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What Do Algebras Form?

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ABSTRACT

What Do Algebras Form?

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Algebras and their bimodules form a 2-category in which 2-morphisms are certain zero-th Hochschild cohomology groups. When we derive this structure (i.e., use Hochschild cochains instead of HH^0 for 2-morphisms), we find that algebras form a category in dg cocategories. The Hochschild-Kostant-Rosenberg theorem and non-commutative calculus give a rich algebraic structure on Hochschild cohomology along with Hochschild homology. When incorporating the structure on Hochschild homology, we find that algebras form a 2-category with a trace functor. Deriving this again, we conclude that algebras form a category in dg cocategories with a trace functor up to homotopy.

Acknowledgements

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Nomenclature

k – a fixed ground field of char 0

1 – the unit in (a vector space isomorphic to) k

$[1]$ – shift operator on complexes, $C^\bullet[1] = C^{\bullet+1}$

Λ – Connes cyclic category, see Appendix A

$\Delta(b) = \sum_{(b)} b_{(1)} \otimes b_{(2)}$ – Sweedler notation for coproducts

${}_f B_g$ – B as an A - C -bimodule with left structure given by

the map of algebras $f : A \rightarrow B$ and right structure

given by the map of algebras $g : C \rightarrow B$

${}_f B := {}_f B_{id_B}$

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CHAPTER 1

Introduction

What do algebras (over a fixed field k of characteristic zero) form? A straight-forward answer is that they form a 2-category as follows:

$$\begin{aligned}
 &\text{Objects: } k\text{-algebras } A, B, \dots \\
 &\text{1-Morphisms: bimodules } {}_A M_B \\
 &\text{1-Composition: } {}_A M_B \otimes_B {}_B N_C \\
 &\text{2-Morphisms: morphisms of bimodules.}
 \end{aligned}
 \tag{1.1}$$

When we restrict the above 1-morphisms to only those bimodules that come from maps of algebras (i.e., bimodules ${}_A M_B$ where ${}_A M_B = {}_{f(A)} B_B =: {}_f B$ for some map of algebras $f : A \rightarrow B$), then 2-morphisms have an additional structure, namely they are certain zero-th Hochschild cohomology groups:

$$\{\text{morphisms of bimodules } {}_f B \rightarrow_g B\} \xrightarrow{1:1} Z_A({}_g B_f) \cong HH^0(A, {}_g B_f)$$

$$M \mapsto M(1)$$

$$(M_b : b' \mapsto b \cdot b') \leftarrow b$$

The question naturally arises: what happens if we use Hochschild cohomology or cochains instead of just HH^0 for 2-morphisms? The answer is that algebras form a

category in dg categories as follows:

$$\begin{aligned}
 &\text{Objects: } k\text{-algebras } A, B, \dots \\
 &\text{Morphisms: dg cocategory } Bar(Hoch(A, B)) \\
 (1.2) \quad &\text{Composition: } \bullet : Bar(Hoch(A, B)) \otimes Bar(Hoch(B, C)) \rightarrow Bar(Hoch(A, C)) \\
 &\text{associative map of dg cocategories}
 \end{aligned}$$

$Bar(Hoch(A, B))$ is the cofree dg cocategory defined in Section ?? and uses Hochschild cochains as morphisms. The composition, \bullet , is defined in Section ?? and uses the brace operator on Hochschild cochains. The fact that \bullet is associative follows from [?, ?].

Thus far, we have used Hochschild cochains to show that algebras form a category in dg cocategories. Non-commutative calculus tells us that the pair, (Hochschild cochains $C^\bullet(A, A)$, Hochschild chains $C_{-\bullet}(A, A)$), is a $Calc_\infty$ -algebra (Reference [1], Corollary 4). In other words, Hochschild cochains is a Gerstenhaber $_\infty$ -algebra and acts on Hochschild chains up to homotopy via (1) an analogue of the Lie derivative, and (2) an analogue of the contraction of a form against a vector field.

Taking advantage of this $Calc_\infty$ structure, we incorporate HH_0 and find that algebras form a 2-category with a trace functor. The definition of a trace functor on a 2-category à la Kaledin is given in Section ??, and in Section ??, we describe a trace functor on the 2-category given in Equation 1.1 that uses the action of HH^0 on HH_0 .

Again, we ask: can we derive this structure? Can we use Hochschild homology or chains instead of HH_0 to get a trace functor on the category in dg cocategories defined in Equation 1.2? We give the definition of a trace functor on a category in dg cocategories in Section ?. Ultimately, we settle on the following language: on \mathcal{C} , a category in dg

cocategories, a trace functor gives a dg functor $\chi(\mathcal{C}) \rightarrow \mathcal{D}$ where $\chi(\mathcal{C})$ and \mathcal{D} are dg categories introduced in Sections ?? and ??, respectively.

For $\mathcal{C} =$ the category in Equation 1.2, we are not able to give a dg functor $\chi(\mathcal{C}) \rightarrow \mathcal{D}$, however, we do give an A_∞ -functor. This is the (precise) sense in which we have a trace functor “up to homotopy”: we give an A_∞ -functor between dg categories $\chi(\mathcal{C})$ and \mathcal{D} .

CHAPTER 2

A category in dg cocategories

2.1. Motivation of this chapter

In this chapter, we show that algebras form a category in dg cocategories. As stated in the introduction, we will construct such a category with

Objects: k -algebras A, B, \dots

Morphisms: dg cocategory $Bar(Hoch(A, B))$

Composition: $\bullet : Bar(Hoch(A, B)) \otimes Bar(Hoch(B, C)) \rightarrow Bar(Hoch(A, C))$

associative map of dg cocategories.

First, we define the dg cocategories $Bar(Hoch(A, B))$ using Hochschild cochains as morphisms, then we define the composition \bullet using the brace operator on Hochschild cochains.

2.2. Dg cocategories $Bar(Hoch(A, B))$

Let A, B be k -algebras. We define a dg category, $Hoch(A, B)$, as follows:

Objects: algebra maps $f : A \rightarrow B$

Morphisms: $Hoch(A)(f, g) = (C^\bullet(A, {}_f B_g), {}_f \delta_g)$

Composition: cup product on cochains.

(See Appendix C for notation and standard operations on Hochschild complexes.) The cup product is an associative map of complexes, so $Hoch(A, B)$ is a dg category.

Now, we will take $Bar(-)$ of $Hoch(A, B)$, which is a categorified bar construction:

$$Bar : DGCat \rightarrow DGCocat.$$

$Bar(Hoch(A, B))$ has the same objects as $Hoch(A, B)$. A morphism in $Bar(Hoch(A, B))$ from object f_0 to object f_n is a sequence of composable morphisms in $Hoch(A, B)$ starting at f_0 and ending at f_n . We can picture such a morphism as follows:

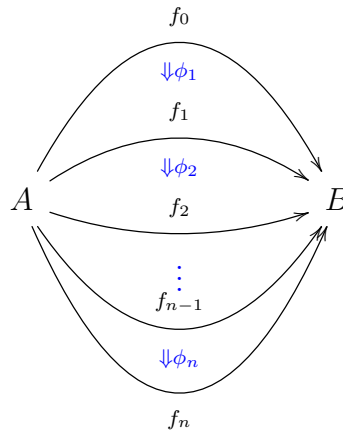


Figure 2.1. A morphism in $Bar(Hoch(A, B))(f_0, f_n)$

As a complex,

$$\begin{aligned} \text{Bar}(\text{Hoch}(A, B))^{\bullet}(f, g) = \\ = \underbrace{k[0]}_{\text{counit}} \oplus \bigoplus_{\substack{n \geq 0, \\ f_i \in \text{Obj}(\text{Hoch}(A, B))}} \text{Hoch}(A, B)^{\bullet}[1](f, f_1) \otimes \text{Hoch}(A, B)^{\bullet}[1](f_1, f_2) \otimes \cdots \otimes \text{Hoch}(A, B)^{\bullet}[1](f_n, g) \end{aligned}$$

$$d_{\text{Bar}(\text{Hoch}(A, B))} = \tilde{d}_{\text{Hoch}(A, B)} + d_{\cup}$$

$$\tilde{d}_{\text{Hoch}(A, B)} = \text{extension of } d_{\text{Hoch}(A, B)} \text{ to a differential on } \text{Bar}$$

$$d_{\cup} = \text{signed sum over composing (cup-producing) two consecutive } \phi_i \text{'s}$$

with cocomposition

$$\Delta(\phi_1 \dots \phi_n) = \sum_{0 \leq i \leq n} \pm (\phi_1 \dots \phi_i) \otimes (\phi_{i+1} \dots \phi_n).$$

For more precise details and explicit signs, see Reference [6], Section 4.6.

2.3. Associative Composition •

Now, we define an associative composition of dg cocategories

$$Bar(Hoch(A, B)) \otimes Bar(Hoch(B, C)) \rightarrow Bar(Hoch(A, C))$$

where A, B, C are k -algebras. To define the composition, we use the fact that $Bar(Hoch(A, C))$ is the cofree dg cocategory over $Hoch(A, C)$. In other words, $Bar(Hoch(A, C))$ satisfies the following universal property:

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\quad} & Hoch(A, C) \\ & \searrow \text{dotted} & \uparrow \\ & & Bar(Hoch(A, C)) \end{array}$$

Figure 2.2. Universal Property of Bar

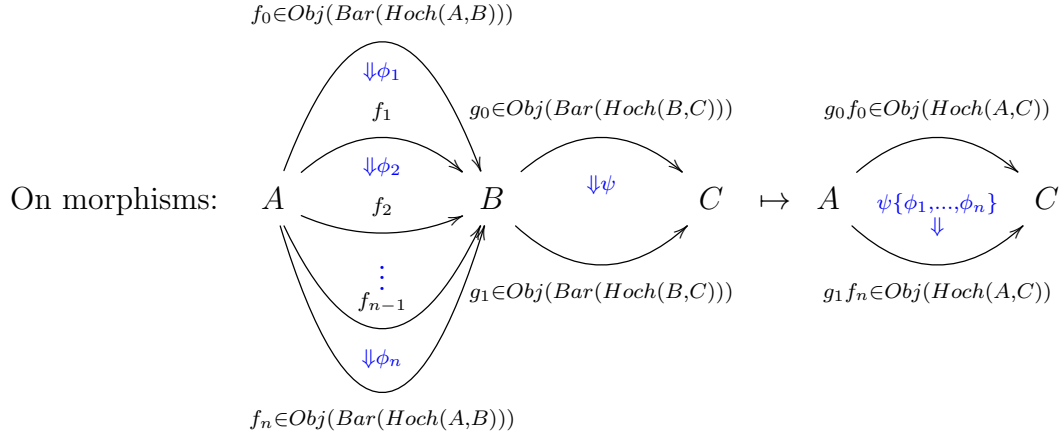
where \mathcal{B} is any dg cocategory, the horizontal map is a map of underlying structure (i.e., an association on objects and maps of complexes of morphisms), and the diagonal lift arrow is a map of dg cocategories. For us, $\mathcal{B} = Bar(Hoch(A, B)) \otimes Bar(Hoch(B, C))$. We will define a map of underlying structure $Bar(Hoch(A, B)) \otimes Bar(Hoch(B, C)) \rightarrow Hoch(A, C)$, which will lift to the map of dg cocategories

$$\bullet : Bar(Hoch(A, B)) \otimes Bar(Hoch(B, C)) \rightarrow Bar(Hoch(A, C)).$$

The map on underlying structure is defined as follows:

$$Bar(Hoch(A, B)) \otimes Bar(Hoch(B, C)) \rightarrow Hoch(A, C)$$

On objects: $f \otimes g \mapsto g \circ f$



$$\begin{array}{ccc}
 A & \xrightarrow{f_0} B & \xrightarrow{g_0} C \\
 \Downarrow \phi & & \Downarrow 1 \in k \\
 A & \xrightarrow{f_n} B & \xrightarrow{g_1} C
 \end{array}
 \mapsto
 \begin{array}{ccc}
 A & \xrightarrow{g_0 f_0} C \\
 \Downarrow \phi & & \\
 A & \xrightarrow{g_1 f_n} C
 \end{array}$$

$$\begin{array}{ccc}
 A & \xrightarrow{f_0} B & \xrightarrow{g_0} C \\
 \Downarrow 1 \in k & & \Downarrow \psi \\
 A & \xrightarrow{f_n} B & \xrightarrow{g_1} C
 \end{array}
 \mapsto
 \begin{array}{ccc}
 A & \xrightarrow{g_0 f_0} C \\
 \Downarrow \psi & & \\
 A & \xrightarrow{g_1 f_n} C
 \end{array}$$

All other non-pictured pairings of a morphism from $Bar(Hoch(A, B))$ and a morphism from $Bar(Hoch(B, C))$ map to zero. The brace operation is given in Reference [6], Equation 4.8, and the fact that it is associative follows from References [?, ?, ?, ?].

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APPENDIX A

Connes cyclic category, Λ

Here, we give generators and relations for the cyclic category, Λ . None of this is new, but we do it to establish notation for the rest of the paper.

Λ has objects $\{[n] : n \in \mathbb{N}\}$ and generating morphisms:

$$\begin{aligned}
 & \text{rotations } \tau_n : [n] \rightarrow [n], \\
 (A.1) \quad & \text{coboundaries } \delta_{j,n} : [n] \rightarrow [n-1], 0 \leq j \leq n-1, \\
 & \text{codegeneracies } \sigma_{i,n} : [n] \rightarrow [n+1], 0 \leq i \leq n
 \end{aligned}$$

subject to relations:

$$\begin{aligned}
 \delta_{i,n-1}\delta_{j,n} &= \delta_{j-1,n-1}\delta_{i,n} \quad 0 \leq i < j \leq n-1 \\
 \sigma_{i,n+1}\sigma_{j,n} &= \sigma_{j+1,n+1}\sigma_{i,n} \quad 0 \leq i \leq j \leq n \\
 \delta_{j,n+1}\sigma_{i,n} &= \begin{cases} \sigma_{i,n-1}\delta_{j-1,n} & 0 \leq i < j \leq n \\ id & j = i, i-1 \\ \sigma_{i-1,n-1}\delta_{j,n} & 0 \leq j < i-1 \leq n-1 \end{cases} \\
 (A.2) \quad \tau_{n+1}\sigma_{i,n} &= \sigma_{i+1,n}\tau_n \quad 0 \leq i \leq n-1 \\
 \tau_{n-1}\delta_{j,n} &= \delta_{j+1,n}\tau_n \quad 0 \leq j \leq n-1 \\
 \tau_n^{n+1} &= id \\
 \delta_{0,1}\tau_1\sigma_{0,0} &= id \\
 \tau_{n+1}\sigma_{n,n} &= \tau_{n+1}^{n+1}\sigma_{0,n}\tau_n \\
 \delta_{0,n}\tau_n^2 &= \tau_{n-1}\delta_{n-1,n}.
 \end{aligned}$$

Some presentations of Λ include an extra coboundary $\delta_{n,n}$ and codegeneracy $\sigma_{n+1,n}$.

In terms of our generators, they are $\delta_{n,n} := \delta_{0,n}\tau_n$ and $\sigma_{n+1,n} := \tau_{n+1}^{n+1}\sigma_{0,n}$.

APPENDIX B

Computations

In this appendix, we give the computational propositions needed to establish the homotopically sheafy-cyclic structure on dg comodules. All the comodules we work with will be cofree, and we will define maps into them by giving maps into cogenerators (see Equation ??).

B.1. Computational notation

For this section's propositions, we establish the following notation:

A_0, A_1 fixed algebras

$$(\vec{\phi}|\vec{\psi}|\alpha) := (\phi_1 \dots \phi_n | \psi_1 \dots \psi_m | \alpha)$$

$$= \begin{array}{c} \begin{array}{ccccc} & f_0 & & g_0 & \\ & \curvearrowright & & \curvearrowright & \\ & \Downarrow \phi_1 & & \Downarrow \psi_1 & \\ & \curvearrowleft & & \curvearrowright & \\ A_0 & & A_1 & & A_0 \\ & \vdots & & \vdots & \\ & f_n & \xrightarrow{\alpha} & g_m & \\ & \curvearrowright & & \curvearrowleft & \\ & id & & id & \end{array} \end{array} \in C(A_0 \rightarrow A_1 \rightarrow A_0)(g_0 f_0)$$

$$\vec{\phi}_{\{i_1, i_2, \dots, i_k\}} := \phi_{i_1} \phi_{i_2} \dots \phi_{i_k}$$

where $\{i_1, i_2, \dots, i_k\}$ is an ordered subset of $\{1, \dots, n\}$

$$\vec{\phi}_{\{\}} := 1 \in k \cong \text{Bar}_0(C^\bullet(A_0, A_1))$$

$$\vec{\psi}_{\{\}} := 1 \in k \cong \text{Bar}_0(C^\bullet(A_1, A_0))$$

$$|I| := \text{number of elements in a set } I$$

$I_1 I_2$:= concatenation as ordered sets of possibly-empty sets I_1 and I_2

$$\epsilon_{I_1, J_1} := (-1)^{\left(\sum_{r \in I_1} |\phi_r| + 1\right) \left(\sum_{s \in J_1} |\psi_s| + 1\right)}$$

when I_1, J_1 are ordered indexing sets

$\lambda(\vec{\psi}), \tilde{\delta}, b', b, \psi\{\vec{\phi}\} \cdot \alpha$ = see Appendix C for operations on Hochschild (co)chains

B.1.1. Notation for elements of Hochschild chains

Let $a_0 \otimes a_1 \otimes \cdots \otimes a_n$ denote a typical element of $C_{-\bullet}(A, A)$ where A is some algebra. At times, we wish to feed a portion of $a_0 \otimes a_1 \otimes \cdots \otimes a_n$ to a Hochschild cochain (or other map on chains) without specifying the degree of the cochain. To do this, we will rewrite $a_0 \otimes a_1 \otimes \cdots \otimes a_n = a_0 \otimes \mathbf{a}_1 \otimes \cdots \otimes \mathbf{a}_r$ where each $\mathbf{a}_i = a_{j_i} \otimes a_{j_i+1} \otimes \cdots \otimes a_{j_{i+1}-1}$ and \mathbf{a}_i is an empty chain if $j_i = j_{i+1}$.

For example, if $\phi \in C^2(A, A)$, then we rewrite

$$\sum_{1 \leq i \leq n-1} a_0 \otimes a_1 \otimes \cdots \otimes a_{i-1} \otimes \phi(a_i, a_{i+1}) \otimes a_{i+2} \otimes \cdots \otimes a_n = \sum a_0 \otimes \mathbf{a}_1 \otimes \phi(\mathbf{a}_2) \otimes \mathbf{a}_3.$$

If $\mathbf{a}_1 = a_1 \otimes \cdots \otimes a_p$, then $|\mathbf{a}_1| = p$. For $a_0 \otimes \mathbf{a}_1 \otimes \mathbf{a}_2$, we write $\eta_{\mathbf{a}_1, \mathbf{a}_2} = (-1)^{|\mathbf{a}_1|(|\mathbf{a}_1|+|\mathbf{a}_2|)}$.

B.2. Computational Propositions

Proposition B.1. *Let $\hat{\tau}_1 : B(A_0 \rightarrow A_1 \rightarrow A_0) \longrightarrow B(A_1 \rightarrow A_0 \rightarrow A_1)$ be as defined in Section ???. Recall from Example ?? that $\hat{\tau}_1^* C(A_1 \rightarrow A_0 \rightarrow A_0) \cong C(A_1 \rightarrow A_0 \rightarrow A_1)$ as complexes. Define a map*

$$\Upsilon_{A_0, A_1} : C(A_0 \rightarrow A_1 \rightarrow A_0) \rightarrow \hat{\tau}_1^* C(A_1 \rightarrow A_0 \rightarrow A_1)$$

of comodules over $B(A_0 \rightarrow A_1 \rightarrow A_0)$ by mapping into cogenerators as follows:

(B.1)

$$v^{f_0, g_0} : C(A_0 \rightarrow A_1 \rightarrow A_0)(f_0, g_0) \rightarrow \hat{\tau}_1^* C(A_1 \rightarrow A_0 \rightarrow A_1)(g_0, f_0)$$

(B.2)

$$\cong C(A_1 \rightarrow A_0 \rightarrow A_1)(g_0, f_0)$$

(B.3)

$$\xrightarrow[\text{cogenerators}]{\text{project onto}} C_{-\bullet}(A_1, f_0 g_0 A_{1id})$$

(B.4)

$$v_{n,m}^{f_0, g_0}(\vec{\phi}|\vec{\psi}|\alpha) = \sum_{\substack{I_1 I_2 = \{2, \dots, n\} \\ \text{as ordered sets}}} \phi_1(\lambda(\vec{\psi})\lambda(\vec{\phi}_{I_2}) \cdot \mathfrak{a}_3, a_0, \mathfrak{a}_1) \otimes \lambda(\vec{\phi}_{I_1}) \cdot \mathfrak{a}_2$$

(B.5)

$$\left(+ f_0 a_0 \otimes \lambda(\vec{\phi})\mathfrak{a}_1 \quad \text{if } m = 0 \right).$$

Then, $\Upsilon_{A_0, A_1} : C(A_0 \rightarrow A_1 \rightarrow A_0) \rightarrow \hat{\tau}^* C(A_1 \rightarrow A_0 \rightarrow A_1)$ is a map of dg comodules over $B(A_0 \rightarrow A_1 \rightarrow A_0)$.

Proof. We must show: (1) Υ is a map of comodules, and (2) Υ commutes with the differentials. (In this proof, we drop the subscripts and write $\Upsilon := \Upsilon_{A_0, A_1}$.)

(1) This proof is standard for cofree comodules. Let $(\vec{\phi}|\vec{\psi}|\alpha)$ be as in the statement of the proposition. We want to show that Υ commutes with the coproducts. On one hand,

$$\begin{aligned}
& [(id_B \otimes \Upsilon) \circ \Delta_{C(A_0 \rightarrow A_1 \rightarrow A_0)}](\vec{\phi}|\vec{\psi}|\alpha) \\
&= [id_B \otimes \Upsilon] \left(\sum_{\substack{I_1 I_2 = \{1, 2, \dots, n\} \text{ and} \\ J_1 J_2 = \{1, 2, \dots, m\} \\ \text{as ordered sets}}} \epsilon_{I_2, J_1} \cdot (\vec{\phi}_{I_1}|\vec{\psi}_{J_1}) \otimes (\vec{\phi}_{I_2}|\vec{\psi}_{J_2}|\alpha) \right) \\
&= \sum_{\substack{I_1 I_2 I_3 = \{1, 2, \dots, n\} \text{ and} \\ J_1 J_2 J_3 = \{1, 2, \dots, m\} \\ \text{as ordered sets}}} \epsilon_{I_2 I_3, J_1} \cdot \epsilon_{I_3, J_2} \cdot (\vec{\phi}_{I_1}|\vec{\psi}_{J_1}) \otimes (\vec{\phi}_{I_2}|\vec{\psi}_{J_2}) \otimes v_{|I_3|, |J_3|}(\vec{\phi}_{I_3}|\vec{\psi}_{J_3}|\alpha).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& [\Delta_{\hat{\tau}^* C(A_1 \rightarrow A_0 \rightarrow A_1)} \circ \Upsilon](\vec{\phi}|\vec{\psi}|\alpha) \\
&= \Delta_{\hat{\tau}^* C(A_1 \rightarrow A_0 \rightarrow A_1)} \left(\sum_{\substack{I_1 I_2 = \{1, 2, \dots, n\} \text{ and} \\ J_1 J_2 = \{1, 2, \dots, m\} \\ \text{as ordered sets}}} \epsilon_{I_2, J_1} \cdot (\vec{\phi}_{I_1}|\vec{\psi}_{J_1}) \otimes v_{|I_2|, |J_2|}(\vec{\phi}_{I_2}|\vec{\psi}_{J_2}|\alpha) \right) \\
&= \sum_{\substack{I_1 I_2 I_3 = \{1, 2, \dots, n\} \text{ and} \\ J_1 J_2 J_3 = \{1, 2, \dots, m\} \\ \text{as ordered sets}}} \epsilon_{I_2 I_3, J_1} \cdot \epsilon_{I_3, J_2} \cdot (\vec{\phi}_{I_1}|\vec{\psi}_{J_1}) \otimes (\vec{\phi}_{I_2}|\vec{\psi}_{J_2}) \otimes v_{|I_3|, |J_3|}(\vec{\phi}_{I_3}|\vec{\psi}_{J_3}|\alpha).
\end{aligned}$$

Clearly $(id_B \otimes \Upsilon) \circ \Delta_{C(A_0 \rightarrow A_1 \rightarrow A_0)} = \Delta_{\hat{\tau}^* C(A_1 \rightarrow A_0 \rightarrow A_1)} \circ \Upsilon$.

(2) We will show that Υ commutes with the differentials by direct computation. Since Υ is a map of cofree comodules, we only need to check that $\pi_1 \circ D(\Upsilon) = 0$ where $D(\Upsilon)$ is the differential applied to Υ as a linear map between complexes and π_1 denotes projection

of a comodule onto its cogenerators. More explicitly, we want to check that

$$\begin{aligned}
& v_{n,m}(\tilde{\delta}(\vec{\phi})|\vec{\psi}|\alpha) + v_{n,m}(\vec{\phi}|\tilde{\delta}(\vec{\psi})|\alpha) + v_{n-1,m}(b'(\vec{\phi})|\vec{\psi}|\alpha) + v_{n,m-1}(\vec{\phi}|b'(\vec{\psi})|\alpha) + \\
& v_{n,m}(\vec{\phi}|\vec{\psi}|b(\alpha)) + b \circ v_{n,m}(\vec{\phi}|\vec{\psi}|\alpha) + \\
& \sum_{\substack{I_1 I_2 = \{1, \dots, n\} \\ \text{as ordered sets}}} \epsilon_{I_2, \{1, \dots, m-1\}} \cdot v_{|I_1|, m-1}(\vec{\phi}_{I_1}|\vec{\psi}_{\{1, \dots, m-1\}}|\psi_m\{\vec{\phi}_{I_2}\} \cdot \alpha) + \\
\text{(B.6)} \quad & \sum_{\substack{J_1 J_2 = \{1, \dots, m\} \\ \text{as ordered sets}}} \epsilon_{\{2, \dots, n\}, J_1} \cdot \phi_1\{\psi_{J_1}\} \cdot v_{n-1, |J_2|}(\phi_{\{2, \dots, n\}}|\psi_{J_2}|\alpha) + \\
& \epsilon_{\{n\}, \{1, \dots, m\}} \cdot v_{n-1, m}(\vec{\phi}_{\{1, \dots, n-1\}}|\vec{\psi}|\phi_n \cdot \alpha) + \\
& \epsilon_{\{1, \dots, n\}, \{1\}} \cdot \psi_1 \cdot v_{n, m-1}(\vec{\phi}|\vec{\psi}_{\{2, \dots, m\}}|\alpha) \\
& = 0.
\end{aligned}$$

In Equation B.6, we will call the terms in rows 1-2 the “standard terms”, and the terms in rows 3-6 the “extra terms”.

We compute the sum of the standard terms. In Table B.1, the leftmost column lists the expressions that don’t cancel in the sum of the standard terms, the middle column gives the standard term from which the expression comes, and the rightmost column gives the term (extra or standard) that cancels the expression.

All of the terms in Table B.1 cancel, so Υ is a map of complexes. \square

Expression (Expansion)	Comes from Standard Term in Equation B.6	Cancelling Term in Equation B.6
$f_0\psi_1(\lambda(\vec{\phi}_{I_2})\mathbf{a}_3) \cdot$ $\phi_1(\lambda(\vec{\psi}_{\{2,\dots,m\}})\lambda(\vec{\phi}_{I_3})\mathbf{a}_4, a_0, \mathbf{a}_1) \otimes \lambda(\vec{\phi}_{I_1})\mathbf{a}_2$	$v_{n,m}(\delta(\phi_1)\phi_2 \cdots \phi_n \vec{\psi} \alpha)$	$f_0\psi_1 \cdot v_{n,m-1}(\vec{\phi} \vec{\psi}_{\{2,\dots,m\}} \alpha)$
$\phi_1(\lambda(\vec{\psi}_{\{1,\dots,m-1\}})\lambda(\vec{\phi}_{I_2})\mathbf{a}_3,$ $\psi_m(\lambda(\vec{\phi}_{I_3})\mathbf{a}_4) \cdot a_0, \mathbf{a}_1) \otimes \lambda(\vec{\phi}_{I_1})\mathbf{a}_2$	$v_{n,m}(\delta(\phi_1)\phi_2 \cdots \phi_n \vec{\psi} \alpha)$	$v_{ I_1 ,m-1}(\vec{\phi}_{I_1} \vec{\psi}_{\{1,\dots,m-1\}} \psi_m\{\vec{\phi}_{I_2}\} \cdot \alpha)$
$\phi_1(\lambda(\vec{\psi})\lambda(\vec{\phi}_{I_2})\mathbf{a}_3, g_m\phi_n(\mathbf{a}_4) \cdot a_0, \mathbf{a}_1) \otimes$ $\otimes \lambda(\vec{\phi}_{I_1})\mathbf{a}_2$	$v_{n,m}(\delta(\phi_1)\phi_2 \cdots \phi_n \vec{\psi} \alpha)$	$v_{n-1,m}(\vec{\phi}_{\{1,\dots,n-1\}} \vec{\psi} g_m\phi_n \cdot \alpha)$
$\phi_1(\lambda(\vec{\psi})\lambda(\vec{\phi}_{I_2})\mathbf{a}_2) \cdot f_1(a_0) \otimes \lambda(\vec{\phi}_{I_1})\mathbf{a}_1$	$v_{n,m}(\delta(\phi_1)\phi_2 \cdots \phi_n \vec{\psi} \alpha)$	$\phi_1 \cdot v_{n-1,0}(\vec{\phi}_{\{2,\dots,n\}} \vec{\psi} \alpha)$
$f_0a_0 \cdot \phi_1(\mathbf{a}_1) \otimes \lambda(\vec{\phi}_{\{1,\dots,n-1\}})\mathbf{a}_2$	$v_{n,m}(\delta(\phi_1)\phi_2 \cdots \phi_n \vec{\psi} \alpha)$ if $\vec{\psi} = 1$	$b \circ v_{n,m}(\vec{\phi} \vec{\psi} \alpha)$ if $\vec{\psi} = 1$
$f_0g_m\phi_n(\mathbf{a}_2)f_0a_0 \otimes \lambda(\vec{\phi}_{\{1,\dots,n-1\}})\mathbf{a}_1$	$b \circ v_{n,m}(\vec{\phi} \vec{\psi} \alpha)$ if $\vec{\psi} = 1$	$v_{n-1,m}(\vec{\phi}_{\{1,\dots,n-1\}} \vec{\psi} g_m\phi_n \cdot \alpha)$ if $\vec{\psi} = 1$
$\phi_1(\lambda(\vec{\psi})\lambda(\vec{\phi}_{I_2})\mathbf{a}_4, a_0, \mathbf{a}_1) \cdot \phi_2(\mathbf{a}_2) \otimes \lambda(\vec{\phi}_{I_1})\mathbf{a}_3$	$b \circ v_{n,m}(\vec{\phi} \vec{\psi} \alpha)$	$v_{n-1,m}(\phi_1 \cup \phi_2\phi_3 \cdots \phi_n \vec{\psi} \alpha)$
$\phi_1(\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_2})\mathbf{a}_3)\phi_2(\lambda(\vec{\psi}_{I_2})\lambda(\vec{\phi}_{I_3})\mathbf{a}_3,$ $a_0, \mathbf{a}_1) \otimes \lambda(\vec{\phi}_{I_1})\mathbf{a}_2$	$v_{n-1,m}(\phi_1 \cup \phi_2\phi_3 \cdots \phi_n \vec{\psi} \alpha)$	$\phi_1\{\vec{\psi}_{J_1}\} \cdot v_{n-1, J_2 }(\vec{\phi}_{\{2,\dots,n\}} \vec{\psi}_{J_2} \alpha)$
$f_0\psi_1(\lambda(\vec{\phi}_{I_2})\mathbf{a}_2) \cdot f_0a_0 \otimes \lambda(\vec{\phi}_{I_1})\mathbf{a}_1$	$f_0\psi_1 \cdot v_{n,0}(\vec{\phi} 1 \alpha)$ if $\vec{\psi} = \psi_1$	$v_{ I_1 ,0}(\vec{\phi}_{I_1} 1 \psi_1\{\vec{\phi}_{I_2}\} \cdot \alpha)$ if $\vec{\psi} = \psi_1$

Table B.1. Expansion of terms in Equation B.6

(Technically, the last term in the middle column is not a standard term, but we include it in the table for convenience.)

Proposition B.2. *Let $B_{A_0, A_1} = B : C(A_0 \rightarrow A_1 \rightarrow A_0) \longrightarrow C(A_0 \rightarrow A_1 \rightarrow A_0)$ be the map of cofree comodules defined by the following maps to cogenerators:*

$$(B.7) \quad B_{n,m}(\vec{\phi}|\vec{\psi}|\alpha) = \eta_{\mathbf{a}_1, \mathbf{a}_2} \cdot 1 \otimes \lambda(\psi)\lambda(\phi)\mathbf{a}_2 \otimes a_0 \otimes \mathbf{a}_1.$$

Then, $D(B_{A_0, A_1}) = \Upsilon_{A_1, A_0} \Upsilon_{A_0, A_1} - id$ where Υ is defined in Proposition B.1.

Proof. We prove the statement by direct computation. Since all of the maps are maps of cofree comodules, we only need to check that $\pi_1(D(B_{A_0, A_1}) - \Upsilon_{A_1, A_0} \Upsilon_{A_0, A_1} - id) = 0$ where π_1 denotes projection of the comodule onto cogenerators. More explicitly, for an element $(\vec{\phi}|\vec{\psi}|\alpha)$, we want to check that

$$(B.8) \quad \begin{aligned} & B_{n,m}(\vec{\delta}(\vec{\phi})|\vec{\psi}|\alpha) + B_{n,m}(\vec{\phi}|\vec{\delta}(\vec{\psi})|\alpha) + B_{n-1,m}(b'(\vec{\phi})|\vec{\psi}|\alpha) + B_{n,m-1}(\vec{\phi}|b'(\vec{\psi})|\alpha) + \\ & B_{n,m}(\vec{\phi}|\vec{\psi}|b(\alpha)) + b \circ B_{n,m}(\vec{\phi}|\vec{\psi}|\alpha) + \\ & \epsilon_{\{n\}, \{1, \dots, m\}} \cdot B_{n-1,m}(\vec{\phi}_{\{1, \dots, n-1\}}|\vec{\psi}_m|\phi_n \cdot \alpha) + \\ & \epsilon_{\{1, \dots, n\}, \{1\}} \cdot \psi_1 \cdot B_{n,m-1}(\vec{\phi}|\vec{\psi}_{\{2, \dots, m\}}|\alpha) + \\ & \sum_{\substack{I_1 I_2 = \{1, \dots, n\} \\ \text{as ordered setts}}} \epsilon_{I_2, \{1, \dots, m-1\}} \cdot B_{|I_1|, m-1}(\vec{\phi}_{I_1}|\vec{\psi}_{\{1, \dots, m-1\}}|\psi_m \{\vec{\phi}_{I_2}\} \cdot \alpha) + \\ & \sum_{\substack{J_1 J_2 = \{1, \dots, m\} \\ \text{as ordered setts}}} \epsilon_{\{2, \dots, n\}, J_1} \cdot \phi_1 \{\psi_{J_1}\} \cdot B_{n-1, |J_2|}(\phi_{\{2, \dots, n\}}|\psi_{J_2}|\alpha) - \\ & \sum_{\substack{I_1 I_2 = \{1, \dots, n\} \\ J_1 J_2 = \{1, \dots, m\} \\ \text{as ordered setts}}} \epsilon_{I_1, J_2} \cdot v_{|J_1|, |I_1|}(\vec{\psi}_{J_1}|\vec{\phi}_{I_1}|v_{|I_2|, |J_2|}(\vec{\phi}_{I_2}|\vec{\psi}_{J_2}|\alpha)) - \pi_1(\vec{\phi}|\vec{\psi}|\alpha) \\ & = 0. \end{aligned}$$

We will call the terms in rows 1-2 the “standard terms” in the computation of $D(B_{A_0,A_1})$, and the terms in rows 3-6 the “extra terms” in the computation of $D(B_{A_0,A_1})$. The seventh row is $\pi_1(\Upsilon_{A_1,A_0}\Upsilon_{A_0,A_1} - id)$.

We compute the sum of the standard terms. In Table B.2, the leftmost column lists the expressions that don’t cancel in the sum of the standard terms, the middle column gives the standard term from which the expression comes, and the rightmost column gives the extra term that cancels the expression. Table B.3 lists the remaining terms from the seventh row that are not already listed in Table B.2. In Table B.3, the left column lists the remaining expressions that don’t cancel in the seventh row, and the right column gives the extra term that cancels the expression.

All of the terms in the tables describing the expansion of equation B.8 cancel, so $D(B_{A_0,A_1}) = \Upsilon_{A_1,A_0}\Upsilon_{A_0,A_1} - id$. □

Expression (Expansion)	Comes from Standard Term in Equation B.8	Cancels with Extra Term in Equation B.8
$\psi_1(\lambda(\vec{\phi}_{I_1})\mathbf{a}_2) \otimes \lambda(\vec{\psi}_{\{2,\dots,m\}})\lambda(\vec{\phi}_{I_2})\mathbf{a}_3 \otimes a_0 \otimes \mathbf{a}_1$	$b \circ B_{n,m}(\vec{\phi} \vec{\psi} \alpha)$	$\psi_1\{\vec{\phi}_{I_1}\} \cdot B_{ I_2 ,m-1}(\vec{\phi}_{I_2} \vec{\psi}_{\{2,\dots,m\}} \alpha)$
$g_0\phi_1(\mathbf{a}_2) \otimes \lambda(\vec{\psi})\lambda(\vec{\phi}_{\{2,\dots,n\}})\mathbf{a}_3 \otimes a_0 \otimes \mathbf{a}_1$	$b \circ B_{n,m}(\vec{\phi} \vec{\psi} \alpha)$	$\phi_1 \cdot B_{n-1,m}(\vec{\phi}_{\{2,\dots,n\}} \vec{\psi} \alpha)$
$1 \otimes \lambda(\vec{\psi})\lambda(\vec{\phi}_{\{1,\dots,n-1\}})\mathbf{a}_2 \otimes g_m\phi_n(\mathbf{a}_3) \cdot a_0 \otimes \mathbf{a}_1$	$b \circ B_{n,m}(\vec{\phi} \vec{\psi} \alpha)$	$B_{n-1,m}(\vec{\phi}_{\{1,\dots,n-1\}} \vec{\psi} \phi_n \cdot \alpha)$
$1 \otimes \lambda(\vec{\psi}_{\{1,\dots,m-1\}})\lambda(\vec{\phi}_{I_1})\mathbf{a}_2 \otimes g_m\psi_m(\lambda(\vec{\phi}_{I_2})\mathbf{a}_3) \cdot a_0 \otimes \mathbf{a}_1$	$b \circ B_{n,m}(\vec{\phi} \vec{\psi} \alpha)$	$B_{ I_1 ,m-1}(\vec{\phi}_{I_2} \vec{\psi}_{\{1,\dots,m-1\}} \psi_m\{\vec{\phi}_{I_2}\} \cdot \alpha)$
$g_0f_0a_0 \otimes \lambda(\vec{\psi})\lambda(\vec{\phi})\mathbf{a}_1$	$b \circ B_{n,m}(\vec{\phi} \vec{\psi} \alpha)$	$v_{ J_1 , I_1 }(\vec{\psi}_{J_1} \vec{\phi}_{I_1} v_{ I_2 , J_2 }(\vec{\phi}_{I_2} \vec{\psi}_{J_2} \alpha))$

Table B.2. Expansion of terms in Equation B.8: “standard terms” and the “extra terms” that cancel them

(Technically, the last term in the right column is not an extra term, but we include it in the table for convenience.)

Expression (Expansion) from Seventh-Row in Equation B.8	Cancels with Extra Term in Equation B.8
$\psi_1(\lambda(\vec{\phi}_{I_1})\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_4})\mathbf{a}_4, \phi_{ I_1 +1}(\lambda(\vec{\psi}_{J_3})\lambda(\vec{\phi}_{I_5})\mathbf{a}_5, a_0, \mathbf{a}_1), \lambda(\vec{\phi}_{I_2 \setminus I_1 +1})\mathbf{a}_2) \otimes \lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_3})\mathbf{a}_3$	$\psi_1\{\vec{\phi}_{I_1}\} \cdot B_{ I_2 ,m-1}(\vec{\phi}_{I_2} \vec{\psi}_{\{2,\dots,m\}} \alpha)$
$\psi_1(\lambda(\vec{\phi}_{I_1})\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_4})\mathbf{a}_4, f_{ I_1 +1}a_0, \lambda(\vec{\phi}_{I_2 \setminus I_1 +1})\mathbf{a}_1) \otimes \lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_3})\mathbf{a}_2$	$\phi_1 \cdot B_{n-1,m}(\vec{\phi}_{\{2,\dots,n\}} \vec{\psi} \alpha)$
$g_0\phi_1(\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_2})\mathbf{a}_3, a_0, \mathbf{a}_1) \otimes \lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_1})\mathbf{a}_2$	$\psi_1\{\vec{\phi}_{I_1}\} \cdot B_{ I_2 ,m-1}(\vec{\phi}_{I_2} \vec{\psi}_{\{2,\dots,m\}} \alpha)$

Table B.3. Expansion of terms in Equation B.8: remaining “seventh-row terms” and the “extra terms” that cancel them

Proposition B.3. *Let $\Upsilon_{A_0, A_1} : C(A_0 \rightarrow A_1 \rightarrow A_0) \longrightarrow C(A_1 \rightarrow A_0 \rightarrow A_1)$ and $B_{A_0, A_1} : C(A_0 \rightarrow A_1 \rightarrow A_0) \longrightarrow C(A_0 \rightarrow A_1 \rightarrow A_0)$ be the maps defined in Propositions B.1 and B.2 above. Then, $[\Upsilon, B] := \Upsilon_{A_0, A_1} B_{A_0, A_1} - B_{A_1, A_0} \Upsilon_{A_0, A_1} = 0$.*

Proof. We show that $[\Upsilon, B] = 0$ by direct computation. Since all of the maps are maps of cofree comodules, we only need to check that $\pi_1([\Upsilon, B]) = 0$ where π_1 denotes projection of the comodule onto cogenerators. We check this directly.

$$\begin{aligned}
\pi_1 \circ \Upsilon_{A_0, A_1} \circ B_{A_0, A_1}(\vec{\phi}|\vec{\psi}|\alpha) &= \sum_{\substack{I_1 I_2 = \{1, \dots, n\} \\ J_1 J_2 = \{1, \dots, m\} \\ \text{as ordered sets}}} \epsilon_{I_1, J_2} \cdot v_{|I_1|, |J_1|}(\vec{\phi}_{I_1}|\vec{\psi}_{J_1}|B_{|I_2|, |J_2|}(\vec{\phi}_{I_2}|\vec{\psi}_{J_2}|\alpha)) \\
&= \sum_{\substack{I_1 I_2 = \{1, \dots, n\} \\ J_1 J_2 = \{1, \dots, m\} \\ \text{as ordered sets}}} \epsilon_{I_1, J_2} \cdot \eta_{\mathbf{a}_1, \mathbf{a}_2} \cdot v_{|I_1|, |J_1|}(\vec{\phi}_{I_1}|\vec{\psi}_{J_1}|1 \otimes \lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_2})\mathbf{a}_2, a_0, \mathbf{a}_1) \\
&= \sum_{\substack{I_1 I_2 = \{1, \dots, n\} \\ J_1 J_2 = \{1, \dots, m\} \\ \text{as ordered sets}}} \epsilon_{I_1, J_2} \cdot \eta_{\mathbf{a}_1, \mathbf{a}_2} \cdot 1 \otimes \lambda(\vec{\phi}_{I_1})(\lambda(\vec{\psi})\lambda(\vec{\phi}_{I_2})\mathbf{a}_2, a_0, \mathbf{a}_1)
\end{aligned}$$

$$\begin{aligned}
& \pi_1 \circ B_{A_1, A_0} \circ \Upsilon_{A_0, A_1}(\vec{\phi} | \vec{\psi} | \alpha) \\
&= \sum_{\substack{I_1 I_2 = \{1, \dots, n\} \\ J_1 J_2 = \{1, \dots, m\} \\ \text{as ordered sets}}} \epsilon_{I_1, J_2} \cdot B_{|J_1|, |I_1|}(\vec{\psi}_{J_1} | \vec{\phi}_{I_1} | v_{|I_2|, |J_2|}(\vec{\phi}_{I_2} | \vec{\psi}_{J_2} | \alpha)) \\
&= \sum_{\substack{I_1 I_2 = \{1, \dots, n\} \\ J_1 J_2 = \{1, \dots, m\} \\ \text{as ordered sets}}} \epsilon_{I_1, J_2} \cdot B_{|J_1|, |I_1|}(\vec{\psi}_{J_1} | \vec{\phi}_{I_1} | \phi_{|I_1|+1}(\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_3})\mathbf{a}_3, a_0, \mathbf{a}_1) \otimes \lambda(\vec{\phi}_{I_2 \setminus |I_1|+1})\mathbf{a}_2 + \\
&\quad + a_0 \otimes \lambda(\vec{\phi}_{I_2 \setminus |I_1|+1})\mathbf{a}_1 \text{ if } J_2 = \emptyset) \\
&= \sum_{\substack{I_1 I_2 = \{1, \dots, n\} \\ J_1 J_2 = \{1, \dots, m\} \\ \text{as ordered sets}}} \epsilon_{I_1, J_2} \cdot \eta_{\mathbf{a}_2, \mathbf{a}_3} \cdot 1 \otimes \lambda(\vec{\phi}_{I_1})\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_3})\mathbf{a}_3 \otimes \phi_{|I_1|+1}(\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_4})\mathbf{a}_4, a_0, \mathbf{a}_1) \otimes \\
&\quad \otimes \lambda(\vec{\phi}_{I_2 \setminus |I_1|+1})\mathbf{a}_2 + \\
&\quad + \epsilon_{I_1, J_2} \cdot \eta_{\mathbf{a}_1, \mathbf{a}_2} \cdot 1 \otimes \lambda(\vec{\phi}_{I_1})\lambda(\vec{\psi})\lambda(\vec{\phi}_{I_3})\mathbf{a}_2 \otimes a_0 \otimes \lambda(\vec{\phi}_{I_2})\mathbf{a}_1
\end{aligned}$$

It's clear that $\pi_1 \circ \Upsilon_{A_0, A_1} \circ B_{A_0, A_1} = \pi_1 \circ B_{A_1, A_0} \circ \Upsilon_{A_0, A_1}$: The final expansion of $\pi_1 \circ \Upsilon_{A_0, A_1} \circ B_{A_0, A_1}$ is the sum of the two terms in the final expansion of $\pi_1 \circ B_{A_1, A_0} \circ \Upsilon_{A_0, A_1}$, which is the sum of terms in which one of the ϕ 's contains a_0 and the terms in which none of the ϕ 's contains a_0). \square

B.3. More notation

For the next two propositions, we will need some more notation. Set

A_0, A_1, A_2 fixed algebras

$$(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha) := (\phi_1 \dots \phi_n | \psi_1 \dots \psi_m | \theta_1 \dots \theta_r | \alpha)$$

$$= \begin{array}{c} \begin{array}{ccccc} & f_0 & & g_0 & & h_0 \\ & \curvearrowright & & \curvearrowright & & \curvearrowright \\ & \Downarrow \phi_1 & & \Downarrow \psi_1 & & \Downarrow \theta_1 \\ & \curvearrowright & & \curvearrowright & & \curvearrowright \\ A_0 & & A_1 & & A_2 & & A_0 \\ & \vdots & & \vdots & & \vdots & \\ & f_n & & g_m & & h_p & \\ & \curvearrowright & & \curvearrowright & & \curvearrowright & \\ & & \alpha & & & & \\ & \curvearrowright & & \curvearrowright & & \curvearrowright & \\ & & id & & & & \end{array} \end{array}$$

$$\in C(A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_0)(h_0 g_0 f_0)$$

$$\epsilon_{I_2, J_1, J_2, K_1} := (-1)^{\left(\sum_{r \in I_1} |\phi_r| + 1 \right) \left(\left(\sum_{s \in J_1} |\psi_s| + 1 \right) + \left(\sum_{t \in K_1} |\theta_t| + 1 \right) \right)}.$$

$$(-1)^{\left(\sum_{s \in J_2} |\psi_s| + 1 \right) \left(\sum_{t \in K_1} |\theta_t| + 1 \right)}$$

when I_1, J_1, J_2, K_1 , are ordered indexing sets

$$\Upsilon_{A_0 \bullet A_1, A_2} : C(A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_0) \rightarrow \hat{\tau}_2^* C(A_2 \rightarrow A_0 \rightarrow A_1 \rightarrow A_2)$$

$$(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha) \mapsto \Upsilon_{A_0, A_2}(\vec{\phi} \bullet \vec{\psi}|\vec{\theta}|\alpha) \quad \text{map of dg comodules}$$

$$\Upsilon_{A_0, A_1 \bullet A_2} : C(A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_0) \rightarrow \hat{\tau}_2^{*2} C(A_1 \rightarrow A_2 \rightarrow A_0 \rightarrow A_1)$$

$$(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha) \mapsto \Upsilon_{A_0, A_1}(\vec{\phi}|\vec{\psi} \bullet \vec{\theta}|\alpha) \quad \text{map of dg comodules}$$

B.4. More Propositions

Proposition B.4. *Let*

$$\mathcal{B}_{A_0, A_1, A_2} = \mathcal{B} : C(A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_0) \rightarrow \hat{\tau}_2^{*2} C(A_1 \rightarrow A_2 \rightarrow A_0 \rightarrow A_1)$$

be a map of comodules over $B(A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_0)$ determined by the following maps to cogenerators:

$$(B.9) \quad \begin{aligned} & \mathcal{B}_{A_0, A_1, A_2}^{f_0, g_0, h_0} : C(A_0 \rightarrow A_1 \rightarrow A_0)(h_0 g_0 f_0) \rightarrow \hat{\tau}_2^{*2} C(A_1 \rightarrow A_2 \rightarrow A_0 \rightarrow A_1)(f_0 h_0 g_0) \\ & \xrightarrow[\text{cogenerators}]{\text{project onto}} C_{-\bullet}(A_1, f_0 h_0 g_0 A_{1id}) \\ & \mathcal{B}_{n, m, p}(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha) = \sum_{\substack{I_1 I_2 = \{1, 2, \dots, n\} \\ \text{as ordered sets}}} \eta_{\mathbf{a}_1, \mathbf{a}_2} \cdot 1 \otimes \lambda(\vec{\phi}_{I_1})(\lambda(\vec{\theta})\lambda(\vec{\psi})\lambda(\vec{\phi}_{I_2})\mathbf{a}_2 \otimes a_0 \otimes \mathbf{a}_1) \end{aligned}$$

Then,

$$(B.10) \quad D(\mathcal{B}_{A_0, A_1, A_2}) = \Upsilon_{A_2 \bullet A_0, A_1} \circ \Upsilon_{A_0 \bullet A_1, A_2} - \Upsilon_{A_0, A_1 \bullet A_2}.$$

Proof. We will show that Equation B.10 holds by direct computation. Since all of the maps are maps of cofree comodules, we only need to check that π_1 (Equation B.10) holds where π_1 denotes projection of the comodule onto cogenerators. More explicitly, we

want to check that

(B.11)

$$\begin{aligned}
& \mathcal{B}_{n,m,p}(\tilde{\delta}(\vec{\phi})|\vec{\psi}|\vec{\theta}|\alpha) + \mathcal{B}_{n,m,p}(\vec{\phi}|\tilde{\delta}(\vec{\psi})|\vec{\theta}|\alpha) + \mathcal{B}_{n,m,p}(\vec{\phi}|\vec{\psi}|\tilde{\delta}(\vec{\theta})|\alpha) + \\
& \mathcal{B}_{n-1,m,p}(b'(\vec{\phi})|\vec{\psi}|\vec{\theta}|\alpha) + \mathcal{B}_{n,m-1,p}(\vec{\phi}|b'(\vec{\psi})|\vec{\theta}|\alpha) + \mathcal{B}_{n,m,p-1}(\vec{\phi}|\vec{\psi}|b'(\vec{\theta})|\alpha) + \\
& \mathcal{B}_{n,m,p}(\vec{\phi}|\vec{\psi}|\vec{\theta}|b(\alpha)) + b \circ \mathcal{B}_{n,m,p}(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha) + \\
& \sum_{\substack{I_1 I_2 = \{1, \dots, n\} \\ J_1 J_2 = \{1, \dots, m\} \\ \text{as ordered sets}}} \epsilon_{I_2, J_1, J_2, \{1, \dots, p-1\}} \cdot \mathcal{B}_{|I_1|, |J_1|, p-1}(\vec{\phi}_{I_1}|\vec{\psi}_{J_1}|\vec{\theta}_{\{1, \dots, p-1\}}|\theta_p\{\vec{\psi}_{J_2}\}\{\vec{\phi}_{I_2}\} \cdot \alpha) + \\
& \sum_{\substack{I_1 I_2 = \{1, \dots, n\} \\ \text{as ordered sets}}} \epsilon_{I_2, \{1, \dots, m-1\}, \{m\}, \{1, \dots, p\}} \cdot \mathcal{B}_{|I_1|, m-1, p}(\vec{\phi}_{I_1}|\vec{\psi}_{\{1, \dots, m-1\}}|\vec{\theta}|\psi_m\{\vec{\phi}_{I_2}\} \cdot \alpha) + \\
& \epsilon_{\{n\}, \{1, \dots, m\}, \{\}, \{1, \dots, p\}} \cdot \mathcal{B}_{n-1, m, p}(\vec{\phi}_{\{1, \dots, n-1\}}|\vec{\psi}_m|\vec{\theta}|\phi_n \cdot \alpha) + \\
& \sum_{\substack{J_1 J_2 = \{1, \dots, m\} \\ K_1 K_2 = \{1, \dots, p\} \\ \text{as ordered sets}}} \epsilon_{\{2, \dots, n\}, J_1, J_2, K_1} \cdot \phi_1\{\vec{\theta}_{K_1}\}\{\vec{\psi}_{J_1}\} \cdot \mathcal{B}_{n-1, |J_2|, |K_2|}(\vec{\phi}_{\{2, \dots, n\}}|\vec{\psi}_{J_2}|\vec{\theta}_{K_2}|\alpha) + \\
& \sum_{\substack{J_1 J_2 = \{1, \dots, m\} \\ \text{as ordered sets}}} \epsilon_{\{1, \dots, n\}, J_1, J_2, \{1\}} \cdot \theta_1\{\vec{\psi}_{J_1}\} \cdot \mathcal{B}_{n, |J_2|, p-1}(\vec{\phi}|\vec{\psi}_{J_2}|\vec{\theta}_{\{2, \dots, p\}}|\alpha) + \\
& \epsilon_{\{1, \dots, n\}, \{1\}, \{2, \dots, m\}, \{\}} \cdot \psi_1 \cdot \mathcal{B}_{n, m-1, p}(\vec{\phi}|\vec{\psi}_{\{2, \dots, m\}}|\vec{\theta}|\alpha) + \\
& v_{n, p \leq * \leq m+p}(\vec{\phi}|\vec{\psi} \bullet \vec{\theta}|\alpha) + \\
& \sum_{\substack{I_1 I_2 = \{1, \dots, n\} \\ J_1 J_2 = \{1, \dots, m\} \\ K_1 K_2 = \{1, \dots, p\} \\ \text{as ordered sets}}} \epsilon_{I_2, J_1, J_2, K_1} \cdot \\
& v_{|I_1| \leq * \leq |I_1| + |K_1|, |J_1|}(\vec{\theta}_{K_1} \bullet \vec{\phi}_{I_1}, \vec{\psi}_{J_1}, v_{|J_2| \leq * \leq |I_2| + |J_2|, |K_2|}(\vec{\phi}_{I_2} \bullet \vec{\psi}_{J_2}|\vec{\theta}_{K_2}|\alpha)) \\
& = 0.
\end{aligned}$$

In Equation B.11 above, we call the terms in rows 1-3 the “standard terms” in the computation of $D(\mathcal{B}_{A_0,A_1,A_2})$, and the terms in rows 4-9 the “extra terms” in the computation of $D(\mathcal{B}_{A_0,A_1,A_2})$. The terms in rows 10-11 are π_1 of the righthand side of Equation B.10; we will call these the “10th- and 11th-row terms”.

We compute the sum of the standard terms. In Table B.4, the leftmost column lists the expressions that don’t cancel in the sum of the standard terms, the middle column gives the standard term from which the expression comes, and the rightmost column gives the term that cancels the expression. Table B.5 lists the remaining ninth row terms that aren’t already listed in Table B.4. In Table B.5, the left column lists the remaining expressions that don’t cancel in the ninth row, and the right column gives the extra term that cancels the expression.

All of the terms in the tables describing the expansion of Equation B.11 cancel, so we’re done. □

Expression (Expansion)	Comes from Standard Term in Equation B.11	Cancelling Term in Equation B.11
$1 \otimes \lambda(\vec{\phi}_{I_1})[\lambda(\vec{\theta}_{\{1, \dots, p-1\}}) \lambda(\vec{\psi}_{J_1}) \lambda(\vec{\phi}_{I_2}) \mathbf{a}_2 \otimes \theta_p(\lambda(\vec{\psi}_{J_2}) \lambda(\vec{\phi}_{I_3}) \mathbf{a}_3) \cdot a_0 \otimes \mathbf{a}_1]$	$b \circ \mathcal{B}_{n,m,p}(\vec{\phi} \vec{\psi} \vec{\theta} \alpha)$	$\mathcal{B}_{ I_1 , J_1 , p-1}(\vec{\phi}_{I_1} \vec{\psi}_{J_1} \vec{\theta}_{\{1, \dots, p-1\}} \theta_p\{\vec{\psi}_{J_2}\}\{\vec{\phi}_{I_3}\} \cdot \alpha)$
$1 \otimes \lambda(\vec{\phi}_{I_1})[\lambda(\theta_{\{1, \dots, m-1\}}) \lambda(\vec{\phi}_{I_2}) \mathbf{a}_2 \otimes \psi_m(\lambda(\vec{\phi}_{I_3}) \mathbf{a}_3) \cdot a_0 \otimes \mathbf{a}_1]$	$b \circ \mathcal{B}_{n,m,p}(\vec{\phi} \vec{\psi} \vec{\theta} \alpha)$	$\mathcal{B}_{ I_1 , m-1, p}(\vec{\phi}_{I_1} \vec{\psi}_{\{1, \dots, m-1\}} \vec{\theta} \psi_m\{\vec{\phi}_{I_3}\} \cdot \alpha)$
$1 \otimes \lambda(\vec{\phi}_{I_1})[\lambda(\vec{\theta}) \lambda(\vec{\psi}_{\{1, \dots, n-1\}}) \mathbf{a}_2 \otimes \psi_n(\mathbf{a}_3) \cdot a_0 \otimes \mathbf{a}_1]$	$b \circ \mathcal{B}_{n,m,p}(\vec{\phi} \vec{\psi} \vec{\theta} \alpha)$	$\mathcal{B}_{n-1, m, p}(\vec{\phi}_{\{1, \dots, n-1\}} \vec{\psi} \vec{\theta} \phi_n \cdot \alpha)$
$\phi_1(\lambda(\vec{\theta}_{K_1}) \lambda(\vec{\psi}_{J_1}) \lambda(\vec{\phi}_{I_2}) \mathbf{a}_2) \otimes \lambda(\vec{\phi}_{I_1 \setminus 1})[\lambda(\vec{\theta}_{K_2}) \lambda(\vec{\psi}_{J_3}) \lambda(\vec{\phi}_{I_3}) \mathbf{a}_3 \otimes a_0 \otimes \mathbf{a}_1]$	$b \circ \mathcal{B}_{n,m,p}(\vec{\phi} \vec{\psi} \vec{\theta} \alpha)$	$\phi_1\{\vec{\theta}_{K_1}\}\{\psi_{J_1}\} \cdot \mathcal{B}_{n-1, J_2 , K_2 }(\vec{\phi}_{\{2, \dots, m\}} \vec{\psi}_{J_2} \vec{\theta}_{K_2} \alpha)$
$f_0 \theta_1(\lambda(\vec{\psi}_{I_1}) \lambda(\vec{\phi}_{I_2}) \mathbf{a}_2) \otimes \lambda(\vec{\phi}_{I_1})[\lambda(\vec{\theta}_{\{2, \dots, p\}}) \lambda(\vec{\psi}_{J_2}) \lambda(\vec{\phi}_{I_3}) \mathbf{a}_3 \otimes a_0 \otimes \mathbf{a}_1]$	$b \circ \mathcal{B}_{n,m,p}(\vec{\phi} \vec{\psi} \vec{\theta} \alpha)$	$\theta_1\{\psi_{J_1}\} \cdot \mathcal{B}_{n, J_2 , p-1}(\vec{\phi} \vec{\psi}_{J_2} \vec{\theta}_{\{2, \dots, p\}} \alpha)$
$f_0 h_0 \psi_1(\lambda(\vec{\phi}_{I_2}) \mathbf{a}_2) \otimes \lambda(\vec{\phi}_{I_1})[\lambda(\vec{\theta}) \lambda(\vec{\psi}_{\{2, \dots, m\}}) \lambda(\vec{\phi}_{I_3}) \mathbf{a}_3 \otimes a_0 \otimes \mathbf{a}_1]$	$b \circ \mathcal{B}_{n,m,p}(\vec{\phi} \vec{\psi} \vec{\theta} \alpha)$	$\psi_1 \cdot \mathcal{B}_{n, m-1, p}(\vec{\phi} \vec{\psi}_{\{2, \dots, m\}} \vec{\theta} \alpha)$
$f_0 h_0 g_0 \phi_{i_1}(\lambda(\vec{\theta}_{K_2}) \lambda(\vec{\psi}_{J_2}) \lambda(\vec{\phi}_{I_3}) \mathbf{a}_3, a_0, \mathbf{a}_1) \otimes \lambda(\vec{\phi}_{I_1}) \lambda(\vec{\theta}_{K_1}) \lambda(\vec{\psi}_{J_1}) \lambda(\vec{\phi}_{I_2 \setminus i_1}) \mathbf{a}_2$	$b \circ \mathcal{B}_{n,m,p}(\vec{\phi} \vec{\psi} \vec{\theta} \alpha)$	11 th row
$f_0 h_0 g_0 f_{i_1} a_0 \otimes \lambda(\vec{\phi}_{I_1}) \lambda(\vec{\theta}) \lambda(\vec{\psi}_{J_1}) \lambda(\vec{\phi}_{I_2}) \mathbf{a}_1$	$b \circ \mathcal{B}_{n,m,p}(\vec{\phi} \vec{\psi} \vec{\theta} \alpha)$	11 th row
$\phi_1(\lambda(\vec{\phi}_{I_1}) \lambda(\vec{\theta}) \lambda(\vec{\psi}_{J_1}) \lambda(\vec{\phi}_{I_2}) \mathbf{a}_3, a_0, \mathbf{a}_1) \otimes \lambda(\vec{\phi}_{I_1 \setminus 1}) \mathbf{a}_2$	$b \circ \mathcal{B}_{n,m,p}(\vec{\phi} \vec{\psi} \vec{\theta} \alpha)$	10 th row

Table B.4. Expansion of terms in Equation B.11: “standard terms” and the terms that cancel them

Expression (expansion) from 11 th -Row Term in Equation B.11	Cancels with Extra Term in Equation B.11
$\phi_1(\lambda(\vec{\theta}_{K_1})\lambda(\vec{\psi}_{I_1})\lambda(\vec{\phi}_{I_2})[\lambda(\vec{\theta}_{K_3})\lambda(\vec{\psi}_{J_4})\lambda(\vec{\phi}_{I_5})\mathbf{a}_3, a_0, \mathbf{a}_1])\otimes$ $\otimes \lambda(\vec{\phi}_{I_1 \setminus 1})\lambda(\vec{\theta}_{K_2})\lambda(\vec{\psi}_{J_3})\lambda(\vec{\phi}_{I_4})\mathbf{a}_2$	$\phi_1\{\vec{\theta}_{K_1}\}\{\vec{\psi}_{J_1}\} \cdot$ $\mathcal{B}_{n-1, J_2 , K_2 }(\vec{\phi}_{\{2, \dots, m\}} \vec{\psi}_{J_2} \vec{\theta}_{K_2} \alpha)$
$f_0\theta_1(\lambda(\vec{\psi}_{I_1})\lambda(\vec{\phi}_{I_2})[\lambda(\vec{\theta}_{K_2})\lambda(\vec{\psi}_{J_3})\lambda(\vec{\phi}_{I_4})\mathbf{a}_3, a_0, \mathbf{a}_1])\otimes$ $\otimes \lambda(\vec{\phi}_{I_1})\lambda(\vec{\theta}_{K_1 \setminus 1})\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_3})\mathbf{a}_2$	$\theta_1\{\vec{\psi}_{J_1}\} \cdot$ $\mathcal{B}_{n, J_2 , p-1}(\vec{\phi} \vec{\psi}_{J_2} \vec{\theta}_{\{2, \dots, p\}} \alpha)$
$f_0h_0\psi_1(\lambda(\vec{\phi}_{I_2})[\lambda(\vec{\theta}_{K_2})\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_4})\mathbf{a}_3, a_0, \mathbf{a}_1])\otimes \lambda(\vec{\phi}_{I_1})\lambda(\vec{\theta}_{K_1})\lambda(\vec{\psi}_{J_1 \setminus 1})\lambda(\vec{\phi}_{I_3})\mathbf{a}_2$	$\psi_1 \cdot \mathcal{B}_{n, m-1, p}(\vec{\phi} \vec{\psi}_{\{2, \dots, m\}} \vec{\theta} \alpha)$

Table B.5. Expansion of terms in Equation B.11: remaining “11th row terms” and the “extra terms” that cancel them

Proposition B.5. *Let Υ and \mathcal{B} be as defined in the previous propositions. Then, $[\Upsilon, \mathcal{B}] := \Upsilon_{A_1 \bullet A_2, A_0} \mathcal{B}_{A_0, A_1, A_2} - \mathcal{B}_{A_2, A_0, A_1} \Upsilon_{A_0 \bullet A_1, A_2} = 0$. (Note that $[\Upsilon, \mathcal{B}]$ is a map from $C(A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_0)$ to itself.)*

Proof. We show the proposition by direct computation. Since all of the maps are maps of cofree comodules, we only need to check that $\pi_1([\Upsilon, \mathcal{B}]) = 0$ where π_1 denotes projection of the comodule onto cogenerators. We check this directly.

$$\begin{aligned}
& \pi_1 \circ \Upsilon_{A_1 \bullet A_2, A_0} \mathcal{B}_{A_0, A_1, A_2}(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha) \\
&= \sum_{\substack{I_1 I_2 = \{1, \dots, n\} \\ J_1 J_2 = \{1, \dots, m\} \\ K_1 K_2 = \{1, \dots, p\} \\ \text{as ordered sets}}} \epsilon_{I_2, J_1, J_2, K_1} \cdot v_{|K_1| \leq * \leq |K_1| + |J_1|, |I_1|}(\vec{\psi}_{J_1} \bullet \vec{\theta}_{K_1} |\vec{\phi}_{I_1}| \mathcal{B}_{|I_2|, |J_2|, |K_2|}(\vec{\phi}_{I_2} |\vec{\psi}_{J_2}| \vec{\theta}_{K_2} |\alpha)) \\
&= \sum_{\substack{I_1 I_2 = \{1, \dots, n\} \\ J_1 J_2 = \{1, \dots, m\} \\ K_1 K_2 = \{1, \dots, p\} \\ \text{as ordered sets}}} \epsilon_{I_2, J_1, J_2, K_1} \cdot \eta_{\mathbf{a}_1, \mathbf{a}_2} \cdot v_{|K_1| \leq * \leq |K_1| + |J_1|, |I_1|}(\vec{\psi}_{J_1} \bullet \vec{\theta}_{K_1} |\vec{\phi}_{I_1}| 1 \otimes \lambda(\vec{\phi}_{I_2})[\lambda(\vec{\theta}_{K_2})\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_3})\mathbf{a}_2, a_0, \mathbf{a}_1]) \\
&= \sum_{\substack{I_1 I_2 = \{1, \dots, n\} \\ J_1 J_2 = \{1, \dots, m\} \\ K_1 K_2 = \{1, \dots, p\} \\ \text{as ordered sets}}} \epsilon_{I_2, J_1, J_2, K_1} \cdot \eta_{\mathbf{a}_1, \mathbf{a}_2} \cdot 1 \otimes \lambda(\vec{\theta}_{K_1})\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_1})[\lambda(\vec{\theta}_{K_2})\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_2})\mathbf{a}_2, a_0, \mathbf{a}_1]
\end{aligned}$$

$$\begin{aligned}
& \pi_1 \circ \mathcal{B}_{A_2, A_0, A_1} \Upsilon_{A_0 \bullet A_1, A_2}(\vec{\phi} | \vec{\psi} | \vec{\theta} | \alpha) \\
&= \sum_{\substack{I_1 I_2 = \{1, \dots, n\} \\ J_1 J_2 = \{1, \dots, m\} \\ K_1 K_2 = \{1, \dots, p\} \\ \text{as ordered sets}}} \epsilon_{I_2, J_1, J_2, K_1} \cdot B_{|K_1|, |I_1|, |J_1|}(\vec{\theta}_{K_1} | \vec{\phi}_{I_1} | \vec{\psi}_{J_1} | v_{|J_2| \leq * \leq |I_2| + |J_2|, |K_2|}(\vec{\phi}_{I_2} \bullet \vec{\psi}_{J_2} | \vec{\theta}_{K_2} | \alpha)) \\
&= \sum_{\substack{I_1 I_2 = \{1, \dots, n\} \\ J_1 J_2 = \{1, \dots, m\} \\ K_1 K_2 = \{1, \dots, p\} \\ \text{as ordered sets}}} \epsilon_{I_2, J_1, J_2, K_1} \cdot \eta_{\mathfrak{a}_1, \mathfrak{a}_2} \cdot 1 \otimes \lambda(\vec{\theta}_{K_1}) \lambda(\vec{\psi}_{J_1}) \lambda(\vec{\phi}_{I_1}) [\lambda(\vec{\theta}_{K_2}) \lambda(\vec{\psi}_{J_2}) \lambda(\vec{\phi}_{I_2}) \mathfrak{a}_2, a_0, \mathfrak{a}_1]
\end{aligned}$$

It's clear that $\pi_1([\Upsilon, \mathcal{B}]) = 0$.

□

APPENDIX C

Background on Hochschild chains and cochains

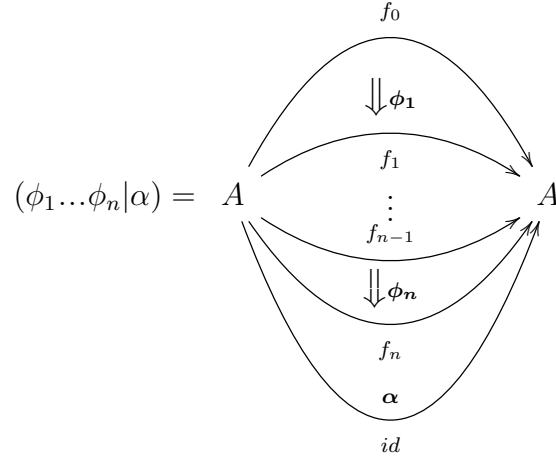
In this appendix, we give some known constructions on Hochschild chains and cochains for the reader's convenience. Let k be a field of characteristic zero, A a flat unital k -algebra, and M be an A - A -bimodule. Then, we can take $(C_\bullet(A, M), b)$, the (reduced or standard) Hochschild chain complex of A with coefficients in M (see Reference [6], Equation 2.1). When $M = B$ is also an algebra over k with left and right module structure given by two maps of algebras $f : A \rightarrow B$ and $g : A \rightarrow B$, respectively, we may write ${}_f B_g$ to clarify the module structure.

Let k, A, M be as above. We can also take $(C^\bullet(A, M), \delta)$, the (reduced) Hochschild cochain complex of A with coefficients in M (see Reference [6], Equations 2.12-13, 2.19-21). When $M = B$ is an algebra, $(C^\bullet(A, B), \delta, \cup)$ is a dga where the cup product \cup is given in Reference [6], Equation 2.14.

Let $f, g, h : A \rightarrow A$ be maps of algebras. We have a contraction operation of Hochschild cochains and chains, which is a map of complexes:

$$\begin{aligned} \iota : C^p(A, {}_f A_g) \otimes C_{-q}(A, {}_g A_h) &\longrightarrow C_{-(q-p)}(A, {}_f A_h) \\ \phi \otimes a_0 \otimes \cdots \otimes a_q &\mapsto \iota(\phi, a_0 \otimes \cdots \otimes a_q) := \phi \cdot (a_0 \otimes \cdots \otimes a_q) := \\ &:= (-1)^{p(q+1)} \phi(a_{q-p+1}, \dots, a_q) \cdot a_0 \otimes a_1 \otimes \cdots \otimes a_{q-p}. \end{aligned}$$

Finally, we have a “Lie derivative like” operation of Hochschild cochains and chains. Fix an algebra A and let $(\phi_1 \dots \phi_n | \alpha) \in C(A \rightarrow A)(f_0)$ (see Equation ??) be the following element



We have a map of complexes

$$C(A \rightarrow A)(f_0) \rightarrow C_{-\bullet}(A,_{f_0} A)$$

$$(\phi_1 \dots \phi_n | a_1 \otimes \dots \otimes a_p) \mapsto \lambda(\phi_1 \dots \phi_n) \cdot (a_1 \otimes \dots \otimes a_p)$$

$$:= \sum_{0 \leq i_1 \leq \dots \leq i_{2n} \leq p} (-1)^{\sum_{j \text{ odd}} i_j (\lfloor \phi_{i_{\frac{j+1}{2}}} \rfloor + 1)}.$$

$$\cdot f_0 a_1 \otimes \dots \otimes f_0 a_{i_1} \otimes \phi_1(a_{i_1+1}, \dots, a_{i_2}) \otimes$$

$$\otimes f_1 a_{i_2+1} \otimes \dots \otimes f_1 a_{i_3} \otimes \phi_2(a_{i_3+1}, \dots, a_{i_4}) \otimes$$

$$\otimes \dots \otimes \phi_n(a_{i_{2n-1}+1}, \dots, a_{i_{2n}}) \otimes f_n a_{i_{2n}+1} \otimes \dots \otimes f_n a_p.$$