

Title

Subtitle

Rebecca Wei

Northwestern University

Date/Event

Background

Theorem

(Hochschild-Kostant-Rosenberg) Let A be a regular, commutative algebra over a field k of characteristic 0. Then,

$$(C_{\bullet}(A, A), b) \xrightarrow{\sim} \Omega_{A/k}^{\bullet}$$

$$(C^{\bullet}(A, A), \delta) \xrightarrow{\sim} \wedge^{\bullet} T_A.$$

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Theorem

(Kontsevich, '97) Let $A = C^\infty(M)$ for M a smooth real manifold. Then, there is an L_∞ map

$$(C^{\bullet+1}(A, A), \delta, [,]_{Ger}) \xrightarrow{\sim} (\wedge^{\bullet+1} T_A, d = 0, [,]_{SN}).$$

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Theorem

(Tamarkin, '98) Dependent on the choice of a Drinfeld associator, there is a Ger_{∞} map

$$(C^{\bullet+1}(A, A), \delta, [,]_{\mathrm{Ger}}, \cup, \dots) \xrightarrow{\sim} (\wedge^{\bullet} T_A, d = 0, [,]_{SN}, \wedge).$$

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Theorem

(Dolgushev-Tamarkin-Tsygan, '08) There is a Calc_{∞} map

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Can we use this calculus structure to create a cyclic object?

A cyclic object in dg cocategories

Fix an algebra, A .

Define a dg category, $Hoch(A)$:

A cyclic object in dg cocategories

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Objects: algebra maps $f : A \rightarrow A$

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$$\begin{aligned} {}_f \delta_g(\phi)(a_1 \otimes \dots \otimes a_n) = & \epsilon_\phi \left(f(a_1) \cdot \phi(a_2, \dots, a_n) + \right. \\ & + \sum_{1 \leq i \leq n-1} (-1)^i \phi(a_1, \dots, a_i a_{i+1}, \dots, a_n) + \\ & \left. + (-1)^n \phi(a_1, \dots, a_{n-1}) \cdot g(a_n) \right) \\ \epsilon_\phi = & (-1)^{|\phi|+1} \end{aligned}$$

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Composition: cup product on cochains

A cyclic object in dg cocategories

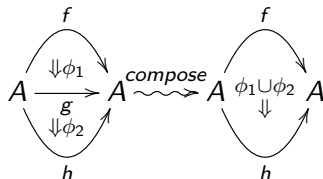
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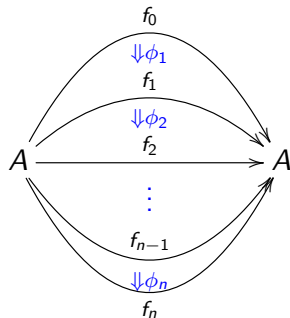
$$Bar(Hoch(A)) \otimes Bar(Hoch(A)) \xrightarrow{\bullet} Hoch(A)$$

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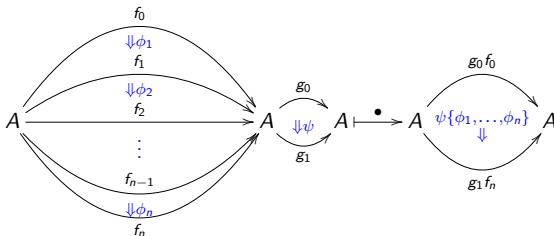
A morphism from f_0 to f_n in $Bar(Hoch(A))$

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In this context, braces give multilinear maps:

$$\begin{array}{ccc} Bar(Hoch(A)) \otimes Bar(Hoch(A)) & \xrightarrow{\bullet} & Hoch(A) \\ & \searrow \bullet & \uparrow \\ & & Bar(Hoch(A)) \end{array}$$

Then, $(Bar(Hoch(A)), \bullet)$ is an algebra in $DGCocats$.

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But we have more...

A cyclic object in dg cocategories

A cyclic object in dg cocategories

Fix algebras, A_0, A_1, \dots, A_n .

We will define a dg cocategory $B(A_0 \rightarrow A_1 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$
where $B(A_0 \rightarrow A_0) := \text{Bar}(\text{Hoch}(A_0))$ for $n=0$.

A cyclic object in dg cocategories

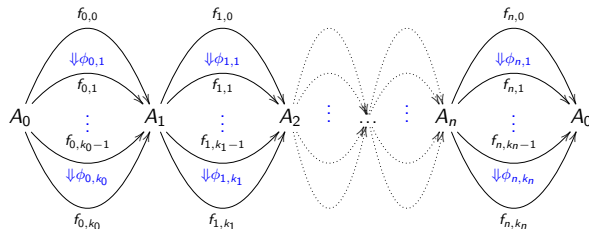
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Objects: $\{A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \dots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_0\}$

A morphism from $(A_0 \xrightarrow{f_{0,0}} \dots \xrightarrow{f_{n,0}} A_0)$ to $(A_0 \xrightarrow{f_{0,k_0}} \dots \xrightarrow{f_{n,k_n}} A_0)$:



$$\phi_{ij} \in C^\bullet(A_{i,f_{i,j-1}} A_{i+1,f_{i,j}})$$

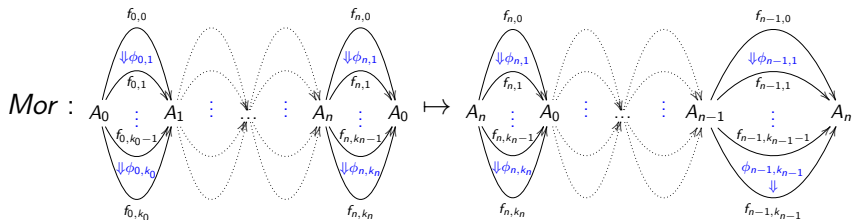
A cyclic object in dg cocategories

Example

We have a dg functor

$$\hat{\tau}_n : B(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0) \rightarrow B(A_n \rightarrow A_0 \rightarrow \dots \rightarrow A_n)$$

$$Obj : (f_0, \dots, f_n) \mapsto (f_n, f_0, \dots, f_{n-1})$$



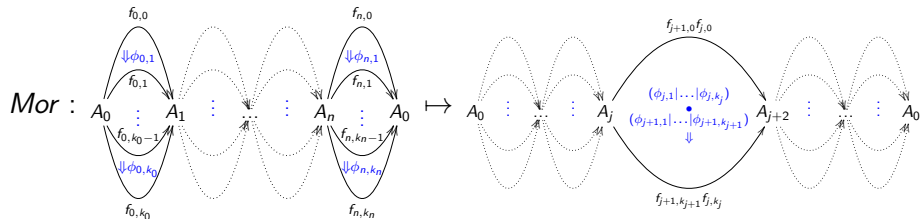
A cyclic object in dg cocategories

Example

For $n \geq 1, 0 \leq j < n$, we have a dg functor

$$\hat{\delta}_{j,n} : B(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0) \rightarrow B(A_0 \rightarrow \dots \rightarrow A_j \rightarrow A_{j+2} \xrightarrow{(mod\ n+1)} \dots \rightarrow A_0)$$

$$Obj : (f_0, \dots, f_n) \mapsto (f_0, \dots, f_{j+1} f_j, \dots, f_n)$$



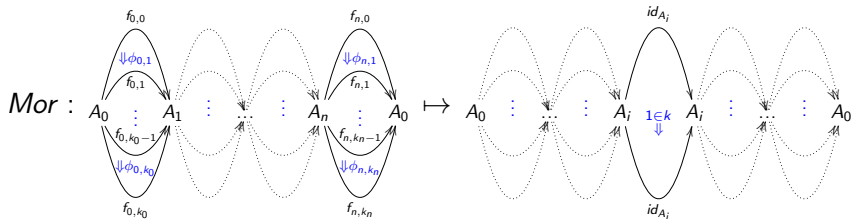
A cyclic object in dg cocategories

Example

For $n \geq 0, 0 \leq i \leq n$, we have a dg functor

$$\hat{\sigma}_{i,n} : B(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0) \rightarrow B(A_0 \rightarrow \dots \rightarrow A_i \rightarrow A_i \rightarrow \dots \rightarrow A_0)$$

$$Obj : (f_0, \dots, f_n) \mapsto (f_0, \dots, f_{i-1}, id_{A_i}, f_i, \dots, f_n)$$



A sheafy-cyclic object in DGCocat

Definition

Let χ be the category with objects $\{A_0 \rightarrow A_1 \rightarrow \dots \rightarrow A_n \rightarrow A_0\}$ and morphisms compositions of

$$\text{rotations } \tau_n : \mathcal{A} \mapsto (A_n \rightarrow A_0 \rightarrow \dots \rightarrow A_n)$$

$$\text{coboundaries } \delta_{j,n} : \mathcal{A} \mapsto (A_0 \rightarrow \dots \rightarrow A_j \rightarrow A_{j+2 \pmod{n+1}} \rightarrow \dots \rightarrow A_0)$$

$$\text{codegeneracies } \sigma_{i,n} : \mathcal{A} \mapsto (A_0 \rightarrow \dots \rightarrow A_i \rightarrow A_i \rightarrow \dots \rightarrow A_0)$$

where $\mathcal{A} := (A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$, subject to the cyclic relations.

A sheafy-cyclic object in $DGCocat$

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Proposition

We have a functor $\chi \rightarrow DGCocat$

$$\text{Objects} : (A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0) \mapsto B(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$$

$$\text{Generating morphisms} : \lambda \mapsto \hat{\lambda}$$

Motivating Question

Each dg cocategory $B(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$ has a dg comodule $C(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$ (which is a categorified version of the bar complex of Hochschild chains as a module over Hochschild cochains via contraction).

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Motivating Question: Can we extend the sheafy-cyclic structure to the comodules? In other words, can we give a functor

$$\chi \rightarrow \mathcal{D} := \{(dg\ cocat, dg\ comod)\}?$$

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$$\chi_\infty \rightarrow \mathcal{D}_\infty \quad \text{dg categories}$$

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Motivating Question: Can we extend the sheafy-cyclic structure to the comodules? In other words, can we give **an A_∞ -functor**

$$\chi \rightarrow \mathcal{D} := \{(\text{dg cocat}, \text{dg comod})\}?$$

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Rest of this talk: Describe our A_∞ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$.

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- Define dg categories χ_∞ and \mathcal{D}_∞
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- Define the A_∞ -functor \mathcal{F}

Dg categories χ_∞ and \mathcal{D}_∞

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χ_∞ :

Objects: same objects as $\chi = \{A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0\}$

$$\chi_\infty^\bullet(X, Y) := \{\sum a_i v_i : a_i \in k, v_i \in \chi(X, Y)\}$$

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\mathcal{D}_∞ :

Objects: same objects as $\mathcal{D} = \{(\text{dg cocategory}, \text{dg comodule})\}$

$$\mathcal{D}_\infty^p((B_1, C_1), (B_0, C_0)) := \left\{ \begin{array}{l} F : B_1 \rightarrow B_0 \text{ dg functor} \\ F_! : C_1 \rightarrow F^* C_0 \text{ degree-}p \text{ linear map} \end{array} \right\}$$

Dg categories χ_∞ and \mathcal{D}_∞

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$F^* C_0$ is the categorified version of co-extension of scalars:

$$F^* C_0 = \ker(B_1 \otimes C_0 \rightrightarrows B_1 \otimes B_0 \otimes C_0)$$

Dg comodules over dg cocategories

Definition

A **dg comodule** C over a dg cocategory B consists of the following data:

- for each object $f \in B$, a complex $C^\bullet(f)$, and
- maps of complexes

$$\Delta_C(f) : C^\bullet(f) \rightarrow \prod_{g \in \text{Obj}(B)} B^\bullet(f, g) \otimes C^\bullet(g).$$

such that the following diagrams for coassociativity and counitality commute:

$$\begin{array}{ccc}
 C^\bullet(f) & \xrightarrow{\Delta_C} & \prod_{g \in \text{Obj}(B)} B^\bullet(f, g) \otimes C^\bullet(g) \\
 \Delta_C \downarrow & & \downarrow id_B \otimes \Delta_C \\
 \prod_{g \in \text{Obj}(B)} B^\bullet(f, g) \otimes C^\bullet(g) & \xrightarrow{\Delta_B \otimes id_C} & \prod_{g, g' \in \text{Obj}(B)} B^\bullet(f, g) \otimes B^\bullet(g, g') \otimes C^\bullet(g')
 \end{array}
 \qquad
 \begin{array}{ccc}
 C^\bullet(f) & \xrightarrow{\Delta_C} & \prod_{g \in \text{Obj}(B)} B^\bullet(f, g) \otimes C^\bullet(g) \\
 \searrow id & & \downarrow \epsilon_B \otimes id_C \\
 & & C^\bullet(f)
 \end{array}$$

Dg comodules over dg cocategories

Fix algebras A_0, \dots, A_n .

Define a dg comodule over $B(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$:

$$C(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)^\bullet(f_{0,0}, \dots, f_{n,0}) :=$$

$$:= \left\{ \begin{array}{c} \begin{array}{c} \text{Diagram: } A_0 \xrightarrow{f_{0,0}} A_1 \xrightarrow{f_{0,1}} \dots \xrightarrow{f_{n,1}} A_n \xrightarrow{f_{n,0}} A_0 \\ \text{with } f_{0,0}, f_{n,0} \text{ solid, } f_{0,1}, f_{n,1} \text{ blue, and } f_{0,k_0}, f_{n,k_n} \text{ curved.} \\ \text{Dotted arrows connect } A_1 \text{ to } A_n \text{ and } A_n \text{ to } A_1. \\ \text{A blue arrow labeled } \alpha \text{ goes from } A_0 \text{ to } A_0. \\ \text{A curved arrow labeled } id_{A_0} \text{ goes from } A_0 \text{ to } A_0. \end{array} \\ \end{array} \right\} = \left\{ \begin{array}{l} (\phi_{0,1} | \dots | \phi_{0,k_0}) \otimes \dots \otimes (\phi_{n,1} | \dots | \phi_{n,k_n}) \otimes \alpha: \\ \text{s.t. } \phi_{i,j} \in C^\bullet(A_i, f_{j-1} A_{i+1} f_j), \\ \alpha \in C_{-\bullet}(A_0, f_{n,k_n} \dots f_{0,k_0} A_0) \end{array} \right\}$$

$$d_{C(A_0 \rightarrow \dots \rightarrow A_0)} = d_B \otimes id_{C_{-\bullet}} + id_B \otimes b + \tilde{t}$$

Dg comodules over dg cocategories

where $\tilde{\iota}$ is given as follows:

$$\begin{array}{ccc} C(A_0 \rightarrow \dots \rightarrow A_0)^\bullet(f_{0,0}, \dots, f_{n,0}) & \xrightarrow{\tilde{\iota}} & C_{-\bullet}(A_0, f_{n,0} \dots f_{0,0} A_0) \\ & \searrow \tilde{\iota} & \uparrow \\ & C(A_0 \rightarrow \dots \rightarrow A_0)^\bullet(f_{0,0}, \dots, f_{n,0}) & \end{array}$$

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 & \searrow \tilde{\iota} & \uparrow \\
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 \end{array}$$

$$\tilde{\iota}((\phi_{0,1} | \dots | \phi_{0,k_0}) \otimes \dots \otimes (\phi_{n,1} | \dots | \phi_{n,k_n}) \otimes \alpha) = \iota(\phi_{0,1} | \dots | \phi_{0,k_0}) \bullet \dots \bullet (\phi_{n,1} | \dots | \phi_{n,k_n}) \alpha$$

An A_∞ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

$$\mathcal{A} := (A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$$

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Generating Morphisms:

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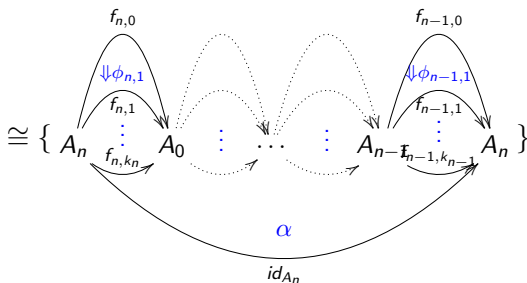
$$\delta_{j,n} \mapsto \left(\begin{array}{c} B(\mathcal{A}) \xrightarrow{\hat{\delta}_{j,n}} B(\delta_{j,n}\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{id} \hat{\delta}_{j,n}^* C(\delta_{j,n}\mathcal{A}) \end{array} \right) \quad \tau_n \mapsto \left(\begin{array}{c} B(\mathcal{A}) \xrightarrow{\hat{\tau}_n} B(\tau_n\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{\tau_n!} \hat{\tau}_n^* C(\tau_n\mathcal{A}) \end{array} \right)$$

An A_∞ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

$$\hat{\tau}_n^* C(\tau_n \mathcal{A})^\bullet(\underbrace{f_0, \dots, f_n}_{\in \text{Obj}(B(\mathcal{A}))}) \cong C(\tau_n \mathcal{A})^\bullet(f_n, f_0, \dots, f_{n-1}) \text{ as complexes.}$$

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$$\hat{\tau}_n^* C(\tau_n \mathcal{A})^\bullet(\underbrace{f_0, \dots, f_n}_{\in \text{Obj}(B(\mathcal{A}))}) \cong C(\tau_n \mathcal{A})^\bullet(f_n, f_0, \dots, f_{n-1}) \text{ as complexes.}$$



An A_∞ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

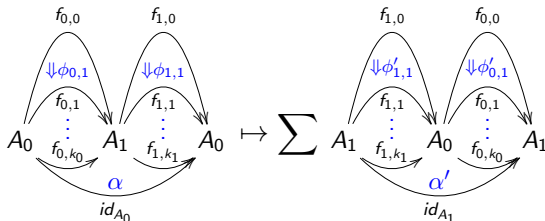
$$\hat{\tau}_n^* C(\tau_n \mathcal{A})^\bullet(\underbrace{f_0, \dots, f_n}_{\in \text{Obj}(B(\mathcal{A}))}) \cong C(\tau_n \mathcal{A})^\bullet(f_n, f_0, \dots, f_{n-1}) \text{ as complexes.}$$

$$n = 1 : \quad C(A_0 \rightarrow A_1 \rightarrow A_0)^\bullet(f, g) \xrightarrow{\tau_{1!}} C(A_1 \rightarrow A_0 \rightarrow A_1)^\bullet(g, f)$$

An A_∞ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

$$\hat{\tau}_n^* C(\tau_n \mathcal{A})^\bullet(\underbrace{f_0, \dots, f_n}_{\in \text{Obj}(B(\mathcal{A}))}) \cong C(\tau_n \mathcal{A})^\bullet(f_n, f_0, \dots, f_{n-1}) \text{ as complexes.}$$

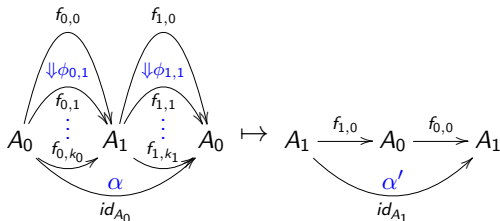
$$n = 1 : \quad C(A_0 \rightarrow A_1 \rightarrow A_0)^\bullet(f, g) \xrightarrow{\tau_{1!}} C(A_1 \rightarrow A_0 \rightarrow A_1)^\bullet(g, f)$$



An A_∞ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

$$\hat{\tau}_n^* C(\tau_n \mathcal{A})^\bullet(\underbrace{f_0, \dots, f_n}_{\in \text{Obj}(B(\mathcal{A}))}) \cong C(\tau_n \mathcal{A})^\bullet(f_n, f_0, \dots, f_{n-1}) \text{ as complexes.}$$

$$n = 1 : \quad C(A_0 \rightarrow A_1 \rightarrow A_0)^\bullet(f, g) \xrightarrow{\tau_{1!}} C(A_1 \rightarrow A_0 \rightarrow A_1)^\bullet(g, f)$$

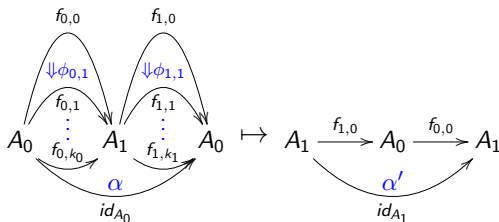


An A_∞ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

Solution: Give these maps to cogenerators to define $\tau_{1!}$, then let

$$\begin{aligned} \tau_{n!} : C(\mathcal{A}) &\cong \hat{\delta}_0^{*n-1} C(A_0 \rightarrow A_n \rightarrow A_0) \xrightarrow{\hat{\delta}_0^{*n-1} \tau_{1!}} \hat{\delta}_0^{*n-1} \hat{\tau}_1^* C(A_n \rightarrow A_0 \rightarrow A_n) \cong \\ &\cong (\widehat{\tau_1 \delta_0^{n-1}})^* C(A_n \rightarrow A_0 \rightarrow A_n) \cong (\widehat{\delta_0^{n-1} \tau_n})^* C(A_n \rightarrow A_0 \rightarrow A_n) \cong \\ &\cong \hat{\tau}_n^* \hat{\delta}_0^{*n-1} C(A_n \rightarrow A_0 \rightarrow A_n) \cong \hat{\tau}_n^* C(\tau_n \mathcal{A}). \end{aligned}$$

$$n = 1 : \quad C(A_0 \rightarrow A_1 \rightarrow A_0)^\bullet(f, g) \xrightarrow{\tau_{1!}} C(A_1 \rightarrow A_0 \rightarrow A_1)^\bullet(g, f)$$



An A_∞ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

Defining $\tau_1!$

An A_∞ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

Defining $\tau_1!$

$$\begin{array}{ccc} A_0 \xrightarrow{f_{0,0}} A_1 \xrightarrow{f_{1,0}} A_0 & \xrightarrow{?} & A_1 \xrightarrow{f_{1,0}} A_0 \xrightarrow{f_{0,0}} A_1 \\ \quad \quad \quad \alpha \quad \quad \quad \nearrow & & \quad \quad \quad \alpha' \quad \quad \quad \nearrow \\ \quad \quad \quad id_{A_0} & & \quad \quad \quad id_{A_1} \end{array}$$

An A_∞ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

Defining $\tau_1!$

$$\begin{array}{c} A_0 \xrightarrow{f_{0,0}} A_1 \xrightarrow{f_{1,0}} A_0 \xrightarrow{?} A_1 \xrightarrow{f_{1,0}} A_0 \xrightarrow{f_{0,0}} A_1 \\ \quad \quad \quad \underbrace{\hspace{1.5cm}}_{\substack{\alpha \\ id_{A_0}}} \hspace{1.5cm} \underbrace{\hspace{1.5cm}}_{\substack{\alpha' \\ id_{A_1}}} \end{array}$$

$$\alpha = a_0 \otimes \dots \otimes a_n \mapsto \alpha' = f_{0,0}(a_0) \otimes \dots \otimes f_{0,0}(a_n)$$

An A_∞ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

Defining $\tau_1!$

$$\begin{array}{c}
 \begin{array}{ccccc}
 & f_{0,0} & & & \\
 & \downarrow \phi & & & \\
 A_0 & \xrightarrow{f_{0,1}} & A_1 & \xrightarrow{f_{1,0}} & A_0 \\
 & \searrow \alpha & & & \\
 & id_{A_0} & & &
 \end{array}
 \quad \mapsto \quad
 \begin{array}{ccccc}
 & f_{1,0} & & f_{0,0} & \\
 A_1 & \xrightarrow{f_{1,0}} & A_0 & \xrightarrow{f_{0,0}} & A_1 \\
 & \searrow \alpha' & & & \\
 & id_{A_1} & & &
 \end{array}
 \end{array}$$

An A_∞ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

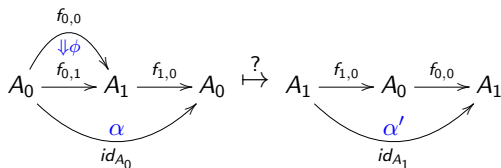
Defining $\tau_{1!}$

$$\begin{array}{c}
 \begin{array}{ccccc}
 & \overset{f_{0,0}}{\curvearrowright} & & & \\
 A_0 & \xrightarrow{f_{0,1}} & A_1 & \xrightarrow{f_{1,0}} & A_0 \\
 & \searrow \scriptstyle \alpha & \nearrow \scriptstyle id_{A_0} & & \\
 & \overset{\downarrow \phi}{\curvearrowright} & & & \\
 & \overset{f_{0,0}}{\curvearrowright} & & &
 \end{array}
 \quad \xrightarrow{?} \quad
 \begin{array}{ccccc}
 & \overset{f_{1,0}}{\curvearrowright} & & & \\
 A_1 & \xrightarrow{f_{1,0}} & A_0 & \xrightarrow{f_{0,0}} & A_1 \\
 & \searrow \scriptstyle \alpha' & \nearrow \scriptstyle id_{A_1} & &
 \end{array}
 \end{array}$$

$$\bar{\tau}_{1!} \circ d(\phi \otimes \alpha) = d \circ \bar{\tau}_{1!}(\phi \otimes \alpha)$$

An A_∞ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

Defining $\tau_{1!}$



$$[b, \bar{\tau}_{1!}](\phi \otimes \alpha) \pm \bar{\tau}_{1!}(\delta\phi \otimes \alpha) = [\bar{\tau}_{1!}, \iota_\phi](\alpha)$$

An A_∞ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

Defining $\tau_{1!}$

$$\begin{array}{c}
 \begin{array}{ccccc}
 & \overset{f_{0,0}}{\curvearrowright} & & & \\
 & \Downarrow \phi & & & \\
 A_0 & \xrightarrow{f_{0,1}} & A_1 & \xrightarrow{f_{1,0}} & A_0 \\
 & \searrow \alpha & \nearrow id_{A_0} & & \\
 & & & &
 \end{array}
 \mapsto
 \begin{array}{ccccc}
 & \overset{f_{1,0}}{\curvearrowright} & & & \\
 & \searrow \alpha' & \nearrow id_{A_1} & & \\
 A_1 & \xrightarrow{f_{1,0}} & A_0 & \xrightarrow{f_{0,0}} & A_1
 \end{array}
 \end{array}$$

$$[b, \bar{\tau}_{1!}](\phi \otimes \alpha) \pm \bar{\tau}_{1!}(\delta\phi \otimes \alpha) = [\bar{\tau}_{1!}, \iota_\phi](\alpha)$$

$$\begin{aligned}
 L_\phi(\alpha) = & \sum_{k \geq 1} \pm a_0 \otimes \dots \otimes \phi(a_k, \dots) \otimes a_r \otimes \dots \otimes a_n + \\
 & \sum \pm \phi(a_k, \dots, a_n, a_0, \dots) \otimes a_s \otimes \dots \otimes a_{k-1} \\
 [b, L_\phi] \pm L_{\delta\phi} = & 0
 \end{aligned}$$

An A_∞ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

Defining $\tau_{1!}$

$$\begin{array}{c}
 \begin{array}{ccccc}
 & f_{0,0} & & & \\
 & \downarrow \phi & & & \\
 A_0 & \xrightarrow{f_{0,1}} & A_1 & \xrightarrow{f_{1,0}} & A_0 \\
 & \searrow \alpha & \nearrow id_{A_0} & & \\
 & & & &
 \end{array}
 \mapsto
 \begin{array}{ccccc}
 & & & & \\
 A_1 & \xrightarrow{f_{1,0}} & A_0 & \xrightarrow{f_{0,0}} & A_1 \\
 & \searrow \alpha' & \nearrow id_{A_1} & &
 \end{array}
 \end{array}$$

$$[b, \bar{\tau}_{1!}](\phi \otimes \alpha) \pm \bar{\tau}_{1!}(\delta\phi \otimes \alpha) = [\bar{\tau}_{1!}, \iota_\phi](\alpha)$$

$$\begin{aligned}
 \bar{\tau}_{1!}(\phi \otimes \alpha) = & \sum_{k \geq 1} \pm f_{0,0} a_0 \otimes \dots \otimes \phi(a_k, \dots) \otimes f_{0,1} a_r \dots \otimes f_{0,1} a_n + \\
 & \sum \pm \phi(f_{1,0} f_{0,1} a_k, \dots, f_{1,0} f_{0,1} a_n, a_0, \dots) \otimes f_{0,1} a_s \otimes \dots \otimes f_{0,1} a_{k-1} \\
 [b, \bar{\tau}_{1!}(\phi, -)] \pm \bar{\tau}_{1!}(\delta\phi, -) = & [\bar{\tau}_{1!}, \iota_\phi]
 \end{aligned}$$

An A_∞ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

Defining $\tau_1!$

$$\begin{aligned}
 & \bar{\tau}_1! \left((\phi_{0,1} | \dots | \phi_{0,k_0}) \otimes (\phi_{1,1} | \dots | \phi_{1,k_1}) \otimes \alpha \right) = \\
 = & \sum_{\substack{1 \leq i \leq n_0 \\ i \leq j_1 \leq \dots \leq j_{2k_1} \leq k_0 \\ p}} \pm \phi_{0,1} \left(\begin{array}{c} f_{1,0} f_{0,i} a_p, \dots, f_{1,0} \phi_{0,i+1}(a_*, \dots), \\ f_{1,0} f_{0,i+1} a_*, \dots, f_{1,0} \phi_{0,j_1}(a_*, \dots), \\ f_{1,0} f_{0,j_1} a_*, \dots, \phi_{1,1}(f_{0,j_1} a_*, \dots, \phi_{0,j_1+1}(a_*, \dots), \dots), \dots, \\ \phi_{1,k_1}(f_{0,j_{2k_1-1}} a_*, \dots, \phi_{0,j_{2k_1-1}+1}(a_*, \dots), \dots), \dots, a_0, \dots \end{array} \right) \otimes \\
 & \otimes f_{0,1} a_* \otimes \dots \otimes \phi_{0,2}(a_*, \dots) \otimes f_{0,2} a_* \otimes \dots \otimes \\
 & \otimes \phi_{0,i}(a_*, \dots) \otimes f_{0,i} a_* \otimes \dots \otimes f_{0,i} a_{p-1} + \\
 & \left(\sum \pm f_{0,0} a_0 \otimes \dots \otimes \phi_{0,1}(a_*, \dots) \otimes \dots \otimes \phi_{0,n_0}(a_*, \dots) \otimes \right. \\
 & \quad \left. \otimes f_{0,n_0} a_* \otimes \dots \otimes f_{0,n_0} a_n \quad \text{if } k_1 = 0 \right)
 \end{aligned}$$

An A_∞ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

$$\mathcal{A} := (A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$$

$$\chi_\infty \rightarrow \mathcal{D}_\infty$$


$$\mathcal{A} \mapsto (B(\mathcal{A}), C(\mathcal{A}))$$

Generating Morphisms: $\sigma_{i,n} \mapsto \left(\begin{array}{c} B(\mathcal{A}) \xrightarrow{\hat{\sigma}_{i,n}} B(\sigma_{i,n}\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{id} \hat{\sigma}_{i,n}^* C(\sigma_{i,n}\mathcal{A}) \end{array} \right)$

$$\delta_{j,n} \mapsto \left(\begin{array}{c} B(\mathcal{A}) \xrightarrow{\hat{\delta}_{j,n}} B(\delta_{j,n}\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{id} \hat{\delta}_{j,n}^* C(\delta_{j,n}\mathcal{A}) \end{array} \right) \quad \tau_n \mapsto \left(\begin{array}{c} B(\mathcal{A}) \xrightarrow{\hat{\tau}_n} B(\tau_n\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{\tau_n! = d_0^{*n-1} \tau_1!} \hat{\tau}_n^* C(\tau_n\mathcal{A}) \end{array} \right)$$

An A_∞ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

n=1:

$$C(A_0 \rightarrow A_1 \rightarrow A_0) \xrightarrow{\tau_1!} \hat{\tau}_1^* C(\tau_1(A_0 \rightarrow A_1 \rightarrow A_0)) \xrightarrow{\hat{\tau}_1^* \tau_1!} \hat{\tau}_1^{*2} C(\tau_1^2(A_0 \rightarrow A_1 \rightarrow A_0))$$


id

An A_∞ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

n=1:

$$C(A_0 \rightarrow A_1 \rightarrow A_0) \xrightarrow{\tau_1!} \hat{\tau}_1^* C(\tau_1(A_0 \rightarrow A_1 \rightarrow A_0)) \xrightarrow{\hat{\tau}_1^* \tau_1!} \hat{\tau}_1^{*2} C(\tau_1^2(A_0 \rightarrow A_1 \rightarrow A_0))$$

$\text{---} id \text{---}$

$$A_0 \xrightarrow{f_{0,0}} A_1 \xrightarrow{f_{1,0}} A_0$$

α
 id_{A_0}

$$\alpha = a_0 \otimes \dots \otimes a_n \xrightarrow{\tau_{1!}} f_{0,0} a_0 \otimes \dots \otimes f_{0,0} a_n \xrightarrow{\hat{\tau}_{1!}^* \tau_{1!}} f_{1,0} f_{0,0} a_0 \otimes \dots \otimes f_{1,0} f_{0,0} a_n$$

An A_∞ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

n=1:

$$C(A_0 \rightarrow A_1 \rightarrow A_0) \xrightarrow{\tau_{1!}} \hat{\tau}_1^* C(\tau_1(A_0 \rightarrow A_1 \rightarrow A_0)) \xrightarrow{\hat{\tau}_1^* \tau_{1!}} \hat{\tau}_1^{*2} C(\tau_1^2(A_0 \rightarrow A_1 \rightarrow A_0))$$

$\searrow id \nearrow$

$$A_0 \xrightarrow{f_{0,0}} A_1 \xrightarrow{f_{1,0}} A_0$$

α
 id_{A_0}

$$\alpha = a_0 \otimes \dots \otimes a_n \xrightarrow{\tau_{1!}} f_{0,0}a_0 \otimes \dots \otimes f_{0,0}a_n \xrightarrow{\hat{\tau}_{1!}^* \tau_{1!}} f_{1,0}f_{0,0}a_0 \otimes \dots \otimes f_{1,0}f_{0,0}a_n$$

$$f_{1,0}f_{0,0}\alpha - \alpha = [b, B](\alpha)$$

$$B(a_0 \otimes \dots \otimes a_n) = \sum_{0 \leq i \leq n} \pm 1 \otimes f_{1,0} f_{0,0} a_i \otimes \dots \otimes f_{1,0} f_{0,0} a_n \otimes a_0 \otimes \dots \otimes a_{i-1}$$

An A_∞ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

n=1:

$$C(A_0 \rightarrow A_1 \rightarrow A_0) \xrightarrow{\tau_{1!}} \hat{\tau}_1^* C(\tau_1(A_0 \rightarrow A_1 \rightarrow A_0)) \xrightarrow{\hat{\tau}_1^* \tau_{1!}} \hat{\tau}_1^{*2} C(\tau_1^2(A_0 \rightarrow A_1 \rightarrow A_0))$$

\xrightarrow{id}

$$\begin{array}{c}
 \begin{array}{ccccc}
 & f_{0,0} & & & \\
 & \downarrow \phi & & & \\
 A_0 & \xrightarrow{f_{0,1}} & A_1 & \xrightarrow{f_{1,0}} & A_0 \\
 & \uparrow \alpha & & & \\
 & id_{A_0} & & &
 \end{array}
 \end{array} = \phi \otimes \alpha$$

$$(\overline{\hat{\tau}^* \tau_{1!} \circ \tau_{1!} - id})(\phi \otimes \alpha) = "f_{1,0} \circ L_\phi(\alpha)"$$

An A_∞ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

n=1:

$$C(A_0 \rightarrow A_1 \rightarrow A_0) \xrightarrow{\tau_1!} \hat{\tau}_1^* C(\tau_1(A_0 \rightarrow A_1 \rightarrow A_0)) \xrightarrow{\hat{\tau}_1^* \tau_1!} \hat{\tau}_1^{*2} C(\tau_1^2(A_0 \rightarrow A_1 \rightarrow A_0))$$

\xrightarrow{id}

$$\begin{aligned} & \overline{(\hat{\tau}^* \tau_{1!} \circ \tau_{1!} - id)}(\phi \otimes \alpha) = "f_{1,0} \circ L_\phi(\alpha)" \\ &= \sum_{k \geq 1} \pm f_{1,0} f_{0,0} a_0 \otimes \dots \otimes f_{1,0} \phi(a_k, \dots) \otimes f_{1,0} f_{0,1} a_r \dots \otimes f_{1,0} f_{0,1} a_n + \\ & \quad \sum \pm f_{1,0} \phi(f_{1,0} f_{0,1} a_k, \dots, f_{1,0} f_{0,1} a_n, a_0, \dots) \otimes f_{1,0} f_{0,1} a_s \otimes \dots \otimes f_{1,0} f_{0,1} a_{k-1} \\ & \stackrel{?}{=} [\iota_\phi, B](\alpha) \pm [b, B](\phi \otimes \alpha) \pm B(\delta \phi \otimes \alpha) \end{aligned}$$

An A_∞ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

n=1:

$$C(A_0 \rightarrow A_1 \rightarrow A_0) \xrightarrow{\tau_{1!}} \hat{\tau}_1^* C(\tau_1(A_0 \rightarrow A_1 \rightarrow A_0)) \xrightarrow{\hat{\tau}_1^* \tau_{1!}} \hat{\tau}_1^{*2} C(\tau_1^2(A_0 \rightarrow A_1 \rightarrow A_0))$$

$\searrow \quad \nearrow$
 id

$$\begin{aligned} & \overline{(\hat{\tau}^* \tau_{1!} \circ \tau_{1!} - id)}(\phi \otimes \alpha) = "f_{1,0} \circ L_\phi(\alpha)" \\ &= \sum_{k \geq 1} \pm f_{1,0} f_{0,0} a_0 \otimes \dots \otimes f_{1,0} \phi(a_k, \dots) \otimes f_{1,0} f_{0,1} a_r \dots \otimes f_{1,0} f_{0,1} a_n + \\ & \quad \sum \pm f_{1,0} \phi(f_{1,0} f_{0,1} a_k, \dots, f_{1,0} f_{0,1} a_n, a_0, \dots) \otimes f_{1,0} f_{0,1} a_s \otimes \dots \otimes f_{1,0} f_{0,1} a_{k-1} \\ & \stackrel{?}{=} [\iota_\phi, B](\alpha) \pm [b, B](\phi \otimes \alpha) \pm B(\delta \phi \otimes \alpha) \end{aligned}$$

$$B(\phi \otimes \alpha) = \sum_{0 \leq j < n} \pm 1 \otimes f_{1,0} f_{0,0} a_i \otimes \dots \otimes f_{1,0} \phi(a_*, \dots) \otimes f_{1,0} f_{0,1} a_* \otimes \dots \otimes f_{1,0} f_{0,1} a_n \otimes a_0 \otimes \dots \otimes a_{i-1}$$

An A_∞ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

n=1:

$$\begin{aligned}
 & B((\phi_{0,1}|\dots|\phi_{0,k_0}) \otimes (\phi_{1,1}|\dots|\phi_{1,k_1}) \otimes \alpha) = \\
 = & \sum_{\substack{0 \leq j_1 \leq \dots \leq j_{2k_1} \leq k_0 \\ p}} \pm 1 \otimes f_{1,0} f_{0,0} a_p \otimes \dots \otimes f_{1,0} \phi_{0,1}(a_*, \dots) \otimes \\
 & \otimes f_{1,0} f_{0,1} a_* \otimes \dots \otimes f_{1,0} \phi_{0,j_1}(a_*, \dots) \otimes \\
 & \otimes f_{1,0} f_{0,j_1} a_* \otimes \dots \otimes \phi_{1,1}(f_{0,j_1} a_*, \dots, \phi_{0,j_1+1}(a_*, \dots), \dots) \otimes \\
 & \otimes \dots \otimes \phi_{1,k_1}(f_{0,j_{2k_1-1}} a_*, \dots, \phi_{0,j_{2k_1-1}+1}(a_*, \dots), \dots) \otimes \dots \otimes \\
 & \otimes a_0 \otimes \dots \otimes a_{p-1}
 \end{aligned}$$

An A_∞ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

$$\chi_\infty \rightarrow \mathcal{D}_\infty$$

$$\mathcal{A} \mapsto (B(\mathcal{A}), C(\mathcal{A}))$$

$$\tau_1 \mapsto \left(\begin{array}{l} \hat{\tau}_1 : B(\mathcal{A}) \rightarrow B(\tau_1 \mathcal{A}) \\ \tau_{1!} : C(\mathcal{A}) \rightarrow \hat{\tau}_1^* C(\tau_1 \mathcal{A}) \end{array} \right)$$

$$(\tau_1, \tau_1) \mapsto \left(\begin{array}{l} id : B(\mathcal{A}) \rightarrow B(\mathcal{A}) \\ B : C(\mathcal{A}) \rightarrow C(\mathcal{A}) \end{array} \right)$$

$$(\tau_1, \tau_1, \tau_1) \mapsto \left(\begin{array}{l} \hat{\tau}_1 : B(\mathcal{A}) \rightarrow B(\tau_1 \mathcal{A}) \\ 0 : C(\mathcal{A}) \rightarrow \hat{\tau}_1^* C(\tau_1 \mathcal{A}) \end{array} \right)$$

$$\vdots$$

An A_∞ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

The A_∞ relations mean:

- $\tau_1!$ is a map of complexes
- $B^2 = 0$
- The following diagram commutes:

$$\begin{array}{ccc} C(\mathcal{A}) & \xrightarrow{\tau_1!} & \hat{\tau}_1^* C(\tau_1 \mathcal{A}) \\ \downarrow B & & \downarrow \hat{\tau}_1^* B \\ C(\mathcal{A}) & \xrightarrow{\tau_1!} & \hat{\tau}_1^* C(\tau_1 \mathcal{A}) \end{array}$$

An A_∞ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

For higher $n > 1$, we want to find a homotopy between “ τ_n^{n+1} ” and id .

An A_∞ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

For higher $n > 1$, we want to find a homotopy between “ τ_n^{n+1} ” and id . However, it is sufficient to find a homotopy between $\hat{\tau}_n^{*2} \delta_{0,n}! \circ \hat{\tau}_n^* \tau_n! \circ \tau_n!$ and $\hat{\delta}_{n-1,n}^* \tau_{n-1}! \circ \delta_{n-1,n}!$.

$$\begin{array}{ccccc} [n] & \xrightarrow{\tau_n} & [n] & \xrightarrow{\tau_n} & [n] \\ \downarrow \delta_{n-1,n} & & & \swarrow \delta_{0,n} & \\ [n-1] & \xrightarrow{\tau_{n-1}} & [n-1] & & \end{array}$$

An A_∞ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

For higher $n > 1$, we want to find a homotopy between “ τ_n^{n+1} ” and id . However, it is sufficient to find a homotopy between $\hat{\tau}_n^{*2} \delta_{0,n}! \circ \hat{\tau}_n^* \tau_n! \circ \tau_n!$ and $\hat{\delta}_{n-1,n}^* \tau_{n-1}! \circ \delta_{n-1,n}!$.

$$\begin{array}{ccccc}
 [n] & \xrightarrow{\tau_n} & [n] & \xrightarrow{\tau_n} & [n] \\
 \downarrow \delta_{n-1,n} & & & \swarrow \delta_{0,n} & \\
 [n-1] & \xrightarrow{\tau_{n-1}} & [n-1] & &
 \end{array}$$

Solution: Find such a homotopy, \mathcal{B} , for $n = 2$, and use $\hat{\delta}_0^{*n-2} \mathcal{B}$ for $n > 2$.

An A_∞ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

For higher n , the A_∞ relations mean:

- $\tau_{n!}$ is a map of complexes
- $\mathcal{B}^2 = 0$
- We have a pair of homotopic maps:

$$\begin{array}{ccc}
 C(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0) & & \\
 \hat{\tau}_n^{*2} \tau_{n!} \circ \hat{\tau}_n^* \tau_{n!} \circ \tau_{n!} & \begin{array}{c} \curvearrowright \\ \downarrow \end{array} & (\widehat{\delta_{n-2, n-1} \delta_{n-1, n}})^* \tau_{n-2!} \\
 \text{"apply } \tau_{n!} \text{ 3 times"} & & \text{"brace together the last 3 algebras,} \\
 & & \text{then apply } \tau_{n-2!} \text{ once"} \\
 C(A_{n-2} \rightarrow A_{n-1} \rightarrow A_n \rightarrow A_0 \rightarrow \dots \rightarrow A_{n-2}). & &
 \end{array}$$

An A_∞ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

For higher n , the A_∞ relations mean:

- $\tau_{n!}$ is a map of complexes
- $\mathcal{B}^2 = 0$
- They are homotopic via two different homotopies:

$$\begin{array}{ccc}
 (\widehat{\delta_{n-2,n-1} \delta_{n-1,n}})^* \tau_{n-2!} & \xrightarrow{\cong} & \hat{\delta}_{n-1,n}^* (\hat{\delta}_{n-2,n-1}^* \tau_{n-2!}) \xrightarrow{\hat{\delta}_{n-1,n}^* \mathcal{B}_{n-1}} \hat{\delta}_{n-1,n}^* (\hat{\tau}_{n-1}^* \tau_{n-1!} \circ \tau_{n-1!}) \\
 \text{"brace together } A_{n-2}, A_{n-1}, A_n, & & \\
 \text{then apply } \tau_{n-2!}" & & \\
 \downarrow \cong & & \downarrow \cong \\
 (\widehat{\delta_{n-2,n-1} \delta_{n-2,n}})^* \tau_{n-2!} & & \hat{\tau}_n^{*2} \tau_{n!} \circ \hat{\delta}_{n-1,n}^* \tau_{n-1!} \\
 \downarrow \hat{\delta}_{n-2,n}^* \mathcal{B}_{n-1} & & \text{"brace together } A_{n-1}, A_n \\
 & & \text{and apply } \tau_{n-1!}, \\
 & & \text{then apply } \tau_{n!}" \\
 \hat{\delta}_{n-2,n}^* (\hat{\tau}_{n-1}^* \tau_{n-1!} \circ \tau_{n-1!}) & \xrightarrow{\cong} & \hat{\tau}_n^* (\hat{\delta}_{n-1,n}^* \tau_{n-1!}) \circ \tau_{n!} \xrightarrow{\hat{\tau}_n^* \mathcal{B}_n \circ \tau_{n!}} \hat{\tau}_n^{*2} \tau_{n!} \circ \hat{\tau}_n^* \tau_{n!} \circ \tau_{n!} \\
 & \text{"apply } \tau_{n!}, & \text{"apply } \tau_{n!} \text{ three times"} \\
 & \text{then brace together } A_{n-1}, A_{n-2} & \\
 & \text{and apply } \tau_{n-1!}" &
 \end{array}$$

Summary:

We have a given a “homotopically sheafy-cyclic object in dg cocategories with a dg comodule”, i.e., an A_∞ -functor between dg categories χ_∞ and \mathcal{D}_∞ .

Time and interest permitting

- Rectify \mathcal{F} to a dg functor
- Give a dg functor $\mathcal{D}_\infty \rightarrow \mathcal{E} = \{(\text{dg cat}, \text{dg mod})\}$

$$U(\chi_\infty) \xrightarrow{\text{rectified}} \mathcal{D}_\infty \rightarrow \mathcal{E}$$

“A homotopically sheafy-cyclic object in dg categories with a dg module”