Title

Subtitle

Rebecca Wei

Northwestern University

Date/Event

Braces, categorically

Fix an algebra, A.
Define a dg category, $Hoch(A)$:

Braces, categorically

Fix an algebra, A.

Define a dg category, $Hoch(A)$:

Objects: algebra maps $f : A \to A$

Morphisms: $Hoch(A)(f, g) = (C^\bullet(A, {}_fA_g), {}_f\delta_g)$

Braces, categorically

Fix an algebra, A.

Define a dg category, $Hoch(A)$:

Objects: algebra maps $f : A \to A$

Morphisms: $Hoch(A)(f, g) = (C^\bullet(A, {}_fA_g), {}_f\delta_g)$

$$
\begin{aligned}
{}_f\delta_g(\phi)(a_1 \otimes \ldots \otimes a_n) = &\epsilon_\phi \Big(f(a_1) \cdot \phi(a_2, \ldots, a_n) + \\
&+ \sum_{1 \le i \le n-1} (-1)^i \phi(a_1, \ldots, a_i a_{i+1}, \ldots, a_n) + \\
&+ (-1)^n \phi(a_1, \ldots, a_{n-1}) \cdot g(a_n) \Big) \\
\epsilon_\phi = &(-1)^{|\phi|+1}
\end{aligned}
$$

Braces, categorically

Fix an algebra, A.

Define a dg category, $Hoch(A)$:

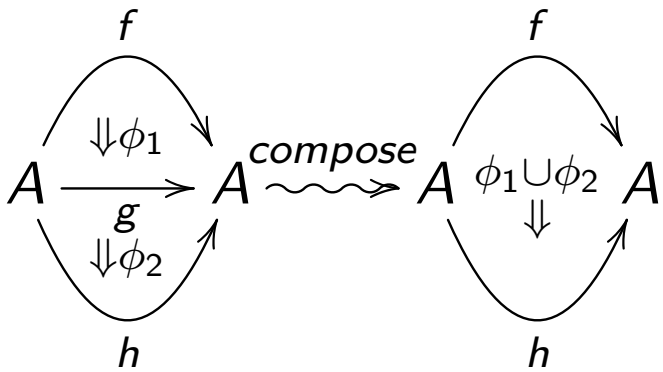
- Objects: algebra maps $f : A \to A$
- Morphisms: $Hoch(A)(f, g) = (C^\bullet(A, {}_fA_g), {}_f\delta_g)$
- Composition: cup product on cochains

Braces, categorically

Fix an algebra, A.

Define a dg category, $Hoch(A)$:

- Objects: algebra maps $f : A \to A$
- Morphisms: $Hoch(A)(f, g) = (C^\bullet(A, {}_fA_g), {}_f\delta_g)$
- Composition: cup product on cochains



Braces, categorically

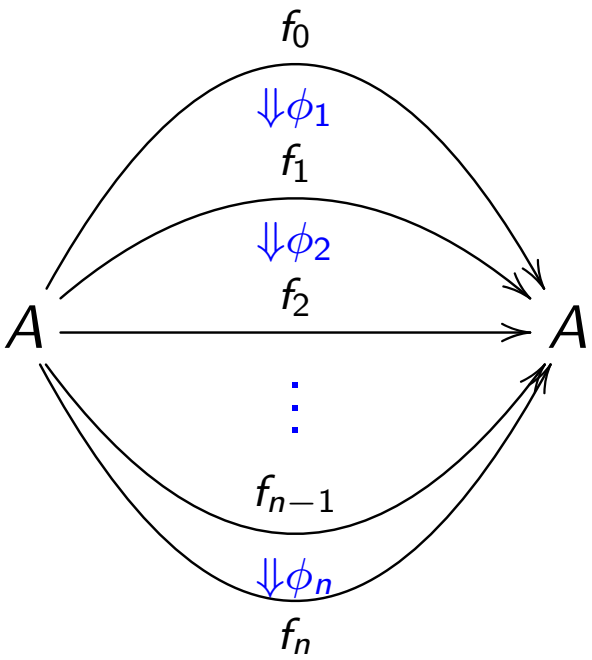
Fix an algebra, A.
Define a dg category, $Hoch(A)$:

$$Bar(Hoch(A)) \otimes Bar(Hoch(A)) \xrightarrow{\bullet} Hoch(A)$$

Braces, categorically

Fix an algebra, A.
Define a dg category, $Hoch(A)$:

$$Bar(Hoch(A)) \otimes Bar(Hoch(A)) \xrightarrow{\bullet} Hoch(A)$$

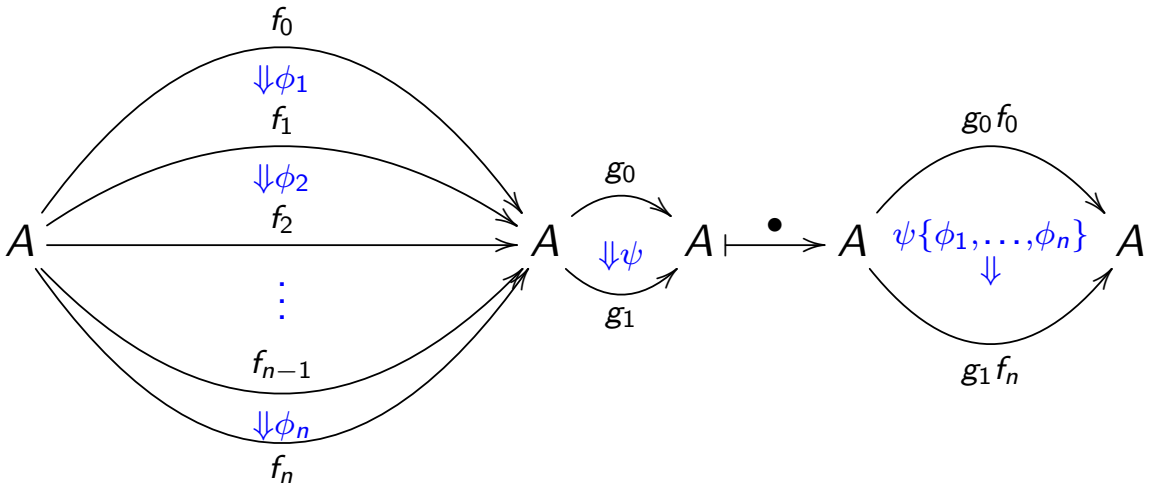


A morphism from f_0 to f_n in Bar(Hoch(A))

Braces, categorically

Fix an algebra, A.
Define a dg category, $Hoch(A)$:

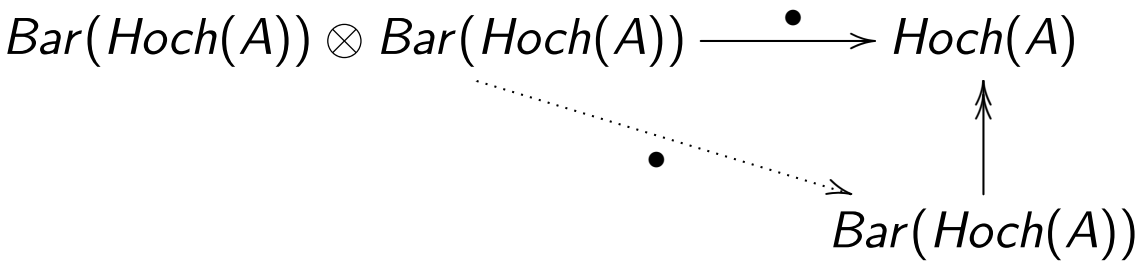
$$Bar(Hoch(A)) \otimes Bar(Hoch(A)) \xrightarrow{\bullet} Hoch(A)$$



Braces, categorically

Fix an algebra, A.
Define a dg category, $Hoch(A)$:

In this context, braces give multilinear maps:

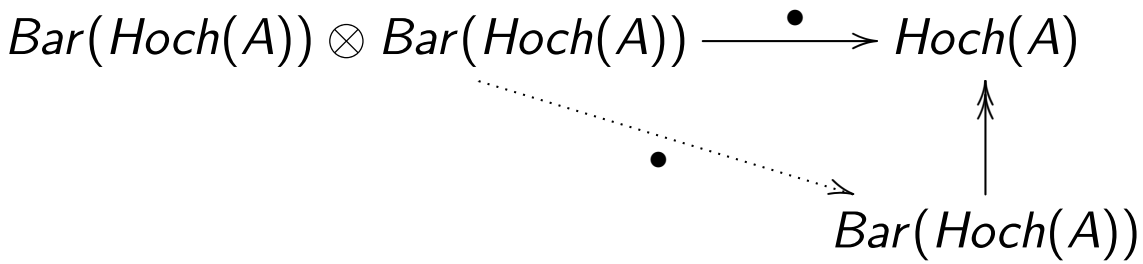


Then, (Bar(Hoch(A)), $\bullet$) is an algebra in DGCocats.

Braces, categorically

Fix an algebra, A.
Define a dg category, $Hoch(A)$:

In this context, braces give multilinear maps:



Then, (Bar(Hoch(A)), •) is an algebra in DGCocats.
But we have more...

More structure

More structure

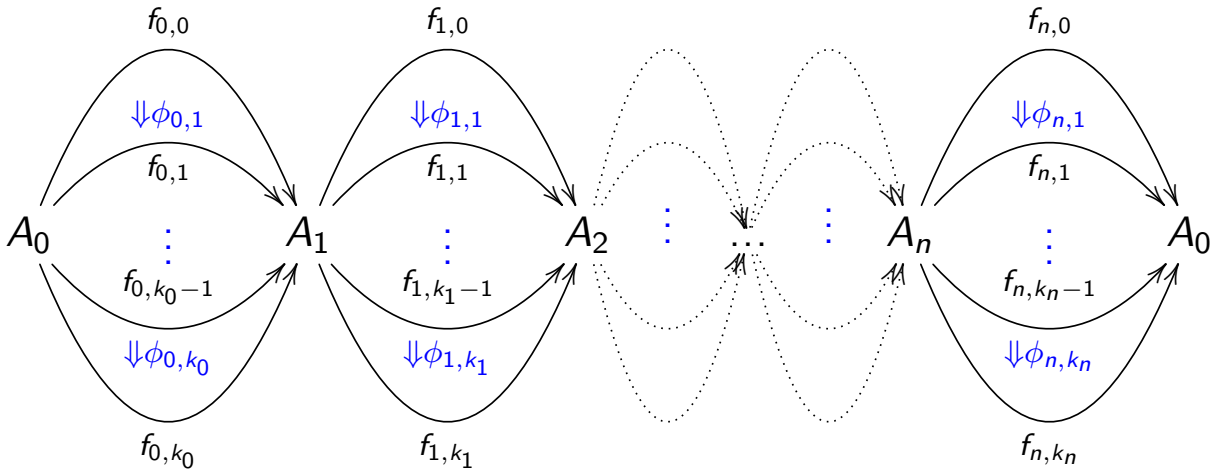
Fix algebras, $A_0, A_1, \ldots, A_n$.
We will define a dg cocategory $B(A_0 \to A_1 \to \ldots \to A_n \to A_0)$
where $B(A_0 \to A_0) := Bar(Hoch(A_0))$ for n=0.

More structure

Fix algebras, $A_0, A_1, \ldots, A_n$.
We will define a dg cocategory $B(A_0 \to A_1 \to \ldots \to A_n \to A_0)$
where $B(A_0 \to A_0) := Bar(Hoch(A_0))$ for n=0.

Objects: $\{A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \ldots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_0\}$

A morphism from $(A_0 \xrightarrow{f_{0,0}} \ldots \xrightarrow{f_{n,0}} A_0)$ to $(A_0 \xrightarrow{f_{0,k_0}} \ldots \xrightarrow{f_{n,k_n}} A_0)$:



$\phi_{i,j} \in C^\bullet(A_{i,f_{i,j-1}} A_{i+1 f_{i,j}})$

Structure among the $B(A_0 \to \ldots \to A_n \to A_0)$'s

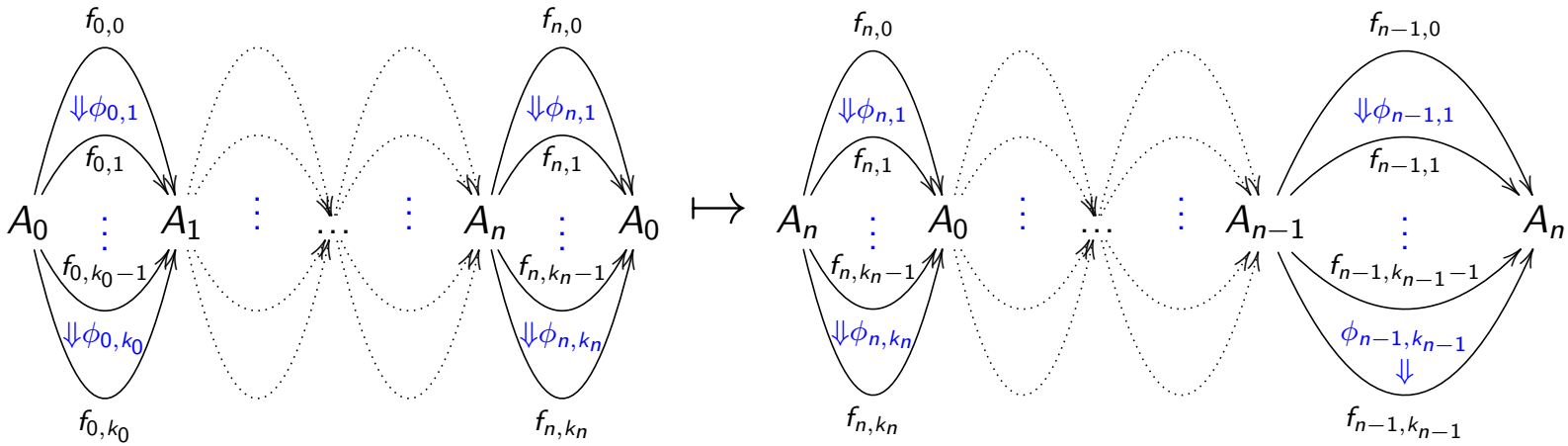
Example

We have a dg functor

$$\hat{\tau}_n : B(A_0 \to \ldots \to A_n \to A_0) \to B(A_n \to A_0 \to \ldots \to A_n)$$

$$Obj : \quad (f_0, \ldots, f_n) \mapsto (f_n, f_0, \ldots, f_{n-1})$$

Mor :



Structure among the $B(A_0 \to \ldots \to A_n \to A_0)$'s

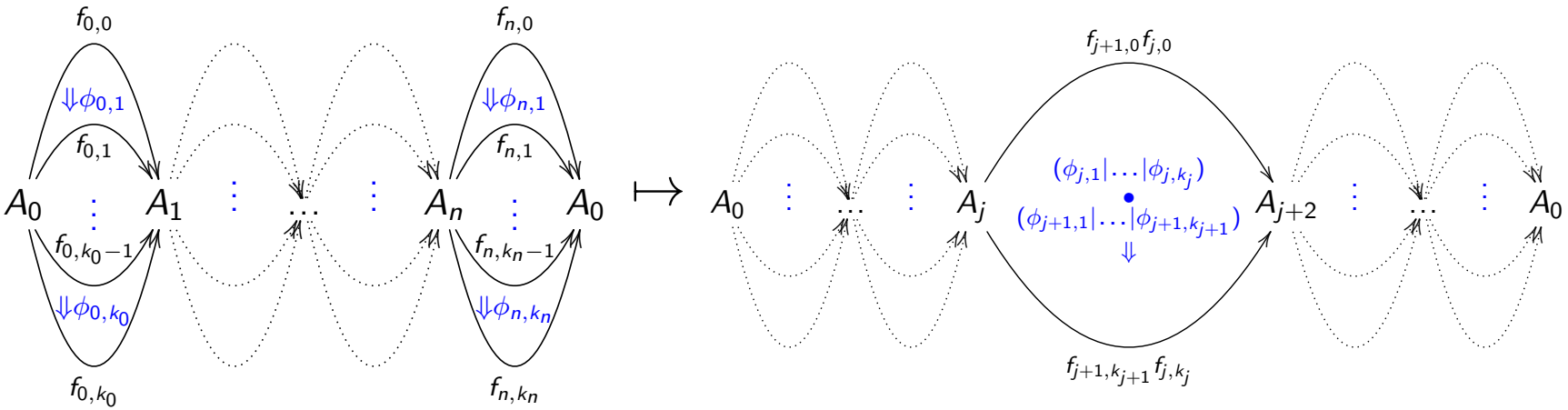
Example

For $n \geq 1, 0 \leq j < n$, we have a dg functor

$$\hat{\delta}_{j,n} : B(A_0 \to \ldots \to A_n \to A_0) \to B(A_0 \to \ldots \to A_j \to A_{j+2} \to A_0)$$

$$Obj : \quad (f_0, \ldots, f_n) \mapsto (f_0, \ldots, f_{j+1} f_j, \ldots, f_n)$$

Mor :



Structure among the $B(A_0 \to \ldots \to A_n \to A_0)$'s

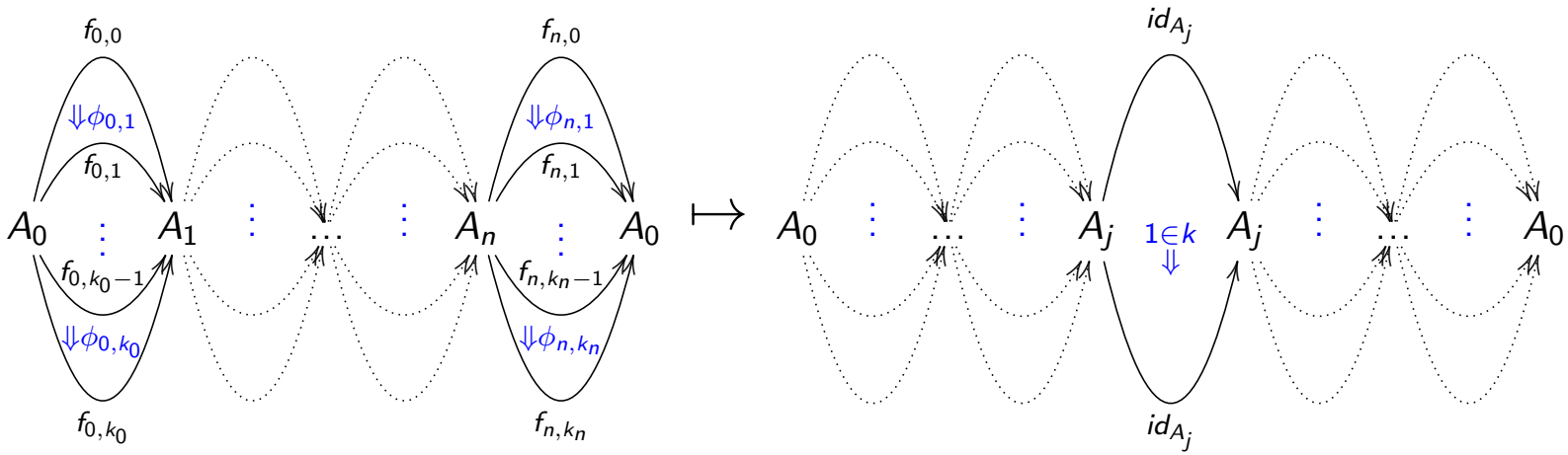
Example

For $n \geq 0, 0 \leq j \leq n$, we have a dg functor

$$\hat{\delta}_{j,n} : B(A_0 \to \ldots \to A_n \to A_0) \to B(A_0 \to \ldots \to A_j \to A_j \to A_0)$$

$$Obj : \quad (f_0, \ldots, f_n) \mapsto (f_0, \ldots, f_{j-1}, id_{A_j}, f_j, \ldots, f_n)$$

Mor :



A sheafy-cyclic object in DGCocat

Definition

Let χ be the category with objects $\{A_0 \to A_1 \to \ldots \to A_n \to A_0\}$ and morphisms compositions of

$$\begin{aligned}
\textit{rotations } \tau_n &: \mathcal{A} \mapsto (A_n \to A_0 \to \ldots \to A_n) \\
\textit{coboundaries } \delta_{j,n} &: \mathcal{A} \mapsto (A_0 \to \ldots \to A_j \to A_{j+2\,(\mathit{mod}\ n+1)} \to \ldots \to A_0) \\
\textit{codegeneracies } \sigma_{i,n} &: \mathcal{A} \mapsto (A_0 \to \ldots \to A_i \to A_i \to \ldots \to A_0)
\end{aligned}$$

where $\mathcal{A} := (A_0 \to \ldots \to A_n \to A_0)$, subject to the cyclic relations.

A sheafy-cyclic object in DGCocat

Definition

Let χ be the category with objects $\{A_0 \to A_1 \to \ldots \to A_n \to A_0\}$ and morphisms compositions of

$$\begin{aligned} \textit{rotations } \tau_n &: \mathcal{A} \mapsto (A_n \to A_0 \to \ldots \to A_n) \\ \textit{coboundaries } \delta_{j,n} &: \mathcal{A} \mapsto (A_0 \to \ldots \to A_j \to A_{j+2\,(\mathit{mod}\ n+1)} \to \ldots \to A_0) \\ \textit{codegeneracies } \sigma_{i,n} &: \mathcal{A} \mapsto (A_0 \to \ldots \to A_i \to A_i \to \ldots \to A_0) \end{aligned}$$

where $\mathcal{A} := (A_0 \to \ldots \to A_n \to A_0)$, subject to the cyclic relations.

Proposition

We have a functor $\chi \to DGCocat$

$$\begin{aligned} \textit{Objects}: \ & (A_0 \to \ldots \to A_n \to A_0) \mapsto B(A_0 \to \ldots \to A_n \to A_0) \\ \textit{Generating morphisms}: \ & \lambda \mapsto \hat{\lambda} \end{aligned}$$