Title

Subtitle

Rebecca Wei

Northwestern University

Date/Event

Background

Theorem

(Hochschild-Kostant-Rosenberg, '62) Let A be a regular, commutative algebra over a field k of characteristic 0. Then,

$$(C_\bullet(A,A), b) \xrightarrow{\sim} \Omega^\bullet_{A/k}$$
$$(C^\bullet(A,A), \delta) \xrightarrow{\sim} \wedge^\bullet T_A = \wedge^\bullet(Der_k(A,A)).$$

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Theorem

(Kontsevich, '97) Let $A = C^\infty(M)$ for M a smooth real manifold. Then, there is an L_∞ map

$$(C^{\bullet+1}(A,A), \delta, [,]_{Ger}) \xrightarrow{\sim} (\wedge^{\bullet+1} T_A, d = 0, [,]_{SN}).$$

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Theorem

(Tamarkin, '98) Dependent on the choice of a Drinfeld associator, there is a Ger_∞ map

$$(C^{\bullet+1}(A,A), \delta, [,]_{Ger}, \cup, \dots) \xrightarrow{\sim} (\wedge^\bullet T_A, d=0, [,]_{SN}, \wedge).$$

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Theorem

(Dolgushev-Tamarkin-Tsygan, '08) There is a $Calc_\infty$ map

$$(C^\bullet(A,A), C_{-\bullet}(A,A)) \xrightarrow{\sim} (\wedge^\bullet T_A, \Omega^\bullet_{A/k}).$$

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Can we use this calculus structure to create a cyclic object?

A cyclic object in dg cocategories

Fix an algebra, A.
Define a dg category, $Hoch(A)$:

A cyclic object in dg cocategories

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Objects: algebra maps $f : A \to A$

Morphisms: $Hoch(A)(f, g) = (C^\bullet(A, {}_fA_g), {}_f\delta_g)$

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$$
\begin{aligned}
{}_f\delta_g(\phi)(a_1 \otimes \ldots \otimes a_n) =& \epsilon_\phi \Big(f(a_1) \cdot \phi(a_2, \ldots, a_n) + \\
&+ \sum_{1 \leq i \leq n-1} (-1)^i \phi(a_1, \ldots, a_i a_{i+1}, \ldots, a_n) + \\
&+ (-1)^n \phi(a_1, \ldots, a_{n-1}) \cdot g(a_n) \Big) \\
\epsilon_\phi =& (-1)^{|\phi|+1}
\end{aligned}
$$

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$$
\begin{tikzcd}
A \arrow[r, bend left=70, "f"] \arrow[r, "g"'] \arrow[r, bend right=70, "h"'] \arrow[r, phantom, shift left=4, "\Downarrow\phi_1"] \arrow[r, phantom, shift right=4, "\Downarrow\phi_2"] & A \arrow[r, squiggly, "compose"] & A \arrow[r, bend left=70, "f"] \arrow[r, bend right=70, "h"'] \arrow[r, phantom, "\phi_1\cup\phi_2\ \Downarrow"] & A
\end{tikzcd}
$$

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$$\begin{tikzcd}[column sep=huge]
A \arrow[r, bend left=80, "f_0"] \arrow[r, bend left=40, "f_1"] \arrow[r, "f_2"] \arrow[r, bend right=50, "f_{n-1}"'] \arrow[r, bend right=80, "f_n"'] \arrow[r, phantom, bend left=60, "\Downarrow\phi_1"] \arrow[r, phantom, bend left=20, "\Downarrow\phi_2"] \arrow[r, phantom, bend right=25, "\vdots"] \arrow[r, phantom, bend right=68, "\Downarrow\phi_n"] & A
\end{tikzcd}$$

A morphism from f_0 to f_n in Bar(Hoch(A))

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$$\begin{tikzcd}
A \arrow[r, bend left=60, "f_0"] \arrow[r, bend left=30, "f_1"] \arrow[r, "f_2"] \arrow[r, bend right=30, "f_{n-1}"'] \arrow[r, bend right=60, "f_n"'] \arrow[r, phantom, shift left=6ex, "\Downarrow\phi_1"] \arrow[r, phantom, shift left=3ex, "\Downarrow\phi_2"] \arrow[r, phantom, shift right=3ex, "\vdots"] \arrow[r, phantom, shift right=7ex, "\Downarrow\phi_n"] & A \arrow[r, bend left, "g_0"] \arrow[r, bend right, "g_1"'] \arrow[r, phantom, "\Downarrow\psi"] & A \arrow[r, mapsto, "\bullet"] & A \arrow[r, bend left, "g_0 f_0"] \arrow[r, bend right, "g_1 f_n"'] \arrow[r, phantom, "\psi\{\phi_1,\dots,\phi_n\} \Downarrow"] & A
\end{tikzcd}$$

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A \arrow[r, bend left=70, "f_0"] \arrow[r, bend left=35, "f_1"] \arrow[r, "f_2"] \arrow[r, bend right=35, "f_{n-1}"'] \arrow[r, bend right=70, "f_n"'] \arrow[r, phantom, bend left=52, "\Downarrow\phi_1"] \arrow[r, phantom, bend left=17, "\Downarrow\phi_2"] \arrow[r, phantom, bend right=17, "\vdots"] \arrow[r, phantom, bend right=52, "\Downarrow\phi_n"] & A \arrow[r, bend left, "g_0"] \arrow[r, bend right, "g_1"'] \arrow[r, phantom, "\Downarrow\psi"] & A \arrow[r, mapsto, "\bullet"] & A \arrow[r, bend left, "g_0 f_0"] \arrow[r, bend right, "g_1 f_n"'] \arrow[r, phantom, "\psi\{\phi_1,\ldots,\phi_n\} \atop \Downarrow"] & A
\end{tikzcd}$$

$$\psi\{\phi_1,\ldots,\phi_n\}(a_1,\ldots,a_q) = \sum_{1\le i_1 \le \ldots \le i_n \le q} \pm\psi(a_1,\ldots,a_{i_1},\phi_1(a_{i_1+1},\ldots),\ldots,\phi_n(a_{i_n+1},\ldots),\ldots)$$

A cyclic object in dg cocategories

Fix an algebra, A.
Define a dg category, $Hoch(A)$:

In this context, braces, $\bullet$, give multilinear maps:

$$\begin{tikzcd}
Bar(Hoch(A)) \otimes Bar(Hoch(A)) \arrow[r, "\bullet"] \arrow[dr, dotted, "\bullet"'] & Hoch(A) \\
& Bar(Hoch(A)) \arrow[u, two heads]
\end{tikzcd}$$

Then, (Bar(Hoch(A)), $\bullet$) is an algebra in DGCocats.

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Then, (Bar(Hoch(A)), $\bullet$) is an algebra in DGCocats.
But we have more...

A cyclic object in dg cocategories

A cyclic object in dg cocategories

Fix algebras, $A_0, A_1, \ldots, A_n$.
We will define a dg cocategory $B(A_0 \to A_1 \to \ldots \to A_n \to A_0)$
where $B(A_0 \to A_0) := Bar(Hoch(A_0))$ for n=0.

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We will define a dg cocategory $B(A_0 \to A_1 \to \ldots \to A_n \to A_0)$
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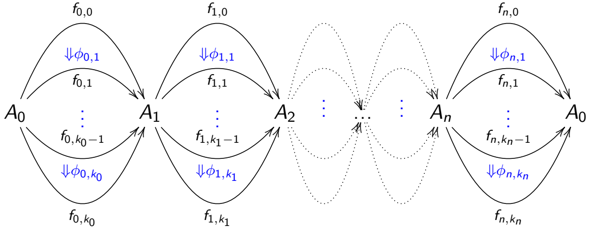
Objects: $\{A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \ldots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_0\}$
A morphism from $(A_0 \xrightarrow{f_{0,0}} \ldots \xrightarrow{f_{n,0}} A_0)$ to $(A_0 \xrightarrow{f_{0,k_0}} \ldots \xrightarrow{f_{n,k_n}} A_0)$:



$\phi_{i,j} \in C^\bullet(A_{i,f_{i,j-1}}\ A_{i+1\,f_{i,j}})$

A cyclic object in dg cocategories

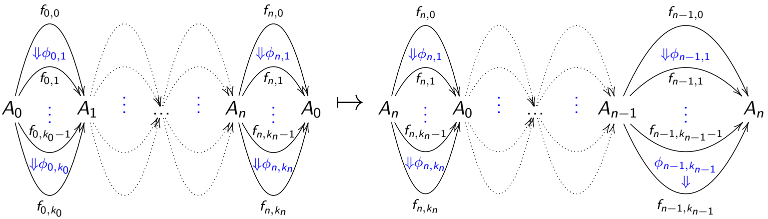
Example

We have a dg functor

$$\hat{\tau}_n : B(A_0 \to \ldots \to A_n \to A_0) \to B(A_n \to A_0 \to \ldots \to A_n)$$

$$Obj : \quad (f_0, \ldots, f_n) \mapsto (f_n, f_0, \ldots, f_{n-1})$$

Mor :



A cyclic object in dg cocategories

Example

For $n \geq 1, 0 \leq j < n$, we have a dg functor

$$\hat{\delta}_{j,n} : B(A_0 \to \ldots \to A_n \to A_0) \to B(A_0 \to \ldots \to A_j \to A_{\substack{j+2 \\ (mod\ n+1)}} \to \ldots \to A_0)$$

$$Obj : \qquad (f_0, \ldots, f_n) \mapsto (f_0, \ldots, f_{j+1} f_j, \ldots, f_n)$$

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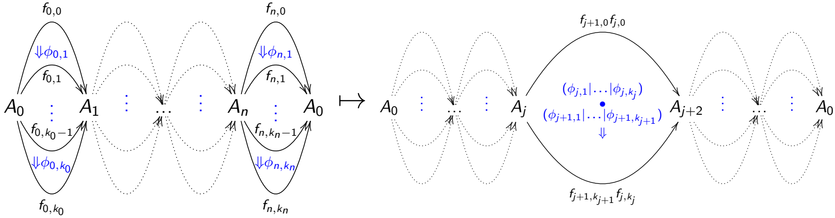


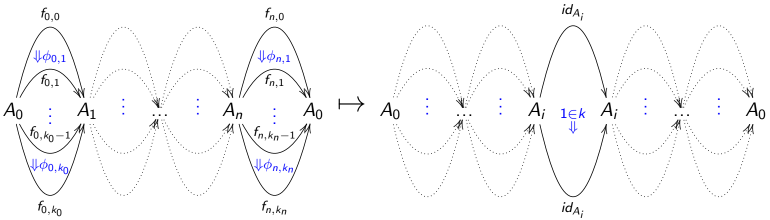
A cyclic object in dg cocategories

Example

For $n \geq 0, 0 \leq i \leq n$, we have a dg functor

$$\hat{\sigma}_{i,n} : B(A_0 \to \ldots \to A_n \to A_0) \to B(A_0 \to \ldots \to A_i \to A_i \to \ldots \to A_0)$$
$$Obj : \quad (f_0, \ldots, f_n) \mapsto (f_0, \ldots, f_{i-1}, id_{A_i}, f_i, \ldots, f_n)$$

Mor :



A **sheafy**-cyclic object in DGCocat

Definition

Let χ be the category with objects $\{A_0 \to A_1 \to \ldots \to A_n \to A_0\}$ and morphisms compositions of

$$\begin{aligned} \textit{rotations } \tau_n &: \mathcal{A} \mapsto (A_n \to A_0 \to \ldots \to A_n) \\ \textit{coboundaries } \delta_{j,n} &: \mathcal{A} \mapsto (A_0 \to \ldots \to A_j \to A_{j+2\,(mod\ n+1)} \to \ldots \to A_0) \\ \textit{codegeneracies } \sigma_{i,n} &: \mathcal{A} \mapsto (A_0 \to \ldots \to A_i \to A_i \to \ldots \to A_0) \end{aligned}$$

where $\mathcal{A} := (A_0 \to \ldots \to A_n \to A_0)$, subject to the cyclic relations.

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where $\mathcal{A} := (A_0 \to \ldots \to A_n \to A_0)$, subject to the cyclic relations.

Proposition

We have a functor $\chi \to DGCocat$

$$\begin{aligned} \textit{Objects}: \; (A_0 \to \ldots \to A_n \to A_0) &\mapsto B(A_0 \to \ldots \to A_n \to A_0) \\ \textit{Generating morphisms}: \; \lambda &\mapsto \hat{\lambda} \end{aligned}$$

Motivating Question

Each dg cocategory $B(A_0 \to \dots \to A_n \to A_0)$ has a dg comodule $C(A_0 \to \dots \to A_n \to A_0)$ (which is a categorified version of the bar complex of Hochschild chains as a module over Hochschild cochains via contraction).

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Motivating Question: Can we extend the sheafy-cyclic structure to the comodules? In other words, can we give a functor

$$\chi \to \mathcal{D} := \{(dg\ cocat,\ dg\ comod)\}?$$

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$$\chi_\infty \to \mathcal{D}_\infty \quad \textit{dg categories}$$

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Rest of this talk: Describe our A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$.

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- Define dg categories χ_∞ and $\mathcal{D}_\infty$
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- Define the A_∞-functor $\mathcal{F}$

Dg categories χ_∞ and $\mathcal{D}_\infty$

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χ_∞:

Objects: same objects as $\chi = \{A_0 \to \ldots \to A_n \to A_0\}$

$\chi_\infty^\bullet(X, Y) := \{\sum a_i v_i : a_i \in k, v_i \in \chi(X, Y)\}$

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$\mathcal{D}_\infty$:

Objects: same objects as $\mathcal{D} = \{(\underset{B}{\text{dg cocategory}}, \underset{C}{\text{dg comodule}})\}$

$$\mathcal{D}^p_\infty\big((B_1, C_1), (B_0, C_0)\big) := \left\{ \begin{matrix} F : B_1 \to B_0 \textit{ dg functor}, \\ F_! : C_1 \to F^* C_0 \textit{ degree-p linear map} \end{matrix} \right\}$$

$$d_{D_\infty}(F, F_!) = (F, [d, F_!] = d_{F^* C_0} \circ F_! \pm F_! \circ d_{C_1})$$

Dg categories χ_∞ and $\mathcal{D}_\infty$

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$F^* C_0$ is the categorified version of co-extension of scalars:

$$F^* C_0 = ker(B_1 \otimes C_0 \rightrightarrows B_1 \otimes B_0 \otimes C_0)$$

Dg comodules over dg cocategories

Definition

A **dg comodule** C over a dg cocategory B consists of the following data:

- for each object $f \in B$, a complex $C^\bullet(f)$, and
- maps of complexes

$$\Delta_C(f) : C^\bullet(f) \to \prod_{g \in Obj(B)} B^\bullet(f,g) \otimes C^\bullet(g).$$

such that the following diagrams for coassociativity and counitality commute:

$$\begin{tikzcd}
C^\bullet(f) \arrow[r, "\Delta_C"] \arrow[d, "\Delta_C"'] & \prod_{g \in Obj(B)} B^\bullet(f,g) \otimes C^\bullet(g) \arrow[d, "id_B \otimes \Delta_C"] \\
\prod_{g \in Obj(B)} B^\bullet(f,g) \otimes C^\bullet(g) \arrow[r, "\Delta_B \otimes id_C"] & \prod_{\substack{g,g' \in \\ Obj(B)}} B^\bullet(f,g) \otimes B^\bullet(g,g') \otimes C^\bullet(g')
\end{tikzcd}
\qquad
\begin{tikzcd}
C^\bullet(f) \arrow[r, "\Delta_C"] \arrow[dr, "id"'] & \prod_{g \in Obj(B)} B^\bullet(f,g) \otimes C^\bullet(g) \arrow[d, "\epsilon_B \otimes id_C"] \\
& C^\bullet(f)
\end{tikzcd}$$

Dg comodules over dg cocategories

Fix algebras $A_0, \ldots, A_n$.
Define a dg comodule over $B(A_0 \to \ldots \to A_n \to A_0)$:

$$C(A_0 \to \ldots \to A_n \to A_0)^\bullet(f_{0,0}, \ldots, f_{n,0}) :=$$

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$$:= \left\{ \begin{tikzcd} A_0 \arrow[r, bend left=60, "f_{0,0}"] \arrow[r, bend left, "f_{0,1}"] \arrow[r, phantom, bend left=45, "\textcolor{blue}{\Downarrow\phi_{0,1}}"] \arrow[r, phantom, "\textcolor{blue}{\vdots}"] \arrow[r, bend right, "f_{0,k_0}"'] \arrow[rrrr, bend right=50, "id_{A_0}"'] \arrow[rrrr, phantom, bend right=35, "\textcolor{blue}{\alpha}"] & A_1 \arrow[r, dotted, bend left=60] \arrow[r, dotted, bend left] \arrow[r, phantom, "\textcolor{blue}{\vdots}"] \arrow[r, dotted, bend right] & \cdots \arrow[r, dotted, bend left=60] \arrow[r, dotted, bend left] \arrow[r, phantom, "\textcolor{blue}{\vdots}"] \arrow[r, dotted, bend right] & A_n \arrow[r, bend left=60, "f_{n,0}"] \arrow[r, bend left, "f_{n,1}"] \arrow[r, phantom, bend left=45, "\textcolor{blue}{\Downarrow\phi_{n,1}}"] \arrow[r, phantom, "\textcolor{blue}{\vdots}"] \arrow[r, bend right, "f_{n,k_n}"'] & A_0 \end{tikzcd} = \begin{array}{c} (\phi_{0,1}|\ldots|\phi_{0,k_0})\otimes\ldots\otimes(\phi_{n,1}|\ldots|\phi_{n,k_n})\otimes\alpha: \\ s.t.\ \phi_{i,j}\in C^{\bullet}({}_{f_{j-1}}A_{i}, {}_{f_j}A_{i+1}), \\ \alpha\in C_{-\bullet}({}_{f_{n,k_n}\ldots f_{0,k_0}}A_0, A_0) \end{array} \right\}$$

Dg comodules over dg cocategories

Fix algebras $A_0, \dots, A_n$.
Define a dg comodule over $B(A_0 \to \dots \to A_n \to A_0)$:

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$$:= \left\{ \begin{tikzcd}[column sep=large]
A_0 \arrow[r, bend left=60, "f_{0,0}"] \arrow[r, bend left=20, "f_{0,1}"] \arrow[r, bend right=20, "f_{0,k_0}"'] \arrow[r, phantom, bend left=40, "\Downarrow\phi_{0,1}"] \arrow[r, phantom, "\vdots"] \arrow[rrrr, bend right=40, "id_{A_0}"', "\alpha"] & A_1 \arrow[r, dotted, bend left=60] \arrow[r, dotted, bend left=20] \arrow[r, dotted, bend right=20] \arrow[r, phantom, "\vdots"] & \cdots \arrow[r, dotted, bend left=60] \arrow[r, dotted, bend left=20] \arrow[r, dotted, bend right=20] \arrow[r, phantom, "\vdots"] & A_n \arrow[r, bend left=60, "f_{n,0}"] \arrow[r, bend left=20, "f_{n,1}"] \arrow[r, bend right=20, "f_{n,k_n}"'] \arrow[r, phantom, bend left=40, "\Downarrow\phi_{n,1}"] \arrow[r, phantom, "\vdots"] & A_0
\end{tikzcd} = \begin{array}{c} (\phi_{0,1}|\dots|\phi_{0,k_0})\otimes\dots\otimes(\phi_{n,1}|\dots|\phi_{n,k_n})\otimes\alpha: \\ s.t.\ \phi_{i,j}\in C^\bullet(A_i, {}_{f_{j-1}}A_{i+1}{}_{f_j}), \\ \alpha\in C_{-\bullet}(A_0, {}_{f_{n,k_n}\dots f_{0,k_0}}A_0) \end{array} \right\}$$

$$d_{C(A_0\to\dots\to A_0)} = d_B \otimes id_{C_{-\bullet}} + id_B \otimes b + \tilde{\iota}$$

where $\tilde{\iota}$ is given as follows:

$$\begin{tikzcd}
C(A_0 \to \ldots \to A_0)^{\bullet}(f_{0,0}, \ldots, f_{n,0}) \arrow[rr, "\tilde{\iota}"] \arrow[drr, "\tilde{\iota}", "\substack{\text{extend as a}\\ \text{coderivation}}"'] & & C_{-\bullet}(A_{0}, {}_{f_{n,0}} \ldots {}_{f_{0,0}} A_0) \\
& & C(A_0 \to \ldots \to A_0)^{\bullet}(f_{0,0}, \ldots, f_{n,0}) \arrow[u, two heads]
\end{tikzcd}$$

Dg comodules over dg cocategories

where $\tilde{\iota}$ is given as follows:

$$\begin{tikzcd}
C(A_0 \to \ldots \to A_0)^\bullet(f_{0,0}, \ldots, f_{n,0}) \arrow[r, "\tilde{\iota}"] \arrow[dr, "\tilde{\iota}", "\text{extend as a coderivation}"'] & C_{-\bullet}(A_0 {}_{f_{n,0}} \ldots {}_{f_{0,0}} A_0) \\
& C(A_0 \to \ldots \to A_0)^\bullet(f_{0,0}, \ldots, f_{n,0}) \arrow[u, two heads]
\end{tikzcd}$$

$$\tilde{\iota}\big((\phi_{0,1}|\ldots|\phi_{0,k_0})\otimes\ldots\otimes(\phi_{n,1}|\ldots|\phi_{n,k_n})\otimes\alpha\big) = \iota_{(\phi_{0,1}|\ldots|\phi_{0,k_0})\bullet\ldots\bullet(\phi_{n,1}|\ldots|\phi_{n,k_n})}\alpha$$

$$\iota_\phi(a_0 \otimes \ldots a_p) = \pm\phi(a_{d+1}, \ldots, a_n) \cdot a_0 \otimes a_1 \otimes \ldots \otimes a_d \quad \text{where } |\phi| = n - d$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

$$\mathcal{A} := (A_0 \to \ldots \to A_n \to A_0)$$

$$\chi_\infty \to \mathcal{D}_\infty$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

$$\mathcal{A} := (A_0 \to \ldots \to A_n \to A_0)$$

$$\begin{aligned} \chi_\infty &\to \mathcal{D}_\infty \\ \mathcal{A} &\mapsto (B(\mathcal{A}), C(\mathcal{A})) \end{aligned}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

$$\mathcal{A} := (A_0 \to \ldots \to A_n \to A_0)$$

$$\begin{aligned} \chi_\infty &\to \mathcal{D}_\infty \\ \mathcal{A} &\mapsto (B(\mathcal{A}), C(\mathcal{A})) \end{aligned}$$

$$\text{Generating Morphisms:} \quad \sigma_{i,n} \mapsto \begin{pmatrix} B(\mathcal{A}) \xrightarrow{\hat{\sigma}_{i,n}} B(\sigma_{i,n}\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{id} \hat{\sigma}^*_{i,n} C(\sigma_{i,n}\mathcal{A}) \end{pmatrix}$$

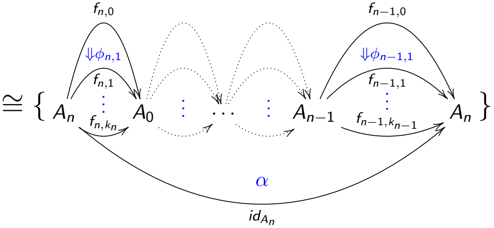
$$\delta_{j,n} \mapsto \begin{pmatrix} B(\mathcal{A}) \xrightarrow{\hat{\delta}_{j,n}} B(\delta_{j,n}\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{id} \hat{\delta}^*_{j,n} C(\delta_{j,n}\mathcal{A}) \end{pmatrix} \quad \tau_n \mapsto \begin{pmatrix} B(\mathcal{A}) \xrightarrow{\hat{\tau}_n} B(\tau_n\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{\tau_{n!}} \hat{\tau}^*_n C(\tau_n\mathcal{A}) \end{pmatrix}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

$$\hat{\tau}_n^* C(\tau_n \mathcal{A})^\bullet(\underbrace{f_0, \ldots, f_n}_{\in Obj(B(\mathcal{A}))}) \cong C(\tau_n \mathcal{A})^\bullet(f_n, f_0, \ldots, f_{n-1}) \text{ as complexes.}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

$$\hat{\tau}_n^* C(\tau_n \mathcal{A})^\bullet(\underbrace{f_0, \ldots, f_n}_{\in Obj(B(\mathcal{A}))}) \cong C(\tau_n \mathcal{A})^\bullet(f_n, f_0, \ldots, f_{n-1}) \text{ as complexes.}$$



An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

$$\hat{\tau}_n^* C(\tau_n \mathcal{A})^\bullet(\underbrace{f_0, \ldots, f_n}_{\in Obj(B(\mathcal{A}))}) \cong C(\tau_n \mathcal{A})^\bullet(f_n, f_0, \ldots, f_{n-1}) \text{ as complexes.}$$

$$n = 1 : \quad C(A_0 \to A_1 \to A_0)^\bullet(f, g) \xrightarrow{\tau_{1!}} C(A_1 \to A_0 \to A_1)^\bullet(g, f)$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

$$\hat{\tau}_n^* C(\tau_n \mathcal{A})^\bullet(\underbrace{f_0, \ldots, f_n}_{\in Obj(B(\mathcal{A}))}) \cong C(\tau_n \mathcal{A})^\bullet(f_n, f_0, \ldots, f_{n-1}) \text{ as complexes.}$$

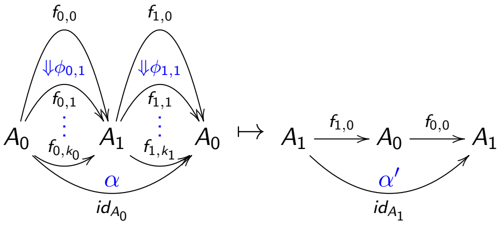
$$n = 1 : \quad C(A_0 \to A_1 \to A_0)^\bullet(f, g) \xrightarrow{\tau_{1!}} C(A_1 \to A_0 \to A_1)^\bullet(g, f)$$

$$\begin{tikzcd}
A_0 \arrow[r, bend left=70, "f_{0,0}"] \arrow[r, bend left=35, "f_{0,1}"'] \arrow[r, phantom, bend left=50, "\Downarrow\phi_{0,1}"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=35, "f_{0,k_0}"] \arrow[rr, bend right=60, "id_{A_0}"'] \arrow[rr, phantom, bend right=40, "\alpha"] & A_1 \arrow[r, bend left=70, "f_{1,0}"] \arrow[r, bend left=35, "f_{1,1}"'] \arrow[r, phantom, bend left=50, "\Downarrow\phi_{1,1}"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=35, "f_{1,k_1}"] & A_0 & \mapsto \sum & A_1 \arrow[r, bend left=70, "f_{1,0}"] \arrow[r, bend left=35, "f_{1,1}"'] \arrow[r, phantom, bend left=50, "\Downarrow\phi'_{1,1}"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=35, "f_{1,k_1}"] \arrow[rr, bend right=60, "id_{A_1}"'] \arrow[rr, phantom, bend right=40, "\alpha'"] & A_0 \arrow[r, bend left=70, "f_{0,0}"] \arrow[r, bend left=35, "f_{0,1}"'] \arrow[r, phantom, bend left=50, "\Downarrow\phi'_{0,1}"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=35, "f_{0,k_0}"] & A_1
\end{tikzcd}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

$$\hat{\tau}_n^* C(\tau_n \mathcal{A})^\bullet(\underbrace{f_0, \ldots, f_n}_{\in Obj(B(\mathcal{A}))}) \cong C(\tau_n \mathcal{A})^\bullet(f_n, f_0, \ldots, f_{n-1}) \text{ as complexes.}$$

$$n = 1 : \quad C(A_0 \to A_1 \to A_0)^\bullet(f, g) \xrightarrow{\tau_{1!}} C(A_1 \to A_0 \to A_1)^\bullet(g, f)$$



An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

Strategy: Give these maps to cogenerators to define $\tau_{1!}$, then let

$$
\begin{aligned}
\tau_{n!} : C(\mathcal{A}) &\cong \hat{\delta}_0^{*\,n-1} C(A_0 \to A_n \to A_0) \xrightarrow{\hat{\delta}_0^{*\,n-1}\tau_{1!}} \hat{\delta}_0^{*\,n-1} \hat{\tau}_1^* C(A_n \to A_0 \to A_n) \cong \\
&\cong (\widehat{\tau_1 \delta_0^{n-1}})^* C(A_n \to A_0 \to A_n) \cong (\widehat{\delta_0^{n-1} \tau_n})^* C(A_n \to A_0 \to A_n) \cong \\
&\cong \hat{\tau}_n^* \hat{\delta}_0^{*\,n-1} C(A_n \to A_0 \to A_n) \cong \hat{\tau}_n^* C(\tau_n \mathcal{A}).
\end{aligned}
$$

$$
n = 1 : \quad C(A_0 \to A_1 \to A_0)^\bullet(f, g) \xrightarrow{\tau_{1!}} C(A_1 \to A_0 \to A_1)^\bullet(g, f)
$$

$$
\begin{tikzcd}
A_0 \arrow[r, bend left=80, "f_{0,0}"] \arrow[r, bend left=40, "f_{0,1}"'] \arrow[r, phantom, bend left=60, "\Downarrow\phi_{0,1}"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=30, "f_{0,k_0}"] \arrow[rr, bend right=50, "id_{A_0}"', "\alpha"] & A_1 \arrow[r, bend left=80, "f_{1,0}"] \arrow[r, bend left=40, "f_{1,1}"'] \arrow[r, phantom, bend left=60, "\Downarrow\phi_{1,1}"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=30, "f_{1,k_1}"] & A_0 & \mapsto & A_1 \arrow[r, "f_{1,0}"] \arrow[rr, bend right=50, "id_{A_1}"', "\alpha'"] & A_0 \arrow[r, "f_{0,0}"] & A_1
\end{tikzcd}
$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

Defining $\tau_{1!}$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

Defining $\tau_{1!}$

$$
\begin{tikzcd}
A_0 \arrow[r, "f_{0,0}"] \arrow[rr, bend right, "id_{A_0}"', "\color{blue}\alpha"] & A_1 \arrow[r, "f_{1,0}"] & A_0
\end{tikzcd}
\overset{?}{\mapsto}
\begin{tikzcd}
A_1 \arrow[r, "f_{1,0}"] \arrow[rr, bend right, "id_{A_1}"', "\color{blue}\alpha'"] & A_0 \arrow[r, "f_{0,0}"] & A_1
\end{tikzcd}
$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

Defining $\tau_{1!}$

$$\begin{tikzcd}
A_0 \arrow[r, "f_{0,0}"] \arrow[rr, bend right, "\alpha", "id_{A_0}"'] & A_1 \arrow[r, "f_{1,0}"] & A_0 \arrow[r, phantom, "\overset{?}{\mapsto}"] & A_1 \arrow[r, "f_{1,0}"] \arrow[rr, bend right, "\alpha'", "id_{A_1}"'] & A_0 \arrow[r, "f_{0,0}"] & A_1
\end{tikzcd}$$

$$\alpha = a_0 \otimes \ldots \otimes a_n \mapsto \alpha' = f_{0,0}(a_0) \otimes \ldots \otimes f_{0,0}(a_n)$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

Defining $\tau_{1!}$

$$
\begin{tikzcd}
A_0 \arrow[r, bend left=60, "f_{0,0}"] \arrow[r, phantom, bend left=30, "\Downarrow\phi"] \arrow[r, "f_{0,1}"] \arrow[rr, bend right=50, "\alpha" description, "id_{A_0}"'] & A_1 \arrow[r, "f_{1,0}"] & A_0
\end{tikzcd}
\overset{?}{\longmapsto}
\begin{tikzcd}
A_1 \arrow[r, "f_{1,0}"] \arrow[rr, bend right=50, "\alpha'" description, "id_{A_1}"'] & A_0 \arrow[r, "f_{0,0}"] & A_1
\end{tikzcd}
$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

Defining $\tau_{1!}$

$$\begin{tikzcd}
A_0 \arrow[r, "f_{0,1}"] \arrow[r, bend left, "f_{0,0}"] \arrow[r, phantom, bend left=20, "\color{blue}\Downarrow\phi"] \arrow[rr, bend right, "id_{A_0}"'] \arrow[rr, phantom, bend right=20, "\color{blue}\alpha"] & A_1 \arrow[r, "f_{1,0}"] & A_0 & \overset{?}{\mapsto} & A_1 \arrow[r, "f_{1,0}"] \arrow[rr, bend right, "id_{A_1}"'] \arrow[rr, phantom, bend right=20, "\color{blue}\alpha'"] & A_0 \arrow[r, "f_{0,0}"] & A_1
\end{tikzcd}$$

$$\overline{\tau_{1!} \circ d}(\phi \otimes \alpha) = \overline{d \circ \tau_{1!}}(\phi \otimes \alpha)$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

Defining $\tau_{1!}$

$$\begin{tikzcd}
A_0 \arrow[r, bend left=60, "f_{0,0}"] \arrow[r, "f_{0,1}"'] \arrow[r, phantom, shift left=3, "\Downarrow\phi"] \arrow[rr, bend right=40, "id_{A_0}"', "\alpha"] & A_1 \arrow[r, "f_{1,0}"] & A_0 & \overset{?}{\mapsto} & A_1 \arrow[r, "f_{1,0}"] \arrow[rr, bend right=40, "id_{A_1}"', "\alpha'"] & A_0 \arrow[r, "f_{0,0}"] & A_1
\end{tikzcd}$$

$$[b, \bar{\tau}_{1!}](\phi \otimes \alpha) \pm \bar{\tau}_{1!}(\delta\phi \otimes \alpha) = [\bar{\tau}_{1!}, \iota_\phi](\alpha)$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

Defining $\tau_{1!}$

$$\begin{tikzcd}
A_0 \arrow[r, "f_{0,1}"] \arrow[r, bend left=60, "f_{0,0}"] \arrow[r, phantom, shift left=3, "\Downarrow\phi"] \arrow[rr, bend right, "\alpha", "id_{A_0}"'] & A_1 \arrow[r, "f_{1,0}"] & A_0 & \overset{?}{\mapsto} & A_1 \arrow[r, "f_{1,0}"] \arrow[rr, bend right, "\alpha'", "id_{A_1}"'] & A_0 \arrow[r, "f_{0,0}"] & A_1
\end{tikzcd}$$

$$[b, \bar{\tau}_{1!}](\phi \otimes \alpha) \pm \bar{\tau}_{1!}(\delta\phi \otimes \alpha) = [\bar{\tau}_{1!}, \iota_\phi](\alpha)$$

$$\begin{aligned}
L_\phi(\alpha) = \sum_{k \geq 1} & \pm a_0 \otimes \ldots \otimes \phi(a_k, \ldots) \otimes a_r \otimes \ldots \otimes a_n + \\
& \sum \pm \phi(a_k, \ldots, a_n, a_0, \ldots) \otimes a_s \otimes \ldots \otimes a_{k-1}
\end{aligned}$$

$$[b, L_\phi] \pm L_{\delta\phi} = 0$$

Defining $\tau_{1!}$

$$\begin{tikzcd}
A_0 \arrow[r, "f_{0,1}"] \arrow[r, bend left=60, "f_{0,0}"] \arrow[r, phantom, shift left=3, "\Downarrow\phi"] \arrow[rr, bend right=50, "id_{A_0}"'] \arrow[rr, phantom, shift right=3, "\alpha"] & A_1 \arrow[r, "f_{1,0}"] & A_0 \arrow[r, mapsto, "?"] & A_1 \arrow[r, "f_{1,0}"] \arrow[rr, bend right=50, "id_{A_1}"'] \arrow[rr, phantom, shift right=3, "\alpha'"] & A_0 \arrow[r, "f_{0,0}"] & A_1
\end{tikzcd}$$

$$[b, \bar{\tau}_{1!}](\phi \otimes \alpha) \pm \bar{\tau}_{1!}(\delta\phi \otimes \alpha) = [\bar{\tau}_{1!}, \iota_\phi](\alpha)$$

$$\bar{\tau}_{1!}(\phi \otimes \alpha) = \sum_{k \geq 1} \pm f_{0,0} a_0 \otimes \ldots \otimes \phi(a_k, \ldots) \otimes f_{0,1} a_r \ldots \otimes f_{0,1} a_n +$$

$$\sum \pm \phi(f_{1,0} f_{0,1} a_k, \ldots, f_{1,0} f_{0,1} a_n, a_0, \ldots) \otimes f_{0,1} a_s \otimes \ldots \otimes f_{0,1} a_{k-1}$$

$$[b, \bar{\tau}_{1!}(\phi, -)] \pm \bar{\tau}_{1!}(\delta\phi, -) = [\bar{\tau}_{1!}, \iota_\phi]$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

Defining $\tau_{1!}$

$$\bar{\tau}_{1!}\big((\phi_{0,1}|\ldots|\phi_{0,k_0}) \otimes (\phi_{1,1}|\ldots|\phi_{1,k_1}) \otimes \alpha\big) =$$

$$= \sum_{\substack{1 \le i \le n_0 \\ i \le j_1 \le \ldots \le j_{2k_1} \le k_0 \\ p}} \pm\phi_{0,1}\left(\begin{matrix} f_{1,0}f_{0,i}a_p, \ldots, f_{1,0}\phi_{0,i+1}(a_*, \ldots), \\ f_{1,0}f_{0,i+1}a_*, \ldots, f_{1,0}\phi_{0,j_1}(a_*, \ldots), \\ f_{1,0}f_{0,j_1}a_*, \ldots, \phi_{1,1}(f_{0,j_1}a_*, \ldots, \phi_{0,j_1+1}(a_*, \ldots), \ldots), \ldots, \\ \phi_{1,k_1}(f_{0,j_{2k_1-1}}a_*, \ldots, \phi_{0,j_{2k_1-1}+1}(a_*, \ldots), \ldots), \ldots, a_0, \ldots \end{matrix} \right)\otimes$$

$$\otimes f_{0,1}a_* \otimes \ldots \otimes \phi_{0,2}(a_*, \ldots) \otimes f_{0,2}a_* \otimes \ldots \otimes$$

$$\otimes \phi_{0,i}(a_*, \ldots) \otimes f_{0,i}a_* \otimes \ldots f_{0,i}a_{p-1} +$$

$$\Big(\sum \pm f_{0,0}a_0 \otimes \ldots \otimes \phi_{0,1}(a_*, \ldots) \otimes \ldots \otimes \phi_{0,n_0}(a_*, \ldots) \otimes$$

$$\otimes f_{0,n_0}a_* \otimes \ldots \otimes f_{0,n_0}a_n \quad \text{if } k_1 = 0 \Big)$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

$$\mathcal{A} := (A_0 \to \ldots \to A_n \to A_0)$$

$$\chi_\infty \to \mathcal{D}_\infty$$

$$\mathcal{A} \mapsto (B(\mathcal{A}), C(\mathcal{A}))$$

Generating Morphisms: $\sigma_{i,n} \mapsto \begin{pmatrix} B(\mathcal{A}) \xrightarrow{\hat{\sigma}_{i,n}} B(\sigma_{i,n}\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{id} \hat{\sigma}^*_{i,n} C(\sigma_{i,n}\mathcal{A}) \end{pmatrix}$

$$\delta_{j,n} \mapsto \begin{pmatrix} B(\mathcal{A}) \xrightarrow{\hat{\delta}_{j,n}} B(\delta_{j,n}\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{id} \hat{\delta}^*_{j,n} C(\delta_{j,n}\mathcal{A}) \end{pmatrix} \quad \tau_n \mapsto \begin{pmatrix} B(\mathcal{A}) \xrightarrow{\hat{\tau}_n} B(\tau_n\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{\tau_{n!} = d_0^{*n-1} \tau_{1!}} \hat{\tau}^*_n C(\tau_n\mathcal{A}) \end{pmatrix}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

n=1:

$$\begin{tikzcd} C(A_0 \to A_1 \to A_0) \arrow[r, "\tau_{1!}"] \arrow[rr, bend right, "id"'] & \hat{\tau}_1^* C(\tau_1(A_0 \to A_1 \to A_0)) \arrow[r, "\hat{\tau}_1^* \tau_{1!}"] & \hat{\tau}_1^{*2} C(\tau_1^2(A_0 \to A_1 \to A_0)) \end{tikzcd}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

n=1:

$$\begin{tikzcd} C(A_0 \to A_1 \to A_0) \arrow[r, "\tau_{1!}"] \arrow[rr, bend right, "id"'] & \hat{\tau}_1^* C(\tau_1(A_0 \to A_1 \to A_0)) \arrow[r, "\hat{\tau}_1^* \tau_{1!}"] & \hat{\tau}_1^{*2} C(\tau_1^2(A_0 \to A_1 \to A_0)) \end{tikzcd}$$

$$\begin{tikzcd} A_0 \arrow[r, "f_{0,0}"] \arrow[rr, bend right, "\alpha", "id_{A_0}"'] & A_1 \arrow[r, "f_{1,0}"] & A_0 \end{tikzcd}$$

$$\alpha = a_0 \otimes \ldots \otimes a_n \overset{\tau_{1!}}{\mapsto} f_{0,0} a_0 \otimes \ldots \otimes f_{0,0} a_n \overset{\hat{\tau}_{1!}^* \tau_{1!}}{\mapsto} f_{1,0} f_{0,0} a_0 \otimes \ldots \otimes f_{1,0} f_{0,0} a_n$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

n=1:

$$\begin{tikzcd} C(A_0 \to A_1 \to A_0) \arrow[r, "\tau_{1!}"] \arrow[rr, bend right, "id"'] & \hat{\tau}_1^* C(\tau_1(A_0 \to A_1 \to A_0)) \arrow[r, "\hat{\tau}_1^* \tau_{1!}"] & \hat{\tau}_1^{*2} C(\tau_1^2(A_0 \to A_1 \to A_0)) \end{tikzcd}$$

$$\begin{tikzcd} A_0 \arrow[r, "f_{0,0}"] \arrow[rr, bend right, "\alpha", "id_{A_0}"'] & A_1 \arrow[r, "f_{1,0}"] & A_0 \end{tikzcd}$$

$$\alpha = a_0 \otimes \ldots \otimes a_n \overset{\tau_{1!}}{\mapsto} f_{0,0}a_0 \otimes \ldots \otimes f_{0,0}a_n \overset{\hat{\tau}_{1!}^* \tau_{1!}}{\mapsto} f_{1,0}f_{0,0}a_0 \otimes \ldots \otimes f_{1,0}f_{0,0}a_n$$

$$f_{1,0}f_{0,0}\alpha - \alpha = [b, B](\alpha)$$

$$B(a_0 \otimes \ldots \otimes a_n) = \sum_{0 \le i \le n} \pm 1 \otimes f_{1,0}f_{0,0}a_i \otimes \ldots \otimes f_{1,0}f_{0,0}a_n \otimes a_0 \otimes \ldots \otimes a_{i-1}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

n=1:

$$\begin{tikzcd}
C(A_0 \to A_1 \to A_0) \arrow[r, "\tau_{1!}"] \arrow[rr, bend right, "id"'] & \hat{\tau}_1^* C(\tau_1(A_0 \to A_1 \to A_0)) \arrow[r, "\hat{\tau}_1^* \tau_{1!}"] & \hat{\tau}_1^{*2} C(\tau_1^2(A_0 \to A_1 \to A_0))
\end{tikzcd}$$

$$\begin{tikzcd}
A_0 \arrow[r, bend left=60, "f_{0,0}"] \arrow[r, "f_{0,1}"'] \arrow[r, phantom, shift left=2, "\Downarrow\phi"] \arrow[rr, bend right=50, "id_{A_0}"'] \arrow[rr, phantom, bend right=30, "\alpha"] & A_1 \arrow[r, "f_{1,0}"] & A_0
\end{tikzcd} = \phi \otimes \alpha$$

$$(\overline{\hat{\tau}^* \tau_{1!} \circ \tau_{1!} - id})(\phi \otimes \alpha) = \text{“}f_{1,0} \circ L_\phi(\alpha)\text{”}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

n=1:

$$
\begin{tikzcd}
C(A_0 \to A_1 \to A_0) \arrow[r, "\tau_{1!}"] \arrow[rr, bend right, "id"'] & \hat{\tau}_1^* C(\tau_1(A_0 \to A_1 \to A_0)) \arrow[r, "\hat{\tau}_1^* \tau_{1!}"] & \hat{\tau}_1^{*2} C(\tau_1^2(A_0 \to A_1 \to A_0))
\end{tikzcd}
$$

$$(\overline{\hat{\tau}^* \tau_{1!} \circ \tau_{1!} - id})(\phi \otimes \alpha) = \text{“}f_{1,0} \circ L_\phi(\alpha)\text{''}$$

$$= \sum_{k \geq 1} \pm f_{1,0} f_{0,0} a_0 \otimes \ldots \otimes f_{1,0} \phi(a_k, \ldots) \otimes f_{1,0} f_{0,1} a_r \ldots \otimes f_{1,0} f_{0,1} a_n +$$

$$\sum \pm f_{1,0} \phi(f_{1,0} f_{0,1} a_k, \ldots, f_{1,0} f_{0,1} a_n, a_0, \ldots) \otimes f_{1,0} f_{0,1} a_s \otimes \ldots \otimes f_{1,0} f_{0,1} a_{k-1}$$

$$\overset{?}{=} [\iota_\phi, B](\alpha) \pm [b, B](\phi \otimes \alpha) \pm B(\delta\phi \otimes \alpha)$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

n=1:

$$\begin{tikzcd} C(A_0 \to A_1 \to A_0) \arrow[r, "\tau_{1!}"] \arrow[rr, bend right, "id"'] & \hat{\tau}_1^* C(\tau_1(A_0 \to A_1 \to A_0)) \arrow[r, "\hat{\tau}_1^* \tau_{1!}"] & \hat{\tau}_1^{*2} C(\tau_1^2(A_0 \to A_1 \to A_0)) \end{tikzcd}$$

$$(\overline{\hat{\tau}^* \tau_{1!} \circ \tau_{1!} - id})(\phi \otimes \alpha) = \text{“} f_{1,0} \circ L_\phi(\alpha)\text{''}$$

$$= \sum_{k \geq 1} \pm f_{1,0} f_{0,0} a_0 \otimes \ldots \otimes f_{1,0} \phi(a_k, \ldots) \otimes f_{1,0} f_{0,1} a_r \ldots \otimes f_{1,0} f_{0,1} a_n +$$

$$\sum \pm f_{1,0} \phi(f_{1,0} f_{0,1} a_k, \ldots, f_{1,0} f_{0,1} a_n, a_0, \ldots) \otimes f_{1,0} f_{0,1} a_s \otimes \ldots \otimes f_{1,0} f_{0,1} a_{k-1}$$

$$\stackrel{?}{=} [\iota_\phi, B](\alpha) \pm [b, B](\phi \otimes \alpha) \pm B(\delta\phi \otimes \alpha)$$

$$B(\phi \otimes \alpha) = \sum_{0 \leq i \leq n} \pm 1 \otimes f_{1,0} f_{0,0} a_i \otimes \ldots \otimes f_{1,0} \phi(a_*, \ldots) \otimes f_{1,0} f_{0,1} a_* \otimes \ldots \otimes f_{1,0} f_{0,1} a_n \otimes a_0 \otimes \ldots \otimes a_{i-1}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

n=1:

$$B\big((\phi_{0,1}|\dots|\phi_{0,k_0}) \otimes (\phi_{1,1}|\dots|\phi_{1,k_1}) \otimes \alpha\big) =$$

$$= \sum_{\substack{0 \le j_1 \le \dots \le j_{2k_1} \le k_0 \\ p}} \pm 1 \otimes f_{1,0} f_{0,0} a_p \otimes \dots \otimes f_{1,0}\phi_{0,1}(a_*, \dots) \otimes$$

$$\otimes f_{1,0} f_{0,1} a_* \otimes \dots \otimes f_{1,0}\phi_{0,j_1}(a_*, \dots) \otimes$$

$$\otimes f_{1,0} f_{0,j_1} a_* \otimes \dots \otimes \phi_{1,1}(f_{0,j_1} a_*, \dots, \phi_{0,j_1+1}(a_*, \dots), \dots) \otimes$$

$$\otimes \dots \otimes \phi_{1,k_1}(f_{0,j_{2k_1-1}} a_*, \dots, \phi_{0,j_{2k_1-1}+1}(a_*, \dots), \dots) \otimes \dots \otimes$$

$$\otimes a_0 \otimes \dots \otimes a_{p-1}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

$$
\begin{aligned}
\chi_\infty &\to \mathcal{D}_\infty \\
\mathcal{A} &\mapsto (B(\mathcal{A}), C(\mathcal{A})) \\
\tau_1 &\mapsto \begin{pmatrix} \hat{\tau}_1 : B(\mathcal{A}) \to B(\tau_1 \mathcal{A}) \\ \tau_{1!} : C(\mathcal{A}) \to \hat{\tau}_1^* C(\tau_1 \mathcal{A}) \end{pmatrix} \\
(\tau_1, \tau_1) &\mapsto \begin{pmatrix} id : B(\mathcal{A}) \to B(\mathcal{A}) \\ B : C(\mathcal{A}) \to C(\mathcal{A}) \end{pmatrix} \\
(\tau_1, \tau_1, \tau_1) &\mapsto \begin{pmatrix} \hat{\tau}_1 : B(\mathcal{A}) \to B(\tau_1 \mathcal{A}) \\ 0 : C(\mathcal{A}) \to \hat{\tau}_1^* C(\tau_1 \mathcal{A}) \end{pmatrix} \\
&\vdots
\end{aligned}
$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

The A_∞ relations mean:

- $\tau_{1!}$ is a map of complexes
- $B^2 = 0$
- The following diagram commutes:

$$\begin{tikzcd}
C(\mathcal{A}) \arrow[r, "\tau_{1!}"] \arrow[d, "B"] & \hat{\tau}_1^* C(\tau_1 \mathcal{A}) \arrow[d, "\hat{\tau}_1^* B"] \\
C(\mathcal{A}) \arrow[r, "\tau_{1!}"'] & \hat{\tau}_1^* C(\tau_1 \mathcal{A})
\end{tikzcd}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

For higher $n > 1$, we want to find a homotopy between "$\tau_{n!}^{n+1}$" and *id*.

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

For higher $n > 1$, we want to find a homotopy between "$\tau_{n!}^{n+1}$" and *id*. However, it is sufficient to find a homotopy between $\hat{\tau}_n^{*2}\delta_{0,n!} \circ \hat{\tau}_n^{*}\tau_{n!} \circ \tau_{n!}$ and $\hat{\delta}_{n-1,n}^{*}\tau_{n-1!} \circ \delta_{n-1,n!}$.

$$\begin{tikzcd}
{[n]} \arrow[r, "\tau_n"] \arrow[d, "\delta_{n-1,n}"] & {[n]} \arrow[r, "\tau_n"] & {[n]} \arrow[dl, "\delta_{0,n}"] \\
{[n-1]} \arrow[r, "\tau_{n-1}"'] & {[n-1]} &
\end{tikzcd}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

For higher $n > 1$, we want to find a homotopy between "$\tau_{n!}^{n+1}$" and *id*. However, it is sufficient to find a homotopy between $\hat{\tau}_n^{*2}\delta_{0,n!} \circ \hat{\tau}_n^{*}\tau_{n!} \circ \tau_{n!}$ and $\hat{\delta}^*_{n-1,n}\tau_{n-1!} \circ \delta_{n-1,n!}$.

$$\begin{tikzcd}
{[n]} \arrow[r, "\tau_n"] \arrow[d, "\delta_{n-1,n}"] & {[n]} \arrow[r, "\tau_n"] & {[n]} \arrow[dl, "\delta_{0,n}"] \\
{[n-1]} \arrow[r, "\tau_{n-1}"'] & {[n-1]} &
\end{tikzcd}$$

Strategy: Find such a homotopy, $\mathcal{B}$, for $n = 2$, and use $\hat{\delta}_0^{*n-2}\mathcal{B}$ for $n > 2$.

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

$$
\begin{aligned}
\chi_\infty &\to \mathcal{D}_\infty \\
\mathcal{A} &\mapsto (B(\mathcal{A}), C(\mathcal{A})) \\
\mu = \tau_{n-1} \circ \delta_{n-1,n} = \delta_{0,n} \circ \tau_n^2 &\mapsto \begin{pmatrix} \hat{\tau}_{n-1} \circ \hat{\delta}_{n-1,n} = \hat{\delta}_{0,n} \circ \hat{\tau}_n^2 \\ \hat{\tau}_n^{*2} \delta_{0,n!} \circ \hat{\tau}_n^* \tau_{n!} \circ \tau_{n!} \end{pmatrix} \\
(\delta_{0,n}, \tau_n^2) &\mapsto \begin{pmatrix} \hat{\delta}_{0,n} \circ \hat{\tau}_n^2 \\ 0 \end{pmatrix} \\
(\tau_{n-1}, \delta_{n-1,n}) &\mapsto \begin{pmatrix} \hat{\tau}_{n-1} \circ \hat{\delta}_{n-1,n} \\ \mathcal{B} \end{pmatrix} \\
(\tau_{n-1}, \delta_{n-1,n}, \lambda) &\mapsto \begin{pmatrix} \hat{\tau}_{n-1} \circ \hat{\delta}_{n-1,n} \circ \hat{\lambda} \\ 0 \end{pmatrix} \\
&\vdots
\end{aligned}
$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

For $n > 1$, the A_∞ relations mean:

- $\tau_{n!}$ is a map of complexes
- $\mathcal{B}^2 = 0$
- We have a pair of homotopic maps:

$$
\begin{tikzcd}
C(A_0 \to \ldots \to A_n \to A_0) \arrow[d, bend right, "{\begin{array}{c}\hat{\tau}_n^{*2}\tau_{n!}\circ\hat{\tau}_n^{*}\tau_{n!}\circ\tau_{n!} \\ \text{“apply } \tau_{n!} \text{ 3 times”}\end{array}}"'] \arrow[d, bend left, "{\begin{array}{c}(\delta_{n-2,\widehat{n-1\delta}_{n-1,n}})^*\tau_{n-2!} \\ \text{“brace together the last 3 algebras,} \\ \text{then apply } \tau_{n-2!} \text{ once”}\end{array}}"] \\
C(A_{n-2} \to A_{n-1} \to A_n \to A_0 \to \ldots \to A_{n-2}).
\end{tikzcd}
$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

For $n > 1$, the A_∞ relations mean:

- $\tau_{n!}$ is a map of complexes
- $\mathcal{B}^2 = 0$
- They are homotopic via two homotopies:

$$
\begin{tikzcd}
(\delta_{n-2,}\widehat{{}_{n-1}\delta}_{n-1,n})^*\tau_{n-2!} \arrow[r, "\cong"] \arrow[dd, "\cong"'] & \hat{\delta}^*_{n-1,n}(\hat{\delta}^*_{n-2,n-1}\tau_{n-2!}) \arrow[r, "\hat{\delta}^*_{n-1,n}\mathcal{B}_{n-1}"] & \hat{\delta}^*_{n-1,n}(\hat{\tau}^*_{n-1}\tau_{n-1!} \circ \tau_{n-1!}) \arrow[dd, "\cong"] \\
\text{“brace together } A_{n-2}, A_{n-1}, A_n, \text{ then apply } \tau_{n-2!}\text{”} & & \\
(\delta_{n-2,}\widehat{{}_{n-1}\delta}_{n-2,n})^*\tau_{n-2!} \arrow[dd, "\hat{\delta}^*_{n-2,n}\mathcal{B}_{n-1}"'] & & \hat{\tau}^{*2}_n\tau_{n!} \circ \hat{\delta}^*_{n-1,n}\tau_{n-1!} \arrow[dd, "\tau^{*2}_n\tau_{n!}\circ\mathcal{B}_n"] \\
& & \text{“brace together } A_{n-1}, A_n \text{ and apply } \tau_{n-1!}, \text{ then apply } \tau_{n!}\text{”} \\
\hat{\delta}^*_{n-2,n}(\hat{\tau}^*_{n-1}\tau_{n-1!} \circ \tau_{n-1!}) \arrow[r, "\cong"'] & \hat{\tau}^*_n(\hat{\delta}^*_{n-1,n}\tau_{n-1!}) \circ \tau_{n!} \arrow[r, "\hat{\tau}^*_n\mathcal{B}_n\circ\tau_{n!}"] & \hat{\tau}^{*2}_n\tau_{n!} \circ \hat{\tau}^*_n\tau_{n!} \circ \tau_{n!} \\
& \text{“apply } \tau_{n!}, \text{ then brace together } A_{n-1}, A_{n-2} \text{ and apply } \tau_{n-1!}\text{”} & \text{“apply } \tau_{n!} \text{ three times”}
\end{tikzcd}
$$

Summary:

We have a given a "homotopically sheafy-cyclic object in dg cocategories with a dg comodule", i.e., an A_∞-functor between dg categories χ_∞ and $\mathcal{D}_\infty$.

Time and interest permitting

- Rectify $\mathcal{F}$ to a dg functor
- Give a dg functor $\mathcal{D}_\infty \to \mathcal{E} = \{(\text{dg cat, dg mod})\}$

$$U(\chi_\infty) \xrightarrow{\textit{rectified}} \mathcal{D}_\infty \to \mathcal{E}$$

"A homotopically sheafy-cyclic object in dg categories with a dg module"