

Title

Subtitle

Rebecca Wei

Northwestern University

Date/Event

Braces, categorically

Fix an algebra, A .

Define a dg category, $Hoch(A)$:

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Objects: algebra maps $f : A \rightarrow A$

Morphisms: $Hoch(A)(f, g) = (C^\bullet(A, {}_f A_g), {}_f \delta_g)$

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$$\begin{aligned} {}_f \delta_g(\phi)(a_1 \otimes \dots \otimes a_n) = & \epsilon_\phi \left(f(a_1) \cdot \phi(a_2, \dots, a_n) + \right. \\ & + \sum_{1 \leq i \leq n-1} (-1)^i \phi(a_1, \dots, a_i a_{i+1}, \dots, a_n) + \\ & \left. + (-1)^n \phi(a_1, \dots, a_{n-1}) \cdot g(a_n) \right) \\ \epsilon_\phi = & (-1)^{|\phi|+1} \end{aligned}$$

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Composition: cup product on cochains

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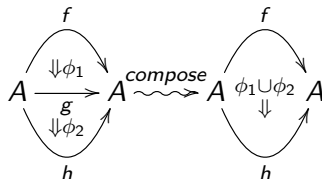
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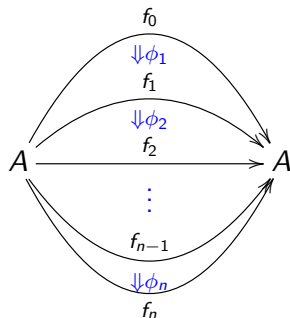
$$Bar(Hoch(A)) \otimes Bar(Hoch(A)) \xrightarrow{\bullet} Hoch(A)$$

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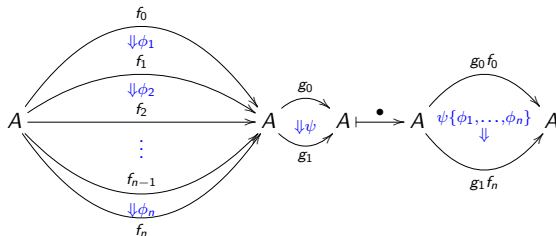
A morphism from f_0 to f_n in $Bar(Hoch(A))$

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In this context, braces give multilinear maps:

$$\begin{array}{ccc} Bar(Hoch(A)) \otimes Bar(Hoch(A)) & \xrightarrow{\bullet} & Hoch(A) \\ & \searrow \bullet & \uparrow \\ & & Bar(Hoch(A)) \end{array}$$

Then, $(Bar(Hoch(A)), \bullet)$ is an algebra in $DGCocats$.

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But we have more...

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Fix algebras, A_0, A_1, \dots, A_n .

We will define a dg cocategory $B(A_0 \rightarrow A_1 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$
where $B(A_0 \rightarrow A_0) := \text{Bar}(\text{Hoch}(A_0))$ for $n=0$.

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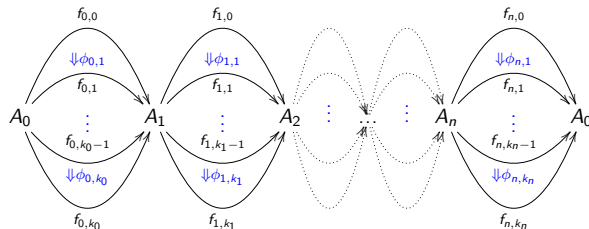
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Objects: $\{A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \dots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_0\}$

A morphism from $(A_0 \xrightarrow{f_{0,0}} \dots \xrightarrow{f_{n,0}} A_0)$ to $(A_0 \xrightarrow{f_{0,k_0}} \dots \xrightarrow{f_{n,k_n}} A_0)$:



$$\phi_{ij} \in C^\bullet(A_{i,f_{i,j-1}} A_{i+1,f_{i,j}})$$

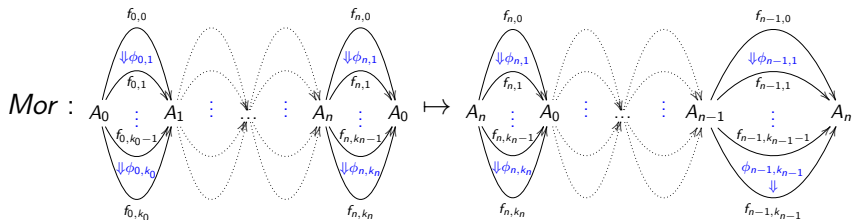
Structure among the $B(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$'s

Example

We have a dg functor

$$\hat{\tau}_n : B(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0) \rightarrow B(A_n \rightarrow A_0 \rightarrow \dots \rightarrow A_n)$$

$$Obj : (f_0, \dots, f_n) \mapsto (f_n, f_0, \dots, f_{n-1})$$



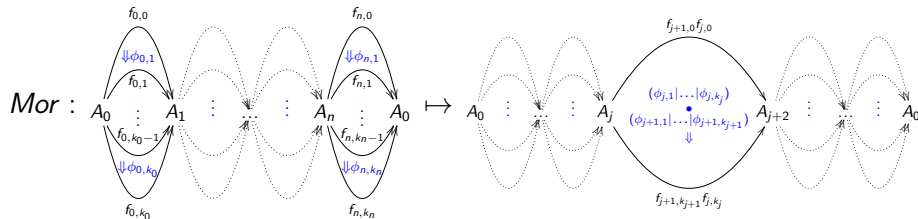
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For $n \geq 1, 0 \leq j < n$, we have a dg functor

$$\hat{\delta}_{j,n} : B(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0) \rightarrow B(A_0 \rightarrow \dots \rightarrow A_j \rightarrow A_{j+2 \pmod{n+1}} \rightarrow \dots \rightarrow A_0)$$

$$\text{Obj} : (f_0, \dots, f_n) \mapsto (f_0, \dots, f_{j+1} f_j, \dots, f_n)$$



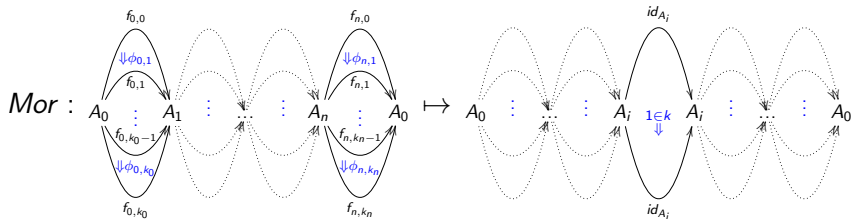
Structure among the $B(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$'s

Example

For $n \geq 0, 0 \leq i \leq n$, we have a dg functor

$$\hat{\sigma}_{i,n} : B(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0) \rightarrow B(A_0 \rightarrow \dots \rightarrow A_i \rightarrow A_i \rightarrow \dots \rightarrow A_0)$$

$$Obj : (f_0, \dots, f_n) \mapsto (f_0, \dots, f_{i-1}, id_{A_i}, f_i, \dots, f_n)$$



A sheafy-cyclic object in DGCocat

Definition

Let χ be the category with objects $\{A_0 \rightarrow A_1 \rightarrow \dots \rightarrow A_n \rightarrow A_0\}$ and morphisms compositions of

$$\text{rotations } \tau_n : \mathcal{A} \mapsto (A_n \rightarrow A_0 \rightarrow \dots \rightarrow A_n)$$

$$\text{coboundaries } \delta_{j,n} : \mathcal{A} \mapsto (A_0 \rightarrow \dots \rightarrow A_j \rightarrow A_{j+2 \pmod{n+1}} \rightarrow \dots \rightarrow A_0)$$

$$\text{codegeneracies } \sigma_{i,n} : \mathcal{A} \mapsto (A_0 \rightarrow \dots \rightarrow A_i \rightarrow A_i \rightarrow \dots \rightarrow A_0)$$

where $\mathcal{A} := (A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$, subject to the cyclic relations.

A sheafy-cyclic object in $DGCocat$

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Proposition

We have a functor $\chi \rightarrow DGCocat$

$$\text{Objects} : (A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0) \mapsto B(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$$

$$\text{Generating morphisms} : \lambda \mapsto \hat{\lambda}$$

Motivating Question

Each dg cocategory $B(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$ has a dg comodule $C(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$ (which is a categorified version of the bar complex of Hochschild chains as a module over Hochschild cochains via contraction).

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Motivating Question: Can we extend the sheafy-cyclic structure to the comodules? In other words, can we give a functor

$$\chi \rightarrow \mathcal{D} := \{(dg\ cocat, dg\ comod)\}?$$

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$$\chi_\infty \rightarrow \mathcal{D}_\infty \quad \text{dg categories}$$

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Motivating Question: Can we extend the sheafy-cyclic structure to the comodules? In other words, can we give **an A_∞ -functor**

$$\chi \rightarrow \mathcal{D} := \{(\text{dg cocat}, \text{dg comod})\}?$$

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Rest of this talk: Describe our A_∞ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$.

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- Define dg categories χ_∞ and \mathcal{D}_∞
- Define dg comodules $C(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$
- Define the A_∞ -functor \mathcal{F}

Motivating Question

Rest of this talk: Describe our A_∞ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$.

- Define dg categories χ_∞ and \mathcal{D}_∞
- Define dg comodules $C(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$
- Define the A_∞ -functor \mathcal{F}
- Time and interest permitting
 - Rectify \mathcal{F} to a dg functor
 - Give a dg functor $\mathcal{D}_\infty \rightarrow \mathcal{E} = \{(\text{dg cat}, \text{dg mod})\}$

$$U(\chi_\infty) \xrightarrow{\text{rectified}} \mathcal{D}_\infty \rightarrow \mathcal{E}$$

“A homotopically sheafy-cyclic object in dg categories with a dg module”

Dg categories χ_∞ and \mathcal{D}_∞

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χ_∞ :

Objects: same objects as $\chi = \{A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0\}$

$$\chi_\infty^\bullet(X, Y) := \{\sum a_i v_i : a_i \in k, v_i \in \chi(X, Y)\}$$

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\mathcal{D}_∞ :

Objects: same objects as $\mathcal{D} = \{(\text{dg cocategory}, \text{dg comodule})\}$

$$\mathcal{D}_\infty^p((B_1, C_1), (B_0, C_0)) := \left\{ \begin{array}{l} F : B_1 \rightarrow B_0 \text{ dg functor} \\ F_! : C_1 \rightarrow F^* C_0 \text{ degree-}p \text{ linear map} \end{array} \right\}$$

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$F^* C_0$ is the categorified version of co-extension of scalars:

$$F^* C_0 = \ker(B_1 \otimes C_0 \rightrightarrows B_1 \otimes B_0 \otimes C_0)$$

Dg comodules over dg cocategories

Definition

A **dg comodule** C over a dg cocategory B consists of the following data:

- for each object $f \in B$, a complex $C^\bullet(f)$, and
- maps of complexes

$$\Delta_C(f) : C^\bullet(f) \rightarrow \prod_{g \in \text{Obj}(B)} B^\bullet(f, g) \otimes C^\bullet(g).$$

such that the following diagrams for coassociativity and counitality commute:

$$\begin{array}{ccc}
 C^\bullet(f) & \xrightarrow{\Delta_C} & \prod_{g \in \text{Obj}(B)} B^\bullet(f, g) \otimes C^\bullet(g) \\
 \Delta_C \downarrow & & \downarrow id_B \otimes \Delta_C \\
 \prod_{g \in \text{Obj}(B)} B^\bullet(f, g) \otimes C^\bullet(g) & \xrightarrow{\Delta_B \otimes id_C} & \prod_{g, g' \in \text{Obj}(B)} B^\bullet(f, g) \otimes B^\bullet(g, g') \otimes C^\bullet(g')
 \end{array}
 \qquad
 \begin{array}{ccc}
 C^\bullet(f) & \xrightarrow{\Delta_C} & \prod_{g \in \text{Obj}(B)} B^\bullet(f, g) \otimes C^\bullet(g) \\
 \searrow id & & \downarrow \epsilon_B \otimes id_C \\
 & & C^\bullet(f)
 \end{array}$$

Dg comodules over dg cocategories

Fix algebras A_0, \dots, A_n .

Define a dg comodule over $B(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$:

$$C(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)^\bullet(f_{0,0}, \dots, f_{n,0}) :=$$

$$:= \{ A_0 \xrightarrow{f_{0,0}} A_1 \xrightarrow{f_{0,1}} \dots \xrightarrow{f_{0,k_0}} A_n \xrightarrow{f_{n,0}} A_0 \mid \begin{array}{l} \text{Dotted arrows } A_i \xrightarrow{f_{i,j}} A_{i+1} \text{ for } j=1, \dots, k_i \\ \text{Dotted arrows } A_i \xrightarrow{f_{i,j}} A_{i+1} \text{ for } j=1, \dots, k_i \end{array} \} =$$

$$d_{C(A_0 \rightarrow \dots \rightarrow A_0)} = d_B \otimes id_{C_{-\bullet}} + id_B \otimes b + \tilde{t}$$

Dg comodules over dg cocategories

where $\tilde{\iota}$ is given as follows:

$$\begin{array}{ccc}
 C(A_0 \rightarrow \dots \rightarrow A_0)^\bullet(f_{0,0}, \dots, f_{n,0}) & \xrightarrow{\tilde{\iota}} & C_{-\bullet}(A_0, f_{n,0} \dots f_{0,0} A_0) \\
 & \searrow \tilde{\iota} & \uparrow \\
 & & C(A_0 \rightarrow \dots \rightarrow A_0)^\bullet(f_{0,0}, \dots, f_{n,0})
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 \end{array}$$

$$\tilde{\iota}((\phi_{0,1} | \dots | \phi_{0,k_0}) \otimes \dots \otimes (\phi_{n,1} | \dots | \phi_{n,k_n}) \otimes \alpha) = \iota(\phi_{0,1} | \dots | \phi_{0,k_0}) \bullet \dots \bullet (\phi_{n,1} | \dots | \phi_{n,k_n}) \alpha$$

An A_∞ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$

$$\mathcal{A} := (A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$$

$$\chi_\infty \rightarrow \mathcal{D}_\infty$$

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Generating Morphisms:

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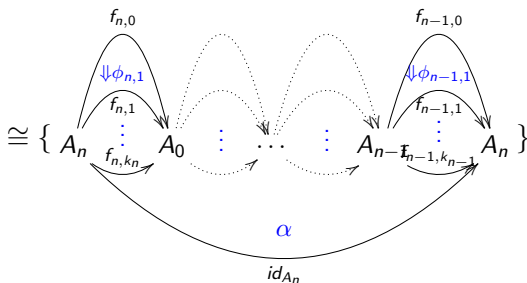
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$$\hat{\tau}_n^* C(\tau_n \mathcal{A})^\bullet(\underbrace{f_0, \dots, f_n}_{\in \text{Obj}(B(\mathcal{A}))}) \cong C(\tau_n \mathcal{A})^\bullet(f_n, f_0, \dots, f_{n-1}) \text{ as complexes.}$$

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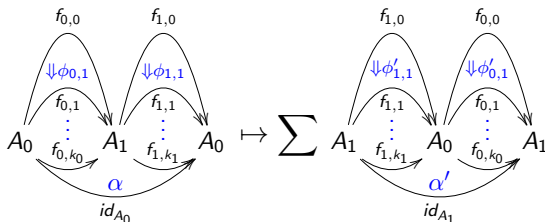
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$$n = 1 : \quad C(A_0 \rightarrow A_1 \rightarrow A_0)^\bullet(f, g) \xrightarrow{\tau_{1!}} C(A_1 \rightarrow A_0 \rightarrow A_1)^\bullet(g, f)$$

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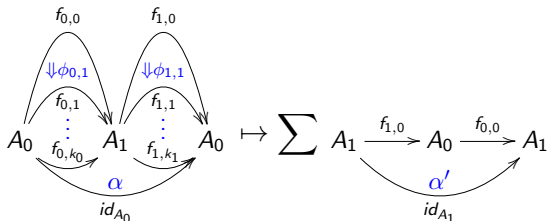
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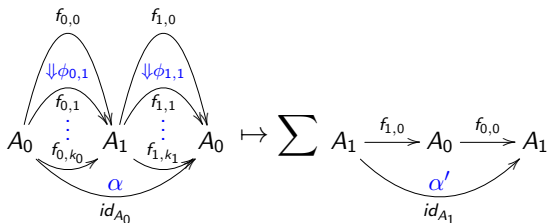
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$$n = 1 : \quad C(A_0 \rightarrow A_1 \rightarrow A_0)^\bullet(f, g) \xrightarrow{\tau_{1!}} C(A_1 \rightarrow A_0 \rightarrow A_1)^\bullet(g, f)$$



Plan: Give these maps to cogenerators to define $\tau_{1!}$, then let

$$\begin{aligned} \tau_{n!} : C(\mathcal{A}) &\cong \hat{\delta}_0^{*n-1} C(A_0 \rightarrow A_n \rightarrow A_0) \xrightarrow{\hat{\delta}_0^{*n-1} \tau_{1!}} \hat{\delta}_0^{*n-1} \hat{\tau}_1^* C(A_n \rightarrow A_0 \rightarrow A_n) \cong \\ &\cong (\widehat{\tau_1 \delta_0^{n-1}})^* C(A_n \rightarrow A_0 \rightarrow A_n) \cong (\widehat{\delta_0^{n-1} \tau_n})^* C(A_n \rightarrow A_0 \rightarrow A_n) \cong \\ &\cong \hat{\tau}_n^* \hat{\delta}_0^{*n-1} C(A_n \rightarrow A_0 \rightarrow A_n) \cong \hat{\tau}_n^* C(\tau_n \mathcal{A}). \end{aligned}$$