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ABSTRACT

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CHAPTER 1

$B(n)$ and $C(n)$

1.1. Motivation of this chapter

In this chapter, we introduce the main characters/objects of study, $B(n)$ and $C(n)$, $n \in \mathbb{N}$. The $B(n)$'s are dg cocategories constructed using Hochschild cochains. Each $C(n)$ is a dg comodule over $B(n)$ constructed using an action of Hochschild cochains on Hochschild chains. We start with definitions for the less-widely-used concepts, and show that the main characters are conilpotent.

See Appendix C for definitions of known operations on Hochschild chains and cochains as well as our notation and conventions.

1.2. Dg cocategories: $B(n)$

1.2.1. Background on dg cocategories

Definition 1.2.1. A **dg cocategory** is a cocategory enriched over chain complexes. More explicitly, a dg cocategory B consists of the following data:

- A collection of objects denoted $Obj(B)$;
- For each pair of objects, $x, z \in Obj(B)$, a complex $B^\bullet(x, z)$ and a morphism of complexes

$$\Delta_B(x, z) : B^\bullet(x, z) \to \prod_{y \in Obj(B)} B^\bullet(x, y) \otimes B^\bullet(y, z)$$

  such that the following diagrams commute (coassociativity):

$$\begin{tikzcd} B^\bullet(x, z) \arrow[r, "\Delta_B(x,z)"] \arrow[d, "\Delta_B(x,z)"'] & \prod_{y \in Obj(B)} B^\bullet(x, y) \otimes B^\bullet(y, z) \arrow[d, "\prod_y id_{B(x,y)} \otimes \Delta_B(y,z)"] \\ \prod_{y \in Obj(B)} B^\bullet(x, y) \otimes B^\bullet(y, z) \arrow[r, "\prod_y \Delta_B(x,y) \otimes id_{B(y,z)}"] & \prod_{y, y' \in Obj(B)} B^\bullet(x, y) \otimes B^\bullet(y, y') \otimes B^\bullet(y', z) \end{tikzcd}$$

- For each pair of objects, $x, z \in Obj(B)$, a morphism of complexes

$$\epsilon_B(x, z) : B^\bullet(x, z) \to k$$

where k is the ground field considered as a chain complex concentrated in degree 0 and $\epsilon_B(x,z) = 0$ if $x \neq z$, such that the following diagrams commute (counitality):

$$\begin{tikzcd}
B^\bullet(x,z) \arrow[r, "\Delta_B(x,z)"] \arrow[d, "\Delta_B(x,z)"'] \arrow[dr, "id"] & \prod\limits_{y \in Obj(B)} B^\bullet(x,y) \otimes B^\bullet(y,z) \arrow[d, "\prod_y \epsilon_B(x,y) \otimes id_{B(y,z)}"] \\
\prod\limits_{y \in Obj(B)} B^\bullet(x,y) \otimes B^\bullet(y,z) \arrow[r, "\prod_y id_{B(x,y)} \otimes \epsilon_B(y,z)"'] & B^\bullet(x,z).
\end{tikzcd}$$

We will denote a dg cocategory with its cocomposition and counit as $(B, \Delta_B, \epsilon_B)$. To make the notation more readable, when the meaning is clear, we will omit references to the objects and write Δ_B instead of $\Delta_B(x,z)$, ϵ_B instead of $\epsilon_B(x,z)$, and for the differentials on morphisms, d_B instead of $d_B(x,z)$.

Definition 1.2.2. A **functor** $F : A \to B$ between two dg cocategories is a functor between the cocategories satisfying $d_B \circ F(f) = F \circ d_A(f)$ for all morphisms f in A.

Definition 1.2.3. A **conilpotent** dg cocategory is a dg cocategory $(B, \Delta_B, \epsilon_B)$ satisfying: for each morphism $f : x \to y$ in B, there exists $n_f \in \mathbb{N}$ such that $\bar{\Delta}_B^{n_f}(f) = 0$ where

$$\begin{aligned}
\bar{\Delta}_B(x,z) : B^\bullet(x,z) \to & \prod_{y \in Obj(B)} B^\bullet(x,y) \otimes B^\bullet(y,z) \\
& f \mapsto \Delta_B(f) - \sum_{e_x \in \epsilon_B(x,x)^{-1}(1)} e_x \otimes f - \sum_{e_z \in \epsilon_B(z,z)^{-1}(1)} f \otimes e_z.
\end{aligned}$$

Fact (needs reference?): If B is a conilpotent dg cocategory, then for all $x \in Obj(B)$, $\epsilon_B(x,x)^{-1}(1)$ has exactly one element, which we will denote e_x.

1.2.2. Structure of $B(n)$

For each sequence of algebras, $A_0, A_1, \cdots, A_n$, we will define a conilpotent dg cocategory, $B(A_0 \to A_1 \to \cdots \to A_n \to A_0)$. In this chapter, we fix the sequence of algebras, and abbreviate

$$B(n) := B(A_0 \to A_1 \to \cdots \to A_n \to A_0).$$

1.2.2.1. Objects. $B(n)$ has objects tuples $(f_0, f_1, \cdots, f_n)$ where $f_i : A_i \to A_{i+1 \text{ (mod n+1)}}$, $0 \leq i \leq n$, are maps of algebras. We can picture an object in $B(n)$ as follows:

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_0$$

1.2.2.2. Morphisms. The graded vector space of morphisms in $B(n)$ between two objects, $(f_0, ..., f_n)$ and $(g_0, ..., g_n)$, is

$$Bar(C^\bullet(A_{0,f_0}\, A_{1g_0})) \otimes Bar(C^\bullet(A_{1,f_1}\, A_{2g_1})) \otimes \cdots \otimes Bar(C^\bullet(A_{n,f_n}\, A_{0g_n}))$$

where $Bar(C^\bullet(A,_f B_g))$ is the following complex:

$$Bar(C^\bullet(A,_f B_g)) := Bar_0(C^\bullet(A,_f B_g)) \oplus \bigoplus_{m\geq 1} Bar_m(C^\bullet(A,_f B_g))$$

$$Bar_0(C^\bullet(A,_f B_g)) := k$$

$$Bar_m(C^\bullet(A,_f B_g)) := \bigoplus_{\substack{h_0=f,\\ h_m=g,\\ h_1,\ldots,h_{m-1}\\ \text{algebra maps}}} C^\bullet(A,\ _{h_0}B_{h_1})[1] \otimes C^\bullet(A,\ _{h_1}B_{h_2})[1] \otimes \cdots \otimes C^\bullet(A,\ _{h_{m-1}}B_{h_m})[1]$$

$(C^\bullet(A,_{h_i} B_{h_j}),\ _{h_i}\delta_{h_j})$ = Hochschild cochain complex, see Appendix C

$$d_{Bar(C^\bullet(A,_f B_g))} = \tilde{\delta} + b'$$

$$\tilde{\delta}(\phi_1 \otimes \ldots \otimes \phi_m) = \sum_{1\leq i\leq m} (-1)^{1+\sum_{j<i} |\phi_i|+1} \phi_1 \otimes \ldots \otimes [_{h_{i-1}}\delta_{h_i}](\phi_i) \otimes \ldots \otimes \phi_m$$

$$b'(\phi_1 \otimes \ldots \otimes \phi_m) = \sum_{1\leq i\leq m-1} (-1)^{\sum_{j\leq i} |\phi_i|+1} \phi_1 \otimes \ldots \otimes \phi_i \cup \phi_{i+1} \otimes \ldots \otimes \phi_m$$

$\cup$ = cup product on Hochschild cochains, see Appendix C.

(This sign convention is consistent with Reference [**1**], Section 4.6.)

1.2.2.3. Aside on notation. When referring to an arbitrary morphism in $B(n)$, we will assume it is a morphism from object $(f_{0,0}, f_{1,0}, \cdots, f_{n,0})$ to object $(f_{0,k_0}, f_{1,k_1}, \cdots, f_{n,k_n})$. We will denote the morphism

$$\phi_{0,1}\ldots\phi_{0,k_0}|\phi_{1,1}\ldots\phi_{1,k_1}|\ldots|\phi_{n,1}\ldots\phi_{n,k_n}$$

where $\phi_{i,j} \in C^\bullet(A_i,_{f_{j-1}} A_{i+1 \text{ (mod n+1)}}{}_{f_j})$. See Figure 1.1 for a picture of this morphism.

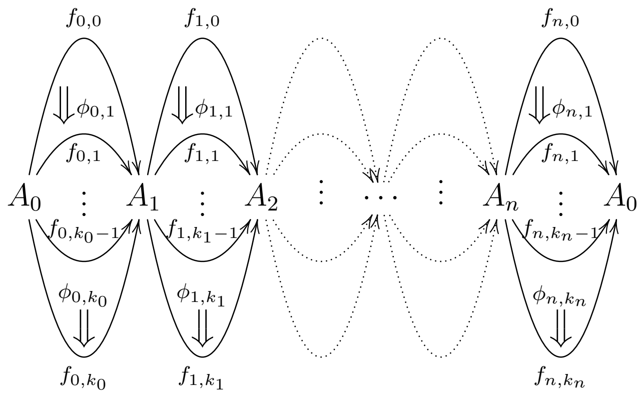

Figure 1.1. A morphism in $B(n)$ from $(f_{0,0}, f_{1,0}, \cdots, f_{n,0})$ to $(f_{0,k_0}, f_{1,k_1}, \cdots, f_{n,k_n})$ where $\phi_{i,j} \in C^\bullet(A_{i,f_{j-1}}\, A_{i+1 \text{ (mod n+1)}}{}_{f_j})$

1.2.2.4. Differential on $B(n)$. Putting everything together, the differential on $B(n)((f_{0,0}, ..., f_{n,0}), (f_{0,k_0}, ..., f_{n,k_n}))$ is

$$
\begin{aligned}
& d_{B(n)}\big(\phi_{0,1}\ldots\phi_{0,k_0}|\ldots|\phi_{n,1}\ldots\phi_{n,k_n}\big) \\
= & \sum_{0\le i\le n} (-1)^{\sum\limits_{p<i;q} |\phi_{p,q}|+1} \phi_{0,1}\ldots|\ldots|d_{Bar(C^\bullet(A_i,A_{i+1}))}(\phi_{i,1}\ldots\phi_{i,k_i})|\ldots|\ldots\phi_{n,k_n}
\end{aligned}
$$

1.2.2.5. Counit. Define

$$
\begin{aligned}
\epsilon_{B(n)}((f_{0,0},\ldots, f_{n,0}), (f_{0,k_0}, \ldots, f_{n,k_n})) &: B(n)((f_{0,0}, \ldots, f_{n,0}), (f_{0,k_0}, \ldots, f_{n,k_n})) = \\
&= Bar(C^\bullet(A_{0,f_{0,0}}\, A_{1}{}_{f_{0,k_0}})) \otimes \cdots \otimes Bar(C^\bullet(A_{0,f_{n,0}}\, A_{1}{}_{f_{n,k_n}})) \to \\
&\xrightarrow{project} Bar_0(C^\bullet(A_{0,f_{0,0}}\, A_{1}{}_{f_{0,k_0}})) \otimes \cdots \otimes Bar_0(C^\bullet(A_{0,f_{n,0}}\, A_{1}{}_{f_{n,k_n}})) \cong k.
\end{aligned}
$$

1.2.2.6. Cocomposition. We have a coassociative map of complexes

$$\Delta_{A,_f B_g} : Bar(C^\bullet(A,_f B_g)) \to \bigoplus_{h:A\to B} Bar(C^\bullet(A,_f B_h)) \otimes Bar(C^\bullet(A,_h B_g))$$

$$\phi_1 \cdots \phi_k \mapsto \sum_{1\leq i\leq k-1} \phi_1 \cdots \phi_i \otimes \phi_{i+1} \cdots \phi_k$$

$$+ e_f \otimes \phi_1 \ldots \phi_k + \phi_1 \ldots \phi_k \otimes e_g$$

where $e_f = 1$ in $Bar_0(C^\bullet(A,_f B_f)) \cong k$. Extend $\Delta_{A,_f B_g}$ to a cocomposition on $B(n)$ by taking (up to signs)

$$\Delta_{B(n)}((f_{0,0}, \ldots, f_{n,0}), (f_{0,k_0}, \ldots, f_{n,k_n})) := \Delta_{A_0,_{f_{0,0}} A_1 {}_{f_{0,k_0}}} \otimes \cdots \otimes \Delta_{A_0,_{f_{n,0}} A_1 {}_{f_{n,k_n}}}.$$

The sign on the term $(\phi_{0,1} \ldots \phi_{0,i_0} | \ldots | \phi_{n,1} \ldots \phi_{n,i_n}) \otimes (\phi_{0,i_0+1} \ldots \phi_{0,k_0} | \ldots | \phi_{n,i_n+1} \ldots \phi_{n,k_n})$ in the cocomposition is:

$$(-1)^{\sum\limits_{\substack{1\leq p\leq n\\ 1\leq r<p}} (\sum\limits_{1\leq q\leq i_p} |\phi_{p,q}|+1)(\sum\limits_{i_r+1\leq s\leq k_r} |\phi_{r,s}|+1)}$$

It's clear from the definitions that $(B(n), \Delta_{B(n)}, \epsilon_{B(n)})$ satisfy the diagrams needed to form a dg cocategory. We also see that $B(n)$ is conilpotent:

$$\bar{\Delta}_{B(n)}^{min(k_0,\ldots,k_n)}(\phi_{0,1} \ldots \phi_{0,k_0} | \ldots | \phi_{n,1} \ldots \phi_{n,k_n}) = 0.$$

1.3. Dg comodules: $C(n)$

1.3.1. Background on dg comodules

Definition 1.3.1. A **dg comodule** C over a dg cocategory B consists of the following data:

- for each object $f \in B$, a complex $C^\bullet(f)$, and
- maps of complexes

$$\Delta_C(f) : C^\bullet(f) \to \prod_{g \in Obj(B)} B^\bullet(f,g) \otimes C^\bullet(g).$$

such that the following diagrams for coassociativity and counitality commute:

$$\begin{tikzcd}
C^\bullet(f) \arrow[r, "\Delta_C(f)"] \arrow[d, "\Delta_C(f)"'] & \prod_{g \in Obj(B)} B^\bullet(f,g) \otimes C^\bullet(g) \arrow[d, "\prod_g id_{B(f,g)} \otimes \Delta_C(g)"] \\
\prod_{g \in Obj(B)} B^\bullet(f,g) \otimes C^\bullet(g) \arrow[r, "\prod_g \Delta_B(f,g) \otimes id_{C(g)}"] & \prod_{g,g' \in Obj(B)} B^\bullet(f,g) \otimes B^\bullet(g,g') \otimes C^\bullet(g')
\end{tikzcd}$$

$$\begin{tikzcd}
C^\bullet(f) \arrow[r, "\Delta_C(f)"] \arrow[dr, "id"'] & \prod_{g \in Obj(B)} B^\bullet(f,g) \otimes C^\bullet(g) \arrow[d, "\prod_g \epsilon_B(f,g) \otimes id_{C(g)}"] \\
 & C^\bullet(f).
\end{tikzcd}$$

To simplify notation, we will write Δ_C instead of $\Delta_C(f)$ when the meaning is clear.

Example 1.3.1. *A dg comodule over a dg cocategory B with one object, $*$, is a dg comodule over the counital dg coalgebra $B^\bullet(*,*)$.*

Definition 1.3.2. A **morphism of dg comodules** $H : C \to D$ over a dg category B consists of maps of complexes $\big(H_f : C^\bullet(f) \to D^\bullet(f)\big)_{f\in Obj(B)}$ such that for each $f \in Obj(B)$, the following diagram commutes:

$$\begin{array}{ccc} C^\bullet(f) & \xrightarrow{H_f} & D^\bullet(f) \\ \Big\downarrow{\scriptstyle\Delta_C} & & \Big\downarrow{\scriptstyle\Delta_D} \\ \prod\limits_{g\in Obj(B)} B^\bullet(f,g)\otimes C^\bullet(g) & \xrightarrow{\prod_g id_B\otimes H_g} & \prod\limits_{g\in Obj(B)} B^\bullet(f,g))\otimes D^\bullet(g). \end{array}$$

Again, when the meaning is clear, we may write H instead of H_f.

Definition 1.3.3. A **conilpotent** dg comodule over a dg cocategory B is a dg comodule (C, Δ_C) over B satisfying: for each $f \in Obj(B)$ and each element $\alpha \in C^\bullet(f)$, there exists $n_\alpha \in \mathbb{N}$ such that $\bar{\Delta}_f^{n_\alpha}(\alpha) = 0$ where

$$\begin{aligned} \bar{\Delta}_C(f) : C^\bullet(f) \to \prod_{g\in Obj(B)} B^\bullet(f,g)\otimes C^\bullet(g) \\ \alpha \mapsto \Delta_B(\alpha) - \sum_{e_f\in\epsilon_B(f,f)^{-1}(1)} e_f\otimes f. \end{aligned}$$

1.3.2. Structure of $C(n)$

Reminder: In this chapter, we fix algebras $A_0, A_1, \cdots, A_n$. $C(n)$ and $B(n)$ are short for $C(A_0 \to A_1 \to \cdots \to A_n \to A_0)$ and $B(A_0 \to A_1 \to \cdots \to A_n \to A_0)$, respectively.

We now give dg comodules $C(n)$ over $B(n)$. First, we will describe the graded comodule

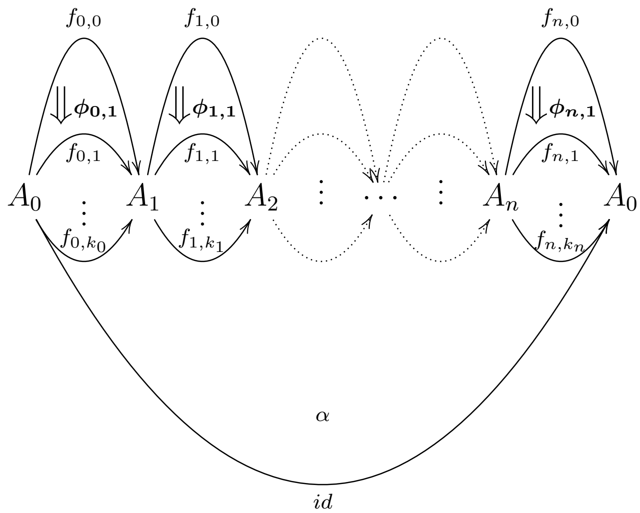

Figure 1.2. Picture of an element of $C(n)^{\bullet}(f)$ where $f = (f_{0,0}, f_{1,0}, \cdots, f_{n,0})$, $\phi_{i,j} \in C^{\bullet}({A_i}_{,f_{j-1}} \, {A_{i+1 \text{ (mod n+1)}}}_{f_j})$, and $\alpha \in C_{-\bullet}({A_0}_{,f_n \cdots f_1 f_0} \, {A_0}_{id})$

structure; then, we will describe the differentials. For an object $f = (f_0, f_1, \cdots, f_n) \in B(n)$, we have

$$C(n)^{\bullet}(f) = \bigoplus_{g \in Obj(B(n))} B(n)^{\bullet}(f, g) \otimes C_{-\bullet}({A_0}_{,comp(g)} \, {A_0}_{id}) \tag{1.1}$$

where, for $g = (g_0, g_1, \cdots, g_n)$, we write $comp(g) = g_n \circ g_{n-1} \circ \cdots \circ g_0$, and $(C_{\bullet}(A, B), \ {}_g b)$ is the Hochschild chain complex (see Appendix C). We will denote a typical element of $C(n)^{\bullet}(f)$ as

$$\phi_{0,1} \ldots \phi_{0,k_0} | \phi_{1,1} \ldots \phi_{1,k_1} | \ldots | \phi_{n,1} \ldots \phi_{n,k_n} | \alpha$$

where $\phi_{0,1} \cdots \phi_{n,k_n}$ is a morphism in $B(n)$ (see Section 1.2.2.3) and $\alpha \in C_{-\bullet}({A_0}_{,f_{k_n} \cdots f_{k_0}} \, {A_0}_{id})$. See Figure 1.2 for a picture of a typical element of $C(n)^{\bullet}(f)$.

1.3.2.1. Comodule structure. The comodule maps on $C(n)^\bullet(f)$ are given by the co-composition maps in $B(n)$:

$$\begin{array}{ccc} C(n)^\bullet(f) & \xrightarrow{\Delta_C} & \bigoplus_{h\in Obj(B(n))} B(n)^\bullet(f,h)\otimes C(n)^\bullet(h) \\ \| & & \| \\ \bigoplus_{g\in Obj(B(n))} B(n)^\bullet(f,g)\otimes C_{-\bullet}(A_{0,g}A_{0id}) & \xrightarrow{\Delta_{B(n)}\otimes id_{C_{-\bullet}}} & \bigoplus_{g,h\in Obj(B(n))} \substack{B(n)^\bullet(f,h)\otimes B(n)^\bullet(h,g) \\ \otimes C_{-\bullet}(A_{0,g}A_{0id})} \end{array}$$

Because $\Delta_{C(n)}$ is induced by $\Delta_{B(n)}$, we have that $\Delta_{C(n)}$ satisfies coassocitivity and counitality and is conilpotent.

$C(n)$ cofree as a comodule in the sense that a morphism to $C(n)$ is determined by projections to its Hochschild-chains component. More precisely, there is a one-to-one correspondence

(1.2)

$$\left\{\begin{matrix}\text{maps of comodules} \\ D\to C(n) \text{ over } B(n)\end{matrix}\right\} \xleftrightarrow{1:1} \left\{\left(\begin{matrix}\text{maps of graded vector spaces} \\ D^\bullet(f)\to C_{-\bullet}(A_{0,f}A_{0id})\end{matrix}\right)_{f\in Obj(B(n))}\right\}$$

$$\Big(F: D\to C(n)\Big) \mapsto \left(\begin{matrix} D(f) \xrightarrow{F_f} C(n)(f) \\ \xrightarrow{project} C_{-\bullet}(A_{0,f}A_{0id})\end{matrix}\right)_f$$

$$\left(\begin{matrix} D(f) & \xrightarrow{\Delta_D} & \bigoplus_{g\in Obj(B(n))} B(n)(f,g)\otimes D(g) \\ & \xrightarrow{\oplus_g id_B\otimes F_g} & \oplus_g B(n)(f,g)\otimes C_{-\bullet}(A_{0,g}A_{0id}) \\ & & \cong C(n)(f)\end{matrix}\right)_f \hookleftarrow \Big(D(f)\to C_{-\bullet}(A_{0,f}A_{0id})\Big)_f$$

Definition 1.3.4. We will call elements of $T(A_0\to\ldots\to A_n\to A_0)(f) := C_{-\bullet}(A_{0,f}A_{0id})$ the **cogenerators** of $C(A_0\to\ldots A_n\to A_0)(f)$. More generally, we will refer to the set $T(A_0\to\ldots\to A_n\to A_0) = \{T(A_0\to\ldots\to A_n\to A_0)(f) | f\in Obj(B(A_0\to\ldots\to A_n\to$

$A_0))\}$ as the **cogenerators** of $C(A_0 \to \dots \to A_n \to A_0)$. When we have fixed a sequence of algebras, $A_0, \dots A_n$, we will use $T(n)$ to denote $T(A_0 \to \dots \to A_n \to A_0)$.

1.3.2.2. Differential. The differential $d_{C(n)}$ on $C(n)$ is:

(1.3)

$$d_{C(n)} = \tilde{d}_{B(n)} + \tilde{b} + \mathfrak{I}$$

$$\tilde{d}_{B(n)}\big(\phi_{0,1}\dots\phi_{0,k_0}|\dots|\phi_{n,1}\dots\phi_{n,k_n}|\alpha\big) = d_{B(n)}(\phi_{0,1}\dots|\dots|\dots\phi_{n,k_n})|\alpha$$

$$\tilde{b}\big(\phi_{0,1}\dots\phi_{0,k_0}|\dots|\phi_{n,1}\dots\phi_{n,k_n}|\alpha\big) = (-1)^{\sum_{i,j}|\phi_{i,j}|+1}\phi_{0,1}\dots|\dots|\dots\phi_{n,k_n}|[_{f_{n,k_n}\dots f_{0,k_0}}b](\alpha)$$

where $d_{B(n)}$ is the differential on $B(n)$, and $\mathfrak{I}$ is a term that captures an action of cochains on chains described by the equations below:

(1.4)

$$\mathfrak{I} = (id_{B(n)} \otimes \eta_{C(n)}) \circ \Delta_{C(n)}$$

$$\eta_{C(n)}(\phi_{0,1}\dots\phi_{0,k_0}|\dots|\phi_{n,1}\dots\phi_{n,k_n}|\alpha) = \iota_{C(0)}(\pi_{B(0)}((\phi_{0,1}\dots\phi_{0,k_0}) \bullet \dots \bullet (\phi_{n,1}\dots\phi_{n,k_n})), \alpha)$$

$$\bullet = \text{brace operation on cochains, see Section C.2}$$

$$\pi_{B(0)} : B(0)^\bullet(f_{n,1}\dots f_{1,1}f_{0,1}, f_{n,k_n}\dots f_{1,k_1}f_{0,k_0}) \xrightarrow[\textit{component}]{\textit{project onto}} C^\bullet(A_{0,f_{n,1}\dots f_{0,1}}\, A_{0f_{n,k_n}\dots f_{0,k_0}})$$

$$\iota = \text{contraction operation, see Section C}$$

Given Equation 1.3, it's easy to check that we can promote Equation 1.2 to a dg statement:

$$\left\{\begin{matrix}\text{maps of dg comodules}\\ D \to C(n) \text{ over } B(n)\end{matrix}\right\} \overset{1:1}{\longleftrightarrow} \left\{\left(\begin{matrix}\text{maps of complexes}\\ D^\bullet(f) \to C_{-\bullet}(A_{0,f}\, A_{0id})\end{matrix}\right)_{f\in Obj(B(n))}\right\}.$$

CHAPTER 2

Pullbacks, Pushforwards and Adjunctions

2.1. A sheafy-cyclic object in dg cocategories

We would like to say that we have a functor from Connes cyclic category Λ (see Appendix A for generators and relations) to the category of dg cocategories where $[n] \mapsto B(n)$, but defining $B(n)$ involved choosing a sequence of algebras $A_0, \dots, A_n$.

Instead, we have the following: Let $X : \Lambda \to Set$ be the functor that sends $[n]$ to the set of diagrams $A_0 \to A_1 \to \cdots \to A_n \to A_0$ where the A_i's are algebras. On generating morphisms in Λ, X acts as follows: Let $\mathcal{A} = (A_0 \to \ldots \to A_n \to A_0) \in X([n])$.

$$X(\tau_n) : \mathcal{A} \mapsto (A_n \to A_0 \to \ldots \to A_{n-1} \to A_n)$$

$$X(\delta_{j,n}) : \mathcal{A} \mapsto (A_0 \to \ldots \to A_j \longrightarrow A_{j+2 \text{ (mod n+1)}} \to \ldots \to A_n \to A_0)$$

$$X(\sigma_{i,n}) : \mathcal{A} \mapsto \begin{cases} (A_0 \to \ldots \to A_i \to A_i \to \ldots A_n \to A_0) & 1 \leq i \leq n \\ (A_0 \to \ldots \to A_n \to A_0 \to A_0) & i = n+1 \end{cases}$$

It's straightforward to check that X respects composition of morphisms. Now, let χ be the category with objects given by diagrams $A_0 \to \ldots \to A_n \to A_0$ where the A_i's are algebras and $n \in \mathbb{N}$. Morphisms in χ are the pointwise images of X. In other words, the set of morphisms in χ is $\{X(\lambda)|_x : \lambda \in \Lambda([n],[m]), x \in X([n])\}$. We will give a functor, $\mathcal{G}$, from χ to the category of dg cocategories; (this is our sheafy-cyclic object, i.e., a **sheafy-cyclic** object in a category $\mathcal{C}$ is a functor $\chi \to \mathcal{C}$).

2.1.1. Aside on notation:

Fix $\lambda : [n] \to [m]$ in Λ and $x \in X([n])$. To define $\mathcal{G}$, we will need to define a functor $\mathcal{G}\big(X(\lambda)|_x\big) : B(x) \to B(X(\lambda)(x))$. To simplify notation, we will denote $\hat{\lambda} := \mathcal{G}\big(X(\lambda)|_x\big)$

and write $\hat{\lambda} : B(x) \to B(\lambda x)$. Technically, we are losing information about the x when we write $\hat{\lambda}$ instead of $\mathcal{G}\big(X(\lambda)|_x\big)$, but we will be clear about the source and target when needed.

2.1.2. Definition of $\mathcal{G}$

Now, we will define $\mathcal{G}$. On objects,

$$\mathcal{G} : (A_0 \to \ldots \to A_n \to A_0) \mapsto B(A_0 \to \ldots \to A_n \to A_0)$$

(see Section 1.2 for definition of $B(\cdot)$)

On generating morphisms in χ, set $\mathcal{A} = (A_0 \to \ldots \to A_n \to A_0) \in Obj(\chi)$, and define

$$\hat{\tau}_n \mapsto \begin{cases} B(\mathcal{A}) \longrightarrow B(A_n \to A_0 \to \ldots \to A_{n-1} \to A_n) \\ \text{objects: } (f_0, f_1, \ldots, f_n) \mapsto (f_n, f_0, \ldots, f_{n-1}) \\ \text{morphisms: } \phi_{0,1} \ldots \phi_{0,k_0} | \ldots | \phi_{n,1} \ldots \phi_{n,k_n} \mapsto \phi_{n,1} \ldots \phi_{n,k_n} | \ldots | \phi_{n-1,1} \ldots \phi_{n-1,k_{n-1}} \end{cases}$$

$$\hat{\delta}_{j,n} \mapsto \begin{cases} B(\mathcal{A}) \longrightarrow B(A_0 \to \ldots \to A_j \to A_{j+2 \text{ (mod n+1)}} \to \ldots \to A_0) \\ \text{objects: } (f_0, f_1, \ldots, f_n) \mapsto (f_0, \ldots, f_{j+1} \circ f_j, \ldots, f_n) \\ \text{morphisms: } \phi_{0,1} \ldots \phi_{0,k_0} | \ldots | \phi_{n,1} \ldots \phi_{n,k_n} \mapsto \\ \qquad \phi_{0,1} \ldots \phi_{0,k_0} | \ldots | (\phi_{j,1} \ldots \phi_{j,k_j}) \bullet (\phi_{j+1,1} \ldots \phi_{j+1,k_{j+1}}) | \ldots | \phi_{n,1} \ldots \phi_{n,k_n} \end{cases}$$

$$\underset{1 \le i \le n}{\hat{\sigma}_{i,n}} \mapsto \begin{cases} B(\mathcal{A}) \longrightarrow B(A_0 \to \ldots \to A_i \to A_i \to \ldots \to A_0) \\ \text{objects: } (f_0, f_1, \ldots, f_n) \mapsto (f_0, \ldots, f_{i-1}, id_{A_i}, f_i, \ldots, f_n) \\ \text{morphisms: } \phi_{0,1} \ldots \phi_{0,k_0} | \ldots | \phi_{n,1} \ldots \phi_{n,k_n} \mapsto \\ \qquad \phi_{0,1} \ldots \phi_{0,k_0} | \ldots | \phi_{i-1,1} \ldots \phi_{i-1,k_{i-1}} | 1 | \phi_{i,1} \ldots \phi_{i,k_i} | \ldots | \phi_{n,1} \ldots \phi_{n,k_n} \\ \qquad 1 \in k = \text{degree 0 component of } Bar(C^\bullet(A_i, A_i)) \end{cases}$$

$$\hat{\sigma}_{n+1,n} \mapsto \begin{cases} B(\mathcal{A}) \longrightarrow B(A_0 \to \ldots \to A_n \to A_0 \to A_0) \\ \text{objects: } (f_0, f_1, \ldots, f_n) \mapsto (f_0, \ldots, f_n, id_{A_0}) \\ \text{morphisms: } \phi_{0,1} \ldots \phi_{0,k_0} | \ldots | \phi_{n,1} \ldots \phi_{n,k_n} \mapsto \\ \qquad \phi_{0,1} \ldots \phi_{0,k_0} | \ldots | \phi_{n,1} \ldots \phi_{n,k_n} | 1 \\ \qquad 1 \in k = \text{degree 0 component of } Bar(C^\bullet(A_0, A_0)) \end{cases}$$

It's straightforward to check that $\mathcal{G}$ is a functor (i.e., that composition of morphisms and the relations are preserved). The only facts we need are that $\bullet$ is an associative map of complexes and that $1 \bullet (\phi_0 \ldots \phi_k) = (\phi_0 \ldots \phi_k) \bullet 1 = (\phi_0 \ldots \phi_k)$ where the 1's are in the degree 0 components of $Bar(C^\bullet(A_i, A_i))$ for the appropriate A_i (see Appendix Section C.2).

2.2. Motivation of this chapter

We would like to extend the sheafy-cyclic structure in Section 2.1 from $B(\mathcal{A})$ to the pair $(B(\mathcal{A}), C(\mathcal{A}))$ where $\mathcal{A} = (A_0 \to \ldots \to A_n \to A_0) \in Obj(\chi)$. However, this presents some complications as $C(\mathcal{A})$ and $C(\lambda\mathcal{A})$ are comodules over different cocategories (where λ is a morphism in Λ inducing a morphism in χ with source $\mathcal{A}$). Instead, we will use the functors $\hat{\lambda} : B(\mathcal{A}) \to B(\lambda\mathcal{A})$ from Section 2.1.2 to define pullbacks $\hat{\lambda}^* C(\lambda\mathcal{A})$ and maps $\lambda_! : C(\mathcal{A}) \to \hat{\lambda}^* C(\lambda\mathcal{A})$ of dg comodules over $B(\mathcal{A})$.

First, we will define functors $\hat{\lambda}^*$ from the category of conilpotent dg comodules over $B(\lambda\mathcal{A})$ to the category of conilpotent dg comodules over $B(\mathcal{A})$. Second, we will give $\hat{\lambda}_\#$, the left adjoint to $\hat{\lambda}^*$. Finally, we will give explicit maps of dg comodules $\lambda_! : C(\mathcal{A}) \to \hat{\lambda}^* C(\lambda\mathcal{A})$, and apply the adjunction to these maps. The following chapter will formalize the relations between the $\lambda_!$'s.

2.3. Pullbacks of dg comodules–theory

Let $\lambda : B_1 \to B_0$ be a functor between conilpotent dg cocategories. In this section, we will define a functor λ^* from the category of conilpotent dg comodules over B_0 to the category of conilpotent dg comodules over B_1. We call λ^* "co-extension of scalars".

2.3.1. Category-theoretic definition of λ^*

Let λ be as above, and let C be a conilpotent dg comodule over B_0. We define $\lambda^* C$ as follows:

$$\lambda^* C := ker\big(B_1 \otimes_\lambda C \underset{(\mathrm{id_{B_1}} \otimes \lambda \otimes \mathrm{id_C}) \circ (\Delta_{B_1} \otimes \mathrm{id_C})}{\overset{\mathrm{id_{B_1}} \otimes \Delta_C}{\rightrightarrows}} B_1 \otimes_\lambda B_0 \otimes C\big) \tag{2.1}$$

where $B_1 \otimes_\lambda C$ and $B_1 \otimes_\lambda B_0 \otimes C$ are dg comodules over B_1 defined below. For $f \in Obj(B_1)$,

$$[B_1 \otimes_\lambda C](f) : = \big(\bigoplus_{h \in Obj(B_1)} B_1^\bullet(f,h) \otimes C^\bullet(\lambda h), \Delta(f) = \bigoplus_h \Delta_{B_1(f,h)} \otimes id_{C(\lambda h)} \big)$$

$$[B_1 \otimes_\lambda B_0 \otimes C](f) : = \big(\bigoplus_{\substack{h_1 \in Obj(B_1), \\ h_2 \in Obj(B_0)}} B_1^\bullet(f,h_1) \otimes B_0^\bullet(\lambda h_1, h_2) \otimes C^\bullet(h_2),$$

$$\Delta(f) = \bigoplus_{h_1, h_2} \Delta_{B_1(f,h_1)} \otimes id_{B_0(\lambda h_1, h_2)} \otimes id_{C(h_2)} \big).$$

The names of the maps in Equation 2.1 are also meant to be suggestive. In full detail, for $f \in Obj(B_1)$,

$$[id_{B_1} \otimes \Delta_C](f) := \bigoplus_h id_{B_1(f,h)} \otimes \Delta_C(\lambda h)$$

and

$$[B_1 \otimes_\lambda C](f) \xrightarrow{[\Delta_{B_1} \otimes id_C](f) := \bigoplus_h \Delta_{B_1}(f,h) \otimes id_{C(\lambda h)}} \bigoplus_{h_1,h_2 \in Obj(B_1)} B_1(f,h_1) \otimes B_1(h_1,h_2) \otimes C(\lambda h_2)$$

$$\xrightarrow{[id_{B_1} \otimes \lambda \otimes id_C](f) := \bigoplus_{h_1,\,h_2} id_{B_1(f,h_1)} \otimes \lambda(h_1,h_2) \otimes id_{C(\lambda h)}} [B_1 \otimes_\lambda B_0 \otimes C](f).$$

That the kernel is well-defined follows formally from the abelianness of the category of chain complexes, but it is also easy to check that the induced differentials from $[B_1 \otimes_\lambda C](f)$ on the kernel are well-defined. Since $\Delta_{\lambda^* C}$ is induced by Δ_{B_1}, we have that $\Delta_{\lambda^* C}$ also satisfies coassociativity, counitality and conilpotency.

Next, we will define λ^* on morphisms. Let $F : C \to D$ be a map of conilpotent dg comodules over B_0. By the universal property of $\lambda^* D$, we can define a morphism $\lambda^* F : \lambda^* C \to \lambda^* D$ by giving a morphism from $(\lambda^* F)' : \lambda^* C \to B_1 \otimes_\lambda D$ such that the two maps

(2.2)

$$(id_{B_1} \otimes \Delta_D) \circ (\lambda^* F)', (id_{B_1} \otimes \lambda \otimes id_D) \circ (\Delta_{B_1} \otimes id_D) \circ (\lambda^* F)' : \lambda^* C \to B_1 \otimes_\lambda D \rightrightarrows B_1 \otimes_\lambda B_0 \otimes D$$

coincide. We define $(\lambda^* F)'$ as follows:

$$(\lambda^* F)' : \lambda^* C \xrightarrow[inclusion]{canonical} B_1 \otimes_\lambda C \xrightarrow{id_{B_1} \otimes F} B_1 \otimes_\lambda D$$

It's easy to check that the two maps in Equation 2.2 coincide: Let $b \otimes c$ be an arbitrary element of $\lambda^* C(f) \hookrightarrow [B_1 \otimes_\lambda C](f)$. Then,

$$\begin{aligned}
[(id_{B_1} \otimes \Delta_D) \circ (\lambda^* F)'](b \otimes c) &= \sum_{(Fc)} b \otimes (Fc)_{(1)} \otimes (Fc)_{(2)} \\
&= \sum_{(c)} b \otimes Fc_{(1)} \otimes Fc_{(2)} \quad \text{(F is a map of comodules)} \\
&= [(id_{B_1} \otimes F \otimes F) \circ (id_{B_1} \otimes \Delta_C)](b \otimes c) \\
&= [(id_{B_1} \otimes F \otimes F) \circ (id_{B_1} \otimes \lambda \otimes id_C) \circ (\Delta_{B_1} \otimes id_C)](b \otimes c) \\
&\quad (b \otimes c \text{ is in the kernel}) \\
&= \sum_{(b)} b_{(1)} \otimes \lambda b_{(2)} \otimes Fc \\
&= [(id_{B_1} \otimes \lambda \otimes id_D) \circ (\Delta_{B_1} \otimes id_D) \circ (\lambda^* F)'](b \otimes c).
\end{aligned}$$

So, $\lambda^* F$ is well-defined. In summary, we have commuting diagrams:

$$\begin{tikzcd}
\lambda^* C \arrow[r, "\substack{\text{canonical}\\ \text{inclusion}}"] \arrow[d, "\lambda^* F"'] & B_1 \otimes_\lambda C \arrow[d, "id_{B_1} \otimes F = \text{ map inducing } \lambda^* F"] \\
\lambda^* D \arrow[r, "\substack{\text{canonical}\\ \text{inclusion}}"] & B_1 \otimes_\lambda D
\end{tikzcd} \tag{2.3}$$

Finally, it is straightforward to see that λ^* is a functor, i.e., that λ^* preserves composition of morphisms: Let $C \xrightarrow{F} D \xrightarrow{G} E$ be composable morphisms of dg comodules over B_0. The maps inducing $\lambda^* F$, $\lambda^* G$ and $\lambda^*(G \circ F)$ are $id_{B_1} \otimes F$, $id_{B_1} \otimes G$ and $id_{B_1} \otimes GF$, respectively. The inducing maps respect composition–$(id_{B_1} \otimes G) \circ (id_{B_1} \otimes F) = id_{B_1} \otimes GF$–and by the commuting diagrams 2.3, the functor λ^* does as well.

Proposition 2.1. *Let $F : B_2 \to B_1$ and $G : B_1 \to B_0$ be functors between dg cocategories B_2, B_1 and B_0. Let M be a dg comodule over B_0. Then,*

$$(GF)^*M \cong F^*G^*M.$$

Proof. We will prove the proposition by showing that F^*G^*M satisfies the universal property of $(GF)^*M$. First, let N be a dg comodule over B_2 and $H : N \to B_2 \otimes_{GF} M$ be a map of dg comodules such that the two maps

(2.4)

$$(id_{B_2} \otimes GF \otimes id_M) \circ (\Delta_{B_2} \otimes id_M) \circ H, (id_{B_2} \otimes \Delta_M) \circ H : N \to B_2 \otimes_{GF} M \rightrightarrows B_2 \otimes_{GF} \otimes B_0 \otimes M$$

coincide. We will show that H determines a map of dg comodules $\tilde{H} : N \to F^*G^*M$. Let $x \in Obj(B_2)$. Define

$$\begin{aligned} H'_x : N(x) &\xrightarrow{H_x} \bigoplus_{y \in Obj(B_2)} B_2(x, y) \otimes M(GFy) \\ &\xrightarrow{F \otimes id_M} \bigoplus_{y \in Obj(B_2)} B_1(Fx, Fy) \otimes M(GFy) \\ &\subset [B_1 \otimes_G M](Fx). \end{aligned}$$

The image of H'_x lands in $G^*M(Fx)$, a subcomplex of $[B_1 \otimes_G M](Fx)$; checking this is straightforward using the universal property of G^*M, the fact that F commutes with the coproducts, and Equation 2.4. So, for each $x \in Obj(B_2)$, we have a map of complexes

$H'_x : N(x) \to G^*M(Fx)$. Now define $\tilde{H}$ as follows:

$$\begin{aligned}
\tilde{H}_x : N(x) &\xrightarrow{\Delta_N} \prod_{y \in Obj(B_2)} B_2(x,y) \otimes N(y) \\
&\xrightarrow{\prod id_{B_2} \otimes H'_y} \prod_{y \in Obj(B_2)} B_2(x,y) \otimes G^*M(Fy) \\
&\subset [B_2 \otimes_F G^*M](x).
\end{aligned}$$

Showing that $\tilde{H}$ lands in G^*F^*M, a subcomodule of $[B_2 \otimes_F G^*M]$, is also straightforward; we only need that F and H commute with the appropriate coproducts, and that the cocomposition on B_2 is coassociative. So, for each $x \in Obj(B_2)$, we have a map $\tilde{H}_x : N(x) \to G^*F^*M(x)$. It's clear that $\tilde{H}$ is a map of dg comodules since all of the maps used to construct $\tilde{H}$ are maps of dg comodules.

Now, let $\tilde{H} : N \to F^*G^*M$ be a map of dg comodules over B_2. We will show that $\tilde{H}$ determines a map of dg comodules $H : N \to B_2 \otimes_G FM$ satisfying Equation 2.4. For $x \in Obj(B_2)$, let H be defined as follows:

$$\begin{aligned}
H_x : N(x) &\xrightarrow{\tilde{H}_x} F^*G^*M(x) \\
&\xrightarrow[inclusion]{canonical} \bigoplus_{\substack{y \in Obj(B_2) \\ z_1 \in Obj(B_1)}} B_2(x,y) \otimes B_1(Fy, z_1) \otimes M(Gz_1) \\
&\xrightarrow{id_{B_2} \otimes \epsilon_{B_1} \otimes id_M} \bigoplus_{y \in Obj(B_2)} B_2(x,y) \otimes M(GFy).
\end{aligned}$$

The universal property of G^*M implies that $(id_{B_2} \otimes \Delta_M) \circ H$ is equal to:

$$N(x) \xrightarrow{\tilde{H}_x} \bigoplus_{\substack{y \in Obj(B_2) \\ z_1 \in Obj(B_1)}} B_2(x,y) \otimes B_1(Fy, z_1) \otimes M(Gz_1)$$

$$\xrightarrow[(id_{B_2} \otimes \Delta_{B_1} \otimes id_M)]{(id_{B_2} \otimes id_{B_1} \otimes G \otimes id_M) \circ} \bigoplus_{\substack{y \in Obj(B_2) \\ y_1, z_1 \in Obj(B_1)}} B_2(x,y) \otimes B_1(Fy, y_1) \otimes B_0(Gy_1, Gz_1) \otimes M(Gz_1)$$

$$\xrightarrow{id_{B_2} \otimes \epsilon_{B_1} \otimes id_{B_0} \otimes id_M} \bigoplus_{\substack{y \in Obj(B_2) \\ z_1 \in Obj(B_1)}} B_2(x,y) \otimes B_0(GFy, Gz_1) \otimes M(Gz_1).$$

On the other hand, the universal property of F^* implies that $(id_{B_2} \otimes GF \otimes id_M) \circ (\Delta_{B_2} \otimes id_M) \circ H$ is equal to:

$$N(x) \xrightarrow{\tilde{H}_x} \bigoplus_{\substack{y \in Obj(B_2) \\ z_1 \in Obj(B_1)}} B_2(x,y) \otimes B_1(Fy, z_1) \otimes M(Gz_1)$$

$$\xrightarrow[(id_{B_2} \otimes \Delta_{B_1} \otimes id_M)]{(id_{B_2} \otimes G \otimes id_{B_1} \otimes id_M) \circ} \bigoplus_{\substack{y \in Obj(B_2) \\ y_1, z_1 \in Obj(B_1)}} B_2(x,y) \otimes B_0(GFy, Gy_1) \otimes B_1(y_1, z_1) \otimes M(Gz_1)$$

$$\xrightarrow{id_{B_2} \otimes id_{B_0} \otimes \epsilon_{B_1} \otimes id_M} \bigoplus_{\substack{y \in Obj(B_2) \\ z_1 \in Obj(B_1)}} B_2(x,y) \otimes B_0(GFy, Gz_1) \otimes M(Gz_1).$$

So, the difference between the two maps in Equation 2.4 comes down to the difference between $(\epsilon_{B_1} \otimes G) \circ \Delta_{B_1}$ and $(G \otimes \epsilon_{B_1}) \circ \Delta_{B_1}$. However, by the counitality of B_1, both of these maps are equal to G. So, H satisfies Equation 2.4. $\square$

2.3.2. Explicit description of $\hat{\lambda}^* C(\mathcal{A}')$

Let λ be a morphism in Λ that induces a morphism in χ with source $\mathcal{A} \in Obj(\chi)$. Recall that from Section 2.1.2 that we have a functor $\hat{\lambda} : B(\mathcal{A}) \to B(\lambda\mathcal{A})$. Applying the constructions in Section 2.3.1 to $\hat{\lambda}$, we get a functor $\hat{\lambda}^*$ from the category of conilpotent dg comodules over $B(\lambda\mathcal{A})$ to the category of conilpotent dg comodules over $B(\mathcal{A})$. Below, we compute explicitly the complexes $[\hat{\lambda}^* C(\lambda\mathcal{A})](f)$ for $f \in Obj(B(\mathcal{A}))$.

Proposition 2.2. *Let λ be a morphism in Λ that induces a morphism in χ with source $\mathcal{A} \in Obj(\chi)$. Fix $f_0 \in Obj(B(\mathcal{A}))$. As comodules,*

$$[\hat{\lambda}^* C(\lambda\mathcal{A})](f_0) \cong [B(\mathcal{A}) \otimes_{\hat{\lambda}} T(\lambda\mathcal{A})](f_0) := \bigoplus_{h \in Obj(B(\mathcal{A}))} B(\mathcal{A})(f_0, h) \otimes T(\lambda\mathcal{A})(\hat{\lambda}h) \tag{2.5}$$

where $T(\lambda\mathcal{A})(\hat{\lambda}h)$ are the cogenerators of $C(\lambda\mathcal{A})(\hat{\lambda}h)$ (see Section 1.3.4).

Remark 2.3.1. Proposition 2.2 holds for any quasi-cofree comodule over $B(\lambda\mathcal{A})$. The proof is the same.

Proof of Proposition 2.2. To simplify notation in this proof, we will drop all references to f_0 and, when unambiguous, references to $\lambda\mathcal{A}$. In other words, in this proof only,

$$C := C(\lambda\mathcal{A}) \text{ will denote } C(\lambda\mathcal{A})(f_0),$$

$$\hat{\lambda}^* C := \hat{\lambda}^* C(\lambda\mathcal{A}) \text{ will denote } [\hat{\lambda}^* C(\lambda\mathcal{A})](f_0),$$

$$B(\mathcal{A}) \otimes_{\hat{\lambda}} T \text{ will denote } [B(\mathcal{A}) \otimes_{\hat{\lambda}} T(\lambda\mathcal{A})](f_0),$$

$$B(\mathcal{A}) \otimes_{\hat{\lambda}} C \text{ will denote } [B(\mathcal{A}) \otimes_{\hat{\lambda}} C(\lambda\mathcal{A})](f_0),$$

$$B(\mathcal{A}) \otimes_{\hat{\lambda}} B(\lambda\mathcal{A}) \otimes C \text{ will denote } [B(\mathcal{A}) \otimes_{\hat{\lambda}} B(\lambda\mathcal{A}) \otimes C(\lambda\mathcal{A})](f_0).$$

To prove the proposition, we will give maps

$$F : \hat{\lambda}^* C \rightleftarrows B(\mathcal{A}) \otimes_{\hat{\lambda}} T : G$$

and show that $F \circ G = id_{B(\mathcal{A}) \otimes_{\hat{\lambda}} T}$ and $G \circ F = id_{\hat{\lambda}^* C}$. We define F as follows:

$$F : \hat{\lambda}^* C \xrightarrow[\text{inclusion}]{\text{canonical}} B(\mathcal{A}) \otimes_{\hat{\lambda}} C \xrightarrow[\text{cogenerators}]{\text{project onto}} B(\mathcal{A}) \otimes_{\hat{\lambda}} T.$$

To define G, we will give a map $G' : B(\mathcal{A}) \otimes_{\hat{\lambda}} T \to B(\mathcal{A}) \otimes_{\hat{\lambda}} C$, and show that the image of G' lands in $\hat{\lambda}^* C$. We define G' as follows:

$$G'(b \otimes t) = \sum_{(b)} b_{(1)} \otimes \hat{\lambda} b_{(2)} \cdot t$$

where $b \otimes t \in B(\mathcal{A}) \otimes_{\hat{\lambda}} T$ and $\hat{\lambda} b_{(2)} \cdot t$ are elements of $C(\lambda\mathcal{A})(\hat{\lambda} h)$ written in terms of cogenerators.

To prove that the image of G' lands in $\hat{\lambda}^* C$, we need to show that the two maps

$$(id_{B(\mathcal{A})} \otimes \Delta_C) \circ G', (id_{B(\mathcal{A})} \otimes \hat{\lambda} \otimes id_C) \circ (\Delta_{B(\mathcal{A})} \otimes id_C) \circ G' : B(\mathcal{A}) \otimes_{\hat{\lambda}} T \to B(\mathcal{A}) \otimes_{\hat{\lambda}} C \rightrightarrows B(\mathcal{A}) \otimes_{\hat{\lambda}} B(\lambda\mathcal{A}) \otimes C$$

coincide. We have

$$\begin{aligned}
[(1 \otimes \Delta_C) \circ G'](b \otimes t) &= \sum_{(b),\, (\hat{\lambda} b)} b_{(1)} \otimes (\hat{\lambda} b_{(2)})_{(1)} \otimes (\hat{\lambda} b_{(2)})_{(2)} \cdot t \\
&= \sum_{(b)} b_{(1)} \otimes \hat{\lambda} b_{(2)} \otimes \hat{\lambda} b_{(3)} \cdot t \\
&= [(id_{B(\mathcal{A})} \otimes \hat{\lambda} \otimes id_C) \circ (\Delta_{B(\mathcal{A})} \otimes id_C) \circ G'](b \otimes t)
\end{aligned}$$

where the second equality holds since $\hat{\lambda}$ is a map of cocategories and $\Delta_{B(\mathcal{A})}$ is coassociative.

It's clear from the definitions that F and G are maps of comodules and that $F \circ G = id_{B(\mathcal{A}) \otimes_{\hat{\lambda}} T}$. All that remains is to show that $G \circ F = id_{\hat{\lambda}^* C}$. Let $\kappa = \Sigma_i b_i \otimes \beta_i \cdot t_i$ be an arbitrary element of $\hat{\lambda}^* C \hookrightarrow B(\mathcal{A}) \otimes_{\hat{\lambda}} C$ where $\beta_i \cdot t_i$ are elements of $C(\lambda\mathcal{A})(\hat{\lambda}h)$ written in terms of cogenerators. Then,

$$GF(\kappa) = GF(\Sigma_i b_i \otimes \beta_i \cdot t_i) = \sum_{\substack{i, \\ \beta_i = 1, \\ (b_i)}} b_{i(1)} \otimes \hat{\lambda} b_{i(2)} \cdot t_i.$$

We can divide the terms in κ into two groups: (a) terms in which $\beta_i = 1 \in k$ and (b) terms in which $\beta_i \neq 1 \in k$. Likewise, we can divide the terms in $GF(\kappa)$ into (a) terms in which $\hat{\lambda} b_{i(2)} = 1$ and (b) terms in which $\hat{\lambda} b_{i(2)} \neq 1$. From the definitions of F and G, it's clear that the Group A terms in κ are exactly the Group A terms in $GF(\kappa)$.

To show that the Group B terms are the same, let $b_i \otimes \beta_i \cdot t_i$ be an arbitrary Group B term in κ. Then, there is a term $b_i \otimes \beta_i \otimes t_i$ in $(id_{B(\mathcal{A})} \otimes \Delta_C)\kappa$. Since $(id_{B(\mathcal{A})} \otimes \Delta_C)\kappa = (id_{B(\mathcal{A})} \otimes \hat{\lambda} \otimes id_C) \circ (\Delta_{B(\mathcal{A})} \otimes id_C)\kappa$, there must be a Group A term, $b_{j_i} \otimes t_{j_i}$, in κ such that $b_i \otimes \beta_i \otimes t_i$ is one of the terms in the sum $[(id_{B(\mathcal{A})} \otimes \hat{\lambda} \otimes id_C) \circ (\Delta_{B(\mathcal{A})} \otimes id_C)](b_{j_i} \otimes t_{j_i}) = \sum_{(b_{j_i})} b_{j_i(1)} \otimes \hat{\lambda} b_{j_i(2)} \otimes t_{j_i}$. Thus, $b_i \otimes \beta_i \cdot t_i$ is a Group B term in $GF(\kappa)$.

Now let $b_{i(1)} \otimes \hat{\lambda} b_{i(2)} \cdot t_i$ be an arbitrary Group B term in $GF(\kappa)$. Then, $b_{i(1)} \otimes \hat{\lambda} b_{i(2)} \otimes t_i$ is a term in $(id_{B(\mathcal{A})} \otimes \hat{\lambda} \otimes id_C) \circ (\Delta_{B(\mathcal{A})} \otimes id_C)\kappa = (id_{B(\mathcal{A})} \otimes \Delta_C)\kappa$. So, there is a Group B term, $b_{j_i} \otimes \beta_{j_i} \cdot t_{j_i}$, in κ such that $b_{i(1)} \otimes \hat{\lambda} b_{i(2)} \otimes t_i$ is one of the terms in the sum $(id_{B(\mathcal{A})} \otimes \Delta_C)(b_{j_i} \otimes \beta_{j_i} \cdot t_{j_i}) = \sum_{(\beta_{j_i})} b_{j_i} \otimes \beta_{j_i(1)} \otimes \beta_{j_i(2)} \cdot t_{j_i}$. Since t_i is a cogenerator, the only term in the sum that could be equal to $b_{i(1)} \otimes \hat{\lambda} b_{i(2)} \otimes t_i$ is $b_{j_i} \otimes \beta_{j_i} \otimes t_{j_i}$. Thus, $b_{i(1)} \otimes \hat{\lambda} b_{i(2)} \cdot t_i$ is a Group B term in κ. $\square$

2.4. Pullbacks of dg comodules–examples

Example 2.4.1 (Another definition of C(1))**.** *Let* $\lambda = \delta_{0,1} \in \Lambda([1],[0])$. *Fix algebras* A_0 *and* A_1, *and set*

$$\hat{\lambda} : B(1) := B(A_0 \to A_1 \to A_0) \to B(A_0 \to A_0) =: B(0)$$

$$C(1) := C(A_0 \to A_1 \to A_0)$$

$$C(0) := C(A_0 \to A_0)$$

($\hat{\lambda}$ *is given by braces, see Section 2.1.2.)*

From Proposition 2.2, we have $\hat{\delta}^*_{0,1}C(0) \cong [B(1) \otimes_{\hat{\delta}_{0,1}} T(0)]$ *as comodules. We will use the isomorphisms* F *and* G *in Proposition 2.2 to induce a differential on* $[B(1) \otimes_{\hat{\delta}_{0,1}} T(0)]$ *from* $\hat{\delta}^*_{0,1}C(0)$. *Let* $\phi_{0,1}\ldots\phi_{0,k_0}|\phi_{1,1}\ldots\phi_{1,k_1}|\alpha$ *be a typical element of* $[B(1) \otimes_{\hat{\delta}_{0,1}} T(0)](f_{0,0}, f_{1,0})$ *(see Figure 1.2 for notational conventions). Then,*

$$\begin{aligned}
&d_{induced}(\phi_{0,1}\ldots\phi_{0,k_0}|\phi_{1,1}\ldots\phi_{1,k_1}|\alpha)\\
&= F d_{\hat{\delta}^*_{0,1}C(0)} G(\phi_{0,1}\ldots\phi_{0,k_0}|\phi_{1,1}\ldots\phi_{1,k_1}|\alpha)\\
&= [F \circ (d_{B(1)} \otimes id_{C(0)} + id_{B(1)} \otimes d_{C(0)})]\\
&\quad \Big(\sum_{\substack{1 \le r_0 \le k_0+1\\ 1 \le r_1 \le k_1+1}} (\phi_{0,1}\ldots\phi_{0,r_0-1}|\phi_{1,1}\ldots\phi_{1,r_1-1}) \otimes ((\phi_{0,r_0}\ldots\phi_{0,k_0}) \bullet (\phi_{1,r_1}\ldots\phi_{1,k_1})|\alpha) \Big)\\
&= d_{C(1)(f_{0,0},f_{1,0})}(\phi_{0,1}\ldots\phi_{0,k_0}|\phi_{1,1}\ldots\phi_{1,k_1}|\alpha)
\end{aligned}$$

where the last equality holds by looking at which terms from $d_{B(1)} \otimes id_{C(0)} + id_{B(1)} \otimes d_{C(0)}$ *are non-zero after projecting to cogenerators, and seeing that those are the same terms as in* $d_{C(1)}$. *So,* $\hat{\delta}^*_{0,1} C(0) \cong C(1)$ *as dg comodules.*

Example 2.4.2 (Another definition of $C(n)$)**.** *Let* $\lambda = \delta_{0,n} \in \Lambda([n],[n-1])$. *Fix algebras* $A_0, \ldots A_n$*, and set*

$$\hat{\lambda} : B(n) := B(A_0 \to A_1 \to \cdots \to A_n \to A_0) \to B(A_0 \to A_2 \to A_3 \to \cdots \to A_n \to A_0)$$

$$C(n) := C(A_0 \to A_1 \to \cdots \to A_n \to A_0)$$

($\hat{\lambda}$ *is given by bracing the first and second terms, see Section 2.1.2.)*

Example 2.4.1 shows

$$C(1) \cong \hat{\delta}^*_{0,1} C(0) \tag{2.6}$$

as dg comodules. Given Equation 2.6 above as a base case, we can show by induction that

$$C(n) \cong \hat{\delta}^*_{0,n} \ldots \hat{\delta}^*_{0,1} C(0)$$

as dg comodules. Suppose that $C(W_0 \to \cdots \to W_{n-1} \to W_0) \cong \hat{\delta}^*_{0,n-1} \ldots \hat{\delta}^*_{0,1} C(W_0 \to W_0)$ *for any choice of algebras* $W_0, \ldots W_{n-1}$. *(inductive hypothesis). Then, as comodules, we* *know*

$$\begin{aligned}
\hat{\delta}^*_{0,n}\ldots\hat{\delta}^*_{0,1}C(0) &\cong \hat{\delta}^*_{0,n}C(A_0 \to A_2 \to \ldots \to A_n \to A_0) \quad \textit{(inductive hypothesis applied to algebras } A_0, A_2, .\\
&\cong B(n) \otimes_{\hat{\delta}_{0,n}} T(A_0 \to A_2 \to \ldots \to A_n \to A_0) \quad \textit{(Proposition 2.2)}\\
&\cong B(n) \otimes_{\hat{\delta}_{0,n}} T(A_0 \to A_0) \quad \textit{(Definition of } T\textit{)}\\
&\cong C(n) \quad \textit{(Definition of } C(n)\textit{)}
\end{aligned}$$

where $T(n)$ are the cogenerators of $C(n)$ (see Definition 1.3.4).

To show that the differentials coincide, we compute

$$\begin{aligned}
&Fd_{\hat{\delta}^*_{0,n}\ldots\hat{\delta}^*_{0,1}C(0)}G(\phi_{0,1}\ldots\phi_{0,k_0}|\ldots|\phi_{n,1}\ldots\phi_{n,k_n}|t)\\
&= [F \circ (d_{B(n)} \otimes id_{\hat{\delta}^*_{0,n-1}\ldots\hat{\delta}^*_{0,1}C(0)} + id_{B(n)} \otimes d_{\hat{\delta}^*_{0,n-1}\ldots\hat{\delta}^*_{0,1}C(0)})]\\
&\quad \Big(\sum_{\substack{1\le j\le n\\ 1\le r_j\le k_j+1}} (\phi_{0,1}\ldots\phi_{0,r_0-1}|\ldots|\phi_{n,1}\ldots\phi_{n,r_1-1})\otimes\\
&\qquad\qquad ((\phi_{0,r_0}\ldots\phi_{0,k_0})\bullet(\phi_{1,r_1}\ldots\phi_{1,k_1})|\phi_{2,r_1}\ldots\phi_{2,k_2}|\ldots|\phi_{n,r_1}\ldots\phi_{n,k_n}|t)\Big)\\
&= [F \circ (d_{B(n)} \otimes id_{C(A_0\to A_2\to\ldots\to A_n\to A_0)} + id_{B(n)} \otimes d_{C(A_0\to A_2\to\ldots\to A_n\to A_0)})]\\
&\quad \Big(\sum_{\substack{1\le j\le n\\ 1\le r_j\le k_j+1}} (\phi_{0,1}\ldots\phi_{0,r_0-1}|\ldots|\phi_{n,1}\ldots\phi_{n,r_1-1})\otimes\\
&\qquad\qquad ((\phi_{0,r_0}\ldots\phi_{0,k_0})\bullet(\phi_{1,r_1}\ldots\phi_{1,k_1})|\phi_{2,r_1}\ldots\phi_{2,k_2}|\ldots|\phi_{n,r_1}\ldots\phi_{n,k_n}|t)\Big)
\end{aligned}$$

where the last equality holds by the inductive hypothesis. The terms from $d_{B(n)}\otimes id_{C(A_0\to A_2\to\ldots\to A_n\to A_0)}+$ $id_{B(n)}\otimes d_{C(A_0\to A_2\to\ldots\to A_n\to A_0)}$ that are non-zero after projecting to cogenerators are exactly the terms in $d_{C(n)}$. So, $C(n) \cong \hat{\delta}^*_{0,n}\ldots\hat{\delta}^*_{0,1}C(0)$ as dg comodules.

Example 2.4.3 (Yet another description of C(n))**.** *Choose a sequence of generating coboundaries* $\delta_{i_1,1}, \ldots, \delta_{i_n,n}$ *with* $0 \leq i_j \leq j-1$, $1 \leq j \leq n$, $n > 0$. *Then,*

$$\delta_{i_1,1} \circ \cdots \circ \delta_{i_n,n} = \delta_{0,1} \circ \cdots \circ \delta_{0,n} = \textit{unique map in } \Delta([n],[0]) \subset \Lambda([n],[0]).$$

This implies that, as functors on categories of comodules,

$$\begin{aligned} \hat{\delta}^*_{i_n,n} \ldots \hat{\delta}^*_{i_1,1} &= (\delta_{i_1,1} \widehat{\circ \cdots \circ} \, \delta_{i_n,n})^* \quad \textit{(Proposition 2.1)} \\ &= (\delta_{0,1} \widehat{\circ \cdots \circ} \, \delta_{0,n})^* \quad \textit{(Computation above)} \\ &= \hat{\delta}^*_{0,n} \ldots \hat{\delta}^*_{0,1} \quad \textit{(Proposition 2.1).} \end{aligned}$$

Since braces are associative, the differentials on $\hat{\delta}^*_{i_n,n} \ldots \hat{\delta}^*_{i_1,1} C(A_0 \to A_0)$ *and* $\hat{\delta}^*_{0,n} \ldots \hat{\delta}^*_{0,1} C(A_0 \to A_0)$ *coincide. So,* $\hat{\delta}^*_{i_n,n} \ldots \hat{\delta}^*_{i_1,1} C(A_0 \to A_0) \cong \hat{\delta}^*_{0,n} \ldots \hat{\delta}^*_{0,1} C(A_0 \to A_0)$ *as dg comodules.*

Example 2.4.4 (Pullbacks along codegeneracies)**.** *Fix algebras* $A_0, \ldots, A_n$ *and let* $\sigma_{i,n} \in \Lambda([n],[n+1])$, $0 \leq i \leq n$ *be a generating codegeneracy. Set*

$$\hat{\sigma}_{i,n} : B(n) := B(A_0 \to \ldots \to A_n \to A_0) \to B(A_0 \to \cdots \to A_i \to A_i \to \ldots A_n \to A_0)$$

$$C(n) := C(A_0 \to \cdots \to A_n \to A_0)$$

From Proposition 2.2, we know that $\hat{\sigma}^*_{i,n} C(A_0 \to \cdots \to A_i \to A_i \to \ldots A_n \to A_0) \cong B(n) \otimes_{\hat{\sigma}_{i,n}} T(A_0 \to \cdots \to A_i \to A_i \to \ldots A_n \to A_0) \cong C(n)$ *as comodules. To show that* *the differentials coincide, we compute*

$$F d_{\hat{\sigma}^*_{i,n} C(A_0 \to \dots \to A_i \to A_i \to \dots A_n \to A_0)} G(\phi_{0,1} \dots \phi_{0,k_0} | \dots | \phi_{n,1} \dots \phi_{n,k_n} | t)$$

$$= [F \circ (d_{B(n)} \otimes id_{\hat{\sigma}^*_{i,n} C(A_0 \to \dots \to A_i \to A_i \to \dots A_n \to A_0)} + id_{B(n)} \otimes d_{\hat{\sigma}^*_{i,n} C(A_0 \to \dots \to A_i \to A_i \to \dots A_n \to A_0)})]$$

$$\Big(\sum_{\substack{1 \leq j \leq n \\ 1 \leq r_j \leq k_j + 1}} (\phi_{0,1} \dots \phi_{0,r_0 - 1} | \dots | \phi_{n,1} \dots \phi_{n,r_1 - 1}) \otimes$$

$$(\phi_{0,r_0} \dots \phi_{0,k_0} | \dots | \phi_{0,r_{i-1}} \dots \phi_{0,k_{i-1}} | 1 | \phi_{0,r_i} \dots \phi_{0,k_i} | \dots | \phi_{n,r_1} \dots \phi_{n,k_n} | t) \Big).$$

Since 1 is a unit for braces, the terms from $d_{B(n)} \otimes id_{\hat{\sigma}^*_{i,n} C(A_0 \to \dots \to A_i \to A_i \to \dots A_n \to A_0)} + id_{B(n)} \otimes d_{\hat{\sigma}^*_{i,n} C(A_0 \to \dots \to A_i \to A_i \to \dots A_n \to A_0)}$ *that are non-zero after projecting to cogenerators are exactly the terms in* $d_{C(n)}$*. So,* $\hat{\sigma}^*_{i,n} C(A_0 \to \dots \to A_i \to A_i \to \dots A_n \to A_0) \cong C(n)$ *as dg comodules.*

Example 2.4.5 (Pullbacks along rotations)**.** *Fix algebras* $A_0, \dots, A_n$ *and let* $\tau_n \in \Lambda([n],[n])$ *be a generating rotation. Set*

$$\hat{\tau}_n : B(n) := B(A_0 \to \dots \to A_n \to A_0) \to B(A_n \to A_0 \to \dots \to A_n)$$

$$C(n) := C(A_0 \to \dots \to A_n \to A_0)$$

From Proposition 2.2, we know that $\hat{\tau}^*_n C(A_n \to A_0 \to \dots \to A_n) \cong B(n) \otimes_{\hat{\tau}_n} T(A_n \to A_0 \to \dots \to A_n)$ *as comodules. Unpacking the righthand side, we see that* $B(n) \otimes_{\hat{\tau}_n} T(A_n \to A_0 \to \dots \to A_n) \cong C(A_n \to A_0 \to \dots \to A_n)$ *as complexes–the isomorphism is given by* $\hat{\tau}_n \otimes id_T$*.*

2.5. Adjunction between λ^* and $\lambda_\#$

In this section, we define $\lambda_\#$, the left adjoint to λ^*. More precisely, for any functor, $\lambda : B_1 \to B_0$ between conilpotent dg cocategories, we define a functor $\lambda_\#$ from the category of conilpotent dg comodules over B_1 to the category of conilpotent dg comodules over B_0. The adjunction will be used to show that structures we've established for $(B(n), C(n))$ still exist after we pass from cocategories and comodules to categories and modules by applying (a categorified) $Cobar(-)$ to $(B(n), C(n))$. If a lesson of this thesis is that working with cocategories is more tractable than with categories, then the reader may skip this section or save it until s/he is ready for Chapter (...).

2.5.1. The functors $\lambda_\#$

Let $\lambda : B_1 \to B_0$ be a functor between conilpotent dg cocategories. Let C be a conilpotent dg comodule over B_1. We define $\lambda_\# C$ as follows: for $f \in Obj(B_0)$,

$$\lambda_\# C(f) := \Big(\bigoplus_{f' \in \lambda^{-1} f} C^\bullet(f'),$$

$$\Delta_{\lambda_\# C}(f) : \bigoplus_{f' \in \lambda^{-1} f} C^\bullet(f') \xrightarrow{\bigoplus_{f'} \Delta_{C^\bullet}(f')} \bigoplus_{\substack{f' \in \lambda^{-1} f \\ h' \in Obj(B_1)}} B_1^\bullet(f', h') \otimes C^\bullet(h')$$

$$\xrightarrow{\bigoplus_{h', f'} \lambda \otimes id_{C^\bullet(h')}} \bigoplus_{h' \in Obj(B_1)} B_0^\bullet(f, \lambda h') \otimes C^\bullet(h')$$

$$\xrightarrow{include} \bigoplus_{h \in Obj(B_0)} B_0^\bullet(f, h) \otimes \Big(\bigoplus_{h' \in \lambda^{-1} h} C^\bullet(h') \Big) \Big).$$

To check that $\Delta_{\lambda_\# C}$ is well-defined, we need that the image of the first map, $\bigoplus_{f'} \Delta_{C^\bullet}(f')$, is a finite sum. This is true since C being conilpotent implies that the image of $\Delta_{C^\bullet}(f')$

is a finite sum for each $f' \in Obj(B_1)$. If $\lambda^{-1}f$ is empty, we set $\lambda_{\#}C(f) := 0$. It is straightforward to check that $(\lambda_{\#}C, \Delta_{\lambda_{\#}C})$ is coassociative, conilpotent and coaugmented. We will call $\lambda_{\#}$ "co-restriction of scalars".

Let $F : C \to D$ be map of dg comodules over B_1. We define $\lambda_{\#}F$ as follows:

$$(\lambda_{\#}F)_f : \lambda_{\#}C(f) = \bigoplus_{f' \in \lambda^{-1}f} C^{\bullet}(f') \xrightarrow{\bigoplus_{f' \in \lambda^{-1}f} F_{f'}} \bigoplus_{f' \in \lambda^{-1}f} D^{\bullet}(f') = \lambda_{\#}D(f).$$

It's straightforward to check that $\lambda_{\#}$ is a functor (i.e., respects composition of morphisms).

2.5.2. Adjunction

Proposition 2.3. *Given a functor between conilpotent dg cocategories, $\lambda : B_1 \to B_0$, let*

$$\lambda^* : \begin{array}{c}\textit{Category of}\\ \textit{conilpotent}\\ \textit{dg comodules over } B_0\end{array} \leftrightarrows \begin{array}{c}\textit{Category of}\\ \textit{conilpotent}\\ \textit{dg comodules over } B_1\end{array} : \lambda_{\#}$$

be the functors defined in Sections 2.3.1 and 2.5.1. Then, $\lambda_{\#}$ is left adjoint to λ^*.

Remark 2.5.1. Proposition 2.3 is a categorified co-version of the adjunction between extension of scalars (left) and restriction of scalars (right) for modules over algebras.

Proof of Proposition 2.3. Let C be a conilpotent dg comodule over B_1 and D be a dg conilpotent dg comodule over B_0. We want to show that

$$Hom_{B_1}(C,\, \lambda^* D) = Hom_{B_0}(\lambda_{\#}C,\, D)$$

as sets.

We will give maps

$$\Phi : Hom_{B_0}(\lambda_{\#}C,\, D) \leftrightarrows Hom_{B_1}(C,\, \lambda^* D) : \Phi^{-1}$$

satisfying $\Phi \circ \Phi^{-1} = id$ and $\Phi^{-1} \circ \Phi = id$.

First, we define Φ. Let F be a morphism from $\lambda_{\#}C$ to D. By defintion, for $f \in Obj(B_0)$, we have maps of complexes

$$F_f : \bigoplus_{f' \in \lambda^{-1} f} C^\bullet(f') \to D^\bullet(f).$$

Define $\Phi F \in Hom_{B_1}(C,\, \lambda^* D)$ as follows: for $f' \in Obj(B_1)$,

$$\begin{aligned} \Phi F_{f'} : C^\bullet(f') &\xrightarrow{\Delta_C} \bigoplus_{h' \in Obj(B_1)} B_1^\bullet(f', h') \otimes C^\bullet(h') \\ &\xrightarrow{\bigoplus_{h'} id_{B_1} \otimes F_{\lambda h'}|_{h'}} \bigoplus_{h' \in Obj(B_1)} B_1^\bullet(f', h') \otimes D^\bullet(\lambda h') \qquad (2.7) \\ &\xrightarrow{include} [B_1 \otimes_\lambda D](f'). \end{aligned}$$

By the universal property of $\lambda^* D$, this defines a morphism $C \to \lambda^* D$ if the two maps

$$(id_{B_1} \otimes \Delta_D) \circ \Phi F,\ (id_{B_1} \otimes \lambda \otimes id_D) \circ (\Delta_{B_1} \otimes id_D) \circ \Phi F : C \rightrightarrows B_1 \otimes_\lambda B_0 \otimes D$$

coincide. In fact, on $f' \in Obj(B_1)$, both maps are equal to:

$$
\begin{aligned}
C^\bullet(f') &\xrightarrow{\Delta_C} \bigoplus_{h' \in Obj(B_1)} B_1^\bullet(f', h') \otimes C^\bullet(h') \\
&\xrightarrow{\bigoplus_{h'} id_{B_1} \otimes \Delta_C} \bigoplus_{g', h' \in Obj(B_1)} B_1^\bullet(f', g') \otimes B_1^\bullet(g', h') \otimes C^\bullet(h') \\
&\xrightarrow{\bigoplus_{h', g'} id_{B_1} \otimes \lambda \otimes 1_C} \bigoplus_{g', h' \in Obj(B_1)} B_1^\bullet(f', g') \otimes B_0^\bullet(\lambda g', \lambda h') \otimes C^\bullet(h') \\
&\xrightarrow{\bigoplus_{h', g'} id_{B_1} \otimes id_{B_0} \otimes F_{\lambda h'}|_{h'}} \bigoplus_{g', h' \in Obj(B_1)} B_1^\bullet(f', g') \otimes B_0^\bullet(\lambda g', \lambda h') \otimes D^\bullet(\lambda h').
\end{aligned}
$$

This fact follows from F being a map of comodules. It's also clear that ΦF commutes with coproducts and differentials. So, we've shown $\Phi F \in Hom_{B_1}(C, \lambda^* D)$.

Second, we define Φ^{-1}. Now, let $F \in Hom_{B_1}(C, \lambda^* D)$. For $f \in Obj(B_0)$, define

$$
\begin{aligned}
\Phi^{-1} F_f : \bigoplus_{f' \in \lambda^{-1} f} C^\bullet(f') &\xrightarrow{\bigoplus_{f'} F_{f'}} \bigoplus_{\substack{f' \in \lambda^{-1} f, \\ h' \in Obj(B_1)}} B_1^\bullet(f', h') \otimes D^\bullet(\lambda h') \\
&\xrightarrow{\bigoplus_{f', h'} \lambda \otimes id_D} \bigoplus_{h \in Obj(B_0)} B_0^\bullet(f, h) \otimes D^\bullet(h) \\
&\xrightarrow{\bigoplus_{h} \epsilon_{B_0} \otimes id_D} D^\bullet(f).
\end{aligned}
$$

It's clear that $\Phi^{-1} F$ commutes with the differentials. We will show that $\Phi^{-1} F$ is a map of comodules. Figure 2.1 gives a diagram showing that

$$
\Delta_D \circ \Phi^{-1} F_f = \Big(\bigoplus_{f', h', r'} \epsilon_{B_0} \lambda \otimes \lambda \otimes id_D\Big) \circ \Big(\bigoplus_{f', h'} \Delta_{B_1} \otimes id_D\Big) \circ \Big(\bigoplus_{f'} F_{f'}\Big). \tag{2.8}
$$

On the other hand, Figure **??** gives a diagram showing that

$$(2.9)\qquad (id_{B_1} \otimes \Phi^{-1}F) \circ \Delta_{\lambda_{\#}C} = (\bigoplus_{f',h',r'} \lambda \otimes \epsilon_{B_0}\lambda \otimes id_D) \circ (\bigoplus_{f',h'} \Delta_{B_1} \otimes id_D) \circ (\bigoplus_{f'} F_{f'}).$$

We see that the righthand sides of Equations 2.8 and 2.9 are the same except for the B_0 factor on which ϵ_{B_0} acts. However, in general, for $\lambda : B_1 \to B_0$ a map of dg cocategories, we have

$$\begin{aligned}
(\lambda \otimes \epsilon_{B_0}\lambda) \circ \Delta_{B_1} &= (id_{B_0} \otimes \epsilon_{B_0}) \circ \Delta_{B_0} \circ \lambda \quad (\lambda \text{ commutes with coproduct}) \\
&= id_{B_0} \circ \lambda \quad (\text{definition of cocategory}) \\
&= (\epsilon_{B_0} \otimes id_{B_0}) \circ (\Delta_{B_0}) \circ \lambda \quad (\text{definition of cocategory}) \\
&= (\epsilon_{B_0}\lambda \otimes \lambda) \circ \Delta_{B_1} \quad (\lambda \text{ commutes with coproduct}).
\end{aligned}$$

So, $(id_{B_1} \otimes \Phi^{-1}F) \circ \Delta_{\lambda_{\#}C} = \Delta_D \circ \Phi^{-1}F$, and $\Phi^{-1}F \in Hom_{B_0}(\lambda_{\#}C, D)$.

For $F : C \to \lambda^*D$ a map of dg comodules and $f' \in B_1$, Figure 2.3 shows that $\Phi\Phi^{-1}F_{f'} = F_{f'}$. For $F : \lambda_{\#}C \to D$ a map of dg comodules and $f \in B_0$, Figure **??** shows that $\Phi^{-1}\Phi F_f = F_f$. Thus, we have $\Phi\Phi^{-1} = id$ and $\Phi^{-1}\Phi = id$. $\square$

$$
\begin{tikzcd}
\bigoplus_{f' \in \lambda^{-1} f} C^\bullet(f') \arrow[d, "\bigoplus_{f'} F_{f'}"] & & \\
\bigoplus_{\substack{f' \in \lambda^{-1} f \\ h' \in Obj(B_1)}} B_1^\bullet(f', h') \otimes D^\bullet(\lambda h') \arrow[r, "\bigoplus_{f',h'} \lambda \otimes id_D"] \arrow[d, shift right, bend right, "\bigoplus_{f',h',r'} \substack{(\Delta_{B_1} \otimes id_D) \circ \\ (id_{B_1} \otimes \lambda \otimes id_D)}"'] \arrow[d, shift left, bend left, "\bigoplus_{f',h'} id_{B_1} \otimes \Delta_D"] & \bigoplus_{h \in Obj(B_0)} B_0^\bullet(f, h) \otimes D^\bullet(h) \arrow[r, "\bigoplus_h \epsilon_{B_0} \otimes id_D"] & D^\bullet(f) \arrow[d, "\Delta_D"] \\
\bigoplus_{\substack{f' \in \lambda^{-1} f, \\ h' \in Obj(B_1), \\ r \in Obj(B_0)}} B_0^\bullet(f', h') \otimes B_1^\bullet(\lambda h', r) \otimes D^\bullet(r) \arrow[rr, "\bigoplus_{f',h',r} \epsilon_{B_0} \lambda \otimes id_{B_1} \otimes id_D"'] & & \bigoplus_{h \in Obj(B_0)} B_0^\bullet(f, h) \otimes D^\bullet(h)
\end{tikzcd}
$$

Figure 2.1. Commuting diagram involving $\Delta_D \circ \Phi^{-1} F =$ composition of red arrows
The fact that $F : C \to \lambda^* D$ and the universal property of $\lambda^* D$ imply that the diagram commutes.

$$\begin{tikzcd}
\bigoplus_{f'\in\lambda^{-1}f} C^\bullet(f') \arrow[r, "\bigoplus_{f'}\Delta_C"] \arrow[d, "\bigoplus_{f'} F_{f'}"] & \bigoplus_{\substack{f'\in\lambda^{-1}f\\ h'\in Obj(B_1)}} B_1^\bullet(f',h')\otimes C^\bullet(h') \arrow[r, "\bigoplus_{f',h'} \lambda\otimes id_C"] \arrow[d, "\bigoplus_{f',h'} id_{B_1}\otimes F_{\lambda h'}|_{h'}"] & \bigoplus_{h'\in Obj(B_1)} B_0^\bullet(f,\lambda h')\otimes C^\bullet(h') \arrow[d, "\bigoplus_{h'} id_{B_0}\otimes F_{\lambda h'}|_{h'}"] \\
\bigoplus_{\substack{f'\in\lambda^{-1}f,\\ r'\in Obj(B_1)}} B_1^\bullet(f',r')\otimes D^\bullet(\lambda r') \arrow[r, "\bigoplus_{f',r'} \substack{\Delta_{\lambda^* D}=\\ \Delta_{B_1}\otimes id_D}"] & \bigoplus_{\substack{f'\in\lambda^{-1}f,\\ h',r'\in Obj(B_1)}} B_1^\bullet(f',h')\otimes B_1^\bullet(h',r')\otimes D^\bullet(\lambda r') \arrow[r, "\bigoplus_{f',h',r'} \lambda\otimes id_{B_0}\otimes id_D"] & \bigoplus_{h',r'\in Obj(B_1)} B_0^\bullet(f,\lambda h')\otimes B_1^\bullet(h',r')\otimes D^\bullet(\lambda r') \arrow[d, "\bigoplus_{h',r'} id_{B_0}\otimes\epsilon_{B_0}\lambda\otimes id_D"] \\
 & & \bigoplus_{h\in Obj(B_0)} B_0^\bullet(f,h)\otimes D^\bullet(h)
\end{tikzcd}$$

Figure 2.2. Commuting diagram involving $(id_{B_1}\otimes\Phi^{-1}F)\circ\Delta_{\lambda_{\#}C}$ = composition of red arrows
The fact that F respects coproducts implies that the left square commutes.

$$
\begin{tikzcd}
C^\bullet(f') \arrow[r, "\Delta_C"] \arrow[d, "F_{f'}"] & \bigoplus\limits_{g' \in Obj(B_1)} B_1^\bullet(f',g') \otimes C^\bullet(g') \arrow[d, "\bigoplus\limits_{g'} id_{B_1} \otimes F_{g'}"] & \\
\bigoplus\limits_{h' \in Obj(B_1)} B_1^\bullet(f',h') \otimes D^\bullet(\lambda h') \arrow[r, "\Delta_{\lambda^* D} = \bigoplus\limits_{h'} \Delta_{B_1} \otimes id_D"] \arrow[rr, bend right, "id"'] & \bigoplus\limits_{g',h' \in Obj(B_1)} B_1^\bullet(f',g') \otimes B_1^\bullet(g',h') \otimes D^\bullet(\lambda h') \arrow[r, "\bigoplus\limits_{g',h'} id_{B_1} \otimes (\epsilon_{B_0} \lambda = \epsilon_{B_1}) \otimes id_D"] & \bigoplus\limits_{g' \in Obj(B_1)} B_1^\bullet(f',g') \otimes D^\bullet(\lambda g')
\end{tikzcd}
$$

Figure 2.3. Commuting diagram involving $\Phi\Phi^{-1}F_{f'}$ = composition of red arrows
The square commutes because F respects coproducts; the composition of the bottom row of horizontal arrows is equal to the identity because $\lambda_\# D$ satisfies counitality.

$$\begin{tikzcd}
\bigoplus_{f'\in\lambda^{-1}f} C^\bullet(f') \arrow[r, "\bigoplus_{f'}\Delta_C"] \arrow[dd, "F_f"'] & \bigoplus_{\substack{f'\in\lambda^{-1}f,\\ g'\in Obj(B_1)}} B_1^\bullet(f',g')\otimes C^\bullet(g') \arrow[r, "\bigoplus_{f',g'} id_{B_1}\otimes F_{\lambda g'}|_{g'}"] \arrow[d, "\bigoplus_{f',g'}\lambda\otimes id_C"] & \bigoplus_{\substack{f'\in\lambda^{-1}f,\\ g'\in Obj(B_1)}} B_1^\bullet(f',g')\otimes D\bullet(\lambda g') \arrow[d, "\bigoplus_{f',g'}\lambda\otimes id_D"] & \\
 & \bigoplus_{g'\in Obj(B_1)} B_0^\bullet(f,\lambda g')\otimes C^\bullet(g') \arrow[r, "\bigoplus_{g'} id_{B_0}\otimes F_{\lambda g'}|_{g'}"] & \bigoplus_{g\in Obj(B_0)} B_0^\bullet(f,g)\otimes D^\bullet(g) \arrow[r, "\bigoplus_{g}\epsilon_{B_0}\otimes id_D"] & D^\bullet(f) \\
D^\bullet(f) \arrow[urr, "\Delta_D"'] \arrow[urrr, bend right, "id"'] & & &
\end{tikzcd}$$

Figure 2.4. Commuting diagram involving $\Phi^{-1}\Phi F_f$ = composition of red arrows
The concave pentagon on the left side commutes because F respects coproducts; the triangle in the bottom right corner commutes because D satisfies counitality.

2.6. Maps $\lambda_!$

In this section, we give maps $\lambda_! : C(\mathcal{A}) \to \hat{\lambda}^* C(\lambda\mathcal{A})$ of dg comodules over $B(\mathcal{A})$ where λ is a generating morphism in Λ that induces a morphsim in χ with source $\mathcal{A} \in Obj(\chi)$. Showing that the $\lambda_!$'s satisfy cyclic relations up to homotopy is the computational heart of this thesis, and will be done in the next chapter. For now, we introduce the $\lambda_!$'s.

Technically, we should write $\lambda_{\mathcal{A}!}$ instead of $\lambda_!$, but we will be clear about the source when needed. In this section, we fix algebras $A_0, \ldots, A_n$ and set $B(n) := B(A_0 \to \ldots \to A_n \to A_0)$, $C(n) := C(A_0 \to \ldots \to A_n \to A_0)$. We define maps $\lambda_! : C(A_0 \to \ldots \to A_n \to A_0) \to \hat{\lambda}^* C(\lambda(A_0 \to \ldots \to A_n \to A_0))$.

2.6.1. Generating coboundaries $\delta_{j,n!}$ for $n \in \mathbb{N}$, $0 \leq j \leq n-1$

From Example 2.4.3, we know that $C(n) \cong \hat{\delta}^*_{j,n} C(A_0 \to \ldots \to A_j \longrightarrow A_{j+2} \to \ldots \to A_0)$. So, define $\delta_{j,n!} : C(n) \xrightarrow{id} C(n) \cong \hat{\delta}^*_{j,n} C(A_0 \to \ldots \to A_j \longrightarrow A_{j+2} \to \ldots \to A_0)$.

2.6.2. Generating codegeneracies $\sigma_{i,n!}$ for $n \in \mathbb{N}$, $0 \leq i \leq n$

From Example 2.4.4, we know that $C(n) \cong \hat{\sigma}^*_{i,n} C(A_0 \to \ldots \to A_i \to A_i \to \ldots \to A_0)$. So, define $\sigma_{i,n!} : C(n) \xrightarrow{id} C(n) \cong \hat{\sigma}^*_{i,n} C(A_0 \to \ldots \to A_i \to A_i \to \ldots \to A_0)$.

2.6.3. Generating rotations $\tau_{n!}$

2.6.3.1. $n = 0$. Let $\tau_{0!} = id : C(0) \xrightarrow{id} C(0) \cong id^* C(0) \cong \hat{\tau}_0^* C(0)$.

2.6.3.2. $n = 1$. We want to define a map of dg comodules over $B(1)$

$$\tau_{1!} : C(1) := C(A_0 \to A_1 \to A_0) \to \hat{\tau}_1^* C(A_1 \to A_0 \to A_1).$$

Example 2.4.5 describes the structure of $\hat{\tau}_1^* C(A_1 \to A_0 \to A_1)$, which is quasi-cofree over $B(1)$. So, we can define $\tau_{1!}$ by giving maps from $C(1)$ to the cogenerators of $\hat{\tau}_1^* C(A_1 \to A_0 \to A_1)$ and checking that the corresponding map of comodules commutes with the differentials.

More explicitly, for $f = (f_{0,0}, f_{1,0}) \in Obj(B(1))$, we will give k-linear maps

$$\upsilon^f : C(1)^\bullet(f) \to C_{-\bullet}(A_{1,f_{0,0}f_{1,0}}\, A_{1id})$$

$$(\phi_{0,1} \ldots \phi_{0,n_0} | \phi_{1,1} \ldots \phi_{0,n_1} | \alpha) \mapsto \upsilon^f_{n_0,n_1}(\phi_{0,1} \ldots \phi_{0,n_0} | \phi_{1,1} \ldots \phi_{1,n_1} | \alpha).$$

Then, we lift $\{\upsilon_f | f \in Obj(B(1))\}$ to a map of comodules in the standard way:

$$\begin{aligned} &\tau_{1!f}(\phi_{0,1} \ldots \phi_{0,n_0} | \phi_{1,1} \ldots \phi_{1,n_1} | \alpha) \\ &= \sum_{\substack{0 \le k_0 \le n_0 \\ 0 \le k_1 \le n_1}} \phi_{0,1} \ldots \phi_{0,k_1} | \phi_{1,1} \ldots \phi_{1,k_0} | \upsilon^{f_{0,k_1}, f_{1,k_0}}_{n_0-k_0, n_1-k_1}(\phi_{0,k_0+1} \ldots \phi_{0,n_0} | \phi_{1,k_1+1} \ldots \phi_{1,n_1} | \alpha) \end{aligned} \tag{2.10}$$

(see Figure 1.2 for notation). Finally, we will check by direct computation that $\tau_{1!}$ defined as such commutes with the differentials. To make the exposition smooth, all of this is done in Appendix Proposition B.1.

2.6.3.3. **$n > 1$.** For $n > 1$, we define $\tau_{n!}$ by pulling back $\tau_{1!}$ along $\delta_{0,*}$'s as follows:

$$
\begin{aligned}
\tau_{n!} : C(n) &\cong (\delta_{0,2} \widehat{\circ \ldots \circ} \delta_{0,n})^* C(A_0 \to A_n \to A_0) \\
&\xrightarrow{(\delta_{0,2} \widehat{\circ \ldots \circ} \delta_{0,n})^* \tau_{1!}} (\delta_{0,2} \widehat{\circ \ldots \circ} \delta_{0,n})^* \hat{\tau}_1^* C(A_n \to A_0 \to A_n) \\
&\cong (\tau_1 \circ \delta_{0,2} \widehat{\circ \ldots} \circ \delta_{0,n})^* C(A_n \to A_0 \to A_n) \\
&\cong (\delta_{1,2} \circ \widehat{\ldots \circ \delta_1}_{,n} \circ \tau_n)^* C(A_n \to A_0 \to A_n) \\
&\cong \hat{\tau}_n^* (\delta_{1,2} \widehat{\circ \ldots \circ} \delta_{1,n})^* C(A_n \to A_0 \to A_n) \\
&\cong \hat{\tau}_n^* C(A_n \to A_0 \to \ldots \to A_n).
\end{aligned}
$$

CHAPTER 3

A homotopically sheafy-cyclic object

3.1. Motivation of this chapter

In Section 2.1, we gave a sheafy-cyclic object in dg cocategories. We would like to extend that construction to a sheafy-cyclic object in the category of dg cocategories with a dg comodule. Namely, we would like to give a functor from χ to $\mathcal{D}$ where $\mathcal{D}$ the following category:

$$Obj(\mathcal{D}) = \{(B, C) | \text{B is a dg cocategory, C is a dg comodule over B}\}$$

$$\mathcal{D}((B_1, C_1), (B_0, C_0)) = \{(f, f_!) | f : B_1 \to B_0 \text{ is a functor,}$$
$$f_! : C_1 \to f^* C_0 \text{ is a map of dg comodules over } B_1\}$$

$$\mathcal{D}((B_2, C_2), (B_1, C_1)) \times \mathcal{D}((B_1, C_1), (B_0, C_0)) \xrightarrow{composition} \mathcal{D}((B_2, C_2), (B_0, C_0))$$
$$(f, f_!) \times (g, g_!) \mapsto (gf, f^*(g_!) \circ f_!)$$

(Proposition 2.1 implies that composition in $\mathcal{D}$ is associative.)

In Section 2.6, we gave maps $\lambda_! : C(\mathcal{A}) \to \hat{\lambda}^* C(\lambda\mathcal{A})$ where λ is a generating morphism in Λ that induces a morphism in χ with source $\mathcal{A} \in Obj(\chi)$. Ideally, $(\hat{\lambda}, \lambda_!)$ would give a functor $\chi \to \mathcal{D}$, however, the $\lambda_!$'s only respect composition up to homotopy. Fortunately, most of the compositions are respected and, for the ones that are not, we have explicit homotopies. Our homotopies commute with the composable $\lambda_!$'s so that no higher homotopies are needed.

In this chapter, we will show which compositions are respected on the nose and which ones need homotopies. We will then give these homotopies and show that no higher homotopies are needed. These sections are the computational heart of this thesis. Finally, we will repackage this "functor up to homotopy" in more abstract terms.

3.2. Homotopies

Here, we will show that the maps of dg comodules given in Section 2.6 satisfy the relations in Λ (Equation A.2) up to homotopy. More precisely, we will show that

(3.1a)
$$\begin{gathered}
\hat{\delta}^*_{j,n}(\delta_{i,n-1!}) \circ \delta_{j,n!} = \hat{\delta}^*_{i,n}(\delta_{j-1,n-1!}) \circ \delta_{i,n!} \quad 0 \leq i < j \leq n-1 \\
\hat{\sigma}^*_{j,n}(\sigma_{i,n+1!}) \circ \sigma_{j,n!} = \hat{\sigma}^*_{i,n}(\sigma_{j+1,n+1!}) \circ \sigma_{i,n!} \quad 0 \leq i \leq j \leq n \\
\hat{\sigma}^*_{i,n}(\delta_{j,n+1!}) \circ \sigma_{i,n!} = \begin{cases} \hat{\delta}^*_{j-1,n}(\sigma_{i,n-1!}) \circ \delta_{j-1,n!} & 0 \leq i < j \leq n \\ id & j = i, i-1 \\ \hat{\delta}^*_{j,n}(\sigma_{i-1,n-1!}) \circ \delta_{j,n!} & 0 \leq j < i-1 \leq n-1 \end{cases}
\end{gathered}$$

(3.1b)
$$\begin{gathered}
\hat{\sigma}^*_{i,n}(\tau_{n+1!}) \circ \sigma_{i,n!} = \hat{\tau}^*_n(\sigma_{i+1,n!}) \circ \tau_{n!} \quad 0 \leq i \leq n-1 \\
\hat{\delta}^*_{j,n}(\tau_{n-1!}) \circ \delta_{j,n!} = \hat{\tau}^*_n(\delta_{j+1,n!}) \circ \tau_{n!} \quad 0 \leq j \leq n-1
\end{gathered}$$

(3.1c)
$$(\widehat{\tau_1\sigma_{0,0}})^*(\delta_{0,1!}) \circ \hat{\sigma}^*_{0,0}(\tau_{1!}) \circ \sigma_{0,0!} = id$$

and

(3.2a)
$$\hat{\tau}^{*2}_n(\delta_{0,n!}) \circ \hat{\tau}^*_n(\tau_{n!}) \circ \tau_{n!} \simeq \hat{\delta}^*_{n-1,n}(\tau_{n-1!}) \circ \delta_{n-1,n!}$$

(3.2b)
$$\hat{\tau}^{*n}_n(\tau_{n!}) \circ \ldots \circ \hat{\tau}^*_n(\tau_{n!}) \circ \tau_{n!} \simeq id$$

(3.2c)
$$\hat{\sigma}^*_{n,n}(\tau_{n+1!}) \circ \sigma_{n,n!} \simeq (\widehat{\tau^n_{n+1}\sigma_{0,n}\tau_n})^*(\tau_{n+1!}) \circ \ldots \circ (\widehat{\tau_{n+1}\sigma_{0,n}\tau_n})^*(\tau_{n+1!}) \circ (\widehat{\sigma_{0,n}\tau_n})^*(\tau_{n+1!}) \circ \hat{\tau}^*_n(\sigma_{0,n!}) \circ \tau_{n!}$$

3.2.1. Aside on composing $\lambda_!$'s

In this section and the next, we will use the following convention for composing $\lambda_!$'s, which we first illustrate with an example. Suppose we have 3 composable generating morphisms in Λ: $\lambda_1 \in \Lambda([m],[n])$, $\lambda_2 \in \Lambda([n],[p])$, and $\lambda_3 \in \Lambda([p],[q])$. Fix a sequence of algebras $A_0, \ldots, A_m$. We can construct the following composition of morphisms of dg comodules over $B(\mathcal{A}) := B(A_0 \to \ldots \to A_m \to A_0)$:

$$C(\mathcal{A}) \xrightarrow{\lambda_{1,\mathcal{A}!}} \hat{\lambda}_1^* C(\lambda_1 \mathcal{A}) \xrightarrow{\hat{\lambda}_1^*(\lambda_{2,\lambda_1\mathcal{A}!})} \hat{\lambda}_1^* \hat{\lambda}_2^* C(\lambda_2 \lambda_1 \mathcal{A}) \xrightarrow{\hat{\lambda}_1^* \hat{\lambda}_2^*(\lambda_{3,\lambda_2\lambda_1\mathcal{A}!})} \hat{\lambda}_1^* \hat{\lambda}_2^* \hat{\lambda}_3^* C(\lambda_3 \lambda_2 \lambda_1 \mathcal{A}).$$

To simplify the notation, we will write

$$\hat{\lambda}_1^* \hat{\lambda}_2^*(\lambda_{3!}) \circ \hat{\lambda}_1^*(\lambda_{2!}) \circ \lambda_{1!} \quad \text{instead of} \quad \hat{\lambda}_1^* \hat{\lambda}_2^*(\lambda_{3,\lambda_2\lambda_1\mathcal{A}!}) \circ \hat{\lambda}_1^*(\lambda_{2,\lambda_1\mathcal{A}!}) \circ \lambda_{1,\mathcal{A}!}.$$

More generally, when the reader sees $\hat{\lambda}_1^* \ldots \hat{\lambda}_{r-1}^*(\lambda_{r!}) \circ \ldots \circ \hat{\lambda}_1^*(\lambda_{2!}) \circ \lambda_{1!}$ for a sequence of composable generating morphisms $\lambda_r \circ \ldots \circ \lambda_1$ in Λ, s/he may decode this notation by choosing a sequence of algebras $A_0, \ldots, A_{m=\text{source of } \lambda_1}$, setting $\mathcal{A} = (A_0 \to \ldots \to A_m \to A_0)$, and letting the composition of $\lambda_!$'s denote $\hat{\lambda}_1^* \ldots \hat{\lambda}_{r-1}^*(\lambda_{r,\lambda_{r-1}\ldots\lambda_1\mathcal{A}!}) \circ \ldots \circ \hat{\lambda}_1^*(\lambda_{2,\lambda_1\mathcal{A}!}) \circ \lambda_{1,\mathcal{A}!}$.

3.2.2. Strict relations: showing Equations 3.1 hold

Equation 3.1a has three relations. All of the $\sigma_!$'s and $\delta_!$'s in Equation 3.1a are identity maps, so it's clear that these relations hold.

Equation 3.1b has two relations. To show that the first one holds, we have

$$\begin{aligned}
\hat{\sigma}^*_{i,n}(\tau_{n+1!}) \circ \sigma_{i,n!} &= \hat{\sigma}^*_{i,n}((\delta_{0,2}\widehat{\ldots\delta_{0,n+1}})^*(\tau_{1!})) \circ \sigma_{i,n!} \quad \text{definition of } \tau_{n+1!} \\
&= (\delta_{0,2}\widehat{\ldots\delta_{0,n+1}}\sigma_{i,n})^*(\tau_{1!}) \circ \sigma_{i,n!} \quad \text{Proposition 2.1} \\
&= (\delta_{0,2}\widehat{\ldots\delta_{0,n}})^*(\tau_{1!}) \circ \sigma_{i,n!} \\
&= \tau_{n!} \circ \sigma_{i,n!} \quad \text{definition of } \tau_{n!} \\
&= \tau_{n!} \circ id = id \circ \tau_{n!} \\
&= \hat{\tau}^*_n(\sigma_{i+1,n!}) \circ \tau_{n!}.
\end{aligned}$$

To show that the second relation holds, the reasoning is the same as above. We have

$$\begin{aligned}
\hat{\delta}^*_{j,n}(\tau_{n-1!}) \circ \delta_{j,n!} &= \hat{\delta}^*_{j,n}((\delta_{0,2}\widehat{\ldots\delta_{0,n-1}})^*(\tau_{1!})) \circ \delta_{j,n!} \\
&= (\delta_{0,2}\widehat{\ldots\delta_{0,n-1}}\delta_{j,n})^*(\tau_{1!}) \circ \delta_{j,n!} \\
&= \tau_{n!} \circ \delta_{j,n!} \\
&= \tau_{n!} \circ id = id \circ \tau_{n!} \\
&= \hat{\tau}^*_n(\delta_{j+1,n!}) \circ \tau_{n!}.
\end{aligned}$$

Equation 3.1c has one relation. The only map in this relation that is not defined to be an identity map is $\hat{\sigma}^*_{0,0}(\tau_{1!})$. We will compute this map and show that it is also an identity. Let $(\phi_{0,1}\ldots\phi_{0,k_0}|\alpha) \in C(A_0 \to A_0)$ (see Figure 1.2 for notation). By Proposition

2.2,

$$C(A_0 \to A_0) \xrightarrow{\cong} \hat{\sigma}^*_{0,0} C(A_0 \to A_0 \to A_0)$$

$$(\phi_{0,1} \ldots \phi_{0,k_0} | \alpha) \mapsto \sum_{0 \le r_0 \le k_0} (\phi_{0,1} \ldots \phi_{0,r_0}) \otimes (1 | \phi_{0,r_0+1} \ldots \phi_{0,k_0} | \alpha).$$

Applying $\hat{\sigma}^*_{0,0}(\tau_{1!})$ to the righthand side, we have

$$\hat{\sigma}^*_{0,0} C(A_0 \to A_0 \to A_0) \xrightarrow{\hat{\sigma}^*_{0,0}(\tau_{1!})} \hat{\sigma}^*_{0,0} \hat{\tau}^*_1 C(A_0 \to A_0 \to A_0)$$

$$\sum_{0 \le r_0 \le k_0} (\phi_{0,1} \ldots \phi_{0,r_0}) \otimes (1 | \phi_{0,r_0+1} \ldots \phi_{0,k_0} | \alpha) \mapsto \sum_{0 \le r_0 \le s_0 \le k_0} (\phi_{0,1} \ldots \phi_{0,r_0}) \otimes$$

$$(\phi_{0,r_0+1} \ldots \phi_{0,s_0} | 1 | \tau_1 (1 | \phi_{0,s_0+1} \ldots \phi_{0,k_0} | \alpha)).$$

The righthand side above is equal to

$$\sum_{0 \le r_0 \le s_0 \le k_0} (\phi_{0,1} \ldots \phi_{0,r_0}) \otimes (\phi_{0,r_0+1} \ldots \phi_{0,s_0} | 1 | \tau_1 (1 | \phi_{0,s_0+1} \ldots \phi_{0,k_0} | \alpha))$$

$$= \sum_{0 \le r_0 \le s_0 \le k_0} (\phi_{0,1} \ldots \phi_{0,r_0}) \otimes (\phi_{0,r_0+1} \ldots \phi_{0,s_0} | 1 | \upsilon_{0,k_0 - s_0} (1 | \phi_{0,s_0+1} \ldots \phi_{0,k_0} | \alpha))$$

(see Proposition B.1 for $\upsilon_{\cdot,\cdot}$)

$$= \sum_{0 \le r_0 \le k_0} (\phi_{0,1} \ldots \phi_{0,r_0}) \otimes (\phi_{0,r_0+1} \ldots \phi_{0,k_0} | 1 | \alpha) \qquad (\upsilon_{0,>0} = 0)$$

$$\in \hat{\sigma}^*_{0,0} \hat{\tau}^*_1 C(A_0 \to A_0 \to A_0).$$

Finally, applying Proposition 2.2 again, we have

$$\hat{\sigma}^*_{0,0}\hat{\tau}^*_1 C(A_0 \to A_0 \to A_0) \xrightarrow[\cong]{\text{project onto cogenerators}} C(A_0 \to A_0)$$

$$\sum_{0 \le r_0 \le k_0} (\phi_{0,1} \ldots \phi_{0,r_0}) \otimes (\phi_{0,r_0+1} \ldots \phi_{0,k_0}|1|\alpha) \mapsto (\phi_{0,1} \ldots \phi_{0,k_0}|\alpha).$$

So, we've shown

$$C(A_0 \to A_0) \cong \hat{\sigma}^*_{0,0} C(A_0 \to A_0 \to A_0) \xrightarrow{\hat{\sigma}^*_{0,0}(\tau_{1!})} \hat{\sigma}^*_{0,0}\hat{\tau}^*_1 C(A_0 \to A_0 \to A_0) \cong C(A_0 \to A_0)$$

is the identity map.

3.2.3. Weak relations: showing Equations 3.2 hold

3.2.3.1. Showing Equation 3.2a holds. For $n = 1$, eliminating the identity maps reduces Equation 3.2a to:

$$\hat{\tau}^*_1(\tau_{1!}) \circ \tau_{1!} \simeq id.$$

We prove the above in Appendix Proposition B.2. (In the appendix, $\tau_{1!} = \Upsilon_{A_0,A_1}$, $\hat{\tau}^*_1(\tau_{1!}) = \Upsilon_{A_1,A_0}$, and the homotopy is denoted B.)

For $n = 2$, eliminating the identity maps and writing $\tau_{2!}$ in terms of $\tau_{1!}$ reduces Equation 3.2a to:

$$(\widehat{\delta_{0,2}\tau_2})^*(\tau_{1!}) \circ \hat{\delta}^*_{0,2}(\tau_{1!}) \simeq \hat{\delta}^*_{1,2}(\tau_{1!}).$$

We prove the above in Appendix Proposition B.4. (In the appendix, $\hat{\delta}^*_{0,2}(\tau_{1!}) = \Upsilon_{A_0 \bullet A_1, A_2}$, $(\widehat{\delta_{0,2}\tau_2})^*(\tau_{1!}) = \Upsilon_{A_2 \bullet A_0, A_1}$, $\hat{\delta}^*_{1,2}(\tau_{1!}) = \Upsilon_{A_0, A_1 \bullet A_2}$, and the homotopy is denoted $\mathcal{B}$.)

For $n > 2$, we reduce Equation 3.2a to the case when $n = 2$. We have

$$
\begin{aligned}
\text{Lefthand side of Equation 3.2a} &= \hat{\tau}_n^{*2}(\delta_{0,n!}) \circ \hat{\tau}_n^{*}(\tau_{n!}) \circ \tau_{n!} \\
&= id \circ \hat{\tau}_n^{*}((\delta_{0,2}\widehat{\ldots\delta_{0,n}})^{*}(\tau_{1!})) \circ \tau_{n!} \\
&= (\delta_{0,2}\widehat{\ldots\delta_{0,n}}\tau_n)^{*}(\tau_{1!}) \circ \tau_{n!} \\
&= (\delta_{0,2}\tau_2\widehat{\delta_{0,3}\ldots}\delta_{0,n})^{*}(\tau_{1!}) \circ \tau_{n!} \\
&= (\delta_{0,2}\tau_2\widehat{\delta_{0,3}\ldots}\delta_{0,n})^{*}(\tau_{1!}) \circ (\delta_{0,2}\widehat{\ldots\delta_{0,n}}\tau_n)^{*}(\tau_{1!}) \\
&= (\delta_{0,3}\widehat{\ldots\delta_{0,n}})^{*}((\widehat{\delta_{0,2}\tau_2})^{*}(\tau_{1!}) \circ \hat{\delta}_{0,2}^{*}\tau_{1!}) \\
\text{Righthand side of Equation 3.2a} &= \hat{\delta}_{n-1,n}^{*}(\tau_{n-1!}) \circ \delta_{n-1,n!} \\
&= \hat{\delta}_{n-1,n}^{*}((\delta_{0,2}\widehat{\ldots\delta_{0,n-1}})^{*}(\tau_{1!})) \circ id \\
&= (\delta_{0,2}\ldots\widehat{\delta_{0,n-1}}\delta_{n-1,n})^{*}(\tau_{1!})) \\
&= (\delta_{1,2}\widehat{\delta_{0,3}\ldots}\delta_{0,n})^{*}(\tau_{1!}) \\
&= (\delta_{0,3}\widehat{\ldots\delta_{0,n}})^{*}(\hat{\delta}_{1,2}^{*}(\tau_{1!})).
\end{aligned}
$$

So, Equation 3.2a $= (\delta_{0,3}\widehat{\ldots\delta_{0,n}})^{*}($ Equation 3.2a, $n = 2)$. If $\mathcal{B}$ is a homotopy giving Equation 3.2a for $n = 2$, then $(\delta_{0,3}\widehat{\ldots\delta_{0,n}})^{*}\mathcal{B}$ is a homotopy giving Equation 3.2a for $n > 2$.

3.2.3.2. Showing Equation 3.2b holds. We prove this by induction on n. For $n = 1$, Equation 3.2b is the same as Equation 3.2a, which we established in the previous section. Now, assume that Equation 3.2b holds for $N = n - 1$. We show that Equation 3.2b holds for $N = n$ below:

$$
\begin{aligned}
\hat{\tau}_n^{*n}(\tau_{n!}) \circ \ldots \circ \hat{\tau}_n^{*}(\tau_{n!}) \circ \tau_{n!} &= \hat{\tau}_n^{*n-1}(\hat{\tau}_n^{*}\tau_{n!} \circ \tau_{n!}) \circ \hat{\tau}_n^{*n-2}\tau_{n!} \circ \ldots \circ \tau_{n!} \\
&\simeq \hat{\tau}_n^{*n-1}(\hat{\delta}^{*}_{n-1,n}\tau_{n-1!}) \circ \hat{\tau}_n^{*n-2}\tau_{n!} \circ \ldots \circ \tau_{n!} \qquad \text{(Equation 3.2a)} \\
&= (\widehat{\tau_{n-1}^{n-1}\delta_{0,n}})^{*}\tau_{n-1!} \circ \big(\hat{\tau}_n^{*n-2}\hat{\delta}^{*}_{n-2,n}\tau_{n-1!} \circ \ldots \circ \hat{\tau}_n^{*}\hat{\delta}^{*}_{1,n}\tau_{n-1!} \circ \hat{\delta}^{*}_{0,n}\tau_{n-1!}\big) \\
&= (\widehat{\tau_{n-1}^{n-1}\delta_{0,n}})^{*}\tau_{n-1!} \circ \hat{\delta}^{*}_{0,n}\big(\hat{\tau}_{n-1}^{*n-2}\tau_{n-1!} \circ \ldots \circ \hat{\tau}_{n-1}^{*}\tau_{n-1!} \circ \tau_{n-1!}\big) \\
&= \hat{\delta}^{*}_{0,n}\big(\hat{\tau}_{n-1}^{*n-1}\tau_{n-1!} \circ \ldots \circ \tau_{n-1!}\big) \\
&\simeq \hat{\delta}^{*}_{0,n}\big(id\big) \qquad \text{(Inductive hypothesis)} \\
&= id.
\end{aligned}
$$

3.2.3.3. Showing Equation 3.2c holds. By manipulating morphisms in Λ, we have

$$
\begin{aligned}
\text{Righthand side of Equation 3.2c} &= \hat{\tau}_n^{*n+1}\tau_{n!} \circ \hat{\tau}_n^{*n}\tau_{n!} \circ \ldots \circ \hat{\tau}_n^{*}\tau_{n!} \circ \hat{\tau}_n^{*n+1}id \circ \tau_{n!} \\
&= \tau_{n!} \circ \big(\hat{\tau}_n^{*n}\tau_{n!} \circ \ldots \circ \hat{\tau}_n^{*}\tau_{n!} \circ \tau_{n!}\big) \\
&\simeq \tau_{n!} \circ \big(id\big) \quad \text{Equation 3.2b.}
\end{aligned}
$$

On the other hand, we have

$$\begin{aligned}\text{Lefthand side of Equation 3.2c} &= \hat{\sigma}^*_{n,n}(\tau_{n+1!}) \circ id \\ &= \hat{\sigma}^*_{n,n}(\hat{\delta}^*_{n,n+1}(\tau_{n+1!})) \\ &= (\widehat{\delta_{n,n+1}\sigma_{n,n}})^*(\tau_{n!}) \\ &= id^*(\tau_{n!}).\end{aligned}$$

So, Equation 3.2c holds.

3.3. Higher Homotopies

In this section, we show that no higher homotopies are needed. First, we will summarize the maps of comodules that we have already given. Let λ be a generating morphism in Λ that induces a morphism in χ with source $\mathcal{A} \in Obj(\chi)$. We have

$$\lambda_! : C(\mathcal{A}) \to \hat{\lambda}^* C(\lambda\mathcal{A}) \quad \text{maps of dg comodules}$$

$$\sigma_! : C(\mathcal{A}) \to \tau^{*2} C(\tau^2\mathcal{A}) \quad \text{deg -1 map of comodules.}$$

where

$$\sigma_! = \begin{cases} B \text{ given in Appendix Proposition B.2} & \text{if } \mathcal{A} = (A_0 \to A_1 \to A_0) \\ \mathcal{B} \text{ given in Appendix Proposition B.4} & \text{if } \mathcal{A} = (A_0 \to A_1 \to A_2 \to A_0) \\ (\widehat{\delta_{0,3} \dots \delta_{0,n}})^* \mathcal{B} & \text{if } \mathcal{A} = (A_0 \to \dots \to A_n \to A_0), n > 2. \end{cases}$$

Using the constructions we've given, a typical map between comodules is one that is freely generated by composable pullbacks of $\lambda_!$'s and $\sigma_!$'s. First, we will establish that there are no such maps of degree ≥ 2. Suppose we have a map $\eta_!$ of degree ≥ 2. Then, $\eta_!$ must contain at least two (pullbacks of) some $\sigma_!$'s. Each $\sigma_!$ involves inserting a 1 into the first slot of the Hochschild chains component (see Equations B.2, B.4). However, since we are working with reduced chains, any chain with two or more 1's is equal to zero. So, $\eta_! = 0$.

Since there are no maps of degree ≥ 2, we know from the classical theory of A_∞ algebras that the only need for higher homotopies will arise from the following situation:

For $n \geq 2$, We have two maps of dg comodules

$$\begin{tikzcd} C(A_0 \to \ldots \to A_n \to A_0) \arrow[d, bend right, "{\substack{\hat{\tau}_n^{*2}\tau_{n!}\circ\hat{\tau}_n^{*}\tau_{n!}\circ\tau_{n!} \\ \text{``apply } \tau_{n!} \text{ 3 times''}}}"'] \arrow[d, bend left, "{\substack{(\delta_{n-2,\widehat{n-1}\delta}{}_{n-1,n})^{*}\tau_{n-2!} \\ \text{``brace together the last 3 algebras,} \\ \text{then apply } \tau_{n-2!} \text{ once''}}}"] \\ C(A_{n-2} \to A_{n-1} \to A_n \to A_0 \to \ldots \to A_{n-2}). \end{tikzcd} \tag{3.3}$$

These two maps are homotopic via two homotopies: $\hat{\delta}^*_{n-1,n}\sigma_! + \tau_n^{*2}\tau_{n!} \circ \sigma_!$ and $\hat{\delta}^*_{n-2,n}\sigma_! + \hat{\tau}_n^*\sigma_! \circ \tau_{n!}$ (see Figure 3.1). If the two homotopies were different, then their difference would be closed and we would desire a higher homotopy (i.e., a degree -2 map of comodules) between them. However, we will show the two homotopies are the same, so that no higher homotopies are needed.

First, we show that $\hat{\delta}^*_{n-1,n}\sigma_! = \hat{\delta}^*_{n-2,n}\sigma_!$. We have

$$\hat{\delta}^*_{n-1,n}\sigma_! = \mathcal{B}_{A_0 \bullet \ldots \bullet A_{n-2}, A_{n-1} \bullet A_n} = \mathcal{B}_{A_0 \bullet \ldots \bullet A_{n-1}, A_n} = \hat{\delta}^*_{n-2,n}\sigma_!$$

where the second equality holds by definition of $\mathcal{B}$ in Appendix Equation B.4.

Second, we show that $\tau_n^{*2}\tau_{n!} \circ \sigma_! = \hat{\tau}_n^*\sigma_! \circ \tau_{n!}$ in Appendix Proposition B.5. In the appendix, $\tau_n^{*2}\tau_{n!} \circ \sigma_! = \Upsilon_{A_1 \bullet A_2, A_0}\mathcal{B}_{A_0,A_1,A_2}$ and $\hat{\tau}_n^*\sigma_! \circ \tau_{n!} = \mathcal{B}_{A_2,A_0,A_1}\Upsilon_{A_0 \bullet A_1, A_2}$.

For $n = 1$, the situation in Equation 3.3 reduces to: We have two maps of dg comodules

$$\begin{tikzcd} C(A_0 \to A_1 \to A_0) \arrow[d, bend right, "\hat{\tau}_1^{*2}\tau_{1!}\circ\hat{\tau}_1^{*}\tau_{1!}\circ\tau_{1!}"'] \arrow[d, bend left, "\tau_{1!}"] \\ C(A_1 \to A_0 \to A_1). \end{tikzcd}$$

$$\begin{tikzcd}
(\delta_{n-2,n-1}\widehat{\delta}_{n-1,n})^*\tau_{n-2!} \arrow[r, "\cong"] \arrow[d, "\cong"'] & \hat{\delta}^*_{n-1,n}(\hat{\delta}^*_{n-2,n-1}\tau_{n-2!}) \arrow[r, "\hat{\delta}^*_{n-1,n}\sigma_!"] & \hat{\delta}^*_{n-1,n}(\hat{\tau}^*_{n-1}\tau_{n-1!} \circ \tau_{n-1!}) \arrow[d, "\cong"] \\
\text{“brace together } A_{n-2}, A_{n-1}, A_n, \text{ then apply } \tau_{n-2!}\text{”} & & \\
(\delta_{n-2,n-1}\widehat{\delta}_{n-2,n})^*\tau_{n-2!} \arrow[d, "\hat{\delta}^*_{n-2,n}\sigma_!"'] & & \hat{\tau}^{*2}_n\tau_{n!} \circ \hat{\delta}^*_{n-1,n}\tau_{n-1!} \arrow[d, "\tau^{*2}_n\tau_{n!}\circ\sigma_!"] \\
 & & \text{“brace together } A_{n-1}, A_n \text{ and apply } \tau_{n-1!}, \text{ then apply } \tau_{n!}\text{”} \\
\hat{\delta}^*_{n-2,n}(\hat{\tau}^*_{n-1}\tau_{n-1!} \circ \tau_{n-1!}) \arrow[r, "\cong"'] & \hat{\tau}^*_n(\hat{\delta}^*_{n-1,n}\tau_{n-1!}) \circ \tau_{n!} \arrow[r, "\hat{\tau}^*_n\sigma_!\circ\tau_{n!}"] & \hat{\tau}^{*2}_n\tau_{n!} \circ \hat{\tau}^*_n\tau_{n!} \circ \tau_{n!} \\
 & \text{“apply } \tau_{n!}, \text{ then brace together } A_{n-1}, A_{n-2} \text{ and apply } \tau_{n-1!}\text{”} & \text{“apply } \tau_{n!} \text{ three times”}
\end{tikzcd}$$

Figure 3.1. Two homotopies between $(\delta_{n-2,n-1}\widehat{\delta}_{n-1,n})^*\tau_{n-2!}$ and $\hat{\tau}^{*2}_n\tau_{n!} \circ \hat{\tau}^*_n\tau_{n!} \circ \tau_{n!}$
Vertices are maps of dg comodules and arrows are chain homotopies.

$$\begin{tikzcd}
id \circ \tau_{1!} = \tau_{1!} = \tau_{1!} \circ id \arrow[d, bend right, "{\substack{\sigma_{A_1\to A_0\to A_1!}\circ\tau_{1!} \\ =\hat{\tau}^*_1(\sigma_{A_0\to A_1\to A_0!})\circ\tau_{1!}}}"'] \arrow[d, bend left, "\tau_{1!}\circ\sigma_{A_0\to A_1\to A_0!}"] \\
\left(\hat{\tau}^{*2}_1\tau_{1!} \circ \hat{\tau}^*_1\tau_{1!}\right) \circ \tau_{1!} = \hat{\tau}^{*2}_1\tau_{1!} \circ \left(\hat{\tau}^*_1\tau_{1!} \circ \tau_{1!}\right)
\end{tikzcd}$$

Figure 3.2. Two homotopies between $\tau_{1!}$ and $\hat{\tau}^{*2}_1\tau_{1!} \circ \hat{\tau}^*_1\tau_{1!} \circ \tau_{1!}$
Vertices are maps of dg comodules and arrows are chain homotopies.

These two maps are homotopic via two homotopies: $\tau_{1!} \circ \sigma_{A_0\to A_1\to A_0!}$ and $\sigma_{A_1\to A_0\to A_1!} \circ \tau_{1!}$ (see Figure 3.2; for clarity, here, we indicate the sources of the $\sigma_!$'s). We show that these two homotopies are the same in Appendix Proposition B.3, so no higher homotopies are needed. In the appendix, $\tau_{1!} \circ \sigma_{A_0\to A_1\to A_0!} = \Upsilon_{A_0,A_1} \circ B_{A_0,A_1}$ and $\sigma_{A_1\to A_0\to A_1!} \circ \tau_{1!} = B_{A_1,A_0} \circ \Upsilon_{A_0,A_1}$.

3.4. An A_∞-functor

We can repackage the work of the previous sections into a concise statement: We have constructed an A_∞-functor from χ to $\mathcal{D}$. This section is devoted to making that concise statement more rigorous. We refer to Reference [**2**], Appendix A, Definitions A.6 and A.8 for the notation and definition of an A_∞-category and A_∞-functor.

First, we must think of χ and $\mathcal{D}$ as A_∞-categories. Let χ_∞ be the (usual) category with the same objects as χ and morphisms linear combinations over k of morphisms in χ. We will think of the morphisms in χ_∞ as complexes concentrated in degree zero. χ_∞ is an A_∞-category with $m_2 =$ (usual) composition of morphisms in χ_∞, $m_1 = m_{\geq 3} = 0$. The relations for an A_∞-category are satisfied because m_2 is associative.

Let $\mathcal{D}_\infty$ be the dg category with the same objects as $\mathcal{D}$ and morphisms

$$\begin{aligned}\mathcal{D}^\bullet_\infty((B_1, C_1), (B_0, C_0)) = \big\{\big(&F : B_1 \to B_0 \quad \text{dg functor},\\ &F_! : C_1 \to F^* C_0 \quad \text{map of comodules of degree } \bullet\big)\big\}\end{aligned}$$

$$d_{\mathcal{D}_\infty}(F, F_!) = (F,\ d_{F^*C_0} \circ F_! - (-1)^{|F_!|} F_! \circ d_{C_1}).$$

Composition in $\mathcal{D}_\infty$ works like composition in $\mathcal{D}$–we can apply the same formulas to pullback a (not necessarily graded) morphism of comodules. We can think of $\mathcal{D}_\infty$ as an A_∞-category with $m_1 = d_{\mathcal{D}_\infty}$, $m_2 =$ composition of morphisms, $m_{\geq 3} = 0$. For $\mathcal{D}_\infty$, the relations for an A_∞-category are precisely that (1) the differentials square to zero, (2) composition is a map of complexes, and (3) composition is associative.

Now, we will show that the constructions given in the previous sections constitute an A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$. Still using the notation in Reference [**2**], Definition A.8, we define $\mathcal{F}$ as follows:

$$f : Obj(\chi_\infty) \to Obj(\mathcal{D}_\infty) \quad \text{map of sets}$$

$$\mathcal{A} \mapsto (B(\mathcal{A}), C(\mathcal{A})) \text{ defined in Chapter 1}$$

$$f_1 : \chi^\bullet_\infty(x_0, x_1) \to \mathcal{D}^\bullet_\infty(fx_0, fx_1) \quad \text{map of graded vector spaces}$$

$$\lambda \mapsto (\hat{\lambda}, \lambda_!) \text{ defined in Sections 2.1, 2.6}$$

$$\text{for } \lambda \text{ a generating morphism in } \Lambda$$

$$id_\mathcal{A} \mapsto (id_{B(\mathcal{A})}, id_{C(\mathcal{A})})$$

$$f_2 : \chi^\bullet_\infty(x_0, x_1) \otimes \chi^\bullet_\infty(x_1, x_2) \to \mathcal{D}^\bullet_\infty(fx_0, fx_2) \quad \text{degree } -1 \text{ map of vector spaces}$$

$$f_{i\geq 3} : \chi^\bullet_\infty(x_0, x_1) \otimes \ldots \otimes \chi^\bullet_\infty(x_{i-1}, x_i) \to \mathcal{D}^\bullet_\infty(fx_0, fx_i) \quad \text{degree } 1-i \text{ map of vector spaces}$$

$$\mu_1 \otimes \ldots \otimes \mu_i \mapsto 0 \text{ for all morphisms } \mu$$

We will show that the f_i's satisfy the relations in Reference [**2**], Definition A.8.

First, we finish defining $\mathcal{F}$, namely, we must define $f_1(\mu)$ for μ not a generating morphism in Λ as well as for linear combinations of morphisms. We also still need to define f_2.

Let μ be a non-generating morphism in Λ that induces a morphsim in χ with source $\mathcal{A}$. Choose (i.e., fix once and for all) a presentation of μ as a composition of generating morphisms. Within the chosen presentation, in the following order, (1) replace all occurrences of $\tau_{n-1}\delta_{n-1,n}$ with $\delta_{0,n}\tau_n^2$, (2) replace all $\tau_{n+1}\sigma_{n,n}$ with $\tau_{n+1}^{n+1}\sigma_{0,n}\tau_n$, (3) replace all decompositions of identity maps with identity maps, (4) remove all identity maps if $\mu \neq id$, (5) call this new presentation "the presentation corresponding to μ", denoted

$\mu = \lambda_{\mu,k_\mu} \ldots \lambda_{\mu,1}$. The presentation corresponding to μ is not unique (i.e., still depends on the original, chosen presentation). However, letting

$$f_1(\mu) := \big(\hat{\mu} : B(\mathcal{A}) \to B(\mu\mathcal{A})$$
$$\hat{\lambda}^*_{\mu,1} \ldots \hat{\lambda}^*_{\mu,k_\mu-1}(\lambda_{\mu,k_\mu !}) \circ \ldots \circ \hat{\lambda}^*_{\mu,1}(\lambda_{\mu,2!}) \circ \lambda_{\mu,1!} : C(\mathcal{A}) \to \hat{\mu}^* C(\mu\mathcal{A})\big)$$

is well-defined because we have made consistent choices for all of the relations among $\lambda_!$'s that only hold up to homotopy (see Equations 3.2). More explicitly, $f_1(\mu)$ would have been well-defined for any choice of presentation if Equations 3.2 were equalities rather than homotopies. Instead, we choose to define $f_1(\mu)$ via a presentation that only uses the lefthand side of Equation 3.2a and only uses the righthand sides of Equations 3.2c and 3.2b. Finally, extend f_1 linearly over k to define f_1 for all linear combinations of morphisms in χ.

Before defining f_2, let's take a look at an A_∞ relation we expect f_2 to satisfy: For $\cdot \xrightarrow{\mu_1} \cdot \xrightarrow{\mu_2} \cdot$ composable morphisms in χ, we expect

$$f_1(\mu_2 \circ \mu_1) = f_1(\mu_2) \circ f_1(\mu_1) + d_{\mathcal{D}_\infty} \circ f_2(\mu_1, \mu_2). \tag{3.4}$$

Given the definition of f_1 above, we require a non-zero f_2 only if: (Condition H) the presentation corresponding to μ_2 composed with the presentation corresponding to μ_1 contains, after removing (decompositions of) identity maps except for τ_n^{n+1}, one or more of the following terms: $\tau_{n-1}\delta_{n-1,n}$, $\tau_{n+1}\sigma_{n,n}$, τ_n^{n+1}. If μ_1, μ_2 satisfy Condition H, homotopies given in Section 3.2.3 can be used to define f_2. If μ_1, μ_2 do not satisfy Condition H, let $f_2(\mu_1, \mu_2) = 0$.

We will give some instructive examples of non-zero f_2 that satisfy Equation 3.4.

Example 3.4.1. *Let $\mu_1 = \delta_{n-1,n}$, $\mu_2 = \tau_{n-1}$. Then, the presentation corresponding to $\mu_2\mu_1$ is $\delta_{0,n}\tau_n^2$. Let $f_2(\mu_1, \mu_2)$ be the homotopy given in Section 3.2.3.1 (also given in Appendix Proposition B.2 or B.4). Then, Equation 3.4 is equivalent to Equation 3.2a.*

Example 3.4.2. *Let $\mu_1 = \sigma_{0,n-1}\delta_{n-1,n}$, $\mu_2 = \tau_{n-1}\delta_{0,n}$. To form the presentation corresponding to $\mu_2\mu_1$, we follow these steps:*

$$\tau_{n-1}\delta_{0,n}\sigma_{0,n-1}\delta_{n-1,n} \xrightarrow[\text{of identities}]{\text{remove decompositions}} \tau_{n-1}\delta_{n-1,n} \xrightarrow{\text{replace}} \delta_{0,n}\tau_n^2.$$

On the other hand,

$$\begin{aligned} f_1(\mu_2)f_1(\mu_1) &= (\delta_{0,n}\widehat{\sigma_{0,n-1}\delta}_{n-1,n})^*(\tau_{n-1!}) \circ (\sigma_{0,\widehat{n-1}\delta_{n-1,n}})^*(\delta_{0,n!}) \circ \hat{\delta}^*_{n-1,n}(\sigma_{0,n-1!}) \circ \delta_{n-1,n!} \\ &= \hat{\delta}^*_{n-1,n}(\tau_{n-1!}) \circ id \circ \delta_{n-1,n!}. \end{aligned}$$

So, we can let $f_2(\mu_1, \mu_2)$ be the homotopy given in Section 3.2.3.1, and Equation 3.4 is equivalent to Equation 3.2a.

Example 3.4.3. *Let $(\mu_1, \mu_2) \in \{(\tau_{n+1}, \sigma_{n,n}), (\tau_n^{n+1-j}, \tau_n^j) : 1 \leq j \leq n, n \in \mathbb{N}\}$. Let $f_2(\mu_1, \mu_2)$ be the homotopy given in 3.2.3.3 if $\mu_2 = \sigma_{n,n}$ and the homotopy given in 3.2.3.2 if $\mu_2 \neq \sigma_{n,n}$. Then, Equation 3.4 is equivalent to either Equation 3.2c ($\mu_2 = \sigma_{n,n}$) or Equation 3.2b ($\mu_2 \neq \sigma_{n,n}$).*

Example 3.4.4. *Let* $\mu_1 = \sigma_{n-1,n-1}\delta_{n-1,n}$*,* $\mu_2 = \tau_n$*. To form the presentation corresponding to* $\mu_2\mu_1$*, we follow these steps:*

$$(\tau_n\sigma_{0,n-1})\delta_{n-1,n} \xrightarrow{\textit{replace } (\cdot)} \tau_n^n\sigma_{0,n-1}(\tau_{n-1}\delta_{n-1,n}) \xrightarrow{\textit{replace } (\cdot)} \tau_n^n\sigma_{0,n-1}\delta_{0,n}\tau_n^2.$$

Let $f_2(\mu_1,\mu_2) = g_1 + g_2$ *where* $g_1 = \hat{\delta}^*_{n-1,n}$*(homotopy in Section 3.2.3.3)* $\circ\, \delta_{n-1,n!}$ *and* $g_2 = (\widehat{\tau_{n-1}\delta_{n-1,n}})^*\big((\widehat{\tau_n^{n-1}\sigma_{0,n-1}})^*(\tau_{n!}) \circ \ldots \circ \hat{\sigma}^*_{0,n-1}(\tau_{n!}) \circ \sigma_{0,n-1!}\big) \circ$ *(homotopy in Section 3.2.3.1). Then, Equation 3.4 reduces to* $\delta^*_{n-1,n}$*(Equation 3.2c) and Equation 3.2a.*

Now, we will check that the f_i's we gave satisfy the rest of the relations for an A_∞-functor from Reference [**2**], Definition A.8: For $\cdot \xrightarrow{\mu_1} \cdot \xrightarrow{\mu_2} \cdot \xrightarrow{\mu_3} \cdot \xrightarrow{\mu_4} \cdot$ composable morphisms in χ, we expect

$$0 = d_{\mathcal{D}_\infty} \circ f_1(\mu_1) \tag{3.5}$$

$$f_2(\mu_3,\mu_2\circ\mu_1) - f_2(\mu_3\circ\mu_2,\mu_1) = f_2(\mu_3,\mu_2)\circ f_1(\mu_1) - f_1(\mu_3)\circ f_2(\mu_2,\mu_1) \tag{3.6}$$

$$0 = f_2(\mu_4,\mu_3)\circ f_2(\mu_2,\mu_1). \tag{3.7}$$

Equation 3.5 is satisfied since the $\lambda_!$'s we defined in Section 2.6 are maps of complexes. Equation 3.7 is satisfied since composing two of our degree -1 homotopies is always equal to zero (see Section 3.3, first paragraph). Finally, Equation 3.6 boils down to showing that the two homotopies in Figure 3.1 and in Figure 3.2 are the same (see Section 3.3).

3.5. Rectify

A homotopically cyclic object in ∞-categories in dg categories with a trace functor

3.6. A functor to dg cocategories

The previous section 3.5 gave a dg functor from $U(\chi_\infty)$ to $\mathcal{D}_\infty$ = "dg cocategories and dg comodules". We would like to have our functor land in the dg category, $\mathcal{E}$ = "dg categories and dg modules". To do so, we will first give a dg functor $\mathcal{D}_\infty \to \mathcal{D}_1$, which makes use of the adjunction in Proposition 2.3. Then, we will give a dg functor *Cobar* : $\mathcal{D}_1 \to \mathcal{E}$.

3.6.1. Using the adjunction

Let $\mathcal{D}_1$ be the dg category with the same objects as $\mathcal{D}$ and morphisms

$$\mathcal{D}_1^\bullet((B_1, C_1), (B_0, C_0)) = \big\{\big(F : B_1 \to B_0 \quad \text{dg functor},$$

$$F_! : F_\# C_1 \to C_0 \quad \text{map of comodules of degree } \bullet\big)\big\}$$

$$d_{\mathcal{D}_\infty}(F, F_!) = (F,\ d_{C_0} \circ F_! - (-1)^{|F_!|} F_! \circ d_{F_\# C_1})$$

with composition

$$\mathcal{D}_1^\bullet((B_2, C_2), (B_1, C_1)) \otimes \mathcal{D}_1^\bullet((B_1, C_1), (B_0, C_0)) \to \mathcal{D}_1^\bullet((B_2, C_2), (B_0, C_0))$$

$$(f, f_!) \otimes (g, g_!) \mapsto (gf, g_! \circ g_\#(f_!)).$$

This composition is well-defined because we can apply the formulas from $g_\#$ to (not necessarily graded) morphisms of comodules. The composition is associative because of the following easy-to-check fact: $g_\# f_\# C = (gf)_\# C$ for $B_2 \xrightarrow{f} B_1 \xrightarrow{g} B_0$ functors of dg cocategories and C a dg comodule over B_2.

Now, we define a dg functor

$$Adj : \mathcal{D}_\infty \to \mathcal{D}_1$$

$$\text{on objects: } (B, C) \mapsto (B, C)$$

$$\text{on morphisms: } \left((B_1, C_1) \xrightarrow{(F, F_!)} (B_0, C_0) \right) \mapsto \left((B_1, C_1) \xrightarrow{(F, \Phi_F^{-1} F)} (B_0, C_0) \right)$$

where $\Phi_F^{-1} : Hom_{\substack{\text{dg comodules} \\ \text{over } B_1}}(C, F^* D) \to Hom_{\substack{\text{dg comodules} \\ \text{over } B_0}}(F_\# C, D)$ is defined in the proof of Proposition 2.3 and makes sense as a function on (not necessarily graded) maps of comodules. To check that Adj commutes with the differentials and respects composition, we need

$$\Phi_F^{-1} \circ d_{Hom_{B_2}(C_2, F^* C_1)} = d_{Hom_{B_1}(F_\# C_2, C_1)} \circ \Phi_F^{-1}$$

$$\Phi_{GF}^{-1}(F^* G_! \circ F_!) = \Phi_G^{-1}(G_!) \circ G_\#(\Phi_F^{-1}(F_!))$$

$$\text{where } (B_2, C_2) \xrightarrow{(F, F_!)} (B_1, C_1) \xrightarrow{(G, G_!)} (B_0, C_0) \text{ in } \mathcal{D}_\infty.$$

The equations above follow straight-forwardly from the definitions.

3.6.2. Applying *Cobar*

References

[1] Tsygan, Boris. Noncommutative Calculus and Operads.

[2] Faonte, Giovanni. A_∞-Functors and Homotopy Theory of DG-Categories

APPENDIX A

Connes cyclic category, Λ

Here, we give generators and relations for the cyclic category, Λ. None of this is new, but we do it to establish notation for the rest of the paper.

Λ has objects $\{[n] : n \in \mathbb{N}\}$ and generating morphisms:

$$\begin{aligned} \text{rotations } \tau_n &: [n] \to [n], \\ \text{coboundaries } \delta_{j,n} &: [n] \to [n-1], 0 \le j \le n-1, \\ \text{codegeneracies } \sigma_{i,n} &: [n] \to [n+1], 0 \le i \le n \end{aligned} \tag{A.1}$$

subject to relations:

$$
\begin{aligned}
\delta_{i,n-1}\delta_{j,n} &= \delta_{j-1,n-1}\delta_{i,n} \quad 0 \le i < j \le n-1 \\
\sigma_{i,n+1}\sigma_{j,n} &= \sigma_{j+1,n+1}\sigma_{i,n} \quad 0 \le i \le j \le n \\
\delta_{j,n+1}\sigma_{i,n} &= \begin{cases} \sigma_{i,n-1}\delta_{j-1,n} & 0 \le i < j \le n \\ id & j = i, i-1 \\ \sigma_{i-1,n-1}\delta_{j,n} & 0 \le j < i-1 \le n-1 \end{cases} \\
\tau_{n+1}\sigma_{i,n} &= \sigma_{i+1,n}\tau_n \quad 0 \le i \le n-1 \\
\tau_{n-1}\delta_{j,n} &= \delta_{j+1,n}\tau_n \quad 0 \le j \le n-1 \\
\tau_n^{n+1} &= id \\
\delta_{0,1}\tau_1\sigma_{0,0} &= id \\
\tau_{n+1}\sigma_{n,n} &= \tau_{n+1}^{n+1}\sigma_{0,n}\tau_n \\
\delta_{0,n}\tau_n^2 &= \tau_{n-1}\delta_{n-1,n}.
\end{aligned}
\tag{A.2}
$$

Some presentations of Λ include an extra coboundary $\delta_{n,n}$ and codegeneracy $\sigma_{n+1,n}$. In terms of our generators, they are $\delta_{n,n} := \delta_{0,n}\tau_n$ and $\sigma_{n+1,n} := \tau_{n+1}^{n+1}\sigma_{0,n}$.

APPENDIX B

Computations

In this appendix, we give the computational propositions needed to establish the homotopically sheafy-cyclic structure on dg comodules. All the comodules we work with will be cofree, and we will define maps into them by giving maps into cogenerators (see Equation 1.2).

B.1. Computational notation

For this section's propositions, we establish the following notation:

$$A_0, A_1 \text{ fixed algebras}$$

$$(\vec{\phi}|\vec{\psi}|\alpha) := (\phi_1 \ldots \phi_n|\psi_1 \ldots \psi_m|\alpha)$$

$$= \begin{tikzcd} A_0 \arrow[r, bend left=70, "f_0"] \arrow[r, phantom, bend left=45, "\Downarrow\phi_1"] \arrow[r, bend left=30, "f_1"'] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=30, "f_n"] \arrow[rr, phantom, bend right=40, "\alpha"] \arrow[rr, bend right=70, "id"'] & A_1 \arrow[r, bend left=70, "g_0"] \arrow[r, phantom, bend left=45, "\Downarrow\psi_1"] \arrow[r, bend left=30, "g_1"'] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=30, "g_m"] & A_0 \end{tikzcd} \in C(A_0 \to A_1 \to A_0)(g_0 f_0)$$

$$\vec{\phi}_{\{i_1,i_2,\ldots,i_k\}} := \phi_{i_1}\phi_{i_2}\ldots\phi_{i_k} \qquad \text{where } \{i_1, i_2, \ldots, i_k\} \text{ is an ordered subset of } \{1, \ldots, n\}$$

$$\vec{\phi}_{\{\}} := 1 \in k \cong Bar_0(C^\bullet(A_0, A_1))$$

$$\vec{\psi}_{\{\}} := 1 \in k \cong Bar_0(C^\bullet(A_1, A_0))$$

$$|I| := \text{number of elements in a set } I$$

$$I_1 I_2 := \text{concatenation as ordered sets of possibly-empty sets } I_1 \text{ and } I_2$$

$$\lambda(\vec{\psi}),\ \tilde{\delta},\ b',\ b,\ \psi\{\vec{\phi}\}\cdot\alpha = \text{see Appendix C for standard operations on Hochschild (co)chains}$$

B.1.1. Notation for elements of Hochschild chains

Let $a_0 \otimes a_1 \otimes \cdots \otimes a_n$ denote a typical element of $C_{-\bullet}(A, A)$ where A is some algebra. At times, we wish to feed a portion of $a_0 \otimes a_1 \otimes \ldots \otimes a_n$ to a Hochschild cochain (or other map on chains) without specifying the degree of the cochain. To do this, we will rewrite $a_0 \otimes a_1 \otimes \ldots \otimes a_n = a_0 \otimes \mathfrak{a}_1 \otimes \ldots \otimes \mathfrak{a}_r$ where each $\mathfrak{a}_i = a_{j_i} \otimes a_{j_i+1} \otimes \ldots \otimes a_{j_{i+1}-1}$ and $\mathfrak{a}_i$ is an empty chain if $j_i = j_{i+1}$.

For example, if $\phi \in C^2(A, A)$, then we rewrite

$$\sum_{1 \leq i \leq n-1} a_0 \otimes a_1 \otimes \ldots a_{i-1} \otimes \phi(a_i, a_{i+1}) \otimes a_{i+2} \otimes \ldots \otimes a_n = \sum a_0 \otimes \mathfrak{a}_1 \otimes \phi(\mathfrak{a}_2) \otimes \mathfrak{a}_3.$$

B.2. Computational Propositions

Proposition B.1. *Let $\hat{\tau}_1 : B(A_0 \to A_1 \to A_0) \longrightarrow B(A_1 \to A_0 \to A_1)$ be as defined in Section 2.1. Recall from Example 2.4.5 that $\hat{\tau}_1^* C(A_1 \to A_0 \to A_0) \cong C(A_1 \to A_0 \to A_1)$ as complexes. Define a map*

$$\Upsilon_{A_0,A_1} : C(A_0 \to A_1 \to A_0) \to \hat{\tau}_1^* C(A_1 \to A_0 \to A_1)$$

of comodules over $B(A_0 \to A_1 \to A_0)$ by mapping into cogenerators as follows:

$$\upsilon^{f_0,g_0} : C(A_0 \to A_1 \to A_0)(f_0, g_0) \to \hat{\tau}_1^* C(A_1 \to A_0 \to A_1)(g_0, f_0) \cong C(A_1 \to A_0 \to A_1)(g_0, f_0)$$

$$\xrightarrow[cogenerators]{project\ onto} C_{-\bullet}(A_1, {}_{f_0 g_0} A_{1id})$$

$$\upsilon^{f_0,g_0}_{n,m}(\vec{\phi}|\vec{\psi}|\alpha) = \sum_{\substack{I_1 I_2 = \{1,2,\cdots,n\} \\ as\ ordered\ sets}} \phi_1(\lambda(\vec{\psi})\lambda(\vec{\phi_{I_2}}) \cdot \mathfrak{a}_3, a_0, \mathfrak{a}_1) \otimes \lambda(\vec{\phi_{I_1}}) \cdot \mathfrak{a}_2$$

$$\Big(+ f_0 a_0 \otimes \lambda(\vec{\phi})\mathfrak{a}_1 \quad if \ \ m = 0 \Big).$$

Then, $\Upsilon_{A_0,A_1} : C(A_0 \to A_1 \to A_0) \to \hat{\tau}^* C(A_1 \to A_0 \to A_1)$ is a map of dg comodules over $B(A_0 \to A_1 \to A_0)$.

Proof. We must show: (1) Υ is a map of comodules, and (2) Υ commutes with the differentials. (In this proof, we drop the subscripts and write $\Upsilon := \Upsilon_{A_0,A_1}$.)

(1) This proof is standard for cofree comodules. Let $(\vec{\phi}|\vec{\psi}|\alpha)$ be as in the statement of the proposition. We want to show that Υ commutes with the coproducts. On one hand,

$$[(id_B \otimes \Upsilon) \circ \Delta_{C(A_0 \to A_1 \to A_0)}](\vec{\phi}|\vec{\psi}|\alpha)$$

$$= [id_B \otimes \Upsilon]\big(\sum_{\substack{I_1 I_2 = \{1,2,\cdots,n\} \text{ and} \\ J_1 J_2 = \{1,2,\cdots,m\} \\ \text{as ordered sets}}} (\vec{\phi}_{I_1}|\vec{\psi}_{J_1}) \otimes (\vec{\phi}_{I_2}|\vec{\psi}_{J_2}|\alpha) \big)$$

$$= \sum_{\substack{I_1 I_2 I_3 = \{1,2,\cdots,n\} \text{ and} \\ J_1 J_2 J_3 = \{1,2,\cdots,m\} \\ \text{as ordered sets}}} (\vec{\phi}_{I_1}|\vec{\psi}_{J_1}) \otimes (\vec{\phi}_{I_2}|\vec{\psi}_{J_2}) \otimes \upsilon_{|I_3|,|J_3|}(\vec{\phi}_{I_3}|\vec{\psi}_{J_3}|\alpha)$$

On the other hand,

$$
\begin{aligned}
&[\Delta_{\hat{\tau}^* C(A_1\to A_0\to A_1)} \circ \Upsilon](\vec{\phi}|\vec{\psi}|\alpha) \\
= {} & \Delta_{\hat{\tau}^* C(A_1\to A_0\to A_1)}\Big(\sum_{\substack{I_1 I_2=\{1,2,\cdots,n\} \text{ and} \\ J_1 J_2=\{1,2,\cdots,m\} \\ \text{as ordered sets}}} (\vec{\phi}_{I_1}|\vec{\psi}_{J_1}) \otimes \upsilon_{|I_2|,|J_2|}(\vec{\phi}_{I_2}|\vec{\psi}_{J_2}|\alpha) \Big) \\
= {} & \sum_{\substack{I_1 I_2 I_3=\{1,2,\cdots,n\} \text{ and} \\ J_1 J_2 J_3=\{1,2,\cdots,m\} \\ \text{as ordered sets}}} (\vec{\phi}_{I_1}|\vec{\psi}_{J_1}) \otimes (\vec{\phi}_{I_2}|\vec{\psi}_{J_2}) \otimes \upsilon_{|I_3|,|J_3|}(\vec{\phi}_{I_3}|\vec{\psi}_{J_3}|\alpha).
\end{aligned}
$$

Clearly $(id_B \otimes \Upsilon) \circ \Delta_{C(A_0\to A_1\to A_0)} = \Delta_{\hat{\tau}^* C(A_1\to A_0\to A_1)} \circ \Upsilon$.

(2) We will show that Υ commutes with the differentials by direct computation. Since Υ is a map of cofree comodules, we only need to check that $\pi_1 \circ D(\Upsilon) = 0$ where $D(\Upsilon)$ is the differential applied to Υ as a linear map between complexes and π_1 denotes projection of a comodule onto its cogenerators. More explicitly, we want to check that

$$
\begin{aligned}
&\upsilon_{n,m}(\tilde{\delta}(\vec{\phi})|\vec{\psi}|\alpha) \ + \upsilon_{n,m}(\vec{\phi}|\tilde{\delta}(\vec{\psi})|\alpha) \ + \upsilon_{n-1,m}(b'(\vec{\phi})|\vec{\psi}|\alpha) \ + \upsilon_{n,m-1}(\vec{\phi}|b'(\vec{\psi})|\alpha) \ + \\
&\upsilon_{n,m}(\vec{\phi}|\vec{\psi}|b(\alpha)) \ + b \circ \upsilon_{n,m}(\vec{\phi}|\vec{\psi}|\alpha) \ + \\
&\upsilon_{|I_1|,m-1}(\vec{\phi}_{I_1}|\vec{\psi}_{\{1,\cdots,m-1\}}|\psi_m\{\vec{\phi}_{I_2}\} \cdot \alpha) \ + \upsilon_{n-1,m}(\vec{\phi}_{\{1,\cdots,n-1\}}|\vec{\psi}_m|\phi_n \cdot \alpha) \ + \\
&\phi_1\{\psi_{J_1}\} \cdot \upsilon_{\vec{\phi}|-1,|J_2|}(\phi_{\{2,\cdots,n\}}|\psi_{J_2}|\alpha) \ + \psi_1 \cdot \upsilon_{n,m-1}(\vec{\phi}|\vec{\psi}_{\{2,\cdots,|\vec{\psi}\}}|\alpha) \\
&= 0.
\end{aligned}
\tag{B.1}
$$

In Equation B.1, we will call the terms in the first two rows the “standard terms”, and the terms in the second two rows the “extra terms”.

We compute the sum of the standard terms. In Table B.1, the leftmost column lists the expressions that don’t cancel in the sum of the standard terms, the middle column gives the standard term from which the expression comes, and the rightmost column gives the term (extra or standard) that cancels the expression.

All of the terms in Table B.1 cancel, so Υ is a map of complexes. $\square$

Expression	Comes from Standard Term	Cancelling Term
$f_0\psi_1(\lambda(\vec{\phi}_{I_2})\mathfrak{a}_3)\phi_1(\lambda(\vec{\psi}_{\{2,\cdots,m\}}\lambda(\vec{\phi}_{I_3})\mathfrak{a}_4, a_0, \mathfrak{a}_1)$ $\otimes\lambda(\vec{\phi}_{I_1})\mathfrak{a}_2$	$\upsilon_{n,m}(\delta(\phi_1)\phi_2\cdots\phi_n\vert\vec{\psi}\vert\alpha)$	$f_0\psi_1\cdot\upsilon_{n,m-1}(\vec{\phi}\vert\vec{\psi}_{\{2,\cdots,m\}}\vert\alpha)$
$\phi_1(\lambda(\vec{\psi}_{\{1,\cdots,m-1\}})\lambda(\vec{\phi}_{I_2})\mathfrak{a}_3, \psi_m(\lambda(\vec{\phi}_{I_3})\mathfrak{a}_4)\cdot a_0, \mathfrak{a}_1)$ $\otimes\lambda(\vec{\phi}_{I_1})\mathfrak{a}_2$	$\upsilon_{n,m}(\delta(\phi_1)\phi_2\cdots\phi_n\vert\vec{\psi}\vert\alpha)$	$\upsilon_{\vert I_1\vert,m-1}(\vec{\phi}_{I_1}\vert\vec{\psi}_{\{1,\cdots,m-1\}}\vert\psi_m\{\vec{\phi}_{I_2}\}\cdot\alpha)$
$\phi_1(\lambda(\vec{\psi})\lambda(\vec{\phi}_{I_2})\mathfrak{a}_3, g_m\phi_n(\mathfrak{a}_4)\cdot a_0, \mathfrak{a}_1)\otimes\lambda(\vec{\phi}_{I_1})\mathfrak{a}_2$	$\upsilon_{n,m}(\delta(\phi_1)\phi_2\cdots\phi_n\vert\vec{\psi}\vert\alpha)$	$\upsilon_{n-1,m}(\vec{\phi}_{\{1,\cdots,n-1\}}\vert\vec{\psi}\vert g_m\phi_n\cdot\alpha)$
$\phi_1(\lambda(\vec{\psi})\lambda(\vec{\phi}_{I_2})\mathfrak{a}_2)\cdot f_1(a_0)\otimes\lambda(\vec{\phi}_{I_1})\mathfrak{a}_1$	$\upsilon_{n,m}(\delta(\phi_1)\phi_2\cdots\phi_n\vert\vec{\psi}\vert\alpha)$	$\phi_1\cdot\upsilon_{n-1,0}(\vec{\phi}_{\{2,\cdots,n\}}\vert\vec{\psi}\vert\alpha)$
$f_0a_0\cdot\phi_1(\mathfrak{a}_1)\otimes\lambda(\vec{\phi}_{\{1,\cdots,n-1\}})\mathfrak{a}_2$	$\upsilon_{n,m}(\delta(\phi_1)\phi_2\cdots\phi_n\vert\vec{\psi}\vert\alpha)$ if $\vec{\psi}=1$	$b\circ\upsilon_{n,m}(\vec{\phi}\vert\vec{\psi}\vert\alpha)$ if $\vec{\psi}=1$
$f_0g_m\phi_n(\mathfrak{a}_2)f_0a_0\otimes\lambda(\vec{\phi}_{\{1,\cdots,n-1\}})\mathfrak{a}_1$	$b\circ\upsilon_{n,m}(\vec{\phi}\vert\vec{\psi}\vert\alpha)$ if $\vec{\psi}=1$	$\upsilon_{n-1,m}(\vec{\phi}_{\{1,\cdots,n-1\}}\vert\vec{\psi}\vert g_m\phi_n\cdot\alpha)$ if $\vec{\psi}=1$
$\phi_1(\lambda(\vec{\psi})\lambda(\vec{\phi}_{I_2})\mathfrak{a}_4, a_0, \mathfrak{a}_1)\cdot\phi_2(\mathfrak{a}_2)\otimes\lambda(\vec{\phi}_{I_1})\mathfrak{a}_3$	$b\circ\upsilon_{n,m}(\vec{\phi}\vert\vec{\psi}\vert\alpha)$	$\upsilon_{n-1,m}(\phi_1\cup\phi_2\phi_3\cdots\phi_n\vert\vec{\psi}\vert\alpha)$
$\phi_1(\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_2})\mathfrak{a}_3)\phi_2(\lambda(\vec{\psi}_{J_2}\lambda(\vec{\phi}_{I_3})\mathfrak{a}_3, a_0, \mathfrak{a}_1)$ $\otimes\lambda(\vec{\phi}_{I_1})\mathfrak{a}_2$	$\upsilon_{n-1,m}(\phi_1\cup\phi_2\phi_3\cdots\phi_n\vert\vec{\psi}\vert\alpha)$	$\phi_1\{\vec{\psi}_{J_1}\}\cdot\upsilon_{n-1,\vert J_2\vert}(\vec{\phi}_{\{2,\cdots,n\}}\vert\vec{\psi}_{J_2}\vert\alpha)$
$f_0\psi_1(\lambda(\vec{\phi}_{I_2})\mathfrak{a}_2)\cdot f_0a_0\otimes\lambda(\vec{\phi}_{I_1})\mathfrak{a}_1$	$f_0\psi_1\cdot\upsilon_{n,0}(\vec{\phi}\vert 1\vert\alpha)$ if $\vec{\psi}=\psi_1$	$\upsilon_{\vert I_1\vert,0}(\vec{\phi}_{I_1}\vert 1\vert\psi_1\{\vec{\phi}_{I_2}\}\cdot\alpha)$ if $\vec{\psi}=\psi_1$

Table B.1. Expansion of terms in Equation B.1

(Technically, the last term in the middle column is not a standard term, but we include it in the table for convenience.)

Proposition B.2. *Let* $B_{A_0,A_1} = B : C(A_0 \to A_1 \to A_0) \longrightarrow C(A_0 \to A_1 \to A_0)$ *be the map of cofree comodules defined by the following maps to cogenerators:*

$$B_{n,m}(\vec{\phi}|\vec{\psi}|\alpha) = 1 \otimes \lambda(\psi)\lambda(\phi)\mathfrak{a}_2 \otimes a_0 \otimes \mathfrak{a}_1. \tag{B.2}$$

Then, $D(B_{A_0,A_1}) = \Upsilon_{A_1,A_0}\Upsilon_{A_0,A_1} - id$ *where* Υ *is defined in Proposition B.1.*

Proof. We prove the statement by direct computation. Since all of the maps are maps of cofree comodules, we only need to check that $\pi_1(D(B_{A_0,A_1}) - \Upsilon_{A_1,A_0}\Upsilon_{A_0,A_1} - id) = 0$ where π_1 denotes projection of the comodule onto cogenerators. More explicitly, for an element $(\vec{\phi}|\vec{\psi}|\alpha)$, we want to check that

$$\begin{aligned}
&B_{n,m}(\tilde{\delta}(\vec{\phi})|\vec{\psi}|\alpha) \ + B_{n,m}(\vec{\phi}|\tilde{\delta}(\vec{\psi})|\alpha) \ + B_{n-1,m}(b'(\vec{\phi})|\vec{\psi}|\alpha) \ + B_{n,m-1}(\vec{\phi}|b'(\vec{\psi})|\alpha) + \\
&B_{n,m}(\vec{\phi}|\vec{\psi}|b(\alpha)) \ + b \circ B_{n,m}(\vec{\phi}|\vec{\psi}|\alpha) + \\
&B_{|I_1|,m-1}(\vec{\phi}_{I_1}|\vec{\psi}_{\{1,\cdots,m-1\}}|\psi_m\{\vec{\phi}_{I_2}\} \cdot \alpha) \ + B_{n-1,m}(\vec{\phi}_{\{1,\cdots,n-1\}}|\vec{\psi}_m|\phi_n \cdot \alpha) + \\
&\phi_1\{\psi_{J_1}\} \cdot B_{\vec{\phi}|-1,|J_2|}(\phi_{\{2,\cdots,n\}}|\psi_{J_2}|\alpha) \ + \psi_1 \cdot B_{n,m-1}(\vec{\phi}|\vec{\psi}_{\{2,\cdots,|\vec{\psi}\}}|\alpha) - \\
&\upsilon_{|J_1|,|I_1|}(\vec{\psi}_{J_1}|\vec{\phi}_{I_1}|\upsilon_{|I_2|,|J_2|}(\vec{\phi}_{I_2}|\vec{\psi}_{J_2}|\alpha)) - \pi_1(\vec{\phi}|\vec{\psi}|\alpha) \\
&= 0.
\end{aligned} \tag{B.3}$$

We will call the terms in the first two rows the “standard terms” in the computaion of $D(B_{A_0,A_1})$, and the terms in the second two rows the “extra terms” in the computation of $D(B_{A_0,A_1})$. The fifth row is $\pi_1(\Upsilon_{A_1,A_0}\Upsilon_{A_0,A_1} - id)$.

We compute the sum of the standard terms. In Table B.2, the leftmost column lists the expressions that don’t cancel in the sum of the standard terms, the middle column gives the standard term from which the expression comes, and the rightmost column gives the extra term that cancels the expression. Table B.3 lists the remaining terms from the fifth row that are not already listed in Table B.2. In Table B.3, the left column lists the remaining expressions that don't cancel in the fifth row, and the right column gives the extra term that cancels the expression.

All of the terms in the tables describing the expansion of equation B.3 cancel, so $D(B_{A_0,A_1}) = \Upsilon_{A_1,A_0}\Upsilon_{A_0,A_1} - id$. $\square$

Expression	Comes from Standard Term	Cancels with Extra Term
$\psi_1(\lambda(\vec{\phi}_{I_1})\mathfrak{a}_2) \otimes \lambda(\vec{\psi}_{\{2,\cdots,m\}})\lambda(\vec{\phi}_{I_2})\mathfrak{a}_3 \otimes a_0 \otimes \mathfrak{a}_1$	$b \circ B_{n,m}(\vec{\phi}\|\vec{\psi}\|\alpha)$	$\psi_1\{\vec{\phi}_{I_1}\} \cdot B_{\|I_2\|,m-1}(\vec{\phi}_{I_2}\|\vec{\psi}_{\{2,\cdots,m\}}\|\alpha)$
$g_0\phi_1(\mathfrak{a}_2) \otimes \lambda(\vec{\psi})\lambda(\vec{\phi}_{\{2,\cdots,n\}})\mathfrak{a}_3 \otimes a_0 \otimes \mathfrak{a}_1$	$b \circ B_{n,m}(\vec{\phi}\|\vec{\psi}\|\alpha)$	$\phi_1 \cdot B_{n-1,m}(\vec{\phi}_{\{2,\cdots,n\}}\|\vec{\psi}\|\alpha)$
$1 \otimes \lambda(\vec{\psi})\lambda(\vec{\phi}_{\{1,\cdots,n-1\}})\mathfrak{a}_2 \otimes g_m\phi_n(\mathfrak{a}_3 \cdot a_0 \otimes \mathfrak{a}_1$	$b \circ B_{n,m}(\vec{\phi}\|\vec{\psi}\|\alpha)$	$B_{n-1,m}(\vec{\phi}_{\{1,\cdots,n-1\}}\|\vec{\psi}\|\phi_n \cdot \alpha)$
$1\otimes\lambda(\vec{\psi}_{\{1,\cdots,m-1\}})\lambda(\vec{\phi}_{I_1})\mathfrak{a}_2\otimes g_m\psi_m(\lambda(\vec{\phi}_{I_2}\mathfrak{a}_3)\cdot a_0\otimes \mathfrak{a}_1$	$b \circ B_{n,m}(\vec{\phi}\|\vec{\psi}\|\alpha)$	$B_{\|I_1\|,m-1}(\vec{\phi}_{I_2}\|\vec{\psi}_{\{1,\cdots,m-1\}}\|\psi_m\{\vec{\phi}_{I_2}\}\cdot\alpha)$
$g_0f_0a_0 \otimes \lambda(\vec{\psi})\lambda(\vec{\phi})\mathfrak{a}_1$	$b \circ B_{n,m}(\vec{\phi}\|\vec{\psi}\|\alpha)$	$\upsilon_{\|J_1\|,\|I_1\|}(\vec{\psi}_{J_1}\|\vec{\phi}_{I_1}\|\upsilon_{\|I_2\|,\|J_2\|}(\vec{\phi}_{I_2}\|\vec{\psi}_{J_2}\|\alpha))$

Table B.2. Expansion of "standard terms" in Equation B.3 and the "extra terms" that cancel them

(Technically, the last term in the right column is not an extra term, but we include it in the table for convenience.)

Expression from Fifth Row	Cancels with Extra Term
$\psi_1(\lambda(\vec{\phi}_{I_1})\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_4})\mathfrak{a}_4, \phi_{\|I_1\|+1}(\lambda(\vec{\psi}_{J_3})\lambda(\vec{\phi}_{I_5})\mathfrak{a}_5, a_0, \mathfrak{a}_1), \lambda(\vec{\phi}_{I_2\setminus\|I_1\|+1})\mathfrak{a}_2) \otimes \lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_3})\mathfrak{a}_3$	$\psi_1\{\vec{\phi}_{I_1}\} \cdot B_{\|I_2\|,m-1}(\vec{\phi}_{I_2}\|\vec{\psi}_{\{2,\cdots,m\}}\|\alpha)$
$\psi_1(\lambda(\vec{\phi}_{I_1})\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_4})\mathfrak{a}_4, f_{\|I_1\|+1}a_0, \lambda(\vec{\phi}_{I_2\setminus\|I_1\|+1})\mathfrak{a}_1) \otimes \lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_3})\mathfrak{a}_2$	$\phi_1 \cdot B_{n-1,m}(\vec{\phi}_{\{2,\cdots,n\}}\|\vec{\psi}\|\alpha)$
$g_0\phi_1(\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_2})\mathfrak{a}_3, a_0, \mathfrak{a}_1) \otimes \lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_1})\mathfrak{a}_2$	$\psi_1\{\vec{\phi}_{I_1}\} \cdot B_{\|I_2\|,m-1}(\vec{\phi}_{I_2}\|\vec{\psi}_{\{2,\cdots,m\}}\|\alpha)$

Table B.3. Expansion of remaining "fifth-row terms" in Equation B.3 and the "extra terms" that cancel them

Proposition B.3. *Let* $\Upsilon_{A_0,A_1} : C(A_0 \to A_1 \to A_0) \longrightarrow C(A_1 \to A_0 \to A_1)$ *and* $B_{A_0,A_1} : C(A_0 \to A_1 \to A_0) \longrightarrow C(A_0 \to A_1 \to A_0)$ *be the maps defined in Propositions B.1 and B.2 above. Then,* $[\Upsilon, B] := \Upsilon_{A_0,A_1} B_{A_0,A_1} - B_{A_1,A_0}\Upsilon_{A_0,A_1} = 0$.

Proof. We show that $[\Upsilon, B] = 0$ by direct computation. Since all of the maps are maps of cofree comodules, we only need to check that $\pi_1([\Upsilon, B]) = 0$ where π_1 denotes projection of the comodule onto cogenerators. We check this directly.

$$\begin{aligned}
\pi_1 \circ \Upsilon_{A_0,A_1} \circ B_{A_0,A_1}(\vec{\phi}|\vec{\psi}|\alpha) &= \upsilon_{|I_1|,|J_1|}(\vec{\phi}_{I_1}|\vec{\psi}_{J_1}|B_{|I_2|,|J_2|}(\vec{\phi}_{I_1}|\vec{\psi}_{J_1}|\alpha)) \\
&= \upsilon_{|I_1|,|J_1|}(\vec{\phi}_{I_1}|\vec{\psi}_{J_1}|1 \otimes \lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_2})\mathfrak{a}_2, a_0, \mathfrak{a}_1) \\
&= 1 \otimes \lambda(\vec{\phi}_{I_1})\big(\lambda(\vec{\psi})\lambda(\vec{\phi}_{I_2})\mathfrak{a}_2, a_0, \mathfrak{a}_1\big)
\end{aligned}$$

$$\begin{aligned}
\pi_1 \circ B_{A_1,A_0} \circ \Upsilon_{A_0,A_1}(\vec{\phi}|\vec{\psi}|\alpha) &= B_{|J_1|,|I_1|}(\vec{\psi}_{J_1}|\vec{\phi}_{I_1}|\upsilon_{|I_2|,|J_2|}(\vec{\phi}_{I_1}|\vec{\psi}_{J_1}|\alpha)) \\
&= B_{|J_1|,|I_1|}\big(\vec{\psi}_{J_1}|\vec{\phi}_{I_1}|\phi_{|I_1|+1}(\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_3})\mathfrak{a}_3, a_0, \mathfrak{a}_1) \otimes \lambda(\vec{\phi}_{I_2\setminus|I_1|+1})\mathfrak{a}_2 + \\
&\qquad + a_0 \otimes \lambda(\vec{\phi}_{I_2\setminus|I_1|+1})\mathfrak{a}_1 \quad \text{if } J_2 = \emptyset\big) \\
&= 1 \otimes \lambda(\vec{\phi}_{I_1})\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_3})\mathfrak{a}_3 \otimes \phi_{|I_1|+1}(\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_4})\mathfrak{a}_4, a_0, \mathfrak{a}_1) \otimes \lambda(\vec{\phi}_{I_2\setminus|I_1|+1})\mathfrak{a}_2 + \\
&\qquad + 1 \otimes \lambda(\vec{\phi}_{I_1})\lambda(\vec{\psi})\lambda(\vec{\phi}_{I_3})\mathfrak{a}_2 \otimes a_0 \otimes \lambda(\vec{\phi}_{I_2})\mathfrak{a}_1
\end{aligned}$$

It's clear that $\pi_1 \circ \Upsilon_{A_0,A_1} \circ B_{A_0,A_1} = \pi_1 \circ B_{A_1,A_0} \circ \Upsilon_{A_0,A_1}$: The final expansion of $\pi_1 \circ \Upsilon_{A_0,A_1} \circ B_{A_0,A_1}$ is the sum of the two terms in the final expansion of $\pi_1 \circ B_{A_1,A_0} \circ \Upsilon_{A_0,A_1}$, which is the sum of terms in which one of the ϕ's contains a_0 and the terms in which none of the ϕ's contains a_0). $\square$

B.3. More notation

For the next two propositions, we will need some more notation. Set

$$A_0, A_1, A_2 \text{ fixed algebras}$$

$$(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha) := (\phi_1 \ldots \phi_n|\psi_1 \ldots \psi_m|\theta_1 \ldots \theta_r|\alpha)$$

$$= \begin{tikzcd}[column sep=huge]
A_0 \arrow[r, bend left=80, "f_0"] \arrow[r, bend left=40, "f_1"'] \arrow[r, phantom, bend left=60, "\Downarrow\phi_1"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=40, "f_n"] \arrow[rrr, bend right=60, "id"'] \arrow[rrr, phantom, bend right=35, "\alpha"] & A_1 \arrow[r, bend left=80, "g_0"] \arrow[r, bend left=40, "g_1"'] \arrow[r, phantom, bend left=60, "\Downarrow\psi_1"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=40, "g_m"] & A_2 \arrow[r, bend left=80, "h_0"] \arrow[r, bend left=40, "h_1"'] \arrow[r, phantom, bend left=60, "\Downarrow\theta_1"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=40, "h_p"] & A_0
\end{tikzcd}$$

$$\in C(A_0 \to A_1 \to A_2 \to A_0)(h_0 g_0 f_0)$$

$$\Upsilon_{A_0 \bullet A_1, A_2} : C(A_0 \to A_1 \to A_2 \to A_0) \to \hat{\tau}_2^* C(A_2 \to A_0 \to A_1 \to A_2) \quad \text{map of dg comodules}$$

$$(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha) \mapsto \Upsilon_{A_0, A_2}(\vec{\phi} \bullet \vec{\psi}|\vec{\theta}|\alpha)$$

$$\Upsilon_{A_0, A_1 \bullet A_2} : C(A_0 \to A_1 \to A_2 \to A_0) \to \hat{\tau}_2^{*2} C(A_1 \to A_2 \to A_0 \to A_1) \quad \text{map of dg comodules}$$

$$(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha) \mapsto \Upsilon_{A_0, A_1}(\vec{\phi}|\vec{\psi} \bullet \vec{\theta}|\alpha)$$

B.4. More Propositions

Proposition B.4. *Let*

$$\mathcal{B}_{A_0,A_1,A_2} = \mathcal{B} : C(A_0 \to A_1 \to A_2 \to A_0) \to \hat{\tau}_2^{*2} C(A_1 \to A_2 \to A_0 \to A_1)$$

be a map of comodules over $B(A_0 \to A_1 \to A_2 \to A_0)$ determined by the following maps to cogenerators:

(B.4)

$$\mathcal{B}^{f_0,g_0,h_0}_{A_0,A_1,A_2} : C(A_0 \to A_1 \to A_0)(h_0 g_0 f_0) \to \hat{\tau}_2^{*2} C(A_1 \to A_2 \to A_0 \to A_1)(f_0 h_0 g_0)$$

$$\xrightarrow[\text{cogenerators}]{\text{project onto}} C_{-\bullet}(A_1,{}_{f_0 h_0 g_0} A_{1id})$$

$$\mathcal{B}_{n,m,p}(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha) = \sum_{\substack{I_1 I_2 = \{1,2,\cdots,n\} \\ \text{as ordered sets}}} 1 \otimes \lambda(\vec{\phi}_{I_1})\big(\lambda(\vec{\theta})\lambda(\vec{\psi})\lambda(\vec{\phi}_{I_2})\mathfrak{a}_2 \otimes a_0 \otimes \mathfrak{a}_1\big)$$

Then,

$$D(\mathcal{B}_{A_0,A_1,A_2}) = \Upsilon_{A_2 \bullet A_0, A_1} \circ \Upsilon_{A_0 \bullet A_1, A_2} - \Upsilon_{A_0, A_1 \bullet A_2}. \tag{B.5}$$

Proof. We will show that Equation B.5 holds by direct computation. Since all of the maps are maps of cofree comodules, we only need to check that π_1(Equation B.5) holds where π_1 denotes projection of the comodule onto cogenerators. More explicitly, we want to check that

$$
\begin{aligned}
&\mathcal{B}_{n,m,p}(\tilde{\delta}(\vec{\phi})|\vec{\psi}|\vec{\theta}|\alpha) \ + \mathcal{B}_{n,m,p}(\vec{\phi}|\tilde{\delta}(\vec{\psi})|\vec{\theta}|\alpha) \ + \mathcal{B}_{n,m,p}(\vec{\phi}|\vec{\psi}|\tilde{\delta}(\vec{\theta})|\alpha) + \\
&\mathcal{B}_{n-1,m,p}(b'(\vec{\phi})|\vec{\psi}|\vec{\theta}|\alpha) \ + \mathcal{B}_{n,m-1,p}(\vec{\phi}|b'(\vec{\psi})|\vec{\theta}|\alpha) \ + \mathcal{B}_{n,m,p-1}(\vec{\phi}|\vec{\psi}|b'(\vec{\theta})|\alpha) + \\
&\mathcal{B}_{n,m,p}(\vec{\phi}|\vec{\psi}|\vec{\theta}|b(\alpha)) \ + b \circ \mathcal{B}_{n,m,p}(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha) + \\
&\mathcal{B}_{|I_1|,|J_1|,p-1}(\vec{\phi}_{I_1}|\vec{\psi}_{J_1}|\vec{\theta}_{\{1,\cdots,p-1\}}|\theta_p\{\vec{\psi}_{J_2}\}\{\vec{\phi}_{I_2}\} \cdot \alpha) + \\
&\mathcal{B}_{|I_1|,m-1,p}(\vec{\phi}_{I_1}|\vec{\psi}_{\{1,\cdots,m-1\}}|\vec{\theta}|\psi_m\{\vec{\phi}_{I_2}\} \cdot \alpha) \ + \mathcal{B}_{n-1,m,p}(\vec{\phi}_{\{1,\cdots,n-1\}}|\vec{\psi}_m|\vec{\theta}|\phi_n \cdot \alpha) + \\
&\phi_1\{\vec{\theta}_{K_1}\}\{\vec{\psi}_{J_1}\} \cdot \mathcal{B}_{n-1,|J_2|,|K_2|}(\vec{\phi}_{\{2,\cdots,n\}}|\vec{\psi}_{J_2}|\vec{\theta}_{K_2}|\alpha) + \\
&\theta_1\{\vec{\psi}_{J_1}\} \cdot \mathcal{B}_{n,|J_2|,p-1}(\vec{\phi}|\vec{\psi}_{J_2}|\vec{\theta}_{\{2,\cdots,p\}}|\alpha) \ + \psi_1 \cdot \mathcal{B}_{n,m-1,p}(\vec{\phi}|\vec{\psi}_{\{2,\cdots,m\}}|\vec{\theta}|\alpha) + \\
&\upsilon_{n,p\leq *\leq m+p}(\vec{\phi}|\vec{\psi} \bullet \vec{\theta}|\alpha) + \\
&\upsilon_{|I_1|\leq*\leq|I_1|+|K_1|,|J_1|}(\vec{\theta}_{K_1} \bullet \vec{\phi}_{I_1}, \vec{\psi}_{J_1}, \upsilon_{|J_2|\leq*\leq|I_2|+|J_2|,|K_2|}(\vec{\phi}_{I_2} \bullet \vec{\psi}_{J_2}|\vec{\theta}_{K_2}|\alpha)) \\
&= 0.
\end{aligned}
\tag{B.6}
$$

In Equation B.6 above, we call the terms in rows 1-3 the "standard terms" in the computation of $D(\mathcal{B}_{A_0,A_1,A_2})$, and the terms in rows 4-7 the "extra terms" in the computation of $D(\mathcal{B}_{A_0,A_1,A_2})$. The terms in rows 8-9 are π_1 of the righthand side of Equation B.5; we will call these the "8^{th}- and 9^{th}-row terms".

We compute the sum of the standard terms. In Table B.4, the leftmost column lists the expressions that don't cancel in the sum of the standard terms, the middle column gives the standard term from which the expression comes, and the rightmost column gives the term that cancels the expression. Table B.5 lists the remaining ninth row terms that aren't already listed in Table B.4. In Table B.5, the left column lists the remaining expressions that don't cancel in the ninth row, and the right column gives the extra term that cancels the expression.

All of the terms in the tables describing the expansion of Equation B.6 cancel, so we're done. □

Expression	Comes from Standard Term	Cancelling Term
$1 \otimes \lambda(\vec{\phi}_{I_1})[\lambda(\vec{\theta}_{\{1,\cdots,p-1\}}\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_2})\mathfrak{a}_2 \otimes \theta_p(\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_3})\mathfrak{a}_3) \cdot a_0 \otimes \mathfrak{a}_1]$	$b \circ \mathcal{B}_{n,m,p}(\vec{\phi}\vert\vec{\psi}\vert\vec{\theta}\vert\alpha)$	$\mathcal{B}_{\vert I_1\vert,\vert J_1\vert,p-1}(\vec{\phi}_{I_1}\vert\vec{\psi}_{J_1}\vert\vec{\theta}_{\{1,\cdots,p-1\}}\vert\theta_p\{\vec{\psi}_{J_2}\}\{\vec{\phi}_{I_2}\}\cdot\alpha)$
$1 \otimes \lambda(\vec{\phi}_{I_1})[\lambda(\vec{\theta}\lambda(\vec{\psi}_{\{1,\cdots,m-1\}})\lambda(\vec{\phi}_{I_2})\mathfrak{a}_2 \otimes \psi_m(\lambda(\vec{\phi}_{I_3})\mathfrak{a}_3) \cdot a_0 \otimes \mathfrak{a}_1]$	$b \circ \mathcal{B}_{n,m,p}(\vec{\phi}\vert\vec{\psi}\vert\vec{\theta}\vert\alpha)$	$\mathcal{B}_{\vert I_1\vert,m-1,p}(\vec{\phi}_{I_1}\vert\vec{\psi}_{\{1,\cdots,m-1\}}\vert\vec{\theta}\vert\psi_m\{\vec{\phi}_{I_2}\}\cdot\alpha)$
$1 \otimes \lambda(\vec{\phi}_{I_1})[\lambda(\vec{\theta}\lambda(\vec{\psi}\lambda(\vec{\phi}_{\{1,\cdots,n-1\}})\mathfrak{a}_2 \otimes \psi_n(\mathfrak{a}_3) \cdot a_0 \otimes \mathfrak{a}_1]$	$b \circ \mathcal{B}_{n,m,p}(\vec{\phi}\vert\vec{\psi}\vert\vec{\theta}\vert\alpha)$	$\mathcal{B}_{n-1,m,p}(\vec{\phi}_{\{1,\cdots,n-1\}}\vert\vec{\psi}\vert\vec{\theta}\vert\phi_n\cdot\alpha)$
$\phi_1(\lambda(\vec{\theta}_{K_1})\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_2})\mathfrak{a}_2) \otimes \lambda(\vec{\phi}_{I_1\setminus 1})[\lambda(\vec{\theta}_{K_2})\lambda(\vec{\psi}_{J_3})\lambda(\vec{\phi}_{I_3})\mathfrak{a}_3 \otimes a_0 \otimes \mathfrak{a}_1]$	$b \circ \mathcal{B}_{n,m,p}(\vec{\phi}\vert\vec{\psi}\vert\vec{\theta}\vert\alpha)$	$\phi_1\{\vec{\theta}_{K_1}\}\{\vec{\psi}_{J_1}\} \cdot \mathcal{B}_{n-1,\vert J_2\vert,\vert K_2\vert}(\vec{\phi}_{\{2,\cdots,n\}}\vert\vec{\psi}_{J_2}\vert\vec{\theta}_{K_2}\vert\alpha)$
$f_0\theta_1(\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_2})\mathfrak{a}_2) \otimes \lambda(\vec{\phi}_{I_1})[\lambda(\vec{\theta}_{\{2,\cdots,p\}})\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_3})\mathfrak{a}_3 \otimes a_0 \otimes \mathfrak{a}_1]$	$b \circ \mathcal{B}_{n,m,p}(\vec{\phi}\vert\vec{\psi}\vert\vec{\theta}\vert\alpha)$	$\theta_1\{\vec{\psi}_{J_1}\} \cdot \mathcal{B}_{n,\vert J_2\vert,p-1}(\vec{\phi}\vert\vec{\psi}_{J_2}\vert\vec{\theta}_{\{2,\cdots,p\}}\vert\alpha)$
$f_0h_0\psi_1(\lambda(\vec{\phi}_{I_2})\mathfrak{a}_2) \otimes \lambda(\vec{\phi}_{I_1})[\lambda(\vec{\theta})\lambda(\vec{\psi}_{\{2,\cdots,m\}})\lambda(\vec{\phi}_{I_3})\mathfrak{a}_3 \otimes a_0 \otimes \mathfrak{a}_1]$	$b \circ \mathcal{B}_{n,m,p}(\vec{\phi}\vert\vec{\psi}\vert\vec{\theta}\vert\alpha)$	$\psi_1 \cdot \mathcal{B}_{n,m-1,p}(\vec{\phi}\vert\vec{\psi}_{\{2,\cdots,m\}}\vert\vec{\theta}\vert\alpha)$
$f_0h_0g_0\phi_{i_1}(\lambda(\vec{\theta}_{K_2})\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_3})\mathfrak{a}_3 \otimes a_0 \otimes \mathfrak{a}_1) \otimes \lambda(\vec{\phi}_{I_1})\lambda(\vec{\theta}_{K_1})\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_2\setminus i_1})\mathfrak{a}_2$	$b \circ \mathcal{B}_{n,m,p}(\vec{\phi}\vert\vec{\psi}\vert\vec{\theta}\vert\alpha)$	9^{th} row
$f_0h_0g_0f_{i_1}a_0 \otimes \lambda(\vec{\phi}_{I_1})\lambda(\vec{\theta})\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_2})\mathfrak{a}_1$	$b \circ \mathcal{B}_{n,m,p}(\vec{\phi}\vert\vec{\psi}\vert\vec{\theta}\vert\alpha)$	9^{th} row
$\phi_1(\lambda(\vec{\phi}_{I_1})\lambda(\vec{\theta})\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_2})\mathfrak{a}_3, a_0, \mathfrak{a}_1) \otimes \lambda(\vec{\phi}_{I_1\setminus 1})\mathfrak{a}_2$	$b \circ \mathcal{B}_{n,m,p}(\vec{\phi}\vert\vec{\psi}\vert\vec{\theta}\vert\alpha)$	8^{th} row

Table B.4. Expansion of "standard terms" in Equation B.6 and the terms that cancel them

Expression from ninth Row	Cancels with Extra Term
$\phi_1(\lambda(\vec{\theta}_{K_1})\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_2})[\lambda(\vec{\theta}_{K_3})\lambda(\vec{\psi}_{J_4})\lambda(\vec{\phi}_{I_5})\mathfrak{a}_3, a_0, \mathfrak{a}_1]) \otimes \lambda(\vec{\phi}_{I_1\setminus 1})\lambda(\vec{\theta}_{K_2})\lambda(\vec{\psi}_{J_3})\lambda(\vec{\phi}_{I_4})\mathfrak{a}_2$	$\phi_1\{\vec{\theta}_{K_1}\}\{\vec{\psi}_{J_1}\} \cdot \mathcal{B}_{n-1,\|J_2\|,\|K_2\|}(\vec{\phi}_{\{2,\cdots,n\}}\|\vec{\psi}_{J_2}\|\vec{\theta}_{K_2}\|\alpha)$
$f_0\theta_1(\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_2})[\lambda(\vec{\theta}_{K_2})\lambda(\vec{\psi}_{J_3})\lambda(\vec{\phi}_{I_4})\mathfrak{a}_3, a_0, \mathfrak{a}_1]) \otimes \lambda(\vec{\phi}_{I_1})\lambda(\vec{\theta}_{K_1\setminus 1})\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_3})\mathfrak{a}_2$	$\theta_1\{\vec{\psi}_{J_1}\} \cdot \mathcal{B}_{n,\|J_2\|,p-1}(\vec{\phi}\|\vec{\psi}_{J_2}\|\vec{\theta}_{\{2,\cdots,p\}}\|\alpha)$
$f_0h_0\psi_1(\lambda(\vec{\phi}_{I_2})[\lambda(\vec{\theta}_{K_2})\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_4})\mathfrak{a}_3, a_0, \mathfrak{a}_1]) \otimes \lambda(\vec{\phi}_{I_1})\lambda(\vec{\theta}_{K_1})\lambda(\vec{\psi}_{J_1\setminus 1})\lambda(\vec{\phi}_{I_3})\mathfrak{a}_2$	$\psi_1 \cdot \mathcal{B}_{n,m-1,p}(\vec{\phi}\|\vec{\psi}_{\{2,\cdots,m\}}\|\vec{\theta}\|\alpha)$

Table B.5. Expansion of remaining "ninth row terms" in Equation B.6 and the "extra terms" that cancel them

Proposition B.5. *Let Υ and $\mathcal{B}$ be as defined in the previous propositions. Then, $[\Upsilon, \mathcal{B}] := \Upsilon_{A_1 \bullet A_2, A_0} \mathcal{B}_{A_0, A_1, A_2} - \mathcal{B}_{A_2, A_0, A_1} \Upsilon_{A_0 \bullet A_1, A_2} = 0$. (Note that $[\Upsilon, \mathcal{B}]$ is a map from $C(A_0 \to A_1 \to A_2 \to A_0)$ to itself.)*

Proof. We show the proposition by direct computation. Since all of the maps are maps of cofree comodules, we only need to check that $\pi_1([\Upsilon, \mathcal{B}]) = 0$ where π_1 denotes projection of the comodule onto cogenerators. We check this directly.

$$\begin{aligned}
\pi_1 \circ \Upsilon_{A_1 \bullet A_2, A_0} \mathcal{B}_{A_0, A_1, A_2}(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha) &= \upsilon_{|K_1| \leq * \leq |K_1|+|J_1|, |I_1|}(\vec{\psi}_{J_1} \bullet \vec{\theta}_{K_1} | \vec{\phi}_{I_1} | \mathcal{B}_{|I_2|, |J_2|, |K_2|}(\vec{\phi}_{I_2}|\vec{\psi}_{J_2}|\vec{\theta}_{K_2}|\alpha)) \\
&= \upsilon_{|K_1| \leq * \leq |K_1|+|J_1|, |I_1|}(\vec{\psi}_{J_1} \bullet \vec{\theta}_{K_1} | \vec{\phi}_{I_1} | 1 \otimes \lambda(\vec{\phi}_{I_2})[\lambda(\vec{\theta}_{K_2})\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_3})\mathfrak{a}_2, a_0, \mathfrak{a} \\
&= 1 \otimes \lambda(\vec{\theta}_{K_1})\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_1})[\lambda(\vec{\theta}_{K_2})\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_2})\mathfrak{a}_2, a_0, \mathfrak{a}_1]
\end{aligned}$$

$$\begin{aligned}
\pi_1 \circ \mathcal{B}_{A_2, A_0, A_1} \Upsilon_{A_0 \bullet A_1, A_2}(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha) &= B_{|K_1|, |I_1|, |J_1|}(\vec{\theta}_{K_1}|\vec{\phi}_{I_1}|\vec{\psi}_{J_1}|\upsilon_{|J_2| \leq * \leq |I_2|+|J_2|, |K_2|}(\vec{\phi}_{I_2} \bullet \vec{\psi}_{J_2}|\vec{\theta}_{K_2}|\alpha)) \\
&= 1 \otimes \lambda(\vec{\theta}_{K_1})\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_1})[\lambda(\vec{\theta}_{K_2})\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_2})\mathfrak{a}_2, a_0, \mathfrak{a}_1]
\end{aligned}$$

It's clear that $\pi_1([\Upsilon, \mathcal{B}]) = 0$. $\square$

APPENDIX C

Background on Hochschild chains and cochains

In this section, we give some known constructions on Hochschild chains and cochains for the reader's convenience.

C.1. Standard constructions and notation

Let k be a field of characteristic zero, A a flat unital k-algebra, and M be an A-A-bimodule. Then, we can take $(C_\bullet(A, M), b)$, the (reduced or standard) Hochschild chain complex of A with coefficients in M (see Reference [**1**], Equation 2.1). When $M = B$ is also an algebra over k with left and right module structure given by two maps of algebras $f : A \to B$ and $g : A \to B$, respectively, we may write ${}_fB_g$ to clarify the module structure.

Let k, A, M be as above. We can also take $(C^\bullet(A, M), \delta)$, the (reduced) Hochschild cochain complex of A with coefficients in M (see Reference [**1**], Equations 2.12-13, 2.19-21). When $M = B$ is an algebra, $(C^\bullet(A, B), \delta, \cup)$ is a dga where the cup product $\cup$ is given in Reference [**1**], Equation 2.14.

Let $f, g, h : A \to A$ be maps of algebras. We have a contraction operation of Hochschild cochains and chains, which is a map of complexes:

$$\iota : C^p(A,_f A_g) \bigotimes C_{-q}(A,_g A_h) \longrightarrow C_{-(q-p)}(A,_f A_h)$$

$$\phi \bigotimes a_0 \otimes \cdots \otimes a_q \mapsto \iota(\phi, a_0 \otimes \cdots \otimes a_q) := \phi \cdot (a_0 \otimes \cdots \otimes a_q) :=$$

$$:= (-1)^{p(q+1)} \phi(a_{q-p+1}, \dots, a_q) \cdot a_0 \otimes a_1 \otimes \cdots \otimes a_{q-p}.$$

Finally, we have a "Lie derivative like" operation of Hochschild cochains and chains. Fix an algebra A and let $(\phi_1 ... \phi_n | \alpha) \in C(A \to A)(f_0)$ (see Equation 1.1) be the following element

$$(\phi_1 ... \phi_n | \alpha) = \begin{tikzcd}[column sep=huge] A \arrow[r, bend left=70, "f_0"] \arrow[r, bend left=20, "f_1"'] \arrow[r, bend right=20, "f_{n-1}"] \arrow[r, bend right=50, "f_n"'] \arrow[r, bend right=80, "id"'] \arrow[r, phantom, shift left=6, "\Downarrow \boldsymbol{\phi_1}"] \arrow[r, phantom, "\vdots"] \arrow[r, phantom, shift right=5, "\Downarrow \boldsymbol{\phi_n}"] \arrow[r, phantom, shift right=9, "\boldsymbol{\alpha}"] & A \end{tikzcd}$$

We have a map of complexes

$$C(A \to A)(f_0) \to C_{-\bullet}(A,_{f_0} A)$$

$$(\phi_1 ... \phi_n | a_1 \otimes ... \otimes a_p) \mapsto \lambda(\phi_1 ... \phi_n) \cdot (a_1 \otimes ... \otimes a_p)$$

$$:= \sum_{0 \le i_1 \le \dots \le i_{2n} \le p} (-1)^{\sum_{\substack{j \text{ odd} \\ j \ge 1}} i_j (|\phi_{i_{\frac{j+1}{2}}}| + 1)} \cdot$$

$$\cdot f_0 a_1 \otimes ... \otimes f_0 a_{i_1} \otimes \phi_1(a_{i_1+1}, ..., a_{i_2}) \otimes$$

$$\otimes f_1 a_{i_2+1} \otimes ... \otimes f_1 a_{i_3} \otimes \phi_2(a_{i_3+1}, ..., a_{i_4}) \otimes$$

$$\otimes ... \otimes \phi_n(a_{i_{2n-1}+1}, ..., a_{i_{2n}}) \otimes f_n a_{i_{2n}+1} \otimes ... \otimes f_n a_p.$$

C.2. Brace operation on Hochschild cochains

Fix algebras A_0, A_1 and maps of algebras $f_0, f_n : A_0 \rightrightarrows A_1$, $A_0 \leftleftarrows A_1 : g_0, g_m$. We will define a map of complexes called **braces**

$$
\begin{aligned}
& C(A_0 \to A_1 \to A_0)((f_0, g_0), (f_n, g_m)) \\
:= & \Bigg(\bigoplus_{\substack{i \in \mathbb{N} \\ f_1, \dots, f_i \text{ maps of algebras} \\ f_{i+1} = f_n}} C^\bullet(A_{0,f_0}\, A_{1f_1}) \otimes \dots \otimes C^\bullet(A_{0,f_i}\, A_{1f_{i+1}}) \Bigg) \otimes \\
& \Bigg(\bigoplus_{\substack{j \in \mathbb{N} \\ g_1, \dots, g_j \text{ maps of algebras} \\ g_{j+1} = g_m}} C^\bullet(A_{1,g_0}\, A_{0g_1}) \otimes \dots \otimes C^\bullet(A_{1,g_j}\, A_{0g_{j+1}}) \Bigg) \\
& \qquad\qquad\qquad \Big\downarrow - \bullet - \\
& C(A_0 \to A_0)(g_0 f_0, g_m f_n) \\
:= & \bigoplus_{\substack{i \in \mathbb{N} \\ h_1, \dots, h_i \text{ maps of algebras} \\ h_{i+1} = g_m f_n}} C^\bullet(A_{0,g_0 f_0}\, A_{0h_1}) \otimes \dots \otimes C^\bullet(A_{0,h_i}\, A_{0h_{i+1}}).
\end{aligned}
$$

First, for

$$
(\phi_1 \dots \phi_n | 1) = \begin{tikzcd} A_0 \arrow[r, bend left=80, "f_0"] \arrow[r, bend left, "f_1"] \arrow[r, phantom, bend left=50, "\Downarrow\phi_1"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right, "f_n"'] & A_0 \arrow[r, bend left, "id"] \arrow[r, bend right, "id"'] \arrow[r, phantom, "\Downarrow 1"] & A_0 \end{tikzcd} \quad \text{and} \quad (1 | \phi_1 \dots \phi_n) = \begin{tikzcd} A_0 \arrow[r, bend left, "id"] \arrow[r, bend right, "id"'] \arrow[r, phantom, "\Downarrow 1"] & A_0 \arrow[r, bend left=80, "f_0"] \arrow[r, bend left, "f_1"] \arrow[r, phantom, bend left=50, "\Downarrow\phi_1"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right, "f_n"'] & A_0, \end{tikzcd}
$$

define $(\phi_1 \ldots \phi_n | 1) \overset{\bullet}{\mapsto} (\phi_1 \ldots \phi_n) \bullet 1 = (\phi_1 \ldots \phi_n)$ and $(1 | \phi_1 \ldots \phi_n) \overset{\bullet}{\mapsto} 1 \bullet (\phi_1 \ldots \phi_n) = (\phi_1 \ldots \phi_n)$. Then, for $n, m \geq 1$, let

$$(\phi_1 \ldots \phi_n | \psi_1 \ldots \psi_m) = \begin{tikzcd} A_0 \arrow[r, bend left=80, "f_0"] \arrow[r, bend left=55, phantom, "\Downarrow\phi_1"] \arrow[r, bend left=30, "f_1"'] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=30, "f_n"] & A_1 \arrow[r, bend left=80, "g_0"] \arrow[r, bend left=55, phantom, "\Downarrow\psi_1"] \arrow[r, bend left=30, "g_1"'] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=30, "g_m"] & A_0 \end{tikzcd}$$

and define $(\phi_1 ... \phi_n) \bullet (\psi_1 ... \psi_m) \in C(A_0 \to A_0)(g_0 f_0, g_m f_n)$ as follows:

$$(\phi_1 ... \phi_n) \bullet (\psi_1 ... \psi_m) = \sum_{0 \le i_1 \le \ldots \le i_{2m} \le n} (-1)^{\sum_{\substack{T \text{ odd} \\ T \ge 1}} \left((\sum_{t \le i_T} |\phi_t|+1)(|\psi_{\frac{T+1}{2}}|+1) \right)} \cdot$$

$$\cdot\, g_0\phi_1 \otimes \ldots \otimes g_0\phi_{i_1} \otimes \psi_1\{\phi_{i_1+1} ... \phi_{i_2}\} \otimes$$

$$\otimes\, g_1\phi_{i_2+1} \otimes \ldots \otimes g_1\phi_{i_3} \otimes \psi_2\{\phi_{i_3+1} ... \phi_{i_4}\} \otimes$$

$$\otimes \cdots \otimes \psi_m\{\phi_{i_{2m-1}+1} ... \phi_{i_{2m}}\} \otimes$$

$$\otimes\, g_m\phi_{i_{2m}+1} \otimes \ldots \otimes g_m\phi_n$$

$$\psi_j\{\phi_{i_r+1} ... \phi_{i_r+s}\} \in C^\bullet(A_{0,g_{j-1}f_{i_r}}\ A_{0 g_j f_{i_r+s}})$$

$$\psi_j\{\phi_{i_r+1} ... \phi_{i_r+s}\}(a_1 \otimes \ldots \otimes a_p)$$

$$= \sum_{0 \le k_1 \le \ldots \le k_{2s} \le p} (-1)^{\sum_{\substack{j \text{ odd} \\ j \ge 1}} k_j(|\phi_{i_r + \frac{j-1}{2}}|+1)} \cdot$$

$$\cdot\, \psi_j \Bigg(f_{i_r} a_1, \ldots, f_{i_r} a_{k_1}, \phi_{i_r+1}(a_{k_1+1}, \ldots, a_{k_2}),$$

$$f_{i_r+1} a_{k_2+1}, \ldots, f_{i_r+1} a_{k_3}, \phi_{i_r+2}(a_{k_3+1}, \ldots, a_{k_4}),$$

$$, \ldots, \phi_{i_r+s}(a_{k_{2s-1}}, \ldots, a_{k_{2s}}),$$

$$f_{i_r+s} a_{k_{2s}+1}, \ldots, f_{i_r+s} a_p \Bigg).$$

We follow the sign convention in Reference [**1**] (Equation 2.25 and Section 4.7.2). In $(\phi_1 ... \phi_n) \bullet (\psi_1 ... \psi_m)$, moving ϕ_i past ψ_j introduces a factor of $(-1)^{(|\phi_i|+1)(|\psi_j|+1)}$. Braces are unital, associative maps of complexes (see Reference [**1**], Proposition 4.7.2). It's also straightforward to check that $\lambda(\psi_1 ... \psi_m)\lambda(\phi_1 ... \phi_n) = \lambda((\phi_1 ... \phi_n) \bullet (\psi_1 ... \psi_m))$.

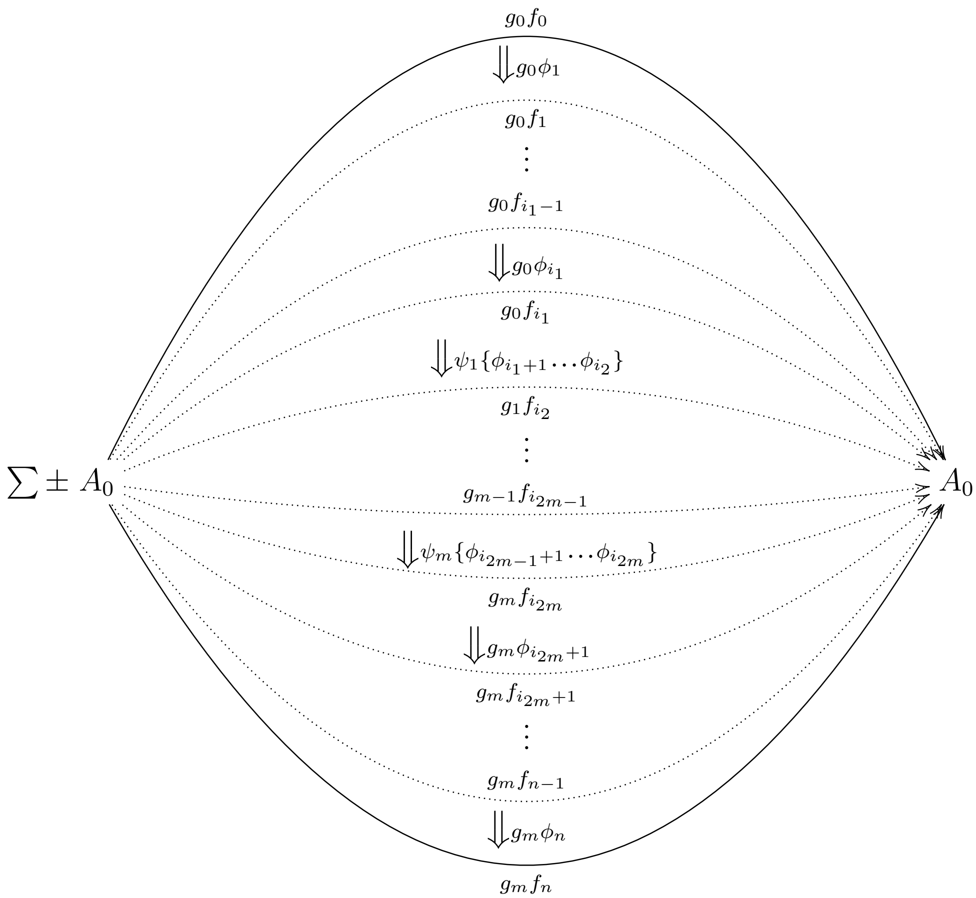

Figure C.1. Picture of the terms in $(\phi_1 \dots \phi_n) \bullet (\psi_1 \dots \psi_m)$