Formality theorems for Hochschild complexes and their applications

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To Giovanni Felder on the occasion of his 50th birthday.

Abstract

We give a popular introduction to formality theorems for Hochschild complexes and their applications. We review some of the recent results and prove that the truncated Hochschild cochain complex of a polynomial algebra is non-formal.

Contents

1 Introduction

The notion of formality was suggested in classical article [29] of P. Deligne, P. Griffiths, J. Morgan, and D. Sullivan. In this article it was shown that the de Rham algebra of a compact Kähler manifold M is quasi-isomorphic to the cohomology ring of M. Using the terminology suggested in this article we can say that de Rham algebra of a compact Kähler manifold M is *formal* as a commutative algebra.

Around 1993-94 M. Kontsevich conjectured (see [58], [90]) that the Hochschild cochain complex for the algebra of functions on a smooth manifold is formal as a Lie algebra with the Gerstenhaber bracket [43]. Then in 1997 M. Kontsevich proved [57] this formality conjecture

for an arbitrary smooth manifold. In the same paper he showed how this result solves a long standing problem on the deformation quantization [7], [9] of a Poisson manifold.

In 1998 the second author proposed a completely different proof of Kontsevich's formality theorem [81] for the case of the affine space over an arbitrary field of characteristic zero. This approach is based on deep results such as the proof of Deligne's conjecture on the Hochschild complex [10], [61], [73], [83], [91] and the formality theorem [60], [65], [82] for the operad of little discs.

In 1999 A.S. Cattaneo and G. Felder described [22] how Kontsevich's star-product formula as well as his formality theorem can be obtained using the correlators of the Poisson sigma model [55], [75].

After M. Kontsevich's celebrated result [57] lots of interesting generalizations and applications of the formality theorem for Hochschild cochain complex were proposed. At this moment all these results can be put under an umbrella of an independent mathematical topic. In this paper we give a popular introduction to this fascinating topic. We review some of the recent results and give an example of a non-formal differential graded (DG) Lie algebra. We hope that our introductory part is accessible to graduate students who are interested in this topic.

The organization of the paper is as follows. In the next section we illustrate the general concept of formality with the example of a DG Lie algebra. We recall Maurer-Cartan elements, the Goldman-Millson groupoid and twisting procedure. We also discuss some consequences of formality for a DG Lie algebra. In Section 3 we recall basic algebraic structures on the Hochschild complexes of an associative algebra A. In this section we also recall the Van den Bergh duality theorem [87]. Section 4 begins with the formulation of Kontsevich's formality theorem [57] and its immediate corollaries. Next we review the alternative approach [81] of the second author. Then we discuss formality theorems for Hochschild and cyclic chains and the formality of the ∞-calculus algebra of Hochschild complexes. We conclude Section 4 with the formality theorems for Hochschild complexes in the algebraic geometry setting. In Section 5 we give a brief outline of few recent applications of formality theorems for Hochschild complexes. Finally, in the concluding section we give an example of a non-formal DG Lie algebra.

Notation. "DG" stands for differential graded. S_n denotes the symmetric group on n letters. By suspension $\mathbf{s}V$ of a graded vector space (or a cochain complex) V we mean $\varepsilon \otimes V$, where ε is a one-dimensional vector space placed in degree $+1$. For a vector $v \in V$ we denote by $|v|$ its degree. We use the Koszul rule of signs which says that a transposition of any two homogeneous vectors v_1 and v_2 yields the sign

$$(-1)^{|v_1||v_2|}.$$

For a groupoid $\mathcal{G}$ we denote by $\pi_0(\mathcal{G})$ the set of isomorphism classes of its objects. This is exactly the set of connected components of the classifying space $B\mathcal{G}$ of $\mathcal{G}$. We assume that the underlying field $\mathbb{K}$ has characteristic zero. $\hbar$ denotes the formal deformation parameter.

2 Formal versus non-formal

2.1 General definitions

Let $\mathcal{L}$ be a DG Lie algebra over the field $\mathbb{K}$. We say that a morphism $\mu : \mathcal{L} \to \widetilde{\mathcal{L}}$ is a *quasi-isomorphism* if μ induces an isomorphism on cohomology groups. In this case we use the tilde over the arrow

$$\mu : \mathcal{L} \xrightarrow{\sim} \widetilde{\mathcal{L}}.$$

We call two DG Lie algebras $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ *quasi-isomorphic* if they can be connected by a sequence of quasi-isomorphisms

$$\mathcal{L} \xrightarrow{\sim} \bullet \xleftarrow{\sim} \bullet \xrightarrow{\sim} \bullet \ldots \bullet \xleftarrow{\sim} \widetilde{\mathcal{L}}.$$

For every DG Lie algebra $\mathcal{L}$ its cohomology $H^\bullet(\mathcal{L})$ is naturally a graded Lie algebra. We think of $H^\bullet(\mathcal{L})$ as the DG Lie algebra with the zero differential.

Definition 1 *A DG Lie algebra $\mathcal{L}$ is called* formal *if it is quasi-isomorphic to its cohomology $H^\bullet(\mathcal{L})$.*

Remark. Similarly, we can talk about formal or non-formal DG associative algebras, DG commutative algebras and DG algebras of other types.

We would like to mention that, since our DG Lie algebras are complexes of vector spaces over a field, we always have a map of cochain complexes

$$F_1 : H^\bullet(\mathcal{L}) \to \mathcal{L}$$

which induces an isomorphism on the level of cohomology. In other words, it is always possible to choose a representative for each cohomology class in such a way that this choice respects linearity.

It is obvious that, in general, F_1 is not compatible with the Lie brackets. However, there exists a bilinear map

$$F_2 : \wedge^2 H^\bullet(\mathcal{L}) \to \mathcal{L}$$

of degree -1 such that

$$F_1([v_1, v_2]) - [F_1(v_1), F_1(v_2)] = \partial F_2(v_1, v_2), \qquad (2.1)$$

where ∂ is the differential on $\mathcal{L}$.

It is convenient to enlarge the category of DG Lie algebras by L_∞ algebras [54], [64]. These more general algebras have two important advantages. First, every sequence of quasi-isomorphisms between L_∞ algebras $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ can be shortened to a single L_∞-quasi-isomorphism

$$F : \mathcal{L} \xrightarrow{\sim} \widetilde{\mathcal{L}}.$$

Second, for every DG Lie algebra (or possibly an L_∞ algebra) $\mathcal{L}$ there exists an L_∞-algebra structure on $H^\bullet(\mathcal{L})$ such that $H^\bullet(\mathcal{L})$ is quasi-isomorphic to $\mathcal{L}$.

Thus if a DG Lie algebra $\mathcal{L}$ is non-formal then this L_∞ algebra structure on $H^\bullet(\mathcal{L})$ "measures" to what extent $\mathcal{L}$ is far from being formal.

The most pedestrian way to introduce the notion of L_∞ algebra is to start with the Chevalley-Eilenberg chain complex $C(\mathcal{L})$ of a DG Lie algebra $\mathcal{L}$.

As a graded vector space the Chevalley-Eilenberg chain complex is the direct sum of all symmetric powers of the desuspension $\mathbf{s}^{-1}\mathcal{L}$ of $\mathcal{L}$

$$C(\mathcal{L}) = \sum_{k=1}^{\infty} \left[(\mathbf{s}^{-1}\mathcal{L})^{\otimes k} \right]^{S_k}. \tag{2.2}$$

To introduce the differential we remark that $C(\mathcal{L})$ is equipped with the following cocommutative comultiplication:

$$\Delta \,:\, C(\mathcal{L}) \mapsto C(\mathcal{L}) \otimes C(\mathcal{L}) \tag{2.3}$$
$$\Delta(v) = 0 \,,$$

$$\Delta(v_1, v_2, \ldots, v_n) = \sum_{k=1}^{n-1} \sum_{\sigma \in \mathrm{Sh}(k,n-k)} \pm (v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \otimes (v_{\sigma(k+1)}, \ldots, v_{\sigma(n)}) \,, \tag{2.4}$$

where $v_1, \ldots, v_n$ are homogeneous elements of $\mathbf{s}^{-1}\mathcal{L}$, $\mathrm{Sh}(k, n-k)$ is the set of $(k, n-k)$-shuffles in S_n, and the signs are determined using the Koszul rule.

To define the boundary operator Q on (2.2) for a DG Lie algebra $\mathcal{L}$ we introduce the natural projection

$$p : C(\mathcal{L}) \to \mathbf{s}^{-1}\mathcal{L} \,. \tag{2.5}$$

It is not hard to see that if Q is coderivation of the coalgebra $C(\mathcal{L})$ in the sense of the equation

$$\Delta Q = (Q \otimes 1 + 1 \otimes Q)\Delta$$

then Q is uniquely determined by its composition $p \circ Q$ with p. This statement follows from the fact that $C(\mathcal{L})$ is a cofree[1] cocommutative coalgebra. The same statement holds for cofree coalgebras of other types. (See Proposition 2.14 in [46].)

Thus we define the coboundary operator Q in terms of the differential ∂ and the Lie bracket $[\,,\,]$ by requiring that it is a coderivation of the coalgebra $C(\mathcal{L})$ and by setting

$$p \circ Q(v) = -\partial v \,, \qquad p \circ Q(v_1, v_2) = (-1)^{|v_1|+1}[v_1, v_2] \,,$$

$$p \circ Q(v_1, v_2, \ldots, v_k) = 0 \,, \qquad k > 2 \,,$$

where $v, v_1, \ldots, v_k$ are homogeneous elements of $\mathcal{L}$.

The equation $Q^2 = 0$ readily follows from the Leibniz rule

$$\partial [v_1, v_2] = [\partial v_1, v_2] + (-1)^{|v_1|}[v_1, \partial v_2] \,,$$

and the Jacobi identity:

$$(-1)^{|v_1||v_3|}[[v_1, v_2], v_3] + \mathrm{c.p.}\{1, 2, 3\} = 0 \,.$$

To define a notion of L_∞ algebra we simply allow the most general degree 1 coderivation Q of the coalgebra $C(\mathcal{L})$ (2.2) satisfying the equation $Q^2 = 0$. More precisely,

Definition 2 *An L_∞-algebra structure* on a graded vector space $\mathcal{L}$ is a degree 1 coderivation Q of the coalgebra $C(\mathcal{L})$ (2.2) with the comultiplication (2.4) such that

$$Q^2 = 0 \,.$$

Furthermore, an L_∞ morphism F from an L_∞ algebra $(\mathcal{L}, Q)$ to an L_∞ algebra $(\widetilde{\mathcal{L}}, \widetilde{Q})$ is a morphism of the corresponding DG cocommutative coalgebras:

$$F : (C(\mathcal{L}), Q) \to (C(\widetilde{\mathcal{L}}), \widetilde{Q}) \,. \tag{2.6}$$

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[1] Strictly speaking $C(\mathcal{L})$ is a cofree cocommutative coalgebra without counit.

The coderivation Q is uniquely determined by the degree 1 maps

$$Q_n = p \circ Q \Big|_{(\mathbf{s}^{-1}\mathcal{L})^{\otimes n}} : (\mathbf{s}^{-1}\mathcal{L})^{\otimes n} \to \widetilde{\mathcal{L}} \qquad (2.7)$$

and we call them the structure maps of the L_∞ algebra $(\mathcal{L}, Q)$.

The equation $Q^2 = 0$ is equivalent to an infinite collection of coherence relations for Q_n's. The first relation says that $Q_1^2 = 0$. The second relation is the Leibniz identity for Q_1 and Q_2. The third relation says that the binary operation

$$[\gamma_1, \gamma_2] = (-1)^{|\gamma_1|+1} Q_2(\gamma_1, \gamma_2)$$

satisfies the Jacobi identity up to homotopy and the corresponding chain homotopy is exactly Q_3. In particular, the L_∞ algebras with the zero higher maps Q_n, $n > 2$ are exactly the DG Lie algebras.

Remark. It is possible to define ∞ or homotopy versions for algebras of other types. Although the intrinsic definition of such ∞-versions requires the language of operads [11], [52], [46], [48], [70], we avoid this language here for sake of accessibility and try to get by using the vague analogy with the case of L_∞ algebras.

By analogy with the coderivations every morphism (2.6) from the coalgebra $(C(\mathcal{L}), Q)$ to the coalgebra $(C(\widetilde{\mathcal{L}}), \widetilde{Q})$ is uniquely determined by its composition $p \circ F$ with the natural projection

$$p : C(\widetilde{\mathcal{L}}) \to \mathbf{s}^{-1}\widetilde{\mathcal{L}}.$$

In other words, an L_∞-morphism is not a map from $\mathcal{L}$ to $\widetilde{\mathcal{L}}$ but a collection of (degree zero) maps:

$$F_n = p \circ F \Big|_{(\mathbf{s}^{-1}\mathcal{L})^{\otimes n}} : (\mathbf{s}^{-1}\mathcal{L})^{\otimes n} \mapsto \mathbf{s}^{-1}\widetilde{\mathcal{L}}, \qquad n \geq 1 \qquad (2.8)$$

compatible with the action of the symmetric groups S_n and satisfying certain equations involving Q and $\widetilde{Q}$.

For this reason we reserve a special arrow $\rightsquigarrow$ for L_∞-morphisms

$$F : \mathcal{L} \rightsquigarrow \widetilde{\mathcal{L}}. \qquad (2.9)$$

We call F_n (2.8) the structure maps of the L_∞ morphism (2.9).

It is not hard to see that the compatibility with with Q and $\widetilde{Q}$ implies that F_1 in (2.8) is a morphism between the cochain complexes $\mathbf{s}^{-1}\mathcal{L}$ and $\mathbf{s}^{-1}\widetilde{\mathcal{L}}$.

Definition 3 An L_∞ quasi-isomorphism *from $\mathcal{L}$ to $\widetilde{\mathcal{L}}$ is an L_∞ morphism (2.9) for which the map $F_1 : \mathbf{s}^{-1}\mathcal{L} \to \mathbf{s}^{-1}\widetilde{\mathcal{L}}$ is a quasi-isomorphism of cochain complexes.*

Following V. Hinich we have[2]

Theorem 1 (Lemma 4.2.1, [53]) *For every DG Lie algebra $\mathcal{L}$ there exists an L_∞ algebra structure Q^H on $H^\bullet(\mathcal{L})$ such that $\mathcal{L}$ is quasi-isomorphic to the L_∞-algebra*

$$(H^\bullet(\mathcal{L}), Q^H).$$

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[2]V. Hinich [53] proved this statement for ∞-versions of algebras over an arbitrary quadratic Koszul operad.

The structure map Q_1^H is zero and

$$Q^H(\gamma_1, \gamma_2) = (-1)^{|\gamma_1|+1}[\gamma_1, \gamma_2],$$

where $\gamma_1, \gamma_2 \in H^\bullet(\mathcal{L})$ and $[\,,\,]$ is the induced Lie bracket on $H^\bullet(\mathcal{L})$.

Thus, even if $\mathcal{L}$ is a non-formal DG Lie algebra, the Lie algebra structure on its cohomology $H^\bullet(\mathcal{L})$ can be corrected to an L_∞ algebra structure Q^H such that the L_∞ algebra $(H^\bullet(\mathcal{L}), Q^H)$ is quasi-isomorphic to $\mathcal{L}$.

The higher structure maps Q_n^H, $n > 2$ depend on various choices and formality of the DG Lie algebra (or more generally L_∞ algebra) $(\mathcal{L}, Q)$ means that these higher maps can be chosen to be all zeros.

In the concluding section of this article we give an example of a non-formal DG Lie algebra.

Remark. Using the higher structure maps Q_n^H one may construct operations which are independent of choices. These operations are known as Massey-Lie products [6], [50], [72], [77] and formality of a DG Lie algebra $\mathcal{L}$ implies that all the Massey-Lie products are zero. A good exposition on Massey-Lie products for the category of DG commutative algebras is given in Section 2 of [6].

2.2 From L_∞ algebras back to DG Lie algebras

As we see from Definition 2 every L_∞ algebra $(\mathcal{L}, Q)$ is defined by the DG cocommutative coalgebra $C(\mathcal{L})$ with the codifferential Q. Using this coalgebra we may construct a DG Lie algebra $\mathcal{R}(\mathcal{L}, Q)$ which is quasi-isomorphic to the L_∞ algebra $(\mathcal{L}, Q)$. As a graded Lie algebra, $\mathcal{R}(\mathcal{L}, Q)$ is the free Lie algebra generated by $C(\mathcal{L})$

$$\mathcal{R}(\mathcal{L}, Q) = \mathrm{Lie}\big(C(\mathcal{L})\big). \tag{2.10}$$

The differential on $\mathcal{R}(\mathcal{L}, Q)$ consists of two parts. To define the first part we use, in the obvious way, the codifferential Q. To define the second part we use the comultiplication Δ (2.4) on $C(\mathcal{L})$ viewing (2.10) as the dual version of the Harrison chain complex.

The construction of the free resolution $\mathcal{R}(\mathcal{L}, Q)$ for an L_∞ algebra $(\mathcal{L}, Q)$ is known in topology as *the rectification* [12]. We describe this construction in more details for a wider class of algebras in [36] (See Proposition 3 therein).

2.3 DG Lie algebras and Maurer-Cartan elements

Given a DG Lie algebra $\mathcal{L}$ with the differential ∂ and the Lie bracket $[\,,\,]$ over the field $\mathbb{K}$ we introduce the DG Lie algebra $\mathcal{L}[[\hbar]]$ over[3] the ring $\mathbb{K}[[\hbar]]$ extending ∂ and $[\,,\,]$ by $\mathbb{K}[[\hbar]]$-linearity.

Definition 4 *A* Maurer-Cartan (or MC) *element* α *of the DG Lie algebra* $\mathcal{L}$ *is a formal series* $\alpha \in \hbar\,\mathcal{L}^1[[\hbar]]$ *of degree* 1 *elements satisfying the equation*

$$\partial\alpha + \frac{1}{2}[\alpha, \alpha] = 0. \tag{2.11}$$

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[3]Here $\hbar$ is a formal deformation parameter.

MC elements may be formally compared to flat connections.

Let us consider the Lie algebra $\hbar \mathcal{L}^0[[\hbar]]$ of formal series of degree zero elements in $\mathcal{L}$. It is easy to see that the Lie algebra $\hbar \mathcal{L}^0[[\hbar]]$ is a projective limit of nilpotent Lie algebras:

$$\hbar \mathcal{L}^0[[\hbar]] \Big/ \hbar^N \mathcal{L}^0[[\hbar]] , \qquad N > 1 .$$

Therefore it can be exponentiated to the group

$$\mathfrak{G} = \exp(\hbar \mathcal{L}^0[[\hbar]]) . \tag{2.12}$$

This group acts on the MC elements of $\mathcal{L}$ according to the formula:

$$\exp(\xi)\, \alpha = \exp([\,,\xi])\, \alpha + f([\,,\xi])\, \partial \xi , \tag{2.13}$$

where f is the power series of the function

$$f(x) = \frac{e^x - 1}{x}$$

at the point $x = 0$. Two MC elements connected by the action of the group $\mathfrak{G}$ may be thought of as equivalent flat connections.

In this way we get the Goldman-Millson groupoid [49] $\mathrm{MC}(\mathcal{L})$ which captures the formal one-parameter deformation theory associated to the DG Lie algebra $\mathcal{L}$. Objects of this groupoid are MC elements of $\mathcal{L}$ and morphisms between two MC elements α_1 and α_2 are elements of the group $\mathfrak{G}$ (2.12) which transform α_1 to α_2.

We denote by $\pi_0(\mathrm{MC}(\mathcal{L}))$ the set of isomorphism classes of the Goldman-Millson groupoid $\mathrm{MC}(\mathcal{L})$.

Every morphism $\mu : \mathcal{L} \to \widetilde{\mathcal{L}}$ of DG Lie algebras gives us an obvious functor

$$\mu_* : \mathrm{MC}(\mathcal{L}) \to \mathrm{MC}(\widetilde{\mathcal{L}}) \tag{2.14}$$

from the groupoid $\mathrm{MC}(\mathcal{L})$ to the groupoid $\mathrm{MC}(\widetilde{\mathcal{L}})$.

According to [45], [49] and [76] we have the following theorem

Theorem 2 *If $\mu : \mathcal{L} \to \widetilde{\mathcal{L}}$ is a quasi-isomorphism of DG Lie algebras then μ_* induces an bijection between $\pi_0(\mathrm{MC}(\mathcal{L}))$ and $\pi_0(\mathrm{MC}(\widetilde{\mathcal{L}}))$.*

Theorem 2 is an immediate corollary of Proposition 4.9 in E. Getzler's paper [45]. According to W. Goldman, J. Millson, M. Schlessinger and J. Stasheff, [49], [76] every quasi-isomorphism μ from $\mathcal{L}$ to $\widetilde{\mathcal{L}}$ induces an equivalence of groupoids $\mathrm{MC}(\mathcal{L})$ and $\mathrm{MC}(\widetilde{\mathcal{L}})$ provided the DG Lie algebras $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ are concentrated in non-negative degrees.

It turns out that if $\mathcal{L}$ has elements in negative degrees then $\mathrm{MC}(\mathcal{L})$ can be upgraded to a higher groupoid [44]. For L_∞ algebras this idea was thoroughly developed by E. Getzler in [45]. Then these results of E. Getzler were generalized by A. Henriques in [51].

2.4 Twisting by a MC element

Given a MC element $\alpha \in \hbar \mathcal{L}[[\hbar]]$ of a DG Lie algebra $\mathcal{L}$ we may modify the DG Lie algebra structure on $\mathcal{L}[[\hbar]]$ by switching to the new differential

$$\partial + [\alpha,\,] . \tag{2.15}$$

It is the MC equation (2.11) which implies the identity $(\partial + [\alpha,\,])^2 = 0$.

We denote the DG Lie algebra $\mathcal{L}[[\hbar]]$ with the differential (2.15) and the original Lie bracket by $\mathcal{L}^\alpha$

$$\mathcal{L}^\alpha = (\mathcal{L}[[\hbar]], \partial + [\alpha, \,], [\,, \,]) \,.$$

Following D. Quillen [74] we call this procedure of modifying the DG Lie algebra *twisting*.

It is obvious that every morphism $\mu : \mathcal{L} \to \widetilde{\mathcal{L}}$ of DG Lie algebras extends by $\mathbb{K}[[\hbar]]$-linearity to the morphism from $\mathcal{L}^\alpha$ to $\widetilde{\mathcal{L}}^{\mu(\alpha)}$. We denote this morphism by μ^α.

We claim that

Proposition 1 (Proposition 1, [32]) *If $\mu : \mathcal{L} \to \widetilde{\mathcal{L}}$ is a quasi-isomorphism between DG Lie algebras then so is the morphism*

$$\mu^\alpha : \mathcal{L}^\alpha \to \widetilde{\mathcal{L}}^{\mu(\alpha)} \,.$$

For L_∞ algebras, the twisting procedure was described in [45].

3 Algebraic structures on the Hochschild complexes

Let us introduce the algebras we are interested in.

First, we recall from [43] that

Definition 5 *A graded vector space V is a* Gerstenhaber algebra *if it is equipped with a graded commutative and associative product $\wedge$ of degree 0 and a graded Lie bracket $[\,,\,]$ of degree -1. These operations have to be compatible in the sense of the following Leibniz rule*

$$[\gamma, \gamma_1 \wedge \gamma_2] = [\gamma, \gamma_1] \wedge \gamma_2 + (-1)^{|\gamma_1|(|\gamma|+1)} \gamma_1 \wedge [\gamma, \gamma_2] \,. \tag{3.1}$$

Second, we recall from [27] that

Definition 6 *A* precalculus *is a pair of a Gerstenhaber algebra $(V, \wedge, [,])$ and a graded vector space W together with*

- *a module structure $i_\bullet : V \otimes W \mapsto W$ of the graded commutative algebra V on W,*

- *an action $l_\bullet : \mathbf{s}^{-1}V \otimes W \mapsto W$ of the Lie algebra $\mathbf{s}^{-1}V$ on W which is compatible with $i_\bullet$ in the sense of the following equations*

$$i_a l_b - (-1)^{|a|(|b|+1)} l_b i_a = i_{[a,b]} \,, \qquad l_{a \wedge b} = l_a i_b + (-1)^{|a|} i_a l_b \,. \tag{3.2}$$

Furthermore,

Definition 7 *A* calculus *is a precalculus $(V, W, [,], \wedge, i_\bullet, l_\bullet)$ with a degree -1 unary operation δ on W such that*

$$\delta \, i_a - (-1)^{|a|} i_a \, \delta = l_a \,, \tag{3.3}$$

and[4] $\delta^2 = 0$.

The simplest examples of these algebraic structures come from geometry. More precisely, if M is smooth real manifold then the graded vector space $T^\bullet_{poly}(M)$ of polyvector fields on M is a Gerstenhaber algebra. The commutative product is simply the exterior product $\wedge$ and the Lie bracket is the Schouten-Nijenhuis bracket $[\,,\,]_{SN}$ [63]. This bracket is defined in the obvious way for vector fields and for functions:

$$[v_1, v_2]_{SN} = [v_1, v_2] \,, \qquad [v, a]_{SN} = v(a) \,, \qquad [a, b]_{SN} = 0 \,,$$

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[4]Although $\delta^2 = 0$, the operation δ is not considered as a part of the differential on W.

$$v, v_1, v_2 \in T_{poly}^1(M)\,, \qquad a, b \in T_{poly}^0(M) = C^\infty(M)\,.$$

Then it is extended by Leibniz rule (3.1) to all polyvector fields.

Adding to polyvector fields the graded vector space $\Omega^{-\bullet}(M)$ of exterior forms with the reversed grading

$$(T_{poly}^\bullet(M), \Omega^{-\bullet}(M)) \tag{3.4}$$

we get a calculus algebra. The module structure on $\Omega^{-\bullet}(M)$ over the commutative algebra $(T_{poly}^\bullet(M), \wedge)$ is defined by the contraction

$$i : T_{poly}^\bullet(M) \otimes \Omega^{-\bullet}(M) \to \Omega^{-\bullet}(M)\,,$$

the unary operation δ is the de Rham differential d and the Lie algebra module structure on $\Omega^{-\bullet}(M)$ over $(T_{poly}^{\bullet+1}(M), [\,,\,]_{SN})$ is given by the Lie derivative

$$l : T_{poly}^{\bullet+1}(M) \otimes \Omega^{-\bullet}(M) \to \Omega^{-\bullet}(M)\,,$$

$$l_\gamma = di_\gamma - (-1)^{|\gamma|} i_\gamma d\,. \tag{3.5}$$

Another example of calculus algebra comes from noncommutative geometry. To give this example we start with an arbitrary unital associative algebra A.

For every bimodule U over the algebra A we introduce the Hochschild cochain complex

$$C^\bullet(A, U) = \mathrm{Hom}(A^{\otimes \bullet}, U) \tag{3.6}$$

and the Hochschild chain complex

$$C_\bullet(A, U) = U \otimes A^{\otimes \bullet} \tag{3.7}$$

of A with coefficients in U.

We reserve the same notation ∂^{Hoch} both for the Hochschild coboundary operator

$$(\partial^{Hoch} P)(a_0, a_1, \ldots, a_k) = a_0 P(a_1, \ldots, a_k) - P(a_0 a_1, \ldots, a_k) + P(a_0, a_1 a_2, a_3, \ldots, a_k) - \ldots \tag{3.8}$$

$$+ (-1)^k P(a_0, \ldots, a_{k-2}, a_{k-1} a_k) + (-1)^{k+1} P(a_0, \ldots, a_{k-2}, a_{k-1}) a_k$$

on $C^\bullet(A, U)$ and for Hochschild boundary operator

$$\partial^{Hoch}(u, a_1, \ldots, a_m) = (u a_1, a_2, \ldots, a_m) - (u, a_1 a_2, a_3, \ldots, a_m) + \ldots \tag{3.9}$$

$$+ (-1)^{m-1}(u, a_1, \ldots, a_{m-2}, a_{m-1} a_m) + (-1)^m (a_m u, a_1, a_2, \ldots, a_{m-1})\,,$$

$$a_i \in A\,, \qquad u \in U$$

on $C_\bullet(A, U)$.

We reserve the notation $HH^\bullet(A, U)$ (resp. $HH_\bullet(A, U)$) for the Hochschild cohomology (resp. homology) groups of A with coefficients in U.

In the case $U = A$ we simplify the notation for the Hochschild complexes and for the (co)homology groups:

$$C^\bullet(A) = C^\bullet(A, A)\,, \tag{3.10}$$

$$C_\bullet(A) = C_{-\bullet}(A, A)\,, \tag{3.11}$$

$$HH^\bullet(A) = HH^\bullet(A, A)\,, \qquad HH_\bullet(A) = HH_{-\bullet}(A, A)\,. \tag{3.12}$$

For our purposes, we use the reversed grading on the Hochschild chains of A with coefficients in A.

Here are the five algebraic operations on the complexes $C^\bullet(A)$ and $C_\bullet(A)$ which play an important role:

- the cup-product $\cup$

$$P_1 \cup P_2(a_1, a_2, \ldots, a_{k_1+k_2}) = P_1(a_1, \ldots, a_{k_1}) P_2(a_{k_1+1}, \ldots, a_{k_1+k_2}), \qquad (3.13)$$

$$P_i \in C^{k_i}(A),$$

- the Gerstenhaber bracket $[\,,\,]_G$

$$[Q_1, Q_2]_G =$$

$$\sum_{i=0}^{k_1} (-1)^{ik_2} Q_1(a_0, \ldots, Q_2(a_i, \ldots, a_{i+k_2}), \ldots, a_{k_1+k_2}) - (-1)^{k_1 k_2} (1 \leftrightarrow 2), \qquad (3.14)$$

$$Q_i \in C^{k_i+1}(A),$$

- the contraction I_P of a Hochschild cochain $P \in C^k(A)$ with Hochschild chains

$$I_P(a_0, a_1, \ldots, a_m) = \begin{cases} (a_0 P(a_1, \ldots, a_k), a_{k+1}, \ldots, a_m), & \text{if } m \geq k, \\ 0, & \text{otherwise}, \end{cases} \qquad (3.15)$$

- the Lie derivative of Hochschild chains along a Hochschild cochain $Q \in C^{k+1}(A)$

$$L_Q(a_0, a_1, \ldots, a_m) = \sum_{i=0}^{m-k} (-1)^{ki}(a_0, \ldots, Q(a_i, \ldots, a_{i+k}), \ldots, a_m) + \qquad (3.16)$$

$$\sum_{j=m-k}^{m-1} (-1)^{m(j+1)} (Q(a_{j+1}, \ldots, a_m, a_0, \ldots, a_{k+j-m}), a_{k+j+1-m}, \ldots, a_j),$$

- and Connes' operator $B : C_\bullet(A) \to C_{\bullet-1}(A)$

$$B(a_0, \ldots, a_m) = \sum_{i=0}^{m} \Big((-1)^{mi} (1, a_i, \ldots, a_m, a_0, \ldots, a_{i-1}) \\ + (-1)^{mi} (a_i, 1, a_{i+1}, \ldots, a_m, a_0, \ldots, a_{i-1}) \Big). \qquad (3.17)$$

All these operations are compatible with the differential ∂^{Hoch} (3.8), (3.9). Therefore they induce the corresponding operations on the level of cohomology.

There are several identities involving the operations $[\,,\,]_G$, L, and B.

First, the Gerstenhaber bracket $[\,,\,]_G$ is a Lie bracket on $C^{\bullet+1}(A)$ and hence $C^{\bullet+1}(A)$ is a DG Lie algebra. The operation L (3.16) gives us an action of the DG Lie algebra $C^{\bullet+1}(A)$ on Hochschild chains. In other words,

$$L_{Q_1} L_{Q_2} - (-1)^{(|Q_1|+1)(|Q_2|+1)} L_{Q_2} L_{Q_1} = L_{[Q_1, Q_2]_G} \qquad (3.18)$$

and hence $C_\bullet(A)$ is a DG Lie algebra module over the DG Lie algebra $C^{\bullet+1}(A)$.

The Connes cyclic operator B (3.17) is used in the definitions of different variants of cyclic chain complex [27], [67]. All these variants have the form[5]

$$CC_\bullet^{\mathcal{W}}(A) = (C_\bullet(A)[[u]] \otimes_{\mathbb{K}[u]} \mathcal{W}, \partial^{Hoch} + uB), \qquad (3.19)$$

---

[5]This notation is due to E. Getzler.

where u is an auxiliary variable of degree 2 and $\mathcal{W}$ is a $\mathbb{K}[u]$-module.

We are interested in two particular cases:

— if $\mathcal{W} = \mathbb{K}[u]$ then $CC_\bullet^{\mathcal{W}}(A)$ is called the negative cyclic complex

$$CC_\bullet^-(A) = (C_\bullet(A)[[u]], \partial^{Hoch} + uB) \,, \tag{3.20}$$

— if $\mathcal{W} = \mathbb{K}$ with u acting by zero then $CC_\bullet^{\mathcal{W}}(A)$ is nothing but the Hochschild chain complex (3.11) of A.

Since the Connes cyclic operator B (3.17) is compatible with the "Lie derivative" L (3.16) in the sense of the equation

$$BL_P - (-1)^{|P|+1} L_P B = 0 \,, \qquad P \in C^\bullet(A)$$

any variant of the cyclic chain complex (3.19) is a DG module over the DG Lie algebra $C^{\bullet+1}(A)$.

Unfortunately, the cup-product $\cup$ and the Gerstenhaber bracket $[\,,\,]_G$ do not satisfy the Leibniz rule. So $\cup$ and $[\,,\,]_G$ do not give us a Gerstenhaber algebra structure on $C^\bullet(A)$. Similarly, the operations $\cup$, $[\,,\,]_G$, I, L, B do not give us a calculus algebra on the pair $(C^\bullet(A), C_\bullet(A))$. However, the required identities hold on the level of cohomology and we have

Proposition 2 (M. Gerstenhaber [43]) *The cup-product (3.13) and the bracket (3.14) induce on $HH^\bullet(A)$ a structure of a Gerstenhaber algebra.*

and

Proposition 3 (Yu. Daletski, I. Gelfand, and B. Tsygan [28]) *The operations (3.13), (3.14), (3.15), (3.16), and (3.17) induce on the pair*

$$(HH^\bullet(A), HH_\bullet(A))$$

a structure of a calculus.

Let us also recall the Van den Bergh duality theorem:

Theorem 3 (M. Van den Bergh, [87]) *If A is a finitely generated bimodule coherent[6] algebra of finite Hochschild dimension d,*

$$HH^m(A, A \otimes A) = \begin{cases} V_A & \text{if} \quad m = d \,, \\ 0 & \text{otherwise} \,, \end{cases} \tag{3.21}$$

where V_A is an invertible[7] A-bimodule then for every A-bimodule U

$$HH^\bullet(A, U) \cong HH_{d-\bullet}(A, V_A \otimes_A U) \,.$$

In Equation (3.21) $A \otimes A$ is considered as a bimodule over A with respect to the external A-bimodule structure. It is the internal A-bimodule structure which equips all the cohomology groups $HH^\bullet(A, A \otimes A)$ with a structure of A-bimodule.

We refer to V_A as *the Van den Bergh dualizing module* of A.

---

[6] An algebra A is called bimodule coherent if every map between finite rank free A-bimodules has a finitely generated kernel (see Definition 3.5.1 in [47]).

[7] A A-bimodule V is called invertible if there is a A-bimodule $\widetilde{V}$ such that $V \otimes_A \widetilde{V} \cong A$.

4 Formality theorems

The famous Kontsevich's formality theorem can be formulated as

Theorem 4 (M. Kontsevich, [57]) *Let M be a smooth real manifold and $A = C^\infty(M)$ be the algebra of smooth functions on M. Then the DG Lie algebra $(C^{\bullet+1}(A), \partial^{Hoch}, [\,,\,]_G)$ of Hochschild cochains is quasi-isomorphic to the graded Lie algebra $(T_{poly}^{\bullet+1}(M), [\,,\,]_{SN})$ of polyvector fields.*

Remark. Since $A = C^\infty(M)$ is a topological algebra the definition of Hochschild cochains for A requires some precaution. By the Hochschild cochains of the algebra $A = C^\infty(M)$ we mean the polydifferential operators on M.

To prove Theorem 4 M. Kontsevich gave [57] an explicit construction of an L_∞ quasi-isomorphism

$$\mathcal{K} : T_{poly}^{\bullet+1}(\mathbb{R}^d) \rightsquigarrow C^{\bullet+1}(C^\infty(\mathbb{R}^d)) \tag{4.1}$$

from the graded Lie algebra $T_{poly}^{\bullet+1}(\mathbb{R}^d)$ of polyvector fields to the DG Lie algebra $C^{\bullet+1}(C^\infty(\mathbb{R}^d))$ of polydifferential operators on $\mathbb{R}^d$. This construction involves very interesting integrals over compactified configuration spaces of points on the upper half plane. Tedious questions about choices of signs were thoroughly addressed in paper [5].

In order to extend this result to an arbitrary smooth manifold M. Kontsevich used what is called the Gelfand-Fuchs trick [41] or the formal geometry [42] in the sense of I.M. Gelfand and D.A. Kazhdan.

This step of globalization was discussed later in more details by A. Cattaneo, G. Felder and Tomassini in [23], by M. Kontsevich in Appendix 3 in [59], by the first author in [30], by A. Yekutieli in [95] and by M. Van den Bergh in [88].

To describe the first immediate corollary of Theorem 4 we recall from [7] and [9] that a star-product on the manifold M is an $\mathbb{R}[[\hbar]]$-linear associative product on $C^\infty(M)[[\hbar]]$ of the form

$$a * b = ab + \sum_{k=1}^{\infty} \hbar^k \Pi_k(a, b), \tag{4.2}$$

$$a, b \in C^\infty(M)[[\hbar]],$$

where Π_k are bidifferential operators.

Since $a * b = ab \mod \hbar$ the star-product (4.2) should be viewed as an associative and not necessarily commutative formal deformation of the ordinary product of functions on M.

Two star-products $*$ and $\tilde{*}$ are called equivalent if there exists a formal series of differential operators

$$T = Id + \hbar T_1 + \hbar^2 T_2 + \dots \tag{4.3}$$

which starts from the identity and intertwines the star-products $*$ and $\tilde{*}$:

$$T(a * b) = T(a) \,\tilde{*}\, T(b). \tag{4.4}$$

It easy to see that the associativity property of the star-product (4.2) is equivalent to the MC equation

$$\partial^{Hoch}\Pi + \frac{1}{2}[\Pi, \Pi]_G = 0$$

for the element

$$\Pi = \sum_{k=1}^{\infty} \hbar^k \Pi_k. \tag{4.5}$$

Thus MC elements of the DG Lie algebra $C^{\bullet+1}(C^\infty(M))$ are exactly the star-products on M.

Furthermore, it is not hard to identify the intertwiners between star-products with morphisms of the Goldman-Millson groupoid $\mathrm{MC}(C^{\bullet+1}(C^\infty(M)))$.

For the graded Lie algebra $T_{poly}^{\bullet+1}(M)$ of polyvector fields MC elements are the formal Poisson structures. These are formal power series of bivectors

$$\pi = \sum_{k=1}^{\infty} \hbar^k \pi_k, \qquad \pi_k \in T_{poly}^2(M) \tag{4.6}$$

satisfying the Jacobi relation

$$[\pi, \pi]_{SN} = 0. \tag{4.7}$$

We say that two formal Poisson structures π and $\widetilde{\pi}$ are equivalent if they are connected by the adjoint action of the group

$$\exp\left(\hbar\, T_{poly}^1(M)[[\hbar]]\right). \tag{4.8}$$

Thus combining Theorem 2 and Theorem 4 we get the following corollary

Corollary 1 *Equivalence classes of star products are in bijection with the equivalence classes of formal Poisson structures.*

Given a star-product $*$ on M, we call the corresponding equivalence class of formal Poisson structures *Kontsevich's class* of the star-product. Notice that the action of the group (4.8) does not change the first term π_1 of the series (4.6). Thus π_1 does not depend on the choice of the representative (4.6) of Kontsevich's class. We refer to π_1 as the Poisson bivector corresponding to the star-product $*$.

Twisting the DG Lie algebra $C^{\bullet+1}(C^\infty(M))$ by the MC element Π (4.5) corresponding to the star-product $*$ (4.2) we get the DG Lie algebra $C^{\bullet+1}(A_\hbar)$ of Hochschild cochains for the deformation quantization algebra $A_\hbar = (C^\infty(M)[[\hbar]], *)$.

Twisting the graded Lie algebra of polyvector fields $T_{poly}^{\bullet+1}(M)$ by the MC element π (4.6) we get the Poisson cochain complex [66]

$$(T_{poly}^{\bullet+1}(M)[[\hbar]], [\pi, \ ]_{SN}) \tag{4.9}$$

of π. Omitting the shift we refer to the cohomology of the complex (4.9) as the Poisson cohomology $HP^\bullet(M, \pi)$ of the formal Poisson structure π

$$HP^\bullet(M, \pi) := H^\bullet(T_{poly}^\bullet(M)[[\hbar]], [\pi, \ ]_{SN}). \tag{4.10}$$

Thus, combining Proposition 1 with Theorems 2 and 4, we get the following corollary

Corollary 2 *If $*$ is a star-product whose Kontsevich's class is represented by the formal Poisson structure π then the Hochschild cohomology of the deformation quantization algebra $A_\hbar = (C^\infty(M)[[\hbar]], *)$ is isomorphic to Poisson cohomology $HP^\bullet(M, \pi)$ of the formal Poisson structure π.*

4.1   Alternative approach to Theorem 4

As we mentioned above, the operations $\cup$ (3.13) and $[\ , \ ]_G$ (3.14) do not equip the Hochschild cochain complex $C^\bullet(A)$ with a Gerstenhaber algebra structure.

However, using the solution of Deligne's Hochschild cohomological conjecture [10], [61], [73], [83], [91] one can show that the operations $\cup$ and $[\,,\,]_G$ can be upgraded to an ∞-Gerstenhaber algebra whose multiplications are expressed in terms of the cup-product $\cup$ and insertions of cochains into a cochain.

This ∞-Gerstenhaber structure depends on the choice of Drinfeld's associator [38]. At this moment it is not known whether the homotopy type of this ∞-Gerstenhaber structure depends on this choice. However,

Theorem 5 (Theorem 2.1, [81]) *The L_∞ algebra part of the ∞-Gerstenhaber structure on Hochschild cochains coincides with the DG Lie algebra structure given by the Hochschild differential and the Gerstenhaber bracket (3.14).*

In 1998 the second author proposed a completely different proof [81] of Theorem 4 in the case when A is the polynomial algebra over an arbitrary field $\mathbb{K}$ of characteristic zero. We would like to refer the reader to excellent V. Hinich's exposition [53] of this approach.

In [81] it was shown that the Gerstenhaber algebra $T^\bullet_{poly}(\mathbb{K}^d)$ of polyvector fields on the affine space $\mathbb{K}^d$ is *intrinsically formal*. In other words, there is no room for cohomological obstructions to the formality of any ∞-Gerstenhaber algebra whose cohomology is the Gerstenhaber algebra $T^\bullet_{poly}(\mathbb{K}^d)$.

Therefore, the Hochschild cochain complex $C^\bullet(\mathbb{K}[x_1, \ldots, x_d])$ with the ∞-Gerstenhaber structure coming from the solution of Deligne's conjecture is formal.

Combining this observation with Theorem 5 we immediately deduce Theorem 4.

At this moment there is a more explicit proof of the formality for the ∞-Gerstenhaber structure for a wider class of algebras. More precisely,

Theorem 6 ([36]) *For every regular commutative algebra A over a field $\mathbb{K}$ of characteristic zero the ∞-Gerstenhaber algebra $C^\bullet(A)$ of Hochschild cochains is formal.*

An analogous statement in the Lie algebroid setting was proved by D. Calaque and M. Van den Bergh in [18].

4.2   Formality theorems for Hochschild and cyclic chains

Hochschild chains (3.11) enter this picture in a very natural way

Theorem 7 *Let M be a smooth real manifold and $A = C^\infty(M)$ be the algebra of smooth functions on M. Then the pair "DG Lie algebra $C^{\bullet+1}(A)$ and its DG module $C_\bullet(A)$" is quasi-isomorphic to the pair "graded Lie algebra $T^{\bullet+1}_{poly}(M)$ and its module $\Omega^{-\bullet}(M)$".*

This statement was formulated as a conjecture by the third author in [86]. It was proved in [78] by B. Shoikhet for the case $M = \mathbb{R}^d$. The step of globalization was performed in the thesis of the first author [31], [32].

As well as Theorem 4, Theorem 7 requires a technical amendment. More precisely, the space $C_k(A)$ of degree k Hochschild chains for the algebra $A = C^\infty(M)$ should be replaced by the space of ∞-jets near the main diagonal of the product $M^{\times k}$.

In [86] the third author also conjectured the formality of the cyclic complexes (3.19) as DG Lie algebra modules over $C^{\bullet+1}(A)$. In paper [84] the second author and the third author proposed a plan on how this cyclic conjecture can be proved.

However, in [93] T. Willwacher showed elegantly that Shoikhet's L_∞ quasi-isomorphism [78] is compatible with the Connes cyclic operator (3.17). This observation readily settled in the positive Tsygan's cyclic formality conjecture:

Theorem 8 (T. Willwacher, [93]) *Let M be a smooth real manifold and $A = C^\infty(M)$ be the algebra of smooth functions on M. If $\mathcal{W}$ is a $\mathbb{R}[u]$-module of finite projective dimension then the pair "DG Lie algebra $C^{\bullet+1}(A)$ and its DG module $CC_\bullet^{\mathcal{W}}(A)$" is quasi-isomorphic to the pair "graded Lie algebra $T_{poly}^{\bullet+1}(M)$ and its DG module $(\Omega^{-\bullet}(M)[[u]] \otimes_{\mathbb{R}[u]} \mathcal{W}, u\,d)$".*

If we set $\mathcal{W} = \mathbb{R}$ with u acting by zero then $CC_\bullet^{\mathcal{W}}(A)$ turns to the Hochschild chain complex and the DG module

$$(\Omega^{-\bullet}(M)[[u]] \otimes_{\mathbb{R}[u]} \mathcal{W}, u\,d)$$

turns to the module $\Omega^{-\bullet}(M)$ with the zero differential. Thus Theorem 7 is a corollary of Theorem 8.

Given a Lie algebra V and its module W, we can form the semi-direct product $V \oplus W$ in which W is an Abelian Lie algebra. It is clear that pairs "Lie algebra V and its module W" can be identified with such semi-direct products.

Using this idea one can generalize the twisting procedure we described in Subsection 2.4 to DG Lie algebra modules.

Thus, if $*$ is a star-product on M then twisting the DG Lie algebra module of negative cyclic chains (3.20) of the algebra $A = C^\infty(M)$ by the corresponding MC element Π (4.5) we get the negative cyclic complex

$$CC_\bullet^-(A_\hbar)$$

for the deformation quantization algebra $A_\hbar = (C^\infty(M)[[\hbar]], *)$.

Let π be a formal Poisson structure (4.6). Regarding π as a MC element of the graded Lie algebra $T_{poly}^\bullet(M)$ and twisting its DG Lie algebra module

$$(\Omega^{-\bullet}(M)[[u]], u\,d)$$

we get the DG Lie algebra module

$$(\Omega^{-\bullet}(M)[[u]][[\hbar]], l_\pi + u\,d) \tag{4.11}$$

over the DG Lie algebra $(T_{poly}^\bullet(M)[[\hbar]], [\pi,]_{SN})$ with the Lichnerowicz differential $[\pi,]_{SN}$.

Generalizing Proposition 1 to DG Lie algebra modules in the obvious way we get the following corollary of Theorem 8

Corollary 3 *If $*$ is a star-product on M whose Kontsevich's class is represented by the formal Poisson structure π then the complex (4.11) computes the negative cyclic homology*

$$HC_\bullet^-(A_\hbar) = H^\bullet(CC_\bullet^-(A_\hbar))$$

of the deformation quantization algebra $A_\hbar = (C^\infty(M)[[\hbar]], *)$.

Remark. Applying similar arguments to the Hochschild chain complex and to the exterior forms we get an isomorphism between Hochschild homology of $A_\hbar$ and the Poisson homology [14], [63] of π.

4.3 Formality of the ∞-calculus algebra $(C^\bullet(A), C_\bullet(A))$

To include the operations $\cup$ (3.13), $[\,,\,]_G$ (3.14), I (3.15), L (3.16), and B (3.17) on the pair

$$(C^\bullet(A), C_\bullet(A)) \tag{4.12}$$

into the picture we need to find a correct algebraic structure on (4.12).

As we already mentioned above the operations $\cup$ (3.13), $[\,,\,]_G$ (3.14), I (3.15), L (3.16), and B (3.17) satisfy the identities of the calculus algebra only up to homotopy. So a calculus algebra is not a correct algebraic structure for this situation.

Luckily the Kontsevich-Soibelman solution [62] of the chain version of Deligne's conjecture implies that the operations $\cup$, $[\,,\,]_G$, I, L, and B can be upgraded to an ∞-calculus algebra whose multiplications are expressed in terms of the cup-product $\cup$, insertions of cochains into a cochain, and insertions of components of a chain into cochains which respect the cyclic order on these components.

In [37] we show that

Theorem 9 (Corollary 4, [37]) *For every smooth real manifold M the ∞-calculus algebra*

$$\left(C^\bullet(C^\infty(M)), C_\bullet(C^\infty(M))\right) \tag{4.13}$$

is quasi-isomorphic to the calculus algebra

$$(T_\bullet^{poly}(M), \Omega^{-\bullet}(M)) \tag{4.14}$$

of polyvector fields and exterior forms on M.

We prove this theorem using our construction from [36] for the ∞-Gerstenhaber algebra on $C^\bullet(C^\infty(M))$ and the Morita equivalence between the algebra of differential operators on exterior forms and the algebra of differential operators on functions. Unlike in [36], we did not produce an explicit sequence of quasi-isomorphisms connecting (4.13) and (4.14). So Theorem 9 has a status of an existence theorem.

Theorem 4 from [37] implies that the algebra structure on (4.12) given by operations $[\,,\,]_G$, L and B do not have higher homotopy corrections inside the ∞-calculus structure. Thus the cyclic formality theorem of T. Willwacher (Theorem 8) is a corollary of Theorem 9.

We should remark that Kontsevich's L_∞ quasi-isomorphism $\mathcal{K}$ (4.1) has a subtle compatibility property with the cup-product. To formulate this property we set π to be a formal Poisson structure (4.6) on $\mathbb{R}^d$ and $A_\hbar = (C^\infty(\mathbb{R}^d)[[\hbar]], *)$ be a deformation quantization algebra whose Kontsevich's class is represented by π. Twisting Kontsevich's L_∞ quasi-isomorphism $\mathcal{K}$ (4.1) by π we get the L_∞ quasi-isomorphism

$$\mathcal{K}^\pi : \left(T_{poly}^{\bullet+1}(\mathbb{R}^d)[[\hbar]], [\pi,\,]_{SN}\right) \rightsquigarrow C^{\bullet+1}(A_\hbar) \tag{4.15}$$

which, in turn, induces an isomorphism from the Poisson cohomology

$$HP^\bullet(\mathbb{R}^d, \pi) := H^\bullet(T_{poly}^\bullet(\mathbb{R}^d)[[\hbar]], [\pi,\,]_{SN}). \tag{4.16}$$

to the Hochschild cohomology

$$HH^\bullet(A_\hbar) \tag{4.17}$$

of $A_\hbar$.

The exterior product $\wedge$ turns the Poisson cohomology (4.16) into a graded commutative algebra. Similarly, the cup-product $\cup$ (3.13) turns the Hochschild cohomology (4.17) into a graded commutative algebra. Due to [57] and [69] we have the following theorem

Theorem 10 *The isomorphism from the Poisson cohomology (4.16) to the Hochschild cohomology (4.17) induced by the L_∞ quasi-isomorphism (4.15) is an isomorphism of graded commutative algebras.*

This property was used in [17] by D. Calaque and M. Van den Bergh to prove Căldăraru's conjecture [20] on Hochschild structure of an algebraic variety.

Shoikhet's quasi-isomorphism of L_∞-modules [78] has a similar subtle compatibility property with the contraction I (3.15). This property is proved by D. Calaque and C. Rossi in [19]. In paper [19] the authors also applied this result to obtain a version of the Duflo isomorphism on coinvariants.

These results indicate that there should be a bridge between Kontsevich's construction [57] (resp. Shoikhet's construction [78]) and the construction of the second author [81] (resp. the construction in [37]).

4.4 Formality theorems for Hochschild and cyclic complexes in the algebraic geometry setting

In the algebraic geometry setting Theorem 4 has a formulation which does not need the amendment about the nature of cochains. In other words, we do not need to restrict ourselves to the polydifferential operators.

Theorem 11 *Let X be a smooth affine variety over a field $\mathbb{K}$ of characteristic zero and $A = \mathcal{O}_X(X)$ be the algebra of regular functions on X. Then the DG Lie algebra $(C^{\bullet+1}(A), \partial^{Hoch}, [\,,\,]_G)$ of Hochschild cochains is quasi-isomorphic to the graded Lie algebra $(\wedge_A^{\bullet+1} \operatorname{Der}(A), [\,,\,]_{SN})$ of polyderivations of A.*

This statement is an immediate corollary of Theorem 2 and Theorem 4 from [36]. It can be also extracted from M. Kontsevich's paper [59] on deformation quantization of algebraic varieties. However, Kontsevich's approach requires that the base field $\mathbb{K}$ contains reals.

Beyond the affine case it no longer makes sense to talk about global sections. Thus we need to reformulate the question for the sheaves of Hochschild cochains.

According to R. Swan [80] and A. Yekutieli [94] an appropriate candidate for the sheaf of Hochschild cochains on an arbitrary smooth algebraic variety X is the sheaf of polydifferential operators with regular coefficients. We denote this sheaf by $C^\bullet(\mathcal{O}_X)$. The Gerstenhaber bracket (3.14) equips $C^{\bullet+1}(\mathcal{O}_X)$ with a structure of a sheaf of DG Lie algebras.

Due to [36], [88], and [95] we have

Theorem 12 *For every smooth algebraic variety X over a field $\mathbb{K}$ of characteristic zero the sheaf of DG Lie algebra $C^{\bullet+1}(\mathcal{O}_X)$ is quasi-isomorphic to the sheaf $\wedge^{\bullet+1} T_X$ of polyvector fields with the Schouten-Nijenhuis bracket.*

For applications of this theorem to deformation quantization in the setting of algebraic geometry we refer the reader to papers [16], [59], and [95].

Hochschild chains can also be added into this picture. An appropriate candidate for the sheaf of Hochschild chains on an algebraic variety X is the sheaf of polyjets.

$$C_\bullet(\mathcal{O}_X) = \mathcal{H}om_{\mathcal{O}_X}(C^{-\bullet}(\mathcal{O}_X), \mathcal{O}_X), \tag{4.18}$$

where $\mathcal{H}om$ denotes the sheaf-Hom and $C^\bullet(\mathcal{O}_X)$ is considered with its natural left $\mathcal{O}_X$-module structure.

Using the isomorphism between X and the main diagonal of the product $X^{\times k}$ we may identify local sections of $C_k(\mathcal{O}_X)$ with ∞-jets on $X^{\times k}$ near its main diagonal.

To introduce sheaves of cyclic chains we introduce an auxiliary variable u of degree 2 and consider a $\mathbb{K}[u]$-module $\mathcal{W}$ as a constant sheaf on X. Then to every such module we assign

a sheaf of cyclic chains

$$CC_\bullet^{\mathcal{W}}(\mathcal{O}_X) = (C_\bullet(\mathcal{O}_X)[[u]] \otimes_{\mathbb{K}[u]} \mathcal{W}, \partial^{Hoch} + uB), \tag{4.19}$$

The operation L (3.16) equips $CC_\bullet^{\mathcal{W}}(\mathcal{O}_X)$ with a structure of a sheaf of DG Lie algebra modules over the sheaf of DG Lie algebras $C^\bullet(\mathcal{O}_X)$.

Theorem 4 and 5 from [37] implies the following statement

Theorem 13 *Let X be a smooth algebraic variety over a field $\mathbb{K}$ of characteristic zero. If $\mathcal{W}$ is a $\mathbb{R}[u]$-module of finite projective dimension then the pair "the sheaf of DG Lie algebras $C^{\bullet+1}(\mathcal{O}_X)$ and the sheaf of its DG modules $CC_\bullet^{\mathcal{W}}(\mathcal{O}_X)$" is quasi-isomorphic to the pair "the sheaf of graded Lie algebras $\wedge^{\bullet+1}T_X$ and the sheaf its DG modules $(\Omega_X^{-\bullet}[[u]] \otimes_{\mathbb{R}[u]} \mathcal{W}, u\, d)$".*

Remark. Using the construction of M. Van den Bergh [88] and A. Yekutieli [94] as well as the results of B. Shoikhet [78] and T. Willwacher [93] it is possible to prove Theorem 13 under the assumption that the base field $\mathbb{K}$ contains reals.

5 More application of formality theorems

Due to Corollary 1 the deformation class of a star product $*$ is uniquely determined by Kontsevich's class which is the equivalence class of a formal Poisson structure. For this reason Kontsevich's class is, sometimes, referred to as the characteristic class of a star-product.

In many cases deformation quantization algebras are not obtained via formality theorems. In such situations it may be hard to find Kontsevich's class of the deformation quantization algebra.

However, it is often possible to extract some information about representatives of this class using the homological properties of the deformation quantization algebra.

An example of this situation is provided by the unimodularity criterion from [33].

To describe this criterion we set X to be a smooth affine variety with the trivial canonical bundle. The triviality of the canonical bundle implies that there exists a nowhere vanishing top degree exterior form Vol on X.

A formal Poisson structure π (4.6) is called *unimodular* [15], [92] if there exists a formal power series

$$\mathrm{Vol}_\hbar = \mathrm{Vol} + \sum_{k=1}^\infty \hbar^k \mathrm{Vol}_k$$

of top degree exterior forms starting with a nowhere vanishing form Vol and such that

$$l_\pi \mathrm{Vol}_\hbar = 0, \tag{5.1}$$

where l_π is the Lie derivative (3.5).

Let $A_\hbar$ be a deformation quantization algebra $(A[[\hbar]], *)$ of X. It turns out that $A_\hbar$ has the Hochschild dimension equal to the dimension of X. Furthermore, $A_\hbar$ satisfies all the conditions of Theorem 3. Due to [33] we have the following homological unimodularity criterion:

Theorem 14 (Theorem 3, [33]) *The Van den Bergh dualizing module*

$$V_{A_\hbar} = HH^d(A_\hbar, A_\hbar \otimes A_\hbar) \tag{5.2}$$

of $A_\hbar = (A[[\hbar]], *)$ is isomorphic to $A_\hbar$ as a bimodule if and only if the formal Poisson structure π (4.6) corresponding to the star-product $*$ is unimodular.

The proof of this criterion is based on the algebraic geometry version of Theorem 7. This criterion was used in recent paper [39] by P. Etingof and V. Ginzburg to show that a certain family of Calabi-Yau algebras associated to del Pezzo surfaces has a non-trivial center.

Another application of formality theorems to quantization of unimodular Poisson structures is given in [25] by A.S. Cattaneo and G. Felder. In this paper they showed that if π is a unimodular Poisson structure on a smooth real manifold M then the deformation quantization algebra $(C^\infty(M)[[\hbar]], *)$ corresponding to π admits the following trace functional

$$f \mapsto \int_M f \, \mathrm{Vol}_\hbar \,, \tag{5.3}$$

where f is a compactly supported function on M and $\mathrm{Vol}_\hbar$ is a formal series of top degree forms starting with the volume form Vol satisfying the equation

$$l_\pi \mathrm{Vol} = 0 \,.$$

Algebraic index theorems [35], [84] for general Poisson manifolds give another tool for extracting information about Kontsevich's class of a star-product. These theorems express the isomorphism between Hochschild (resp. periodic cyclic) homology of a deformation quantization algebra and the Poisson homology (resp. de Rham cohomology) of the manifold or variety in terms of characteristic classes. For the lack of space we do not give more details about these theorems here and instead refer the reader to [35] and [84].

Many interesting examples of Poisson manifolds are obtained via reduction [21], [79]. In papers [24], [68] a "super"-version of Kontsevich's formality theorem is considered with the application to deformation quantization of reduced spaces in a fairly general situation. In both papers [24], [68] it was noticed that in general there may be obstructions to the construction of the star-product on the reduced space. This is not surprising because, in general, reduction procedure gives a geometric object which is not even a manifold. The presence of this obstruction does not mean that certain reduced spaces should be discarded. It is rather an indication that our formulation of the quantization problem for such "spaces" should be modified.

The question of functoriality in deformation quantization is closely related to the above question on the reduction. We suspect that the ideas from paper [13] may shed some light on this question.

There are also very interesting applications of formality theorems for Hochschild complexes to Lie theory. It is Kontsevich's formality theorem [57] which helped to solve [1], [2], [3], [4], [85] the long standing Kashiwara-Vergne conjecture [56]. Furthermore, using Kontsevich's formality theorem in [26] A.S. Cattaneo and C. Torossian generalized some of Lichnerowicz's results for the commutativity of the algebra of invariant differential operators and solved a long standing problem posed by M. Duflo for the expression of invariant differential operators on any symmetric spaces in exponential coordinates. They also developed a new method to construct characters for algebras of invariant differential operators.

6   An example of a non-formal DG Lie algebra

Let $A = \mathbb{K}[x^1, \ldots, x^d]$ be a polynomial algebra in d variables over $\mathbb{K}$. It is obvious that the DG Lie algebra structure on $C^{\bullet+1}(A)$ restricts to the truncated Hochschild complex $C^{\geq 1}(A)$ of A. Furthermore, due to commutativity of A the Hochschild differential vanishes

on the degree zero cochains. Therefore, the cohomology of the truncated Hochschild complex $C^{\geq 1}(A)$ is the vector space

$$\wedge^{\geq 1}\mathrm{Der}(A)$$

of polynomial polyvector fields on $\mathbb{K}^d$ of degrees ≥ 1.

In this subsection we prove that

Theorem 15 *If d is even then the DG Lie algebra $C^{\geq 1}(A)$ is non-formal.*

Remark 1. This theorem answers a question of the referee of our paper [36].

Remark 2. We suspect that the case of odd dimension can be considered similarly with a help of a regular constant Poisson structure of maximal rank.

Proof goes by contradiction. The formality of the DG Lie algebra $C^{\geq 1}(A)$ would still imply a bijection between the equivalence classes of star-products and the equivalence classes of formal Poisson structures on the affine space $\mathbb{K}^d$.

Let

$$F : \wedge^{\geq 1}\mathrm{Der}(A) \rightsquigarrow C^{\geq 1}(A) \tag{6.1}$$

be an L_∞ quasi-isomorphism from the graded Lie algebra $\wedge^{\geq 1}\mathrm{Der}(A)$ to the DG Lie algebra $C^{\geq 1}(A)$.

As we mentioned in Section 2 the structure map

$$F_1 : \wedge^{\geq 1}\mathrm{Der}(A) \rightarrow C^{\geq 1}(A) \tag{6.2}$$

is a quasi-isomorphism of cochain complexes.

In general, this quasi-isomorphism may differ from the standard Hochschild-Kostant-Rosenberg inclusion

$$I_{HKR} : \wedge^{\geq 1}\mathrm{Der}(A) \hookrightarrow C^{\geq 1}(A) \,. \tag{6.3}$$

by a coboundary term.

If this coboundary term is non-zero then applying Lemma 1 from [34] we modify the L_∞ quasi-isomorphism (6.1) in such a way that F_1 will coincide with (6.3). Thus, we may assume, without loss of generality, that

$$F_1 = I_{HKR} \,. \tag{6.4}$$

The correspondence between the formal Poisson structures (4.6) and the star-products is given by the assignment

$$\pi \;\mapsto\; *$$

$$a * b = ab + \sum_{k=1}^{\infty} \frac{1}{k!} F_k(\pi, \pi, \dots, \pi)(a, b) \,. \tag{6.5}$$

Furthermore, π allows us to twist the quasi-isomorphism (6.1) to the L_∞ morphism

$$F^\pi : \wedge^{\geq 1}\mathrm{Der}(A)[[\hbar]] \rightsquigarrow C^{\geq 1}(A_\hbar) \,, \tag{6.6}$$

where $A_\hbar$ is the algebra $A[[\hbar]]$ with the star-product (6.5) and the DG Lie algebra $\wedge^{\geq 1}\mathrm{Der}(A)[[\hbar]]$ is considered with the Lichnerowicz differential $[\pi, \,]_{SN}$.

According to [32] or [45] the structure maps of the twisted L_∞ quasi-isomorphism F^π are given by the formula

$$F^\pi_n(\gamma_1, \ldots, \gamma_n) = \sum_{k=1}^\infty \frac{1}{k!} F_{n+k}(\pi, \pi, \ldots, \pi, \gamma_1, \gamma_2, \ldots, \gamma_n). \tag{6.7}$$

An obvious analog of Proposition 1 allows us to conclude that the structure map of the first level

$$F^\pi_1 : (\wedge^{\geq 1} \mathrm{Der}(A)[[\hbar]], [\pi, \ ]_{SN}) \xrightarrow{\sim} C^{\geq 1}(A_\hbar) \tag{6.8}$$

is a quasi-isomorphism of cochain complexes.

Thus localizing (6.6) in $\hbar$ we get the following L_∞ quasi-isomorphism of DG Lie algebras

$$F^\pi : (\wedge^{\geq 1} \mathrm{Der}(A)((\hbar)), [\pi, \ ]_{SN}) \rightsquigarrow C^{\geq 1}(A_\hbar[\hbar^{-1}]). \tag{6.9}$$

The zeroth cohomology of the DG Lie algebra $(\wedge^{\geq 1} \mathrm{Der}(A)((\hbar)), [\pi, \ ]_{SN})$ is the Lie algebra of Poisson vector fields and the zeroth cohomology of the DG Lie algebra $C^{\geq 1}(A_\hbar[\hbar^{-1}])$ is the Lie algebra of derivations of $A_\hbar[\hbar^{-1}]$.

Therefore, the L_∞ quasi-isomorphism (6.9) gives us an isomorphism from the Lie algebra of Poisson vector fields of π to the Lie algebra $\mathrm{Der}(A_\hbar[\hbar^{-1}])$ of derivations of $A_\hbar[\hbar^{-1}]$.

Since d is even we may choose $\pi = \hbar\theta$ where θ is a non-degenerate constant bivector.

Let us denote by $*_W$ the Moyal-Weyl star-product which quantizes θ

$$a *_W b = \exp\left(\frac{\hbar}{2} \theta^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial y^j}\right) a(x) b(y) \Big|_{x=y}. \tag{6.10}$$

Due to (6.4) the equivalence class of $*_W$ corresponds to an equivalence class of a formal Poisson structure $\widetilde{\pi}$ which starts with $\hbar\theta$:

$$\widetilde{\pi} = \hbar\theta + \hbar^2 \widetilde{\pi}_2 + \hbar^3 \widetilde{\pi}_3 + \ldots. \tag{6.11}$$

Since θ is a non-degenerate bivector the formal Poisson structure $\widetilde{\pi}$ is equivalent to the original Poisson structure $\pi = \hbar\theta$. This statement can be easily deduced from the fact that, in the symplectic case, the Poisson cohomology is isomorphic to the de Rham cohomology.

Therefore, the Moyal-Weyl star-product $*_W$ is equivalent to $*$ (6.5). Hence the Lie algebra $\mathrm{Der}(A_\hbar{}^W)$ of derivations of the Weyl algebra

$$A_\hbar{}^W = (A((\hbar)), *_W) \tag{6.12}$$

is isomorphic to the Lie algebra of Poisson vector fields of $\hbar\theta$.

It is known that the Lie algebra $\mathrm{Der}(A_\hbar{}^W)$ of derivations of the Weyl algebra $A_\hbar{}^W$ is isomorphic to

$$\left((A/\mathbb{K})((\hbar)), [, \ ]_{*_W} \right), \tag{6.13}$$

where $[a, b]_{*_W} = a *_W b - b *_W a$. The desired isomorphism is defined by assigning to an element $a \in (A/\mathbb{K})((\hbar))$ the corresponding inner derivation:

$$a \ \mapsto \ [a, \ ]_{*_W}.$$

Here we identify the quotient $A/\mathbb{K}$ with the ideal of polynomials vanishing at the origin.

Similarly, the Lie algebra of Poisson vector fields of $\hbar\theta$ is isomorphic to

$$\left((A/\mathbb{K})((\hbar)), \{, \ \}_{\hbar\theta} \right), \tag{6.14}$$

where $\{a, b\}_{\hbar\theta} = \hbar\, i_\theta\, da\, db$. The desired isomorphism is defined by assigning to an element $a \in (A/\mathbb{K})((\hbar))$ the corresponding Hamiltonian vector field:

$$a \quad \mapsto \quad \{a,\ \}_{\hbar\theta}\,.$$

Thus we conclude that an L_∞ quasi-isomorphism F (6.1) would give us an isomorphism T from the Lie algebra (6.14) to the Lie algebra (6.13) over the field $\mathbb{K}((\hbar))$. Furthermore, equations (6.4) and (6.7) imply that this isomorphism T satisfies the following property

$$T(a) = a \mod \hbar\,, \qquad \forall \ a \in A/\mathbb{K}\,. \tag{6.15}$$

The Lie bracket in (6.13) has the form

$$[a, b]_{*W} = \hbar\theta^{ij}\partial_{x_i}a\,\partial_{x^j}b + \hbar^3 V(a, b) \mod \hbar^5\,, \tag{6.16}$$

where

$$V(a, b) = \frac{1}{24}\,\theta_1^{i_1 j_1}\theta_1^{i_2 j_2}\theta_1^{i_3 j_3}\,\partial_{x^{i_1}}\partial_{x^{i_2}}\partial_{x^{i_3}}\,a\,\,\partial_{x^{j_1}}\partial_{x^{j_2}}\partial_{x^{j_3}}\,b\,, \tag{6.17}$$

$$a, b \in (A/\mathbb{K})((\hbar))\,.$$

From deformation theory it follows that map

$$V : A/\mathbb{K} \otimes A/\mathbb{K} \to A/\mathbb{K}$$

given by the formula (6.17) is a cocycle for the Lie algebra $A/\mathbb{K}$ with the bracket

$$\{a, b\} = \theta^{ij}\,\partial_{x^i}a\,\partial_{x^j}b\,. \tag{6.18}$$

The existence of the isomorphism T satisfying the property (6.15) would imply that this cocycle is trivial. In other words, there should exist a linear map

$$P : A/\mathbb{K} \to A/\mathbb{K}$$

such that

$$V(a, b) = P(\{a, b\}) - \{P(a), b\} - \{a, P(b)\}\,, \tag{6.19}$$

$$a, b \in A/\mathbb{K}\,.$$

It is not hard to see that if the cocycle V (6.17) is trivial in the general case then it is trivial in the two-dimensional case with the canonical Poisson bracket

$$\{a, b\}(x, y) = \partial_x a(x, y)\partial_y b(x, y) - \partial_y a(x, y)\partial_x b(x, y)\,, \tag{6.20}$$

$$a, b \in \mathbb{K}[x, y]/\mathbb{K}\,.$$

Thus we may restrict ourselves to the two-dimensional case with the canonical Poisson bracket (6.20).

In this case the cocycle V reads

$$V(a, b) = \frac{1}{24}\left(\partial_x^3(a)\partial_y^3 b - 3\partial_x^2\partial_y(a)\partial_x\partial_y^2(b) + 3\partial_x\partial_y^2(a)\partial_x^2\partial_y(b) - \partial_y^3(a)\partial_x^3(b)\right)\,, \tag{6.21}$$

$$a, b \in \mathbb{K}[x, y]/\mathbb{K}\,.$$

For $a = x$ and $b = y$ equation (6.19) implies that

$$\partial_x P(x) + \partial_y P(y) \in \mathbb{K}\,.$$

Let $c_0 = \frac{1}{2}(\partial_x P(x) + \partial_y P(y))$. Then, setting $Q_1 = P(x) - c_0 x$ and $Q_2 = P(y) - c_0 y$, we get

$$\partial_x Q_1 + \partial_y Q_2 = 0.$$

Hence, there exists a polynomial $Q \in \mathbb{K}[x, y]$ such that $Q_1 = \{Q, x\}$ and $Q_2 = \{Q, y\}$.

Thus, adjusting P by a Hamiltonian vector field, we reduce it to the form in which

$$P(x) = c_0 x, \qquad P(y) = c_0 y. \tag{6.22}$$

Substituting to equation (6.19) quadratic monomials for a and linear for b we deduce that

$$P(x^2) = c_{11} x + c_{12} y,$$

$$P(xy) = c_{21} x + c_{22} y,$$

$$P(y^2) = c_{31} x + c_{32} y,$$

where $c_{ij} \in \mathbb{K}$. Next substituting quadratic monomials for a and b we deduce that $c_{12} = c_{31} = 0$, $c_{11} = 2c_{22}$ and $c_{32} = 2c_{21}$. Thus

$$P(x^2) = 2c_y x,$$

$$P(xy) = c_x x + c_y y, \tag{6.23}$$

$$P(y^2) = 2c_x y$$

for some constants $c_x, c_y \in \mathbb{K}$.

Adjusting P by the Hamiltonian vector field

$$\{c_x x - c_y y, \,\}$$

we kill the right hand sides in (6.23).

Thus we may assume that

$$P(x) = c_0 x, \quad P(y) = c_0 y,$$
$$P(x^2) = P(xy) = P(y^2) = 0. \tag{6.24}$$

Next, plugging in cubic monomials x^3, $x^2 y$, xy^2, y^3 for a and linear monomials for b in (6.19), we deduce that

$$P(x^3) = b_{11} x + b_{12} y - c_0 x^3,$$

$$P(x^2 y) = b_{21} x + b_{22} y - c_0 x^2 y,$$

$$P(xy^2) = b_{31} x + b_{32} y - c_0 xy^2, \tag{6.25}$$

$$P(y^3) = b_{41} x + b_{42} y - c_0 y^3,$$

where $b_{ij} \in \mathbb{K}$.

To get a further restriction we substitute cubic monomials for a and quadratic for b in (6.19). We get that all the coefficients b_{ij} should vanish. Thus

$$P(x^3) = -c_0 x^3, \qquad P(x^2 y) = -c_0 x^2 y,$$
$$P(xy^2) = -c_0 xy^2, \qquad P(y^3) = -c_0 y^3, \tag{6.26}$$

Every quartic monomial can be written as a Poisson bracket of two cubic monomials. Using this observation and equation (6.19) for a and b being cubic monomials we deduce that
$$P(x^n y^k) = -2c_0 x^n y^k$$
whenever $n + k = 4$.

To get the desired contradiction we, first, set $a = x^4$ and $b = y^3$ in (6.19) and get
$$6x - (12P(x^3 y^2) + 2c_0\{x^4, y^3\} + c_0\{x^4, y^3\}) \in \mathbb{K}$$
or equivalently
$$12P(x^3 y^2) - 6x + 3c_0 x^3 y^2 \in \mathbb{K}. \tag{6.27}$$

Second, plugging $a = x^3 y$ and $b = xy^2$ into (6.19) we get
$$-\frac{3}{2}x - (5P(x^3 y^2) + 2c_0\{x^3 y, xy^2\} + c_0\{x^3 y, xy^2\})$$
or equivalently
$$5P(x^3 y^2) - \frac{3}{2}x + 3c_0 x^3 y^2 \in \mathbb{K}. \tag{6.28}$$

The inclusion in (6.27) clearly contradicts to the inclusion in (6.28) and the theorem follows.
$\square$

Remark 1. The same expression (6.17) defines a cocycle for the Lie algebra A with the bracket $\{\,,\,\}_\theta$. It was shown by J. Vey in [89] that this cocycle is non-trivial. Here we had to work with the quotient $A/\mathbb{K}$ and this is why we had to redo the computation of J. Vey taking into account this modification.

Remark 2. It makes sense to consider a modification of the Weyl algebra which is defined over $\mathbb{K}$. This is the algebra A^W generated by x^i's satisfying the relations
$$x^i\,x^j - x^j\,x^i = \theta^{ij}\,,$$
where θ^{ij} is as above a non-degenerate antisymmetric constant matrix. It is known [8] that the Lie algebra of derivation of A^W is not isomorphic to the Lie algebra of the derivation of the corresponding Poisson algebra $\mathbb{K}[x^1, \ldots, x^d]$ with the bracket $\{\,,\,\}_\theta$. This fact is mentioned in [8] as a negative evidence for the conjecture about the automorphisms of the Weyl algebra.

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