Rebecca Wei

Northwestern University

Jan 25, 2017

Outline

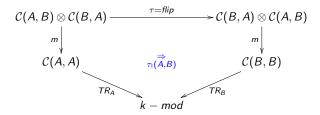
- Question: What do algebras form?
- **Answer 1:** A category in categories (*HH*⁰)
- Derived Answer 1: A category in dg cocategories (Hochschild cochains...)
- **Answer 2:** A 2-category with a trace functor (HH_0)
- Derived Answer 2: A category in dg cocategories with a trace functor (Hochschild chains...)

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- **Answer 2:** A 2-category with a trace functor (HH_0)
- Derived Answer 2: A category in dg cocategories with a trace functor (Hochschild chains...) up to homotopy

(Kaledin): A <u>trace functor</u> on a 2-category C is:

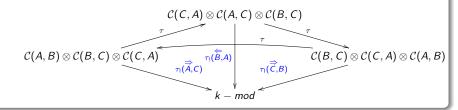
- for each $A \in Obj(\mathcal{C})$, a functor $TR_A : \mathcal{C}(A,A) \to k mod$
- for each pair $A, B \in Obj(\mathcal{C})$, a natural transformation $\tau_!(A, B)$



• such that $\tau_!(B,A) \circ \tau_!(C,B) \circ \tau_!(A,C) = id$

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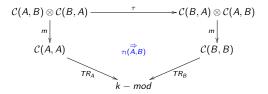


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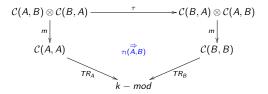
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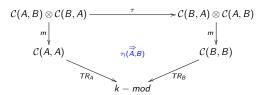
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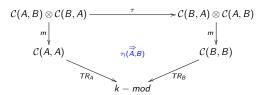
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Let $\mathcal C$ be a category in k-linear categories. Let $\chi(\mathcal C)$ be the k-linear category with

- Objects = $\{A \to A, \ A \to B \to A, \ A \to B \to C \to A : A, B, C \in Obj(\mathcal{C})\}$
- Morphisms = {linear combinations of compositions of rotations τ_i s.t. $\tau_i^{i+1} = id$, i = 0, 1, 2}.

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A trace functor on C gives a functor

$$\chi(\mathcal{C}) \to \mathcal{D} = \{ (k\text{-lin category, module}) \}$$

$$(A \to A) \mapsto (\mathcal{C}(A, A), T(A))$$

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$$(A \to B \to C \to A) \mapsto (\mathcal{C}(A, B) \otimes \mathcal{C}(B, C) \otimes \mathcal{C}(C, A), m^{*2}T(A))$$

$$\tau_1 : (A \to B \to A) \to (B \to A \to B) \mapsto \tau_1(A, B)$$

$$\tau_2 : (A \to B \to C \to A) \to (C \to A \to B \to C) \mapsto m^*\tau_1(A, C)$$

 $au_1^2=id,\; au_2^3=id\mapsto {
m relations}\; {
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m definition}\; {
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m trace}\; {
m functor}\;$

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- Morphisms = {linear combinations of compositions of rotations τ_i s.t. $\tau_i^{i+1}=id$, i=0,1,2}. Why stop at n=2? What about δ , σ ?

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$$\tau_1^2 = id, \ \tau_2^3 = id \mapsto \text{relations in definition of trace functor}$$

- - Long Mr. (New York and The Long A)

Let $\mathcal C$ be a category in dg cocategories. Let $\chi_\infty(\mathcal C)$ be the dg category with

- Objects = $\{A_0 \rightarrow ... \rightarrow A_n \rightarrow A_0 : A_i \in Obj(\mathcal{C}), n \geq 0\}$
- Morphisms = {linear combinations of compositions of

rotations
$$\tau_n: \mathcal{A} \mapsto (A_n \to A_0 \to ... \to A_n)$$

coboundaries $\delta_{j,n}: \mathcal{A} \mapsto (A_0 \to ... \to A_j \to A_{j+2 \pmod{n+1}} \to ... \to A_0)$
codegeneracies $\sigma_{i,n}: \mathcal{A} \mapsto (A_0 \to ... \to A_i \to A_i \to ... \to A_0)$
where $\mathcal{A}:=(A_0 \to ... \to A_n \to A_0)$, subject to the cyclic relations} [0]

Let \mathcal{D}_{∞} be the dg category with

- Objects = $\{(\text{dg cocategory}, \text{dg comodule})\}$
- Morphisms:

$$\mathcal{D}^p_{\infty}\big((B_1,C_1),(B_0,C_0)\big) := \begin{cases} F:B_1 \to B_0 \ dg \ functor, \\ F_!:C_1 \to F^*C_0 \ degree-p \ linear \ map \end{cases}$$

$$d_{D_{\infty}}(F,F_!) = (F,[d,F_!] = d_{F^*C_0} \circ F_! \pm F_! \circ d_{C_1})$$

 F^*C_0 is the categorified version of co-extension of scalars:

$$F^*C_0 = ker(B_1 \otimes C_0 \Rightarrow B_1 \otimes B_0 \otimes C_0)$$

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Can we give a dg functor $\chi(\mathcal{C}) \to \mathcal{D}_{\infty}$?

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No, but we can give an A_{∞} -functor.

