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# Outline

- **Question:** What do algebras form?

- **Answer 1:** A category in categories ($HH^0$)
- **Derived Answer 1:** A category in dg cocategories (Hochschild cochains...)

- **Answer 2:** A 2-category with a trace functor ($HH_0$)
- **Derived Answer 2:** A category in dg cocategories with a trace functor (Hochschild chains...)

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- **Answer 2:** A 2-category with a trace functor ($HH_0$)
- **Derived Answer 2:** A category in dg cocategories with a trace functor (Hochschild chains...) up to homotopy

**Answer 1:** Algebras form a **2-category**.

- Objects: algebras $A, B, \ldots$
- 1-Morphisms: bimodules ${}_AM_B$
- 1-Composition: ${}_AM_B \otimes_B {}_BN_C$
- 2-Morphisms: morphisms of bimodules

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$$\{\text{maps of bimodules } {}_fB \to_g B\} \cong Z_A({}_gB_f) \cong HH^0(A, {}_gB_f)$$
$$M \mapsto M(1)$$
$$(M_b : b' \mapsto b \cdot b') \leftarrow\!\shortmid b$$

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$$\begin{aligned} \{\text{maps of bimodules } {}_fB \to_g B\} \cong Z_A({}_gB_f) &\cong HH^0(A, {}_gB_f) \\ M &\mapsto M(1) \\ (M_b : b' \mapsto b \cdot b') &\mapsfrom b \end{aligned}$$

Can we use Hochschild cohomology or cochains instead of $HH^0$?

**Derived Answer 1:** Algebras form a category in dg cocategories.

- Objects: algebras $A, B, \dots$
- Morphisms: a dg cocategory $Bar(Hoch(A, B))$
- Composition:
  $\bullet : Bar(Hoch(A, B)) \otimes Bar(Hoch(B, C)) \to Bar(Hoch(A, C))$
  associative map of dg cocategories

## Defining $Bar(Hoch(A, B))$

1. $Hoch(A, B)$ is a dg category with
   - Objects: algebra maps $f : A \to B$
   - Morphisms: $Hoch(A)(f, g) = (C^\bullet(A, {}_fB_g), {}_f\delta_g)$
   - Composition: cup product on cochains

$$\begin{aligned} \phi &\in C^p(A,_f B_g) \\ \psi &\in C^q(A,_g B_h) \\ (\phi \cup \psi)(a_1, \ldots, a_{p+q}) &= \pm\phi(a_1, \ldots, a_p)\psi(a_{p+1}, \ldots, a_q) \end{aligned}$$

## **Defining** $Bar(Hoch(A, B))$

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2. $Bar : DGCat \to DGCocat$

## Defining $Bar(Hoch(A, B))$

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   - Composition: cup product on cochains
2. $Bar : DGCat \to DGCocat$

   Bar(Hoch(A,B)) has the same objects as Hoch(A,B).

$$\begin{tikzcd}[column sep=huge]
A \arrow[r, bend left=70, "f_0"] \arrow[r, phantom, bend left=55, "\Downarrow\phi_1"] \arrow[r, bend left=40, "f_1"] \arrow[r, phantom, bend left=20, "\Downarrow\phi_2"] \arrow[r, "f_2"] \arrow[r, phantom, bend right=20, "\vdots"] \arrow[r, bend right=40, "f_{n-1}"'] \arrow[r, phantom, bend right=55, "\Downarrow\phi_n"] \arrow[r, bend right=70, "f_n"'] & B
\end{tikzcd}$$

A morphism from $f_0$ to $f_n$ in Bar(Hoch(A,B))

$$\Delta(\phi_1 \dots \phi_n) = \sum_{0 \le i \le n} \pm \phi_1 \dots \phi_i \otimes \phi_{i+1} \dots \phi_n$$

$$|\phi_1 \dots \phi_n| = \sum_{1 \le i \le n} |\phi_i| - n$$

$$d_{Bar(Hoch(A,B))} = \tilde{d}_{Hoch(A,B)} + d_\cup$$

**Derived Answer 1:** Algebra form a category in dg cocategories.

- Objects: algebras $A, B, \dots$
- Morphisms: a dg cocategory $Bar(Hoch(A, B))$
- Composition:
  $\bullet : Bar(Hoch(A, B)) \otimes Bar(Hoch(B, C)) \to Bar(Hoch(A, C))$
  associative map of dg cocategories

$Bar(Hoch(A, C))$ is the cofree dg cocategory on $Hoch(A, C)$:

$$\begin{tikzcd}
Bar(Hoch(A,B)) \otimes Bar(Hoch(B,C)) \arrow[r, "\bullet"] \arrow[dr, dotted, "\bullet"'] & Hoch(A,C) \\
& Bar(Hoch(A,C)) \arrow[u, two heads]
\end{tikzcd}$$

## $Bar(Hoch(A,C))$ is the cofree dg cocategory on $Hoch(A,C)$:

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\end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge]
A \arrow[r, bend left=70, "f_0\in Obj(Bar(Hoch(A,B)))"] \arrow[r, phantom, bend left=55, "\Downarrow\phi_1"] \arrow[r, bend left=35, "f_1"] \arrow[r, phantom, bend left=22, "\Downarrow\phi_2"] \arrow[r, "f_2"] \arrow[r, phantom, bend right=20, "\vdots"] \arrow[r, bend right=45, "f_{n-1}"] \arrow[r, phantom, bend right=60, "\Downarrow\phi_n"] \arrow[r, bend right=70, "f_n\in Obj(Bar(Hoch(A,B)))"'] & B \arrow[r, bend left, "g_0\in Obj(Bar(Hoch(B,C)))"] \arrow[r, phantom, "\Downarrow\psi"] \arrow[r, bend right, "g_1\in Obj(Bar(Hoch(B,C)))"'] & C \arrow[r, mapsto, "\bullet"] & A \arrow[r, bend left=60, "g_0f_0\in Obj(Hoch(A,C))"] \arrow[r, phantom, "\substack{\psi\{\phi_1,\dots,\phi_n\}\\ \Downarrow}"] \arrow[r, bend right=60, "g_1f_n\in Obj(Hoch(A,C))"'] & C
\end{tikzcd}$$

# $Bar(Hoch(A,C))$ is the cofree dg cocategory on $Hoch(A,C)$:

$$\begin{tikzcd}
Bar(Hoch(A,B)) \otimes Bar(Hoch(B,C)) \arrow[r, "\bullet"] \arrow[dr, dotted, "\bullet"'] & Hoch(A,C) \\
& Bar(Hoch(A,C)) \arrow[u, two heads]
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge]
A \arrow[r, bend left=70, "f_0\in Obj(Bar(Hoch(A,B)))"] \arrow[r, bend left=70, phantom, "\Downarrow\phi_1"' near start] \arrow[r, bend left=35, "f_1"] \arrow[r, bend left=35, phantom, "\Downarrow\phi_2"'] \arrow[r, "f_2"] \arrow[r, bend right=35, phantom, "\vdots"] \arrow[r, bend right=35, "f_{n-1}"] \arrow[r, bend right=70, phantom, "\Downarrow\phi_n"] \arrow[r, bend right=70, "f_n\in Obj(Bar(Hoch(A,B)))"'] & B \arrow[r, bend left, "g_0\in Obj(Bar(Hoch(B,C)))"] \arrow[r, phantom, "\Downarrow\psi"] \arrow[r, bend right, "g_1\in Obj(Bar(Hoch(B,C)))"'] & C \arrow[r, mapsto, "\bullet"] & A \arrow[r, bend left=60, "g_0f_0\in Obj(Hoch(A,C))"] \arrow[r, phantom, "\psi\{\phi_1,\ldots,\phi_n\}\Downarrow"] \arrow[r, bend right=60, "g_1f_n\in Obj(Hoch(A,C))"'] & C
\end{tikzcd}$$

$$\psi\{\phi_1,\ldots,\phi_n\}(a_1,\ldots,a_q)=\sum\pm\psi(f_0a_1,\ldots,f_0a_{i_1},\phi_1(a_{i_1+1},\ldots),f_1a_*,\ldots,f_1a_*,$$
$$\phi_2(a_*,\ldots),f_2a_*,\ldots,f_na_q)$$

## $Bar(Hoch(A,C))$ is the cofree dg cocategory on $Hoch(A,C)$:

$$\begin{tikzcd}
Bar(Hoch(A,B)) \otimes Bar(Hoch(B,C)) \arrow[r, "\bullet"] \arrow[dr, dotted, "\bullet"'] & Hoch(A,C) \\
& Bar(Hoch(A,C)) \arrow[u, two heads]
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge]
A \arrow[r, bend left=80, "f_0\in Obj(Bar(Hoch(A,B)))"] \arrow[r, bend left=40, "f_1"] \arrow[r, bend left=80, phantom, shift right=3, "\Downarrow\phi_1"] \arrow[r, bend left=40, phantom, shift right=3, "\Downarrow\phi_2"] \arrow[r, "f_2"] \arrow[r, phantom, shift right=5, "\vdots"] \arrow[r, bend right=40, "f_{n-1}"] \arrow[r, bend right=40, phantom, shift right=3, "\Downarrow\phi_n"] \arrow[r, bend right=80, "f_n\in Obj(Bar(Hoch(A,B)))"'] & B \arrow[r, bend left=30, "g_0\in Obj(Bar(Hoch(B,C)))"] \arrow[r, bend right=30, "g_1\in Obj(Bar(Hoch(B,C)))"'] \arrow[r, phantom, "\Downarrow\psi"] & C \arrow[r, mapsto, "\bullet"] & A \arrow[r, bend left=50, "g_0f_0\in Obj(Hoch(A,C))"] \arrow[r, bend right=50, "g_1f_n\in Obj(Hoch(A,C))"'] \arrow[r, phantom, "\psi\{\phi_1,\ldots,\phi_n\}\Downarrow"] & C
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, bend left=50, "f_0"] \arrow[r, bend right=50, "f_n"'] \arrow[r, phantom, "\Downarrow\phi"] & B \arrow[r, bend left=50, "g_0"] \arrow[r, bend right=50, "g_1"'] \arrow[r, phantom, "\Downarrow1\in k"] & C \arrow[r, mapsto, "\bullet"] & A \arrow[r, bend left=50, "g_0f_0"] \arrow[r, bend right=50, "g_1f_n"'] \arrow[r, phantom, "\Downarrow\phi"] & C
\end{tikzcd}
\qquad
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A \arrow[r, bend left=50, "f_0"] \arrow[r, bend right=50, "f_n"'] \arrow[r, phantom, "\Downarrow1\in k"] & B \arrow[r, bend left=50, "g_0"] \arrow[r, bend right=50, "g_1"'] \arrow[r, phantom, "\Downarrow\psi"] & C \arrow[r, mapsto, "\bullet"] & A \arrow[r, bend left=50, "g_0f_0"] \arrow[r, bend right=50, "g_1f_n"'] \arrow[r, phantom, "\Downarrow\psi"] & C
\end{tikzcd}$$

# $Bar(Hoch(A, C))$ is the cofree dg cocategory on $Hoch(A, C)$:

$$\begin{tikzcd}
Bar(Hoch(A,B)) \otimes Bar(Hoch(B,C)) \arrow[r, "\bullet"] \arrow[dr, dotted, "\bullet"'] & Hoch(A,C) \\
& Bar(Hoch(A,C)) \arrow[u, two heads]
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, bend left=60, "f_0 \in Obj(Bar(Hoch(A,B)))"] \arrow[r, phantom, bend left=45, "\Downarrow\phi_1"] \arrow[r, bend left=25, "f_1"] \arrow[r, phantom, bend left=15, "\Downarrow\phi_2"] \arrow[r, "f_2"] \arrow[r, phantom, bend right=15, "\vdots"] \arrow[r, bend right=30, "f_{n-1}"] \arrow[r, phantom, bend right=45, "\Downarrow\phi_n"] \arrow[r, bend right=60, "f_n \in Obj(Bar(Hoch(A,B)))"'] & B \arrow[r, bend left, "g_0 \in Obj(Bar(Hoch(B,C)))"] \arrow[r, phantom, "\Downarrow\psi"] \arrow[r, bend right, "g_1 \in Obj(Bar(Hoch(B,C)))"'] & C \arrow[r, mapsto, "\bullet"] & A \arrow[r, bend left, "g_0 f_0 \in Obj(Hoch(A,C))"] \arrow[r, phantom, "\psi\{\phi_1,\ldots,\phi_n\} \Downarrow"] \arrow[r, bend right, "g_1 f_n \in Obj(Hoch(A,C))"'] & C
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A \arrow[r, bend left, "f_0"] \arrow[r, phantom, "\Downarrow 1 \in k"] \arrow[r, bend right, "f_n"'] & B \arrow[r, bend left, "g_0"] \arrow[r, phantom, "\Downarrow\psi"] \arrow[r, bend right, "g_1"'] & C \arrow[r, mapsto, "\bullet"] & A \arrow[r, bend left, "g_0 f_0"] \arrow[r, phantom, "\Downarrow\psi"] \arrow[r, bend right, "g_1 f_n"'] & C
\end{tikzcd}$$

Braces are associative. (...)

# Outline

- **Question:** What do algebras form?

- **Answer 1:** A category in categories ($HH^0$)
- **Derived Answer 1:** A category in dg cocategories (Hochschild cochains...)

- **Answer 2:** A 2-category with a trace functor ($HH_0$)
- **Derived Answer 2:** A category in dg cocategories with a trace functor (Hochschild chains...) up to homotopy

## **Answer 2:** Algebras form a 2-category with a trace functor

### Definition

(Kaledin): A trace functor on a 2-category $\mathcal{C}$ is:

- for each $A \in Obj(\mathcal{C})$, a functor $TR_A : \mathcal{C}(A, A) \to k - mod$
- for each pair $A, B \in Obj(\mathcal{C})$, a natural transformation $\tau_!(A, B)$

$$\begin{tikzcd}
\mathcal{C}(A,B)\otimes\mathcal{C}(B,A) \arrow[rr, "\tau=flip"] \arrow[d, "m"'] & & \mathcal{C}(B,A)\otimes\mathcal{C}(A,B) \arrow[d, "m"'] \\
\mathcal{C}(A,A) \arrow[dr, "TR_A"'] & \tau_!(\overset{\Rightarrow}{A},B) & \mathcal{C}(B,B) \arrow[dl, "TR_B"] \\
& k-mod &
\end{tikzcd}$$

- such that $\tau_!(B, A) \circ \tau_!(C, B) \circ \tau_!(A, C) = id$

## **Answer 2:** Algebras form a 2-category with a trace functor

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(Kaledin): A <u>trace functor</u> on a 2-category $\mathcal{C}$ is:

- for each $A \in Obj(\mathcal{C})$, a functor $TR_A : \mathcal{C}(A,A) \to k - mod$
- for each pair $A, B \in Obj(\mathcal{C})$, a natural transformation $\tau_!(A,B)$
- such that $\tau_!(B,A) \circ \tau_!(C,B) \circ \tau_!(A,C) = id$

$$\begin{tikzcd}
 & & \mathcal{C}(C,A) \otimes \mathcal{C}(A,C) \otimes \mathcal{C}(B,C) \arrow[dd] \arrow[drr, "\tau"] & & \\
\mathcal{C}(A,B) \otimes \mathcal{C}(B,C) \otimes \mathcal{C}(C,A) \arrow[urr, "\tau"] \arrow[drr] & \tau_!(\overset{\Rightarrow}{A},C) & \tau_!(\overset{\Leftarrow}{B},A) & \tau_!(\overset{\Rightarrow}{C},B) & \mathcal{C}(B,C) \otimes \mathcal{C}(C,A) \otimes \mathcal{C}(A,B) \arrow[llll, bend right=10, "\tau"'] \arrow[dll] \\
 & & k - mod & &
\end{tikzcd}$$

**Answer 2:** Algebras form 2-category with a trace functor

Definition

(Kaledin): A trace functor on a 2-category $\mathcal{C}$ is:

- for each $A \in Obj(\mathcal{C})$, a functor $TR_A : \mathcal{C}(A, A) \to k - mod$
  $TR_A : \text{bimodule } {}_A M_A \mapsto M/[A, M] \cong HH_0(A, M)$

## **Answer 2:** Algebras form 2-category with a trace functor

### Definition

(Kaledin): A trace functor on a 2-category $\mathcal{C}$ is:

- for each $A \in Obj(\mathcal{C})$, a functor $TR_A : \mathcal{C}(A, A) \to k - mod$
  $TR_A$ : bimodule ${}_AM_A \mapsto M/[A, M] \cong HH_0(A, M)$
- for each pair $A, B \in Obj(\mathcal{C})$, a natural transformation $\tau_!(A, B)$

$$\begin{tikzcd}
\mathcal{C}(A,B) \otimes \mathcal{C}(B,A) \arrow[rr, "\tau=flip"] \arrow[d, "m"'] & & \mathcal{C}(B,A) \otimes \mathcal{C}(A,B) \arrow[d, "m"] \\
\mathcal{C}(A,A) \arrow[dr, "TR_A"'] & \tau_!(\overset{\Rightarrow}{A},B) & \mathcal{C}(B,B) \arrow[dl, "TR_B"] \\
& k - mod &
\end{tikzcd}$$

$$\frac{{}_AM_B \otimes_B {}_BN_A}{[A, M \otimes_B N]} \xrightarrow{\tau_!(A,B)} \frac{{}_BN_A \otimes_A {}_AM_B}{[B, N \otimes_A M]}$$

$$m \otimes n \mapsto n \otimes m$$

such that

$$\tau_!(B, A) \circ \tau_!(C, B) \circ \tau_!(A, C) = id$$

## **Answer 2:** Algebras form 2-category with a trace functor

### Definition

(Kaledin): A trace functor on a 2-category $\mathcal{C}$ is:

- for each $A \in Obj(\mathcal{C})$, a functor $TR_A : \mathcal{C}(A, A) \to k - mod$
  $TR_A : \text{bimodule } {}_AM_A \mapsto M/[A, M] \cong HH_0(A, M)$
- for each pair $A, B \in Obj(\mathcal{C})$, a natural transformation $\tau_!(A, B)$

$$\begin{tikzcd}
\mathcal{C}(A,B) \otimes \mathcal{C}(B,A) \arrow[rr, "\tau=flip"] \arrow[d, "m"'] & & \mathcal{C}(B,A) \otimes \mathcal{C}(A,B) \arrow[d, "m"'] \\
\mathcal{C}(A,A) \arrow[dr, "TR_A"'] & \overset{\Rightarrow}{\tau_!(A,B)} & \mathcal{C}(B,B) \arrow[dl, "TR_B"] \\
& k - mod &
\end{tikzcd}$$

$$\frac{{}_AM_B \otimes_B {}_BN_A}{[A, M \otimes_B N]} \xrightarrow{\tau_!(A,B)} \frac{{}_BN_A \otimes_A {}_AM_B}{[B, N \otimes_A M]}$$

$$m \otimes n \mapsto n \otimes m$$

such that
$\tau_!(B, A) \circ \tau_!(C, B) \circ \tau_!(A, C) = id$

Can we use Hochschild homology or chains instead of $HH_0$ to extend this to a trace functor on the category in dg cocategories?

# Massaging the definition of a trace functor

## Definition

(Kaledin): A <u>trace functor</u> on a 2-category $\mathcal{C}$ is:

- for each $A \in Obj(\mathcal{C})$, a functor $TR_A : \mathcal{C}(A,A) \to k - mod$

$$\mathcal{C}(A,A)(f,g) \otimes_k TR_A(g) \to TR_A(f)$$

- for each pair $A, B \in Obj(\mathcal{C})$, a natural transformation $\tau_!(A,B)$

$$\begin{tikzcd}
\mathcal{C}(A,B) \otimes \mathcal{C}(B,A) \arrow[rr, "\tau"] \arrow[d, "m"'] & & \mathcal{C}(B,A) \otimes \mathcal{C}(A,B) \arrow[d, "m"'] \\
\mathcal{C}(A,A) \arrow[dr, "TR_A"'] & \tau_!(\overset{\Rightarrow}{A},B) & \mathcal{C}(B,B) \arrow[dl, "TR_B"] \\
& k - mod &
\end{tikzcd}$$

- such that $\tau_!(B,A) \circ \tau_!(C,B) \circ \tau_!(A,C) = id$

## Massaging the definition of a trace functor

**Definition**

(Kaledin): A trace functor on a category in k-linear categories $\mathcal{C}$ is:

- for each $A \in Obj(\mathcal{C})$, a functor $TR_A : \mathcal{C}(A,A) \to k - mod$

$$\mathcal{C}(A,A)(f,g) \otimes_k TR_A(g) \to TR_A(f)$$

- for each pair $A, B \in Obj(\mathcal{C})$, a natural transformation $\tau_!(A,B)$

$$\begin{tikzcd}
\mathcal{C}(A,B) \otimes \mathcal{C}(B,A) \arrow[rr, "\tau"] \arrow[d, "m"'] & & \mathcal{C}(B,A) \otimes \mathcal{C}(A,B) \arrow[d, "m"'] \\
\mathcal{C}(A,A) \arrow[dr, "TR_A"'] & \tau_!(\overset{\Rightarrow}{A},B) & \mathcal{C}(B,B) \arrow[dl, "TR_B"] \\
& k - mod &
\end{tikzcd}$$

- such that $\tau_!(B,A) \circ \tau_!(C,B) \circ \tau_!(A,C) = id$

## Massaging the definition of a trace functor

### Definition

(Kaledin): A <u>trace functor</u> on a category in k-linear categories $\mathcal{C}$ is:

- for each $A \in Obj(\mathcal{C})$, a left module $T(A)$ over $\mathcal{C}(A, A)$

$$\mathcal{C}(A,A)(f,g) \otimes_k TR_A(g) \to TR_A(f)$$

- for each pair $A, B \in Obj(\mathcal{C})$, a natural transformation $\tau_!(A, B)$

$$\begin{tikzcd}
\mathcal{C}(A,B) \otimes \mathcal{C}(B,A) \arrow[rr, "\tau"] \arrow[d, "m"'] & & \mathcal{C}(B,A) \otimes \mathcal{C}(A,B) \arrow[d, "m"'] \\
\mathcal{C}(A,A) \arrow[dr, "TR_A"'] & \tau_!(\overset{\Rightarrow}{A},B) & \mathcal{C}(B,B) \arrow[dl, "TR_B"] \\
& k - mod &
\end{tikzcd}$$

- such that $\tau_!(B, A) \circ \tau_!(C, B) \circ \tau_!(A, C) = id$

# Massaging the definition of a trace functor

## Definition

(Kaledin): A trace functor on a category in k-linear categories $\mathcal{C}$ is:

- for each $A \in Obj(\mathcal{C})$, a left module $T(A)$ over $\mathcal{C}(A,A)$

$$\mathcal{C}(A,A)(f,g) \otimes_k TR_A(g) \to TR_A(f)$$

- for each pair $A, B \in Obj(\mathcal{C})$, a map of modules $\tau_!(A,B) : m^* T(A) \to \tau^* m^* T(B)$ over $\mathcal{C}(A,B) \otimes \mathcal{C}(B,A)$

$$\begin{tikzcd}
\mathcal{C}(A,B) \otimes \mathcal{C}(B,A) \arrow[rr, "\tau"] \arrow[d, "m"'] & & \mathcal{C}(B,A) \otimes \mathcal{C}(A,B) \arrow[d, "m"] \\
\mathcal{C}(A,A) \arrow[dr, "TR_A"'] & \tau_!(\overset{\Rightarrow}{A},B) & \mathcal{C}(B,B) \arrow[dl, "TR_B"] \\
& k - mod &
\end{tikzcd}$$

- such that $\tau_!(B,A) \circ \tau_!(C,B) \circ \tau_!(A,C) = id$

## Massaging the definition of a trace functor

Definition

(Kaledin): A trace functor on a category in k-linear categories $\mathcal{C}$ is:

- for each $A \in Obj(\mathcal{C})$, a left module $T(A)$ over $\mathcal{C}(A, A)$

$$\mathcal{C}(A,A)(f,g) \otimes_k TR_A(g) \to TR_A(f)$$

- for each pair $A, B \in Obj(\mathcal{C})$, a map of modules $\tau_!(A,B) : m^* T(A) \to \tau^* m^* T(B)$ over $\mathcal{C}(A,B) \otimes \mathcal{C}(B,A)$

$$\begin{tikzcd}
\mathcal{C}(A,B) \otimes \mathcal{C}(B,A) \arrow[rr, "\tau"] \arrow[d, "m"'] & & \mathcal{C}(B,A) \otimes \mathcal{C}(A,B) \arrow[d, "m"'] \\
\mathcal{C}(A,A) \arrow[dr, "TR_A"'] & \tau_!(\overset{\Rightarrow}{A},B) & \mathcal{C}(B,B) \arrow[dl, "TR_B"] \\
& k - mod &
\end{tikzcd}$$

- such that $\tau^{*2}\tau_!(B,A) \circ \tau^*\tau_!(C,B) \circ \tau_!(A,C) = id$

**Definition**

Let $\mathcal{C}$ be a category in $k$-linear categories. Let $\chi(\mathcal{C})$ be the $k$-linear category with

- Objects $=$ $\{A \to A,\ A \to B \to A,\ A \to B \to C \to A : A, B, C \in Obj(\mathcal{C})\}$
- Morphisms $= \{$linear combinations of compositions of rotations $\tau_i$ s.t. $\tau_i^{i+1} = id,\ i = 0, 1, 2\}$.

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A <u>trace functor</u> on a category $\mathcal{C}$ in k-linear categories gives a functor

$$\chi(\mathcal{C}) \to \mathcal{D} = \{(k\text{-linear category, module})\}$$

$$\begin{aligned}
(A \to A) &\mapsto (\mathcal{C}(A, A), T(A)) \\
(A \to B \to A) &\mapsto (\mathcal{C}(A, B) \otimes \mathcal{C}(B, A), m^* T(A)) \\
(A \to B \to C \to A) &\mapsto (\mathcal{C}(A, B) \otimes \mathcal{C}(B, C) \otimes \mathcal{C}(C, A), m^{*2} T(A)) \\
\tau_1 : (A{\to}B{\to}A){\to}(B{\to}A{\to}B) &\mapsto \tau_!(A, B) \\
\tau_2 : (A{\to}B{\to}C{\to}A){\to}(C{\to}A{\to}B{\to}C) &\mapsto m^*\tau_!(A, C),\ m : \mathcal{C}(A, B) \otimes \mathcal{C}(B, C) \to \mathcal{C}(A, C) \\
\tau_1^2 = id,\ \tau_2^3 = id &\mapsto \text{relations in the definition of trace functor}
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- Morphisms $=$ $\{$linear combinations of compositions of rotations $\tau_i$ s.t. $\tau_i^{i+1} = id,\ i = 0, 1, 2\}$. Why stop at n=2? What about $\delta$, $\sigma$?

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### Definition

Let $\mathcal{C}$ be a category in dg cocategories. Let $\chi_\infty(\mathcal{C})$ be the dg category with

- Objects $= \{A_0 \to \ldots \to A_n \to A_0 : A_i \in Obj(\mathcal{C}), n \geq 0\}$
- Morphisms $= \{$linear combinations of compositions of

$$\begin{aligned} \textit{rotations } \tau_n &: \mathcal{A} \mapsto (A_n \to A_0 \to \ldots \to A_n) \\ \textit{coboundaries } \delta_{j,n} &: \mathcal{A} \mapsto (A_0 \to \ldots \to A_j \to A_{j+2\,(mod\ n+1)} \to \ldots \to A_0) \\ \textit{codegeneracies } \sigma_{i,n} &: \mathcal{A} \mapsto (A_0 \to \ldots \to A_i \to A_i \to \ldots \to A_0) \end{aligned}$$

where $\mathcal{A} := (A_0 \to \ldots \to A_n \to A_0)$, subject to the cyclic relations$\}[0]$

### Definition

Let $\mathcal{D}_\infty$ be the dg category with

- Objects $= \{(\underset{B}{\text{dg cocategory}}, \underset{C}{\text{dg comodule}})\}$
- Morphisms:

$$\mathcal{D}^p_\infty\big((B_1, C_1), (B_0, C_0)\big) := \left\{ \begin{matrix} F : B_1 \to B_0 \textit{ dg functor}, \\ F_! : C_1 \to F^* C_0 \textit{ degree-p linear map} \end{matrix} \right\}$$

$$d_{D_\infty}(F, F_!) = (F, [d, F_!] = {\scriptstyle d_{F^* C_0} \circ F_! \pm F_! \circ d_{C_1}})$$

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For us, $F^* C_0$ is the categorified version of co-extension of scalars:

$$F^* C_0 = ker(B_1 \otimes C_0 \rightrightarrows B_1 \otimes B_0 \otimes C_0)$$

**Question:** Can we give a dg functor

$$\chi_\infty(\mathcal{C}) \to \mathcal{D}_\infty$$

$$\text{where } (A_0{\to}\ldots{\to}A_n{\to}A_0) \mapsto \left(\begin{array}{c} B(A_0{\to}\ldots{\to}A_n{\to}A_0){:=} \\ {:=}Bar(Hoch(A_0,A_1)){\otimes}\ldots{\otimes}Bar(Hoch(A_n,A_0)), \\ C(A_0{\to}\ldots{\to}A_n{\to}A_0) \end{array}\right)?$$

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**Answer:** No, but we can give an $A_\infty$-functor.
This will imply that algebras form a category in dg cocategories with a trace functor up to homotopy.

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This will imply that algebras form a category in dg cocategories with a trace functor up to homotopy.

**Rest of this talk:**

- Define dg comodules $C(A_0 \to \ldots \to A_0)$ using Hochschild chains
- Describe the $A_\infty$-functor, and in particular the role of homotopies

## Definition

A **dg comodule** $C$ over a dg cocategory $B$ consists of the following data:

- for each object $f \in B$, a complex $C^\bullet(f)$, and
- maps of complexes

$$\Delta_C(f) : C^\bullet(f) \to \prod_{g \in Obj(B)} B^\bullet(f,g) \otimes C^\bullet(g).$$

such that the following diagrams for coassociativity and counitality commute:

$$\begin{tikzcd}
C^\bullet(f) \arrow[r, "\Delta_C"] \arrow[d, "\Delta_C"'] & \prod\limits_{g \in Obj(B)} B^\bullet(f,g) \otimes C^\bullet(g) \arrow[d, "id_B \otimes \Delta_C"] \\
\prod\limits_{g \in Obj(B)} B^\bullet(f,g) \otimes C^\bullet(g) \arrow[r, "\Delta_B \otimes id_C"] & \prod\limits_{\substack{g,g' \in \\ Obj(B)}} B^\bullet(f,g) \otimes B^\bullet(g,g') \otimes C^\bullet(g')
\end{tikzcd}$$

$$\begin{tikzcd}
C^\bullet(f) \arrow[r, "\Delta_C"] \arrow[dr, "id"'] & \prod\limits_{g \in Obj(B)} B^\bullet(f,g) \otimes C^\bullet(g) \arrow[d, "\epsilon_B \otimes id_C"] \\
& C^\bullet(f)
\end{tikzcd}$$

Fix algebras $A_0, \ldots, A_n$. Let $\mathcal{A} = (A_0 \to \ldots \to A_n \to A_0)$
Define a dg comodule $C(\mathcal{A})$ over $B(\mathcal{A})$:

$$C(\mathcal{A})^\bullet(\underbrace{A_0 \overset{f_{0,0}}{\to} \ldots \to A_n \overset{f_{n,0}}{\to} A_0}_{\in Obj(B(\mathcal{A}))}) :=$$

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$$:= \left\{ \begin{tikzcd} A_0 \arrow[r, bend left=70, "f_{0,0}"] \arrow[r, bend left=30, "f_{0,1}"] \arrow[r, bend right=30, "f_{0,k_0}"'] \arrow[r, phantom, shift left=4, "\Downarrow\phi_{0,1}"] \arrow[r, phantom, "\vdots"] \arrow[rrrr, bend right=40, "id_{A_0}"'] \arrow[rrrr, phantom, shift right=5, "\alpha"] & A_1 \arrow[r, dotted, bend left=70] \arrow[r, dotted, bend left=30] \arrow[r, dotted, bend right=30] \arrow[r, phantom, "\vdots"] & \cdots \arrow[r, dotted, bend left=70] \arrow[r, dotted, bend left=30] \arrow[r, dotted, bend right=30] \arrow[r, phantom, "\vdots"] & A_n \arrow[r, bend left=70, "f_{n,0}"] \arrow[r, bend left=30, "f_{n,1}"] \arrow[r, bend right=30, "f_{n,k_n}"'] \arrow[r, phantom, shift left=4, "\Downarrow\phi_{n,1}"] \arrow[r, phantom, "\vdots"] & A_0 \end{tikzcd} = \begin{matrix} (\phi_{0,1}|\ldots|\phi_{0,k_0})\otimes\ldots\otimes(\phi_{n,1}|\ldots|\phi_{n,k_n})\otimes\alpha: \\ s.t.\ \phi_{i,j}\in C^{\bullet}(A_{i}, {}_{f_{j-1}}A_{i+1}{}_{f_j}), \\ \alpha\in C_{-\bullet}(A_0, {}_{f_{n,k_n}}\ldots{}_{f_{0,k_0}}A_0) \end{matrix} \right\}$$

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$$d_{C(A_0\to\dots\to A_0)} = d_B \otimes id_{C_{-\bullet}} + id_B \otimes b + \tilde{\iota}$$

where $\tilde{\iota}$ is given as follows:

$$\begin{tikzcd}
C(A_0 \to \dots \to A_0)^\bullet(f_{0,0}, \dots, f_{n,0}) \arrow[r, "\tilde{\iota}"] \arrow[dr, "\tilde{\iota}", "\text{extend as a coderivation}"'] & C_{-\bullet}(A_0{}_{,f_{n,0}\dots f_{0,0}}A_0) \\
& C(A_0 \to \dots \to A_0)^\bullet(f_{0,0}, \dots, f_{n,0}) \arrow[u, two heads]
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\end{tikzcd}$$

$$\tilde{\iota}\big((\phi_{0,1}|\dots|\phi_{0,k_0})\otimes\dots\otimes(\phi_{n,1}|\dots|\phi_{n,k_n})\otimes\alpha\big) = \iota_{(\phi_{0,1}|\dots|\phi_{0,k_0})\bullet\dots\bullet(\phi_{n,1}|\dots|\phi_{n,k_n})}\alpha$$

$$\iota_\phi\big(a_0 \otimes \dots a_p\big) = \pm\phi\big(a_{d+1}, \dots, a_p\big) \cdot a_0 \otimes a_1 \otimes \dots \otimes a_d \quad \text{where } |\phi| = p - d$$

Give an $A_\infty$-functor

$$\chi_\infty(\mathcal{C}) \to \mathcal{D}_\infty$$

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$$\text{Generating Morphisms:} \quad \sigma_{i,n} \mapsto \begin{pmatrix} B(\mathcal{A}) \xrightarrow{\hat{\sigma}_{i,n}} B(\sigma_{i,n}\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{id} \hat{\sigma}^*_{i,n} C(\sigma_{i,n}\mathcal{A}) \end{pmatrix}$$

$$\delta_{j,n} \mapsto \begin{pmatrix} B(\mathcal{A}) \xrightarrow{\hat{\delta}_{j,n}} B(\delta_{j,n}\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{id} \hat{\delta}^*_{j,n} C(\delta_{j,n}\mathcal{A}) \end{pmatrix} \quad \tau_n \mapsto \begin{pmatrix} B(\mathcal{A}) \xrightarrow{\hat{\tau}_n} B(\tau_n\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{\tau_{n!}} \hat{\tau}^*_n C(\tau_n\mathcal{A}) \end{pmatrix}$$