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What Do Algebras Form?

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ABSTRACT

What Do Algebras Form?

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Algebras and their bimodules form a 2-category in which 2-morphisms are certain zero-th Hochschild cohomology groups. When we derive this structure (i.e., use Hochschild cochains instead of HH^0 for 2-morphisms), we find that algebras form a category in dg cocategories. The Hochschild-Kostant-Rosenberg theorem and non-commutative calculus give a rich algebraic structure on Hochschild cohomology along with Hochschild homology. When incorporating the structure on Hochschild homology, we find that algebras form a 2-category with a trace functor. Deriving this again, we conclude that algebras form a category in dg cocategories with a trace functor up to homotopy.

Acknowledgements

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Nomenclature

k – a fixed ground field of char 0

$k - mods$ – the category of modules over k

1 – the unit in (a vector space isomorphic to) k

$[1]$ – shift operator on complexes, $C^\bullet[1] = C^{\bullet+1}$

Λ – Connes cyclic category, see Appendix A

$\Delta(b) = \sum_{(b)} b_{(1)} \otimes b_{(2)}$ – Sweedler notation for coproducts

${}_fB_g$ – B as an A-C-bimodule with left structure given by the map of algebras $f : A \to B$ and right structure given by the map of algebras $g : C \to B$

${}_fB :={}_f B_{id_B}$

CHAPTER 1

Introduction

What do algebras (over a fixed field k of characteristic zero) form? A straight-forward answer is that they form a 2-category as follows:

Objects: k-algebras $A, B, \ldots$

1-Morphisms: bimodules ${}_AM_B$

1-Composition: ${}_AM_B \otimes_B {}_BN_C$

2-Morphisms: morphisms of bimodules.

When we restrict the above 1-morphisms to only those bimodules that come from maps of algebras (i.e., bimodules ${}_AM_B$ where ${}_AM_B =_{f(A)} B_B =:_f B$ for some map of algebras $f : A \to B$), then 2-morphisms have an additional structure, namely they are certain zero-th Hochschild cohomology groups:

$$\{\text{morphisms of bimodules } {}_fB \to_g B\} \overset{1:1}{\leftrightarrow} Z_A({}_gB_f) \cong HH^0(A, {}_gB_f)$$

$$M \mapsto M(1)$$

$$(M_b : b' \mapsto b \cdot b') \leftarrow\!\shortmid b$$

In summary, we have the following 2-category $\underline{C}$:

$$
(1.1)\quad
\begin{aligned}
&\text{Objects: } k\text{-algebras } A, B, \ldots\\
&\text{1-Morphisms: bimodules } {}_fB,\ f: A \to B \text{ map of algebras}\\
&\text{1-Composition: } {}_fB \otimes_B {}_gC,\ A \xrightarrow{f} B \xrightarrow{g} C\\
&\text{2-Morphisms: } HH^0(A, {}_fB_g) \cong Z_A({}_fB_g)
\end{aligned}
$$

The question naturally arises: what happens if we use Hochschild cohomology or cochains instead of just HH^0 for 2-morphisms? The answer is that algebras form a category, $\mathcal{C}$, in dg categories as follows:

$$
(1.2)\quad
\begin{aligned}
&\text{Objects: } k\text{-algebras } A, B, \ldots\\
&\text{Morphisms: dg cocategory } Bar(Hoch(A, B)\\
&\text{Composition: } \bullet : Bar(Hoch(A, B)) \otimes Bar(Hoch(B, C)) \to Bar(Hoch(A, C))\\
&\qquad\qquad\text{associative map of dg cocategories}
\end{aligned}
$$

$Bar(Hoch(A, B))$ is the cofree dg cocategory defined in Section 2.2 and uses Hochschild cochains as morphisms. The composition, $\bullet$, is defined in Section 2.3 and uses the brace operator on Hochschild cochains (Reference [**6**], Equation 4.8). The fact that $\bullet$ is associative follows from [**?, ?**].

Thus far, we have used Hochschild cochains to show that algebras form a category in dg cocategories. Non-commutative calculus tells us that the pair, (Hochschild cochains $C^\bullet(A, A)$, Hochschild chains $C_{-\bullet}(A, A)$), is a $Calc_\infty$-algebra (Reference [**1**], Corollary 4). In other words, Hochschild cochains is a Gerstenhaber$_\infty$-algebra and acts on Hochschild chains up to homotopy via (1) an analogue of the Lie derivative, and (2) an analogue of the contraction of a form against a vector field.

Taking advantage of this $Calc_\infty$ structure, we incorporate HH_0 and find that algebras form a 2-category with a trace functor. In Section 3.2, we give the definition of a trace functor on a 2-category à la Kaledin, and in Section **??**, we describe a trace functor on $\underline{C}$ (the 2-category given in Equation 1.1) that uses the action of HH^0 on HH_0.

Again, we ask: can we derive this structure? Can we use Hochschild homology or chains instead of HH_0 to get a trace functor on $\mathcal{C}$ (the category given in Equation 1.2)? We give the definition of a trace functor on a category in dg cocategories in Section 3.3. Ultimately, we settle on the following language: on $\mathcal{C}'$, a category in dg cocategories, a trace functor gives a dg functor $\chi(\mathcal{C}') \to \mathcal{D}$ where $\chi(\mathcal{C}')$ and $\mathcal{D}$ are dg categories introduced in Section 4.2.

For our category $\mathcal{C}$, we are not able to give a dg functor $\chi(\mathcal{C}) \to \mathcal{D}$, however, we do give an A_∞-functor (Section **??**). This is the (precise) sense in which we have a trace functor "up to homotopy" (see Definition 4.2.3). Finally, we apply $Cobar(-)$ functor to everything to get a category in dg *categories* with a trace functor up to homotopy (Chapter 6); however, understanding of all of the structures after applying $Cobar(-)$ is still evolving.

CHAPTER 2

A category in dg cocategories

2.1. Motivation of this chapter

In this chapter, we show that algebras form a category in dg cocategories. As stated in the introduction, we will construct such a category with

Objects: k-algebras $A, B, \ldots$

Morphisms: dg cocategory $Bar(Hoch(A, B)$

Composition: $\bullet : Bar(Hoch(A, B)) \otimes Bar(Hoch(B, C)) \to Bar(Hoch(A, C))$

associative map of dg cocategories.

First, we define the dg cocategories $Bar(Hoch(A, B)$ using Hochschild cochains as morphisms, then we define the composition $\bullet$ using the brace operator on Hochschild cochains.

2.2. Dg cocategories $Bar(Hoch(A, B))$

Let A, B be k-algebras. We define a dg category, $Hoch(A, B)$, as follows:

Objects: algebra maps $f : A \to B$

Morphisms: $Hoch(A)(f, g) = (C^{\bullet}(A,\ {}_fB_g),\ {}_f\delta_g)$

Composition: cup product on cochains.

(See Appendix B for notation and standard operations on Hochschild complexes.) The cup product is an associative map of complexes, so $Hoch(A, B)$ is a dg category.

Now, we will take $Bar(-)$ of $Hoch(A, B)$, which is a categorified bar construction:

$$Bar : DGCat \to DGCocat.$$

$Bar(Hoch(A, B))$ has the same objects as $Hoch(A, B)$. A morphism in $Bar(Hoch(A, B))$ from object f_0 to object f_n is a sequence of composable morphisms in $Hoch(A, B)$ starting at f_0 and ending at f_n. We can picture such a morphism as follows:

$(\phi_1 \dots \phi_n) =$

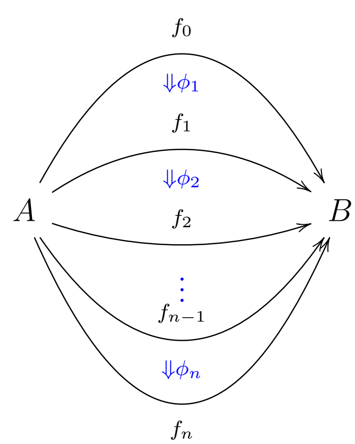

Figure 2.1. A morphism in $Bar(Hoch(A, B))(f_0, f_n)$

where $\phi_i \in C^\bullet(A,_{f_{i-1}} B_{f_i})$. As a complex,

$$Bar(Hoch(A,B))^\bullet(f,g) =$$

$$= \underbrace{k[0]}_{\text{counit}} \oplus \bigoplus_{\substack{n\geq 0,\\ f_i \in Obj(Hoch(A,B))}} Hoch(A,B)^\bullet[1](f,f_1)\otimes Hoch(A,B)^\bullet[1](f_1,f_2)\otimes\cdots\otimes Hoch(A,B)^\bullet[1](f_n,g)$$

$$d_{Bar(Hoch(A,B))} = \tilde{d}_{Hoch(A,B)} + d_\cup$$

$\tilde{d}_{Hoch(A,B)} =$ extension of $d_{Hoch(A,B)}$ to a differential on Bar

$d_\cup =$ signed sum over composing (cup-producting) two consecutive ϕ_i's

with cocomposition

$$\Delta(\phi_1 \dots \phi_n) = \sum_{0\leq i\leq n} \pm(\phi_1 \dots \phi_i) \otimes (\phi_{i+1} \dots \phi_n).$$

For more precise details and explicit signs, see Reference [**6**], Section 4.6.

2.3. Associative Composition $\bullet$

Now, we define an associative composition of dg cocategories

$$Bar(Hoch(A,B)) \otimes Bar(Hoch(B,C)) \to Bar(Hoch(A,C))$$

where A, B, C are k-algebras. To define the composition, we use the fact that $Bar(Hoch(A,C))$ is the cofree dg cocategory over $Hoch(A,C)$. In other words, $Bar(Hoch(A,C))$ satisfies the following universal property:

$$\begin{tikzcd} \mathcal{B} \arrow[r] \arrow[dr, dotted] & Hoch(A,C) \\ & Bar(Hoch(A,C)) \arrow[u, two heads] \end{tikzcd}$$

Figure 2.2. Universal Property of Bar

where $\mathcal{B}$ is any dg cocategory, the horizontal map is a map of underlying structure (i.e., an association on objects and maps of complexes of morphisms), and the diagonal lift arrow is a map of dg cocategories. For us, $\mathcal{B} = Bar(Hoch(A,B)) \otimes Bar(Hoch(B,C))$. We will define a map of underlying structure $Bar(Hoch(A,B)) \otimes Bar(Hoch(B,C)) \to Hoch(A,C)$, which will lift to the map of dg cocategories

$$\bullet : Bar(Hoch(A,B)) \otimes Bar(Hoch(B,C)) \to Bar(Hoch(A,C)).$$

The map on underlying structure is defined as follows:

$$Bar(Hoch(A,B)) \otimes Bar(Hoch(B,C)) \to Hoch(A,C)$$

On objects: $f \otimes g \mapsto g \circ f$

$$\text{On morphisms:} \quad \begin{tikzcd}[column sep=huge]
A \arrow[r, bend left=80, "f_0 \in Obj(Bar(Hoch(A,B)))"{name=F0}] \arrow[r, bend left=40, "f_1"{name=F1, description}] \arrow[r, bend left=10, "f_2"{name=F2, description}] \arrow[r, bend right=40, "f_{n-1}"{name=Fn1, description}] \arrow[r, bend right=80, "f_n \in Obj(Bar(Hoch(A,B)))"'{name=Fn}] \arrow[Rightarrow, from=F0, to=F1, phantom, "\Downarrow\phi_1"] \arrow[from=F1, to=F2, phantom, "\Downarrow\phi_2"] \arrow[from=F2, to=Fn1, phantom, "\vdots"] \arrow[from=Fn1, to=Fn, phantom, "\Downarrow\phi_n"] & B \arrow[r, bend left, "g_0 \in Obj(Bar(Hoch(B,C)))"{name=G0}] \arrow[r, bend right, "g_1 \in Obj(Bar(Hoch(B,C)))"'{name=G1}] \arrow[from=G0, to=G1, phantom, "\Downarrow\psi"] & C & \mapsto & A \arrow[r, bend left, "g_0 f_0 \in Obj(Hoch(A,C))"{name=H0}] \arrow[r, bend right, "g_1 f_n \in Obj(Hoch(A,C))"'{name=H1}] \arrow[from=H0, to=H1, phantom, "\psi\{\phi_1,\ldots,\phi_n\} \Downarrow"] & C
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, bend left, "f_0"{name=a}] \arrow[r, bend right, "f_n"'{name=b}] \arrow[from=a, to=b, phantom, "\Downarrow\phi"] & B \arrow[r, bend left, "g_0"{name=c}] \arrow[r, bend right, "g_1"'{name=d}] \arrow[from=c, to=d, phantom, "\Downarrow 1 \in k"] & C & \mapsto & A \arrow[r, bend left, "g_0 f_0"{name=e}] \arrow[r, bend right, "g_1 f_n"'{name=f}] \arrow[from=e, to=f, phantom, "\Downarrow\phi"] & C
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, bend left, "f_0"{name=a}] \arrow[r, bend right, "f_n"'{name=b}] \arrow[from=a, to=b, phantom, "\Downarrow 1 \in k"] & B \arrow[r, bend left, "g_0"{name=c}] \arrow[r, bend right, "g_1"'{name=d}] \arrow[from=c, to=d, phantom, "\Downarrow\psi"] & C & \mapsto & A \arrow[r, bend left, "g_0 f_0"{name=e}] \arrow[r, bend right, "g_1 f_n"'{name=f}] \arrow[from=e, to=f, phantom, "\Downarrow\psi"] & C
\end{tikzcd}$$

All other non-pictured pairings of a morphism from $Bar(Hoch(A,B))$ and a morphism from $Bar(Hoch(B,C))$ map to zero. The brace operation is given in Reference [**6**], Equation 4.8, and the fact that it is associative follows from References [**?, ?, ?, ?**].

CHAPTER 3

A 2-category with a trace functor

3.1. Motivation of this chapter

In this chapter, we give a trace functor on $\underline{C}$, the 2-category introduced in Equation 1.1. This trace functor enriches the categorical structure on algebras by incorporating the action on Hochschild cohomology (HH^0) on Hochschild homology (HH_0). We start with Kaledin's definition of a trace functor on a 2-category.

In preparation of Section **??**, we generalize Kaledin's definition to a trace functor on a category in dg cocategories in Section .

3.2. A trace on C

Definition 3.2.1. (Kaledin): A trace functor on a 2-category $\underline{C}$ is:

- for each $A \in Obj(\underline{C})$, a functor $TR_A : \underline{C}(A,A) \to k-mod$
- for each pair $A, B \in Obj(\underline{C})$, a natural transformation $\tau_!(A,B)$:

$$\begin{tikzcd}
\underline{C}(A,B)\otimes\underline{C}(B,A) \arrow[rr, "\tau=flip"] \arrow[d, "m"'] & & \underline{C}(B,A)\otimes\underline{C}(A,B) \arrow[d, "m"'] \\
\underline{C}(A,A) \arrow[dr, "TR_A"'] & \tau_!(\overrightarrow{A,B}) & \underline{C}(B,B) \arrow[dl, "TR_B"] \\
& k-mod &
\end{tikzcd}$$

such that, for $A, B, C \in Obj(\underline{C})$,

$$\tau_!(B,A)\circ\tau_!(C,B)\circ\tau_!(A,C) = id.$$

$$\begin{tikzcd}
& & \underline{C}(C,A)\otimes\underline{C}(A,C)\otimes\underline{C}(B,C) \arrow[drr, "\tau"] \arrow[dd, "TR_C\circ m^2"] & & \\
\underline{C}(A,B)\otimes\underline{C}(B,C)\otimes\underline{C}(C,A) \arrow[urr, "\tau"] \arrow[drr, "TR_A\circ m^2"'] & \tau_!(\overrightarrow{A,C}) & \tau_!(\overleftarrow{B,A}) & \tau_!(\overrightarrow{C,B}) & \underline{C}(B,C)\otimes\underline{C}(C,A)\otimes\underline{C}(A,B) \arrow[llll, bend right, "\tau"'] \arrow[dll, "TR_B\circ m^2"] \\
& & k-mod & &
\end{tikzcd}$$

Now, we will give a trace functor on the 2-category, $\underline{C}$, define in Equation 1.1. Let $A \in Obj(\underline{C})$ be an algebra and $f : A \to A$ a map of algebras. Then, we set

$$TR_A({}_fA) := \frac{A}{[A,_f A]} = \frac{A}{(f(a)\cdot a' - a'\cdot a)}.$$

And for morphisms,

$$\underline{C}(A,A)(f,g) \otimes \frac{A}{[A,_g A]} \cong Z_A({}_fA_g) \otimes \frac{A}{[A,_g A]} \to \frac{A}{[A,_f A]}$$

$$b \otimes a \mapsto b \cdot a$$

is a well-defined map on k-modules. For algebra maps $f, f' : A \leftrightarrows B : g, g'$, we define the natural transformation $\tau_!(A,B)$ as follows:

$$\begin{tikzcd}
{}_fB \underset{B}{\otimes} {}_gA/[A,_f B \underset{B}{\otimes} {}_gA] \arrow[rrr, "\tau_!(A,B)(f,g)"] \arrow[ddd] & & & {}_gA \underset{A}{\otimes} {}_fB/[B,_g A \underset{A}{\otimes} {}_fB] \arrow[ddd] \\
& {[b \otimes a]} \arrow[r, mapsto] \arrow[d, mapsto, "(b'\cdot, a'\cdot)"'] & {[a \otimes b]} \arrow[d, mapsto, "(a'\cdot, b'\cdot)"] & \\
& {[b' \cdot b \otimes a' \cdot a]} \arrow[r, mapsto] & {[a' \cdot a \otimes b' \cdot b]} & \\
{}_{f'}B \underset{B}{\otimes} {}_{g'}A/[A,_{f'} B \underset{B}{\otimes} {}_{g'}A] \arrow[rrr, "\tau_!(A,B)(f',g')"'] & & & {}_{g'}A \underset{A}{\otimes} {}_{f'}B/[B,_{g'} A \underset{A}{\otimes} {}_{f'}B]
\end{tikzcd}$$

where $b' \in Z_A({}_{f'}B_f)$, $a' \in Z_B({}_{g'}A_g)$, $a \in A$, $b \in B$. Clearly, this flip map $\tau_!$ satisfies Equation 3.1.

3.3. Redefining the trace functor

In this section, we generalize Kaledin's definition of a trace functor on a 2-category to a trace functor on dg cocategories. First, we transform the definition from the language from functors and natural transformations to the language of modules.

Definition 3.3.1. Let $\mathcal{C}$ be a k-linear category. A left module over $\mathcal{C}$ is a k-linear functor $\mathcal{C} \to k - mods$.

Given the definition above, we can rewrite the definition of a trace functor on a 2-category in the language of modules.

Definition 3.3.2. (Kaledin, reformulated): Let $\mathcal{C}$ be a category in k-linear categories. A trace functor on $\mathcal{C}$ is:

- for each $A \in Obj(\mathcal{C})$, a left module $T(A)$ over $\mathcal{C}(A, A)$
- for each pair $A, B \in Obj(\mathcal{C})$, a map of modules over $\mathcal{C}(A, B) \otimes \mathcal{C}(B, A)$

  $$\tau_!(A, B) : m^*_{ABA}T(A) \to \tau^* m^*_{BAB}T(B)$$

  where m_{ABA} is the composition functor $m_{ABA} : \mathcal{C}(A, B) \otimes \mathcal{C}(B, A) \to \mathcal{C}(A, A)$, τ is a flip functor, and pulling back along a functor means pre-composition.
- for $A, B, C \in Obj(\mathcal{C})$,

  $$\tau^{*2}\tau_!(B, A) \circ \tau^*\tau_!(C, B) \circ \tau_!(A, C) = id.$$

Now, we will translate from modules to dg comodules. Reversing the arrows in Definition 3.3.1, we have the following definition for a dg comodule over a category in dg cocategories.

Definition 3.3.3. Let $\mathcal{C}$ be a dg cocategory. A dg comodule over $\mathcal{C}$ is: for each $f \in Obj(\mathcal{C})$, a complex $T^\bullet(f)$ and map of complexes

$$\Delta_f : T^\bullet(f) \to \prod_{g \in Obj(\mathcal{C})} \mathcal{C}^\bullet(f,g) \otimes T^\bullet(g)$$

such that the following two maps coincide (coassociativity):

$$\begin{tikzcd}
T^\bullet(f) \arrow[d, "\Delta(f)"'] \\
\prod\limits_{g \in Obj(\mathcal{C})} \mathcal{C}^\bullet(f,g) \otimes T^\bullet(g) \arrow[d, bend right, "\Delta_{\mathcal{C}(}\otimes id"'] \arrow[d, bend left, "id\otimes\Delta(g)"] \\
\prod\limits_{g,g' \in Obj(\mathcal{C})} \mathcal{C}^\bullet(f,g) \otimes \mathcal{C}^\bullet(g,g') \otimes T^\bullet(g')
\end{tikzcd}$$

and the following diagram commutes (counitality):

$$\begin{tikzcd}
T^\bullet(f) \arrow[r, "\Delta(f)"] \arrow[dr, "id"'] & \prod\limits_{g \in Obj(\mathcal{C})} \mathcal{C}^\bullet(f,g) \otimes T^\bullet(g) \arrow[d, "\epsilon_{\mathcal{C}}\otimes id"] \\
& T^\bullet(f).
\end{tikzcd}$$

Finally, we can rewrite Definition 3.3.2 in terms of dg comodules.

Definition 3.3.4. Let $\mathcal{C}$ be a category in dg cocategories. A trace functor on $\mathcal{C}$ is:

- for each $A \in Obj(\mathcal{C})$, a dg comodule $T(A)$ over $\mathcal{C}(A, A)$
- for each pair $A, B \in Obj(\mathcal{C})$, a map of dg comodules over $\mathcal{C}(A, B) \otimes \mathcal{C}(B, A)$

$$\tau_!(A, B) : m^*_{ABA}T(A) \to \tau^* m^*_{BAB}T(B)$$

  where m_{ABA} is the composition functor $m_{ABA} : \mathcal{C}(A, B) \otimes \mathcal{C}(B, A) \to \mathcal{C}(A, A)$, τ is a flip functor. We can take any definition for the pullback that is a natural and satisifies

$$F^*G^* = (GF)^*.$$

- for $A, B, C \in Obj(\mathcal{C})$,

$$\tau^{*2}\tau_!(B, A) \circ \tau^*\tau_!(C, B) \circ \tau_!(A, C) = id. \tag{3.1}$$

CHAPTER 4

Interlude

4.1. Motivation of this chapter

The purpose of this chapter is to show that a trace functor T on a category $\mathcal{C}$ in dg cocategories gives a dg functor $\mathcal{F}_T : \chi(\mathcal{C}) \to \mathcal{D}$ where $\chi(\mathcal{C})$ and $\mathcal{D}$ are dg categories introduced in Defintions 4.2.1 and 4.2.2, respectively. We switch from the trace functor T to the dg functor $\mathcal{F}_T$ so that we can make precise the notion of a "trace functor up to homotopy". Namely, a trace functor on $\mathcal{C}$ up to homotopy is an A_∞-functor from $\chi(\mathcal{C})$ to $\mathcal{D}$ (see Definition 4.2.3). We will then give such an A_∞-functor for $\mathcal{C}$ being the category given in Equation 1.2 (see Section **??**).

4.2. From a trace functor to a dg functor

We begin this section by defining two dg categories.

Definition 4.2.1. Let $\mathcal{C}$ be a category in dg cocategories. Let $\chi(\mathcal{C})$ be the dg category with

- Objects $= \{A_0 \to \cdots \to A_n \to A_0 : A_i \in Obj(\mathcal{C}), n \geq 0\}$
- Morphisms $= \{$linear combinations of compositions of

$$\text{rotations } \tau_n : \mathcal{A} \mapsto (A_n \to A_0 \to \cdots \to A_n)$$

$$\text{coboundaries } \delta_{j,n} : \mathcal{A} \mapsto (A_0 \to \cdots \to A_j \to A_{j+2\,(mod\, n+1)} \to \cdots \to A_0)$$

$$\text{codegeneracies: } \sigma_{i,n} : \mathcal{A} \mapsto (A_0 \to \cdots \to A_i \to A_i \to \cdots \to A_0)$$

where $\mathcal{A} := (A_0 \to \cdots \to A_n \to A_0)$, subject to the cyclic relations in Appendix $\}[0]$

Definition 4.2.2. Let $\mathcal{D}$ be the dg category with

- Objects $= \{(\underset{B}{\text{dg cocategory}}, \underset{C}{\text{dg comodule}})\}$
- Morphisms:

$$\mathcal{D}^p\big((B_1, C_1), (B_0, C_0)\big) := \left\{ \begin{matrix} F : B_1 \to B_0 \text{ dg functor,} \\ F_! : C_1 \to F^*C_0 \text{ degree-p linear map} \end{matrix} \right\}$$

$$d_{\mathcal{D}}(F, F_!) = (F, [d, F_!] = d_{F^*C_0} \circ F_! \pm F_! \circ d_{C_1})$$

- Composition: $(G, G_!) \underset{D}{\circ} (F, F_!) = (GF, F^*G_! \circ F_!)$

Composition in $\mathcal{D}$ will be well-defined and associative for any choice of a natural pullback that satisfies

$$F^*G^* \cong (GF)^*. \tag{4.1}$$

For consistency, we will choose the same pullback of dg comodules for Definitions 3.3.4 and 4.2.2. (See Section **??** for an explicit description of the pullback we've chosen for dg comodules over the endomorphism dg cocategories given in Equation 1.2.)

Now, let $\mathcal{C}$ be a category in dg cocategories and T be a trace functor on $\mathcal{C}$ (Definition 3.3.4). We will show that T gives a dg functor $\mathcal{F}_T : \chi(\mathcal{C}) \to \mathcal{D}$. On objects,

$$\underbrace{(A_0 \to \cdots \to A_n \to A_0)}_{\in Obj(\chi(\mathcal{C}))} \underset{\mathcal{F}_T}{\mapsto} \begin{pmatrix} \mathcal{C}(A_0, A_1) \otimes \cdots \otimes \mathcal{C}(A_n, A_0) \text{ dg cocategory,} \\ m^{*n}T(A_0) \text{ dg comodule where} \\ m^n : \mathcal{C}(A_0, A_1) \otimes \cdots \otimes \mathcal{C}(A_n, A_0) \to \mathcal{C}(A_0, A_0) \end{pmatrix}$$

On generating morphisms in $\chi(\mathcal{C})$,

(4.2)

$$\delta_{j,n} \underset{\mathcal{F}_T}{\mapsto} \begin{pmatrix} \hat{\delta}_{j,n} := \text{composition functor over } (j+1)^{th} \text{ factor} \\ \cdots \otimes \mathcal{C}(A_j, A_{j+1}) \otimes \mathcal{C}(A_{j+1}, A_{j+2}) \otimes \ldots \xrightarrow{\hat{\delta}_{j,n}=m} \cdots \otimes \mathcal{C}(A_j, A_{j+2}) \otimes \ldots, \\ m^{*n}T(A_0) \xrightarrow{\delta_{j,n!}:=id} \hat{\delta}^*_{j,n} m^{*n-1} T(A_0) \cong (m^{n-1}\hat{\delta}_{j,n})^* T(A_0) \cong m^{*n} T(A_0) \end{pmatrix}$$

$$\sigma_{i,n} \underset{\mathcal{F}_T}{\mapsto} \begin{pmatrix} \hat{\sigma}_{i,n} := \text{insert } id_{A_i} \text{ and } 1 \in k \text{ into the } i^{th} \text{ slot} \\ \cdots \otimes \mathcal{C}(A_i, A_{i+1}) \otimes \ldots \xrightarrow{\hat{\sigma}_{i,n}} \cdots \otimes \mathcal{C}(A_i, A_i) \otimes \mathcal{C}(A_i, A_{i+1}) \otimes \ldots, \\ m^{*n}T(A_0) \xrightarrow{\sigma_{i,n!}:=id} \hat{\sigma}^*_{i,n} m^{*n+1} T(A_0) \cong (m^{n+1}\hat{\sigma}_{i,n})^* T(A_0) \cong m^{*n} T(A_0) \end{pmatrix}$$

$$\tau_n \underset{\mathcal{F}_T}{\mapsto} \begin{pmatrix} \hat{\tau}_n := \text{rotate factors} \\ \mathcal{C}(A_0, A_1) \otimes \cdots \otimes \mathcal{C}(A_n, A_0) \xrightarrow{\hat{\tau}_n} \mathcal{C}(A_n, A_0) \otimes \cdots \otimes \mathcal{C}(A_{n-1}, A_n), \\ m^{*n}T(A_0) \xrightarrow{\tau_{n!}:=m^{*n-1}\tau_!(A_0,A_n)} \hat{\tau}^*_n m^{*n} T(A_n) \text{ where} \\ m^{n-1} : \big(\mathcal{C}(A_0, A_1) \otimes \cdots \otimes \mathcal{C}(A_{n-1}, A_n)\big) \otimes \mathcal{C}(A_n, A_0) \to \mathcal{C}(A_0, A_n) \otimes \mathcal{C}(A_n, A_0) \end{pmatrix}.$$

To show that this association on generating morphisms gives a functor, we should check that $\mathcal{F}_T$ preserves the cyclic relations in Equation A.2. All of the relations involving δ's and σ's are straightforward to check and follow from (1) the associativity of the composition functor m in $\mathcal{C}$, and (2) the general fact that $f \circ id = id \circ f = f$ for a map f. The remaining relation, $\tau_n^{n+1} = id$, is preserved:

- for $n = 2$ because this is Equation 3.1 from the definition of a trace functor,
- for $n > 2$ because these are pullbacks of Equation 3.1,
- and for $n = 1$ because this follows from Equation 3.1 with $B = C$ and the fact that $\sigma_{1,1!}$ is an identity map on comodules.

$\mathcal{F}_T$ is dg because $\delta_{j,n!} := id$, $\sigma_{i,n!} := id$ and $\tau_{n!} := m^{*n-1}\tau_!$ commute with the differentials.

Now, we are ready to define a "trace functor up to homotopy".

Definition 4.2.3. Let $\mathcal{C}$ be a category in dg cocategories. A trace functor up to homotopy on $\mathcal{C}$ is an A_∞-functor

$$\mathcal{F} : \chi(\mathcal{C}) \to \mathcal{D}$$

where $\chi(\mathcal{C})$ and $\mathcal{D}$ are dg categories defined in Defintions 4.2.1 and 4.2.2, respectively, (and we use the notation and conventions from Reference [**2**], Appendix A, Definition A.8 for the definition of an A_∞-functor,) satisfying

- $\mathcal{F}(A_0 \to A_0) \cong \begin{pmatrix} \mathcal{C}(A_0, A_0), \\ T(A_0) \text{ any dg comodule over } \mathcal{C}(A_0, A_0) \end{pmatrix}$
- for $n > 0$, $\mathcal{F}(A_0 \to \cdots \to A_n \to A_0) \cong \begin{pmatrix} \mathcal{C}(A_0, A_1) \otimes \cdots \otimes \mathcal{C}(A_n, A_0), \\ m^{*n}T(A_0) \text{ where} \\ m^n : \mathcal{C}(A_0, A_1) \otimes \cdots \otimes \mathcal{C}(A_n, A_0) \to \mathcal{C}(A_0, A_0) \end{pmatrix}$
- for $\lambda = \delta_{j,n}, \sigma_{i,n}$, $\mathcal{F}(\lambda) \cong \mathcal{F}_T(\lambda)$ given in Equation 4.2
- $\mathcal{F}(\tau_1) \cong \begin{pmatrix} \hat{\tau}_1 := \text{rotate factors} \\ \mathcal{C}(A_0, A_1) \otimes \mathcal{C}(A_1, A_0) \xrightarrow{\hat{\tau}_1} \mathcal{C}(A_1, A_0) \otimes \mathcal{C}(A_0, A_1), \\ T(A_0) \xrightarrow{\tau_{1!}} \hat{\tau}_1^* T(A_1) \text{ any map of dg comodules} \end{pmatrix}$
- for $n > 1$, $\mathcal{F}(\tau_n) \cong \mathcal{F}_T(\tau_n)$ given in Equation 4.2.

There are many stipulations in the definition above because not every functor $\chi(\mathcal{C}) \to \mathcal{D}$ comes from a trace functor. However, an dg functor satsifying Definition 4.2.3 does come from a trace functor.

CHAPTER 5

A trace functor up to homotopy

5.1. Motivation of this chapter

In this chapter, we give a trace functor up to homotopy on the category $\mathcal{C}$ defined in Equation 1.2. To do so, we give an A_∞-functor $\mathcal{F} : \chi(\mathcal{C}) \to \mathcal{D}$ satisfying certain requirements (see Definition 4.2.3). Applying the definition of an A_∞-functor (from Reference [**2**], Appendix A, Definition A.8), the only choices we need to make to define $\mathcal{F}$ are:

(1) for each algebra A, a dg comodule $T(A)$ over $\mathcal{C}(A, A)$,

(2) for a functor of dg cocategories $F : C_1 \to C_0$ and a dg comodule T_0 over C_0, a definition of a pullback F^*T_0 that is natural in T_0 and satisfies Equation 4.1,

(3) for each pair of algebras A, B, a map of dg comodules over $\mathcal{C}(A, B) \otimes \mathcal{C}(B, A)$

$$\tau_{1!}(A, B) : T(A) \to \hat{\tau}_1^* T(B)$$

where $\hat{\tau}_1 : \mathcal{C}(A, B) \otimes \mathcal{C}(B, A) \to \mathcal{C}(B, A) \otimes \mathcal{C}(A, B)$ is rotation,

(4) for each non-generating morphism $\mu \in \chi(\mathcal{C})$, a map of dg comodules $\mathcal{F}(\mu) \in \mathcal{D}$,

(5) for each pair of morphisms $\mu_1, \mu_2 \in \chi(\mathcal{C})$, a degree-1 map of comodules $\mathcal{F}(\mu_1, \mu_2) \in \mathcal{D}$,

(6) for each sequence of morphisms $\mu_1, \ldots, \mu_n \in \chi(\mathcal{C})$ where $n > 2$, a degree-(n-1) map of comodules $\mathcal{F}(\mu_1, \ldots, \mu_n) \in \mathcal{D}$.

In Section 5.2, we define item (1), the dg comodule $T(A)$, which is a (categorified) bar construction of the module $C_\bullet(A, A)$ over the algebra $C^\bullet(A, A)$ acting via contraction. In Appendix **??**, we give item (2) as well as compute some examples of pullbacks for later use. In Proposition C.1, we define item (3) by adapting known equations for the Lie derivative of a Hochschild cochain against a chain. In Section 5.3.1, we give a prescription for defining item (4). We see that $\mathcal{F}$ respects composition except for a few cases (Section 5.4), and we give a prescription for defining the few non-zero $\mathcal{F}(\mu_1, \mu_2)$'s in item (5) (Section 5.3.2). Finally, for item (6), we set $\mathcal{F}(\mu_1, \ldots, \mu_n) =$ (zero map on comodules) for all composable $mu_1, \ldots, \mu_n$, $n > 2$: this is the claim that we have no higher homotopies, justified in Section 5.5.

5.2. Dg comodules $T(A)$

Let A be an algebra and $Hoch(A, A)$ be the dg category defined in Section 2.2. First, we will define a dg module, $\underline{T}(A)$ over $Hoch(A, A)$:

$$\underline{T}(A)^{\bullet}(f) := \big(C_{-\bullet}(A,_f A), b\big)$$

$$Hoch(A, A)^{\bullet}(f, g) \otimes T(A)^{\bullet}(g) \cong C^{\bullet}(A,_f A_g) \otimes C_{-\bullet}(A,_g A) \xrightarrow{\iota} C_{-\bullet}(A,_f A) \cong T(A)^{\bullet}(f)$$

where $f : A \to A$ is a map of algebras, $(C_{-\bullet}(A,_f A), b)$ is the Hochschild chain complex (see Appendix B) and ι is the contraction operation from Equation B.1.

Now, let $B(A) := \mathcal{C}(A, A) = Bar(Hoch(A, A))$ be the endomorphism dg cocategory defined in Section 2.2. Then, we set $T(A) := Bar_{mod}(Hoch(A, A), \underline{T}(A))$, a dg comodule over $B(A)$. Bar_{mod} is a functor

$$Bar_{mod} : \{\text{dg modules over } Hoch(A, A)\} \to \{\text{dg comodules over } B(A)\}.$$

More explicitly,

$$\begin{aligned} T(A)^{\bullet}(f) :=& \bigoplus_{\substack{n \geq 0, \\ f_i \in Obj(Hoch(A,A)) \\ f_0 = f}} Hoch(A, A)^{\bullet}[1](f_0, f_1) \otimes \cdots \otimes Hoch(A, A)^{\bullet}[1](f_{n-1}, f_n) \otimes \underline{T}^{\bullet}(f_n) \\ =& \bigoplus_{\substack{n \geq 0, \\ f_i : A \to A \\ f_0 = f}} C^{\bullet}(A,_{f_0} A_{f_1})[1] \otimes \cdots \otimes C^{\bullet}(A,_{f_{n-1}} A_{f_n})[1] \otimes C_{-\bullet}(A,_{f_n} A). \end{aligned}$$

We can picture an element of $T(A)^{\bullet}(f)$ as follows:

$$
(\phi_1 \dots \phi_n|\alpha) = \quad
\begin{tikzcd}[column sep=huge]
A \arrow[r, bend left=80, "f_0"] \arrow[r, bend left=80, phantom, ""{name=U0, below}] \arrow[r, bend left=40, "f_1"] \arrow[r, bend left=40, phantom, ""{name=U1}] \arrow[r, "f_2"] \arrow[r, bend right=40, "f_{n-1}"'] \arrow[r, bend right=70, "f_n"'] \arrow[r, bend right=80, "id_A"'] & A
\arrow[from=U0, to=U1, phantom, "\Downarrow\phi_1"]
\end{tikzcd}
$$

$$
\begin{array}{c}
f_0 \\ \Downarrow\phi_1 \\ f_1 \\ \Downarrow\phi_2 \\ f_2 \\ \vdots \\ f_{n-1} \\ \Downarrow\phi_n \\ f_n \\ \alpha \\ id_A
\end{array}
$$

Figure 5.1. An element of $T(A)^\bullet(f = f_0)$

where $\phi_i \in C^\bullet(A,_{f_{i-1}} A_{f_i})$ and $\alpha \in C_{-\bullet}(A,_{f_n} A)$. The differential on $T(A)$ is:

$$d_{T(A)} = \tilde{d}_{Hoch(A,A)} + \tilde{b} + \tilde{\iota}$$

$$\tilde{d}_{Hoch(A,A)} = \text{extension of } d_{Hoch(A,A)} \text{ to a differential on } T(A)$$

$$\tilde{b} = \text{extension of the Hochschild chain differential } b \text{ to a differential on } T(A)$$

$$\tilde{\iota}(\phi_1 \dots \phi_n|\alpha) := (\phi_1 \dots \phi_{n-1}|\iota(\phi_n, \alpha)).$$

The coproduct on $T(A)$ is induced by the coproduct on $B(A)$:

$$\Delta(\phi_1 \dots \phi_n|n) = \sum_{0 \le i \le n} \pm(\phi_1 \dots \phi_i) \otimes (\phi_{i+1} \dots \phi_n|\alpha).$$

For more precise details and explicit signs, see Reference [**6**], Section 4.6. $T(A)$ is the cofree dg comodule over $B(A)$ with cogenerators given by Hochschild chains. In other words,

$$
\begin{aligned}
&\left\{\begin{matrix}\text{maps of dg comodules}\\ D \to T(A) \text{ over } B(A)\end{matrix}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{pmatrix}\text{maps of complexes}\\ D^\bullet(f) \to C_{-\bullet}(A,_f A)\end{pmatrix}_{f\in Obj(B(A))}\right\} \\
&\Big(F : D \to T(A)\Big) \mapsto \Big(D^\bullet(f) \xrightarrow{F_f} T(A)^\bullet(f) \xrightarrow{project} C_{-\bullet}(A,_f A)\Big)_f
\end{aligned} \tag{5.1}
$$

$$
\begin{pmatrix}
D(f) \xrightarrow{\Delta_D} \bigoplus_{g\in Obj(B(A))} B(A)^\bullet(f,g)\otimes D(g) \\
\xrightarrow{id\otimes F} \bigoplus_g B(A)^\bullet(f,g)\otimes C_{-\bullet}(A,_g A) \\
\cong T(A)(f)
\end{pmatrix}_f \hookleftarrow \Big(D^\bullet(f) \xrightarrow{F} C_{-\bullet}(A,_f A)\Big)_f
$$

5.3. Prescriptions for $\mathcal{F}(\mu_1, \dots, \mu_n)$

5.3.1. Prescription for $\mathcal{F}(\mu)$

Now, we will define $\mathcal{F}(\mu)$ for μ not a generating morphism in Λ. (A general morphism in $\chi(\mathcal{C})$ is a linear combination of morphisms in Λ, so we extend $\mathcal{F}$ linearly to define $\mathcal{F}$ on any morphism in $\chi(\mathcal{C})$, see Definition 4.2.1.)

Let μ be a non-generating morphism in Λ that induces a morphsim in $\chi(\mathcal{C})$ with source $\mathcal{A} := (A_0 \to \cdots \to A_n \to A_0)$ for some algebras A_i, $0 \leq i \leq n$, $n \geq 0$. Choose (i.e., fix once and for all) a presentation of μ as a composition of generating morphisms. Within the chosen presentation, in the following order, (1) replace all occurrences of $\tau_{n-1}\delta_{n-1,n}$ with $\delta_{0,n}\tau_n^2$, (2) replace all $\tau_{n+1}\sigma_{n,n}$ with $\tau_{n+1}^{n+1}\sigma_{0,n}\tau_n$, (3) replace all decompositions of identity maps with identity maps, (4) remove all identity maps if $\mu \neq id$, (5) call this new presentation "the presentation corresponding to μ", denoted $\mu = \lambda_{\mu,k_\mu} \dots \lambda_{\mu,1}$. The presentation corresponding to μ is not unique (i.e., still depends on the original chosen presentation). However, letting $\mathcal{F}(\mu)$ act on comodules via

$$\mathcal{F}(\mu) := \hat{\lambda}^*_{\mu,1} \dots \hat{\lambda}^*_{\mu,k_\mu-1}(\lambda_{\mu,k_\mu!}) \circ \dots \circ \hat{\lambda}^*_{\mu,1}(\lambda_{\mu,2!}) \circ \lambda_{\mu,1!} : T(\mathcal{A}) \to \hat{\mu}^* T(\mu\mathcal{A})$$

is well-defined because we have made consistent choices. More explicitly, we show in Section 5.4 that the choices we have made for $\mathcal{F}(\{\text{generating morphisms}\})$ respect all of the relations in Λ (Equation A.2) except for Equations 5.4. The above steps ensure that the presentation corresponding to μ only uses the lefthand side of Equation 5.4a and the righthand sides of Equations 5.4c and 5.4b.

5.3.2. Prescription for $\mathcal{F}(\mu_1, \mu_2)$

Before defining $\mathcal{F}$ on pairs of composable morphisms, let's take a look at an A_∞ relation we expect $\mathcal{F}$ to satisfy: For $\cdot \xrightarrow{\mu_1} \cdot \xrightarrow{\mu_2} \cdot$ composable morphisms in $\chi(\mathcal{C})$, we expect

$$\mathcal{F}(\mu_2 \circ \mu_1) = \mathcal{F}(\mu_2) \circ \mathcal{F}(\mu_1) + d_{\mathcal{D}_\infty} \circ \mathcal{F}(\mu_1, \mu_2). \tag{5.2}$$

Given the definition of $\mathcal{F}(\mu)$ above, we require a non-zero $\mathcal{F}(\mu_1, \mu_2)$ if and only if: (Condition H) the presentation corresponding to μ_2 composed with the presentation corresponding to μ_1 contains, after removing (decompositions of) identity maps except for τ_n^{n+1}, one or more of the following terms: $\tau_{n-1}\delta_{n-1,n}$, $\tau_{n+1}\sigma_{n,n}$, τ_n^{n+1}. If μ_1, μ_2 satisfy Condition H, homotopies given in Section 5.4.2 can be used to define $\mathcal{F}(\mu_1, \mu_2)$. If μ_1, μ_2 do not satisfy Condition H, let $\mathcal{F}(\mu_1, \mu_2) = 0$ on comodules.

We will give some instructive examples of non-zero $\mathcal{F}(\mu_1, \mu_2)$ that satisfy Equation 5.2.

Example 5.3.1. *Let $\mu_1 = \delta_{n-1,n}$, $\mu_2 = \tau_{n-1}$. Then, the presentation corresponding to $\mu_2\mu_1$ is $\delta_{0,n}\tau_n^2$. Let $\mathcal{F}(\mu_1, \mu_2)$ be the homotopy given in Section 5.4.2.1. Then, Equation 5.2 holds because it is equivalent to Equation 5.4a.*

Example 5.3.2. *Let $\mu_1 = \sigma_{0,n-1}\delta_{n-1,n}$, $\mu_2 = \tau_{n-1}\delta_{0,n}$. To form the presentation corresponding to $\mu_2\mu_1$, we follow these steps:*

$$\tau_{n-1}\delta_{0,n}\sigma_{0,n-1}\delta_{n-1,n} \xrightarrow[\text{of identities}]{\text{remove decompositions}} \tau_{n-1}\delta_{n-1,n} \xrightarrow{\text{replace}} \delta_{0,n}\tau_n^2.$$

On the other hand,

$$\mathcal{F}(\mu_2)\mathcal{F}(\mu_1) = (\delta_{0,n}\widehat{\sigma_{0,n-1}\delta}_{n-1,n})^*(\tau_{n-1!}) \circ (\sigma_{0,\widehat{n-1}\delta_{n-1,n}})^*(\delta_{0,n!}) \circ \hat{\delta}^*_{n-1,n}(\sigma_{0,n-1!}) \circ \delta_{n-1,n!}$$

$$= \hat{\delta}^*_{n-1,n}(\tau_{n-1!}) \circ id \circ \delta_{n-1,n!}.$$

So, we can let $\mathcal{F}(\mu_1,\mu_2)$ be the homotopy given in Section 5.4.2.1, and Equation 5.2 holds because it is equivalent to Equation 5.4a.

Example 5.3.3. *Let $(\mu_1,\mu_2) \in \{(\tau_{n+1},\sigma_{n,n}),(\tau_n^{n+1-j},\tau_n^j) : 1 \leq j \leq n, n \in \mathbb{N}\}$. Let $\mathcal{F}(\mu_1,\mu_2)$ be the homotopy given in 5.4.2.3 if $\mu_2 = \sigma_{n,n}$ and the homotopy given in 5.4.2.2 if $\mu_2 \neq \sigma_{n,n}$. Then, Equation 5.2 holds because it is equivalent to either Equation 5.4c ($\mu_2 = \sigma_{n,n}$) or Equation 5.4b ($\mu_2 \neq \sigma_{n,n}$).*

Example 5.3.4. *Let $\mu_1 = \sigma_{n-1,n-1}\delta_{n-1,n}$, $\mu_2 = \tau_n$. To form the presentation corresponding to $\mu_2\mu_1$, we follow these steps:*

$$(\tau_n\sigma_{0,n-1})\delta_{n-1,n} \xrightarrow{\text{replace } (\cdot)} \tau_n^n\sigma_{0,n-1}(\tau_{n-1}\delta_{n-1,n}) \xrightarrow{\text{replace } (\cdot)} \tau_n^n\sigma_{0,n-1}\delta_{0,n}\tau_n^2.$$

Let $\mathcal{F}(\mu_1,\mu_2) = g_1 + g_2$ where $g_1 = \hat{\delta}^*_{n-1,n}$(homotopy in Section 5.4.2.3)$\circ \delta_{n-1,n!}$ and $g_2 = (\widehat{\tau_{n-1}\delta_{n-1,n}})^*\big((\widehat{\tau_n^{n-1}\sigma_{0,n-1}})^*(\tau_{n!}) \circ \ldots \circ \hat{\sigma}^*_{0,n-1}(\tau_{n!}) \circ \sigma_{0,n-1!}\big) \circ$ (homotopy in Section 5.4.2.1). Then, Equation 5.2 holds because it reduces to $\delta^*_{n-1,n}$(Equation 5.4c) and Equation 5.4a.

5.4. Computational: Composition of maps of dg comodules

In Equations 4.2 and C.1, we gave the maps of dg comodules re-stated below:

$$\delta_{j,n!} : m^{*n}T(A_0) \xrightarrow[\cong]{id} \hat{\delta}^*_{j,n} m^{*n-1}T(A_0) \qquad \text{Equation 4.2}$$

$$\sigma_{i,n!} : m^{*n}T(A_0) \xrightarrow[\cong]{id} \hat{\sigma}^*_{i,n} m^{*n+1}T(A_0) \qquad \text{Equation 4.2}$$

$$\tau_{n!} : m^{*n}T(A_0) \xrightarrow{m^{*n-1}\tau_!(A_0,A_n)} \hat{\tau}^*_n m^{*n}T(A_n) \qquad \text{Equation 4.2}$$

$$\tau_{1!} : m^{*}T(A_0) \to \hat{\tau}^*_1 m^{*}T(A_1) \qquad \text{Equation C.1 for } A = A_0,\ B = A_1$$

Here, we show that these maps satisfy the relations in Λ (Equation A.2) up to homotopy. More precisely, we will show that

$$\hat{\delta}^*_{j,n}(\delta_{i,n-1!}) \circ \delta_{j,n!} = \hat{\delta}^*_{i,n}(\delta_{j-1,n-1!}) \circ \delta_{i,n!} \quad 0 \le i < j \le n-1$$

$$\hat{\sigma}^*_{j,n}(\sigma_{i,n+1!}) \circ \sigma_{j,n!} = \hat{\sigma}^*_{i,n}(\sigma_{j+1,n+1!}) \circ \sigma_{i,n!} \quad 0 \le i \le j \le n$$

(5.3a)

$$\hat{\sigma}^*_{i,n}(\delta_{j,n+1!}) \circ \sigma_{i,n!} = \begin{cases} \hat{\delta}^*_{j-1,n}(\sigma_{i,n-1!}) \circ \delta_{j-1,n!} & 0 \le i < j \le n \\ id & j = i, i-1 \\ \hat{\delta}^*_{j,n}(\sigma_{i-1,n-1!}) \circ \delta_{j,n!} & 0 \le j < i-1 \le n-1 \end{cases}$$

(5.3b)

$$\hat{\sigma}^*_{i,n}(\tau_{n+1!}) \circ \sigma_{i,n!} = \hat{\tau}^*_n(\sigma_{i+1,n!}) \circ \tau_{n!} \quad 0 \le i \le n-1$$

$$\hat{\delta}^*_{j,n}(\tau_{n-1!}) \circ \delta_{j,n!} = \hat{\tau}^*_n(\delta_{j+1,n!}) \circ \tau_{n!} \quad 0 \le j \le n-1$$

(5.3c)

$$(\widehat{\tau_1\sigma_{0,0}})^*(\delta_{0,1!}) \circ \hat{\sigma}^*_{0,0}(\tau_{1!}) \circ \sigma_{0,0!} = id$$

and

$$\hat{\tau}_n^{*2}(\delta_{0,n!}) \circ \hat{\tau}_n^{*}(\tau_{n!}) \circ \tau_{n!} \simeq \hat{\delta}_{n-1,n}^{*}(\tau_{n-1!}) \circ \delta_{n-1,n!} \tag{5.4a}$$

$$\hat{\tau}_n^{*n}(\tau_{n!}) \circ \ldots \circ \hat{\tau}_n^{*}(\tau_{n!}) \circ \tau_{n!} \simeq id \tag{5.4b}$$

$$\begin{aligned} &\hat{\sigma}_{n,n}^{*}(\tau_{n+1!}) \circ \sigma_{n,n!} \\ &\simeq (\widehat{\tau_{n+1}^{n}\sigma_{0,n}}\tau_n)^{*}(\tau_{n+1!}) \circ \ldots \circ (\widehat{\tau_{n+1}\sigma_{0,n}}\tau_n)^{*}(\tau_{n+1!}) \circ (\widehat{\sigma_{0,n}\tau_n})^{*}(\tau_{n+1!}) \circ \hat{\tau}_n^{*}(\sigma_{0,n!}) \circ \tau_{n!} \end{aligned} \tag{5.4c}$$

5.4.1. Strict relations: showing Equations 5.3 hold

Equation 5.3a has three relations. All of the $\sigma_!$'s and $\delta_!$'s in Equation 5.3a are identity maps, so it's clear that these relations hold.

Equation 5.3b has two relations. To show that the first one holds, we have

$$\begin{aligned} \hat{\sigma}_{i,n}^{*}(\tau_{n+1!}) \circ \sigma_{i,n!} &= \hat{\sigma}_{i,n}^{*}((\widehat{\delta_{0,2}\ldots\delta_{0,n+1}})^{*}(\tau_{1!})) \circ \sigma_{i,n!} \quad \text{definitions of } \tau_{n+1!} \text{ and } \hat{\delta}_{\cdot,\cdot} \\ &= (\widehat{\delta_{0,2}\ldots\delta_{0,n+1}}\sigma_{i,n})^{*}(\tau_{1!}) \circ \sigma_{i,n!} \quad \text{Proposition } D.1 \\ &= (\widehat{\delta_{0,2}\ldots\delta_{0,n}})^{*}(\tau_{1!}) \circ \sigma_{i,n!} \\ &= \tau_{n!} \circ \sigma_{i,n!} \quad \text{definitions of } \tau_{n!} \text{ and } \hat{\delta}_{\cdot,\cdot} \\ &= \tau_{n!} \circ id = id \circ \tau_{n!} \\ &= \hat{\tau}_n^{*}(\sigma_{i+1,n!}) \circ \tau_{n!}. \end{aligned}$$

To show that the second relation holds, the reasoning is the same as above. We have

$$\begin{aligned}
\hat{\delta}^*_{j,n}(\tau_{n-1!}) \circ \delta_{j,n!} &= \hat{\delta}^*_{j,n}((\delta_{0,2}\widehat{\ldots\delta_{0,n}}_{-1})^*(\tau_{1!})) \circ \delta_{j,n!} \\
&= (\delta_{0,2}\widehat{\ldots\delta_{0,n-1}}\delta_{j,n})^*(\tau_{1!}) \circ \delta_{j,n!} \\
&= \tau_{n!} \circ \delta_{j,n!} \\
&= \tau_{n!} \circ id = id \circ \tau_{n!} \\
&= \hat{\tau}^*_n(\delta_{j+1,n!}) \circ \tau_{n!}.
\end{aligned}$$

Equation 5.3c has one relation. The only map in this relation that is not defined to be an identity map is $\hat{\sigma}^*_{0,0}(\tau_{1!})$. We will compute this map and show that it is also an identity. Let $(\phi_1\ldots\phi_k|\alpha) \in T(A_0) =: T(A_0 \to A_0)$ (see Figure 5.1 for notation). By Proposition D.2,

$$T(A_0 \to A_0) \xrightarrow{\cong} \hat{\sigma}^*_{0,0}T(A_0 \to A_0 \to A_0)$$

$$(\phi_1\ldots\phi_k|\alpha) \mapsto \sum_{0\le r\le k} (\phi_1\ldots\phi_r) \otimes (1|\phi_{r+1}\ldots\phi_k|\alpha).$$

Applying $\hat{\sigma}^*_{0,0}(\tau_{1!})$ to the righthand side, we have

$$\hat{\sigma}^*_{0,0}T(A_0 \to A_0 \to A_0) \xrightarrow{\hat{\sigma}^*_{0,0}(\tau_{1!})} \hat{\sigma}^*_{0,0}\hat{\tau}^*_1 T(A_0 \to A_0 \to A_0)$$

$$\sum_{0\le r\le k} (\phi_1\ldots\phi_r) \otimes (1|\phi_{r+1}\ldots\phi_k|\alpha) \mapsto \sum_{0\le r\le s\le k} (\phi_1\ldots\phi_r)\otimes$$

$$(\phi_{r+1}\ldots\phi_s|1|\tau_{1!}(1|\phi_{s+1}\ldots\phi_k|\alpha)).$$

The righthand side above is equal to

$$\begin{aligned}
&\sum_{0\le r\le s\le k} (\phi_1\ldots\phi_r)\otimes(\phi_{r+1}\ldots\phi_s|1|\tau_{1!}(1|\phi_{s+1}\ldots\phi_k|\alpha)) \\
&= \sum_{0\le r\le s\le k} (\phi_1\ldots\phi_r)\otimes(\phi_{r+1}\ldots\phi_s|1|\tau_{1!}^{0,k-s}(1|\phi_{0,s_0+1}\ldots\phi_{0,k_0}|\alpha)) \\
&\qquad\qquad\text{(see Proposition C.1 for definition of } \tau_{1!}^{0,k-s}) \\
&= \sum_{0\le r\le k} (\phi_1\ldots\phi_r)\otimes(\phi_{r+1}\ldots\phi_k|1|\alpha) \qquad (\tau_{1!}^{0,>0}=0) \\
&\in \hat{\sigma}_{0,0}^*\hat{\tau}_1^*T(A_0\to A_0\to A_0).
\end{aligned}$$

Finally, applying Proposition D.2 again, we have

$$\hat{\sigma}_{0,0}^*\hat{\tau}_1^*T(A_0\to A_0\to A_0)\xrightarrow[\cong]{\text{project onto cogenerators}} T(A_0\to A_0)$$

$$\sum_{0\le r\le k} (\phi_1\ldots\phi_r)\otimes(\phi_{r+1}\ldots\phi_k|1|\alpha)\mapsto(\phi_1\ldots\phi_k|\alpha).$$

So, we've shown

$$T(A_0\to A_0)\cong\hat{\sigma}_{0,0}^*T(A_0\to A_0\to A_0)\xrightarrow{\hat{\sigma}_{0,0}^*(\tau_{1!})}\hat{\sigma}_{0,0}^*\hat{\tau}_1^*T(A_0\to A_0\to A_0)\cong T(A_0\to A_0)$$

is the identity map.

5.4.2. Weak relations: showing Equations 5.4 hold

5.4.2.1. Showing Equation 5.4a holds. For $n = 1$, eliminating the identity maps reduces Equation 5.4a to:

$$\hat{\tau}_1^*(\tau_{1!}) \circ \tau_{1!} \simeq id.$$

We prove the above in Appendix Proposition C.2. (In the appendix, we fix algebras A_0, A_1, and $\tau_{1!} = \tau_{1!}(A_0, A_1)$, $\hat{\tau}_1^*(\tau_{1!}) = \tau_{1!}(A_1, A_0)$, and the homotopy is denoted $B(A_0, A_1)$.)

For $n = 2$, eliminating the identity maps and writing $\tau_{2!}$ in terms of $\tau_{1!}$ reduces Equation 5.4a to:

$$(\widehat{\delta_{0,2}\tau_2})^*(\tau_{1!}) \circ \hat{\delta}_{0,2}^*(\tau_{1!}) \simeq \hat{\delta}_{1,2}^*(\tau_{1!}).$$

We prove the above in Appendix Proposition C.4. (In the appendix, we fix algebras A_0, A_1, A_2, and $\hat{\delta}_{0,2}^*(\tau_{1!}) = \tau_{1!}(A_0 \bullet A_1, A_2)$, $(\widehat{\delta_{0,2}\tau_2})^*(\tau_{1!}) = \tau_{1!}(A_2 \bullet A_0, A_1)$, $\hat{\delta}_{1,2}^*(\tau_{1!}) = \tau_{1!}(A_0, A_1 \bullet A_2)$, and the homotopy is denoted $\mathcal{B}(A_0, A_1, A_2)$.)

For $n > 2$, we reduce Equation 5.4a to the case when $n = 2$. We have

$$\begin{aligned}
\text{Lefthand side of Equation 5.4a} &= \hat{\tau}_n^{*2}(\delta_{0,n!}) \circ \hat{\tau}_n^{*}(\tau_{n!}) \circ \tau_{n!} \\
&= id \circ \hat{\tau}_n^{*}((\delta_{0,2}\widehat{\ldots\delta_{0,n}})^{*}(\tau_{1!})) \circ \tau_{n!} \\
&= (\delta_{0,2}\widehat{\ldots\delta_{0,n}}\tau_n)^{*}(\tau_{1!}) \circ \tau_{n!} \\
&= (\delta_{0,2}\tau_2\widehat{\delta_{0,3}\ldots}\delta_{0,n})^{*}(\tau_{1!}) \circ \tau_{n!} \\
&= (\delta_{0,2}\tau_2\widehat{\delta_{0,3}\ldots}\delta_{0,n})^{*}(\tau_{1!}) \circ (\delta_{0,2}\widehat{\ldots\delta_{0,n}}\tau_n)^{*}(\tau_{1!}) \\
&= (\delta_{0,3}\widehat{\ldots\delta_{0,n}})^{*}((\widehat{\delta_{0,2}\tau_2})^{*}(\tau_{1!}) \circ \hat{\delta}_{0,2}^{*}\tau_{1!}) \\
\text{Righthand side of Equation 5.4a} &= \hat{\delta}_{n-1,n}^{*}(\tau_{n-1!}) \circ \delta_{n-1,n!} \\
&= \hat{\delta}_{n-1,n}^{*}((\delta_{0,2}\widehat{\ldots\delta_{0,n}}_{-1})^{*}(\tau_{1!})) \circ id \\
&= (\delta_{0,2}\ldots\widehat{\delta_{0,n-1}}\delta_{n-1,n})^{*}(\tau_{1!})) \\
&= (\delta_{1,2}\widehat{\delta_{0,3}\ldots}\delta_{0,n})^{*}(\tau_{1!}) \\
&= (\delta_{0,3}\widehat{\ldots\delta_{0,n}})^{*}(\hat{\delta}_{1,2}^{*}(\tau_{1!})).
\end{aligned}$$

So, Equation 5.4a $= (\delta_{0,3}\widehat{\ldots\delta_{0,n}})^{*}($ Equation 5.4a, $n = 2)$. If $\mathcal{B}$ is a homotopy giving Equation 5.4a for $n = 2$, then $(\delta_{0,3}\widehat{\ldots\delta_{0,n}})^{*}\mathcal{B}$ is a homotopy giving Equation 5.4a for $n > 2$.

5.4.2.2. Showing Equation 5.4b holds. We prove this by induction on n. For $n = 1$, Equation 5.4b is the same as Equation 5.4a, which we established in the previous section. Now, assume that Equation 5.4b holds for $N = n - 1$. We show that Equation 5.4b holds for $N = n$ below:

$$\begin{aligned}
\hat{\tau}_n^{*n}(\tau_{n!}) \circ \ldots \circ \hat{\tau}_n^{*}(\tau_{n!}) \circ \tau_{n!} &= \hat{\tau}_n^{*n-1}(\hat{\tau}_n^{*}\tau_{n!} \circ \tau_{n!}) \circ \hat{\tau}_n^{*n-2}\tau_{n!} \circ \ldots \circ \tau_{n!} \\
&\simeq \hat{\tau}_n^{*n-1}(\hat{\delta}^{*}_{n-1,n}\tau_{n-1!}) \circ \hat{\tau}_n^{*n-2}\tau_{n!} \circ \ldots \circ \tau_{n!} \qquad \text{(Equation 5.4a)} \\
&= (\widehat{\tau_{n-1}^{n-1}\delta_{0,n}})^{*}\tau_{n-1!} \circ \\
&\quad \circ \big(\hat{\tau}_n^{*n-2}\hat{\delta}^{*}_{n-2,n}\tau_{n-1!} \circ \ldots \circ \hat{\tau}_n^{*}\hat{\delta}^{*}_{1,n}\tau_{n-1!} \circ \hat{\delta}^{*}_{0,n}\tau_{n-1!}\big) \\
&= (\widehat{\tau_{n-1}^{n-1}\delta_{0,n}})^{*}\tau_{n-1!} \circ \hat{\delta}^{*}_{0,n}\big(\hat{\tau}_{n-1}^{*n-2}\tau_{n-1!} \circ \ldots \circ \hat{\tau}_{n-1}^{*}\tau_{n-1!} \circ \tau_{n-1!}\big) \\
&= \hat{\delta}^{*}_{0,n}\big(\hat{\tau}_{n-1}^{*n-1}\tau_{n-1!} \circ \ldots \circ \tau_{n-1!}\big) \\
&\simeq \hat{\delta}^{*}_{0,n}\big(id\big) \qquad \text{(Inductive hypothesis)} \\
&= id.
\end{aligned}$$

5.4.2.3. Showing Equation 5.4c holds. By manipulating morphisms in Λ, we have

$$\begin{aligned}
\text{Righthand side of Equation 5.4c} &= \hat{\tau}_n^{*n+1}\tau_{n!} \circ \hat{\tau}_n^{*n}\tau_{n!} \circ \ldots \circ \hat{\tau}_n^{*}\tau_{n!} \circ \hat{\tau}_n^{*n+1} id \circ \tau_{n!} \\
&= \tau_{n!} \circ \big(\hat{\tau}_n^{*n}\tau_{n!} \circ \ldots \circ \hat{\tau}_n^{*}\tau_{n!} \circ \tau_{n!}\big) \\
&\simeq \tau_{n!} \circ \big(id\big) \quad \text{Equation 5.4b.}
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\text{Lefthand side of Equation 5.4c} &= \hat{\sigma}^*_{n,n}(\tau_{n+1!}) \circ id \\
&= \hat{\sigma}^*_{n,n}(\hat{\delta}^*_{n,n+1}(\tau_{n+1!})) \\
&= (\delta_{n,\widehat{n+1}\sigma_{n,n}})^*(\tau_{n!}) \\
&= id^*(\tau_{n!}).
\end{aligned}$$

So, Equation 5.4c holds.

5.5. Verification of A_∞ relations

Now, we will check that our choices for $\mathcal{F}$ satisfy the rest of the relations for an A_∞-functor from Reference [**2**], Definition A.8: For $\cdot \xrightarrow{\mu_1} \cdot \xrightarrow{\mu_2} \cdot \xrightarrow{\mu_3} \cdot \xrightarrow{\mu_4} \cdot$ composable morphisms in $\chi(\mathcal{C})$, we expect

$$0 = d_{\mathcal{D}} \circ \mathcal{F}(\mu_1) \tag{5.5}$$

$$\mathcal{F}(\mu_3, \mu_2 \circ \mu_1) - \mathcal{F}(\mu_3 \circ \mu_2, \mu_1) = \mathcal{F}(\mu_3, \mu_2) \circ \mathcal{F}(\mu_1) - \mathcal{F}(\mu_3) \circ \mathcal{F}(\mu_2, \mu_1) \tag{5.6}$$

$$0 = \mathcal{F}(\mu_4, \mu_3) \circ \mathcal{F}(\mu_2, \mu_1). \tag{5.7}$$

Equation 5.5 is satisfied since, for $\lambda \in \Lambda$ a generating morphism, the $\lambda_!$'s we gave at the beginning of Section 5.4 are maps of complexes. Equation 5.7 requires that composing two of our degree -1 homotopies is always equal to zero. This is true because we use reduced Hochschild chains (Section B) and each homotopy (Equations C.3, C.5) inserts a 1 into the first slot of the Hochschild chains component.

We check that Equation 5.6 holds for $n = 1$ and $n \geq 2$ separately. For $n \geq 2$, checking Equation 5.6 boils down to the following situation: We have two maps of dg comodules

$$\begin{tikzcd} T(A_0 \to \dots \to A_n \to A_0) \arrow[d, bend right, "{\substack{\hat{\tau}_n^{*2}\tau_{n!} \circ \hat{\tau}_n^{*}\tau_{n!} \circ \tau_{n!} \\ \text{“apply } \tau_{n!} \text{ 3 times”}}}"'] \arrow[d, bend left, "{\substack{(\delta_{n-2,\widehat{n-1}\delta}_{n-1,n})^*\tau_{n-2!} \\ \text{“brace together the last 3 algebras,} \\ \text{then apply } \tau_{n-2!} \text{ once”}}}"] \\ T(A_{n-2} \to A_{n-1} \to A_n \to A_0 \to \dots \to A_{n-2}). \end{tikzcd} \tag{5.8}$$

These two maps are homotopic via two homotopies: $\hat{\delta}^*_{n-1,n}\mathcal{B}(A_0 \bullet \cdots \bullet A_{n-3}, A_{n-2}, A_{n-1} \bullet A_n) + \tau_n^{*2}\tau_{n!} \circ \mathcal{B}(A_0 \bullet \cdots \bullet A_{n-2}, A_{n-1}, A_n)$ and $\hat{\delta}^*_{n-2,n}\mathcal{B}(A_0 \bullet \cdots \bullet A_{n-3}, A_{n-2} \bullet A_{n-1}, A_n) +$

$\hat{\tau}_n^*\mathcal{B}(A_n \bullet \cdots \bullet A_{n-3}, A_{n-2}, A_{n-1}) \circ \tau_{n!}$ (see Figure 5.2). If the two homotopies were different, then their difference would be closed and we would desire a higher homotopy (i.e., a degree -2 map of comodules) between them. However, we will show the two homotopies are the same, so that no higher homotopies are needed.

First, it follows directly from the definition of $\mathcal{B}$ (Appendix Equation C.5) that

$$\hat{\delta}^*_{n-1,n}\mathcal{B}(A_0 \bullet \cdots \bullet A_{n-3}, A_{n-2}, A_{n-1} \bullet A_n) = \hat{\delta}^*_{n-2,n}\mathcal{B}(A_0 \bullet \cdots \bullet A_{n-3}, A_{n-2} \bullet A_{n-1}, A_n).$$

Second, for $n = 2$, we show that

$$\tau_2^{*2}\tau_{2!} \circ \mathcal{B}(A_0, A_1, A_2) = \hat{\tau}_2^*\mathcal{B}(A_2, A_0, A_1) \circ \tau_{2!} \tag{5.9}$$

in Appendix Proposition C.5. (In the appendix, $\tau_2^{*2}\tau_{2!} = \tau_{1!}(A_1 \bullet A_2, A_0)$ and $\tau_{21} = \tau_{1!}(A_0 \bullet A_1, A_2)$.) For $n > 2$, the equation $\tau_n^{*2}\tau_{n!} \circ \mathcal{B}(A_0 \bullet \cdots \bullet A_{n-2}, A_{n-1}, A_n) = \hat{\tau}_n^*\mathcal{B}(A_n \bullet \cdots \bullet A_{n-3}, A_{n-2}, A_{n-1}) \circ \tau_{n!}$ is a pullback along $\hat{\delta}_0$'s of Equation 5.9.

For $n = 1$, the situation in Equation 5.8 reduces to: We have two maps of dg comodules

$$\begin{tikzcd} T(A_0 \to A_1 \to A_0) \arrow[d, bend right, "\hat{\tau}_1^{*2}\tau_{1!}\circ\hat{\tau}_1^*\tau_{1!}\circ\tau_{1!}"'] \arrow[d, bend left, "\tau_{1!}"] \\ T(A_1 \to A_0 \to A_1). \end{tikzcd}$$

These two maps are homotopic via two homotopies: $\tau_{1!}(A_0, A_1) \circ B(A_0, A_1)$ and $B(A_1, A_0) \circ \tau_{1!}(A_0, A_1)$ (see Figure 5.3). We show that these two homotopies are the same in Appendix Proposition C.3, so no higher homotopies are needed.

$$
\begin{tikzcd}
\begin{array}{c}(\widehat{\delta_{n-2,n-1}\delta_{n-1,n}})^*\tau_{n-2!} \\ \text{“brace together } A_{n-2}, A_{n-1}, A_n, \\ \text{then apply } \tau_{n-2!}\text{”}\end{array} \arrow[r, "\cong"] \arrow[d, "\cong"'] & \hat{\delta}^*_{n-1,n}\big(\hat{\delta}^*_{n-2,n-1}\tau_{n-2!}\big) \arrow[r] & \hat{\delta}^*_{n-1,n}\big(\hat{\tau}^*_{n-1}\tau_{n-1!}\circ\tau_{n-1!}\big) \arrow[d, "\hat{\delta}^*_{n-1,n}\mathcal{B}(A_0\bullet\cdots\bullet A_{n-3},A_{n-2},A_{n-1}\bullet A_n)"', "\cong"] \\
(\widehat{\delta_{n-2,n-1}\delta_{n-2,n}})^*\tau_{n-2!} \arrow[d, "\hat{\delta}^*_{n-2,n}\mathcal{B}(A_0\bullet\cdots\bullet A_{n-3},A_{n-2}\bullet A_{n-1},A_n)"] & & \begin{array}{c}\hat{\tau}^{*2}_n\tau_{n!}\circ\hat{\delta}^*_{n-1,n}\tau_{n-1!} \\ \text{“brace together } A_{n-1}, A_n \\ \text{and apply } \tau_{n-1!}, \\ \text{then apply } \tau_{n!}\text{”}\end{array} \arrow[d, "\tau^{*2}_n\tau_{n!}\circ\mathcal{B}(A_0\bullet\cdots\bullet A_{n-2},A_{n-1},A_n)"'] \\
\hat{\delta}^*_{n-2,n}\big(\hat{\tau}^*_{n-1}\tau_{n-1!}\circ\tau_{n-1!}\big) \arrow[r, "\cong"'] & \begin{array}{c}\hat{\tau}^*_n\big(\hat{\delta}^*_{n-1,n}\tau_{n-1!}\big)\circ\tau_{n!} \\ \text{“apply } \tau_{n!}, \\ \text{then brace together } A_{n-1}, A_{n-2} \\ \text{and apply } \tau_{n-1!}\text{”}\end{array} \arrow[r, "\substack{\hat{\tau}^*_n\mathcal{B}(A_n\bullet\cdots\bullet A_{n-3},A_{n-2},A_{n-1}) \\ \circ\tau_{n!}}"] & \begin{array}{c}\hat{\tau}^{*2}_n\tau_{n!}\circ\hat{\tau}^*_n\tau_{n!}\circ\tau_{n!} \\ \text{“apply } \tau_{n!} \text{ three times”}\end{array}
\end{tikzcd}
$$

Figure 5.2. Two homotopies between $(\widehat{\delta_{n-2,n-1}\delta_{n-1,n}})^*\tau_{n-2!}$ and $\hat{\tau}^{*2}_n\tau_{n!}\circ\hat{\tau}^*_n\tau_{n!}\circ\tau_{n!}$
Vertices are maps of dg comodules and arrows are chain homotopies.

$$
\begin{tikzcd}
id\circ\tau_{1!} = \tau_{1!} = \tau_{1!}\circ id \arrow[d, bend right, "B(A_1,A_0)\circ\tau_{1!}(A_0,A_1)"'] \arrow[d, bend left, "\tau_{1!}(A_0,A_1)\circ B(A_0,A_1)"] \\
\big(\hat{\tau}_1^{*2}\tau_{1!}\circ\hat{\tau}_1^*\tau_{1!}\big)\circ\tau_{1!} = \hat{\tau}_1^{*2}\tau_{1!}\circ\big(\hat{\tau}_1^*\tau_{1!}\circ\tau_{1!}\big)
\end{tikzcd}
$$

Figure 5.3. Two homotopies between $\tau_{1!}$ and $\hat{\tau}_1^{*2}\tau_{1!}\circ\hat{\tau}_1^*\tau_{1!}\circ\tau_{1!}$
Vertices are maps of dg comodules and arrows are chain homotopies.

CHAPTER 6

Coda: other directions

6.1. Motivation of this chapter

In Chapter 5, we gave an A_∞-functor $\mathcal{F} : \chi(\mathcal{C}) \to \mathcal{D}$ where $\mathcal{C}$ is the category defined in Equation 1.2. Applying Reference [**2**], Remark A.27, we can rectify $\mathcal{F}$ to a dg functor $\tilde{\mathcal{F}} : U(\chi(\mathcal{C})) \to \mathcal{D}$ where $U(\chi(\mathcal{C}))$ is the enveloping dg category of χ (see Reference [**2**], Definition A.25).

In other words, we have shown that algebras form a "category in dg cocategories with a trace functor up to homotopy". In this chapter, we show that algebras form a category in dg *categories* with a trace functor up to homotopy. In other words, we give a dg functor $U(\chi(\mathcal{C})) \to \mathcal{E}$ where $\mathcal{E}$ is a dg category with objects pairs (dg category, dg module).

This chapter is not central to the narrative of this thesis, especially since understanding of what happens after applying $Cobar(-)$ is still evolving.

6.2. A functor to dg categories

In this section, we first give a dg functor $\mathcal{D} \to \mathcal{D}_1$, which makes use of the adjunction in Proposition D.3. Then, we will give a dg functor $\Omega : \mathcal{D}_1 \to \mathcal{E}$.

6.2.1. Using the adjunction

Let $\mathcal{D}_1$ be the dg category with the same objects as $\mathcal{D}$ and morphisms

$$\begin{aligned} \mathcal{D}_1^\bullet((B_1, C_1), (B_0, C_0)) = \big\{\big(&F : B_1 \to B_0 \quad \text{dg functor}, \\ &F_! : F_\# C_1 \to C_0 \quad \text{map of comodules of degree } \bullet\big)\big\} \end{aligned}$$

$$d_{\mathcal{D}}(F, F_!) = (F,\ d_{C_0} \circ F_! - (-1)^{|F_!|} F_! \circ d_{F_\# C_1})$$

with composition

$$\mathcal{D}_1^\bullet((B_2,C_2),(B_1,C_1)) \otimes \mathcal{D}_1^\bullet((B_1,C_1),(B_0,C_0)) \to \mathcal{D}_1^\bullet((B_2,C_2),(B_0,C_0))$$
$$(f,f_!) \otimes (g,g_!) \mapsto (gf, g_! \circ g_\#(f_!)).$$

This composition is well-defined because we can apply the formulas from $g_\#$ to (not necessarily graded) morphisms of comodules. The composition is associative because of the following easy-to-check fact: $g_\# f_\# C = (gf)_\# C$ for $B_2 \xrightarrow{f} B_1 \xrightarrow{g} B_0$ functors of dg cocategories and C a dg comodule over B_2.

Now, we define a dg functor

$$Adj : \mathcal{D} \to \mathcal{D}_1$$
$$\text{on objects: } (B,C) \mapsto (B,C)$$
$$\text{on morphisms: } \left((B_1,C_1) \xrightarrow{(F,F_!)} (B_0,C_0)\right) \mapsto \left((B_1,C_1) \xrightarrow{(F,\Phi_F^{-1}F)} (B_0,C_0)\right)$$

where $\Phi_F^{-1} : Hom_{\substack{\text{dg comodules}\\ \text{over } B_1}}(C, F^*D) \to Hom_{\substack{\text{dg comodules}\\ \text{over } B_0}}(F_\# C, D)$ is defined in the proof of Proposition D.3 and makes sense as a function on (not necessarily graded) maps of comodules. To check that Adj commutes with the differentials and respects composition, we need

$$\Phi_F^{-1} \circ d_{Hom_{B_2}(C_2,F^*C_1)} = d_{Hom_{B_1}(F_\# C_2, C_1)} \circ \Phi_F^{-1}$$
$$\Phi_{GF}^{-1}(F^*G_! \circ F_!) = \Phi_G^{-1}(G_!) \circ G_\#(\Phi_F^{-1}(F_!))$$
$$\text{where } (B_2,C_2) \xrightarrow{(F,F_!)} (B_1,C_1) \xrightarrow{(G,G_!)} (B_0,C_0) \text{ in } \mathcal{D}.$$

The equations above follow straight-forwardly from the definitions.

6.2.2. Applying $Cobar$

In this section, we will use the notion of a dg module over a dg category. This is dual to a dg comodule over a dg cocategory (Definition 3.3.3). Given a dg functor between dg categories $F : A_1 \to A_0$, we define "restriction of scalars", F^*, a functor from the category of dg comodules over A_0 to the category of dg comodules over A_1. For M_0 a dg comodule over A_0 and $f \in Obj(B_1)$, $F^*M_0(f) := M_0(Ff)$.

Let $\mathcal{E}$ be the dg category defined below:

$$Obj(\mathcal{E}) = \{(A, M) | \text{A is a dg category, M is a dg module over A}\}$$

$$\mathcal{E}^p((A_1, M_1), (A_0, M_0)) = \{(f, f_!) | f : A_1 \to A_0 \text{ is a dg functor,}$$
$$f_! : M_1 \to f^*M_0 \text{ is a degree-}p\text{ map of modules over } A_1\}$$

$$d_{\mathcal{E}}(f, f_!) = (f,\ d_{f^*M_0} \circ f_! - (-1)^{|f_!|} f_! \circ d_{C_1})$$

$$\mathcal{E}^\bullet((A_2, M_2), (A_1, M_1)) \times \mathcal{E}^\bullet((A_1, M_1), (A_0, M_0)) \xrightarrow{composition} \mathcal{E}^\bullet((A_2, M_2), (A_0, M_0))$$
$$(f, f_!) \times (g, g_!) \mapsto (gf, f^*(g_!) \circ f_!).$$

We will define a dg functor $\Omega : \mathcal{D}_1 \to \mathcal{E}$. On objects,

$$\Omega(B, C) := (Cobar(B), Cobar(B, C))$$

where the first Cobar is a dg functor from the category of dg cocategories to the category of dg categories, and the second Cobar sends dg comodules over B to dg modules over $Cobar(B)$ (see [**6**], Section 4.6). On morphisms,

$$\mathcal{D}_1 \ni \begin{pmatrix} B_1 \xrightarrow{F} B_0 \\ F_\# C_1 \xrightarrow{F_!} C_0 \end{pmatrix} \mapsto \begin{pmatrix} Cobar(B_1) \xrightarrow{Cobar(F)} Cobar(B_0) \\ Cobar(B_1, C_1) \xrightarrow{\Omega(F_!)} (Cobar(F))^* Cobar(B_0, C_0) \end{pmatrix} \in \mathcal{E}$$

where $\Omega(F_!) : Cobar(B_1, C_1) \to (Cobar(F))^* Cobar(B_0, C_0)$

$$(b_1|\dots|b_n|c) \mapsto (Fb_1|\dots|Fb_n|F_!c)$$

for $b_i \in B_1^\bullet(f_{i-1}, f_i), c \in C_1^\bullet(f_n)$, and $f_i \in Obj(B_1), 0 \leq i \leq n$.

It's straightforward from the definitions to check that Ω commutes with the differentials and respects composition.

6.2.3. The end: putting everything together

We have dg functors

$$U(\chi(\mathcal{C})) \xrightarrow{\tilde{\mathcal{F}}} \mathcal{D} \xrightarrow{Adj} \mathcal{D}_1 \xrightarrow{\Omega} \mathcal{E}.$$

This gives our category in dg categories with a trace functor up to homotopy.

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APPENDIX A

Connes cyclic category, Λ

Here, we give generators and relations for the cyclic category, Λ. None of this is new, but we do it to establish notation for the rest of the paper.

Λ has objects $\{[n] : n \in \mathbb{N}\}$ and generating morphisms:

$$\begin{aligned} &\text{rotations } \tau_n : [n] \to [n], \\ &\text{coboundaries } \delta_{j,n} : [n] \to [n-1], 0 \leq j \leq n-1, \\ &\text{codegeneracies } \sigma_{i,n} : [n] \to [n+1], 0 \leq i \leq n \end{aligned} \tag{A.1}$$

subject to relations:

$$
\begin{aligned}
\delta_{i,n-1}\delta_{j,n} &= \delta_{j-1,n-1}\delta_{i,n} \quad 0 \leq i < j \leq n-1 \\
\sigma_{i,n+1}\sigma_{j,n} &= \sigma_{j+1,n+1}\sigma_{i,n} \quad 0 \leq i \leq j \leq n \\
\delta_{j,n+1}\sigma_{i,n} &= \begin{cases} \sigma_{i,n-1}\delta_{j-1,n} & 0 \leq i < j \leq n \\ id & j = i, i-1 \\ \sigma_{i-1,n-1}\delta_{j,n} & 0 \leq j < i-1 \leq n-1 \end{cases} \\
\tau_{n+1}\sigma_{i,n} &= \sigma_{i+1,n}\tau_n \quad 0 \leq i \leq n-1 \\
\tau_{n-1}\delta_{j,n} &= \delta_{j+1,n}\tau_n \quad 0 \leq j \leq n-1 \\
\tau_n^{n+1} &= id \\
\delta_{0,1}\tau_1\sigma_{0,0} &= id \\
\tau_{n+1}\sigma_{n,n} &= \tau_{n+1}^{n+1}\sigma_{0,n}\tau_n \\
\delta_{0,n}\tau_n^2 &= \tau_{n-1}\delta_{n-1,n}.
\end{aligned} \tag{A.2}
$$

Some presentations of Λ include an extra coboundary $\delta_{n,n}$ and codegeneracy $\sigma_{n+1,n}$. In terms of our generators, they are $\delta_{n,n} := \delta_{0,n}\tau_n$ and $\sigma_{n+1,n} := \tau_{n+1}^{n+1}\sigma_{0,n}$.

APPENDIX B

Background on Hochschild chains and cochains

In this appendix, we give some known constructions on Hochschild chains and cochains for the reader's convenience. Let k be a field of characteristic zero, A a flat unital k-algebra, and M be an A-A-bimodule. Then, we can take $(C_\bullet(A, M), b)$, the (reduced or standard) Hochschild chain complex of A with coefficients in M (see Reference [**6**], Equation 2.1). When $M = B$ is also an algebra over k with left and right module structure given by two maps of algebras $f : A \to B$ and $g : A \to B$, respectively, we may write ${}_fB_g$ to clarify the module structure.

Let k, A, M be as above. We can also take $(C^\bullet(A, M), \delta)$, the (reduced) Hochschild cochain complex of A with coefficients in M (see Reference [**6**], Equations 2.12-13, 2.19-21). When $M = B$ is an algebra, $(C^\bullet(A, B), \delta, \cup)$ is a dga where the cup product $\cup$ is given in Reference [**6**], Equation 2.14.

Let $f, g, h : A \to A$ be maps of algebras. We have a contraction operation of Hochschild cochains and chains, which is a map of complexes:

(B.1)

$$\iota : C^p(A,_f A_g) \bigotimes C_{-q}(A,_g A_h) \longrightarrow C_{-(q-p)}(A,_f A_h)$$

$$\phi \bigotimes a_0 \otimes \cdots \otimes a_q \mapsto \iota(\phi, a_0 \otimes \cdots \otimes a_q) := \phi \cdot (a_0 \otimes \cdots \otimes a_q) :=$$

$$:= (-1)^{p(q+1)} \phi(a_{q-p+1}, \ldots, a_q) \cdot a_0 \otimes a_1 \otimes \cdots \otimes a_{q-p}.$$

Finally, we have a "Lie derivative like" operation of Hochschild cochains and chains. Fix an algebra A and let $(\phi_1 \ldots \phi_n|\alpha) \in C(A \to A)(f_0)$ (see Equation **??**) be the following element

$$(\phi_1 \ldots \phi_n|\alpha) = \begin{tikzcd}[column sep=huge] A \arrow[r, bend left=70, "f_0"] \arrow[r, phantom, bend left=45, "\Downarrow \phi_1"] \arrow[r, bend left=25, "f_1"'] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=25, "f_{n-1}"] \arrow[r, phantom, bend right=45, "\Downarrow \phi_n"] \arrow[r, bend right=55, "f_n"'] \arrow[r, phantom, bend right=65, "\alpha"] \arrow[r, bend right=80, "id"'] & A \end{tikzcd}$$

We have a map of complexes

$$C(A \to A)(f_0) \to C_{-\bullet}(A,_{f_0} A)$$

$$(\phi_1 ... \phi_n | a_1 \otimes ... \otimes a_p) \mapsto \lambda(\phi_1 ... \phi_n) \cdot (a_1 \otimes ... \otimes a_p)$$

$$:= \sum_{0 \le i_1 \le \ldots \le i_{2n} \le p} (-1)^{\sum_{\substack{j \text{ odd} \\ j \ge 1}} i_j (|\phi_{i_{\frac{j+1}{2}}}| + 1)} \cdot$$

$$\cdot f_0 a_1 \otimes ... \otimes f_0 a_{i_1} \otimes \phi_1(a_{i_1+1}, ..., a_{i_2}) \otimes$$

$$\otimes f_1 a_{i_2+1} \otimes ... \otimes f_1 a_{i_3} \otimes \phi_2(a_{i_3+1}, ..., a_{i_4}) \otimes$$

$$\otimes ... \otimes \phi_n(a_{i_{2n-1}+1}, ..., a_{i_{2n}}) \otimes f_n a_{i_{2n}+1} \otimes ... \otimes f_n a_p.$$

APPENDIX C

Computations

In this appendix, we give the computational propositions needed to establish the homotopically sheafy-cyclic structure on dg comodules. All the comodules we work with will be cofree, and we will define maps into them by giving maps into cogenerators (see Equation **??**).

C.1. Computational notation

For this section's propositions, we establish the following notation:

$$A_0, A_1 \text{ fixed algebras}$$

$$(\vec{\phi}|\vec{\psi}|\alpha) := (\phi_1 \ldots \phi_n|\psi_1 \ldots \psi_m|\alpha)$$

$$= \begin{tikzcd}[column sep=huge] A_0 \arrow[r, bend left=70, "f_0"] \arrow[r, phantom, bend left=45, "\Downarrow\phi_1"] \arrow[r, bend left=25, "f_1"'] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=25, "f_n"] \arrow[rr, bend right=70, "id"'] \arrow[rr, phantom, bend right=30, "\alpha"] & A_1 \arrow[r, bend left=70, "g_0"] \arrow[r, phantom, bend left=45, "\Downarrow\psi_1"] \arrow[r, bend left=25, "g_1"'] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=25, "g_m"] & A_0 \end{tikzcd} \in T(A_0 \to A_1 \to A_0)(g_0 f_0)$$

$$\vec{\phi}_{\{i_1,i_2,\ldots,i_k\}} := \phi_{i_1}\phi_{i_2}\ldots\phi_{i_k}$$

where $\{i_1, i_2, \ldots, i_k\}$ is an ordered subset of $\{1, \ldots, n\}$

$$\vec{\phi}_{\{\}} := 1 \in k \cong Bar_0(C^\bullet(A_0, A_1))$$

$$\vec{\psi}_{\{\}} := 1 \in k \cong Bar_0(C^\bullet(A_1, A_0))$$

$$|I| := \text{number of elements in a set } I$$

$$I_1 I_2 := \text{concatenation as ordered sets of possibly-empty sets } I_1 \text{ and } I_2$$

$$\epsilon_{I_1,J_1} := (-1)^{(\sum_{r \in I_1} |\phi_r|+1)(\sum_{s \in J_1} |\psi_s|+1)}$$

when I_1, J_1 are ordered indexing sets

$$\lambda(\vec{\psi}),\ \tilde{\delta},\ b',\ b,\ \psi\{\vec{\phi}\} \cdot \alpha = \text{see Appendix B for operations on Hochschild (co)chains}$$

C.1.1. Notation for elements of Hochschild chains

Let $a_0 \otimes a_1 \otimes \cdots \otimes a_n$ denote a typical element of $C_{-\bullet}(A, A)$ where A is some algebra. At times, we wish to feed a portion of $a_0 \otimes a_1 \otimes \ldots \otimes a_n$ to a Hochschild cochain (or other map on chains) without specifying the degree of the cochain. To do this, we will rewrite $a_0 \otimes a_1 \otimes \ldots \otimes a_n = a_0 \otimes \mathfrak{a}_1 \otimes \ldots \otimes \mathfrak{a}_r$ where each $\mathfrak{a}_i = a_{j_i} \otimes a_{j_i+1} \otimes \ldots \otimes a_{j_{i+1}-1}$ and $\mathfrak{a}_i$ is an empty chain if $j_i = j_{i+1}$.

For example, if $\phi \in C^2(A, A)$, then we rewrite

$$\sum_{1 \le i \le n-1} a_0 \otimes a_1 \otimes \ldots a_{i-1} \otimes \phi(a_i, a_{i+1}) \otimes a_{i+2} \otimes \ldots \otimes a_n = \sum a_0 \otimes \mathfrak{a}_1 \otimes \phi(\mathfrak{a}_2) \otimes \mathfrak{a}_3.$$

If $\mathfrak{a}_1 = a_1 \otimes \ldots \otimes a_p$, then $|\mathfrak{a}_1| = p$. For $a_0 \otimes \mathfrak{a}_1 \otimes \mathfrak{a}_2$, we write $\eta_{\mathfrak{a}_1, \mathfrak{a}_2} = (-1)^{|\mathfrak{a}_1|(|\mathfrak{a}_1| + |\mathfrak{a}_2|)}$.

C.2. Computational Propositions

Proposition C.1. *Fix algebras A, B, and let $\hat{\tau}_1 : \mathcal{C}(A, B) \otimes \mathcal{C}(B, A) \to \mathcal{C}(B, A) \otimes \mathcal{C}(A, B)$ be the rotation functor. Recall from Example D.2.2 the descriptions of the cofree dg comodules*

$$m^* T(A) \cong T(A \to B \to A)$$

$$\hat{\tau}^* m^* T(B) \cong T(B \to A \to B).$$

Define a map

$$\tau_{1!}(A, B) : m^* T(A) \cong T(A \to B \to A) \longrightarrow T(B \to A \to B) \cong \hat{\tau}^* m^* T(B)$$

of comodules over $\mathcal{C}(A,B)\otimes\mathcal{C}(B,A)$ *by mapping into cogenerators as follows: for* $(A\overset{f_0}{\to}B\overset{g_0}{\to}A)\in Obj(\mathcal{C}(A,B)\otimes\mathcal{C}(B,A))$,

$$\tau_{1!}(f_0,g_0):T(A\overset{f_0}{\to}B\overset{g_0}{\to}A)^{\bullet}\to T(B\overset{g_0}{\to}A\overset{f_0}{\to}B)^{\bullet}\xrightarrow[\text{cogenerators}]{\text{project onto}}C_{-\bullet}(B,{}_{f_0g_0}B)$$

$$[\tau_{1!}(f_0,g_0)]^{n,m}(\vec{\phi}|\vec{\psi}|\alpha)=\sum_{\substack{I_1I_2=\{2,\cdots,n\}\\ \text{as ordered sets}}}\phi_1(\lambda(\vec{\psi})\lambda(\vec{\phi}_{I_2})\cdot\mathfrak{a}_3,a_0,\mathfrak{a}_1)\otimes\lambda(\vec{\phi}_{I_1})\cdot\mathfrak{a}_2 \tag{C.1}$$
$$\Big(+f_0a_0\otimes\lambda(\vec{\phi})\mathfrak{a}_1\quad if\ \ m=0\Big).$$

where $\vec{\phi}$ *is an element of length* n *and* $\vec{\psi}$ *is an element of length* m *(see Section C.1). Then,* $\tau_{1!}(A,B):m^*T(A)\to\hat{\tau}^*m^*T(B)$ *is a map of dg comodules over* $\mathcal{C}(A,B)\otimes\mathcal{C}(B,A)$.

Proof. We must show: (1) $\tau_{1!}$ is a map of comodules, and (2) $\tau_{1!}$ commutes with the differentials. (In this proof, we drop the subscripts and write $\tau_{1!}:=\tau_{1!}(A,B)$.)

(1) This proof is standard for cofree comodules. Let $(\vec{\phi}|\vec{\psi}|\alpha)$ be as in the statement of the proposition. We want to show that $\tau_{1!}$ commutes with the coproducts. On one hand,

$$\begin{aligned}
&[(id_B\otimes\tau_{1!})\circ\Delta_{m^*T(A)}](\vec{\phi}|\vec{\psi}|\alpha)\\
&=[id_B\otimes\tau_{1!}]\Big(\sum_{\substack{I_1I_2=\{1,2,\cdots,n\}\text{ and}\\ J_1J_2=\{1,2,\cdots,m\}\\ \text{as ordered sets}}}\epsilon_{I_2,J_1}\cdot(\vec{\phi}_{I_1}|\vec{\psi}_{J_1})\otimes(\vec{\phi}_{I_2}|\vec{\psi}_{J_2}|\alpha)\Big)\\
&=\sum_{\substack{I_1I_2I_3=\{1,2,\cdots,n\}\text{ and}\\ J_1J_2J_3=\{1,2,\cdots,m\}\\ \text{as ordered sets}}}\epsilon_{I_2I_3,J_1}\cdot\epsilon_{I_3,J_2}\cdot(\vec{\phi}_{I_1}|\vec{\psi}_{J_1})\otimes(\vec{\phi}_{I_2}|\vec{\psi}_{J_2})\otimes\tau_{1!}^{|I_3|,|J_3|}(\vec{\phi}_{I_3}|\vec{\psi}_{J_3}|\alpha).
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
&[\Delta_{\hat{\tau}^* m^* T(B)} \circ \tau_{1!}](\vec{\phi}|\vec{\psi}|\alpha)\\
&= \Delta_{\hat{\tau}^* m^* T(B)}\Big(\sum_{\substack{I_1 I_2=\{1,2,\cdots,n\} \text{ and}\\ J_1 J_2=\{1,2,\cdots,m\}\\ \text{as ordered sets}}} \epsilon_{I_2,J_1} \cdot (\vec{\phi}_{I_1}|\vec{\psi}_{J_1}) \otimes \tau_{1!}^{|I_2|,|J_2|}(\vec{\phi}_{I_2}|\vec{\psi}_{J_2}|\alpha)\Big)\\
&= \sum_{\substack{I_1 I_2 I_3=\{1,2,\cdots,n\} \text{ and}\\ J_1 J_2 J_3=\{1,2,\cdots,m\}\\ \text{as ordered sets}}} \epsilon_{I_2 I_3,J_1} \cdot \epsilon_{I_3,J_2} \cdot (\vec{\phi}_{I_1}|\vec{\psi}_{J_1}) \otimes (\vec{\phi}_{I_2}|\vec{\psi}_{J_2}) \otimes \tau_{1!}^{|I_3|,|J_3|}(\vec{\phi}_{I_3}|\vec{\psi}_{J_3}|\alpha).
\end{aligned}
$$

Clearly $(id_B \otimes \tau_{1!}) \circ \Delta_{m^*T(A)} = \Delta_{\hat{\tau}^* m^* T(B)} \circ \tau_{1!}$.

(2) We will show that $\tau_{1!}$ commutes with the differentials by direct computation. Since $\tau_{1!}$ is a map of cofree comodules, we only need to check that $\pi_1 \circ D(\tau_{1!}) = 0$ where $D(\tau_{1!})$ is the differential applied to $\tau_{1!}$ as a linear map between complexes and π_1 denotes projection of a comodule onto its cogenerators. More explicitly, we want to check that

$$
\begin{aligned}
&\tau_{1!}^{n,m}(\tilde{\delta}(\vec{\phi})|\vec{\psi}|\alpha) \ + \tau_{1!}^{n,m}(\vec{\phi}|\tilde{\delta}(\vec{\psi})|\alpha) \ + \tau_{1!}^{n-1,m}(b'(\vec{\phi})|\vec{\psi}|\alpha) \ + \tau_{1!}^{n,m-1}(\vec{\phi}|b'(\vec{\psi})|\alpha) +\\
&\tau_{1!}^{n,m}(\vec{\phi}|\vec{\psi}|b(\alpha)) \ + b \circ \tau_{1!}^{n,m}(\vec{\phi}|\vec{\psi}|\alpha) +\\
&\sum_{\substack{I_1 I_2=\{1,\ldots,n\}\\ \text{as ordered sets}}} \epsilon_{I_2,\{1,\ldots,m-1\}} \cdot \tau_{1!}^{|I_1|,m-1}(\vec{\phi}_{I_1}|\vec{\psi}_{\{1,\cdots,m-1\}}|\psi_m\{\vec{\phi}_{I_2}\} \cdot \alpha) +\\
&\sum_{\substack{J_1 J_2=\{1,\ldots,m\}\\ \text{as ordered sets}}} \epsilon_{\{2,\ldots,n\},J_1} \cdot \phi_1\{\psi_{J_1}\} \cdot \tau_{1!}^{n-1,|J_2|}(\phi_{\{2,\cdots,n\}}|\psi_{J_2}|\alpha) +\\
&\epsilon_{\{n\},\{1,\ldots,m\}} \cdot \tau_{1!}^{n-1,m}(\vec{\phi}_{\{1,\cdots,n-1\}}|\vec{\psi}|\phi_n \cdot \alpha) +\\
&\epsilon_{\{1,\ldots,n\},\{1\}} \cdot \psi_1 \cdot \tau_{1!}^{n,m-1}(\vec{\phi}|\vec{\psi}_{\{2,\cdots,m\}}|\alpha)\\
&= 0.
\end{aligned}
\tag{C.2}
$$

In Equation C.2, we will call the terms in rows 1-2 the “standard terms”, and the terms in rows 3-6 the “extra terms”.

We compute the sum of the standard terms. In Table C.1, the leftmost column lists the expressions that don’t cancel in the sum of the standard terms, the middle column gives the standard term from which the expression comes, and the rightmost column gives the term (extra or standard) that cancels the expression.

All of the terms in Table C.1 cancel, so $\tau_{1!}$ is a map of complexes. $\square$

Expression (Expansion)	Comes from Standard Term in Equation C.2	Cancelling Term in Equation C.2
$f_0\psi_1(\lambda(\vec{\phi}_{I_2})\mathfrak{a}_3)\cdot$ $\phi_1(\lambda(\vec{\psi}_{\{2,\cdots,m\}}\lambda(\vec{\phi}_{I_3})\mathfrak{a}_4, a_0, \mathfrak{a}_1) \otimes \lambda(\vec{\phi}_{I_1})\mathfrak{a}_2$	$\tau_{1!}^{n,m}(\delta(\phi_1)\phi_2\cdots\phi_n\|\vec{\psi}\|\alpha)$	$f_0\psi_1 \cdot \tau_{1!}^{n,m-1}(\vec{\phi}\|\vec{\psi}_{\{2,\cdots,m\}}\|\alpha)$
$\phi_1(\lambda(\vec{\psi}_{\{1,\cdots,m-1\}})\lambda(\vec{\phi}_{I_2})\mathfrak{a}_3,$ $\psi_m(\lambda(\vec{\phi}_{I_3})\mathfrak{a}_4)\cdot a_0, \mathfrak{a}_1) \otimes \lambda(\vec{\phi}_{I_1})\mathfrak{a}_2$	$\tau_{1!}^{n,m}(\delta(\phi_1)\phi_2\cdots\phi_n\|\vec{\psi}\|\alpha)$	$\tau_{1!}^{\|I_1\|,m-1}(\vec{\phi}_{I_1}\|\vec{\psi}_{\{1,\cdots,m-1\}}\|\psi_m\{\vec{\phi}_{I_2}\}\cdot\alpha)$
$\phi_1(\lambda(\vec{\psi})\lambda(\vec{\phi}_{I_2})\mathfrak{a}_3, g_m\phi_n(\mathfrak{a}_4)\cdot a_0, \mathfrak{a}_1)\otimes$ $\otimes\,\lambda(\vec{\phi}_{I_1})\mathfrak{a}_2$	$\tau_{1!}^{n,m}(\delta(\phi_1)\phi_2\cdots\phi_n\|\vec{\psi}\|\alpha)$	$\tau_{1!}^{n-1,m}(\vec{\phi}_{\{1,\cdots,n-1\}}\|\vec{\psi}\|g_m\phi_n\cdot\alpha)$
$\phi_1(\lambda(\vec{\psi})\lambda(\vec{\phi}_{I_2})\mathfrak{a}_2)\cdot f_1(a_0)\otimes\lambda(\vec{\phi}_{I_1})\mathfrak{a}_1$	$\tau_{1!}^{n,m}(\delta(\phi_1)\phi_2\cdots\phi_n\|\vec{\psi}\|\alpha)$	$\phi_1\cdot\tau_{1!}^{n-1,0}(\vec{\phi}_{\{2,\cdots,n\}}\|\vec{\psi}\|\alpha)$
$f_0a_0\cdot\phi_1(\mathfrak{a}_1)\otimes\lambda(\vec{\phi}_{\{1,\cdots,n-1\}})\mathfrak{a}_2$	$\tau_{1!}^{n,m}(\delta(\phi_1)\phi_2\cdots\phi_n\|\vec{\psi}\|\alpha)$ if $\vec{\psi}=1$	$b\circ\tau_{1!}^{n,m}(\vec{\phi}\|\vec{\psi}\|\alpha)$ if $\vec{\psi}=1$
$f_0g_m\phi_n(\mathfrak{a}_2)f_0a_0\otimes\lambda(\vec{\phi}_{\{1,\cdots,n-1\}})\mathfrak{a}_1$	$b\circ\tau_{1!}^{n,m}(\vec{\phi}\|\vec{\psi}\|\alpha)$ if $\vec{\psi}=1$	$\tau_{1!}^{n-1,m}(\vec{\phi}_{\{1,\cdots,n-1\}}\|\vec{\psi}\|g_m\phi_n\cdot\alpha)$ if $\vec{\psi}=1$
$\phi_1(\lambda(\vec{\psi})\lambda(\vec{\phi}_{I_2})\mathfrak{a}_4, a_0, \mathfrak{a}_1)\cdot\phi_2(\mathfrak{a}_2)\otimes\lambda(\vec{\phi}_{I_1})\mathfrak{a}_3$	$b\circ\tau_{1!}^{n,m}(\vec{\phi}\|\vec{\psi}\|\alpha)$	$\tau_{1!}^{n-1,m}(\phi_1\cup\phi_2\phi_3\cdots\phi_n\|\vec{\psi}\|\alpha)$
$\phi_1(\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_2})\mathfrak{a}_3)\phi_2(\lambda(\vec{\psi}_{J_2}\lambda(\vec{\phi}_{I_3})\mathfrak{a}_3,$ $a_0, \mathfrak{a}_1)\otimes\lambda(\vec{\phi}_{I_1})\mathfrak{a}_2$	$\tau_{1!}^{n-1,m}(\phi_1\cup\phi_2\phi_3\cdots\phi_n\|\vec{\psi}\|\alpha)$	$\phi_1\{\vec{\psi}_{J_1}\}\cdot\tau_{1!}^{n-1,\|J_2\|}(\vec{\phi}_{\{2,\cdots,n\}}\|\vec{\psi}_{J_2}\|\alpha)$
$f_0\psi_1(\lambda(\vec{\phi}_{I_2})\mathfrak{a}_2)\cdot f_0a_0\otimes\lambda(\vec{\phi}_{I_1})\mathfrak{a}_1$	$f_0\psi_1\cdot\tau_{1!}^{n,0}(\vec{\phi}\|1\|\alpha)$ if $\vec{\psi}=\psi_1$	$\tau_{1!}^{\|I_1\|,0}(\vec{\phi}_{I_1}\|1\|\psi_1\{\vec{\phi}_{I_2}\}\cdot\alpha)$ if $\vec{\psi}=\psi_1$

Table C.1. Expansion of terms in Equation C.2

(Technically, the last term in the middle column is not a standard term, but we include it in the table for convenience.)

Proposition C.2. *Let* $B(A_0, A_1) = B : T(A_0 \to A_1 \to A_0) \longrightarrow T(A_0 \to A_1 \to A_0)$ *be the map of cofree comodules defined by the following maps to cogenerators:*

$$B^{n,m}(\vec{\phi}|\vec{\psi}|\alpha) = \eta_{\mathfrak{a}_1,\mathfrak{a}_2} \cdot 1 \otimes \lambda(\psi)\lambda(\phi)\mathfrak{a}_2 \otimes a_0 \otimes \mathfrak{a}_1. \tag{C.3}$$

Then, $D(B(A_0, A_1)) = \tau_{1!}(A_1, A_0) \circ \tau_{1!}(A_0, A_1) - id$ *where* $\tau_{1!}$ *is defined in Proposition C.1.*

Proof. We prove the statement by direct computation. Since all of the maps are maps of cofree comodules, we only need to check that $\pi_1(D(B(A_0, A_1)) - \tau_{1!}(A_1, A_0) \circ \tau_{1!}(A_0, A_1) - id) = 0$ where π_1 denotes projection of the comodule onto cogenerators. More explicitly, for an element $(\vec{\phi}|\vec{\psi}|\alpha)$, we want to check that

$$\begin{aligned}
&B^{n,m}(\tilde{\delta}(\vec{\phi})|\vec{\psi}|\alpha) \ + B^{n,m}(\vec{\phi}|\tilde{\delta}(\vec{\psi})|\alpha) \ + B^{n-1,m}(b'(\vec{\phi})|\vec{\psi}|\alpha) \ + B^{n,m-1}(\vec{\phi}|b'(\vec{\psi})|\alpha) + \\
&B^{n,m}(\vec{\phi}|\vec{\psi}|b(\alpha)) \ + b \circ B^{n,m}(\vec{\phi}|\vec{\psi}|\alpha) + \\
&\epsilon_{\{n\},\{1,\dots,m\}} \cdot B^{n-1,m}(\vec{\phi}_{\{1,\cdots,n-1\}}|\vec{\psi}_m|\phi_n \cdot \alpha) + \\
&\epsilon_{\{1,\dots,n\},\{1\}} \cdot \psi_1 \cdot B^{n,m-1}(\vec{\phi}|\vec{\psi}_{\{2,\cdots,m\}}|\alpha) + \\
&\sum_{\substack{I_1 I_2=\{1,\dots,n\} \\ \text{as ordered setts}}} \epsilon_{I_2,\{1,\dots,m-1\}} \cdot B^{|I_1|,m-1}(\vec{\phi}_{I_1}|\vec{\psi}_{\{1,\cdots,m-1\}}|\psi_m\{\vec{\phi}_{I_2}\} \cdot \alpha) + \\
&\sum_{\substack{J_1 J_2=\{1,\dots,m\} \\ \text{as ordered setts}}} \epsilon_{\{2,\dots,n\},J_1} \cdot \phi_1\{\psi_{J_1}\} \cdot B^{n-1,|J_2|}(\phi_{\{2,\cdots,n\}}|\psi_{J_2}|\alpha) - \\
&\sum_{\substack{I_1 I_2=\{1,\dots,n\} \\ J_1 J_2=\{1,\dots,m\} \\ \text{as ordered setts}}} \epsilon_{I_1,J_2} \cdot \tau_{1!}^{|J_1|,|I_1|}(\vec{\psi}_{J_1}|\vec{\phi}_{I_1}|\tau_{1!}^{|I_2|,|J_2|}(\vec{\phi}_{I_2}|\vec{\psi}_{J_2}|\alpha)) - \pi_1(\vec{\phi}|\vec{\psi}|\alpha) \\
&= 0.
\end{aligned} \tag{C.4}$$

We will call the terms in rows 1-2 the “standard terms” in the computation of $D(B(A_0, A_1))$, and the terms in rows 3-6 the “extra terms” in the computation of $D(B(A_0, A_1))$. The seventh row is $\pi_1(\tau_{1!}(A_1, A_0) \circ \tau_{1!}(A_0, A_1) - id)$.

We compute the sum of the standard terms. In Table C.2, the leftmost column lists the expressions that don’t cancel in the sum of the standard terms, the middle column gives the standard term from which the expression comes, and the rightmost column gives the extra term that cancels the expression. Table C.3 lists the remaining terms from the seventh row that are not already listed in Table C.2. In Table C.3, the left column lists the remaining expressions that don’t cancel in the seventh row, and the right column gives the extra term that cancels the expression.

All of the terms in the tables describing the expansion of equation C.4 cancel, so $D(B(A_0, A_1)) = \tau_{1!}(A_1, A_0) \circ \tau_{1!}(A_0, A_1) - id.$ $\square$

Expression (Expansion)	Comes from Standard Term in Equation C.4	Cancels with Extra Term in Equation C.4
$\psi_1(\lambda(\vec{\phi}_{I_1})\mathfrak{a}_2) \otimes \lambda(\vec{\psi}_{\{2,\cdots,m\}})\lambda(\vec{\phi}_{I_2})\mathfrak{a}_3 \otimes a_0 \otimes \mathfrak{a}_1$	$b \circ B^{n,m}(\vec{\phi}\|\vec{\psi}\|\alpha)$	$\psi_1\{\vec{\phi}_{I_1}\} \cdot B^{\|I_2\|,m-1}(\vec{\phi}_{I_2}\|\vec{\psi}_{\{2,\cdots,m\}}\|\alpha)$
$g_0\phi_1(\mathfrak{a}_2) \otimes \lambda(\vec{\psi})\lambda(\vec{\phi}_{\{2,\cdots,n\}})\mathfrak{a}_3 \otimes a_0 \otimes \mathfrak{a}_1$	$b \circ B^{n,m}(\vec{\phi}\|\vec{\psi}\|\alpha)$	$\phi_1 \cdot B^{n-1,m}(\vec{\phi}_{\{2,\cdots,n\}}\|\vec{\psi}\|\alpha)$
$1 \otimes \lambda(\vec{\psi})\lambda(\vec{\phi}_{\{1,\cdots,n-1\}})\mathfrak{a}_2 \otimes g_m\phi_n(\mathfrak{a}_3) \cdot a_0 \otimes \mathfrak{a}_1$	$b \circ B^{n,m}(\vec{\phi}\|\vec{\psi}\|\alpha)$	$B^{n-1,m}(\vec{\phi}_{\{1,\cdots,n-1\}}\|\vec{\psi}\|\phi_n \cdot \alpha)$
$1 \otimes \lambda(\vec{\psi}_{\{1,\cdots,m-1\}})\lambda(\vec{\phi}_{I_1})\mathfrak{a}_2 \otimes g_m\psi_m(\lambda(\vec{\phi}_{I_2}\mathfrak{a}_3)) \cdot a_0 \otimes \mathfrak{a}_1$	$b \circ B^{n,m}(\vec{\phi}\|\vec{\psi}\|\alpha)$	$B^{\|I_1\|,m-1}(\vec{\phi}_{I_2}\|\vec{\psi}_{\{1,\cdots,m-1\}}\|\psi_m\{\vec{\phi}_{I_2}\} \cdot \alpha)$
$g_0 f_0 a_0 \otimes \lambda(\vec{\psi})\lambda(\vec{\phi})\mathfrak{a}_1$	$b \circ B^{n,m}(\vec{\phi}\|\vec{\psi}\|\alpha)$	$\tau_{1!}^{\|J_1\|,\|I_1\|}(\vec{\psi}_{J_1}\|\vec{\phi}_{I_1}\|\tau_{1!}^{\|I_2\|,\|J_2\|}(\vec{\phi}_{I_2}\|\vec{\psi}_{J_2}\|\alpha))$

Table C.2. Expansion of terms in Equation C.4: "standard terms" and the "extra terms" that cancel them

(Technically, the last term in the right column is not an extra term, but we include it in the table for convenience.)

Expression (Expansion) from Seventh-Row in Equation C.4	Cancels with Extra Term in Equation C.4
$\psi_1(\lambda(\vec{\phi}_{I_1})\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_4})\mathfrak{a}_4, \phi_{\|I_1\|+1}(\lambda(\vec{\psi}_{J_3})\lambda(\vec{\phi}_{I_5})\mathfrak{a}_5, a_0, \mathfrak{a}_1),$ $\lambda(\vec{\phi}_{I_2\setminus\|I_1\|+1})\mathfrak{a}_2) \otimes \lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_3})\mathfrak{a}_3$	$\psi_1\{\vec{\phi}_{I_1}\} \cdot B^{\|I_2\|,m-1}(\vec{\phi}_{I_2}\|\vec{\psi}_{\{2,\cdots,m\}}\|\alpha)$
$\psi_1(\lambda(\vec{\phi}_{I_1})\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_4})\mathfrak{a}_4, f_{\|I_1\|+1}a_0, \lambda(\vec{\phi}_{I_2\setminus\|I_1\|+1})\mathfrak{a}_1) \otimes \lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_3})\mathfrak{a}_2$	$\phi_1 \cdot B^{n-1,m}(\vec{\phi}_{\{2,\cdots,n\}}\|\vec{\psi}\|\alpha)$
$g_0\phi_1(\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_2})\mathfrak{a}_3, a_0, \mathfrak{a}_1) \otimes \lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_1})\mathfrak{a}_2$	$\psi_1\{\vec{\phi}_{I_1}\} \cdot B^{\|I_2\|,m-1}(\vec{\phi}_{I_2}\|\vec{\psi}_{\{2,\cdots,m\}}\|\alpha)$

Table C.3. Expansion of terms in Equation C.4: remaining "seventh-row terms" and the "extra terms" that cancel them

Proposition C.3. *Let* $\tau_{1!}(A_0, A_1) : T(A_0 \to A_1 \to A_0) \longrightarrow T(A_1 \to A_0 \to A_1)$ *and* $B(A_0, A_1) : T(A_0 \to A_1 \to A_0) \longrightarrow T(A_0 \to A_1 \to A_0)$ *be the maps defined in Propositions C.1 and C.2 above. Then,*

$$[\tau_{1!}, B] := \tau_{1!}(A_0, A_1) \circ B(A_0, A_1) - B(A_1, A_0) \circ \tau_{1!}(A_0, A_1) = 0.$$

Proof. We show that $[\tau_{1!}, B] = 0$ by direct computation. Since all of the maps are maps of cofree comodules, we only need to check that $\pi_1([\tau_{1!}, B]) = 0$ where π_1 denotes projection of the comodule onto cogenerators. We check this directly.

$$\begin{aligned}
[\pi_1 \circ \tau_{1!}(A_0, A_1) \circ B(A_0, A_1)](\vec{\phi}|\vec{\psi}|\alpha) &= \sum_{\substack{I_1 I_2 = \{1,\ldots,n\} \\ J_1 J_2 = \{1,\ldots,m\} \\ \text{as ordered sets}}} \epsilon_{I_1,J_2} \cdot \tau_{1!}^{|I_1|,|J_1|}(\vec{\phi}_{I_1}|\vec{\psi}_{J_1}|B^{|I_2|,|J_2|}(\vec{\phi}_{I_2}|\vec{\psi}_{J_2}|\alpha)) \\
&= \sum_{\substack{I_1 I_2 = \{1,\ldots,n\} \\ J_1 J_2 = \{1,\ldots,m\} \\ \text{as ordered sets}}} \begin{matrix} \epsilon_{I_1,J_2} \cdot \eta_{\mathfrak{a}_1,\mathfrak{a}_2} \cdot \\ \tau_{1!}^{|I_1|,|J_1|}(\vec{\phi}_{I_1}|\vec{\psi}_{J_1}|1 \otimes \lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_2})\mathfrak{a}_2, a_0, \mathfrak{a}_1) \end{matrix} \\
&= \sum_{\substack{I_1 I_2 = \{1,\ldots,n\} \\ J_1 J_2 = \{1,\ldots,m\} \\ \text{as ordered sets}}} \epsilon_{I_1,J_2} \cdot \eta_{\mathfrak{a}_1,\mathfrak{a}_2} \cdot 1 \otimes \lambda(\vec{\phi}_{I_1})\big(\lambda(\vec{\psi})\lambda(\vec{\phi}_{I_2})\mathfrak{a}_2, a_0, \mathfrak{a}_1\big)
\end{aligned}$$

$$
\begin{aligned}
&[\pi_1 \circ B(A_1, A_0) \circ \tau_{1!}(A_0, A_1)](\vec{\phi}|\vec{\psi}|\alpha) \\
&= \sum_{\substack{I_1 I_2 = \{1,\dots,n\} \\ J_1 J_2 = \{1,\dots,m\} \\ \text{as ordered sets}}} \epsilon_{I_1,J_2} \cdot B^{|J_1|,|I_1|}(\vec{\psi}_{J_1}|\vec{\phi}_{I_1}|\tau_{1!}^{|I_2|,|J_2|}(\vec{\phi}_{I_2}|\vec{\psi}_{J_2}|\alpha)) \\
&= \sum_{\substack{I_1 I_2 = \{1,\dots,n\} \\ J_1 J_2 = \{1,\dots,m\} \\ \text{as ordered sets}}} \epsilon_{I_1,J_2} \cdot B^{|J_1|,|I_1|}\big(\vec{\psi}_{J_1}|\vec{\phi}_{I_1}|\phi_{|I_1|+1}(\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_3})\mathfrak{a}_3, a_0, \mathfrak{a}_1) \otimes \lambda(\vec{\phi}_{I_2 \setminus |I_1|+1})\mathfrak{a}_2 + \\
&\qquad\qquad + a_0 \otimes \lambda(\vec{\phi}_{I_2 \setminus |I_1|+1})\mathfrak{a}_1 \quad \text{if } J_2 = \emptyset\big) \\
&= \sum_{\substack{I_1 I_2 = \{1,\dots,n\} \\ J_1 J_2 = \{1,\dots,m\} \\ \text{as ordered sets}}} \epsilon_{I_1,J_2} \cdot \eta_{\mathfrak{a}_2,\mathfrak{a}_3} \cdot 1 \otimes \lambda(\vec{\phi}_{I_1})\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_3})\mathfrak{a}_3 \otimes \phi_{|I_1|+1}(\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_4})\mathfrak{a}_4, a_0, \mathfrak{a}_1) \otimes \\
&\qquad\qquad \otimes \lambda(\vec{\phi}_{I_2 \setminus |I_1|+1})\mathfrak{a}_2 + \\
&\qquad + \epsilon_{I_1,J_2} \cdot \eta_{\mathfrak{a}_1,\mathfrak{a}_2} \cdot 1 \otimes \lambda(\vec{\phi}_{I_1})\lambda(\vec{\psi})\lambda(\vec{\phi}_{I_3})\mathfrak{a}_2 \otimes a_0 \otimes \lambda(\vec{\phi}_{I_2})\mathfrak{a}_1
\end{aligned}
$$

It's clear that $\pi_1 \circ \tau_{1!}(A_0, A_1) \circ B(A_0, A_1) = \pi_1 \circ B(A_1, A_0) \circ \tau_{1!}(A_0, A_1)$: The final expansion of $\pi_1 \circ \tau_{1!}(A_0, A_1) \circ B(A_0, A_1)$ is the sum of the two terms in the final expansion of $\pi_1 \circ B(A_1, A_0) \circ \tau_{1!}(A_0, A_1)$, which is the sum of terms in which one of the ϕ's contains a_0 and the terms in which none of the ϕ's contains a_0). $\square$

C.3. More notation

For the next two propositions, we will need some more notation. Set

$$A_0, A_1, A_2 \text{ fixed algebras}$$

$$(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha) := (\phi_1 \ldots \phi_n|\psi_1 \ldots \psi_m|\theta_1 \ldots \theta_r|\alpha)$$

$$= \begin{tikzcd}[column sep=huge]
A_0 \arrow[r, bend left=70, "f_0"] \arrow[r, bend left=70, phantom, "\Downarrow\phi_1"'] \arrow[r, bend left, "f_1"'] \arrow[r, phantom, "\vdots"] \arrow[r, bend right, "f_n"] \arrow[rrr, bend right=50, "id"'] \arrow[rrr, bend right=25, phantom, "\alpha"] & A_1 \arrow[r, bend left=70, "g_0"] \arrow[r, bend left=70, phantom, "\Downarrow\psi_1"'] \arrow[r, bend left, "g_1"'] \arrow[r, phantom, "\vdots"] \arrow[r, bend right, "g_m"] & A_2 \arrow[r, bend left=70, "h_0"] \arrow[r, bend left=70, phantom, "\Downarrow\theta_1"'] \arrow[r, bend left, "h_1"'] \arrow[r, phantom, "\vdots"] \arrow[r, bend right, "h_p"] & A_0
\end{tikzcd}$$

$$\in T(A_0 \to A_1 \to A_2 \to A_0)(h_0 g_0 f_0)$$

$$\epsilon_{I_2,J_1,J_2,K_1} := (-1)^{(\sum_{r \in I_1} |\phi_r|+1)((\sum_{s \in J_1} |\psi_s|+1)+(\sum_{t \in K_1} |\theta_t|+1))} \cdot$$

$$(-1)^{(\sum_{s \in J_2} |\psi_s|+1)(\sum_{t \in K_1} |\theta_t|+1)}$$

when I_1, J_1, J_2, K_1, are ordered indexing sets

$$\tau_{1!}(A_0 \bullet A_1, A_2) : T(A_0 \to A_1 \to A_2 \to A_0) \to \hat{\tau}_2^{*} T(A_2 \to A_0 \to A_1 \to A_2)$$

$$(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha) \mapsto \tau_{1!}(A_0, A_2)(\vec{\phi} \bullet \vec{\psi}|\vec{\theta}|\alpha) \quad \text{map of dg comodules}$$

$$\tau_{1!}(A_0, A_1 \bullet A_2) : T(A_0 \to A_1 \to A_2 \to A_0) \to \hat{\tau}_2^{*2} T(A_1 \to A_2 \to A_0 \to A_1)$$

$$(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha) \mapsto \tau_{1!}(A_0, A_1)(\vec{\phi}|\vec{\psi} \bullet \vec{\theta}|\alpha) \quad \text{map of dg comodules}$$

C.4. More Propositions

Proposition C.4. *Let*

$$\mathcal{B}(A_0, A_1, A_2) = \mathcal{B} : T(A_0 \to A_1 \to A_2 \to A_0) \to \hat{\tau}_2^{*2} T(A_1 \to A_2 \to A_0 \to A_1)$$

be a map of comodules over $\mathcal{C}(A_0, A_1) \otimes \mathcal{C}(A_1, A_2) \otimes \mathcal{C}(A_2, A_0)$ *determined by the following maps to cogenerators: for* $(A_0 \xrightarrow{f_0} A_1 \xrightarrow{g_0} A_2 \xrightarrow{h_0} A_0) \in Obj(\mathcal{C}(A_0, A_1) \otimes \mathcal{C}(A_1, A_2) \otimes \mathcal{C}(A_2, A_0))$

$$\begin{aligned} \mathcal{B}(f_0, g_0, h_0) : T(A_0 \xrightarrow{f_0} A_1 \xrightarrow{g_0} A_2 \xrightarrow{h_0} A_0)^{\bullet} &\to \hat{\tau}_2^{*2} T(A_1 \xrightarrow{g_0} A_2 \xrightarrow{h_0} A_0 \xrightarrow{f_0} A_1)^{\bullet} \\ &\xrightarrow[cogenerators]{project\ onto} C_{-\bullet}(A_{1, f_0 h_0 g_0} A_{1id}) \end{aligned} \tag{C.5}$$

$$\mathcal{B}^{n,m,p}(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha) = \sum_{\substack{I_1 I_2 = \{1,2,\cdots,n\} \\ as\ ordered\ sets}} \eta_{\mathfrak{a}_1, \mathfrak{a}_2} \cdot 1 \otimes \lambda(\vec{\phi}_{I_1})\big(\lambda(\vec{\theta})\lambda(\vec{\psi})\lambda(\vec{\phi}_{I_2})\mathfrak{a}_2 \otimes a_0 \otimes \mathfrak{a}_1\big)$$

Then,

$$D(\mathcal{B}(A_0, A_1, A_2)) = \tau_{1!}(A_2 \bullet A_0, A_1) \circ \tau_{1!}(A_0 \bullet A_1, A_2) - \tau_{1!}(A_0, A_1 \bullet A_2). \tag{C.6}$$

Proof. We will show that Equation C.6 holds by direct computation. Since all of the maps are maps of cofree comodules, we only need to check that $\pi_1($ Equation C.6 $)$ holds where π_1 denotes projection of the comodule onto cogenerators. More explicitly, we want to check that

(C.7)

$$\mathcal{B}^{n,m,p}(\tilde{\delta}(\vec{\phi})|\vec{\psi}|\vec{\theta}|\alpha) \ + \mathcal{B}^{n,m,p}(\vec{\phi}|\tilde{\delta}(\vec{\psi})|\vec{\theta}|\alpha) \ + \mathcal{B}^{n,m,p}(\vec{\phi}|\vec{\psi}|\tilde{\delta}(\vec{\theta})|\alpha) +$$

$$\mathcal{B}^{n-1,m,p}(b'(\vec{\phi})|\vec{\psi}|\vec{\theta}|\alpha) \ + \mathcal{B}^{n,m-1,p}(\vec{\phi}|b'(\vec{\psi})|\vec{\theta}|\alpha) \ + \mathcal{B}^{n,m,p-1}(\vec{\phi}|\vec{\psi}|b'(\vec{\theta})|\alpha) +$$

$$\mathcal{B}^{n,m,p}(\vec{\phi}|\vec{\psi}|\vec{\theta}|b(\alpha)) \ + b \circ \mathcal{B}^{n,m,p}(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha) +$$

$$\sum_{\substack{I_1 I_2=\{1,\ldots,n\} \\ J_1 J_2=\{1,\ldots,m\} \\ \text{as ordered sets}}} \epsilon_{I_2,J_1,J_2,\{1,\ldots,p-1\}} \cdot \mathcal{B}^{|I_1|,|J_1|,p-1}(\vec{\phi}_{I_1}|\vec{\psi}_{J_1}|\vec{\theta}_{\{1,\cdots,p-1\}}|\theta_p\{\vec{\psi}_{J_2}\}\{\vec{\phi}_{I_2}\} \cdot \alpha) +$$

$$\sum_{\substack{I_1 I_2=\{1,\ldots,n\} \\ \text{as ordered sets}}} \epsilon_{I_2,\{1,\ldots,m-1\},\{m\},\{1,\ldots,p\}} \cdot \mathcal{B}^{|I_1|,m-1,p}(\vec{\phi}_{I_1}|\vec{\psi}_{\{1,\cdots,m-1\}}|\vec{\theta}|\psi_m\{\vec{\phi}_{I_2}\} \cdot \alpha) +$$

$$\epsilon_{\{n\},\{1,\ldots,m\},\{\},\{1,\ldots,p\}} \cdot \mathcal{B}^{n-1,m,p}(\vec{\phi}_{\{1,\cdots,n-1\}}|\vec{\psi}_m|\vec{\theta}|\phi_n \cdot \alpha) +$$

$$\sum_{\substack{J_1 J_2=\{1,\ldots,m\} \\ K_1 K_2=\{1,\ldots,p\} \\ \text{as ordered sets}}} \epsilon_{\{2,\ldots,n\},J_1,J_2,K_1} \cdot \phi_1\{\vec{\theta}_{K_1}\}\{\vec{\psi}_{J_1}\} \cdot \mathcal{B}^{n-1,|J_2|,|K_2|}(\vec{\phi}_{\{2,\cdots,n\}}|\vec{\psi}_{J_2}|\vec{\theta}_{K_2}|\alpha) +$$

$$\sum_{\substack{J_1 J_2=\{1,\ldots,m\} \\ \text{as ordered sets}}} \epsilon_{\{1,\ldots,n\},J_1,J_2,\{1\}} \cdot \theta_1\{\vec{\psi}_{J_1}\} \cdot \mathcal{B}^{n,|J_2|,p-1}(\vec{\phi}|\vec{\psi}_{J_2}|\vec{\theta}_{\{2,\cdots,p\}}|\alpha) +$$

$$\epsilon_{\{1,\ldots,n\},\{1\},\{2,\ldots,m\},\{\}} \cdot \psi_1 \cdot \mathcal{B}^{n,m-1,p}(\vec{\phi}|\vec{\psi}_{\{2,\cdots,m\}}|\vec{\theta}|\alpha) +$$

$$\tau_{1!}^{n,p\le * \le m+p}(\vec{\phi}|\vec{\psi} \bullet \vec{\theta}|\alpha) +$$

$$\sum_{\substack{I_1 I_2=\{1,\ldots,n\} \\ J_1 J_2=\{1,\ldots,m\} \\ K_1 K_2=\{1,\ldots,p\} \\ \text{as ordered sets}}} \begin{array}{l} \epsilon_{I_2,J_1,J_2,K_1} \cdot \\ \tau_{1!}^{|I_1|\le * \le |I_1|+|K_1|,|J_1|}(\vec{\theta}_{K_1} \bullet \vec{\phi}_{I_1}, \vec{\psi}_{J_1}, \tau_{1!}^{|J_2|\le * \le |I_2|+|J_2|,|K_2|}(\vec{\phi}_{I_2} \bullet \vec{\psi}_{J_2}|\vec{\theta}_{K_2}|\alpha)) \end{array}$$

$$= 0.$$

In Equation C.7 above, we call the terms in rows 1-3 the "standard terms" in the computation of $D(\mathcal{B}(A_0, A_1, A_2))$, and the terms in rows 4-9 the "extra terms" in the computation of $D(\mathcal{B}(A_0, A_1, A_2))$. The terms in rows 10-11 are π_1 of the righthand side of Equation C.6; we will call these the "10^{th}- and 11^{th}-row terms".

We compute the sum of the standard terms. In Table C.4, the leftmost column lists the expressions that don't cancel in the sum of the standard terms, the middle column gives the standard term from which the expression comes, and the rightmost column gives the term that cancels the expression. Table C.5 lists the remaining ninth row terms that aren't already listed in Table C.4. In Table C.5, the left column lists the remaining expressions that don't cancel in the ninth row, and the right column gives the extra term that cancels the expression.

All of the terms in the tables describing the expansion of Equation C.7 cancel, so we're done. $\square$

Expression (Expansion)	Comes from Standard Term in Equation C.7	Cancelling Term in Equation C.7
$1 \otimes \lambda(\vec{\phi}_{I_1})[\lambda(\vec{\theta}_{\{1,\cdots,p-1\}}\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_2})\mathfrak{a}_2\otimes$ $\otimes\, \theta_p(\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_3})\mathfrak{a}_3)\cdot a_0 \otimes \mathfrak{a}_1]$	$b \circ \mathcal{B}^{n,m,p}(\vec{\phi}\vert\vec{\psi}\vert\vec{\theta}\vert\alpha)$	$\mathcal{B}^{\vert I_1\vert,\vert J_1\vert,p-1}(\vec{\phi}_{I_1}\vert\vec{\psi}_{J_1}\vert\vec{\theta}_{\{1,\cdots,p-1\}}\vert$ $\theta_p\{\vec{\psi}_{J_2}\}\{\vec{\phi}_{I_2}\}\cdot\alpha)$
$1 \otimes \lambda(\vec{\phi}_{I_1})[\lambda(\vec{\theta}\lambda(\vec{\psi}_{\{1,\cdots,m-1\}})\lambda(\vec{\phi}_{I_2})\mathfrak{a}_2\otimes$ $\otimes\, \psi_m(\lambda(\vec{\phi}_{I_3})\mathfrak{a}_3)\cdot a_0 \otimes \mathfrak{a}_1]$	$b \circ \mathcal{B}^{n,m,p}(\vec{\phi}\vert\vec{\psi}\vert\vec{\theta}\vert\alpha)$	$\mathcal{B}^{\vert I_1\vert,m-1,p}(\vec{\phi}_{I_1}\vert\vec{\psi}_{\{1,\cdots,m-1\}}\vert\vec{\theta}\vert$ $\psi_m\{\vec{\phi}_{I_2}\}\cdot\alpha)$
$1 \otimes \lambda(\vec{\phi}_{I_1})[\lambda(\vec{\theta}\lambda(\vec{\psi}\lambda(\vec{\phi}_{\{1,\cdots,n-1\}})\mathfrak{a}_2 \otimes \psi_n(\mathfrak{a}_3)\cdot a_0 \otimes \mathfrak{a}_1]$	$b \circ \mathcal{B}^{n,m,p}(\vec{\phi}\vert\vec{\psi}\vert\vec{\theta}\vert\alpha)$	$\mathcal{B}^{n-1,m,p}(\vec{\phi}_{\{1,\cdots,n-1\}}\vert\vec{\psi}\vert\vec{\theta}\vert\phi_n\cdot\alpha)$
$\phi_1(\lambda(\vec{\theta}_{K_1})\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_2})\mathfrak{a}_2)\otimes$ $\otimes\, \lambda(\vec{\phi}_{I_1\setminus 1})[\lambda(\vec{\theta}_{K_2})\lambda(\vec{\psi}_{J_3})\lambda(\vec{\phi}_{I_3})\mathfrak{a}_3 \otimes a_0 \otimes \mathfrak{a}_1]$	$b \circ \mathcal{B}^{n,m,p}(\vec{\phi}\vert\vec{\psi}\vert\vec{\theta}\vert\alpha)$	$\phi_1\{\vec{\theta}_{K_1}\}\{\vec{\psi}_{J_1}\}\cdot$ $\mathcal{B}^{n-1,\vert J_2\vert,\vert K_2\vert}(\vec{\phi}_{\{2,\cdots,n\}}\vert\vec{\psi}_{J_2}\vert\vec{\theta}_{K_2}\vert\alpha)$
$f_0\theta_1(\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_2})\mathfrak{a}_2)\otimes$ $\otimes\, \lambda(\vec{\phi}_{I_1})[\lambda(\vec{\theta}_{\{2,\cdots,p\}})\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_3})\mathfrak{a}_3 \otimes a_0 \otimes \mathfrak{a}_1]$	$b \circ \mathcal{B}^{n,m,p}(\vec{\phi}\vert\vec{\psi}\vert\vec{\theta}\vert\alpha)$	$\theta_1\{\vec{\psi}_{J_1}\}\cdot$ $\mathcal{B}^{n,\vert J_2\vert,p-1}(\vec{\phi}\vert\vec{\psi}_{J_2}\vert\vec{\theta}_{\{2,\cdots,p\}}\vert\alpha)$
$f_0h_0\psi_1(\lambda(\vec{\phi}_{I_2})\mathfrak{a}_2)\otimes$ $\otimes\, \lambda(\vec{\phi}_{I_1})[\lambda(\vec{\theta})\lambda(\vec{\psi}_{\{2,\cdots,m\}})\lambda(\vec{\phi}_{I_3})\mathfrak{a}_3 \otimes a_0 \otimes \mathfrak{a}_1]$	$b \circ \mathcal{B}^{n,m,p}(\vec{\phi}\vert\vec{\psi}\vert\vec{\theta}\vert\alpha)$	$\psi_1 \cdot \mathcal{B}^{n,m-1,p}(\vec{\phi}\vert\vec{\psi}_{\{2,\cdots,m\}}\vert\vec{\theta}\vert\alpha)$
$f_0h_0g_0\phi_{i_1}(\lambda(\vec{\theta}_{K_2})\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_3})\mathfrak{a}_3, a_0, \mathfrak{a}_1)\otimes$ $\otimes\, \lambda(\vec{\phi}_{I_1})\lambda(\vec{\theta}_{K_1})\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_2\setminus i_1})\mathfrak{a}_2$	$b \circ \mathcal{B}^{n,m,p}(\vec{\phi}\vert\vec{\psi}\vert\vec{\theta}\vert\alpha)$	11^{th} row
$f_0h_0g_0f_{i_1}a_0 \otimes \lambda(\vec{\phi}_{I_1})\lambda(\vec{\theta})\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_2})\mathfrak{a}_1$	$b \circ \mathcal{B}^{n,m,p}(\vec{\phi}\vert\vec{\psi}\vert\vec{\theta}\vert\alpha)$	11^{th} row
$\phi_1(\lambda(\vec{\phi}_{I_1})\lambda(\vec{\theta})\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_2})\mathfrak{a}_3, a_0, \mathfrak{a}_1) \otimes \lambda(\vec{\phi}_{I_1\setminus 1})\mathfrak{a}_2$	$b \circ \mathcal{B}^{n,m,p}(\vec{\phi}\vert\vec{\psi}\vert\vec{\theta}\vert\alpha)$	10^{th} row

Table C.4. Expansion of terms in Equation C.7: “standard terms” and the terms that cancel them

Expression (expansion) from 11^{th}-Row Term in Equation C.7	Cancels with Extra Term in Equation C.7
$\phi_1(\lambda(\vec{\theta}_{K_1})\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_2})[\lambda(\vec{\theta}_{K_3})\lambda(\vec{\psi}_{J_4})\lambda(\vec{\phi}_{I_5})\mathfrak{a}_3, a_0, \mathfrak{a}_1])\otimes$ $\otimes\, \lambda(\vec{\phi}_{I_1\setminus 1})\lambda(\vec{\theta}_{K_2})\lambda(\vec{\psi}_{J_3})\lambda(\vec{\phi}_{I_4})\mathfrak{a}_2$	$\phi_1\{\vec{\theta}_{K_1}\}\{\vec{\psi}_{J_1}\}\cdot$ $\mathcal{B}^{n-1,\lvert J_2\rvert,\lvert K_2\rvert}(\vec{\phi}_{\{2,\cdots,n\}}\lvert\vec{\psi}_{J_2}\lvert\vec{\theta}_{K_2}\lvert\alpha)$
$f_0\theta_1(\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_2})[\lambda(\vec{\theta}_{K_2})\lambda(\vec{\psi}_{J_3})\lambda(\vec{\phi}_{I_4})\mathfrak{a}_3, a_0, \mathfrak{a}_1])\otimes$ $\otimes\, \lambda(\vec{\phi}_{I_1})\lambda(\vec{\theta}_{K_1\setminus 1})\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_3})\mathfrak{a}_2$	$\theta_1\{\vec{\psi}_{J_1}\}\cdot$ $\mathcal{B}^{n,\lvert J_2\rvert,p-1}(\vec{\phi}\lvert\vec{\psi}_{J_2}\lvert\vec{\theta}_{\{2,\cdots,p\}}\lvert\alpha)$
$f_0h_0\psi_1(\lambda(\vec{\phi}_{I_2})[\lambda(\vec{\theta}_{K_2})\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_4})\mathfrak{a}_3, a_0, \mathfrak{a}_1])\otimes\lambda(\vec{\phi}_{I_1})\lambda(\vec{\theta}_{K_1})\lambda(\vec{\psi}_{J_1\setminus 1})\lambda(\vec{\phi}_{I_3})\mathfrak{a}_2$	$\psi_1\cdot\mathcal{B}^{n,m-1,p}(\vec{\phi}\lvert\vec{\psi}_{\{2,\cdots,m\}}\lvert\vec{\theta}\lvert\alpha)$

Table C.5. Expansion of terms in Equation C.7: remaining "11^{th} row terms" and the "extra terms" that cancel them

Proposition C.5. *Let $\tau_{1!}$ and $\mathcal{B}$ be as defined in the previous propositions. Then, $[\tau_{1!}, \mathcal{B}] := \tau_{1!}(A_1 \bullet A_2, A_0) \circ \mathcal{B}(A_0, A_1, A_2) - \mathcal{B}(A_2, A_0, A_1) \circ \tau_{1!}(A_0 \bullet A_1, A_2) = 0$. (Note that $[\tau_{1!}, \mathcal{B}]$ is a map from $T(A_0 \to A_1 \to A_2 \to A_0)$ to itself.)*

Proof. We show the proposition by direct computation. Since all of the maps are maps of cofree comodules, we only need to check that $\pi_1([\tau_{1!}, \mathcal{B}]) = 0$ where π_1 denotes projection of the comodule onto cogenerators. We check this directly.

$$[\pi_1 \circ \tau_{1!}(A_1 \bullet A_2, A_0) \circ \mathcal{B}(A_0, A_1, A_2)](\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha)$$

$$= \sum_{\substack{I_1 I_2 = \{1,\ldots,n\} \\ J_1 J_2 = \{1,\ldots,m\} \\ K_1 K_2 = \{1,\ldots,p\} \\ \text{as ordered sets}}} \epsilon_{I_2,J_1,J_2,K_1} \cdot \tau_{1!}^{|K_1| \le * \le |K_1|+|J_1|,|I_1|}(\vec{\psi}_{J_1} \bullet \vec{\theta}_{K_1}|\vec{\phi}_{I_1}|\mathcal{B}^{|I_2|,|J_2|,|K_2|}(\vec{\phi}_{I_2}|\vec{\psi}_{J_2}|\vec{\theta}_{K_2}|\alpha))$$

$$= \sum_{\substack{I_1 I_2 = \{1,\ldots,n\} \\ J_1 J_2 = \{1,\ldots,m\} \\ K_1 K_2 = \{1,\ldots,p\} \\ \text{as ordered sets}}} \begin{array}{l} \epsilon_{I_2,J_1,J_2,K_1} \cdot \eta_{\mathfrak{a}_1,\mathfrak{a}_2} \cdot \\ \tau_{1!}^{|K_1| \le * \le |K_1|+|J_1|,|I_1|}(\vec{\psi}_{J_1} \bullet \vec{\theta}_{K_1}|\vec{\phi}_{I_1}|1 \otimes \lambda(\vec{\phi}_{I_2})[\lambda(\vec{\theta}_{K_2})\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_3})\mathfrak{a}_2, a_0, \mathfrak{a}_1]) \end{array}$$

$$= \sum_{\substack{I_1 I_2 = \{1,\ldots,n\} \\ J_1 J_2 = \{1,\ldots,m\} \\ K_1 K_2 = \{1,\ldots,p\} \\ \text{as ordered sets}}} \epsilon_{I_2,J_1,J_2,K_1} \cdot \eta_{\mathfrak{a}_1,\mathfrak{a}_2} \cdot 1 \otimes \lambda(\vec{\theta}_{K_1})\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_1})[\lambda(\vec{\theta}_{K_2})\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_2})\mathfrak{a}_2, a_0, \mathfrak{a}_1]$$

$$
\begin{aligned}
&[\pi_1 \circ \mathcal{B}(A_2, A_0, A_1) \circ \tau_{1!}(A_0 \bullet A_1, A_2)](\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha) \\
= &\sum_{\substack{I_1 I_2 = \{1,\dots,n\} \\ J_1 J_2 = \{1,\dots,m\} \\ K_1 K_2 = \{1,\dots,p\} \\ \text{as ordered sets}}} \epsilon_{I_2,J_1,J_2,K_1} \cdot \mathcal{B}^{|K_1|,|I_1|,|J_1|}(\vec{\theta}_{K_1}|\vec{\phi}_{I_1}|\vec{\psi}_{J_1}|\tau_{1!}^{|J_2|\le * \le |I_2|+|J_2|,|K_2|}(\vec{\phi}_{I_2} \bullet \vec{\psi}_{J_2}|\vec{\theta}_{K_2}|\alpha)) \\
= &\sum_{\substack{I_1 I_2 = \{1,\dots,n\} \\ J_1 J_2 = \{1,\dots,m\} \\ K_1 K_2 = \{1,\dots,p\} \\ \text{as ordered sets}}} \epsilon_{I_2,J_1,J_2,K_1} \cdot \eta_{\mathfrak{a}_1,\mathfrak{a}_2} \cdot 1 \otimes \lambda(\vec{\theta}_{K_1})\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_1})[\lambda(\vec{\theta}_{K_2})\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_2})\mathfrak{a}_2, a_0, \mathfrak{a}_1]
\end{aligned}
$$

It's clear that $\pi_1([\tau_{1!}, \mathcal{B}]) = 0$. $\square$

APPENDIX D

Pullbacks, Pushforwards and an Adjunction

In the first section of this appendix, we give the definition of the natural pullback used for dg comodules and show that it satisfies Equation 4.1 (Proposition D.1). We also prove a useful Proposition D.2 describing the pullbacks of cofree dg comodules in terms of cogenerators. We then use Proposition D.2 to compute some examples of pullbacks.

In Section D.3, we show that our pullback is right adjoint to a pushforward. This adjunction is used in Section **??** when passing from dg cocategories and dg comodules to dg categories and dg modules. Use of this adjunction is not central to our narrative, and may perhaps become unnecessary as understanding of the structure on dg categories and dg modules evolves.

A technical detail in all of this is that we work with conilpotent dg comodules over conilpotent dg cocategories. We discuss these details in Section D.4.

D.1. Pullbacks of dg comodules

Let $\lambda : B_1 \to B_0$ be a functor between conilpotent dg cocategories. In this section, we will define a functor λ^* from the category of conilpotent dg comodules over B_0 to the category of conilpotent dg comodules over B_1. We call λ^* "co-extension of scalars".

D.1.1. Category-theoretic definition of λ^*

Let λ be as above, and let C be a conilpotent dg comodule over B_0. We define λ^*C as follows:

$$\lambda^*C := ker\Big(B_1 \otimes_\lambda C \underset{(\mathrm{id_{B_1}}\otimes\lambda\otimes\mathrm{id_C})\circ(\Delta_{\mathrm{B_1}}\otimes\mathrm{id_C})}{\overset{\mathrm{id_{B_1}}\otimes\Delta_{\mathrm{C}}}{\rightrightarrows}} B_1 \otimes_\lambda B_0 \otimes C\Big) \tag{D.1}$$

where $B_1 \otimes_\lambda C$ and $B_1 \otimes_\lambda B_0 \otimes C$ are dg comodules over B_1 defined below. For $f \in Obj(B_1)$,

$$[B_1 \otimes_\lambda C](f) := \Big(\bigoplus_{h \in Obj(B_1)} B_1^\bullet(f,h) \otimes C^\bullet(\lambda h), \Delta(f) = \bigoplus_h \Delta_{B_1(f,h)} \otimes id_{C(\lambda h)}\Big)$$

$$[B_1 \otimes_\lambda B_0 \otimes C](f) := \Big(\bigoplus_{\substack{h_1 \in Obj(B_1),\\ h_2 \in Obj(B_0)}} B_1^\bullet(f,h_1) \otimes B_0^\bullet(\lambda h_1, h_2) \otimes C^\bullet(h_2),$$

$$\Delta(f) = \bigoplus_{h_1,h_2} \Delta_{B_1(f,h_1)} \otimes id_{B_0(\lambda h_1, h_2)} \otimes id_{C(h_2)}\Big).$$

The names of the maps in Equation D.1 are also meant to be suggestive. In full detail, for $f \in Obj(B_1)$,

$$[id_{B_1} \otimes \Delta_C](f) := \bigoplus_h id_{B_1(f,h)} \otimes \Delta_C(\lambda h)$$

and

$$[B_1 \otimes_\lambda C](f) \xrightarrow{[\Delta_{B_1} \otimes id_C](f) := \bigoplus_h \Delta_{B_1}(f,h) \otimes id_{C(\lambda h)}} \bigoplus_{h_1,h_2 \in Obj(B_1)} B_1(f,h_1) \otimes B_1(h_1,h_2) \otimes C(\lambda h_2)$$

$$\xrightarrow{[id_{B_1} \otimes \lambda \otimes id_C](f) := \bigoplus_{h_1,h_2} id_{B_1(f,h_1)} \otimes \lambda(h_1,h_2) \otimes id_{C(\lambda h)}} [B_1 \otimes_\lambda B_0 \otimes C](f).$$

That the kernel is well-defined follows formally from the abelianness of the category of chain complexes, but it is also easy to check that the induced differentials from $[B_1 \otimes_\lambda C](f)$

on the kernel are well-defined. Since $\Delta_{\lambda^* C}$ is induced by Δ_{B_1}, we have that $\Delta_{\lambda^* C}$ also satisfies coassociativity, counitality and conilpotency.

Next, we will define λ^* on morphisms. Let $F : C \to D$ be a map of conilpotent dg comodules over B_0. By the universal property of $\lambda^* D$, we can define a morphism $\lambda^* F : \lambda^* C \to \lambda^* D$ by giving a morphism from $(\lambda^* F)' : \lambda^* C \to B_1 \otimes_\lambda D$ such that the two maps

(D.2)

$$(id_{B_1} \otimes \Delta_D) \circ (\lambda^* F)', (id_{B_1} \otimes \lambda \otimes id_D) \circ (\Delta_{B_1} \otimes id_D) \circ (\lambda^* F)' : \lambda^* C \to B_1 \otimes_\lambda D \rightrightarrows B_1 \otimes_\lambda B_0 \otimes D$$

coincide. We define $(\lambda^* F)'$ as follows:

$$(\lambda^* F)' : \lambda^* C \xrightarrow[inclusion]{canonical} B_1 \otimes_\lambda C \xrightarrow{id_{B_1} \otimes F} B_1 \otimes_\lambda D$$

It's easy to check that the two maps in Equation D.2 coincide: Let $b \otimes c$ be an arbitrary element of $\lambda^* C(f) \hookrightarrow [B_1 \otimes_\lambda C](f)$. Then,

$$\begin{aligned}
[(id_{B_1} \otimes \Delta_D) \circ (\lambda^* F)'](b \otimes c) &= \sum_{(Fc)} b \otimes (Fc)_{(1)} \otimes (Fc)_{(2)} \\
&= \sum_{(c)} b \otimes Fc_{(1)} \otimes Fc_{(2)} \quad \text{(F is a map of comodules)} \\
&= [(id_{B_1} \otimes F \otimes F) \circ (id_{B_1} \otimes \Delta_C)](b \otimes c) \\
&= [(id_{B_1} \otimes F \otimes F) \circ (id_{B_1} \otimes \lambda \otimes id_C) \circ (\Delta_{B_1} \otimes id_C)](b \otimes c) \\
&\quad (b \otimes c \text{ is in the kernel}) \\
&= \sum_{(b)} b_{(1)} \otimes \lambda b_{(2)} \otimes Fc \\
&= [(id_{B_1} \otimes \lambda \otimes id_D) \circ (\Delta_{B_1} \otimes id_D) \circ (\lambda^* F)'](b \otimes c).
\end{aligned}$$

So, $\lambda^* F$ is well-defined. In summary, we have commuting diagrams:

(D.3)
$$\begin{tikzcd}
\lambda^* C \arrow[r, "\substack{\text{canonical} \\ \text{inclusion}}"] \arrow[d, "\lambda^* F"'] & B_1 \otimes_\lambda C \arrow[d, "id_{B_1} \otimes F = \text{ map inducing } \lambda^* F"] \\
\lambda^* D \arrow[r, "\substack{\text{canonical} \\ \text{inclusion}}"] & B_1 \otimes_\lambda D
\end{tikzcd}$$

Finally, it is straightforward to see that λ^* is a functor, i.e., that λ^* preserves composition of morphisms: Let $C \xrightarrow{F} D \xrightarrow{G} E$ be composable morphisms of dg comodules over B_0. The maps inducing $\lambda^* F$, $\lambda^* G$ and $\lambda^*(G \circ F)$ are $id_{B_1} \otimes F$, $id_{B_1} \otimes G$ and $id_{B_1} \otimes GF$, respectively. The inducing maps respect composition–$(id_{B_1} \otimes G) \circ (id_{B_1} \otimes F) = id_{B_1} \otimes GF$–and by the commuting diagrams D.3, the functor λ^* does as well.

Proposition D.1. *Let $F : B_2 \to B_1$ and $G : B_1 \to B_0$ be functors between dg cocategories B_2, B_1 and B_0. Let M be a dg comodule over B_0. Then,*

$$(GF)^*M \cong F^*G^*M.$$

Proof. We will prove the proposition by showing that F^*G^*M satisfies the universal property of $(GF)^*M$. First, let N be a dg comodule over B_2 and $H : N \to B_2 \otimes_{GF} M$ be a map of dg comodules such that the two maps

(D.4)

$$(id_{B_2} \otimes GF \otimes id_M) \circ (\Delta_{B_2} \otimes id_M) \circ H, (id_{B_2} \otimes \Delta_M) \circ H : N \to B_2 \otimes_{GF} M \rightrightarrows B_2 \otimes_{GF} \otimes B_0 \otimes M$$

coincide. We will show that H determines a map of dg comodules $\tilde{H} : N \to F^*G^*M$. Let $x \in Obj(B_2)$. Define

$$\begin{aligned} H'_x : N(x) &\xrightarrow{H_x} \bigoplus_{y \in Obj(B_2)} B_2(x, y) \otimes M(GFy) \\ &\xrightarrow{F \otimes id_M} \bigoplus_{y \in Obj(B_2)} B_1(Fx, Fy) \otimes M(GFy) \\ &\subset [B_1 \otimes_G M](Fx). \end{aligned}$$

The image of H'_x lands in $G^*M(Fx)$, a subcomplex of $[B_1 \otimes_G M](Fx)$; checking this is straightforward using the universal property of G^*M, the fact that F commutes with the coproducts, and Equation D.4. So, for each $x \in Obj(B_2)$, we have a map of complexes

$H'_x : N(x) \to G^*M(Fx)$. Now define $\tilde{H}$ as follows:

$$
\begin{aligned}
\tilde{H}_x : N(x) &\xrightarrow{\Delta_N} \bigoplus_{y \in Obj(B_2)} B_2(x,y) \otimes N(y) \\
&\xrightarrow{\prod id_{B_2} \otimes H'_y} \bigoplus_{y \in Obj(B_2)} B_2(x,y) \otimes G^*M(Fy) \\
&\subset [B_2 \otimes_F G^*M](x).
\end{aligned}
$$

Showing that $\tilde{H}$ lands in G^*F^*M, a subcomodule of $[B_2 \otimes_F G^*M]$, is also straightforward; we only need that F and H commute with the appropriate coproducts, and that the cocomposition on B_2 is coassociative. So, for each $x \in Obj(B_2)$, we have a map $\tilde{H}_x : N(x) \to G^*F^*M(x)$. It's clear that $\tilde{H}$ is a map of dg comodules since all of the maps used to construct $\tilde{H}$ are maps of dg comodules.

Now, let $\tilde{H} : N \to F^*G^*M$ be a map of dg comodules over B_2. We will show that $\tilde{H}$ determines a map of dg comodules $H : N \to B_2 \otimes_G FM$ satisfying Equation D.4. For $x \in Obj(B_2)$, let H be defined as follows:

$$
\begin{aligned}
H_x : N(x) &\xrightarrow{\tilde{H}_x} F^*G^*M(x) \\
&\xrightarrow[\text{inclusion}]{\text{canonical}} \bigoplus_{\substack{y \in Obj(B_2) \\ z_1 \in Obj(B_1)}} B_2(x,y) \otimes B_1(Fy, z_1) \otimes M(Gz_1) \\
&\xrightarrow{id_{B_2} \otimes \epsilon_{B_1} \otimes id_M} \bigoplus_{y \in Obj(B_2)} B_2(x,y) \otimes M(GFy).
\end{aligned}
$$

The universal property of G^*M implies that $(id_{B_2} \otimes \Delta_M) \circ H$ is equal to:

$$N(x) \xrightarrow{\tilde{H}_x} \bigoplus_{\substack{y \in Obj(B_2) \\ z_1 \in Obj(B_1)}} B_2(x,y) \otimes B_1(Fy, z_1) \otimes M(Gz_1)$$

$$\xrightarrow[(id_{B_2} \otimes \Delta_{B_1} \otimes id_M)]{(id_{B_2} \otimes id_{B_1} \otimes G \otimes id_M) \circ} \bigoplus_{\substack{y \in Obj(B_2) \\ y_1, z_1 \in Obj(B_1)}} B_2(x,y) \otimes B_1(Fy, y_1) \otimes B_0(Gy_1, Gz_1) \otimes M(Gz_1)$$

$$\xrightarrow{id_{B_2} \otimes \epsilon_{B_1} \otimes id_{B_0} \otimes id_M} \bigoplus_{\substack{y \in Obj(B_2) \\ z_1 \in Obj(B_1)}} B_2(x,y) \otimes B_0(GFy, Gz_1) \otimes M(Gz_1).$$

On the other hand, the universal property of F^* implies that $(id_{B_2} \otimes GF \otimes id_M) \circ (\Delta_{B_2} \otimes id_M) \circ H$ is equal to:

$$N(x) \xrightarrow{\tilde{H}_x} \bigoplus_{\substack{y \in Obj(B_2) \\ z_1 \in Obj(B_1)}} B_2(x,y) \otimes B_1(Fy, z_1) \otimes M(Gz_1)$$

$$\xrightarrow[(id_{B_2} \otimes \Delta_{B_1} \otimes id_M)]{(id_{B_2} \otimes G \otimes id_{B_1} \otimes id_M) \circ} \bigoplus_{\substack{y \in Obj(B_2) \\ y_1, z_1 \in Obj(B_1)}} B_2(x,y) \otimes B_0(GFy, Gy_1) \otimes B_1(y_1, z_1) \otimes M(Gz_1)$$

$$\xrightarrow{id_{B_2} \otimes id_{B_0} \otimes \epsilon_{B_1} \otimes id_M} \bigoplus_{\substack{y \in Obj(B_2) \\ z_1 \in Obj(B_1)}} B_2(x,y) \otimes B_0(GFy, Gz_1) \otimes M(Gz_1).$$

So, the difference between the two maps in Equation D.4 comes down to the difference between $(\epsilon_{B_1} \otimes G) \circ \Delta_{B_1}$ and $(G \otimes \epsilon_{B_1}) \circ \Delta_{B_1}$. However, by the counitality of B_1, both of these maps are equal to G. So, H satisfies Equation D.4. $\square$

Proposition D.2. *Let $\lambda : B_1 \to B_0$ be a functor between conilpotent dg cocategories and C a conilpotent cofree dg comodule over B_0. Then, as comodules,*

$$\lambda^* C \cong B_1 \otimes_\lambda T \tag{D.5}$$

where righthand side is the following cofree comodule over B_1:

$$[B_1 \otimes_\lambda T](f) := \bigoplus_{h \in Obj(B_0)} B_1(f,h) \otimes T(\lambda h)$$

$$T(\lambda h) = \textit{cogenerators of } C(\lambda h)$$

(See Equation 5.1 for an explanation of cogenerators.)

Proof of Proposition D.2. To prove the proposition, we will give maps

$$F : \lambda^* C \rightleftarrows B_1 \otimes_\lambda T : G$$

and show that $F \circ G = id_{B_1 \otimes_\lambda T}$ and $G \circ F = id_{\lambda^* C}$. We define F as follows:

$$F : \lambda^* C \xrightarrow[\textit{inclusion}]{\textit{canonical}} B_1 \otimes_\lambda C \xrightarrow[\textit{cogenerators}]{\textit{project onto}} B_1 \otimes_\lambda T.$$

To define G, we will give a map $G' : B_1 \otimes_\lambda T \to B_1 \otimes_\lambda C$, and show that the image of G' lands in $\lambda^* C$. We define G' as follows:

$$G'(b \otimes t) = \sum_{(b)} b_{(1)} \otimes \lambda b_{(2)} \cdot t$$

where $b \otimes t \in B_1 \otimes_\lambda T$ and $\lambda b_{(2)} \cdot t$ are elements of the appropriate components of C written in terms of cogenerators.

To prove that the image of G' lands in $\lambda^* C$, we need to show that the two maps

$$(id_{B_1} \otimes \Delta_C) \circ G',\ (id_{B_1} \otimes \lambda \otimes id_C) \circ (\Delta_{B_1} \otimes id_C) \circ G' : B_1 \otimes_\lambda T \to B_1 \otimes_\lambda C \rightrightarrows B_1 \otimes_\lambda B_0 \otimes C$$

coincide. We have

$$
\begin{aligned}
[(1 \otimes \Delta_C) \circ G'](b \otimes t) &= \sum_{(b),\,(\lambda b)} b_{(1)} \otimes (\lambda b_{(2)})_{(1)} \otimes (\lambda b_{(2)})_{(2)} \cdot t \\
&= \sum_{(b)} b_{(1)} \otimes \lambda b_{(2)} \otimes \lambda b_{(3)} \cdot t \\
&= [(id_{B_1} \otimes \lambda \otimes id_C) \circ (\Delta_{B_1} \otimes id_C) \circ G'](b \otimes t)
\end{aligned}
$$

where the second equality holds since λ is a map of cocategories and Δ_{B_1} is coassociative.

It's clear from the definitions that F and G are maps of comodules and that $F \circ G = id_{B_1 \otimes_\lambda T}$. All that remains is to show that $G \circ F = id_{\lambda^* C}$. Let $\kappa = \Sigma_i b_i \otimes \beta_i \cdot t_i$ be an arbitrary element of $\lambda^* C \hookrightarrow B_1 \otimes_\lambda C$ where $\beta_i \cdot t_i$ are elements of C written in terms of cogenerators. Then,

$$
GF(\kappa) = GF(\Sigma_i b_i \otimes \beta_i \cdot t_i) = \sum_{\substack{i,\\ \beta_i = 1,\\ (b_i)}} b_{i(1)} \otimes \lambda b_{i(2)} \cdot t_i .
$$

We can divide the terms in κ into two groups: (a) terms in which $\beta_i = 1 \in k$ and (b) terms in which $\beta_i \neq 1 \in k$. Likewise, we can divide the terms in $GF(\kappa)$ into (a) terms in which $\lambda b_{i(2)} = 1$ and (b) terms in which $\lambda b_{i(2)} \neq 1$. From the definitions of F and G, it's clear that the Group A terms in κ are exactly the Group A terms in $GF(\kappa)$.

To show that the Group B terms are the same, let $b_i \otimes \beta_i \cdot t_i$ be an arbitrary Group B term in κ. Then, there is a term $b_i \otimes \beta_i \otimes t_i$ in $(id_{B_1} \otimes \Delta_C)\kappa$. Since $(id_{B_1} \otimes \Delta_C)\kappa = (id_{B_1} \otimes \lambda \otimes id_C) \circ (\Delta_{B_1} \otimes id_C)\kappa$, there must be a Group A term, $b_{j_i} \otimes t_{j_i}$, in κ such that $b_i \otimes \beta_i \otimes t_i$ is one of the terms in the sum $[(id_{B_1} \otimes \lambda \otimes id_C) \circ (\Delta_{B_1} \otimes id_C)](b_{j_i} \otimes t_{j_i}) = \sum_{(b_{j_i})} b_{j_i(1)} \otimes \lambda b_{j_i(2)} \otimes t_{j_i}$. Thus, $b_i \otimes \beta_i \cdot t_i$ is a Group B term in $GF(\kappa)$.

Now let $b_{i(1)} \otimes \lambda b_{i(2)} \cdot t_i$ be an arbitrary Group B term in $GF(\kappa)$. Then, $b_{i(1)} \otimes \lambda b_{i(2)} \otimes t_i$ is a term in $(id_{B_1} \otimes \lambda \otimes id_C) \circ (\Delta_{B_1} \otimes id_C)\kappa = (id_{B_1} \otimes \Delta_C)\kappa$. So, there is a Group B term, $b_{j_i} \otimes \beta_{j_i} \cdot t_{j_i}$, in κ such that $b_{i(1)} \otimes \lambda b_{i(2)} \otimes t_i$ is one of the terms in the sum $(id_{B_1} \otimes \Delta_C)(b_{j_i} \otimes \beta_{j_i} \cdot t_{j_i}) = \sum_{(\beta_{j_i})} b_{j_i} \otimes \beta_{j_i(1)} \otimes \beta_{j_i(2)} \cdot t_{j_i}$. Since t_i is a cogenerator, the only term in the sum that could be equal to $b_{i(1)} \otimes \lambda b_{i(2)} \otimes t_i$ is $b_{j_i} \otimes \beta_{j_i} \otimes t_{j_i}$. Thus, $b_{i(1)} \otimes \lambda b_{i(2)} \cdot t_i$ is a Group B term in κ. $\square$

D.2. Examples of pullbacks

Now, we use Proposition D.2 to compute some examples of pullbacks of dg comodules. For the examples below, let $\mathcal{C}$ be the category in dg cocategories defined in Equation 1.2 and $T(A)$ be the dg comodule defined in Section 5.2.

Example D.2.1. *Let* $m : \mathcal{C}(A_0, A_1) \otimes \cdots \otimes \mathcal{C}(A_n, A_0) \to \mathcal{C}(A_0, A_0)$ *be the composition functor. Then,* $T(A_0 \to \cdots \to A_n \to A_0) := m^*T(A_0)$ *is a cofree dg comodule with the following structure. Let* $(A_0 \xrightarrow{f_0} A_1 \to \cdots \to A_n \xrightarrow{f_n} A_0)$ *be an object in* $\mathcal{C}(A_0, A_1) \otimes \cdots \otimes$

$\mathcal{C}(A_n, A_0)$. *Then,*

$T(A_0 \xrightarrow{f_0} A_1 \to \cdots \to A_n \xrightarrow{f_n} A_0)^\bullet =$

$$= \{(\phi_{0,1}|\ldots|\phi_{0,k_0}) \otimes \cdots \otimes (\phi_{n,1}|\ldots|\phi_{n,k_n}) \otimes \alpha = \begin{tikzcd}
A_0 \arrow[r, bend left=70, "f_{0,0}"] \arrow[r, bend left=25, "f_{0,1}"] \arrow[r, bend right=25, "f_{0,k_0}"'] \arrow[r, phantom, shift left=4, "\Downarrow\phi_{0,1}"] \arrow[r, phantom, "\vdots"] \arrow[rrrr, bend right=40, "id_{A_0}"'] \arrow[rrrr, phantom, shift right=6, "\alpha"] & A_1 \arrow[r, dotted, bend left=70] \arrow[r, dotted, bend left=25] \arrow[r, dotted, bend right=25] \arrow[r, phantom, "\vdots"] & \cdots \arrow[r, dotted, bend left=70] \arrow[r, dotted, bend left=25] \arrow[r, dotted, bend right=25] \arrow[r, phantom, "\vdots"] & A_n \arrow[r, bend left=70, "f_{n,0}"] \arrow[r, bend left=25, "f_{n,1}"] \arrow[r, bend right=25, "f_{n,k_n}"'] \arrow[r, phantom, shift left=4, "\Downarrow\phi_{n,1}"] \arrow[r, phantom, "\vdots"] & A_0
\end{tikzcd}$$

$$s.t.\ \phi_{i,j} \in C^\bullet(A_{i,f_{j-1}}\, A_{i+1\,f_j}), \alpha \in C_{-\bullet}(A_{0,f_{n,k_n}\cdots f_{0,k_0}}\, A_0)\}$$

$d_T = \tilde{d}_{\mathcal{C}} + \tilde{b} + \tilde{\iota}$ *where*

$\tilde{d}_{\mathcal{C}} =$ *extension of the differentials on* $\mathcal{C}(A_i, A_{i+1\,(mod n+1)}), 0 \leq i \leq n$ *to* T

$\tilde{b} =$ *extension of the Hochschild chain differential to* T

$\tilde{\iota} =$ *extension of* $\iota_{(\phi_{0,1}|\ldots|\phi_{0,k_0})\bullet\cdots\bullet(\phi_{n,1}|\ldots|\phi_{n,k_n})}\alpha$ *as a coderivation to* T *(see Equation B.1)*

Example D.2.2 (Pullbacks along rotations)**.** *Fix algebras* $A_0, \ldots, A_n$ *and let* $\tau_n \in \Lambda([n],[n])$ *be a generating rotation. Set*

$$\hat{\tau}_n : \mathcal{C}(A_0, A_1) \otimes \ldots \mathcal{C}(A_n, A_0) \xrightarrow{\textit{rotation functor}} \mathcal{C}(A_n, A_0) \otimes \cdots \otimes \mathcal{C}(A_{n-1}, A_n)$$

$$\tau_{n!} : T(A_0 \to \cdots \to A_n \to A_0) \to \hat{\tau}_n^* T(A_n \to A_0 \cdots \to A_n) \textit{ map of dg comodules.}$$

Then, the target of $\tau_{n!}$, $\hat{\tau}_n^* T(A_n \to A_0 \cdots \to A_n)$ *is a cofree dg comodule with the following structure. Let* $(A_0 \xrightarrow{f_0} A_1 \to \cdots \to A_n \xrightarrow{f_n} A_0)$ *be an object in* $\mathcal{C}(A_0, A_1) \otimes \cdots \otimes \mathcal{C}(A_n, A_0)$.

Then,

$$\hat{\tau}_n^* T(A_n \xrightarrow{f_n} A_0 \to \dots \xrightarrow{f_{n-1}} A_n)^\bullet =$$

$$= \{(\phi_{0,1}|\dots|\phi_{0,k_0}) \otimes \cdots \otimes (\phi_{n,1}|\dots|\phi_{n,k_n}) \otimes \alpha = \begin{tikzcd}[column sep=large] A_n \arrow[r, bend left=80, "f_{n,0}"] \arrow[r, bend left=30, "f_{n,1}"'] \arrow[r, bend right=30, "f_{n,k_n}"] \arrow[r, phantom, shift left=5ex, "\Downarrow\phi_{n,1}"] \arrow[r, phantom, "\vdots"] \arrow[rrrr, bend right=40, "id_{A_n}"'] \arrow[rrrr, phantom, shift right=4ex, "\alpha"] & A_0 \arrow[r, dotted, bend left=80] \arrow[r, dotted, bend left=30] \arrow[r, dotted, bend right=30] \arrow[r, phantom, "\vdots"] & \cdots \arrow[r, dotted, bend left=80] \arrow[r, dotted, bend left=30] \arrow[r, dotted, bend right=30] \arrow[r, phantom, "\vdots"] & A_{n-1} \arrow[r, bend left=80, "f_{n-1,0}"] \arrow[r, bend left=30, "f_{n-1,1}"'] \arrow[r, bend right=30, "f_{n-1,k_{n-1}}"] \arrow[r, phantom, shift left=5ex, "\Downarrow\phi_{n-1,1}"] \arrow[r, phantom, "\vdots"] & A_n \end{tikzcd}$$

$$s.t.\ \phi_{i,j} \in C^\bullet(A_i,{}_{f_{j-1}} A_{i+1}{}_{f_j}), \alpha \in C_{-\bullet}(A_n,{}_{f_{n-1,k_{n-1}} \cdots f_{n,k_n}} A_n)\}$$

$d_T = \tilde{d}_{\mathcal{C}} + \tilde{b} + \tilde{\iota}$ *where*

$\tilde{d}_{\mathcal{C}} =$ *extension of the differentials on* $\mathcal{C}(A_i, A_{i+1\,(mod n+1)}), 0 \leq i \leq n$ *to* T

$\tilde{b} =$ *extension of the Hochschild chain differential to* T

$\tilde{\iota} =$ *extension of* $\iota_{(\phi_{n,1}|\dots|\phi_{n,k_n})\bullet(\phi_{0,1}|\dots|\phi_{0,k_0})\bullet\cdots\bullet(\phi_{n-1,1}|\dots|\phi_{n-1,k_{n-1}})}\alpha$ *as a coderivation to* T.

D.3. Adjunction between λ^* and $\lambda_\#$

In this section, we define $\lambda_\#$, the left adjoint to λ^*. More precisely, for any functor, $\lambda : B_1 \to B_0$ between conilpotent dg cocategories, we define a functor $\lambda_\#$ from the category of conilpotent dg comodules over B_1 to the category of conilpotent dg comodules over B_0.

D.3.1. The functors $\lambda_\#$

Let $\lambda : B_1 \to B_0$ be a functor between conilpotent dg cocategories. Let C be a conilpotent dg comodule over B_1. We define $\lambda_\# C$ as follows: for $f \in Obj(B_0)$,

$$\lambda_\# C(f) := \Big(\bigoplus_{f' \in \lambda^{-1} f} C^\bullet(f'),$$

$$\Delta_{\lambda_\# C}(f) : \bigoplus_{f' \in \lambda^{-1} f} C^\bullet(f') \xrightarrow{\bigoplus_{f'} \Delta_{C^\bullet}(f')} \bigoplus_{\substack{f' \in \lambda^{-1} f \\ h' \in Obj(B_1)}} B_1^\bullet(f', h') \otimes C^\bullet(h')$$

$$\xrightarrow{\bigoplus_{h', f'} \lambda \otimes id_{C^\bullet(h')}} \bigoplus_{h' \in Obj(B_1)} B_0^\bullet(f, \lambda h') \otimes C^\bullet(h')$$

$$\xrightarrow{include} \bigoplus_{h \in Obj(B_0)} B_0^\bullet(f, h) \otimes \big(\bigoplus_{h' \in \lambda^{-1} h} C^\bullet(h') \big) \Big).$$

To check that $\Delta_{\lambda_\# C}$ is well-defined, we need that the image of the first map, $\bigoplus_{f'} \Delta_{C^\bullet}(f')$, is a finite sum. This is true since C being conilpotent implies that the image of $\Delta_{C^\bullet}(f')$ is a finite sum for each $f' \in Obj(B_1)$. If $\lambda^{-1} f$ is empty, we set $\lambda_\# C(f) := 0$. It is straightforward to check that $(\lambda_\# C, \Delta_{\lambda_\# C})$ is coassociative, conilpotent and coaugmented. We will call $\lambda_\#$ "co-restriction of scalars".

Let $F : C \to D$ be map of dg comodules over B_1. We define $\lambda_\# F$ as follows:

$$(\lambda_\# F)_f : \lambda_\# C(f) = \bigoplus_{f' \in \lambda^{-1} f} C^\bullet(f') \xrightarrow{\bigoplus_{f' \in \lambda^{-1} f} F_{f'}} \bigoplus_{f' \in \lambda^{-1} f} D^\bullet(f') = \lambda_\# D(f).$$

It's straightforward to check that $\lambda_\#$ is a functor (i.e., respects composition of morphisms).

D.3.2. Adjunction

Proposition D.3. *Given a functor between conilpotent dg cocategories, $\lambda : B_1 \to B_0$, let*

$$\lambda^* : \begin{matrix} \textit{Category of} \\ \textit{conilpotent} \\ \textit{dg comodules over } B_0 \end{matrix} \leftrightarrows \begin{matrix} \textit{Category of} \\ \textit{conilpotent} \\ \textit{dg comodules over } B_1 \end{matrix} : \lambda_{\#}$$

be the functors defined in Sections D.1.1 and D.3.1. Then, $\lambda_{\#}$ is left adjoint to λ^*.

Remark D.3.1. Proposition D.3 is a categorified co-version of the adjunction between extension of scalars (left) and restriction of scalars (right) for modules over algebras.

Proof of Proposition D.3. Let C be a conilpotent dg comodule over B_1 and D be a dg conilpotent dg comodule over B_0. We want to show that

$$Hom_{B_1}(C,\, \lambda^* D) = Hom_{B_0}(\lambda_{\#} C,\, D)$$

as sets.

We will give maps

$$\Phi : Hom_{B_0}(\lambda_{\#} C,\, D) \leftrightarrows Hom_{B_1}(C,\, \lambda^* D) : \Phi^{-1}$$

satisfying $\Phi \circ \Phi^{-1} = id$ and $\Phi^{-1} \circ \Phi = id$.

First, we define Φ. Let F be a morphism from $\lambda_{\#} C$ to D. By defintion, for $f \in Obj(B_0)$, we have maps of complexes

$$F_f : \bigoplus_{f' \in \lambda^{-1} f} C^\bullet(f') \to D^\bullet(f).$$

Define $\Phi F \in Hom_{B_1}(C,\, \lambda^* D)$ as follows: for $f' \in Obj(B_1)$,

$$\begin{aligned}
\Phi F_{f'} : C^\bullet(f') &\xrightarrow{\Delta_C} \bigoplus_{h' \in Obj(B_1)} B_1^\bullet(f', h') \otimes C^\bullet(h') \\
&\xrightarrow{\bigoplus_{h'} id_{B_1} \otimes F_{\lambda h'}|_{h'}} \bigoplus_{h' \in Obj(B_1)} B_1^\bullet(f', h') \otimes D^\bullet(\lambda h') \\
&\xrightarrow{include} [B_1 \otimes_\lambda D](f').
\end{aligned} \tag{D.6}$$

By the universal property of $\lambda^* D$, this defines a morphism $C \to \lambda^* D$ if the two maps

$$(id_{B_1} \otimes \Delta_D) \circ \Phi F,\ (id_{B_1} \otimes \lambda \otimes id_D) \circ (\Delta_{B_1} \otimes id_D) \circ \Phi F : C \rightrightarrows B_1 \otimes_\lambda B_0 \otimes D$$

coincide. In fact, on $f' \in Obj(B_1)$, both maps are equal to:

$$\begin{aligned}
C^\bullet(f') &\xrightarrow{\Delta_C} \bigoplus_{h' \in Obj(B_1)} B_1^\bullet(f', h') \otimes C^\bullet(h') \\
&\xrightarrow{\bigoplus_{h'} id_{B_1} \otimes \Delta_C} \bigoplus_{g', h' \in Obj(B_1)} B_1^\bullet(f', g') \otimes B_1^\bullet(g', h') \otimes C^\bullet(h') \\
&\xrightarrow{\bigoplus_{h', g'} id_{B_1} \otimes \lambda \otimes 1_C} \bigoplus_{g', h' \in Obj(B_1)} B_1^\bullet(f', g') \otimes B_0^\bullet(\lambda g', \lambda h') \otimes C^\bullet(h') \\
&\xrightarrow{\bigoplus_{h', g'} id_{B_1} \otimes id_{B_0} \otimes F_{\lambda h'}|_{h'}} \bigoplus_{g', h' \in Obj(B_1)} B_1^\bullet(f', g') \otimes B_0^\bullet(\lambda g', \lambda h') \otimes D^\bullet(\lambda h').
\end{aligned}$$

This fact follows from F being a map of comodules. It's also clear that ΦF commutes with coproducts and differentials. So, we've shown $\Phi F \in Hom_{B_1}(C,\, \lambda^* D)$.

Second, we define Φ^{-1}. Now, let $F \in Hom_{B_1}(C, \lambda^* D)$. For $f \in Obj(B_0)$, define

$$\begin{aligned}
\Phi^{-1}F_f : \bigoplus_{f' \in \lambda^{-1} f} C^\bullet(f') &\xrightarrow{\bigoplus_{f'} F_{f'}} \bigoplus_{\substack{f' \in \lambda^{-1} f,\\ h' \in Obj(B_1)}} B_1^\bullet(f', h') \otimes D^\bullet(\lambda h') \\
&\xrightarrow{\bigoplus_{f',h'} \lambda \otimes id_D} \bigoplus_{h \in Obj(B_0)} B_0^\bullet(f, h) \otimes D^\bullet(h) \\
&\xrightarrow{\bigoplus_{h} \epsilon_{B_0} \otimes id_D} D^\bullet(f).
\end{aligned}$$

It's clear that $\Phi^{-1}F$ commutes with the differentials. We will show that $\Phi^{-1}F$ is a map of comodules. Figure D.1 gives a diagram showing that

$$\Delta_D \circ \Phi^{-1}F_f = (\bigoplus_{f',h',r'} \epsilon_{B_0}\lambda \otimes \lambda \otimes id_D) \circ (\bigoplus_{f',h'} \Delta_{B_1} \otimes id_D) \circ (\bigoplus_{f'} F_{f'}). \tag{D.7}$$

On the other hand, Figure D.2 gives a diagram showing that

$$(id_{B_1} \otimes \Phi^{-1}F) \circ \Delta_{\lambda_{\#}C} = (\bigoplus_{f',h',r'} \lambda \otimes \epsilon_{B_0}\lambda \otimes id_D) \circ (\bigoplus_{f',h'} \Delta_{B_1} \otimes id_D) \circ (\bigoplus_{f'} F_{f'}). \tag{D.8}$$

We see that the righthand sides of Equations D.7 and D.8 are the same except for the B_0 factor on which ϵ_{B_0} acts. However, in general, for $\lambda : B_1 \to B_0$ a map of dg cocategories, we have

$$
\begin{aligned}
(\lambda \otimes \epsilon_{B_0}\lambda) \circ \Delta_{B_1} &= (id_{B_0} \otimes \epsilon_{B_0}) \circ \Delta_{B_0} \circ \lambda \quad (\lambda \text{ commutes with coproduct}) \\
&= id_{B_0} \circ \lambda \quad (\text{definition of cocategory}) \\
&= (\epsilon_{B_0} \otimes id_{B_0}) \circ (\Delta_{B_0}) \circ \lambda \quad (\text{definition of cocategory}) \\
&= (\epsilon_{B_0}\lambda \otimes \lambda) \circ \Delta_{B_1} \quad (\lambda \text{ commutes with coproduct}).
\end{aligned}
$$

So, $(id_{B_1} \otimes \Phi^{-1}F) \circ \Delta_{\lambda_\# C} = \Delta_D \circ \Phi^{-1}F$, and $\Phi^{-1}F \in Hom_{B_0}(\lambda_\# C, D)$.

For $F : C \to \lambda^* D$ a map of dg comodules and $f' \in B_1$, Figure D.3 shows that $\Phi\Phi^{-1}F_{f'} = F_{f'}$. For $F : \lambda_\# C \to D$ a map of dg comodules and $f \in B_0$, Figure D.4 shows that $\Phi^{-1}\Phi F_f = F_f$. Thus, we have $\Phi\Phi^{-1} = id$ and $\Phi^{-1}\Phi = id$. $\square$

$$
\begin{tikzcd}
\bigoplus_{f'\in\lambda^{-1}f} C^\bullet(f') \arrow[d, "\bigoplus_{f'} F_{f'}"] & & \\
\bigoplus_{\substack{f'\in\lambda^{-1}f\\ h'\in Obj(B_1)}} B_1^\bullet(f',h')\otimes D^\bullet(\lambda h') \arrow[r, "\bigoplus_{f',h'} \lambda\otimes id_D"] \arrow[d, shift right, bend right=5, "\bigoplus_{f',h',r'} {\substack{(\Delta_{B_1}\otimes id_D)\circ\\ (id_{B_1}\otimes\lambda\otimes id_D)}}"'] \arrow[d, shift left, bend left=5, "\bigoplus_{f',h'} id_{B_1}\otimes\Delta_D"] & \bigoplus_{h\in Obj(B_0)} B_0^\bullet(f,h)\otimes D^\bullet(h) \arrow[r, "\bigoplus_h \epsilon_{B_0}\otimes id_D"] & D^\bullet(f) \arrow[d, "\Delta_D"] \\
\bigoplus_{\substack{f'\in\lambda^{-1}f,\\ h'\in Obj(B_1),\\ r\in Obj(B_0)}} B_0^\bullet(f',h')\otimes B_1^\bullet(\lambda h',r)\otimes D^\bullet(r) \arrow[rr, "\bigoplus_{f',h',r} \epsilon_{B_0}\lambda\otimes id_{B_1}\otimes id_D"'] & & \bigoplus_{h\in Obj(B_0)} B_0^\bullet(f,h)\otimes D^\bullet(h)
\end{tikzcd}
$$

Figure D.1. Commuting diagram involving $\Delta_D \circ \Phi^{-1}F$
$\Delta_D \circ \Phi^{-1}F =$ composition of red arrows. The fact that $F : C \to \lambda^*D$ and the universal property of λ^*D imply that the diagram commutes.

$$\begin{tikzcd}
\bigoplus_{f' \in \lambda^{-1} f} C^\bullet(f') \arrow[r, "\bigoplus_{f'} \Delta_C"] \arrow[d, "\bigoplus_{f'} F_{f'}"] & \bigoplus_{\substack{f' \in \lambda^{-1} f \\ h' \in Obj(B_1)}} B_1^\bullet(f', h') \otimes C^\bullet(h') \arrow[r, "\bigoplus_{f',h'} \lambda \otimes id_C"] \arrow[d, "\bigoplus_{f',h'} id_{B_1} \otimes F_{\lambda h'}|_{h'}"] & \bigoplus_{h' \in Obj(B_1)} B_0^\bullet(f, \lambda h') \otimes C^\bullet(h') \arrow[d, "\bigoplus_{h'} id_{B_0} \otimes F_{\lambda h'}|_{h'}"] \\
\bigoplus_{\substack{f' \in \lambda^{-1} f, \\ r' \in Obj(B_1)}} B_1^\bullet(f', r') \otimes D^\bullet(\lambda r') \arrow[r, "\bigoplus_{f',r'} \substack{\Delta_{\lambda^* D} = \\ \Delta_{B_1} \otimes id_D}"] & \bigoplus_{\substack{f' \in \lambda^{-1} f, \\ h', r' \in Obj(B_1)}} B_1^\bullet(f', h') \otimes B_1^\bullet(h', r') \otimes D^\bullet(\lambda r') \arrow[r, "\bigoplus_{f',h',r'} \lambda \otimes id_{B_0} \otimes id_D"] & \bigoplus_{h', r' \in Obj(B_1)} B_0^\bullet(f, \lambda h') \otimes B_1^\bullet(h', r') \otimes D^\bullet(\lambda r') \arrow[d, "\bigoplus_{h',r'} id_{B_0} \otimes \epsilon_{B_0} \lambda \otimes id_D"] \\
& & \bigoplus_{h \in Obj(B_0)} B_0^\bullet(f, h) \otimes D^\bullet(h)
\end{tikzcd}$$

Figure D.2. Commuting diagram involving $(id_{B_1} \otimes \Phi^{-1} F) \circ \Delta_{\lambda_{\#} C}$
$(id_{B_1} \otimes \Phi^{-1} F) \circ \Delta_{\lambda_{\#} C}$ = composition of red arrows. The fact that F respects coproducts implies that the left square commutes.

$$
\begin{tikzcd}
C^\bullet(f') \arrow[r, red, "\Delta_C"] \arrow[d, "F_{f'}"] & \bigoplus_{g' \in Obj(B_1)} B_1^\bullet(f', g') \otimes C^\bullet(g') \arrow[d, red, "\bigoplus_{g'} id_{B_1} \otimes F_{g'}"] & \\
\bigoplus_{h' \in Obj(B_1)} B_1^\bullet(f', h') \otimes D^\bullet(\lambda h') \arrow[r, "\Delta_{\lambda^* D} = \bigoplus_{h'} \Delta_{B_1} \otimes id_D"] \arrow[rr, bend right, "id"] & \bigoplus_{g', h' \in Obj(B_1)} B_1^\bullet(f', g') \otimes B_1^\bullet(g', h') \otimes D^\bullet(\lambda h') \arrow[r, red, "\bigoplus_{g', h'} id_{B_1} \otimes (\epsilon_{B_0} \lambda = \epsilon_{B_1}) \otimes id_D"] & \bigoplus_{g' \in Obj(B_1)} B_1^\bullet(f', g') \otimes D^\bullet(\lambda g')
\end{tikzcd}
$$

Figure D.3. Commuting diagram involving $\Phi\Phi^{-1}F_{f'}$

$\Phi\Phi^{-1}F_{f'}$ = composition of red arrows. The square commutes because F respects coproducts; the composition of the bottom row of horizontal arrows is equal to the identity because $\lambda_{\#}D$ satisfies counitality.

$$
\begin{tikzcd}
\bigoplus_{f' \in \lambda^{-1} f} C^\bullet(f') \arrow[r, red, "\bigoplus_{f'} \Delta_C"] \arrow[dd, "F_f"'] & \bigoplus_{\substack{f' \in \lambda^{-1} f, \\ g' \in Obj(B_1)}} B_1^\bullet(f', g') \otimes C^\bullet(g') \arrow[r, red, "\bigoplus_{f',g'} id_{B_1} \otimes F_{\lambda g'}|_{g'}"] \arrow[d, "\bigoplus_{f',g'} \lambda \otimes id_C"] & \bigoplus_{\substack{f' \in \lambda^{-1} f, \\ g' \in Obj(B_1)}} B_1^\bullet(f', g') \otimes D \bullet (\lambda g') \arrow[d, red, "\bigoplus_{f',g'} \lambda \otimes id_D"] & \\
& \bigoplus_{g' \in Obj(B_1)} B_0^\bullet(f, \lambda g') \otimes C^\bullet(g') \arrow[r, "\bigoplus_{g'} id_{B_0} \otimes F_{\lambda g'}|_{g'}"] & \bigoplus_{g \in Obj(B_0)} B_0^\bullet(f, g) \otimes D^\bullet(g) \arrow[r, red, "\bigoplus_{g} \epsilon_{B_0} \otimes id_D"] & D^\bullet(f) \\
D^\bullet(f) \arrow[urr, "\Delta_D"'] \arrow[urrr, bend right, "id"'] & & &
\end{tikzcd}
$$

Figure D.4. Commuting diagram involving $\Phi^{-1}\Phi F_f$

$\Phi^{-1}\Phi F_f$ = composition of red arrows. The concave pentagon on the left side commutes because F respects coproducts; the triangle in the bottom right corner commutes because D satisfies counitality.

D.4. Conilpotence

In this section, we show that the dg categories and dg comodules we have been working with are conilpotent. For completeness, we start with the definition of a dg cocategory.

Definition D.4.1. A dg cocategory is a cocategory enriched over chain complexes. More explicitly, a dg cocategory B consists of the following data:

- A collection of objects denoted $Obj(B)$;
- For each pair of objects, $x, z \in Obj(B)$, a complex $B^\bullet(x,z)$ and a morphism of complexes

$$\Delta_B(x,z) : B^\bullet(x,z) \to \prod_{y \in Obj(B)} B^\bullet(x,y) \otimes B^\bullet(y,z)$$

  such that the following diagrams commute (coassociativity):

$$\begin{tikzcd}
B^\bullet(x,z) \arrow[rr, "\Delta_B(x,z)"] \arrow[d, "\Delta_B(x,z)"'] & & \prod\limits_{y \in Obj(B)} B^\bullet(x,y) \otimes B^\bullet(y,z) \arrow[d, "\prod_y id_{B(x,y)} \otimes \Delta_B(y,z)"] \\
\prod\limits_{y \in Obj(B)} B^\bullet(x,y) \otimes B^\bullet(y,z) \arrow[rr, "\prod_y \Delta_B(x,y) \otimes id_{B(y,z)}"] & & \prod\limits_{y,y' \in Obj(B)} B^\bullet(x,y) \otimes B^\bullet(y,y') \otimes B^\bullet(y',z)
\end{tikzcd}$$

- For each pair of objects, $x, z \in Obj(B)$, a morphism of complexes

$$\epsilon_B(x,z) : B^\bullet(x,z) \to k$$

where k is the ground field considered as a chain complex concentrated in degree 0 and $\epsilon_B(x,z) = 0$ if $x \neq z$, such that the following diagrams commute (counitality):

$$\begin{tikzcd}
B^\bullet(x,z) \arrow[rr, "\Delta_B(x,z)"] \arrow[d, "\Delta_B(x,z)"'] \arrow[drr, "id"] & & \prod\limits_{y \in Obj(B)} B^\bullet(x,y) \otimes B^\bullet(y,z) \arrow[d, "\prod\limits_y \epsilon_B(x,y) \otimes id_{B(y,z)}"] \\
\prod\limits_{y \in Obj(B)} B^\bullet(x,y) \otimes B^\bullet(y,z) \arrow[rr, "\prod\limits_y id_{B(x,y)} \otimes \epsilon_B(y,z)"'] & & B^\bullet(x,z).
\end{tikzcd}$$

We will denote a dg cocategory with its cocomposition and counit as $(B, \Delta_B, \epsilon_B)$. To make the notation more readable, when the meaning is clear, we will omit references to the objects and write Δ_B instead of $\Delta_B(x,z)$, ϵ_B instead of $\epsilon_B(x,z)$, and for the differentials on morphisms, d_B instead of $d_B(x,z)$.

Definition D.4.2. A (dg) functor $F : A \to B$ between two dg cocategories is a functor between the cocategories satisfying $d_B \circ F(f) = F \circ d_A(f)$ for all morphisms f in A.

Definition D.4.3. A conilpotent dg cocategory is a dg cocategory $(B, \Delta_B, \epsilon_B)$ satisfying: for each morphism $f : x \to y$ in B, there exists $n_f \in \mathbb{N}$ such that $\bar{\Delta}_B^{n_f}(f) = 0$ where

$$\bar{\Delta}_B(x,z) : B^\bullet(x,z) \to \prod_{y \in Obj(B)} B^\bullet(x,y) \otimes B^\bullet(y,z)$$

$$f \mapsto \Delta_B(f) - \sum_{e_x \in \epsilon_B(x,x)^{-1}(1)} e_x \otimes f - \sum_{e_z \in \epsilon_B(z,z)^{-1}(1)} f \otimes e_z.$$

The following fact follows from the definitions: If B is a conilpotent dg cocategory, then for all $x \in Obj(B)$, $\epsilon_B(x,x)^{-1}(1)$ has exactly one element, which we will denote e_x.

Example D.4.1. *Let $\mathcal{C}$ be the category in dg cocategories defined in Equation 1.2 and $A_0, \ldots, A_n$ be algebras. Then, $\mathcal{C}(A_0, A_1) \otimes \cdots \otimes \mathcal{C}(A_n, A_0)$ is conilpotent:*

$$\bar{\Delta}^{min(k_0,\ldots,k_n)}(\phi_{0,1}\ldots\phi_{0,k_0}|\ldots|\phi_{n,1}\ldots\phi_{n,k_n}) = 0.$$

Now, we will discuss conilpotence of the dg comodules. Recall the definition of a dg comodule in Definition 3.3.3.

Definition D.4.4. A conilpotent dg comodule over a dg cocategory B is a dg comodule (C, Δ_C) over B satisfying: for each $f \in Obj(B)$ and each element $\alpha \in C^\bullet(f)$, there exists $n_\alpha \in \mathbb{N}$ such that $\bar{\Delta}_f^{n_\alpha}(\alpha) = 0$ where

$$\bar{\Delta}_C(f) : C^\bullet(f) \to \prod_{g \in Obj(B)} B^\bullet(f,g) \otimes C^\bullet(g)$$
$$\alpha \mapsto \Delta_B(\alpha) - \sum_{e_f \in \epsilon_B(f,f)^{-1}(1)} e_f \otimes f.$$

Example D.4.2. *Since all of the dg comodules we use are cofree, their comodule structure maps are induced by the cocompositions of the dg cocategories. Any cofree dg comodule over a conilpotent dg cocategory is conilpotent.*