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Outline

- **Question:** What do algebras form?

- **Answer 1:** A category in categories (HH^0)
- **Derived Answer 1:** A category in dg cocategories (Hochschild cochains...)

- **Answer 2:** A 2-category with a trace functor (HH_0)
- **Derived Answer 2:** A category in dg cocategories with a trace functor (Hochschild chains...)

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- **Answer 1:** A category in categories (HH^0)
- **Derived Answer 1:** A category in dg cocategories (Hochschild cochains...)

- **Answer 2:** A 2-category with a trace functor (HH_0)
- **Derived Answer 2:** A category in dg cocategories with a trace functor (Hochschild chains...) up to homotopy

Definition

(Kaledin): A <u>trace functor</u> on a 2-category $\mathcal{C}$ is:

- for each $A \in Obj(\mathcal{C})$, a functor $TR_A : \mathcal{C}(A, A) \to k - mod$
- for each pair $A, B \in Obj(\mathcal{C})$, a natural transformation $\tau_!(A, B)$

$$\begin{tikzcd}
\mathcal{C}(A,B) \otimes \mathcal{C}(B,A) \arrow[rr, "\tau = flip"] \arrow[d, "m"'] & & \mathcal{C}(B,A) \otimes \mathcal{C}(A,B) \arrow[d, "m"'] \\
\mathcal{C}(A,A) \arrow[dr, "TR_A"'] \arrow[rr, phantom, "\tau_!(\overset{\Rightarrow}{A},B)"] & & \mathcal{C}(B,B) \arrow[dl, "TR_B"] \\
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- such that $\tau_!(B, A) \circ \tau_!(C, B) \circ \tau_!(A, C) = id$

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\begin{tikzcd}
 & & \mathcal{C}(C, A) \otimes \mathcal{C}(A, C) \otimes \mathcal{C}(B, C) \arrow[drr, "\tau"] \arrow[dd] & & \\
\mathcal{C}(A, B) \otimes \mathcal{C}(B, C) \otimes \mathcal{C}(C, A) \arrow[urr, "\tau"] \arrow[drr] \arrow[r, phantom, "\tau_!(\overset{\Rightarrow}{A}, C)"] & & \tau_!(\overset{\Leftarrow}{B}, A) & \arrow[r, phantom, "\tau_!(\overset{\Rightarrow}{C}, B)"] & \mathcal{C}(B, C) \otimes \mathcal{C}(C, A) \otimes \mathcal{C}(A, B) \arrow[llll, bend right, "\tau"'] \arrow[dll] \\
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- such that $\tau^{*2}\tau_!(B, A) \circ \tau^*\tau_!(C, B) \circ \tau_!(A, C) = id$

Definition

Let $\mathcal{C}$ be a category in k-linear categories. Let $\chi(\mathcal{C})$ be the k-linear category with

- Objects $=$ $\{A \to A,\ A \to B \to A,\ A \to B \to C \to A : A, B, C \in Obj(\mathcal{C})\}$
- Morphisms $= \{$linear combinations of compositions of rotations τ_i s.t. $\tau_i^{i+1} = id,\ i = 0, 1, 2\}$.

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A <u>trace functor</u> on $\mathcal{C}$ gives a functor

$$\chi(\mathcal{C}) \to \mathcal{D} = \{(k\text{-lin category, module})\}$$

$$(A \to A) \mapsto (\mathcal{C}(A, A), T(A))$$
$$(A \to B \to A) \mapsto (\mathcal{C}(A, B) \otimes \mathcal{C}(B, A), m^* T(A))$$
$$(A \to B \to C \to A) \mapsto (\mathcal{C}(A, B) \otimes \mathcal{C}(B, C) \otimes \mathcal{C}(C, A), m^{*2} T(A))$$
$$\tau_1 : (A \to B \to A) \to (B \to A \to B) \mapsto \tau_!(A, B)$$
$$\tau_2 : (A \to B \to C \to A) \to (C \to A \to B \to C) \mapsto m^* \tau_!(A, C)$$
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- Morphisms = {linear combinations of compositions of rotations τ_i s.t. $\tau_i^{i+1} = id$, $i = 0, 1, 2\}$. Why stop at n=2? What about δ, σ?

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Let $\mathcal{C}$ be a category in dg cocategories. Let $\chi_\infty(\mathcal{C})$ be the dg category with

- Objects $= \{A_0 \to \ldots \to A_n \to A_0 : A_i \in Obj(\mathcal{C}), n \geq 0\}$
- Morphisms $= \{$linear combinations of compositions of

$$\begin{aligned} \textit{rotations } \tau_n &: \mathcal{A} \mapsto (A_n \to A_0 \to \ldots \to A_n) \\ \textit{coboundaries } \delta_{j,n} &: \mathcal{A} \mapsto (A_0 \to \ldots \to A_j \to A_{j+2\,(mod\ n+1)} \to \ldots \to A_0) \\ \textit{codegeneracies } \sigma_{i,n} &: \mathcal{A} \mapsto (A_0 \to \ldots \to A_i \to A_i \to \ldots \to A_0) \end{aligned}$$

  where $\mathcal{A} := (A_0 \to \ldots \to A_n \to A_0)$, subject to the cyclic relations$\}[0]$

Definition

Let $\mathcal{D}_\infty$ be the dg category with

- Objects $= \{(\underset{B}{\text{dg cocategory}}, \underset{C}{\text{dg comodule}})\}$
- Morphisms:

$$\mathcal{D}^p_\infty((B_1, C_1), (B_0, C_0)) := \left\{ \begin{matrix} F : B_1 \to B_0 \textit{ dg functor}, \\ F_! : C_1 \to F^* C_0 \textit{ degree-p linear map} \end{matrix} \right\}$$

$$d_{D_\infty}(F, F_!) = (F, [d, F_!] = {\scriptstyle d_{F^* C_0} \circ F_! \pm F_! \circ d_{C_1}})$$

$F^* C_0$ is the categorified version of co-extension of scalars:

$$F^* C_0 = ker(B_1 \otimes C_0 \rightrightarrows B_1 \otimes B_0 \otimes C_0)$$

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No, but we can give an A_∞-functor.