Title

Subtitle

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Date/Event

Braces, categorically

Fix an algebra, A.
Define a dg category, $Hoch(A)$:

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Morphisms: $Hoch(A)(f, g) = (C^\bullet(A, {}_fA_g), {}_f\delta_g)$

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$$
\begin{aligned}
{}_f\delta_g(\phi)(a_1 \otimes \ldots \otimes a_n) =& \epsilon_\phi \Big(f(a_1) \cdot \phi(a_2, \ldots, a_n) + \\
&+ \sum_{1 \le i \le n-1} (-1)^i \phi(a_1, \ldots, a_i a_{i+1}, \ldots, a_n) + \\
&+ (-1)^n \phi(a_1, \ldots, a_{n-1}) \cdot g(a_n) \Big) \\
\epsilon_\phi =& (-1)^{|\phi|+1}
\end{aligned}
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  Composition: cup product on cochains

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$$\begin{tikzcd} A \arrow[r, bend left=60, "f"] \arrow[r, "g"] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=3, "\Downarrow\phi_1"] \arrow[r, phantom, shift right=3, "\Downarrow\phi_2"] & A \arrow[r, rightsquigarrow, "\mathit{compose}"] & A \arrow[r, bend left=60, "f"] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, "\overset{\phi_1\cup\phi_2}{\Downarrow}"] & A \end{tikzcd}$$

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$$\begin{tikzcd}[column sep=huge]
A \arrow[r, bend left=60, "f_0"] \arrow[r, phantom, bend left=45, "\Downarrow\phi_1"] \arrow[r, bend left=30, "f_1"] \arrow[r, phantom, bend left=15, "\Downarrow\phi_2"] \arrow[r, "f_2"] \arrow[r, phantom, bend right=15, "\vdots"] \arrow[r, bend right=30, "f_{n-1}"'] \arrow[r, phantom, bend right=45, "\Downarrow\phi_n"] \arrow[r, bend right=60, "f_n"'] & A
\end{tikzcd}$$

A morphism from f_0 to f_n in Bar(Hoch(A))

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$$\begin{tikzcd}[column sep=huge]
A \arrow[r, bend left=70, "f_0"] \arrow[r, bend left=35, "f_1"] \arrow[r, "f_2"] \arrow[r, bend right=35, "f_{n-1}"'] \arrow[r, bend right=70, "f_n"'] \arrow[r, phantom, bend left=52, "\Downarrow\phi_1"] \arrow[r, phantom, bend left=18, "\Downarrow\phi_2"] \arrow[r, phantom, bend right=18, "\vdots"] \arrow[r, phantom, bend right=52, "\Downarrow\phi_n"] & A \arrow[r, bend left, "g_0"] \arrow[r, bend right, "g_1"'] \arrow[r, phantom, "\Downarrow\psi"] & A
\end{tikzcd}
\overset{\bullet}{\longmapsto}
\begin{tikzcd}[column sep=huge]
A \arrow[r, bend left=50, "g_0 f_0"] \arrow[r, bend right=50, "g_1 f_n"'] \arrow[r, phantom, "\psi\{\phi_1,\ldots,\phi_n\} \Downarrow"] & A
\end{tikzcd}$$

Braces, categorically

Fix an algebra, A.
Define a dg category, $Hoch(A)$:

In this context, braces give multilinear maps:

$$\begin{tikzcd}
Bar(Hoch(A)) \otimes Bar(Hoch(A)) \arrow[r, "\bullet"] \arrow[dr, dotted, "\bullet"'] & Hoch(A) \\
& Bar(Hoch(A)) \arrow[u, two heads]
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Then, (Bar(Hoch(A)), $\bullet$) is an algebra in DGCocats.

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Then, (Bar(Hoch(A)), $\bullet$) is an algebra in DGCocats.
But we have more...

More structure

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Fix algebras, $A_0, A_1, \ldots, A_n$.
We will define a dg cocategory $B(A_0 \to A_1 \to \ldots \to A_n \to A_0)$
where $B(A_0 \to A_0) := Bar(Hoch(A_0))$ for n=0.

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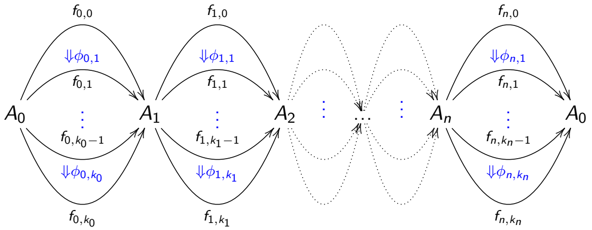
Objects: $\{A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \ldots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_0\}$
A morphism from $(A_0 \xrightarrow{f_{0,0}} \ldots \xrightarrow{f_{n,0}} A_0)$ to $(A_0 \xrightarrow{f_{0,k_0}} \ldots \xrightarrow{f_{n,k_n}} A_0)$:



$\phi_{i,j} \in C^{\bullet}({A_i}_{f_{i,j-1}} {A_{i+1}}_{f_{i,j}})$

Structure among the $B(A_0 \to \ldots \to A_n \to A_0)$'s

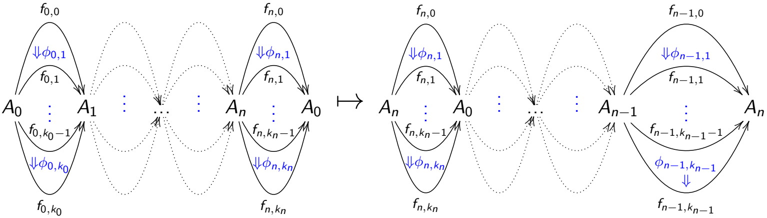
Example

We have a dg functor

$$\hat{\tau}_n : B(A_0 \to \ldots \to A_n \to A_0) \to B(A_n \to A_0 \to \ldots \to A_n)$$

$$Obj : \quad (f_0, \ldots, f_n) \mapsto (f_n, f_0, \ldots, f_{n-1})$$

Mor :

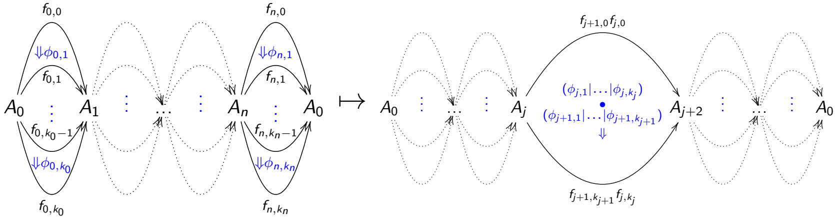


Example

For $n \geq 1, 0 \leq j < n$, we have a dg functor

$$\begin{aligned}
\hat{\delta}_{j,n} : B(A_0 \to \ldots \to A_n \to A_0) &\to B(A_0 \to \ldots \to A_j \to A_{j+2} \to A_0) \\
Obj : \quad (f_0, \ldots, f_n) &\mapsto (f_0, \ldots, f_{j+1}f_j, \ldots, f_n)
\end{aligned}$$

Mor :



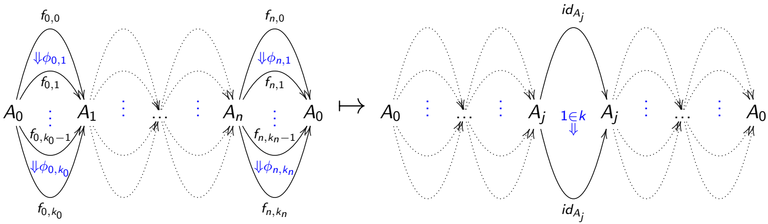
Structure among the $B(A_0 \to \ldots \to A_n \to A_0)$'s

Example

For $n \geq 0, 0 \leq j \leq n$, we have a dg functor

$$
\begin{aligned}
\hat{\delta}_{j,n} : B(A_0 \to \ldots \to A_n \to A_0) &\to B(A_0 \to \ldots \to A_j \to A_j \to A_0) \\
Obj : \qquad (f_0, \ldots, f_n) &\mapsto (f_0, \ldots, f_{j-1}, id_{A_j}, f_j, \ldots, f_n)
\end{aligned}
$$

Mor :



A sheafy-cyclic object in DGCocat

Definition

Let χ be the category with objects $\{A_0 \to A_1 \to \dots \to A_n \to A_0\}$ and morphisms compositions of

$$\begin{aligned} \textit{rotations } \tau_n &: \mathcal{A} \mapsto (A_n \to A_0 \to \dots \to A_n) \\ \textit{coboundaries } \delta_{j,n} &: \mathcal{A} \mapsto (A_0 \to \dots \to A_j \to A_{j+2\,(mod\ n+1)} \to \dots \to A_0) \\ \textit{codegeneracies } \sigma_{i,n} &: \mathcal{A} \mapsto (A_0 \to \dots \to A_i \to A_i \to \dots \to A_0) \end{aligned}$$

where $\mathcal{A} := (A_0 \to \dots \to A_n \to A_0)$, subject to the cyclic relations.

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where $\mathcal{A} := (A_0 \to \ldots \to A_n \to A_0)$, subject to the cyclic relations.

Proposition

We have a functor $\chi \to DGCocat$

$$\begin{aligned} \textit{Objects}: &\ (A_0 \to \ldots \to A_n \to A_0) \mapsto B(A_0 \to \ldots \to A_n \to A_0) \\ \textit{Generating morphisms}: &\ \lambda \mapsto \hat{\lambda} \end{aligned}$$

Each dg cocategory $B(A_0 \to \dots \to A_n \to A_0)$ has a dg comodule $C(A_0 \to \dots \to A_n \to A_0)$ (which is a categorified version of the bar complex of Hochschild chains as a module over Hochschild cochains via contraction).

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Motivating Question: Can we extend the sheafy-cyclic structure to the comodules? In other words, can we give a functor

$$\chi \to \mathcal{D} := \{(dg\ cocat,\ dg\ comod)\}?$$

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$$\chi_\infty \to \mathcal{D}_\infty \quad \textit{dg categories}$$

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$$\mathcal{D}^\bullet_\infty((B_1, C_1), (B_0, C_0)) = \left\{ \begin{matrix} F : B_1 \to B_0\ dg\ functor \\ F_! : C_1 \to F^* C_0\ linear\ map \end{matrix} \right\}$$

or

$$= \left\{ \begin{matrix} F : B_1 \to B_0\ dg\ functor \\ F_\# : F_\# C_1 \to C_0\ linear\ map \end{matrix} \right\}$$