \xrightarrow{\text{rectified } \mathcal{F}} \mathcal{D}_\infty \xrightarrow{Adj} \mathcal{D}_1 \xrightarrow{Cobar} \mathcal{E}$$