

Table of Contents

Chapter 1. $B(n)$ and $C(n)$	3
1.1. Motivation of this chapter	4
1.2. Dg cocategories: $B(n)$	5
1.3. Dg comodules: $C(n)$	11
Chapter 2. Pullbacks, Pushforwards and Adjunctions	16
2.1. A sheafy-cyclic object in dg cocategories	17
2.2. Motivation of this chapter	21
2.3. Pullbacks of dg comodules–theory	22
2.4. Pullbacks of dg comodules–examples	31
2.5. Adjunction between λ^* and $\lambda_\#$	36
2.6. Maps λ_l	45
Chapter 3. A homotopically sheafy-cyclic object	48
3.1. Motivation of this chapter	49
3.2. Homotopies	51
3.3. Higher Homotopies	59
3.4. An A_∞ -functor	62
3.5. A functor to dg categories	67
References	71

Appendix A. Connes cyclic category, Λ	72
Appendix B. Computations	74
B.1. Computational notation	75
B.2. Computational Propositions	76
B.3. More notation	85
B.4. More Propositions	86
Appendix C. Background on Hochschild chains and cochains	92
C.1. Standard constructions and notation	92
C.2. Brace operation on Hochschild cochains	95

CHAPTER 1

 $B(n)$ **and** $C(n)$

1.1. Motivation of this chapter

In this chapter, we introduce the main characters/objects of study, $B(n)$ and $C(n)$, $n \in \mathbb{N}$. The $B(n)$'s are dg cocategories constructed using Hochschild cochains. Each $C(n)$ is a dg comodule over $B(n)$ constructed using an action of Hochschild cochains on Hochschild chains. We start with definitions for the less-widely-used concepts, and show that the main characters are conilpotent.

See Appendix C for definitions of known operations on Hochschild chains and cochains as well as our notation and conventions.

1.2. Dg cocategories: $B(n)$

1.2.1. Background on dg cocategories

Definition 1.2.1. A **dg cocategory** is a cocategory enriched over chain complexes.

More explicitly, a dg cocategory B consists of the following data:

- A collection of objects denoted $Obj(B)$;
- For each pair of objects, $x, z \in Obj(B)$, a complex $B^\bullet(x, z)$ and a morphism of complexes

$$\Delta_B(x, z) : B^\bullet(x, z) \rightarrow \prod_{y \in Obj(B)} B^\bullet(x, y) \otimes B^\bullet(y, z)$$

such that the following diagrams commute (coassociativity):

$$\begin{array}{ccc}
 B^\bullet(x, z) & \xrightarrow{\Delta_B(x, z)} & \prod_{y \in Obj(B)} B^\bullet(x, y) \otimes B^\bullet(y, z) \\
 \Delta_B(x, z) \downarrow & & \downarrow \prod_y id_{B(x, y)} \otimes \Delta_B(y, z) \\
 \prod_{y \in Obj(B)} B^\bullet(x, y) \otimes B^\bullet(y, z) & \xrightarrow{\prod_y \Delta_B(x, y) \otimes id_{B(y, z)}} & \prod_{y, y' \in Obj(B)} B^\bullet(x, y) \otimes B^\bullet(y, y') \otimes B^\bullet(y', z)
 \end{array}$$

- For each pair of objects, $x, z \in Obj(B)$, a morphism of complexes

$$\epsilon_B(x, z) : B^\bullet(x, z) \rightarrow k$$

where k is the ground field considered as a chain complex concentrated in degree 0 and $\epsilon_B(x, z) = 0$ if $x \neq z$, such that the following diagrams commute (counitality):

$$\begin{array}{ccc}
 B^\bullet(x, z) & \xrightarrow{\Delta_B(x, z)} & \prod_{y \in \text{Obj}(B)} B^\bullet(x, y) \otimes B^\bullet(y, z) \\
 \downarrow \Delta_B(x, z) & \searrow id & \downarrow \prod_y \epsilon_B(x, y) \otimes id_{B(y, z)} \\
 \prod_{y \in \text{Obj}(B)} B^\bullet(x, y) \otimes B^\bullet(y, z) & \xrightarrow{\prod_y id_{B(x, y)} \otimes \epsilon_B(y, z)} & B^\bullet(x, z).
 \end{array}$$

We will denote a dg cocategory with its cocomposition and counit as $(B, \Delta_B, \epsilon_B)$. To make the notation more readable, when the meaning is clear, we will omit references to the objects and write Δ_B instead of $\Delta_B(x, z)$, ϵ_B instead of $\epsilon_B(x, z)$, and for the differentials on morphisms, d_B instead of $d_B(x, z)$.

Definition 1.2.2. A **functor** $F : A \rightarrow B$ between two dg cocategories is a functor between the cocategories satisfying $d_B \circ F(f) = F \circ d_A(f)$ for all morphisms f in A .

Definition 1.2.3. A **conilpotent** dg cocategory is a dg cocategory $(B, \Delta_B, \epsilon_B)$ satisfying: for each morphism $f : x \rightarrow y$ in B , there exists $n_f \in \mathbb{N}$ such that $\bar{\Delta}_B^{n_f}(f) = 0$ where

$$\begin{aligned}
 \bar{\Delta}_B(x, z) : B^\bullet(x, z) &\rightarrow \prod_{y \in \text{Obj}(B)} B^\bullet(x, y) \otimes B^\bullet(y, z) \\
 f &\mapsto \Delta_B(f) - \sum_{e_x \in \epsilon_B(x, x)^{-1}(1)} e_x \otimes f - \sum_{e_z \in \epsilon_B(z, z)^{-1}(1)} f \otimes e_z.
 \end{aligned}$$

Fact (needs reference?): If B is a conilpotent dg cocategory, then for all $x \in \text{Obj}(B)$, $\epsilon_B(x, x)^{-1}(1)$ has exactly one element, which we will denote e_x .

1.2.2. Structure of $B(n)$

For each sequence of algebras, A_0, A_1, \dots, A_n , we will define a conilpotent dg cocategory, $B(A_0 \rightarrow A_1 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$. In this chapter, we fix the sequence of algebras, and abbreviate

$$B(n) := B(A_0 \rightarrow A_1 \rightarrow \dots \rightarrow A_n \rightarrow A_0).$$

1.2.2.1. Objects. $B(n)$ has objects tuples (f_0, f_1, \dots, f_n) where $f_i : A_i \rightarrow A_{i+1 \pmod{n+1}}$, $0 \leq i \leq n$, are maps of algebras. We can picture an object in $B(n)$ as follows:

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_0$$

1.2.2.2. Morphisms. The graded vector space of morphisms in $B(n)$ between two objects, (f_0, \dots, f_n) and (g_0, \dots, g_n) , is

$$\text{Bar}(C^\bullet(A_{0,f_0} A_{1g_0})) \otimes \text{Bar}(C^\bullet(A_{1,f_1} A_{2g_1})) \otimes \dots \otimes \text{Bar}(C^\bullet(A_{n,f_n} A_{0g_n}))$$

where $\text{Bar}(C^\bullet(A, {}_f B_g))$ is the following complex:

$$\text{Bar}(C^\bullet(A, {}_f B_g)) := \text{Bar}_0(C^\bullet(A, {}_f B_g)) \oplus \bigoplus_{m \geq 1} \text{Bar}_m(C^\bullet(A, {}_f B_g))$$

$$\text{Bar}_0(C^\bullet(A, {}_f B_g)) := k$$

$$\text{Bar}_m(C^\bullet(A, {}_f B_g)) := \bigoplus_{\substack{h_0=f, \\ h_m=g, \\ h_1, \dots, h_{m-1} \\ \text{algebra maps}}} C^\bullet(A, {}_{h_0} B_{h_1})[1] \otimes C^\bullet(A, {}_{h_1} B_{h_2})[1] \otimes \cdots \otimes C^\bullet(A, {}_{h_{m-1}} B_{h_m})[1]$$

$(C^\bullet(A, {}_{h_i} B_{h_j}), {}_{h_i} \delta_{h_j}) = \text{Hochschild cochain complex, see Appendix C}$

$$d_{\text{Bar}(C^\bullet(A, {}_f B_g))} = \tilde{\delta} + b'$$

$$\tilde{\delta}(\phi_1 \otimes \cdots \otimes \phi_m) = \sum_{1 \leq i \leq m} (-1)^{1 + \sum_{j < i} |\phi_j| + 1} \phi_1 \otimes \cdots \otimes [{}_{h_{i-1}} \delta_{h_i}](\phi_i) \otimes \cdots \otimes \phi_m$$

$$b'(\phi_1 \otimes \cdots \otimes \phi_m) = \sum_{1 \leq i \leq m-1} (-1)^{\sum_{j \leq i} |\phi_j| + 1} \phi_1 \otimes \cdots \otimes \phi_i \cup \phi_{i+1} \otimes \cdots \otimes \phi_m$$

$\cup = \text{cup product on Hochschild cochains, see Appendix C.}$

(This sign convention is consistent with Reference [1], Section 4.6.)

1.2.2.3. Aside on notation. When referring to an arbitrary morphism in $B(n)$, we will assume it is a morphism from object $(f_{0,0}, f_{1,0}, \dots, f_{n,0})$ to object $(f_{0,k_0}, f_{1,k_1}, \dots, f_{n,k_n})$.

We will denote the morphism

$$\phi_{0,1} \cdots \phi_{0,k_0} | \phi_{1,1} \cdots \phi_{1,k_1} | \cdots | \phi_{n,1} \cdots \phi_{n,k_n}$$

where $\phi_{i,j} \in C^\bullet(A_{i,f_{j-1}}, A_{i+1 \pmod{n+1}} f_j)$. See Figure 1.1 for a picture of this morphism.

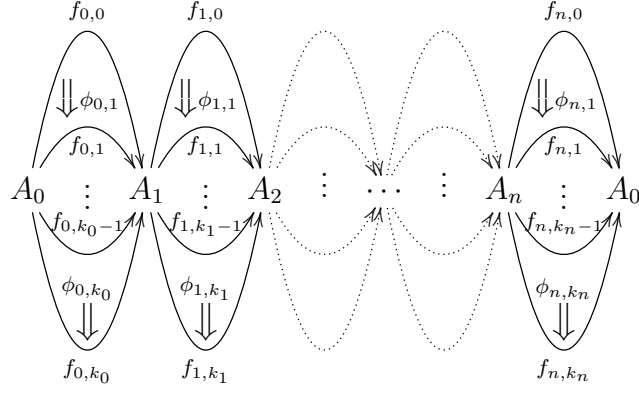


Figure 1.1. A morphism in $B(n)$ from $(f_{0,0}, f_{1,0}, \dots, f_{n,0})$ to $(f_{0,k_0}, f_{1,k_1}, \dots, f_{n,k_n})$ where $\phi_{i,j} \in C^\bullet(A_i, A_{i+1 \pmod{n+1}})_{f_j}$

1.2.2.4. Differential on $B(n)$. Putting everything together, the differential on $B(n)((f_{0,0}, \dots, f_{n,0}), (f_{0,k_0}, \dots, f_{n,k_n}))$ is

$$\begin{aligned} & d_{B(n)}(\phi_{0,1} \dots \phi_{0,k_0} | \dots | \phi_{n,1} \dots \phi_{n,k_n}) \\ &= \sum_{0 \leq i \leq n} (-1)^{\sum_{p < i; q} |\phi_{p,q}| + 1} \phi_{0,1} \dots | \dots | d_{\text{Bar}(C^\bullet(A_i, A_{i+1}))}(\phi_{i,1} \dots \phi_{i,k_i}) | \dots | \phi_{n,k_n} \end{aligned}$$

1.2.2.5. Cunit. Define

$$\begin{aligned} & \epsilon_{B(n)}((f_{0,0}, \dots, f_{n,0}), (f_{0,k_0}, \dots, f_{n,k_n})) : B(n)((f_{0,0}, \dots, f_{n,0}), (f_{0,k_0}, \dots, f_{n,k_n})) = \\ &= \text{Bar}(C^\bullet(A_{0,f_{0,0}} A_{1f_{0,k_0}})) \otimes \dots \otimes \text{Bar}(C^\bullet(A_{0,f_{n,0}} A_{1f_{n,k_n}})) \rightarrow \\ & \xrightarrow{\text{project}} \text{Bar}_0(C^\bullet(A_{0,f_{0,0}} A_{1f_{0,k_0}})) \otimes \dots \otimes \text{Bar}_0(C^\bullet(A_{0,f_{n,0}} A_{1f_{n,k_n}})) \cong k. \end{aligned}$$

1.2.2.6. Cocomposition. We have a coassociative map of complexes

$$\begin{aligned} \Delta_{A,fB_g} : Bar(C^\bullet(A,f B_g)) &\rightarrow \bigoplus_{h:A \rightarrow B} Bar(C^\bullet(A,f B_h)) \otimes Bar(C^\bullet(A,h B_g)) \\ \phi_1 \cdots \phi_k &\mapsto \sum_{1 \leq i \leq k-1} \phi_1 \cdots \phi_i \otimes \phi_{i+1} \cdots \phi_k \\ &\quad + e_f \otimes \phi_1 \cdots \phi_k + \phi_1 \cdots \phi_k \otimes e_g \end{aligned}$$

where $e_f = 1$ in $Bar_0(C^\bullet(A,f B_f)) \cong k$. Extend Δ_{A,fB_g} to a cocomposition on $B(n)$ by taking (up to signs)

$$\Delta_{B(n)}((f_{0,0}, \dots, f_{n,0}), (f_{0,k_0}, \dots, f_{n,k_n})) := \Delta_{A_0,f_{0,0} A_1 f_{0,k_0}} \otimes \cdots \otimes \Delta_{A_0,f_{n,0} A_1 f_{n,k_n}}.$$

The sign on the term $(\phi_{0,1} \cdots \phi_{0,i_0} | \cdots | \phi_{n,1} \cdots \phi_{n,i_n}) \otimes (\phi_{0,i_0+1} \cdots \phi_{0,k_0} | \cdots | \phi_{n,i_n+1} \cdots \phi_{n,k_n})$ in the cocomposition is:

$$(-1)^{\sum_{1 \leq p \leq n} (\sum_{1 \leq q \leq i_p} |\phi_{p,q}| + 1) (\sum_{i_r+1 \leq s \leq k_r} |\phi_{r,s}| + 1)}$$

It's clear from the definitions that $(B(n), \Delta_{B(n)}, \epsilon_{B(n)})$ satisfy the diagrams needed to form a dg cocategory. We also see that $B(n)$ is conilpotent:

$$\bar{\Delta}_{B(n)}^{min(k_0, \dots, k_n)}(\phi_{0,1} \cdots \phi_{0,k_0} | \cdots | \phi_{n,1} \cdots \phi_{n,k_n}) = 0.$$

1.3. Dg comodules: $C(n)$

1.3.1. Background on dg comodules

Definition 1.3.1. A dg comodule C over a dg cocategory B consists of the following data:

- for each object $f \in B$, a complex $C^\bullet(f)$, and
- maps of complexes

$$\Delta_C(f) : C^\bullet(f) \rightarrow \prod_{g \in \text{Obj}(B)} B^\bullet(f, g) \otimes C^\bullet(g).$$

such that the following diagrams for coassociativity and counitality commute:

$$\begin{array}{ccc} C^\bullet(f) & \xrightarrow{\Delta_C(f)} & \prod_{g \in \text{Obj}(B)} B^\bullet(f, g) \otimes C^\bullet(g) \\ \Delta_C(f) \downarrow & & \downarrow \prod_g id_{B(f, g)} \otimes \Delta_C(g) \\ \prod_{g \in \text{Obj}(B)} B^\bullet(f, g) \otimes C^\bullet(g) & \xrightarrow{\prod_g \Delta_B(f, g) \otimes id_{C(g)}} & \prod_{g, g' \in \text{Obj}(B)} B^\bullet(f, g) \otimes B^\bullet(g, g') \otimes C^\bullet(g') \end{array}$$

$$\begin{array}{ccc} C^\bullet(f) & \xrightarrow{\Delta_C(f)} & \prod_{g \in \text{Obj}(B)} B^\bullet(f, g) \otimes C^\bullet(g) \\ & \searrow id & \downarrow \prod_g \epsilon_B(f, g) \otimes id_{C(g)} \\ & & C^\bullet(f). \end{array}$$

To simplify notation, we will write Δ_C instead of $\Delta_C(f)$ when the meaning is clear.

Example 1.3.1. A dg comodule over a dg cocategory B with one object, $*$, is a dg comodule over the counital dg coalgebra $B^\bullet(*, *)$.

Definition 1.3.2. A morphism of dg comodules $H : C \rightarrow D$ over a dg category B consists of maps of complexes $(H_f : C^\bullet(f) \rightarrow D^\bullet(f))_{f \in \text{Obj}(B)}$ such that for each $f \in \text{Obj}(B)$, the following diagram commutes:

$$\begin{array}{ccc} C^\bullet(f) & \xrightarrow{H_f} & D^\bullet(f) \\ \downarrow \Delta_C & & \downarrow \Delta_D \\ \prod_{g \in \text{Obj}(B)} B^\bullet(f, g) \otimes C^\bullet(g) & \xrightarrow{\prod_g \text{id}_B \otimes H_g} & \prod_{g \in \text{Obj}(B)} B^\bullet(f, g) \otimes D^\bullet(g). \end{array}$$

Again, when the meaning is clear, we may write H instead of H_f .

Definition 1.3.3. A **conilpotent** dg comodule over a dg cocategory B is a dg comodule (C, Δ_C) over B satisfying: for each $f \in \text{Obj}(B)$ and each element $\alpha \in C^\bullet(f)$, there exists $n_\alpha \in \mathbb{N}$ such that $\bar{\Delta}_f^{n_\alpha}(\alpha) = 0$ where

$$\begin{aligned} \bar{\Delta}_C(f) : C^\bullet(f) &\rightarrow \prod_{g \in \text{Obj}(B)} B^\bullet(f, g) \otimes C^\bullet(g) \\ \alpha &\mapsto \Delta_B(\alpha) - \sum_{e_f \in \epsilon_B(f, f)^{-1}(1)} e_f \otimes f. \end{aligned}$$

1.3.2. Structure of $C(n)$

Reminder: In this chapter, we fix algebras A_0, A_1, \dots, A_n . $C(n)$ and $B(n)$ are short for $C(A_0 \rightarrow A_1 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$ and $B(A_0 \rightarrow A_1 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$, respectively.

We now give dg comodules $C(n)$ over $B(n)$. First, we will describe the graded comodule

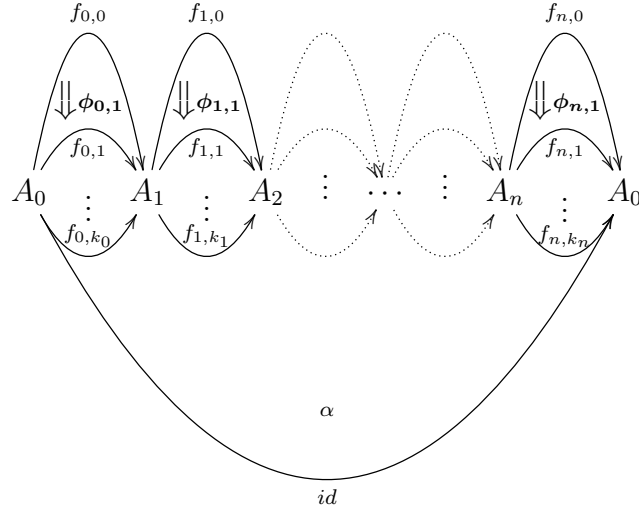


Figure 1.2. Picture of an element of $C(n)^\bullet(f)$ where $f = (f_{0,0}, f_{1,0}, \dots, f_{n,0})$, $\phi_{i,j} \in C^\bullet(A_i, f_{j-1} A_{i+1 \pmod{n+1}} f_j)$, and $\alpha \in C_{-\bullet}(A_0, f_n \dots f_1 f_0 A_{0id})$

structure; then, we will describe the differentials. For an object $f = (f_0, f_1, \dots, f_n) \in B(n)$, we have

$$(1.1) \quad C(n)^\bullet(f) = \bigoplus_{g \in \text{Obj}(B(n))} B(n)^\bullet(f, g) \otimes C_{-\bullet}(A_0, \text{comp}(g) A_{0id})$$

where, for $g = (g_0, g_1, \dots, g_n)$, we write $\text{comp}(g) = g_n \circ g_{n-1} \circ \dots \circ g_0$, and $(C_\bullet(A, B), {}_g b)$ is the Hochschild chain complex (see Appendix C). We will denote a typical element of $C(n)^\bullet(f)$ as

$$\phi_{0,1} \dots \phi_{0,k_0} | \phi_{1,1} \dots \phi_{1,k_1} | \dots | \phi_{n,1} \dots \phi_{n,k_n} | \alpha$$

where $\phi_{0,1} \dots \phi_{n,k_n}$ is a morphism in $B(n)$ (see Section 1.2.2.3) and $\alpha \in C_{-\bullet}(A_0, f_{k_n} \dots f_{k_0} A_{0id})$.

See Figure 1.2 for a picture of a typical element of $C(n)^\bullet(f)$.

1.3.2.1. Comodule structure. The comodule maps on $C(n)^\bullet(f)$ are given by the co-composition maps in $B(n)$:

$$\begin{array}{ccc}
C(n)^\bullet(f) & \xrightarrow{\Delta_C} & \bigoplus_{h \in \text{Obj}(B(n))} B(n)^\bullet(f, h) \otimes C(n)^\bullet(h) \\
\parallel & & \parallel \\
\bigoplus_{g \in \text{Obj}(B(n))} B(n)^\bullet(f, g) \otimes C_{-\bullet}(A_{0,g} A_{0id}) & \xrightarrow{\Delta_{B(n)} \otimes id_{C_{-\bullet}}} & \bigoplus_{g, h \in \text{Obj}(B(n))} \begin{matrix} B(n)^\bullet(f, h) \otimes B(n)^\bullet(h, g) \\ \otimes C_{-\bullet}(A_{0,g} A_{0id}) \end{matrix}
\end{array}$$

Because $\Delta_{C(n)}$ is induced by $\Delta_{B(n)}$, we have that $\Delta_{C(n)}$ satisfies coassociativity and counitality and is conilpotent.

$C(n)$ cofree as a comodule in the sense that a morphism to $C(n)$ is determined by projections to its Hochschild-chains component. More precisely, there is a one-to-one correspondence

(1.2)

$$\begin{aligned}
& \left\{ \begin{array}{l} \text{maps of comodules} \\ D \rightarrow C(n) \text{ over } B(n) \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{l} \text{maps of graded vector spaces} \\ D^\bullet(f) \rightarrow C_{-\bullet}(A_{0,f} A_{0id}) \end{array} \right\}_{f \in \text{Obj}(B(n))} \\
& \left(F : D \rightarrow C(n) \right) \mapsto \left(\begin{array}{c} D(f) \xrightarrow{F_f} C(n)(f) \\ \xrightarrow{\text{project}} C_{-\bullet}(A_{0,f} A_{0id}) \end{array} \right)_f \\
& \left(\begin{array}{c} D(f) \xrightarrow{\Delta_D} \bigoplus_{g \in \text{Obj}(B(n))} B(n)(f, g) \otimes D(g) \\ \xrightarrow{\oplus_g id_B \otimes F_g} \bigoplus_g B(n)(f, g) \otimes C_{-\bullet}(A_{0,g} A_{0id}) \\ \cong C(n)(f) \end{array} \right)_f \leftarrow \left(D(f) \rightarrow C_{-\bullet}(A_{0,f} A_{0id}) \right)_f
\end{aligned}$$

Definition 1.3.4. We will call elements of $T(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)(f) := C_{-\bullet}(A_{0,f} A_{0id})$ the **cogenerators** of $C(A_0 \rightarrow \dots A_n \rightarrow A_0)(f)$. More generally, we will refer to the set $T(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0) = \{T(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)(f) | f \in \text{Obj}(B(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0))\}$

$A_0))\}$ as the **cogenerators** of $C(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$. When we have fixed a sequence of algebras, A_0, \dots, A_n , we will use $T(n)$ to denote $T(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$.

1.3.2.2. Differential. The differential $d_{C(n)}$ on $C(n)$ is:

(1.3)

$$d_{C(n)} = \tilde{d}_{B(n)} + \tilde{b} + \mathcal{J}$$

$$\tilde{d}_{B(n)}(\phi_{0,1} \dots \phi_{0,k_0} | \dots | \phi_{n,1} \dots \phi_{n,k_n} | \alpha) = d_{B(n)}(\phi_{0,1} \dots | \dots | \dots | \phi_{n,k_n}) | \alpha$$

$$\tilde{b}(\phi_{0,1} \dots \phi_{0,k_0} | \dots | \phi_{n,1} \dots \phi_{n,k_n} | \alpha) = (-1)^{\sum_{i,j} |\phi_{i,j}| + 1} \phi_{0,1} \dots | \dots | \phi_{n,k_n} | [f_{n,k_n} \dots f_{0,k_0} b](\alpha)$$

where $d_{B(n)}$ is the differential on $B(n)$, and \mathcal{J} is a term that captures an action of cochains on chains described by the equations below:

(1.4)

$$\mathcal{J} = (id_{B(n)} \otimes \eta_{C(n)}) \circ \Delta_{C(n)}$$

$$\eta_{C(n)}(\phi_{0,1} \dots \phi_{0,k_0} | \dots | \phi_{n,1} \dots \phi_{n,k_n} | \alpha) = \iota_{C(0)}(\pi_{B(0)}((\phi_{0,1} \dots \phi_{0,k_0}) \bullet \dots \bullet (\phi_{n,1} \dots \phi_{n,k_n})), \alpha)$$

\bullet = brace operation on cochains, see Section C.2

$$\pi_{B(0)} : B(0)^\bullet(f_{n,1} \dots f_{1,1} f_{0,1}, f_{n,k_n} \dots f_{1,k_1} f_{0,k_0}) \xrightarrow[\text{component}]{\text{project onto}} C^\bullet(A_{0,f_{n,1} \dots f_{0,1}} A_{0,f_{n,k_n} \dots f_{0,k_0}})$$

ι = contraction operation, see Section C

Given Equation 1.3, it's easy to check that we can promote Equation 1.2 to a dg statement:

$$\left\{ \begin{array}{c} \text{maps of dg comodules} \\ D \rightarrow C(n) \text{ over } B(n) \end{array} \right\} \xleftrightarrow{1:1} \left\{ \left(\begin{array}{c} \text{maps of complexes} \\ D^\bullet(f) \rightarrow C_{-\bullet}(A_{0,f} A_{0,id}) \end{array} \right)_{f \in \text{Obj}(B(n))} \right\}.$$

CHAPTER 2

Pullbacks, Pushforwards and Adjunctions

2.1. A sheafy-cyclic object in dg cocategories

We would like to say that we have a functor from Connes cyclic category Λ (see Appendix A for generators and relations) to the category of dg cocategories where $[n] \mapsto B(n)$, but defining $B(n)$ involved choosing a sequence of algebras A_0, \dots, A_n .

Instead, we have the following: Let $X : \Lambda \rightarrow \text{Set}$ be the functor that sends $[n]$ to the set of diagrams $A_0 \rightarrow A_1 \rightarrow \dots \rightarrow A_n \rightarrow A_0$ where the A_i 's are algebras. On generating morphisms in Λ , X acts as follows: Let $\mathcal{A} = (A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0) \in X([n])$.

$$X(\tau_n) : \mathcal{A} \mapsto (A_n \rightarrow A_0 \rightarrow \dots \rightarrow A_{n-1} \rightarrow A_n)$$

$$X(\delta_{j,n}) : \mathcal{A} \mapsto (A_0 \rightarrow \dots \rightarrow A_j \rightarrow A_{j+2 \pmod{n+1}} \rightarrow \dots \rightarrow A_n \rightarrow A_0)$$

$$X(\sigma_{i,n}) : \mathcal{A} \mapsto \begin{cases} (A_0 \rightarrow \dots \rightarrow A_i \rightarrow A_i \rightarrow \dots \rightarrow A_n \rightarrow A_0) & 1 \leq i \leq n \\ (A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0 \rightarrow A_0) & i = n + 1 \end{cases}$$

It's straightforward to check that X respects composition of morphisms. Now, let χ be the category with objects given by diagrams $A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0$ where the A_i 's are algebras and $n \in \mathbb{N}$. Morphisms in χ are the pointwise images of X . In other words, the set of morphisms in χ is $\{X(\lambda)|_x : \lambda \in \Lambda([n], [m]), x \in X([n])\}$. We will give a functor, \mathcal{G} , from χ to the category of dg cocategories; (this is our sheafy-cyclic object, i.e., a **sheafy-cyclic** object in a category \mathcal{C} is a functor $\chi \rightarrow \mathcal{C}$).

2.1.1. Aside on notation:

Fix $\lambda : [n] \rightarrow [m]$ in Λ and $x \in X([n])$. To define \mathcal{G} , we will need to define a functor $\mathcal{G}(X(\lambda)|_x) : B(x) \rightarrow B(X(\lambda)(x))$. To simplify notation, we will denote $\hat{\lambda} := \mathcal{G}(X(\lambda)|_x)$

and write $\hat{\lambda} : B(x) \rightarrow B(\lambda x)$. Technically, we are losing information about the x when we write $\hat{\lambda}$ instead of $\mathcal{G}(X(\lambda)|_x)$, but we will be clear about the source and target when needed.

2.1.2. Definition of \mathcal{G}

Now, we will define \mathcal{G} . On objects,

$$\mathcal{G} : (A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0) \mapsto B(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$$

(see Section 1.2 for definition of $B(\cdot)$)

On generating morphisms in χ , set $\mathcal{A} = (A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0) \in \text{Obj}(\chi)$, and define

$$\begin{aligned}
 \hat{\tau}_n &\mapsto \left\{ \begin{array}{l} B(\mathcal{A}) \longrightarrow B(A_n \rightarrow A_0 \rightarrow \dots \rightarrow A_{n-1} \rightarrow A_n) \\ \text{objects: } (f_0, f_1, \dots, f_n) \mapsto (f_n, f_0, \dots, f_{n-1}) \\ \text{morphisms: } \phi_{0,1} \dots \phi_{0,k_0} | \dots | \phi_{n,1} \dots \phi_{n,k_n} \mapsto \phi_{n,1} \dots \phi_{n,k_n} | \dots | \phi_{n-1,1} \dots \phi_{n-1,k_{n-1}} \end{array} \right. \\
 \hat{\delta}_{j,n} &\mapsto \left\{ \begin{array}{l} B(\mathcal{A}) \longrightarrow B(A_0 \rightarrow \dots \rightarrow A_j \rightarrow A_{j+2 \pmod{n+1}} \rightarrow \dots \rightarrow A_0) \\ \text{objects: } (f_0, f_1, \dots, f_n) \mapsto (f_0, \dots, f_{j+1} \circ f_j, \dots, f_n) \\ \text{morphisms: } \phi_{0,1} \dots \phi_{0,k_0} | \dots | \phi_{n,1} \dots \phi_{n,k_n} \mapsto \\ \phi_{0,1} \dots \phi_{0,k_0} | \dots | (\phi_{j,1} \dots \phi_{j,k_j}) \bullet (\phi_{j+1,1} \dots \phi_{j+1,k_{j+1}}) | \dots | \phi_{n,1} \dots \phi_{n,k_n} \end{array} \right. \\
 \hat{\sigma}_{i,n} \mapsto \left\{ \begin{array}{l} B(\mathcal{A}) \longrightarrow B(A_0 \rightarrow \dots \rightarrow A_i \rightarrow A_i \rightarrow \dots \rightarrow A_0) \\ \text{objects: } (f_0, f_1, \dots, f_n) \mapsto (f_0, \dots, f_{i-1}, id_{A_i}, f_i, \dots, f_n) \\ \text{morphisms: } \phi_{0,1} \dots \phi_{0,k_0} | \dots | \phi_{n,1} \dots \phi_{n,k_n} \mapsto \\ \phi_{0,1} \dots \phi_{0,k_0} | \dots | \phi_{i-1,1} \dots \phi_{i-1,k_{i-1}} | 1 | \phi_{i,1} \dots \phi_{i,k_i} | \dots | \phi_{n,1} \dots \phi_{n,k_n} \\ 1 \in k = \text{degree 0 component of } Bar(C^\bullet(A_i, A_i)) \end{array} \right. \\
 \hat{\sigma}_{n+1,n} &\mapsto \left\{ \begin{array}{l} B(\mathcal{A}) \longrightarrow B(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0 \rightarrow A_0) \\ \text{objects: } (f_0, f_1, \dots, f_n) \mapsto (f_0, \dots, f_n, id_{A_0}) \\ \text{morphisms: } \phi_{0,1} \dots \phi_{0,k_0} | \dots | \phi_{n,1} \dots \phi_{n,k_n} \mapsto \\ \phi_{0,1} \dots \phi_{0,k_0} | \dots | \phi_{n,1} \dots \phi_{n,k_n} | 1 \\ 1 \in k = \text{degree 0 component of } Bar(C^\bullet(A_0, A_0)) \end{array} \right.
 \end{aligned}$$

It's straightforward to check that \mathcal{G} is a functor (i.e., that composition of morphisms and the relations are preserved). The only facts we need are that \bullet is an associative map of complexes and that $1 \bullet (\phi_0 \dots \phi_k) = (\phi_0 \dots \phi_k) \bullet 1 = (\phi_0 \dots \phi_k)$ where the 1's are in the degree 0 components of $Bar(C^\bullet(A_i, A_i))$ for the appropriate A_i (see Appendix Section C.2).

2.2. Motivation of this chapter

We would like to extend the sheafy-cyclic structure in Section 2.1 from $B(\mathcal{A})$ to the pair $(B(\mathcal{A}), C(\mathcal{A}))$ where $\mathcal{A} = (A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0) \in \text{Obj}(\chi)$. However, this presents some complications as $C(\mathcal{A})$ and $C(\lambda\mathcal{A})$ are comodules over different cocategories (where λ is a morphism in Λ inducing a morphism in χ with source \mathcal{A}). Instead, we will use the functors $\hat{\lambda} : B(\mathcal{A}) \rightarrow B(\lambda\mathcal{A})$ from Section 2.1.2 to define pullbacks $\hat{\lambda}^*C(\lambda\mathcal{A})$ and maps $\lambda_! : C(\mathcal{A}) \rightarrow \hat{\lambda}^*C(\lambda\mathcal{A})$ of dg comodules over $B(\mathcal{A})$.

First, we will define functors $\hat{\lambda}^*$ from the category of conilpotent dg comodules over $B(\lambda\mathcal{A})$ to the category of conilpotent dg comodules over $B(\mathcal{A})$. Second, we will give $\hat{\lambda}_\#$, the left adjoint to $\hat{\lambda}^*$. Finally, we will give explicit maps of dg comodules $\lambda_! : C(\mathcal{A}) \rightarrow \hat{\lambda}^*C(\lambda\mathcal{A})$, and apply the adjunction to these maps. The following chapter will formalize the relations between the $\lambda_!$'s.

2.3. Pullbacks of dg comodules–theory

Let $\lambda : B_1 \rightarrow B_0$ be a functor between conilpotent dg cocategories. In this section, we will define a functor λ^* from the category of conilpotent dg comodules over B_0 to the category of conilpotent dg comodules over B_1 . We call λ^* “co-extension of scalars”.

2.3.1. Category-theoretic definition of λ^*

Let λ be as above, and let C be a conilpotent dg comodule over B_0 . We define λ^*C as follows:

$$(2.1) \quad \lambda^*C := \ker \left(B_1 \otimes_\lambda C \xrightarrow[(\text{id}_{B_1} \otimes \lambda \otimes \text{id}_C) \circ (\Delta_{B_1} \otimes \text{id}_C)]{\text{id}_{B_1} \otimes \Delta_C} B_1 \otimes_\lambda B_0 \otimes C \right)$$

where $B_1 \otimes_\lambda C$ and $B_1 \otimes_\lambda B_0 \otimes C$ are dg comodules over B_1 defined below. For $f \in \text{Obj}(B_1)$,

$$\begin{aligned} [B_1 \otimes_\lambda C](f) &:= \left(\bigoplus_{h \in \text{Obj}(B_1)} B_1^\bullet(f, h) \otimes C^\bullet(\lambda h), \Delta(f) = \bigoplus_h \Delta_{B_1(f, h)} \otimes \text{id}_{C(\lambda h)} \right) \\ [B_1 \otimes_\lambda B_0 \otimes C](f) &:= \left(\bigoplus_{\substack{h_1 \in \text{Obj}(B_1), \\ h_2 \in \text{Obj}(B_0)}} B_1^\bullet(f, h_1) \otimes B_0^\bullet(\lambda h_1, h_2) \otimes C^\bullet(h_2), \right. \\ &\quad \left. \Delta(f) = \bigoplus_{h_1, h_2} \Delta_{B_1(f, h_1)} \otimes \text{id}_{B_0(\lambda h_1, h_2)} \otimes \text{id}_{C(h_2)} \right). \end{aligned}$$

The names of the maps in Equation 2.1 are also meant to be suggestive. In full detail, for $f \in \text{Obj}(B_1)$,

$$[\text{id}_{B_1} \otimes \Delta_C](f) := \bigoplus_h \text{id}_{B_1(f, h)} \otimes \Delta_C(\lambda h)$$

and

$$\begin{aligned}
[B_1 \otimes_\lambda C](f) &\xrightarrow{[\Delta_{B_1} \otimes id_C](f) := \bigoplus_h \Delta_{B_1}(f, h) \otimes id_{C(\lambda h)}} \bigoplus_{h_1, h_2 \in Obj(B_1)} B_1(f, h_1) \otimes B_1(h_1, h_2) \otimes C(\lambda h_2) \\
&\xrightarrow{[id_{B_1} \otimes \lambda \otimes id_C](f) := \bigoplus_{h_1, h_2} id_{B_1}(f, h_1) \otimes \lambda(h_1, h_2) \otimes id_{C(\lambda h)}} [B_1 \otimes_\lambda B_0 \otimes C](f).
\end{aligned}$$

That the kernel is well-defined follows formally from the abelianness of the category of chain complexes, but it is also easy to check that the induced differentials from $[B_1 \otimes_\lambda C](f)$ on the kernel are well-defined. Since Δ_{λ^*C} is induced by Δ_{B_1} , we have that Δ_{λ^*C} also satisfies coassociativity, counitality and conilpotency.

Next, we will define λ^* on morphisms. Let $F : C \rightarrow D$ be a map of conilpotent dg comodules over B_0 . By the universal property of λ^*D , we can define a morphism $\lambda^*F : \lambda^*C \rightarrow \lambda^*D$ by giving a morphism from $(\lambda^*F)' : \lambda^*C \rightarrow B_1 \otimes_\lambda D$ such that the two maps

(2.2)

$$(id_{B_1} \otimes \Delta_D) \circ (\lambda^*F)', (id_{B_1} \otimes \lambda \otimes id_D) \circ (\Delta_{B_1} \otimes id_D) \circ (\lambda^*F)' : \lambda^*C \rightarrow B_1 \otimes_\lambda D \rightrightarrows B_1 \otimes_\lambda B_0 \otimes D$$

coincide. We define $(\lambda^*F)'$ as follows:

$$(\lambda^*F)' : \lambda^*C \xrightarrow[\text{inclusion}]{\text{canonical}} B_1 \otimes_\lambda C \xrightarrow{id_{B_1} \otimes F} B_1 \otimes_\lambda D$$

It's easy to check that the two maps in Equation 2.2 coincide: Let $b \otimes c$ be an arbitrary element of $\lambda^*C(f) \hookrightarrow [B_1 \otimes_\lambda C](f)$. Then,

$$\begin{aligned}
[(id_{B_1} \otimes \Delta_D) \circ (\lambda^*F)'](b \otimes c) &= \sum_{(Fc)} b \otimes (Fc)_{(1)} \otimes (Fc)_{(2)} \\
&= \sum_{(c)} b \otimes Fc_{(1)} \otimes Fc_{(2)} \quad (F \text{ is a map of comodules}) \\
&= [(id_{B_1} \otimes F \otimes F) \circ (id_{B_1} \otimes \Delta_C)](b \otimes c) \\
&= [(id_{B_1} \otimes F \otimes F) \circ (id_{B_1} \otimes \lambda \otimes id_C) \circ (\Delta_{B_1} \otimes id_C)](b \otimes c) \\
&\quad (b \otimes c \text{ is in the kernel}) \\
&= \sum_{(b)} b_{(1)} \otimes \lambda b_{(2)} \otimes Fc \\
&= [(id_{B_1} \otimes \lambda \otimes id_D) \circ (\Delta_{B_1} \otimes id_D) \circ (\lambda^*F)'](b \otimes c).
\end{aligned}$$

So, λ^*F is well-defined. In summary, we have commuting diagrams:

$$(2.3) \quad \begin{array}{ccc}
\lambda^*C & \xrightarrow{\text{canonical inclusion}} & B_1 \otimes_\lambda C \\
\lambda^*F \downarrow & & \downarrow id_{B_1} \otimes F = \text{map inducing } \lambda^*F \\
\lambda^*D & \xrightarrow{\text{canonical inclusion}} & B_1 \otimes_\lambda D
\end{array}$$

Finally, it is straightforward to see that λ^* is a functor, i.e., that λ^* preserves composition of morphisms: Let $C \xrightarrow{F} D \xrightarrow{G} E$ be composable morphisms of dg comodules over B_0 . The maps inducing λ^*F , λ^*G and $\lambda^*(G \circ F)$ are $id_{B_1} \otimes F$, $id_{B_1} \otimes G$ and $id_{B_1} \otimes GF$, respectively. The inducing maps respect composition— $(id_{B_1} \otimes G) \circ (id_{B_1} \otimes F) = id_{B_1} \otimes GF$ —and by the commuting diagrams 2.3, the functor λ^* does as well.

Proposition 2.1. *Let $F : B_2 \rightarrow B_1$ and $G : B_1 \rightarrow B_0$ be functors between dg cocategories B_2 , B_1 and B_0 . Let M be a dg comodule over B_0 . Then,*

$$(GF)^*M \cong F^*G^*M.$$

Proof. We will prove the proposition by showing that F^*G^*M satisfies the universal property of $(GF)^*M$. First, let N be a dg comodule over B_2 and $H : N \rightarrow B_2 \otimes_{GF} M$ be a map of dg comodules such that the two maps

(2.4)

$$(id_{B_2} \otimes GF \otimes id_M) \circ (\Delta_{B_2} \otimes id_M) \circ H, (id_{B_2} \otimes \Delta_M) \circ H : N \rightarrow B_2 \otimes_{GF} M \rightrightarrows B_2 \otimes_{GF} B_0 \otimes M$$

coincide. We will show that H determines a map of dg comodules $\tilde{H} : N \rightarrow F^*G^*M$. Let $x \in Obj(B_2)$. Define

$$\begin{aligned} H'_x : N(x) &\xrightarrow{H_x} \bigoplus_{y \in Obj(B_2)} B_2(x, y) \otimes M(GFy) \\ &\xrightarrow{F \otimes id_M} \bigoplus_{y \in Obj(B_2)} B_1(Fx, Fy) \otimes M(GFy) \\ &\subset [B_1 \otimes_G M](Fx). \end{aligned}$$

The image of H'_x lands in $G^*M(Fx)$, a subcomplex of $[B_1 \otimes_G M](Fx)$; checking this is straightforward using the universal property of G^*M , the fact that F commutes with the coproducts, and Equation 2.4. So, for each $x \in Obj(B_2)$, we have a map of complexes

$H'_x : N(x) \rightarrow G^*M(Fx)$. Now define \tilde{H} as follows:

$$\begin{aligned} \tilde{H}_x : N(x) &\xrightarrow{\Delta_N} \prod_{y \in \text{Obj}(B_2)} B_2(x, y) \otimes N(y) \\ &\xrightarrow{\prod id_{B_2} \otimes H'_y} \prod_{y \in \text{Obj}(B_2)} B_2(x, y) \otimes G^*M(Fy) \\ &\subset [B_2 \otimes_F G^*M](x). \end{aligned}$$

Showing that \tilde{H} lands in G^*F^*M , a subcomodule of $[B_2 \otimes_F G^*M]$, is also straightforward; we only need that F and H commute with the appropriate coproducts, and that the cocomposition on B_2 is coassociative. So, for each $x \in \text{Obj}(B_2)$, we have a map $\tilde{H}_x : N(x) \rightarrow G^*F^*M(x)$. It's clear that \tilde{H} is a map of dg comodules since all of the maps used to construct \tilde{H} are maps of dg comodules.

Now, let $\tilde{H} : N \rightarrow F^*G^*M$ be a map of dg comodules over B_2 . We will show that \tilde{H} determines a map of dg comodules $H : N \rightarrow B_2 \otimes_G FM$ satisfying Equation 2.4. For $x \in \text{Obj}(B_2)$, let H be defined as follows:

$$\begin{aligned} H_x : N(x) &\xrightarrow{\tilde{H}_x} F^*G^*M(x) \\ &\xrightarrow[\text{inclusion}]{\text{canonical}} \bigoplus_{\substack{y \in \text{Obj}(B_2) \\ z_1 \in \text{Obj}(B_1)}} B_2(x, y) \otimes B_1(Fy, z_1) \otimes M(Gz_1) \\ &\xrightarrow{id_{B_2} \otimes \epsilon_{B_1} \otimes id_M} \bigoplus_{y \in \text{Obj}(B_2)} B_2(x, y) \otimes M(GFy). \end{aligned}$$

The universal property of G^*M implies that $(id_{B_2} \otimes \Delta_M) \circ H$ is equal to:

$$\begin{aligned}
N(x) &\xrightarrow{\tilde{H}_x} \bigoplus_{\substack{y \in \text{Obj}(B_2) \\ z_1 \in \text{Obj}(B_1)}} B_2(x, y) \otimes B_1(Fy, z_1) \otimes M(Gz_1) \\
&\xrightarrow[(id_{B_2} \otimes \Delta_{B_1} \otimes id_M)]{(id_{B_2} \otimes id_{B_1} \otimes G \otimes id_M) \circ} \bigoplus_{\substack{y \in \text{Obj}(B_2) \\ y_1, z_1 \in \text{Obj}(B_1)}} B_2(x, y) \otimes B_1(Fy, y_1) \otimes B_0(Gy_1, Gz_1) \otimes M(Gz_1) \\
&\xrightarrow{id_{B_2} \otimes \epsilon_{B_1} \otimes id_{B_0} \otimes id_M} \bigoplus_{\substack{y \in \text{Obj}(B_2) \\ z_1 \in \text{Obj}(B_1)}} B_2(x, y) \otimes B_0(GFy, Gz_1) \otimes M(Gz_1).
\end{aligned}$$

On the other hand, the universal property of F^* implies that $(id_{B_2} \otimes GF \otimes id_M) \circ (\Delta_{B_2} \otimes id_M) \circ H$ is equal to:

$$\begin{aligned}
N(x) &\xrightarrow{\tilde{H}_x} \bigoplus_{\substack{y \in \text{Obj}(B_2) \\ z_1 \in \text{Obj}(B_1)}} B_2(x, y) \otimes B_1(Fy, z_1) \otimes M(Gz_1) \\
&\xrightarrow[(id_{B_2} \otimes \Delta_{B_1} \otimes id_M)]{(id_{B_2} \otimes G \otimes id_{B_1} \otimes id_M) \circ} \bigoplus_{\substack{y \in \text{Obj}(B_2) \\ y_1, z_1 \in \text{Obj}(B_1)}} B_2(x, y) \otimes B_0(GFy, Gy_1) \otimes B_1(y_1, z_1) \otimes M(Gz_1) \\
&\xrightarrow{id_{B_2} \otimes id_{B_0} \otimes \epsilon_{B_1} \otimes id_M} \bigoplus_{\substack{y \in \text{Obj}(B_2) \\ z_1 \in \text{Obj}(B_1)}} B_2(x, y) \otimes B_0(GFy, Gz_1) \otimes M(Gz_1).
\end{aligned}$$

So, the difference between the two maps in Equation 2.4 comes down to the difference between $(\epsilon_{B_1} \otimes G) \circ \Delta_{B_1}$ and $(G \otimes \epsilon_{B_1}) \circ \Delta_{B_1}$. However, by the counitality of B_1 , both of these maps are equal to G . So, H satisfies Equation 2.4. \square

2.3.2. Explicit description of $\hat{\lambda}^*C(\mathcal{A}')$

Let λ be a morphism in Λ that induces a morphism in χ with source $\mathcal{A} \in \text{Obj}(\chi)$.

Recall that from Section 2.1.2 that we have a functor $\hat{\lambda} : B(\mathcal{A}) \rightarrow B(\lambda\mathcal{A})$. Applying the

constructions in Section 2.3.1 to $\hat{\lambda}$, we get a functor $\hat{\lambda}^*$ from the category of conilpotent dg comodules over $B(\lambda\mathcal{A})$ to the category of conilpotent dg comodules over $B(\mathcal{A})$. Below, we compute explicitly the complexes $[\hat{\lambda}^*C(\lambda\mathcal{A})](f)$ for $f \in \text{Obj}(B(\mathcal{A}))$.

Proposition 2.2. *Let λ be a morphism in Λ that induces a morphism in χ with source $\mathcal{A} \in \text{Obj}(\chi)$. Fix $f_0 \in \text{Obj}(B(\mathcal{A}))$. As comodules,*

$$(2.5) \quad [\hat{\lambda}^*C(\lambda\mathcal{A})](f_0) \cong [B(\mathcal{A}) \otimes_{\hat{\lambda}} T(\lambda\mathcal{A})](f_0) := \bigoplus_{h \in \text{Obj}(B(\mathcal{A}))} B(\mathcal{A})(f_0, h) \otimes T(\lambda\mathcal{A})(\hat{\lambda}h)$$

where $T(\lambda\mathcal{A})(\hat{\lambda}h)$ are the cogenerators of $C(\lambda\mathcal{A})(\hat{\lambda}h)$ (see Section 1.3.4).

Remark 2.3.1. Proposition 2.2 holds for any quasi-cofree comodule over $B(\lambda\mathcal{A})$. The proof is the same.

PROOF OF PROPOSITION 2.2. To simplify notation in this proof, we will drop all references to f_0 and, when unambiguous, references to $\lambda\mathcal{A}$. In other words, in this proof only,

$$C := C(\lambda\mathcal{A}) \text{ will denote } C(\lambda\mathcal{A})(f_0),$$

$$\hat{\lambda}^*C := \hat{\lambda}^*C(\lambda\mathcal{A}) \text{ will denote } [\hat{\lambda}^*C(\lambda\mathcal{A})](f_0),$$

$$B(\mathcal{A}) \otimes_{\hat{\lambda}} T \text{ will denote } [B(\mathcal{A}) \otimes_{\hat{\lambda}} T(\lambda\mathcal{A})](f_0),$$

$$B(\mathcal{A}) \otimes_{\hat{\lambda}} C \text{ will denote } [B(\mathcal{A}) \otimes_{\hat{\lambda}} C(\lambda\mathcal{A})](f_0),$$

$$B(\mathcal{A}) \otimes_{\hat{\lambda}} B(\lambda\mathcal{A}) \otimes C \text{ will denote } [B(\mathcal{A}) \otimes_{\hat{\lambda}} B(\lambda\mathcal{A}) \otimes C(\lambda\mathcal{A})](f_0).$$

To prove the proposition, we will give maps

$$F : \hat{\lambda}^* C \rightleftharpoons B(\mathcal{A}) \otimes_{\hat{\lambda}} T : G$$

and show that $F \circ G = id_{B(\mathcal{A}) \otimes_{\hat{\lambda}} T}$ and $G \circ F = id_{\hat{\lambda}^* C}$. We define F as follows:

$$F : \hat{\lambda}^* C \xrightarrow[\text{inclusion}]{\text{canonical}} B(\mathcal{A}) \otimes_{\hat{\lambda}} C \xrightarrow[\text{cogenerators}]{\text{project onto}} B(\mathcal{A}) \otimes_{\hat{\lambda}} T.$$

To define G , we will give a map $G' : B(\mathcal{A}) \otimes_{\hat{\lambda}} T \rightarrow B(\mathcal{A}) \otimes_{\hat{\lambda}} C$, and show that the image of G' lands in $\hat{\lambda}^* C$. We define G' as follows:

$$G'(b \otimes t) = \sum_{(b)} b_{(1)} \otimes \hat{\lambda} b_{(2)} \cdot t$$

where $b \otimes t \in B(\mathcal{A}) \otimes_{\hat{\lambda}} T$ and $\hat{\lambda} b_{(2)} \cdot t$ are elements of $C(\lambda \mathcal{A})(\hat{\lambda} h)$ written in terms of cogenerators.

To prove that the image of G' lands in $\hat{\lambda}^* C$, we need to show that the two maps

$$(id_{B(\mathcal{A})} \otimes \Delta_C) \circ G', (id_{B(\mathcal{A})} \otimes \hat{\lambda} \otimes id_C) \circ (\Delta_{B(\mathcal{A})} \otimes id_C) \circ G' : B(\mathcal{A}) \otimes_{\hat{\lambda}} T \rightarrow B(\mathcal{A}) \otimes_{\hat{\lambda}} C \rightrightarrows B(\mathcal{A}) \otimes_{\hat{\lambda}} B(\lambda \mathcal{A}) \otimes C$$

coincide. We have

$$\begin{aligned} [(1 \otimes \Delta_C) \circ G'](b \otimes t) &= \sum_{(b), (\hat{\lambda} b)} b_{(1)} \otimes (\hat{\lambda} b_{(2)})_{(1)} \otimes (\hat{\lambda} b_{(2)})_{(2)} \cdot t \\ &= \sum_{(b)} b_{(1)} \otimes \hat{\lambda} b_{(2)} \otimes \hat{\lambda} b_{(3)} \cdot t \\ &= [(id_{B(\mathcal{A})} \otimes \hat{\lambda} \otimes id_C) \circ (\Delta_{B(\mathcal{A})} \otimes id_C) \circ G'](b \otimes t) \end{aligned}$$

where the second equality holds since $\hat{\lambda}$ is a map of cocategories and $\Delta_{B(\mathcal{A})}$ is coassociative.

It's clear from the definitions that F and G are maps of comodules and that $F \circ G = id_{B(\mathcal{A}) \otimes_{\hat{\lambda}} T}$. All that remains is to show that $G \circ F = id_{\hat{\lambda}^* C}$. Let $\kappa = \sum_i b_i \otimes \beta_i \cdot t_i$ be an arbitrary element of $\hat{\lambda}^* C \hookrightarrow B(\mathcal{A}) \otimes_{\hat{\lambda}} C$ where $\beta_i \cdot t_i$ are elements of $C(\lambda \mathcal{A})(\hat{\lambda} h)$ written in terms of cogenerators. Then,

$$GF(\kappa) = GF(\sum_i b_i \otimes \beta_i \cdot t_i) = \sum_{\substack{i, \\ \beta_i=1, \\ (b_i)}} b_{i(1)} \otimes \hat{\lambda} b_{i(2)} \cdot t_i.$$

We can divide the terms in κ into two groups: (a) terms in which $\beta_i = 1 \in k$ and (b) terms in which $\beta_i \neq 1 \in k$. Likewise, we can divide the terms in $GF(\kappa)$ into (a) terms in which $\hat{\lambda} b_{i(2)} = 1$ and (b) terms in which $\hat{\lambda} b_{i(2)} \neq 1$. From the definitions of F and G , it's clear that the Group A terms in κ are exactly the Group A terms in $GF(\kappa)$.

To show that the Group B terms are the same, let $b_i \otimes \beta_i \cdot t_i$ be an arbitrary Group B term in κ . Then, there is a term $b_i \otimes \beta_i \otimes t_i$ in $(id_{B(\mathcal{A})} \otimes \Delta_C)\kappa$. Since $(id_{B(\mathcal{A})} \otimes \Delta_C)\kappa = (id_{B(\mathcal{A})} \otimes \hat{\lambda} \otimes id_C) \circ (\Delta_{B(\mathcal{A})} \otimes id_C)\kappa$, there must be a Group A term, $b_{j_i} \otimes t_{j_i}$, in κ such that $b_i \otimes \beta_i \otimes t_i$ is one of the terms in the sum $[(id_{B(\mathcal{A})} \otimes \hat{\lambda} \otimes id_C) \circ (\Delta_{B(\mathcal{A})} \otimes id_C)](b_{j_i} \otimes t_{j_i}) = \sum_{(b_{j_i})} b_{j_i(1)} \otimes \hat{\lambda} b_{j_i(2)} \otimes t_{j_i}$. Thus, $b_i \otimes \beta_i \cdot t_i$ is a Group B term in $GF(\kappa)$.

Now let $b_{i(1)} \otimes \hat{\lambda} b_{i(2)} \cdot t_i$ be an arbitrary Group B term in $GF(\kappa)$. Then, $b_{i(1)} \otimes \hat{\lambda} b_{i(2)} \otimes t_i$ is a term in $(id_{B(\mathcal{A})} \otimes \hat{\lambda} \otimes id_C) \circ (\Delta_{B(\mathcal{A})} \otimes id_C)\kappa = (id_{B(\mathcal{A})} \otimes \Delta_C)\kappa$. So, there is a Group B term, $b_{j_i} \otimes \beta_{j_i} \cdot t_{j_i}$, in κ such that $b_{i(1)} \otimes \hat{\lambda} b_{i(2)} \otimes t_i$ is one of the terms in the sum $(id_{B(\mathcal{A})} \otimes \Delta_C)(b_{j_i} \otimes \beta_{j_i} \cdot t_{j_i}) = \sum_{(\beta_{j_i})} b_{j_i} \otimes \beta_{j_i(1)} \otimes \beta_{j_i(2)} \cdot t_{j_i}$. Since t_i is a cogenerator, the only term in the sum that could be equal to $b_{i(1)} \otimes \hat{\lambda} b_{i(2)} \otimes t_i$ is $b_{j_i} \otimes \beta_{j_i} \otimes t_{j_i}$. Thus, $b_{i(1)} \otimes \hat{\lambda} b_{i(2)} \cdot t_i$ is a Group B term in κ . \square

2.4. Pullbacks of dg comodules—examples

Example 2.4.1 (Another definition of $C(1)$). *Let $\lambda = \delta_{0,1} \in \Lambda([1], [0])$. Fix algebras A_0 and A_1 , and set*

$$\hat{\lambda} : B(1) := B(A_0 \rightarrow A_1 \rightarrow A_0) \rightarrow B(A_0 \rightarrow A_0) =: B(0)$$

$$C(1) := C(A_0 \rightarrow A_1 \rightarrow A_0)$$

$$C(0) := C(A_0 \rightarrow A_0)$$

($\hat{\lambda}$ is given by braces, see Section 2.1.2.)

From Proposition 2.2, we have $\hat{\delta}_{0,1}^* C(0) \cong [B(1) \otimes_{\hat{\delta}_{0,1}} T(0)]$ as comodules. We will use the isomorphisms F and G in Proposition 2.2 to induce a differential on $[B(1) \otimes_{\hat{\delta}_{0,1}} T(0)]$ from $\hat{\delta}_{0,1}^* C(0)$. Let $\phi_{0,1} \dots \phi_{0,k_0} | \phi_{1,1} \dots \phi_{1,k_1} | \alpha$ be a typical element of $[B(1) \otimes_{\hat{\delta}_{0,1}} T(0)](f_{0,0}, f_{1,0})$ (see Figure 1.2 for notational conventions). Then,

$$\begin{aligned} & d_{\text{induced}}(\phi_{0,1} \dots \phi_{0,k_0} | \phi_{1,1} \dots \phi_{1,k_1} | \alpha) \\ &= F d_{\hat{\delta}_{0,1}^* C(0)} G(\phi_{0,1} \dots \phi_{0,k_0} | \phi_{1,1} \dots \phi_{1,k_1} | \alpha) \\ &= [F \circ (d_{B(1)} \otimes id_{C(0)} + id_{B(1)} \otimes d_{C(0)})] \\ & \quad \left(\sum_{\substack{1 \leq r_0 \leq k_0+1 \\ 1 \leq r_1 \leq k_1+1}} (\phi_{0,1} \dots \phi_{0,r_0-1} | \phi_{1,1} \dots \phi_{1,r_1-1}) \otimes ((\phi_{0,r_0} \dots \phi_{0,k_0}) \bullet (\phi_{1,r_1} \dots \phi_{1,k_1}) | \alpha) \right) \\ &= d_{C(1)(f_{0,0}, f_{1,0})}(\phi_{0,1} \dots \phi_{0,k_0} | \phi_{1,1} \dots \phi_{1,k_1} | \alpha) \end{aligned}$$

where the last equality holds by looking at which terms from $d_{B(1)} \otimes id_{C(0)} + id_{B(1)} \otimes d_{C(0)}$ are non-zero after projecting to cogenerators, and seeing that those are the same terms as in $d_{C(1)}$. So, $\hat{\delta}_{0,1}^* C(0) \cong C(1)$ as dg comodules.

Example 2.4.2 (Another definition of $C(n)$). Let $\lambda = \delta_{0,n} \in \Lambda([n], [n-1])$. Fix algebras A_0, \dots, A_n , and set

$$\hat{\lambda} : B(n) := B(A_0 \rightarrow A_1 \rightarrow \dots \rightarrow A_n \rightarrow A_0) \rightarrow B(A_0 \rightarrow A_2 \rightarrow A_3 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$$

$$C(n) := C(A_0 \rightarrow A_1 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$$

($\hat{\lambda}$ is given by bracing the first and second terms, see Section 2.1.2.)

Example 2.4.1 shows

$$(2.6) \quad C(1) \cong \hat{\delta}_{0,1}^* C(0)$$

as dg comodules. Given Equation 2.6 above as a base case, we can show by induction that

$$C(n) \cong \hat{\delta}_{0,n}^* \dots \hat{\delta}_{0,1}^* C(0)$$

as dg comodules. Suppose that $C(W_0 \rightarrow \dots \rightarrow W_{n-1} \rightarrow W_0) \cong \hat{\delta}_{0,n-1}^* \dots \hat{\delta}_{0,1}^* C(W_0 \rightarrow W_0)$ for any choice of algebras W_0, \dots, W_{n-1} . (inductive hypothesis). Then, as comodules, we

know

$$\begin{aligned}
\hat{\delta}_{0,n}^* \dots \hat{\delta}_{0,1}^* C(0) &\cong \hat{\delta}_{0,n}^* C(A_0 \rightarrow A_2 \rightarrow \dots \rightarrow A_n \rightarrow A_0) \quad (\text{inductive hypothesis applied to algebras } A_0, A_2, \dots) \\
&\cong B(n) \otimes_{\hat{\delta}_{0,n}} T(A_0 \rightarrow A_2 \rightarrow \dots \rightarrow A_n \rightarrow A_0) \quad (\text{Proposition 2.2}) \\
&\cong B(n) \otimes_{\hat{\delta}_{0,n}} T(A_0 \rightarrow A_0) \quad (\text{Definition of } T) \\
&\cong C(n) \quad (\text{Definition of } C(n))
\end{aligned}$$

where $T(n)$ are the cogenerators of $C(n)$ (see Definition 1.3.4).

To show that the differentials coincide, we compute

$$\begin{aligned}
&Fd_{\hat{\delta}_{0,n}^* \dots \hat{\delta}_{0,1}^* C(0)} G(\phi_{0,1} \dots \phi_{0,k_0} | \dots | \phi_{n,1} \dots \phi_{n,k_n} | t) \\
&= [F \circ (d_{B(n)} \otimes id_{\hat{\delta}_{0,n-1}^* \dots \hat{\delta}_{0,1}^* C(0)} + id_{B(n)} \otimes d_{\hat{\delta}_{0,n-1}^* \dots \hat{\delta}_{0,1}^* C(0)})] \\
&\quad \left(\sum_{\substack{1 \leq j \leq n \\ 1 \leq r_j \leq k_j + 1}} (\phi_{0,1} \dots \phi_{0,r_0-1} | \dots | \phi_{n,1} \dots \phi_{n,r_1-1}) \otimes \right. \\
&\quad \left. ((\phi_{0,r_0} \dots \phi_{0,k_0}) \bullet (\phi_{1,r_1} \dots \phi_{1,k_1}) | \phi_{2,r_1} \dots \phi_{2,k_2} | \dots | \phi_{n,r_1} \dots \phi_{n,k_n} | t) \right) \\
&= [F \circ (d_{B(n)} \otimes id_{C(A_0 \rightarrow A_2 \rightarrow \dots \rightarrow A_n \rightarrow A_0)} + id_{B(n)} \otimes d_{C(A_0 \rightarrow A_2 \rightarrow \dots \rightarrow A_n \rightarrow A_0)})] \\
&\quad \left(\sum_{\substack{1 \leq j \leq n \\ 1 \leq r_j \leq k_j + 1}} (\phi_{0,1} \dots \phi_{0,r_0-1} | \dots | \phi_{n,1} \dots \phi_{n,r_1-1}) \otimes \right. \\
&\quad \left. ((\phi_{0,r_0} \dots \phi_{0,k_0}) \bullet (\phi_{1,r_1} \dots \phi_{1,k_1}) | \phi_{2,r_1} \dots \phi_{2,k_2} | \dots | \phi_{n,r_1} \dots \phi_{n,k_n} | t) \right)
\end{aligned}$$

where the last equality holds by the inductive hypothesis. The terms from $d_{B(n)} \otimes id_{C(A_0 \rightarrow A_2 \rightarrow \dots \rightarrow A_n \rightarrow A_0)} + id_{B(n)} \otimes d_{C(A_0 \rightarrow A_2 \rightarrow \dots \rightarrow A_n \rightarrow A_0)}$ that are non-zero after projecting to cogenerators are exactly the terms in $d_{C(n)}$. So, $C(n) \cong \hat{\delta}_{0,n}^* \dots \hat{\delta}_{0,1}^* C(0)$ as dg comodules.

Example 2.4.3 (Yet another description of $C(n)$). *Choose a sequence of generating coboundaries $\delta_{i_1,1}, \dots, \delta_{i_n,n}$ with $0 \leq i_j \leq j-1$, $1 \leq j \leq n$, $n > 0$. Then,*

$$\delta_{i_1,1} \circ \dots \circ \delta_{i_n,n} = \delta_{0,1} \circ \dots \circ \delta_{0,n} = \text{unique map in } \Delta([n], [0]) \subset \Lambda([n], [0]).$$

This implies that, as functors on categories of comodules,

$$\begin{aligned} \hat{\delta}_{i_n,n}^* \dots \hat{\delta}_{i_1,1}^* &= (\delta_{i_1,1} \circ \dots \circ \delta_{i_n,n})^* \quad (\text{Proposition 2.1}) \\ &= (\delta_{0,1} \circ \dots \circ \delta_{0,n})^* \quad (\text{Computation above}) \\ &= \hat{\delta}_{0,n}^* \dots \hat{\delta}_{0,1}^* \quad (\text{Proposition 2.1}). \end{aligned}$$

Since braces are associative, the differentials on $\hat{\delta}_{i_n,n}^ \dots \hat{\delta}_{i_1,1}^* C(A_0 \rightarrow A_0)$ and $\hat{\delta}_{0,n}^* \dots \hat{\delta}_{0,1}^* C(A_0 \rightarrow A_0)$ coincide. So, $\hat{\delta}_{i_n,n}^* \dots \hat{\delta}_{i_1,1}^* C(A_0 \rightarrow A_0) \cong \hat{\delta}_{0,n}^* \dots \hat{\delta}_{0,1}^* C(A_0 \rightarrow A_0)$ as dg comodules.*

Example 2.4.4 (Pullbacks along codegeneracies). *Fix algebras A_0, \dots, A_n and let $\sigma_{i,n} \in \Lambda([n], [n+1])$, $0 \leq i \leq n$ be a generating codegeneracy. Set*

$$\hat{\sigma}_{i,n} : B(n) := B(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0) \rightarrow B(A_0 \rightarrow \dots \rightarrow A_i \rightarrow A_i \rightarrow \dots \rightarrow A_n \rightarrow A_0)$$

$$C(n) := C(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$$

From Proposition 2.2, we know that $\hat{\sigma}_{i,n}^ C(A_0 \rightarrow \dots \rightarrow A_i \rightarrow A_i \rightarrow \dots \rightarrow A_n \rightarrow A_0) \cong B(n) \otimes_{\hat{\sigma}_{i,n}} T(A_0 \rightarrow \dots \rightarrow A_i \rightarrow A_i \rightarrow \dots \rightarrow A_n \rightarrow A_0) \cong C(n)$ as comodules. To show that*

the differentials coincide, we compute

$$\begin{aligned}
& Fd_{\hat{\sigma}_{i,n}^*} C(A_0 \rightarrow \dots \rightarrow A_i \rightarrow A_i \rightarrow \dots A_n \rightarrow A_0) G(\phi_{0,1} \dots \phi_{0,k_0} | \dots | \phi_{n,1} \dots \phi_{n,k_n} | t) \\
&= [F \circ (d_{B(n)} \otimes id_{\hat{\sigma}_{i,n}^*} C(A_0 \rightarrow \dots \rightarrow A_i \rightarrow A_i \rightarrow \dots A_n \rightarrow A_0) + id_{B(n)} \otimes d_{\hat{\sigma}_{i,n}^*} C(A_0 \rightarrow \dots \rightarrow A_i \rightarrow A_i \rightarrow \dots A_n \rightarrow A_0))] \\
&\quad \left(\sum_{\substack{1 \leq j \leq n \\ 1 \leq r_j \leq k_j + 1}} (\phi_{0,1} \dots \phi_{0,r_0-1} | \dots | \phi_{n,1} \dots \phi_{n,r_1-1}) \otimes \right. \\
&\quad \left. (\phi_{0,r_0} \dots \phi_{0,k_0} | \dots | \phi_{0,r_{i-1}} \dots \phi_{0,k_{i-1}} | 1 | \phi_{0,r_i} \dots \phi_{0,k_i} | \dots | \phi_{n,r_1} \dots \phi_{n,k_n} | t) \right).
\end{aligned}$$

Since 1 is a unit for braces, the terms from $d_{B(n)} \otimes id_{\hat{\sigma}_{i,n}^*} C(A_0 \rightarrow \dots \rightarrow A_i \rightarrow A_i \rightarrow \dots A_n \rightarrow A_0) + id_{B(n)} \otimes d_{\hat{\sigma}_{i,n}^*} C(A_0 \rightarrow \dots \rightarrow A_i \rightarrow A_i \rightarrow \dots A_n \rightarrow A_0)$ that are non-zero after projecting to cogenerators are exactly the terms in $d_{C(n)}$. So, $\hat{\sigma}_{i,n}^* C(A_0 \rightarrow \dots \rightarrow A_i \rightarrow A_i \rightarrow \dots A_n \rightarrow A_0) \cong C(n)$ as dg comodules.

Example 2.4.5 (Pullbacks along rotations). Fix algebras A_0, \dots, A_n and let $\tau_n \in \Lambda([n], [n])$ be a generating rotation. Set

$$\hat{\tau}_n : B(n) := B(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0) \rightarrow B(A_n \rightarrow A_0 \rightarrow \dots \rightarrow A_n)$$

$$C(n) := C(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$$

From Proposition 2.2, we know that $\hat{\tau}_n^* C(A_n \rightarrow A_0 \rightarrow \dots \rightarrow A_n) \cong B(n) \otimes_{\hat{\tau}_n} T(A_n \rightarrow A_0 \rightarrow \dots \rightarrow A_n)$ as comodules. Unpacking the righthand side, we see that $B(n) \otimes_{\hat{\tau}_n} T(A_n \rightarrow A_0 \rightarrow \dots \rightarrow A_n) \cong C(A_n \rightarrow A_0 \rightarrow \dots \rightarrow A_n)$ as complexes—the isomorphism is given by $\hat{\tau}_n \otimes id_T$.

2.5. Adjunction between λ^* and $\lambda_\#$

In this section, we define $\lambda_\#$, the left adjoint to λ^* . More precisely, for any functor, $\lambda : B_1 \rightarrow B_0$ between conilpotent dg cocategories, we define a functor $\lambda_\#$ from the category of conilpotent dg comodules over B_1 to the category of conilpotent dg comodules over B_0 . The adjunction will be used to show that structures we've established for $(B(n), C(n))$ still exist after we pass from cocategories and comodules to categories and modules by applying (a categorified) $Cobar(-)$ to $(B(n), C(n))$. If a lesson of this thesis is that working with cocategories is more tractable than with categories, then the reader may skip this section or save it until s/he is ready for Chapter (...).

2.5.1. The functors $\lambda_\#$

Let $\lambda : B_1 \rightarrow B_0$ be a functor between conilpotent dg cocategories. Let C be a conilpotent dg comodule over B_1 . We define $\lambda_\# C$ as follows: for $f \in \text{Obj}(B_0)$,

$$\begin{aligned}
 \lambda_\# C(f) &:= \left(\bigoplus_{f' \in \lambda^{-1}f} C^\bullet(f'), \right. \\
 \Delta_{\lambda_\# C}(f) : \bigoplus_{f' \in \lambda^{-1}f} C^\bullet(f') &\xrightarrow{\bigoplus_{f'} \Delta_{C^\bullet}(f')} \bigoplus_{\substack{f' \in \lambda^{-1}f \\ h' \in \text{Obj}(B_1)}} B_1^\bullet(f', h') \otimes C^\bullet(h') \\
 &\xrightarrow{\bigoplus_{h', f'} \lambda \otimes \text{id}_{C^\bullet(h')}} \bigoplus_{h' \in \text{Obj}(B_1)} B_0^\bullet(f, \lambda h') \otimes C^\bullet(h') \\
 &\xrightarrow{\text{include}} \bigoplus_{h \in \text{Obj}(B_0)} B_0^\bullet(f, h) \otimes \left(\bigoplus_{h' \in \lambda^{-1}h} C^\bullet(h') \right).
 \end{aligned}$$

To check that $\Delta_{\lambda_\# C}$ is well-defined, we need that the image of the first map, $\bigoplus_{f'} \Delta_{C^\bullet}(f')$, is a finite sum. This is true since C being conilpotent implies that the image of $\Delta_{C^\bullet}(f')$

is a finite sum for each $f' \in \text{Obj}(B_1)$. If $\lambda^{-1}f$ is empty, we set $\lambda_{\#}C(f) := 0$. It is straightforward to check that $(\lambda_{\#}C, \Delta_{\lambda_{\#}C})$ is coassociative, conilpotent and coaugmented. We will call $\lambda_{\#}$ “co-restriction of scalars”.

Let $F : C \rightarrow D$ be map of dg comodules over B_1 . We define $\lambda_{\#}F$ as follows:

$$(\lambda_{\#}F)_f : \lambda_{\#}C(f) = \bigoplus_{f' \in \lambda^{-1}f} C^{\bullet}(f') \xrightarrow{\bigoplus_{f' \in \lambda^{-1}f} F_{f'}} \bigoplus_{f' \in \lambda^{-1}f} D^{\bullet}(f') = \lambda_{\#}D(f).$$

It's straightforward to check that $\lambda_{\#}$ is a functor (i.e., respects composition of morphisms).

2.5.2. Adjunction

Proposition 2.3. *Given a functor between conilpotent dg cocategories, $\lambda : B_1 \rightarrow B_0$,*

let

$$\lambda^* : \begin{array}{c} \text{Category of} \\ \text{conilpotent} \\ \text{dg comodules over } B_0 \end{array} \rightleftarrows \begin{array}{c} \text{Category of} \\ \text{conilpotent} \\ \text{dg comodules over } B_1 \end{array} : \lambda_{\#}$$

be the functors defined in Sections 2.3.1 and 2.5.1. Then, $\lambda_{\#}$ is left adjoint to λ^ .*

Remark 2.5.1. Proposition 2.3 is a categorified co-version of the adjunction between extension of scalars (left) and restriction of scalars (right) for modules over algebras.

PROOF OF PROPOSITION 2.3. Let C be a conilpotent dg comodule over B_1 and D be a dg conilpotent dg comodule over B_0 . We want to show that

$$\text{Hom}_{B_1}(C, \lambda^*D) = \text{Hom}_{B_0}(\lambda_{\#}C, D)$$

as sets.

We will give maps

$$\Phi : Hom_{B_0}(\lambda_{\#}C, D) \xrightarrow{\sim} Hom_{B_1}(C, \lambda^*D) : \Phi^{-1}$$

satisfying $\Phi \circ \Phi^{-1} = id$ and $\Phi^{-1} \circ \Phi = id$.

First, we define Φ . Let F be a morphism from $\lambda_{\#}C$ to D . By definition, for $f \in Obj(B_0)$, we have maps of complexes

$$F_f : \bigoplus_{f' \in \lambda^{-1}f} C^{\bullet}(f') \rightarrow D^{\bullet}(f).$$

Define $\Phi F \in Hom_{B_1}(C, \lambda^*D)$ as follows: for $f' \in Obj(B_1)$,

$$\begin{aligned} \Phi F_{f'} : C^{\bullet}(f') &\xrightarrow{\Delta_C} \bigoplus_{h' \in Obj(B_1)} B_1^{\bullet}(f', h') \otimes C^{\bullet}(h') \\ (2.7) \quad &\xrightarrow{\bigoplus_{h'} id_{B_1} \otimes F_{\lambda h'}|_{h'}} \bigoplus_{h' \in Obj(B_1)} B_1^{\bullet}(f', h') \otimes D^{\bullet}(\lambda h') \\ &\xrightarrow{include} [B_1 \otimes_{\lambda} D](f'). \end{aligned}$$

By the universal property of λ^*D , this defines a morphism $C \rightarrow \lambda^*D$ if the two maps

$$(id_{B_1} \otimes \Delta_D) \circ \Phi F, (id_{B_1} \otimes \lambda \otimes id_D) \circ (\Delta_{B_1} \otimes id_D) \circ \Phi F : C \rightrightarrows B_1 \otimes_{\lambda} B_0 \otimes D$$

coincide. In fact, on $f' \in \text{Obj}(B_1)$, both maps are equal to:

$$\begin{aligned}
C^\bullet(f') &\xrightarrow{\Delta_C} \bigoplus_{h' \in \text{Obj}(B_1)} B_1^\bullet(f', h') \otimes C^\bullet(h') \\
&\xrightarrow{\bigoplus_{h'} id_{B_1} \otimes \Delta_C} \bigoplus_{g', h' \in \text{Obj}(B_1)} B_1^\bullet(f', g') \otimes B_1^\bullet(g', h') \otimes C^\bullet(h') \\
&\xrightarrow{\bigoplus_{h', g'} id_{B_1} \otimes \lambda \otimes 1_C} \bigoplus_{g', h' \in \text{Obj}(B_1)} B_1^\bullet(f', g') \otimes B_0^\bullet(\lambda g', \lambda h') \otimes C^\bullet(h') \\
&\xrightarrow{\bigoplus_{h', g'} id_{B_1} \otimes id_{B_0} \otimes F_{\lambda h'}|_{h'}} \bigoplus_{g', h' \in \text{Obj}(B_1)} B_1^\bullet(f', g') \otimes B_0^\bullet(\lambda g', \lambda h') \otimes D^\bullet(\lambda h').
\end{aligned}$$

This fact follows from F being a map of comodules. It's also clear that ΦF commutes with coproducts and differentials. So, we've shown $\Phi F \in \text{Hom}_{B_1}(C, \lambda^* D)$.

Second, we define Φ^{-1} . Now, let $F \in \text{Hom}_{B_1}(C, \lambda^* D)$. For $f \in \text{Obj}(B_0)$, define

$$\begin{aligned}
\Phi^{-1} F_f : \bigoplus_{f' \in \lambda^{-1} f} C^\bullet(f') &\xrightarrow{\bigoplus_{f'} F_{f'}} \bigoplus_{\substack{f' \in \lambda^{-1} f, \\ h' \in \text{Obj}(B_1)}} B_1^\bullet(f', h') \otimes D^\bullet(\lambda h') \\
&\xrightarrow{\bigoplus_{f', h'} \lambda \otimes id_D} \bigoplus_{h \in \text{Obj}(B_0)} B_0^\bullet(f, h) \otimes D^\bullet(h) \\
&\xrightarrow{\bigoplus_h \epsilon_{B_0} \otimes id_D} D^\bullet(f).
\end{aligned}$$

It's clear that $\Phi^{-1} F$ commutes with the differentials. We will show that $\Phi^{-1} F$ is a map of comodules. Figure 2.1 gives a diagram showing that

$$(2.8) \quad \Delta_D \circ \Phi^{-1} F_f = \left(\bigoplus_{f', h', r'} \epsilon_{B_0} \lambda \otimes \lambda \otimes id_D \right) \circ \left(\bigoplus_{f', h'} \Delta_{B_1} \otimes id_D \right) \circ \left(\bigoplus_{f'} F_{f'} \right).$$

On the other hand, Figure ?? gives a diagram showing that

$$(2.9) \quad (id_{B_1} \otimes \Phi^{-1}F) \circ \Delta_{\lambda_{\#}C} = \left(\bigoplus_{f', h', r'} \lambda \otimes \epsilon_{B_0} \lambda \otimes id_D \right) \circ \left(\bigoplus_{f', h'} \Delta_{B_1} \otimes id_D \right) \circ \left(\bigoplus_{f'} F_{f'} \right).$$

We see that the righthand sides of Equations 2.8 and 2.9 are the same except for the B_0 factor on which ϵ_{B_0} acts. However, in general, for $\lambda : B_1 \rightarrow B_0$ a map of dg cocategories, we have

$$\begin{aligned} (\lambda \otimes \epsilon_{B_0} \lambda) \circ \Delta_{B_1} &= (id_{B_0} \otimes \epsilon_{B_0}) \circ \Delta_{B_0} \circ \lambda \quad (\lambda \text{ commutes with coproduct}) \\ &= id_{B_0} \circ \lambda \quad (\text{definition of cocategory}) \\ &= (\epsilon_{B_0} \otimes id_{B_0}) \circ (\Delta_{B_0}) \circ \lambda \quad (\text{definition of cocategory}) \\ &= (\epsilon_{B_0} \lambda \otimes \lambda) \circ \Delta_{B_1} \quad (\lambda \text{ commutes with coproduct}). \end{aligned}$$

So, $(id_{B_1} \otimes \Phi^{-1}F) \circ \Delta_{\lambda_{\#}C} = \Delta_D \circ \Phi^{-1}F$, and $\Phi^{-1}F \in Hom_{B_0}(\lambda_{\#}C, D)$.

For $F : C \rightarrow \lambda^*D$ a map of dg comodules and $f' \in B_1$, Figure 2.3 shows that $\Phi\Phi^{-1}F_{f'} = F_{f'}$. For $F : \lambda_{\#}C \rightarrow D$ a map of dg comodules and $f \in B_0$, Figure ?? shows that $\Phi^{-1}\Phi F_f = F_f$. Thus, we have $\Phi\Phi^{-1} = id$ and $\Phi^{-1}\Phi = id$. \square

$$\begin{array}{c}
\bigoplus_{f' \in \lambda^{-1}f} C^\bullet(f') \\
\downarrow \bigoplus_{f'} F_{f'} \\
\bigoplus_{\substack{f' \in \lambda^{-1}f \\ h' \in \text{Obj}(B_1)}} B_1^\bullet(f', h') \otimes D^\bullet(\lambda h') \xrightarrow{\bigoplus_{f', h'} \lambda \otimes id_D} \bigoplus_{h \in \text{Obj}(B_0)} B_0^\bullet(f, h) \otimes D^\bullet(h) \xrightarrow{\bigoplus \epsilon_{B_0} \otimes id_D} D^\bullet(f) \\
\downarrow \bigoplus_{\substack{f', h', r' \\ r \in \text{Obj}(B_0)}} (\Delta_{B_1} \otimes id_D) \circ (id_{B_1} \otimes \lambda \otimes id_D) \left(\bigoplus_{f', h'} id_{B_1} \otimes \Delta_D \right) \\
\bigoplus_{\substack{f' \in \lambda^{-1}f, \\ h' \in \text{Obj}(B_1), \\ r \in \text{Obj}(B_0)}} B_0^\bullet(f', h') \otimes B_1^\bullet(\lambda h', r) \otimes D^\bullet(r) \xrightarrow{\bigoplus_{f', h', r} \epsilon_{B_0} \lambda \otimes id_{B_1} \otimes id_D} \bigoplus_{h \in \text{Obj}(B_0)} B_0^\bullet(f, h) \otimes D^\bullet(h) \\
\downarrow \Delta_D \\
\bigoplus_{h \in \text{Obj}(B_0)} B_0^\bullet(f, h) \otimes D^\bullet(h)
\end{array}$$

Figure 2.1. Commuting diagram involving $\Delta_D \circ \Phi^{-1}F =$ composition of red arrows

The fact that $F : C \rightarrow \lambda^*D$ and the universal property of λ^*D imply that the diagram commutes.

$$\begin{array}{ccccc}
\bigoplus_{f' \in \lambda^{-1}f} C^\bullet(f') & \xrightarrow{\bigoplus_{f'} \Delta_C} & \bigoplus_{\substack{f' \in \lambda^{-1}f \\ h' \in \text{Obj}(B_1)}} B_1^\bullet(f', h') \otimes C^\bullet(h') & \xrightarrow{\bigoplus_{f', h'} \lambda \otimes id_C} & \bigoplus_{h' \in \text{Obj}(B_1)} B_0^\bullet(f, \lambda h') \otimes C^\bullet(h') \\
\downarrow \bigoplus_{f'} F_{f'} & & \downarrow \bigoplus_{f', h'} id_{B_1} \otimes F_{\lambda h'}|_{h'} & & \downarrow \bigoplus_{h'} id_{B_0} \otimes F_{\lambda h'}|_{h'} \\
\bigoplus_{\substack{f' \in \lambda^{-1}f \\ r' \in \text{Obj}(B_1)}} B_1^\bullet(f', r') \otimes D^\bullet(\lambda r') & \xrightarrow{\bigoplus_{f', r'} \Delta_{B_1}^{*\Delta} = \Delta_{B_1}^* \otimes id_D} & \bigoplus_{\substack{f' \in \lambda^{-1}f \\ h', r' \in \text{Obj}(B_1)}} B_1^\bullet(f', h') \otimes B_1^\bullet(h', r') \otimes D^\bullet(\lambda r') & \xrightarrow{\bigoplus_{h', r'} \lambda \otimes id_{B_0} \otimes id_D} & \bigoplus_{h', r' \in \text{Obj}(B_1)} B_0^\bullet(f, \lambda h') \otimes B_1^\bullet(h', r') \otimes D^\bullet(\lambda r') \\
& & & & \downarrow \bigoplus_{h', r'} id_{B_0} \otimes \epsilon_{B_0} \lambda \otimes id_D \\
& & & & \bigoplus_{h \in \text{Obj}(B_0)} B_0^\bullet(f, h) \otimes D^\bullet(h)
\end{array}$$

Figure 2.2. Commuting diagram involving $(id_{B_1} \otimes \Phi^{-1}F) \circ \Delta_{\lambda^\#C}$ = composition of red arrows
The fact that F respects coproducts implies that the left square commutes.

$$\begin{array}{ccc}
C^\bullet(f') & \xrightarrow{\Delta_C} & \bigoplus_{g' \in \text{Obj}(B_1)} B_1^\bullet(f', g') \otimes C^\bullet(g') \\
\downarrow F_{f'} & & \uparrow \bigoplus_{g'} id_{B_1} \otimes F_{g'} \\
\bigoplus_{h' \in \text{Obj}(B_1)} B_1^\bullet(f', h') \otimes D^\bullet(\lambda h') & \xrightarrow{\Delta_{\lambda * D} = \bigoplus_{h'} \Delta_{B_1} \otimes id_D} & \bigoplus_{g', h' \in \text{Obj}(B_1)} B_1^\bullet(f', g') \otimes B_1^\bullet(g', h') \otimes D^\bullet(\lambda h') \xrightarrow{\bigoplus id_{B_1} \otimes (\epsilon_{B_0} \lambda = \epsilon_{B_1}) \otimes id_D} \bigoplus_{g' \in \text{Obj}(B_1)} B_1^\bullet(f', g') \otimes D^\bullet(\lambda g') \\
& \searrow id &
\end{array}$$

Figure 2.3. Commuting diagram involving $\Phi \Phi^{-1} F_{f'} =$ composition of red arrows

The square commutes because F respects coproducts; the composition of the bottom row of horizontal arrows is equal to the identity because $\lambda_\# D$ satisfies counitality.

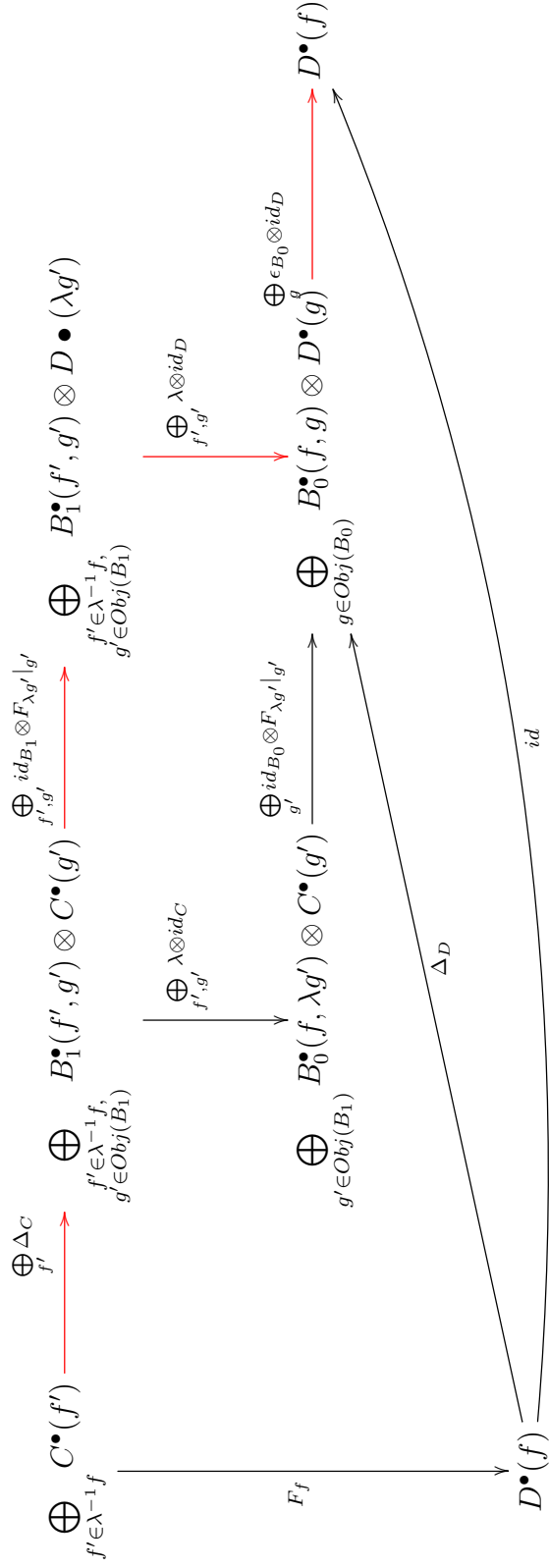


Figure 2.4. Commuting diagram involving $\Phi^{-1}\Phi F_f = \text{composition of red arrows}$
The concave pentagon on the left side commutes because F respects coproducts; the triangle in the bottom right corner commutes because D satisfies counitality.

2.6. Maps $\lambda_!$

In this section, we give maps $\lambda_! : C(\mathcal{A}) \rightarrow \hat{\lambda}^*C(\lambda\mathcal{A})$ of dg comodules over $B(\mathcal{A})$ where λ is a generating morphism in Λ that induces a morphism in χ with source $\mathcal{A} \in \text{Obj}(\chi)$. Showing that the $\lambda_!$'s satisfy cyclic relations up to homotopy is the computational heart of this thesis, and will be done in the next chapter. For now, we introduce the $\lambda_!$'s.

Technically, we should write $\lambda_{\mathcal{A}!}$ instead of $\lambda_!$, but we will be clear about the source when needed. In this section, we fix algebras A_0, \dots, A_n and set $B(n) := B(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$, $C(n) := C(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$. We define maps $\lambda_! : C(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0) \rightarrow \hat{\lambda}^*C(\lambda(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0))$.

2.6.1. Generating coboundaries $\delta_{j,n!}$ for $n \in \mathbb{N}$, $0 \leq j \leq n-1$

From Example 2.4.3, we know that $C(n) \cong \hat{\delta}_{j,n}^*C(A_0 \rightarrow \dots \rightarrow A_j \rightarrow A_{j+2} \rightarrow \dots \rightarrow A_0)$. So, define $\delta_{j,n!} : C(n) \xrightarrow{id} C(n) \cong \hat{\delta}_{j,n}^*C(A_0 \rightarrow \dots \rightarrow A_j \rightarrow A_{j+2} \rightarrow \dots \rightarrow A_0)$.

2.6.2. Generating codegeneracies $\sigma_{i,n!}$ for $n \in \mathbb{N}$, $0 \leq i \leq n$

From Example 2.4.4, we know that $C(n) \cong \hat{\sigma}_{i,n}^*C(A_0 \rightarrow \dots \rightarrow A_i \rightarrow A_i \rightarrow \dots \rightarrow A_0)$. So, define $\sigma_{i,n!} : C(n) \xrightarrow{id} C(n) \cong \hat{\sigma}_{i,n}^*C(A_0 \rightarrow \dots \rightarrow A_i \rightarrow A_i \rightarrow \dots \rightarrow A_0)$.

2.6.3. Generating rotations $\tau_{n!}$

2.6.3.1. $n = 0$. Let $\tau_{0!} = id : C(0) \xrightarrow{id} C(0) \cong id^*C(0) \cong \hat{\tau}_0^*C(0)$.

2.6.3.2. $n = 1$. We want to define a map of dg comodules over $B(1)$

$$\tau_{1!} : C(1) := C(A_0 \rightarrow A_1 \rightarrow A_0) \rightarrow \hat{\tau}_1^*C(A_1 \rightarrow A_0 \rightarrow A_1).$$

Example 2.4.5 describes the structure of $\hat{\tau}_1^* C(A_1 \rightarrow A_0 \rightarrow A_1)$, which is quasi-cofree over $B(1)$. So, we can define $\tau_{1!}$ by giving maps from $C(1)$ to the cogenerators of $\hat{\tau}_1^* C(A_1 \rightarrow A_0 \rightarrow A_1)$ and checking that the corresponding map of comodules commutes with the differentials.

More explicitly, for $f = (f_{0,0}, f_{1,0}) \in \text{Obj}(B(1))$, we will give k -linear maps

$$v^f : C(1)^\bullet(f) \rightarrow C_{-\bullet}(A_{1,f_{0,0}f_{1,0}} A_{1id})$$

$$(\phi_{0,1} \dots \phi_{0,n_0} | \phi_{1,1} \dots \phi_{1,n_1} | \alpha) \mapsto v_{n_0,n_1}^f(\phi_{0,1} \dots \phi_{0,n_0} | \phi_{1,1} \dots \phi_{1,n_1} | \alpha).$$

Then, we lift $\{v_f | f \in \text{Obj}(B(1))\}$ to a map of comodules in the standard way:

$$\begin{aligned} & \tau_{1!} f(\phi_{0,1} \dots \phi_{0,n_0} | \phi_{1,1} \dots \phi_{1,n_1} | \alpha) \\ (2.10) \quad &= \sum_{\substack{0 \leq k_0 \leq n_0 \\ 0 \leq k_1 \leq n_1}} \phi_{0,1} \dots \phi_{0,k_1} | \phi_{1,1} \dots \phi_{1,k_0} | v_{n_0-k_0, n_1-k_1}^{f_{0,k_1}, f_{1,k_0}}(\phi_{0,k_0+1} \dots \phi_{0,n_0} | \phi_{1,k_1+1} \dots \phi_{1,n_1} | \alpha) \end{aligned}$$

(see Figure 1.2 for notation). Finally, we will check by direct computation that $\tau_{1!}$ defined as such commutes with the differentials. To make the exposition smooth, all of this is done in Appendix Proposition B.1.

2.6.3.3. $n > 1$. For $n > 1$, we define $\tau_{n!}$ by pulling back $\tau_{1!}$ along $\delta_{0,*}$'s as follows:

$$\begin{aligned}
\tau_{n!} : C(n) &\cong (\delta_{0,2} \circ \dots \circ \delta_{0,n})^* C(A_0 \rightarrow A_n \rightarrow A_0) \\
&\xrightarrow{(\delta_{0,2} \circ \dots \circ \delta_{0,n})^* \tau_{1!}} (\delta_{0,2} \circ \dots \circ \delta_{0,n})^* \hat{\tau}_1^* C(A_n \rightarrow A_0 \rightarrow A_n) \\
&\cong (\tau_1 \circ \delta_{0,2} \circ \dots \circ \delta_{0,n})^* C(A_n \rightarrow A_0 \rightarrow A_n) \\
&\cong (\delta_{1,2} \circ \dots \circ \delta_{1,n} \circ \tau_n)^* C(A_n \rightarrow A_0 \rightarrow A_n) \\
&\cong \hat{\tau}_n^* (\delta_{1,2} \circ \dots \circ \delta_{1,n})^* C(A_n \rightarrow A_0 \rightarrow A_n) \\
&\cong \hat{\tau}_n^* C(A_n \rightarrow A_0 \rightarrow \dots \rightarrow A_n).
\end{aligned}$$

CHAPTER 3

A homotopically sheafy-cyclic object

3.1. Motivation of this chapter

In Section 2.1, we gave a sheafy-cyclic object in dg cocategories. We would like to extend that construction to a sheafy-cyclic object in the category of dg cocategories with a dg comodule. Namely, we would like to give a functor from χ to \mathcal{D} where \mathcal{D} the following category:

$$Obj(\mathcal{D}) = \{(B, C) | B \text{ is a dg cocategory, } C \text{ is a dg comodule over } B\}$$

$$\mathcal{D}((B_1, C_1), (B_0, C_0)) = \{(f, f_!) | f : B_1 \rightarrow B_0 \text{ is a functor,}$$

$$f_! : C_1 \rightarrow f^*C_0 \text{ is a map of dg comodules over } B_1\}$$

$$\mathcal{D}((B_2, C_2), (B_1, C_1)) \times \mathcal{D}((B_1, C_1), (B_0, C_0)) \xrightarrow{\text{composition}} \mathcal{D}((B_2, C_2), (B_0, C_0))$$

$$(f, f_!) \times (g, g_!) \mapsto (gf, f^*(g_!) \circ f_!)$$

(Proposition 2.1 implies that composition in \mathcal{D} is associative.)

In Section 2.6, we gave maps $\lambda_! : C(\mathcal{A}) \rightarrow \hat{\lambda}^*C(\lambda\mathcal{A})$ where λ is a generating morphism in Λ that induces a morphism in χ with source $\mathcal{A} \in Obj(\chi)$. Ideally, $(\hat{\lambda}, \lambda_!)$ would give a functor $\chi \rightarrow \mathcal{D}$, however, the $\lambda_!$'s only respect composition up to homotopy. Fortunately, most of the compositions are respected and, for the ones that are not, we have explicit homotopies. Our homotopies commute with the composable $\lambda_!$'s so that no higher homotopies are needed.

In this chapter, we will show which compositions are respected on the nose and which ones need homotopies. We will then give these homotopies and show that no higher

homotopies are needed. These sections are the computational heart of this thesis. Finally, we will repackage this “functor up to homotopy” in more abstract terms.

3.2. Homotopies

Here, we will show that the maps of dg comodules given in Section 2.6 satisfy the relations in Λ (Equation A.2) up to homotopy. More precisely, we will show that

$$\begin{aligned}
 (3.1a) \quad & \hat{\delta}_{j,n}^*(\delta_{i,n-1!}) \circ \delta_{j,n!} = \hat{\delta}_{i,n}^*(\delta_{j-1,n-1!}) \circ \delta_{i,n!} \quad 0 \leq i < j \leq n-1 \\
 & \hat{\sigma}_{j,n}^*(\sigma_{i,n+1!}) \circ \sigma_{j,n!} = \hat{\sigma}_{i,n}^*(\sigma_{j+1,n+1!}) \circ \sigma_{i,n!} \quad 0 \leq i \leq j \leq n \\
 & \hat{\sigma}_{i,n}^*(\delta_{j,n+1!}) \circ \sigma_{i,n!} = \begin{cases} \hat{\delta}_{j-1,n}^*(\sigma_{i,n-1!}) \circ \delta_{j-1,n!} & 0 \leq i < j \leq n \\ id & j = i, i-1 \\ \hat{\delta}_{j,n}^*(\sigma_{i-1,n-1!}) \circ \delta_{j,n!} & 0 \leq j < i-1 \leq n-1 \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 (3.1b) \quad & \hat{\sigma}_{i,n}^*(\tau_{n+1!}) \circ \sigma_{i,n!} = \hat{\tau}_n^*(\sigma_{i+1,n!}) \circ \tau_n! \quad 0 \leq i \leq n-1 \\
 & \hat{\delta}_{j,n}^*(\tau_{n-1!}) \circ \delta_{j,n!} = \hat{\tau}_n^*(\delta_{j+1,n!}) \circ \tau_n! \quad 0 \leq j \leq n-1
 \end{aligned}$$

$$(3.1c) \quad (\widehat{\tau_1 \sigma_{0,0}})^*(\delta_{0,1!}) \circ \hat{\sigma}_{0,0}^*(\tau_{1!}) \circ \sigma_{0,0!} = id$$

and

$$(3.2a) \quad \hat{\tau}_n^{*2}(\delta_{0,n!}) \circ \hat{\tau}_n^*(\tau_n!) \circ \tau_n! \simeq \hat{\delta}_{n-1,n}^*(\tau_{n-1!}) \circ \delta_{n-1,n!}$$

$$(3.2b) \quad \hat{\tau}_n^{*n}(\tau_n!) \circ \dots \circ \hat{\tau}_n^*(\tau_n!) \circ \tau_n! \simeq id$$

$$(3.2c) \quad \hat{\sigma}_{n,n}^*(\tau_{n+1!}) \circ \sigma_{n,n!} \simeq (\widehat{\tau_{n+1}^n \sigma_{0,n} \tau_n})^*(\tau_{n+1!}) \circ \dots \circ (\widehat{\tau_{n+1} \sigma_{0,n} \tau_n})^*(\tau_{n+1!}) \circ (\widehat{\sigma_{0,n} \tau_n})^*(\tau_{n+1!}) \circ \hat{\tau}_n^*(\sigma_{0,n!}) \circ \tau_n!$$

3.2.1. Aside on composing λ_i 's

In this section and the next, we will use the following convention for composing λ_i 's, which we first illustrate with an example. Suppose we have 3 composable generating morphisms in Λ : $\lambda_1 \in \Lambda([m], [n])$, $\lambda_2 \in \Lambda([n], [p])$, and $\lambda_3 \in \Lambda([p], [q])$. Fix a sequence of algebras A_0, \dots, A_m . We can construct the following composition of morphisms of dg comodules over $B(\mathcal{A}) := B(A_0 \rightarrow \dots \rightarrow A_m \rightarrow A_0)$:

$$C(\mathcal{A}) \xrightarrow{\lambda_{1,\mathcal{A}!}} \hat{\lambda}_1^* C(\lambda_1 \mathcal{A}) \xrightarrow{\hat{\lambda}_1^*(\lambda_2, \lambda_{1,\mathcal{A}!})} \hat{\lambda}_1^* \hat{\lambda}_2^* C(\lambda_2 \lambda_1 \mathcal{A}) \xrightarrow{\hat{\lambda}_1^* \hat{\lambda}_2^*(\lambda_3, \lambda_{2,\lambda_1 \mathcal{A}!})} \hat{\lambda}_1^* \hat{\lambda}_2^* \hat{\lambda}_3^* C(\lambda_3 \lambda_2 \lambda_1 \mathcal{A}).$$

To simplify the notation, we will write

$$\hat{\lambda}_1^* \hat{\lambda}_2^*(\lambda_{3!}) \circ \hat{\lambda}_1^*(\lambda_{2!}) \circ \lambda_{1!} \quad \text{instead of} \quad \hat{\lambda}_1^* \hat{\lambda}_2^*(\lambda_{3,\lambda_2 \lambda_1 \mathcal{A}!}) \circ \hat{\lambda}_1^*(\lambda_{2,\lambda_1 \mathcal{A}!}) \circ \lambda_{1,\mathcal{A}!}.$$

More generally, when the reader sees $\hat{\lambda}_1^* \dots \hat{\lambda}_{r-1}^*(\lambda_{r!}) \circ \dots \circ \hat{\lambda}_1^*(\lambda_{2!}) \circ \lambda_{1!}$ for a sequence of composable generating morphisms $\lambda_r \circ \dots \circ \lambda_1$ in Λ , s/he may decode this notation by choosing a sequence of algebras $A_0, \dots, A_{m=\text{source of } \lambda_1}$, setting $\mathcal{A} = (A_0 \rightarrow \dots \rightarrow A_m \rightarrow A_0)$, and letting the composition of λ_i 's denote $\hat{\lambda}_1^* \dots \hat{\lambda}_{r-1}^*(\lambda_{r,\lambda_{r-1} \dots \lambda_1 \mathcal{A}!}) \circ \dots \circ \hat{\lambda}_1^*(\lambda_{2,\lambda_1 \mathcal{A}!}) \circ \lambda_{1,\mathcal{A}!}$.

3.2.2. Strict relations: showing Equations 3.1 hold

Equation 3.1a has three relations. All of the σ_i 's and δ_i 's in Equation 3.1a are identity maps, so it's clear that these relations hold.

Equation 3.1b has two relations. To show that the first one holds, we have

$$\begin{aligned}
\hat{\sigma}_{i,n}^*(\tau_{n+1}!) \circ \sigma_{i,n}! &= \hat{\sigma}_{i,n}^*((\widehat{\delta_{0,2} \dots \delta_{0,n+1}})^*(\tau_1!)) \circ \sigma_{i,n}! && \text{definition of } \tau_{n+1}! \\
&= (\widehat{\delta_{0,2} \dots \delta_{0,n+1} \sigma_{i,n}})^*(\tau_1!) \circ \sigma_{i,n}! && \text{Proposition 2.1} \\
&= (\widehat{\delta_{0,2} \dots \delta_{0,n}})^*(\tau_1!) \circ \sigma_{i,n}! \\
&= \tau_n! \circ \sigma_{i,n}! && \text{definition of } \tau_n! \\
&= \tau_n! \circ id = id \circ \tau_n! \\
&= \hat{\tau}_n^*(\sigma_{i+1,n}!) \circ \tau_n!.
\end{aligned}$$

To show that the second relation holds, the reasoning is the same as above. We have

$$\begin{aligned}
\hat{\delta}_{j,n}^*(\tau_{n-1}!) \circ \delta_{j,n}! &= \hat{\delta}_{j,n}^*((\widehat{\delta_{0,2} \dots \delta_{0,n-1}})^*(\tau_1!)) \circ \delta_{j,n}! \\
&= (\widehat{\delta_{0,2} \dots \delta_{0,n-1} \delta_{j,n}})^*(\tau_1!) \circ \delta_{j,n}! \\
&= \tau_n! \circ \delta_{j,n}! \\
&= \tau_n! \circ id = id \circ \tau_n! \\
&= \hat{\tau}_n^*(\delta_{j+1,n}!) \circ \tau_n!.
\end{aligned}$$

Equation 3.1c has one relation. The only map in this relation that is not defined to be an identity map is $\hat{\sigma}_{0,0}^*(\tau_1!)$. We will compute this map and show that it is also an identity. Let $(\phi_{0,1} \dots \phi_{0,k_0} | \alpha) \in C(A_0 \rightarrow A_0)$ (see Figure 1.2 for notation). By Proposition

2.2,

$$C(A_0 \rightarrow A_0) \xrightarrow{\cong} \hat{\sigma}_{0,0}^* C(A_0 \rightarrow A_0 \rightarrow A_0)$$

$$(\phi_{0,1} \dots \phi_{0,k_0} | \alpha) \mapsto \sum_{0 \leq r_0 \leq k_0} (\phi_{0,1} \dots \phi_{0,r_0}) \otimes (1 | \phi_{0,r_0+1} \dots \phi_{0,k_0} | \alpha).$$

Applying $\hat{\sigma}_{0,0}^*(\tau_{1!})$ to the righthand side, we have

$$\begin{aligned} \hat{\sigma}_{0,0}^* C(A_0 \rightarrow A_0 \rightarrow A_0) &\xrightarrow{\hat{\sigma}_{0,0}^*(\tau_{1!})} \hat{\sigma}_{0,0}^* \hat{\tau}_1^* C(A_0 \rightarrow A_0 \rightarrow A_0) \\ \sum_{0 \leq r_0 \leq k_0} (\phi_{0,1} \dots \phi_{0,r_0}) \otimes (1 | \phi_{0,r_0+1} \dots \phi_{0,k_0} | \alpha) &\mapsto \sum_{0 \leq r_0 \leq s_0 \leq k_0} (\phi_{0,1} \dots \phi_{0,r_0}) \otimes \\ &(\phi_{0,r_0+1} \dots \phi_{0,s_0} | 1 | \tau_1(1 | \phi_{0,s_0+1} \dots \phi_{0,k_0} | \alpha)). \end{aligned}$$

The righthand side above is equal to

$$\begin{aligned} &\sum_{0 \leq r_0 \leq s_0 \leq k_0} (\phi_{0,1} \dots \phi_{0,r_0}) \otimes (\phi_{0,r_0+1} \dots \phi_{0,s_0} | 1 | \tau_1(1 | \phi_{0,s_0+1} \dots \phi_{0,k_0} | \alpha)) \\ &= \sum_{0 \leq r_0 \leq s_0 \leq k_0} (\phi_{0,1} \dots \phi_{0,r_0}) \otimes (\phi_{0,r_0+1} \dots \phi_{0,s_0} | 1 | v_{0,k_0-s_0}(1 | \phi_{0,s_0+1} \dots \phi_{0,k_0} | \alpha)) \\ &\quad \text{(see Proposition B.1 for } v_{\cdot,\cdot}) \\ &= \sum_{0 \leq r_0 \leq k_0} (\phi_{0,1} \dots \phi_{0,r_0}) \otimes (\phi_{0,r_0+1} \dots \phi_{0,k_0} | 1 | \alpha) \quad (v_{0,>0} = 0) \\ &\in \hat{\sigma}_{0,0}^* \hat{\tau}_1^* C(A_0 \rightarrow A_0 \rightarrow A_0). \end{aligned}$$

Finally, applying Proposition 2.2 again, we have

$$\hat{\sigma}_{0,0}^* \hat{\tau}_1^* C(A_0 \rightarrow A_0 \rightarrow A_0) \xrightarrow[\cong]{\text{project onto cogenerators}} C(A_0 \rightarrow A_0)$$

$$\sum_{0 \leq r_0 \leq k_0} (\phi_{0,1} \dots \phi_{0,r_0}) \otimes (\phi_{0,r_0+1} \dots \phi_{0,k_0} | 1 | \alpha) \mapsto (\phi_{0,1} \dots \phi_{0,k_0} | \alpha).$$

So, we've shown

$$C(A_0 \rightarrow A_0) \cong \hat{\sigma}_{0,0}^* C(A_0 \rightarrow A_0 \rightarrow A_0) \xrightarrow{\hat{\sigma}_{0,0}^*(\tau_{1!})} \hat{\sigma}_{0,0}^* \hat{\tau}_1^* C(A_0 \rightarrow A_0 \rightarrow A_0) \cong C(A_0 \rightarrow A_0)$$

is the identity map.

3.2.3. Weak relations: showing Equations 3.2 hold

3.2.3.1. Showing Equation 3.2a holds. For $n = 1$, eliminating the identity maps reduces Equation 3.2a to:

$$\hat{\tau}_1^*(\tau_{1!}) \circ \tau_{1!} \simeq id.$$

We prove the above in Appendix Proposition B.2. (In the appendix, $\tau_{1!} = \Upsilon_{A_0, A_1}$, $\hat{\tau}_1^*(\tau_{1!}) = \Upsilon_{A_1, A_0}$, and the homotopy is denoted B .)

For $n = 2$, eliminating the identity maps and writing $\tau_{2!}$ in terms of $\tau_{1!}$ reduces Equation 3.2a to:

$$(\widehat{\delta_{0,2}\tau_2})^*(\tau_{1!}) \circ \hat{\delta}_{0,2}^*(\tau_{1!}) \simeq \hat{\delta}_{1,2}^*(\tau_{1!}).$$

We prove the above in Appendix Proposition B.4. (In the appendix, $\hat{\delta}_{0,2}^*(\tau_{1!}) = \Upsilon_{A_0 \bullet A_1, A_2}$, $(\widehat{\delta_{0,2}\tau_2})^*(\tau_{1!}) = \Upsilon_{A_2 \bullet A_0, A_1}$, $\hat{\delta}_{1,2}^*(\tau_{1!}) = \Upsilon_{A_0, A_1 \bullet A_2}$, and the homotopy is denoted \mathcal{B} .)

For $n > 2$, we reduce Equation 3.2a to the case when $n = 2$. We have

$$\begin{aligned}
\text{Lefthand side of Equation 3.2a} &= \hat{\tau}_n^{*2}(\delta_{0,n!}) \circ \hat{\tau}_n^*(\tau_{n!}) \circ \tau_{n!} \\
&= id \circ \hat{\tau}_n^*((\widehat{\delta_{0,2} \dots \delta_{0,n}})^*(\tau_{1!})) \circ \tau_{n!} \\
&= (\widehat{\delta_{0,2} \dots \delta_{0,n} \tau_n})^*(\tau_{1!}) \circ \tau_{n!} \\
&= (\widehat{\delta_{0,2} \tau_2 \delta_{0,3} \dots \delta_{0,n}})^*(\tau_{1!}) \circ \tau_{n!} \\
&= (\widehat{\delta_{0,2} \tau_2 \delta_{0,3} \dots \delta_{0,n}})^*(\tau_{1!}) \circ (\widehat{\delta_{0,2} \dots \delta_{0,n} \tau_n})^*(\tau_{1!}) \\
&= (\widehat{\delta_{0,3} \dots \delta_{0,n}})^*((\widehat{\delta_{0,2} \tau_2})^*(\tau_{1!}) \circ \hat{\delta}_{0,2}^* \tau_{1!})
\end{aligned}$$

$$\begin{aligned}
\text{Righthand side of Equation 3.2a} &= \hat{\delta}_{n-1,n}^*(\tau_{n-1!}) \circ \delta_{n-1,n!} \\
&= \hat{\delta}_{n-1,n}^*((\widehat{\delta_{0,2} \dots \delta_{0,n-1}})^*(\tau_{1!})) \circ id \\
&= (\widehat{\delta_{0,2} \dots \delta_{0,n-1} \delta_{n-1,n}})^*(\tau_{1!}) \\
&= (\widehat{\delta_{1,2} \delta_{0,3} \dots \delta_{0,n}})^*(\tau_{1!}) \\
&= (\widehat{\delta_{0,3} \dots \delta_{0,n}})^*(\hat{\delta}_{1,2}^*(\tau_{1!})).
\end{aligned}$$

So, Equation 3.2a = $(\widehat{\delta_{0,3} \dots \delta_{0,n}})^*(\text{Equation 3.2a, } n = 2)$. If \mathcal{B} is a homotopy giving Equation 3.2a for $n = 2$, then $(\widehat{\delta_{0,3} \dots \delta_{0,n}})^*\mathcal{B}$ is a homotopy giving Equation 3.2a for $n > 2$.

3.2.3.2. Showing Equation 3.2b holds. We prove this by induction on n . For $n = 1$, Equation 3.2b is the same as Equation 3.2a, which we established in the previous section. Now, assume that Equation 3.2b holds for $N = n - 1$. We show that Equation 3.2b holds

for $N = n$ below:

$$\begin{aligned}
\hat{\tau}_n^{*n}(\tau_{n!}) \circ \dots \circ \hat{\tau}_n^*(\tau_{n!}) \circ \tau_{n!} &= \hat{\tau}_n^{*n-1}(\hat{\tau}_n^* \tau_{n!} \circ \tau_{n!}) \circ \hat{\tau}_n^{*n-2} \tau_{n!} \circ \dots \circ \tau_{n!} \\
&\simeq \hat{\tau}_n^{*n-1}(\hat{\delta}_{n-1,n}^* \tau_{n-1!}) \circ \hat{\tau}_n^{*n-2} \tau_{n!} \circ \dots \circ \tau_{n!} \quad (\text{Equation 3.2a}) \\
&= (\widehat{\tau_{n-1}^{n-1} \delta_{0,n}})^* \tau_{n-1!} \circ (\hat{\tau}_n^{*n-2} \hat{\delta}_{n-2,n}^* \tau_{n-1!} \circ \dots \circ \hat{\tau}_n^* \hat{\delta}_{1,n}^* \tau_{n-1!} \circ \hat{\delta}_{0,n}^* \tau_{n-1!}) \\
&= (\widehat{\tau_{n-1}^{n-1} \delta_{0,n}})^* \tau_{n-1!} \circ \hat{\delta}_{0,n}^* (\hat{\tau}_{n-1}^{*n-2} \tau_{n-1!} \circ \dots \circ \hat{\tau}_{n-1}^* \tau_{n-1!} \circ \tau_{n-1!}) \\
&= \hat{\delta}_{0,n}^* (\hat{\tau}_{n-1}^{*n-1} \tau_{n-1!} \circ \dots \circ \tau_{n-1!}) \\
&\simeq \hat{\delta}_{0,n}^*(id) \quad (\text{Inductive hypothesis}) \\
&= id.
\end{aligned}$$

3.2.3.3. Showing Equation 3.2c holds. By manipulating morphisms in Λ , we have

$$\begin{aligned}
\text{Righthand side of Equation 3.2c} &= \hat{\tau}_n^{*n+1} \tau_{n!} \circ \hat{\tau}_n^{*n} \tau_{n!} \circ \dots \circ \hat{\tau}_n^* \tau_{n!} \circ \hat{\tau}_n^{*n+1} id \circ \tau_{n!} \\
&= \tau_{n!} \circ (\hat{\tau}_n^{*n} \tau_{n!} \circ \dots \circ \hat{\tau}_n^* \tau_{n!} \circ \tau_{n!}) \\
&\simeq \tau_{n!} \circ (id) \quad \text{Equation 3.2b.}
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
 \text{Lefthand side of Equation 3.2c} &= \hat{\sigma}_{n,n}^*(\tau_{n+1!}) \circ id \\
 &= \hat{\sigma}_{n,n}^*(\hat{\delta}_{n,n+1}^*(\tau_{n+1!})) \\
 &= (\widehat{\delta_{n,n+1} \sigma_{n,n}})^*(\tau_{n!}) \\
 &= id^*(\tau_{n!}).
 \end{aligned}$$

So, Equation 3.2c holds.

3.3. Higher Homotopies

In this section, we show that no higher homotopies are needed. First, we will summarize the maps of comodules that we have already given. Let λ be a generating morphism in Λ that induces a morphism in χ with source $\mathcal{A} \in \text{Obj}(\chi)$. We have

$$\lambda_! : C(\mathcal{A}) \rightarrow \hat{\lambda}^* C(\lambda \mathcal{A}) \quad \text{maps of dg comodules}$$

$$\sigma_! : C(\mathcal{A}) \rightarrow \tau^{*2} C(\tau^2 \mathcal{A}) \quad \text{deg -1 map of comodules.}$$

where

$$\sigma_! = \begin{cases} B \text{ given in Appendix Proposition B.2} & \text{if } \mathcal{A} = (A_0 \rightarrow A_1 \rightarrow A_0) \\ \mathcal{B} \text{ given in Appendix Proposition B.4} & \text{if } \mathcal{A} = (A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_0) \\ \widehat{(\delta_{0,3} \dots \delta_{0,n})}^* \mathcal{B} & \text{if } \mathcal{A} = (A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0), n > 2. \end{cases}$$

Using the constructions we've given, a typical map between comodules is one that is freely generated by composable pullbacks of $\lambda_!$'s and $\sigma_!$'s. First, we will establish that there are no such maps of degree ≥ 2 . Suppose we have a map $\eta_!$ of degree ≥ 2 . Then, $\eta_!$ must contain at least two (pullbacks of) some $\sigma_!$'s. Each $\sigma_!$ involves inserting a 1 into the first slot of the Hochschild chains component (see Equations B.2, B.4). However, since we are working with reduced chains, any chain with two or more 1's is equal to zero. So, $\eta_! = 0$.

Since there are no maps of degree ≥ 2 , we know from the classical theory of A_∞ algebras that the only need for higher homotopies will arise from the following situation:

For $n \geq 2$, We have two maps of dg comodules

$$(3.3) \quad \begin{array}{ccc} C(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0) & & \\ \begin{array}{c} \hat{\tau}_n^{*2} \tau_{n!} \circ \hat{\tau}_n^* \tau_{n!} \circ \tau_{n!} \\ \text{"apply } \tau_{n!} \text{ 3 times"} \end{array} \left(\begin{array}{c} \downarrow \\ \downarrow \end{array} \right) & \begin{array}{c} (\widehat{\delta_{n-2,n-1} \delta_{n-1,n}})^* \tau_{n-2!} \\ \text{"brace together the last 3 algebras,} \\ \text{then apply } \tau_{n-2!} \text{ once"} \end{array} & \\ C(A_{n-2} \rightarrow A_{n-1} \rightarrow A_n \rightarrow A_0 \rightarrow \dots \rightarrow A_{n-2}). & & \end{array}$$

These two maps are homotopic via two homotopies: $\hat{\delta}_{n-1,n}^* \sigma! + \tau_n^{*2} \tau_{n!} \circ \sigma!$ and $\hat{\delta}_{n-2,n}^* \sigma! + \hat{\tau}_n^* \sigma! \circ \tau_{n!}$ (see Figure 3.1). If the two homotopies were different, then their difference would be closed and we would desire a higher homotopy (i.e., a degree -2 map of comodules) between them. However, we will show the two homotopies are the same, so that no higher homotopies are needed.

First, we show that $\hat{\delta}_{n-1,n}^* \sigma! = \hat{\delta}_{n-2,n}^* \sigma!$. We have

$$\hat{\delta}_{n-1,n}^* \sigma! = \mathcal{B}_{A_0 \bullet \dots \bullet A_{n-2}, A_{n-1} \bullet A_n} = \mathcal{B}_{A_0 \bullet \dots \bullet A_{n-1}, A_n} = \hat{\delta}_{n-2,n}^* \sigma!$$

where the second equality holds by definition of \mathcal{B} in Appendix Equation B.4.

Second, we show that $\tau_n^{*2} \tau_{n!} \circ \sigma! = \hat{\tau}_n^* \sigma! \circ \tau_{n!}$ in Appendix Proposition B.5. In the appendix, $\tau_n^{*2} \tau_{n!} \circ \sigma! = \Upsilon_{A_1 \bullet A_2, A_0} \mathcal{B}_{A_0, A_1, A_2}$ and $\hat{\tau}_n^* \sigma! \circ \tau_{n!} = \mathcal{B}_{A_2, A_0, A_1} \Upsilon_{A_0 \bullet A_1, A_2}$.

For $n = 1$, the situation in Equation 3.3 reduces to: We have two maps of dg comodules

$$\begin{array}{ccc} C(A_0 \rightarrow A_1 \rightarrow A_0) & & \\ \hat{\tau}_1^{*2} \tau_{1!} \circ \hat{\tau}_1^* \tau_{1!} \circ \tau_{1!} \left(\begin{array}{c} \downarrow \\ \downarrow \end{array} \right) \tau_{1!} & & \\ C(A_1 \rightarrow A_0 \rightarrow A_1). & & \end{array}$$

$$\begin{array}{ccc}
(\widehat{\delta_{n-2,n-1} \delta_{n-1,n}})^* \tau_{n-2}! & \xrightarrow{\cong} & \hat{\delta}_{n-1,n}^* (\hat{\delta}_{n-2,n-1}^* \tau_{n-2}!) \xrightarrow{\hat{\delta}_{n-1,n}^* \sigma!} \hat{\delta}_{n-1,n}^* (\hat{\tau}_{n-1}^* \tau_{n-1}! \circ \tau_{n-1}!) \\
\text{"brace together } A_{n-2}, A_{n-1}, A_n, \\ \text{then apply } \tau_{n-2}!" & & \downarrow \cong \\
(\widehat{\delta_{n-2,n-1} \delta_{n-2,n}})^* \tau_{n-2}! & & \hat{\tau}_n^{*2} \tau_n! \circ \hat{\delta}_{n-1,n}^* \tau_{n-1}! \\
\downarrow \hat{\delta}_{n-2,n}^* \sigma! & & \text{"brace together } A_{n-1}, A_n \\ \text{and apply } \tau_{n-1}!, \\ \text{then apply } \tau_n!" \\
\hat{\delta}_{n-2,n}^* (\hat{\tau}_{n-1}^* \tau_{n-1}! \circ \tau_{n-1}!) & \xrightarrow{\cong} & \hat{\tau}_n^* (\hat{\delta}_{n-1,n}^* \tau_{n-1}!) \circ \tau_n! \xrightarrow{\hat{\tau}_n^* \sigma! \circ \tau_n!} \hat{\tau}_n^{*2} \tau_n! \circ \hat{\tau}_n^* \tau_n! \circ \tau_n! \\
& \text{"apply } \tau_n!, \\ \text{then brace together } A_{n-1}, A_{n-2} \\ \text{and apply } \tau_{n-1}!" & & \downarrow \tau_n^{*2} \tau_n! \circ \sigma! \\
& & \text{"apply } \tau_n! \text{ three times"}
\end{array}$$

Figure 3.1. Two homotopies between $(\widehat{\delta_{n-2,n-1} \delta_{n-1,n}})^* \tau_{n-2}!$ and $\hat{\tau}_n^{*2} \tau_n! \circ \hat{\tau}_n^* \tau_n! \circ \tau_n!$

Vertices are maps of dg comodules and arrows are chain homotopies.

$$\begin{array}{c}
id \circ \tau_1! = \tau_1! = \tau_1! \circ id \\
\begin{array}{ccc}
\sigma_{A_1 \rightarrow A_0 \rightarrow A_1! \circ \tau_1!} & \left(\begin{array}{c} \downarrow \\ \downarrow \end{array} \right) & \tau_1! \circ \sigma_{A_0 \rightarrow A_1 \rightarrow A_0!} \\
= \hat{\tau}_1^* (\sigma_{A_0 \rightarrow A_1 \rightarrow A_0!}) \circ \tau_1! & & \\
(\hat{\tau}_1^{*2} \tau_1! \circ \hat{\tau}_1^* \tau_1!) \circ \tau_1! & = & \hat{\tau}_1^{*2} \tau_1! \circ (\hat{\tau}_1^* \tau_1! \circ \tau_1!)
\end{array}
\end{array}$$

Figure 3.2. Two homotopies between $\tau_1!$ and $\hat{\tau}_1^{*2} \tau_1! \circ \hat{\tau}_1^* \tau_1! \circ \tau_1!$

Vertices are maps of dg comodules and arrows are chain homotopies.

These two maps are homotopic via two homotopies: $\tau_1! \circ \sigma_{A_0 \rightarrow A_1 \rightarrow A_0!}$ and $\sigma_{A_1 \rightarrow A_0 \rightarrow A_1!} \circ \tau_1!$ (see Figure 3.2; for clarity, here, we indicate the sources of the σ_i 's). We show that these two homotopies are the same in Appendix Proposition B.3, so no higher homotopies are needed. In the appendix, $\tau_1! \circ \sigma_{A_0 \rightarrow A_1 \rightarrow A_0!} = \Upsilon_{A_0, A_1} \circ B_{A_0, A_1}$ and $\sigma_{A_1 \rightarrow A_0 \rightarrow A_1!} \circ \tau_1! = B_{A_1, A_0} \circ \Upsilon_{A_0, A_1}$.

3.4. An A_∞ -functor

We can repackage the work of the previous sections into a concise statement: We have constructed an A_∞ -functor from χ to \mathcal{D} . This section is devoted to making that concise statement more rigorous. We refer to Reference [2], Appendix A, Definitions A.6 and A.8 for the notation and definition of an A_∞ -category and A_∞ -functor.

First, we must think of χ and \mathcal{D} as A_∞ -categories. Let χ_∞ be the (usual) category with the same objects as χ and morphisms linear combinations over k of morphisms in χ . We will think of the morphisms in χ_∞ as complexes concentrated in degree zero. χ_∞ is an A_∞ -category with $m_2 =$ (usual) composition of morphisms in χ_∞ , $m_1 = m_{\geq 3} = 0$. The relations for an A_∞ -category are satisfied because m_2 is associative.

Let \mathcal{D}_∞ be the dg category with the same objects as \mathcal{D} and morphisms

$$\begin{aligned} \mathcal{D}_\infty^\bullet((B_1, C_1), (B_0, C_0)) &= \{ (F : B_1 \rightarrow B_0 \text{ dg functor,} \\ &\quad F_! : C_1 \rightarrow F^* C_0 \text{ map of comodules of degree } \bullet) \} \\ d_{\mathcal{D}_\infty}(F, F_!) &= (F, d_{F^* C_0} \circ F_! - (-1)^{|F_!|} F_! \circ d_{C_1}). \end{aligned}$$

Composition in \mathcal{D}_∞ works like composition in \mathcal{D} —we can apply the same formulas to pullback a (not necessarily graded) morphism of comodules. We can think of \mathcal{D}_∞ as an A_∞ -category with $m_1 = d_{\mathcal{D}_\infty}$, $m_2 =$ composition of morphisms, $m_{\geq 3} = 0$. For \mathcal{D}_∞ , the relations for an A_∞ -category are precisely that (1) the differentials square to zero, (2) composition is a map of complexes, and (3) composition is associative.

Now, we will show that the constructions given in the previous sections constitute an A_∞ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$. Still using the notation in Reference [2], Definition A.8, we

define \mathcal{F} as follows:

$$f : \text{Obj}(\chi_\infty) \rightarrow \text{Obj}(\mathcal{D}_\infty) \quad \text{map of sets}$$

$$\mathcal{A} \mapsto (B(\mathcal{A}), C(\mathcal{A})) \text{ defined in Chapter 1}$$

$$f_1 : \chi_\infty^\bullet(x_0, x_1) \rightarrow \mathcal{D}_\infty^\bullet(fx_0, fx_1) \quad \text{map of graded vector spaces}$$

$$\lambda \mapsto (\hat{\lambda}, \lambda_!) \text{ defined in Sections 2.1, 2.6}$$

for λ a generating morphism in Λ

$$id_{\mathcal{A}} \mapsto (id_{B(\mathcal{A})}, id_{C(\mathcal{A})})$$

$$f_2 : \chi_\infty^\bullet(x_0, x_1) \otimes \chi_\infty^\bullet(x_1, x_2) \rightarrow \mathcal{D}_\infty^\bullet(fx_0, fx_2) \quad \text{degree } -1 \text{ map of vector spaces}$$

$$f_{i \geq 3} : \chi_\infty^\bullet(x_0, x_1) \otimes \dots \otimes \chi_\infty^\bullet(x_{i-1}, x_i) \rightarrow \mathcal{D}_\infty^\bullet(fx_0, fx_i) \quad \text{degree } 1 - i \text{ map of vector spaces}$$

$$\mu_1 \otimes \dots \otimes \mu_i \mapsto 0 \text{ for all morphisms } \mu$$

We will show that the f_i 's satisfy the relations in Reference [2], Definition A.8.

First, we finish defining \mathcal{F} , namely, we must define $f_1(\mu)$ for μ not a generating morphism in Λ as well as for linear combinations of morphisms. We also still need to define f_2 .

Let μ be a non-generating morphism in Λ that induces a morphism in χ with source \mathcal{A} . Choose (i.e., fix once and for all) a presentation of μ as a composition of generating morphisms. Within the chosen presentation, in the following order, (1) replace all occurrences of $\tau_{n-1}\delta_{n-1,n}$ with $\delta_{0,n}\tau_n^2$, (2) replace all $\tau_{n+1}\sigma_{n,n}$ with $\tau_{n+1}^{n+1}\sigma_{0,n}\tau_n$, (3) replace all decompositions of identity maps with identity maps, (4) remove all identity maps if $\mu \neq id$, (5) call this new presentation “the presentation corresponding to μ ”, denoted

$\mu = \lambda_{\mu, k_\mu} \dots \lambda_{\mu, 1}$. The presentation corresponding to μ is not unique (i.e., still depends on the original, chosen presentation). However, letting

$$f_1(\mu) := (\hat{\mu} : B(\mathcal{A}) \rightarrow B(\mu\mathcal{A}))$$

$$\hat{\lambda}_{\mu, 1}^* \dots \hat{\lambda}_{\mu, k_\mu - 1}^* (\lambda_{\mu, k_\mu!}) \circ \dots \circ \hat{\lambda}_{\mu, 1}^* (\lambda_{\mu, 2!}) \circ \lambda_{\mu, 1!} : C(\mathcal{A}) \rightarrow \hat{\mu}^* C(\mu\mathcal{A}))$$

is well-defined because we have made consistent choices for all of the relations among λ_i 's that only hold up to homotopy (see Equations 3.2). More explicitly, $f_1(\mu)$ would have been well-defined for any choice of presentation if Equations 3.2 were equalities rather than homotopies. Instead, we choose to define $f_1(\mu)$ via a presentation that only uses the lefthand side of Equation 3.2a and only uses the righthand sides of Equations 3.2c and 3.2b. Finally, extend f_1 linearly over k to define f_1 for all linear combinations of morphisms in χ .

Before defining f_2 , let's take a look at an A_∞ relation we expect f_2 to satisfy: For $\cdot \xrightarrow{\mu_1} \cdot \xrightarrow{\mu_2} \cdot$ composable morphisms in χ , we expect

$$(3.4) \quad f_1(\mu_2 \circ \mu_1) = f_1(\mu_2) \circ f_1(\mu_1) + d_{\mathcal{D}_\infty} \circ f_2(\mu_1, \mu_2).$$

Given the definition of f_1 above, we require a non-zero f_2 only if: (Condition H) the presentation corresponding to μ_2 composed with the presentation corresponding to μ_1 contains, after removing (decompositions of) identity maps except for τ_n^{n+1} , one or more of the following terms: $\tau_{n-1}\delta_{n-1, n}$, $\tau_{n+1}\sigma_{n, n}$, τ_n^{n+1} . If μ_1, μ_2 satisfy Condition H, homotopies given in Section 3.2.3 can be used to define f_2 . If μ_1, μ_2 do not satisfy Condition H, let $f_2(\mu_1, \mu_2) = 0$.

We will give some instructive examples of non-zero f_2 that satisfy Equation 3.4.

Example 3.4.1. Let $\mu_1 = \delta_{n-1,n}$, $\mu_2 = \tau_{n-1}$. Then, the presentation corresponding to $\mu_2\mu_1$ is $\delta_{0,n}\tau_n^2$. Let $f_2(\mu_1, \mu_2)$ be the homotopy given in Section 3.2.3.1 (also given in Appendix Proposition B.2 or B.4). Then, Equation 3.4 is equivalent to Equation 3.2a.

Example 3.4.2. Let $\mu_1 = \sigma_{0,n-1}\delta_{n-1,n}$, $\mu_2 = \tau_{n-1}\delta_{0,n}$. To form the presentation corresponding to $\mu_2\mu_1$, we follow these steps:

$$\tau_{n-1}\delta_{0,n}\sigma_{0,n-1}\delta_{n-1,n} \xrightarrow[\text{of identities}]{\text{remove decompositions}} \tau_{n-1}\delta_{n-1,n} \xrightarrow{\text{replace}} \delta_{0,n}\tau_n^2.$$

On the other hand,

$$\begin{aligned} f_1(\mu_2)f_1(\mu_1) &= (\delta_{0,n}\widehat{\sigma_{0,n-1}\delta_{n-1,n}})^*(\tau_{n-1}!) \circ (\sigma_{0,n-1}\widehat{\delta_{n-1,n}})^*(\delta_{0,n}!) \circ \hat{\delta}_{n-1,n}^*(\sigma_{0,n-1}!) \circ \delta_{n-1,n}! \\ &= \hat{\delta}_{n-1,n}^*(\tau_{n-1}!) \circ id \circ \delta_{n-1,n}!. \end{aligned}$$

So, we can let $f_2(\mu_1, \mu_2)$ be the homotopy given in Section 3.2.3.1, and Equation 3.4 is equivalent to Equation 3.2a.

Example 3.4.3. Let $(\mu_1, \mu_2) \in \{(\tau_{n+1}, \sigma_{n,n}), (\tau_n^{n+1-j}, \tau_n^j) : 1 \leq j \leq n, n \in \mathbb{N}\}$. Let $f_2(\mu_1, \mu_2)$ be the homotopy given in 3.2.3.3 if $\mu_2 = \sigma_{n,n}$ and the homotopy given in 3.2.3.2 if $\mu_2 \neq \sigma_{n,n}$. Then, Equation 3.4 is equivalent to either Equation 3.2c ($\mu_2 = \sigma_{n,n}$) or Equation 3.2b ($\mu_2 \neq \sigma_{n,n}$).

Example 3.4.4. Let $\mu_1 = \sigma_{n-1,n-1}\delta_{n-1,n}$, $\mu_2 = \tau_n$. To form the presentation corresponding to $\mu_2\mu_1$, we follow these steps:

$$(\tau_n\sigma_{0,n-1})\delta_{n-1,n} \xrightarrow{\text{replace } (\cdot)} \tau_n^n\sigma_{0,n-1}(\tau_{n-1}\delta_{n-1,n}) \xrightarrow{\text{replace } (\cdot)} \tau_n^n\sigma_{0,n-1}\delta_{0,n}\tau_n^2.$$

Let $f_2(\mu_1, \mu_2) = g_1 + g_2$ where $g_1 = \hat{\delta}_{n-1,n}^*(\text{homotopy in Section 3.2.3.3}) \circ \delta_{n-1,n}!$ and $g_2 = (\widehat{\tau_{n-1}\delta_{n-1,n}})^*((\widehat{\tau_n^{n-1}\sigma_{0,n-1}})^*(\tau_n!) \circ \dots \circ \hat{\sigma}_{0,n-1}^*(\tau_n!) \circ \sigma_{0,n-1}!) \circ (\text{homotopy in Section 3.2.3.1})$. Then, Equation 3.4 reduces to $\delta_{n-1,n}^*(\text{Equation 3.2c})$ and Equation 3.2a.

Now, we will check that the f_i 's we gave satisfy the rest of the relations for an A_∞ -functor from Reference [2], Definition A.8: For $\cdot \xrightarrow{\mu_1} \cdot \xrightarrow{\mu_2} \cdot \xrightarrow{\mu_3} \cdot \xrightarrow{\mu_4} \cdot$ composable morphisms in χ , we expect

$$(3.5) \quad 0 = d_{\mathcal{D}_\infty} \circ f_1(\mu_1)$$

$$(3.6) \quad f_2(\mu_3, \mu_2 \circ \mu_1) - f_2(\mu_3 \circ \mu_2, \mu_1) = f_2(\mu_3, \mu_2) \circ f_1(\mu_1) - f_1(\mu_3) \circ f_2(\mu_2, \mu_1)$$

$$(3.7) \quad 0 = f_2(\mu_4, \mu_3) \circ f_2(\mu_2, \mu_1).$$

Equation 3.5 is satisfied since the λ_i 's we defined in Section 2.6 are maps of complexes. Equation 3.7 is satisfied since composing two of our degree -1 homotopies is always equal to zero (see Section 3.3, first paragraph). Finally, Equation 3.6 boils down to showing that the two homotopies in Figure 3.1 and in Figure 3.2 are the same (see Section 3.3).

Thus, we have an A_∞ -functor $\mathcal{F} : \chi_\infty \rightarrow \mathcal{D}_\infty$. Applying Reference [2], Remark A.27, we can rectify \mathcal{F} to a dg functor $\tilde{\mathcal{F}} : U(\chi_\infty) \rightarrow \mathcal{D}_\infty$ where $U(\chi_\infty)$ is the enveloping dg category of χ (see Reference [2], Definition A.25).

3.5. A functor to dg categories

In the previous section 3.4, we gave a dg functor from $U(\chi_\infty)$ to $\mathcal{D}_\infty =$ “dg cocategories and dg comodules”. We would like to have our functor land in the dg category, $\mathcal{E} =$ “dg categories and dg modules”. To do so, we will first give a dg functor $\mathcal{D}_\infty \rightarrow \mathcal{D}_1$, which makes use of the adjunction in Proposition 2.3. Then, we will give a dg functor $\Omega : \mathcal{D}_1 \rightarrow \mathcal{E}$.

3.5.1. Using the adjunction

Let \mathcal{D}_1 be the dg category with the same objects as \mathcal{D} and morphisms

$$\begin{aligned} \mathcal{D}_1^\bullet((B_1, C_1), (B_0, C_0)) &= \left\{ (F : B_1 \rightarrow B_0 \text{ dg functor,} \right. \\ &\quad \left. F_! : F_\# C_1 \rightarrow C_0 \text{ map of comodules of degree } \bullet) \right\} \\ d_{\mathcal{D}_\infty}(F, F_!) &= (F, d_{C_0} \circ F_! - (-1)^{|F_!|} F_! \circ d_{F_\# C_1}) \end{aligned}$$

with composition

$$\begin{aligned} \mathcal{D}_1^\bullet((B_2, C_2), (B_1, C_1)) \otimes \mathcal{D}_1^\bullet((B_1, C_1), (B_0, C_0)) &\rightarrow \mathcal{D}_1^\bullet((B_2, C_2), (B_0, C_0)) \\ (f, f_!) \otimes (g, g_!) &\mapsto (gf, g_! \circ g_\#(f_!)). \end{aligned}$$

This composition is well-defined because we can apply the formulas from $g_\#$ to (not necessarily graded) morphisms of comodules. The composition is associative because of the following easy-to-check fact: $g_\# f_\# C = (gf)_\# C$ for $B_2 \xrightarrow{f} B_1 \xrightarrow{g} B_0$ functors of dg cocategories and C a dg comodule over B_2 .

Now, we define a dg functor

$$Adj : \mathcal{D}_\infty \rightarrow \mathcal{D}_1$$

$$\text{on objects: } (B, C) \mapsto (B, C)$$

$$\text{on morphisms: } \left((B_1, C_1) \xrightarrow{(F, F_!)} (B_0, C_0) \right) \mapsto \left((B_1, C_1) \xrightarrow{(F, \Phi_F^{-1} F)} (B_0, C_0) \right)$$

where $\Phi_F^{-1} : Hom_{\text{dg comodules over } B_1}(C, F^* D) \rightarrow Hom_{\text{dg comodules over } B_0}(F_\# C, D)$ is defined in the proof of Proposition 2.3 and makes sense as a function on (not necessarily graded) maps of comodules. To check that Adj commutes with the differentials and respects composition, we need

$$\Phi_F^{-1} \circ d_{Hom_{B_2}(C_2, F^* C_1)} = d_{Hom_{B_1}(F_\# C_2, C_1)} \circ \Phi_F^{-1}$$

$$\Phi_{GF}^{-1}(F^* G_! \circ F_!) = \Phi_G^{-1}(G_!) \circ G_\#(\Phi_F^{-1}(F_!))$$

$$\text{where } (B_2, C_2) \xrightarrow{(F, F_!)} (B_1, C_1) \xrightarrow{(G, G_!)} (B_0, C_0) \text{ in } \mathcal{D}_\infty.$$

The equations above follow straight-forwardly from the definitions.

3.5.2. Applying *Cobar*

In this section, we will use the notion of a dg module over a dg category. This is dual to a dg comodule over a dg cocategory (Definition 1.3.1). Given a dg functor between dg categories $F : A_1 \rightarrow A_0$, we define “restriction of scalars”, F^* , a functor from the category of dg comodules over A_0 to the category of dg comodules over A_1 . For M_0 a dg comodule over A_0 and $f \in Obj(B_1)$, $F^* M_0(f) := M_0(Ff)$.

Let \mathcal{E} be the dg category defined below:

$$Obj(\mathcal{E}) = \{(A, M) | A \text{ is a dg category, } M \text{ is a dg module over } A\}$$

$$\mathcal{E}^p((A_1, M_1), (A_0, M_0)) = \{(f, f_!) | f : A_1 \rightarrow A_0 \text{ is a dg functor,}$$

$$f_! : M_1 \rightarrow f^* M_0 \text{ is a degree-} p \text{ map of modules over } A_1\}$$

$$d_{\mathcal{E}}(f, f_!) = (f, d_{f^* M_0} \circ f_! - (-1)^{|f_!|} f_! \circ d_{C_1})$$

$$\mathcal{E}^\bullet((A_2, M_2), (A_1, M_1)) \times \mathcal{E}^\bullet((A_1, M_1), (A_0, M_0)) \xrightarrow{\text{composition}} \mathcal{E}^\bullet((A_2, M_2), (A_0, M_0))$$

$$(f, f_!) \times (g, g_!) \mapsto (gf, f^*(g_!) \circ f_!).$$

We will define a dg functor $\Omega : \mathcal{D}_1 \rightarrow \mathcal{E}$. On objects, $\Omega(B, C) := (Cobar(B), Cobar(B, C))$ where the first Cobar is a dg functor from the category of dg cocategories to the category of dg categories, and the second Cobar sends dg comodules over B to dg modules over $Cobar(B)$ (see [1], Section 4.6). On morphisms,

$$\mathcal{D}_1 \ni \left(\begin{array}{c} B_1 \xrightarrow{F} B_0 \\ \downarrow F_! \\ C_1 \xrightarrow{\quad} F^* C_0 \end{array} \right) \mapsto \left(\begin{array}{c} Cobar(B_1) \xrightarrow{Cobar(F)} Cobar(B_0) \\ \downarrow \Omega(F_!) \\ Cobar(B_1, C_1) \xrightarrow{\quad} (Cobar(F))^* Cobar(B_0, C_0) \end{array} \right) \in \mathcal{E}$$

$$\text{where } \Omega(F_!) : Cobar(B_1, C_1) \rightarrow (Cobar(F))^* Cobar(B_0, C_0)$$

$$(b_1 | \dots | b_n | c) \mapsto (Fb_1 | \dots | Fb_n | F_! c)$$

$$\text{for } b_i \in B_1^\bullet(f_{i-1}, f_i), c \in C_1^\bullet(f_n), \text{ and } f_i \in Obj(B_1), 0 \leq i \leq n.$$

It's straightforward from the definitions to check that Ω commutes with the differentials and respects composition.

3.5.3. The end: putting everything together

We have dg functors

$$U(\chi_\infty) \xrightarrow{\tilde{\mathcal{F}}} \mathcal{D}_\infty \xrightarrow{Adj} \mathcal{D}_1 \xrightarrow{\Omega} \mathcal{E}.$$

This is our homotopically sheafy-cyclic object in dg categories with a trace functor.

References

- [1] Tsygan, Boris. Noncommutative Calculus and Operads.
- [2] Faonte, Giovanni. A_∞ -Functors and Homotopy Theory of DG-Categories

APPENDIX A

Connes cyclic category, Λ

Here, we give generators and relations for the cyclic category, Λ . None of this is new, but we do it to establish notation for the rest of the paper.

Λ has objects $\{[n] : n \in \mathbb{N}\}$ and generating morphisms:

$$\begin{aligned}
 & \text{rotations } \tau_n : [n] \rightarrow [n], \\
 (A.1) \quad & \text{coboundaries } \delta_{j,n} : [n] \rightarrow [n-1], 0 \leq j \leq n-1, \\
 & \text{codegeneracies } \sigma_{i,n} : [n] \rightarrow [n+1], 0 \leq i \leq n
 \end{aligned}$$

subject to relations:

$$\begin{aligned}
 \delta_{i,n-1}\delta_{j,n} &= \delta_{j-1,n-1}\delta_{i,n} \quad 0 \leq i < j \leq n-1 \\
 \sigma_{i,n+1}\sigma_{j,n} &= \sigma_{j+1,n+1}\sigma_{i,n} \quad 0 \leq i \leq j \leq n \\
 \delta_{j,n+1}\sigma_{i,n} &= \begin{cases} \sigma_{i,n-1}\delta_{j-1,n} & 0 \leq i < j \leq n \\ id & j = i, i-1 \\ \sigma_{i-1,n-1}\delta_{j,n} & 0 \leq j < i-1 \leq n-1 \end{cases} \\
 (A.2) \quad \tau_{n+1}\sigma_{i,n} &= \sigma_{i+1,n}\tau_n \quad 0 \leq i \leq n-1 \\
 \tau_{n-1}\delta_{j,n} &= \delta_{j+1,n}\tau_n \quad 0 \leq j \leq n-1 \\
 \tau_n^{n+1} &= id \\
 \delta_{0,1}\tau_1\sigma_{0,0} &= id \\
 \tau_{n+1}\sigma_{n,n} &= \tau_{n+1}^{n+1}\sigma_{0,n}\tau_n \\
 \delta_{0,n}\tau_n^2 &= \tau_{n-1}\delta_{n-1,n}.
 \end{aligned}$$

Some presentations of Λ include an extra coboundary $\delta_{n,n}$ and codegeneracy $\sigma_{n+1,n}$.

In terms of our generators, they are $\delta_{n,n} := \delta_{0,n}\tau_n$ and $\sigma_{n+1,n} := \tau_{n+1}^{n+1}\sigma_{0,n}$.

APPENDIX B

Computations

In this appendix, we give the computational propositions needed to establish the homotopically sheafy-cyclic structure on dg comodules. All the comodules we work with will be cofree, and we will define maps into them by giving maps into cogenerators (see Equation 1.2).

B.1. Computational notation

For this section's propositions, we establish the following notation:

A_0, A_1 fixed algebras

$$(\vec{\phi}|\vec{\psi}|\alpha) := (\phi_1 \dots \phi_n | \psi_1 \dots \psi_m | \alpha)$$

$$= \begin{array}{c} \begin{array}{ccccc} & f_0 & & g_0 & \\ & \curvearrowright & & \curvearrowright & \\ & \Downarrow \phi_1 & & \Downarrow \psi_1 & \\ & \curvearrowright & & \curvearrowright & \\ A_0 & & A_1 & & A_0 \\ & \vdots & & \vdots & \\ & f_n & & g_m & \\ & \curvearrowright & & \curvearrowright & \\ & \alpha & & \alpha & \\ & \curvearrowright & & \curvearrowright & \\ & id & & id & \end{array} \end{array} \in C(A_0 \rightarrow A_1 \rightarrow A_0)(g_0 f_0)$$

$$\vec{\phi}_{\{i_1, i_2, \dots, i_k\}} := \phi_{i_1} \phi_{i_2} \dots \phi_{i_k} \quad \text{where } \{i_1, i_2, \dots, i_k\} \text{ is an ordered subset of } \{1, \dots, n\}$$

$$\vec{\phi}_{\emptyset} := 1 \in k \cong \text{Bar}_0(C^\bullet(A_0, A_1))$$

$$\vec{\psi}_{\emptyset} := 1 \in k \cong \text{Bar}_0(C^\bullet(A_1, A_0))$$

$$|I| := \text{number of elements in a set } I$$

$$I_1 I_2 := \text{concatenation as ordered sets of possibly-empty sets } I_1 \text{ and } I_2$$

$$\lambda(\vec{\psi}), \tilde{\delta}, b', b, \psi\{\vec{\phi}\} \cdot \alpha = \text{see Appendix C for standard operations on Hochschild (co)chains}$$

B.1.1. Notation for elements of Hochschild chains

Let $a_0 \otimes a_1 \otimes \cdots \otimes a_n$ denote a typical element of $C_{-\bullet}(A, A)$ where A is some algebra. At times, we wish to feed a portion of $a_0 \otimes a_1 \otimes \cdots \otimes a_n$ to a Hochschild cochain (or other map on chains) without specifying the degree of the cochain. To do this, we will rewrite $a_0 \otimes a_1 \otimes \cdots \otimes a_n = a_0 \otimes \mathbf{a}_1 \otimes \cdots \otimes \mathbf{a}_r$ where each $\mathbf{a}_i = a_{j_i} \otimes a_{j_i+1} \otimes \cdots \otimes a_{j_{i+1}-1}$ and \mathbf{a}_i is an empty chain if $j_i = j_{i+1}$.

For example, if $\phi \in C^2(A, A)$, then we rewrite

$$\sum_{1 \leq i \leq n-1} a_0 \otimes a_1 \otimes \cdots \otimes a_{i-1} \otimes \phi(a_i, a_{i+1}) \otimes a_{i+2} \otimes \cdots \otimes a_n = \sum a_0 \otimes \mathbf{a}_1 \otimes \phi(\mathbf{a}_2) \otimes \mathbf{a}_3.$$

B.2. Computational Propositions

Proposition B.1. *Let $\hat{\tau}_1 : B(A_0 \rightarrow A_1 \rightarrow A_0) \longrightarrow B(A_1 \rightarrow A_0 \rightarrow A_1)$ be as defined in Section 2.1. Recall from Example 2.4.5 that $\hat{\tau}_1^* C(A_1 \rightarrow A_0 \rightarrow A_0) \cong C(A_1 \rightarrow A_0 \rightarrow A_1)$ as complexes. Define a map*

$$\Upsilon_{A_0, A_1} : C(A_0 \rightarrow A_1 \rightarrow A_0) \rightarrow \hat{\tau}_1^* C(A_1 \rightarrow A_0 \rightarrow A_1)$$

of comodules over $B(A_0 \rightarrow A_1 \rightarrow A_0)$ by mapping into cogenerators as follows:

$$\begin{aligned}
 v^{f_0, g_0} : C(A_0 \rightarrow A_1 \rightarrow A_0)(f_0, g_0) &\rightarrow \hat{\tau}_1^* C(A_1 \rightarrow A_0 \rightarrow A_1)(g_0, f_0) \cong C(A_1 \rightarrow A_0 \rightarrow A_1)(g_0, f_0) \\
 &\xrightarrow[\text{cogenerators}]{\text{project onto}} C_{-\bullet}(A_{1, f_0 g_0} A_{1 id}) \\
 v_{n, m}^{f_0, g_0}(\vec{\phi} | \vec{\psi} | \alpha) &= \sum_{\substack{I_1 I_2 = \{1, 2, \dots, n\} \\ \text{as ordered sets}}} \phi_1(\lambda(\vec{\psi}) \lambda(\vec{\phi}_{I_2}) \cdot \mathbf{a}_3, a_0, \mathbf{a}_1) \otimes \lambda(\vec{\phi}_{I_1}) \cdot \mathbf{a}_2 \\
 &\quad \left(+ f_0 a_0 \otimes \lambda(\vec{\phi}) \mathbf{a}_1 \quad \text{if } m = 0 \right).
 \end{aligned}$$

Then, $\Upsilon_{A_0, A_1} : C(A_0 \rightarrow A_1 \rightarrow A_0) \rightarrow \hat{\tau}^* C(A_1 \rightarrow A_0 \rightarrow A_1)$ is a map of dg comodules over $B(A_0 \rightarrow A_1 \rightarrow A_0)$.

Proof. We must show: (1) Υ is a map of comodules, and (2) Υ commutes with the differentials. (In this proof, we drop the subscripts and write $\Upsilon := \Upsilon_{A_0, A_1}$.)

(1) This proof is standard for cofree comodules. Let $(\vec{\phi} | \vec{\psi} | \alpha)$ be as in the statement of the proposition. We want to show that Υ commutes with the coproducts. On one hand,

$$\begin{aligned}
 &[(id_B \otimes \Upsilon) \circ \Delta_{C(A_0 \rightarrow A_1 \rightarrow A_0)}](\vec{\phi} | \vec{\psi} | \alpha) \\
 &= [id_B \otimes \Upsilon] \left(\sum_{\substack{I_1 I_2 = \{1, 2, \dots, n\} \text{ and} \\ J_1 J_2 = \{1, 2, \dots, m\} \\ \text{as ordered sets}}} (\vec{\phi}_{I_1} | \vec{\psi}_{J_1}) \otimes (\vec{\phi}_{I_2} | \vec{\psi}_{J_2} | \alpha) \right) \\
 &= \sum_{\substack{I_1 I_2 I_3 = \{1, 2, \dots, n\} \text{ and} \\ J_1 J_2 J_3 = \{1, 2, \dots, m\} \\ \text{as ordered sets}}} (\vec{\phi}_{I_1} | \vec{\psi}_{J_1}) \otimes (\vec{\phi}_{I_2} | \vec{\psi}_{J_2}) \otimes v_{|I_3|, |J_3|}(\vec{\phi}_{I_3} | \vec{\psi}_{J_3} | \alpha)
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
& [\Delta_{\hat{\tau}^* C(A_1 \rightarrow A_0 \rightarrow A_1)} \circ \Upsilon](\vec{\phi}|\vec{\psi}|\alpha) \\
&= \Delta_{\hat{\tau}^* C(A_1 \rightarrow A_0 \rightarrow A_1)} \left(\sum_{\substack{I_1 I_2 = \{1, 2, \dots, n\} \text{ and} \\ J_1 J_2 = \{1, 2, \dots, m\} \\ \text{as ordered sets}}} (\vec{\phi}_{I_1}|\vec{\psi}_{J_1}) \otimes v_{|I_2|, |J_2|}(\vec{\phi}_{I_2}|\vec{\psi}_{J_2}|\alpha) \right) \\
&= \sum_{\substack{I_1 I_2 I_3 = \{1, 2, \dots, n\} \text{ and} \\ J_1 J_2 J_3 = \{1, 2, \dots, m\} \\ \text{as ordered sets}}} (\vec{\phi}_{I_1}|\vec{\psi}_{J_1}) \otimes (\vec{\phi}_{I_2}|\vec{\psi}_{J_2}) \otimes v_{|I_3|, |J_3|}(\vec{\phi}_{I_3}|\vec{\psi}_{J_3}|\alpha).
\end{aligned}$$

Clearly $(id_B \otimes \Upsilon) \circ \Delta_{C(A_0 \rightarrow A_1 \rightarrow A_0)} = \Delta_{\hat{\tau}^* C(A_1 \rightarrow A_0 \rightarrow A_1)} \circ \Upsilon$.

(2) We will show that Υ commutes with the differentials by direct computation. Since Υ is a map of cofree comodules, we only need to check that $\pi_1 \circ D(\Upsilon) = 0$ where $D(\Upsilon)$ is the differential applied to Υ as a linear map between complexes and π_1 denotes projection of a comodule onto its cogenerators. More explicitly, we want to check that

$$\begin{aligned}
& v_{n,m}(\tilde{\delta}(\vec{\phi})|\vec{\psi}|\alpha) + v_{n,m}(\vec{\phi}|\tilde{\delta}(\vec{\psi})|\alpha) + v_{n-1,m}(b'(\vec{\phi})|\vec{\psi}|\alpha) + v_{n,m-1}(\vec{\phi}|b'(\vec{\psi})|\alpha) + \\
& v_{n,m}(\vec{\phi}|\vec{\psi}|b(\alpha)) + b \circ v_{n,m}(\vec{\phi}|\vec{\psi}|\alpha) + \\
\text{(B.1)} \quad & v_{|I_1|, m-1}(\vec{\phi}_{I_1}|\vec{\psi}_{\{1, \dots, m-1\}}|\psi_m\{\vec{\phi}_{I_2}\} \cdot \alpha) + v_{n-1, m}(\vec{\phi}_{\{1, \dots, n-1\}}|\vec{\psi}_m|\phi_n \cdot \alpha) + \\
& \phi_1\{\psi_{J_1}\} \cdot v_{\vec{\phi}_{[-1, |J_2|]}(\phi_{\{2, \dots, n\}}|\psi_{J_2}|\alpha) + \psi_1 \cdot v_{n, m-1}(\vec{\phi}|\vec{\psi}_{\{2, \dots, |\vec{\psi}\}}|\alpha) \\
& = 0.
\end{aligned}$$

In Equation B.1, we will call the terms in the first two rows the “standard terms”, and the terms in the second two rows the “extra terms”.

We compute the sum of the standard terms. In Table B.1, the leftmost column lists the expressions that don’t cancel in the sum of the standard terms, the middle column

gives the standard term from which the expression comes, and the rightmost column gives the term (extra or standard) that cancels the expression.

All of the terms in Table B.1 cancel, so Υ is a map of complexes. \square

Expression	Comes from Standard Term	Cancelling Term
$f_0\psi_1(\lambda(\vec{\phi}_{I_2})\mathbf{a}_3)\phi_1(\lambda(\psi_{I_2,\dots,m})\lambda(\vec{\phi}_{I_3})\mathbf{a}_4, a_0, \mathbf{a}_1)$ $\otimes \lambda(\vec{\phi}_{I_1})\mathbf{a}_2$	$v_{n,m}(\delta(\phi_1)\phi_2 \cdots \phi_n \vec{\psi} \alpha)$	$f_0\psi_1 \cdot v_{n,m-1}(\vec{\phi}_{I_1} \vec{\psi}_{\{1,\dots,m-1\}} \psi_m \{\vec{\phi}_{I_2}\} \cdot \alpha)$
$\phi_1(\lambda(\psi_{\{1,\dots,m-1\}})\lambda(\vec{\phi}_{I_2})\mathbf{a}_3, \psi_m(\lambda(\vec{\phi}_{I_3})\mathbf{a}_4) \cdot a_0, \mathbf{a}_1)$ $\otimes \lambda(\vec{\phi}_{I_1})\mathbf{a}_2$	$v_{n,m}(\delta(\phi_1)\phi_2 \cdots \phi_n \vec{\psi} \alpha)$	$v_{I_1 m-1}(\vec{\phi}_{I_1} \vec{\psi}_{\{1,\dots,m-1\}} \psi_m \{\vec{\phi}_{I_2}\} \cdot \alpha)$
$\phi_1(\lambda(\vec{\psi})\lambda(\vec{\phi}_{I_2})\mathbf{a}_3, g_m\phi_n(\mathbf{a}_4) \cdot a_0, \mathbf{a}_1) \otimes \lambda(\vec{\phi}_{I_1})\mathbf{a}_2$	$v_{n,m}(\delta(\phi_1)\phi_2 \cdots \phi_n \vec{\psi} \alpha)$	$v_{n-1,m}(\vec{\phi}_{\{1,\dots,n-1\}} \vec{\psi} g_m\phi_n \cdot \alpha)$
$\phi_1(\lambda(\vec{\psi})\lambda(\vec{\phi}_{I_2})\mathbf{a}_2) \cdot f_1(a_0) \otimes \lambda(\vec{\phi}_{I_1})\mathbf{a}_1$	$v_{n,m}(\delta(\phi_1)\phi_2 \cdots \phi_n \vec{\psi} \alpha)$	$\phi_1 \cdot v_{n-1,0}(\vec{\phi}_{\{2,\dots,n\}} \vec{\psi} \alpha)$
$f_0a_0 \cdot \phi_1(\mathbf{a}_1) \otimes \lambda(\vec{\phi}_{\{1,\dots,n-1\}})\mathbf{a}_2$	$v_{n,m}(\delta(\phi_1)\phi_2 \cdots \phi_n \vec{\psi} \alpha)$ if $\vec{\psi} = 1$	$b \circ v_{n,m}(\vec{\phi} \vec{\psi} \alpha)$ if $\vec{\psi} = 1$
$f_0g_m\phi_n(\mathbf{a}_2)f_0a_0 \otimes \lambda(\vec{\phi}_{\{1,\dots,n-1\}})\mathbf{a}_1$	$b \circ v_{n,m}(\vec{\phi} \vec{\psi} \alpha)$ if $\vec{\psi} = 1$	$v_{n-1,m}(\vec{\phi}_{\{1,\dots,n-1\}} \vec{\psi} g_m\phi_n \cdot \alpha)$ if $\vec{\psi} = 1$
$\phi_1(\lambda(\vec{\psi})\lambda(\vec{\phi}_{I_2})\mathbf{a}_4, a_0, \mathbf{a}_1) \cdot \phi_2(\mathbf{a}_2) \otimes \lambda(\vec{\phi}_{I_1})\mathbf{a}_3$	$b \circ v_{n,m}(\vec{\phi} \vec{\psi} \alpha)$	$v_{n-1,m}(\phi_1 \cup \phi_2\phi_3 \cdots \phi_n \vec{\psi} \alpha)$
$\phi_1(\lambda(\psi_{I_1})\lambda(\vec{\phi}_{I_2})\mathbf{a}_3)\phi_2(\lambda(\psi_{I_2})\lambda(\vec{\phi}_{I_3})\mathbf{a}_3, a_0, \mathbf{a}_1)$ $\otimes \lambda(\vec{\phi}_{I_1})\mathbf{a}_2$	$v_{n-1,m}(\phi_1 \cup \phi_2\phi_3 \cdots \phi_n \vec{\psi} \alpha)$	$\phi_1\{\vec{\psi}_{I_1}\} \cdot v_{n-1,I_2}(\vec{\phi}_{\{2,\dots,n\}} \vec{\psi}_{I_2} \alpha)$
$f_0\psi_1(\lambda(\vec{\phi}_{I_2})\mathbf{a}_2) \cdot f_0a_0 \otimes \lambda(\vec{\phi}_{I_1})\mathbf{a}_1$	$f_0\psi_1 \cdot v_{n,0}(\vec{\phi} 1 \alpha)$ if $\vec{\psi} = \psi_1$	$v_{I_1 0}(\vec{\phi}_{I_1} 1 \psi_1 \{\vec{\phi}_{I_2}\} \cdot \alpha)$ if $\vec{\psi} = \psi_1$

Table B.1. Expansion of terms in Equation B.1

(Technically, the last term in the middle column is not a standard term, but we include it in the table for convenience.)

Proposition B.2. *Let $B_{A_0, A_1} = B : C(A_0 \rightarrow A_1 \rightarrow A_0) \longrightarrow C(A_0 \rightarrow A_1 \rightarrow A_0)$ be the map of cofree comodules defined by the following maps to cogenerators:*

$$(B.2) \quad B_{n,m}(\vec{\phi}|\vec{\psi}|\alpha) = 1 \otimes \lambda(\psi)\lambda(\phi)\mathfrak{a}_2 \otimes a_0 \otimes \mathfrak{a}_1.$$

Then, $D(B_{A_0, A_1}) = \Upsilon_{A_1, A_0} \Upsilon_{A_0, A_1} - id$ where Υ is defined in Proposition B.1.

Proof. We prove the statement by direct computation. Since all of the maps are maps of cofree comodules, we only need to check that $\pi_1(D(B_{A_0, A_1}) - \Upsilon_{A_1, A_0} \Upsilon_{A_0, A_1} - id) = 0$ where π_1 denotes projection of the comodule onto cogenerators. More explicitly, for an element $(\vec{\phi}|\vec{\psi}|\alpha)$, we want to check that

$$(B.3) \quad \begin{aligned} & B_{n,m}(\tilde{\delta}(\vec{\phi})|\vec{\psi}|\alpha) + B_{n,m}(\vec{\phi}|\tilde{\delta}(\vec{\psi})|\alpha) + B_{n-1,m}(b'(\vec{\phi})|\vec{\psi}|\alpha) + B_{n,m-1}(\vec{\phi}|b'(\vec{\psi})|\alpha) + \\ & B_{n,m}(\vec{\phi}|\vec{\psi}|b(\alpha)) + b \circ B_{n,m}(\vec{\phi}|\vec{\psi}|\alpha) + \\ & B_{|I_1|, m-1}(\vec{\phi}_{I_1}|\vec{\psi}_{\{1, \dots, m-1\}}|\psi_m\{\vec{\phi}_{I_2}\} \cdot \alpha) + B_{n-1, m}(\vec{\phi}_{\{1, \dots, n-1\}}|\vec{\psi}_m|\phi_n \cdot \alpha) + \\ & \phi_1\{\psi_{J_1}\} \cdot B_{\vec{\phi}_{-1}, |J_2|}(\phi_{\{2, \dots, n\}}|\psi_{J_2}|\alpha) + \psi_1 \cdot B_{n, m-1}(\vec{\phi}|\vec{\psi}_{\{2, \dots, |\vec{\psi}\}}|\alpha) - \\ & v_{|J_1|, |I_1|}(\vec{\psi}_{J_1}|\vec{\phi}_{I_1}|v_{|I_2|, |J_2|}(\vec{\phi}_{I_2}|\vec{\psi}_{J_2}|\alpha)) - \pi_1(\vec{\phi}|\vec{\psi}|\alpha) \\ & = 0. \end{aligned}$$

We will call the terms in the first two rows the “standard terms” in the computaion of $D(B_{A_0, A_1})$, and the terms in the second two rows the “extra terms” in the computation of $D(B_{A_0, A_1})$. The fifth row is $\pi_1(\Upsilon_{A_1, A_0} \Upsilon_{A_0, A_1} - id)$.

We compute the sum of the standard terms. In Table B.2, the leftmost column lists the expressions that don’t cancel in the sum of the standard terms, the middle column

gives the standard term from which the expression comes, and the rightmost column gives the extra term that cancels the expression. Table B.3 lists the remaining terms from the fifth row that are not already listed in Table B.2. In Table B.3, the left column lists the remaining expressions that don't cancel in the fifth row, and the right column gives the extra term that cancels the expression.

All of the terms in the tables describing the expansion of equation B.3 cancel, so

$$D(B_{A_0,A_1}) = \Upsilon_{A_1,A_0} \Upsilon_{A_0,A_1} - id. \quad \square$$

Expression	Comes from Standard Term	Cancels with Extra Term
$\psi_1(\lambda(\vec{\phi}_{I_1})\mathbf{a}_2) \otimes \lambda(\vec{\psi}_{I_2, \dots, m})\lambda(\vec{\phi}_{I_2})\mathbf{a}_3 \otimes a_0 \otimes \mathbf{a}_1$	$b \circ B_{n,m}(\vec{\phi} \vec{\psi} \alpha)$	$\psi_1\{\vec{\phi}_{I_1}\} \cdot B_{ I_2 ,m-1}(\vec{\phi}_{I_2} \vec{\psi}_{\{2, \dots, m\}} \alpha)$
$g_0\phi_1(\mathbf{a}_2) \otimes \lambda(\vec{\psi})\lambda(\vec{\phi}_{\{2, \dots, n\}})\mathbf{a}_3 \otimes a_0 \otimes \mathbf{a}_1$	$b \circ B_{n,m}(\vec{\phi} \vec{\psi} \alpha)$	$\phi_1 \cdot B_{n-1,m}(\vec{\phi}_{\{2, \dots, n\}} \vec{\psi} \alpha)$
$1 \otimes \lambda(\vec{\psi})\lambda(\vec{\phi}_{\{1, \dots, n-1\}})\mathbf{a}_2 \otimes g_m\phi_n(\mathbf{a}_3 \cdot a_0 \otimes \mathbf{a}_1$	$b \circ B_{n,m}(\vec{\phi} \vec{\psi} \alpha)$	$B_{n-1,m}(\vec{\phi}_{\{1, \dots, n-1\}} \vec{\psi} \phi_n \cdot \alpha)$
$1 \otimes \lambda(\vec{\psi}_{\{1, \dots, m-1\}})\lambda(\vec{\phi}_{I_1})\mathbf{a}_2 \otimes g_m\psi_m(\lambda(\vec{\phi}_{I_2})\mathbf{a}_3) \cdot a_0 \otimes \mathbf{a}_1$	$b \circ B_{n,m}(\vec{\phi} \vec{\psi} \alpha)$	$B_{ I_1 ,m-1}(\vec{\phi}_{I_2} \vec{\psi}_{\{1, \dots, m-1\}} \psi_m\{\vec{\phi}_{I_2}\} \cdot \alpha)$
$g_0f_0a_0 \otimes \lambda(\vec{\psi})\lambda(\vec{\phi})\mathbf{a}_1$	$b \circ B_{n,m}(\vec{\phi} \vec{\psi} \alpha)$	$v_{ J_1 , I_1 }(\vec{\psi}_{J_1} \vec{\phi}_{I_1} v_{ I_2 , J_2 }(\vec{\phi}_{I_2} \vec{\psi}_{J_2} \alpha))$

Table B.2. Expansion of “standard terms” in Equation B.3 and the “extra terms” that cancel them

(Technically, the last term in the right column is not an extra term, but we include it in the table for convenience.)

Expression from Fifth Row	Cancels with Extra Term
$\psi_1(\lambda(\vec{\phi}_{I_1})\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_4})\mathbf{a}_4, \phi_{ I_1 +1}(\lambda(\vec{\psi}_{J_3})\lambda(\vec{\phi}_{I_5})\mathbf{a}_5, a_0, \mathbf{a}_1), \lambda(\vec{\phi}_{I_2 \setminus I_1 +1})\mathbf{a}_2) \otimes \lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_3})\mathbf{a}_3$	$\psi_1\{\vec{\phi}_{I_1}\} \cdot B_{ I_2 ,m-1}(\vec{\phi}_{I_2} \vec{\psi}_{\{2, \dots, m\}} \alpha)$
$\psi_1(\lambda(\vec{\phi}_{I_1})\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_4})\mathbf{a}_4, f_{ I_1 +1}a_0, \lambda(\vec{\phi}_{I_2 \setminus I_1 +1})\mathbf{a}_1) \otimes \lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_3})\mathbf{a}_2$	$\phi_1 \cdot B_{n-1,m}(\vec{\phi}_{\{2, \dots, n\}} \vec{\psi} \alpha)$
$g_0\phi_1(\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_2})\mathbf{a}_3, a_0, \mathbf{a}_1) \otimes \lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_1})\mathbf{a}_2$	$\psi_1\{\vec{\phi}_{I_1}\} \cdot B_{ I_2 ,m-1}(\vec{\phi}_{I_2} \vec{\psi}_{\{2, \dots, m\}} \alpha)$

Table B.3. Expansion of remaining “fifth-row terms” in Equation B.3 and the “extra terms” that cancel them

Proposition B.3. *Let $\Upsilon_{A_0, A_1} : C(A_0 \rightarrow A_1 \rightarrow A_0) \longrightarrow C(A_1 \rightarrow A_0 \rightarrow A_1)$ and $B_{A_0, A_1} : C(A_0 \rightarrow A_1 \rightarrow A_0) \longrightarrow C(A_0 \rightarrow A_1 \rightarrow A_0)$ be the maps defined in Propositions B.1 and B.2 above. Then, $[\Upsilon, B] := \Upsilon_{A_0, A_1} B_{A_0, A_1} - B_{A_1, A_0} \Upsilon_{A_0, A_1} = 0$.*

Proof. We show that $[\Upsilon, B] = 0$ by direct computation. Since all of the maps are maps of cofree comodules, we only need to check that $\pi_1([\Upsilon, B]) = 0$ where π_1 denotes projection of the comodule onto cogenerators. We check this directly.

$$\begin{aligned} \pi_1 \circ \Upsilon_{A_0, A_1} \circ B_{A_0, A_1}(\vec{\phi}|\vec{\psi}|\alpha) &= v_{|I_1|, |J_1|}(\vec{\phi}_{I_1}|\vec{\psi}_{J_1}|B_{|I_2|, |J_2|}(\vec{\phi}_{I_1}|\vec{\psi}_{J_1}|\alpha)) \\ &= v_{|I_1|, |J_1|}(\vec{\phi}_{I_1}|\vec{\psi}_{J_1}|1 \otimes \lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_2})\mathbf{a}_2, a_0, \mathbf{a}_1) \\ &= 1 \otimes \lambda(\vec{\phi}_{I_1})(\lambda(\vec{\psi})\lambda(\vec{\phi}_{I_2})\mathbf{a}_2, a_0, \mathbf{a}_1) \end{aligned}$$

$$\begin{aligned} \pi_1 \circ B_{A_1, A_0} \circ \Upsilon_{A_0, A_1}(\vec{\phi}|\vec{\psi}|\alpha) &= B_{|J_1|, |I_1|}(\vec{\psi}_{J_1}|\vec{\phi}_{I_1}|v_{|I_2|, |J_2|}(\vec{\phi}_{I_1}|\vec{\psi}_{J_1}|\alpha)) \\ &= B_{|J_1|, |I_1|}(\vec{\psi}_{J_1}|\vec{\phi}_{I_1}|\phi_{|I_1|+1}(\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_3})\mathbf{a}_3, a_0, \mathbf{a}_1) \otimes \lambda(\vec{\phi}_{I_2 \setminus |I_1|+1})\mathbf{a}_2 + \\ &\quad + a_0 \otimes \lambda(\vec{\phi}_{I_2 \setminus |I_1|+1})\mathbf{a}_1 \text{ if } J_2 = \emptyset) \\ &= 1 \otimes \lambda(\vec{\phi}_{I_1})\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_3})\mathbf{a}_3 \otimes \phi_{|I_1|+1}(\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_4})\mathbf{a}_4, a_0, \mathbf{a}_1) \otimes \lambda(\vec{\phi}_{I_2 \setminus |I_1|+1})\mathbf{a}_2 + \\ &\quad + 1 \otimes \lambda(\vec{\phi}_{I_1})\lambda(\vec{\psi})\lambda(\vec{\phi}_{I_3})\mathbf{a}_2 \otimes a_0 \otimes \lambda(\vec{\phi}_{I_2})\mathbf{a}_1 \end{aligned}$$

It's clear that $\pi_1 \circ \Upsilon_{A_0, A_1} \circ B_{A_0, A_1} = \pi_1 \circ B_{A_1, A_0} \circ \Upsilon_{A_0, A_1}$: The final expansion of $\pi_1 \circ \Upsilon_{A_0, A_1} \circ B_{A_0, A_1}$ is the sum of the two terms in the final expansion of $\pi_1 \circ B_{A_1, A_0} \circ \Upsilon_{A_0, A_1}$, which is the sum of terms in which one of the ϕ 's contains a_0 and the terms in which none of the ϕ 's contains a_0 . \square

B.3. More notation

For the next two propositions, we will need some more notation. Set

A_0, A_1, A_2 fixed algebras

$$(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha) := (\phi_1 \dots \phi_n | \psi_1 \dots \psi_m | \theta_1 \dots \theta_r | \alpha)$$

$$= \begin{array}{c} \begin{array}{ccccc} & f_0 & & g_0 & & h_0 \\ & \curvearrowright & & \curvearrowright & & \curvearrowright \\ & \Downarrow \phi_1 & & \Downarrow \psi_1 & & \Downarrow \theta_1 \\ & \curvearrowright & & \curvearrowright & & \curvearrowright \\ A_0 & & A_1 & & A_2 & & A_0 \\ & \vdots & & \vdots & & \vdots & \\ & f_n & & g_m & & h_p & \\ & \curvearrowright & & \curvearrowright & & \curvearrowright & \\ & & \alpha & & & & \end{array} \\ id \end{array}$$

$$\in C(A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_0)(h_0 g_0 f_0)$$

$$\Upsilon_{A_0 \bullet A_1, A_2} : C(A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_0) \rightarrow \hat{\tau}_2^* C(A_2 \rightarrow A_0 \rightarrow A_1 \rightarrow A_2) \quad \text{map of dg comodules}$$

$$(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha) \mapsto \Upsilon_{A_0, A_2}(\vec{\phi} \bullet \vec{\psi}|\vec{\theta}|\alpha)$$

$$\Upsilon_{A_0, A_1 \bullet A_2} : C(A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_0) \rightarrow \hat{\tau}_2^{*2} C(A_1 \rightarrow A_2 \rightarrow A_0 \rightarrow A_1) \quad \text{map of dg comodules}$$

$$(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha) \mapsto \Upsilon_{A_0, A_1}(\vec{\phi}|\vec{\psi} \bullet \vec{\theta}|\alpha)$$

B.4. More Propositions

Proposition B.4. *Let*

$$\mathcal{B}_{A_0, A_1, A_2} = \mathcal{B} : C(A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_0) \rightarrow \hat{\tau}_2^{*2} C(A_1 \rightarrow A_2 \rightarrow A_0 \rightarrow A_1)$$

be a map of comodules over $B(A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_0)$ determined by the following maps to cogenerators:

(B.4)

$$\mathcal{B}_{A_0, A_1, A_2}^{f_0, g_0, h_0} : C(A_0 \rightarrow A_1 \rightarrow A_0)(h_0 g_0 f_0) \rightarrow \hat{\tau}_2^{*2} C(A_1 \rightarrow A_2 \rightarrow A_0 \rightarrow A_1)(f_0 h_0 g_0)$$

$$\xrightarrow[\text{cogenerators}]{\text{project onto}} C_{-\bullet}(A_1, f_0 h_0 g_0 A_1 \text{id})$$

$$\mathcal{B}_{n, m, p}(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha) = \sum_{\substack{I_1 I_2 = \{1, 2, \dots, n\} \\ \text{as ordered sets}}} 1 \otimes \lambda(\vec{\phi}_{I_1})(\lambda(\vec{\theta})\lambda(\vec{\psi})\lambda(\vec{\phi}_{I_2})\mathbf{a}_2 \otimes a_0 \otimes \mathbf{a}_1)$$

Then,

$$(B.5) \quad D(\mathcal{B}_{A_0, A_1, A_2}) = \Upsilon_{A_2 \bullet A_0, A_1} \circ \Upsilon_{A_0 \bullet A_1, A_2} - \Upsilon_{A_0, A_1 \bullet A_2}.$$

Proof. We will show that Equation B.5 holds by direct computation. Since all of the maps are maps of cofree comodules, we only need to check that π_1 (Equation B.5) holds where π_1 denotes projection of the comodule onto cogenerators. More explicitly, we want

to check that

$$\begin{aligned}
& \mathcal{B}_{n,m,p}(\tilde{\delta}(\vec{\phi})|\vec{\psi}|\vec{\theta}|\alpha) + \mathcal{B}_{n,m,p}(\vec{\phi}|\tilde{\delta}(\vec{\psi})|\vec{\theta}|\alpha) + \mathcal{B}_{n,m,p}(\vec{\phi}|\vec{\psi}|\tilde{\delta}(\vec{\theta})|\alpha) + \\
& \mathcal{B}_{n-1,m,p}(b'(\vec{\phi})|\vec{\psi}|\vec{\theta}|\alpha) + \mathcal{B}_{n,m-1,p}(\vec{\phi}|b'(\vec{\psi})|\vec{\theta}|\alpha) + \mathcal{B}_{n,m,p-1}(\vec{\phi}|\vec{\psi}|b'(\vec{\theta})|\alpha) + \\
& \mathcal{B}_{n,m,p}(\vec{\phi}|\vec{\psi}|\vec{\theta}|b(\alpha)) + b \circ \mathcal{B}_{n,m,p}(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha) + \\
& \mathcal{B}_{|I_1|,|J_1|,p-1}(\vec{\phi}_{I_1}|\vec{\psi}_{J_1}|\vec{\theta}_{\{1,\dots,p-1\}}|\theta_p\{\vec{\psi}_{J_2}\}\{\vec{\phi}_{I_2}\} \cdot \alpha) + \\
\text{(B.6)} \quad & \mathcal{B}_{|I_1|,m-1,p}(\vec{\phi}_{I_1}|\vec{\psi}_{\{1,\dots,m-1\}}|\vec{\theta}|\psi_m\{\vec{\phi}_{I_2}\} \cdot \alpha) + \mathcal{B}_{n-1,m,p}(\vec{\phi}_{\{1,\dots,n-1\}}|\vec{\psi}_m|\vec{\theta}|\phi_n \cdot \alpha) + \\
& \phi_1\{\vec{\theta}_{K_1}\}\{\vec{\psi}_{J_1}\} \cdot \mathcal{B}_{n-1,|J_2|,|K_2|}(\vec{\phi}_{\{2,\dots,n\}}|\vec{\psi}_{J_2}|\vec{\theta}_{K_2}|\alpha) + \\
& \theta_1\{\vec{\psi}_{J_1}\} \cdot \mathcal{B}_{n,|J_2|,p-1}(\vec{\phi}|\vec{\psi}_{J_2}|\vec{\theta}_{\{2,\dots,p\}}|\alpha) + \psi_1 \cdot \mathcal{B}_{n,m-1,p}(\vec{\phi}|\vec{\psi}_{\{2,\dots,m\}}|\vec{\theta}|\alpha) + \\
& v_{n,p \leq * \leq m+p}(\vec{\phi}|\vec{\psi} \bullet \vec{\theta}|\alpha) + \\
& v_{|I_1| \leq * \leq |I_1|+|K_1|,|J_1|}(\vec{\theta}_{K_1} \bullet \vec{\phi}_{I_1}, \vec{\psi}_{J_1}, v_{|J_2| \leq * \leq |I_2|+|J_2|,|K_2|}(\vec{\phi}_{I_2} \bullet \vec{\psi}_{J_2}|\vec{\theta}_{K_2}|\alpha)) \\
& = 0.
\end{aligned}$$

In Equation B.6 above, we call the terms in rows 1-3 the “standard terms” in the computation of $D(\mathcal{B}_{A_0,A_1,A_2})$, and the terms in rows 4-7 the “extra terms” in the computation of $D(\mathcal{B}_{A_0,A_1,A_2})$. The terms in rows 8-9 are π_1 of the righthand side of Equation B.5; we will call these the “8th- and 9th-row terms”.

We compute the sum of the standard terms. In Table B.4, the leftmost column lists the expressions that don’t cancel in the sum of the standard terms, the middle column gives the standard term from which the expression comes, and the rightmost column gives the term that cancels the expression. Table B.5 lists the remaining ninth row terms that aren’t already listed in Table B.4. In Table B.5, the left column lists the remaining

expressions that don't cancel in the ninth row, and the right column gives the extra term that cancels the expression.

All of the terms in the tables describing the expansion of Equation B.6 cancel, so we're done. □

Expression	Comes from Standard Term	Cancelling Term
$1 \otimes \lambda(\vec{\phi}_{I_1})[\lambda(\vec{\theta}_{\{1,\dots,p-1\}}\lambda(\vec{\psi}_{I_1})\lambda(\vec{\phi}_{I_2})\mathbf{a}_2) \otimes \theta_p(\lambda(\vec{\psi}_{I_2})\lambda(\vec{\phi}_{I_3})\mathbf{a}_3) \cdot a_0 \otimes \mathbf{a}_1]$	$b \circ \mathcal{B}_{n,m,p}(\vec{\phi} \vec{\psi} \vec{\theta} \alpha)$	$\mathcal{B}_{ I_1 , I_1 ,p-1}(\vec{\phi}_{I_1} \vec{\psi}_{I_1} \vec{\theta}_{\{1,\dots,p-1\}} \theta_p\{\vec{\psi}_{I_2}\}\{\vec{\phi}_{I_2}\} \cdot \alpha)$
$1 \otimes \lambda(\vec{\phi}_{I_1})[\lambda(\vec{\theta}\lambda(\vec{\psi}_{\{1,\dots,m-1\}})\lambda(\vec{\phi}_{I_2})\mathbf{a}_2) \otimes \psi_m(\lambda(\vec{\phi}_{I_3})\mathbf{a}_3) \cdot a_0 \otimes \mathbf{a}_1]$	$b \circ \mathcal{B}_{n,m,p}(\vec{\phi} \vec{\psi} \vec{\theta} \alpha)$	$\mathcal{B}_{ I_1 ,m-1,p}(\vec{\phi}_{I_1} \vec{\psi}_{\{1,\dots,m-1\}} \vec{\theta} \psi_m\{\vec{\phi}_{I_2}\} \cdot \alpha)$
$1 \otimes \lambda(\vec{\phi}_{I_1})[\lambda(\vec{\theta}\lambda(\vec{\psi}_{\{1,\dots,n-1\}})\mathbf{a}_2 \otimes \psi_n(\mathbf{a}_3) \cdot a_0 \otimes \mathbf{a}_1)]$	$b \circ \mathcal{B}_{n,m,p}(\vec{\phi} \vec{\psi} \vec{\theta} \alpha)$	$\mathcal{B}_{n-1,m,p}(\vec{\phi}_{\{1,\dots,n-1\}} \vec{\psi} \vec{\theta} \phi_n \cdot \alpha)$
$\phi_1(\vec{\lambda}(\vec{\theta}_{K_1})\lambda(\vec{\psi}_{I_1})\lambda(\vec{\phi}_{I_2})\mathbf{a}_2) \otimes \lambda(\vec{\phi}_{I_1 \setminus 1})[\lambda(\vec{\theta}_{K_2})\lambda(\vec{\psi}_{I_3})\lambda(\vec{\phi}_{I_3})\mathbf{a}_3 \otimes a_0 \otimes \mathbf{a}_1]$	$b \circ \mathcal{B}_{n,m,p}(\vec{\phi} \vec{\psi} \vec{\theta} \alpha)$	$\phi_1\{\vec{\theta}_{K_1}\}\{\vec{\psi}_{I_1}\} \cdot \mathcal{B}_{n-1, J_2 ,K_2}(\vec{\phi}_{\{2,\dots,n\}} \vec{\psi}_{I_2} \vec{\theta}_{K_2} \alpha)$
$f_0\theta_1(\lambda(\vec{\psi}_{I_1})\lambda(\vec{\phi}_{I_2})\mathbf{a}_2) \otimes \lambda(\vec{\phi}_{I_1})[\lambda(\vec{\theta}_{\{I_2,\dots,p\}}\lambda(\vec{\psi}_{I_2})\lambda(\vec{\phi}_{I_3})\mathbf{a}_3 \otimes a_0 \otimes \mathbf{a}_1)]$	$b \circ \mathcal{B}_{n,m,p}(\vec{\phi} \vec{\psi} \vec{\theta} \alpha)$	$\theta_1\{\vec{\psi}_{I_1}\} \cdot \mathcal{B}_{n, I_2 ,p-1}(\vec{\phi} \vec{\psi}_{I_2} \vec{\theta}_{\{2,\dots,p\}} \alpha)$
$f_0h_0\psi_1(\lambda(\vec{\phi}_{I_2})\mathbf{a}_2) \otimes \lambda(\vec{\phi}_{I_1})[\lambda(\vec{\theta})\lambda(\vec{\psi}_{\{2,\dots,m\}})\lambda(\vec{\phi}_{I_3})\mathbf{a}_3 \otimes a_0 \otimes \mathbf{a}_1]$	$b \circ \mathcal{B}_{n,m,p}(\vec{\phi} \vec{\psi} \vec{\theta} \alpha)$	$\psi_1 \cdot \mathcal{B}_{n,m-1,p}(\vec{\phi} \vec{\psi}_{\{2,\dots,m\}} \vec{\theta} \alpha)$
$f_0h_0g_0\phi_{i_1}(\lambda(\vec{\theta}_{K_2})\lambda(\vec{\psi}_{I_2})\lambda(\vec{\phi}_{I_3})\mathbf{a}_3 \otimes a_0 \otimes \mathbf{a}_1) \otimes \lambda(\vec{\phi}_{I_1})\lambda(\vec{\theta}_{K_1})\lambda(\vec{\psi}_{I_1})\lambda(\vec{\phi}_{I_2 \setminus i_1})\mathbf{a}_2$	$b \circ \mathcal{B}_{n,m,p}(\vec{\phi} \vec{\psi} \vec{\theta} \alpha)$	9 th row
$f_0h_0g_0f_{i_1}a_0 \otimes \lambda(\vec{\phi}_{I_1})\lambda(\vec{\theta})\lambda(\vec{\psi}_{I_1})\lambda(\vec{\phi}_{I_2})\mathbf{a}_1$	$b \circ \mathcal{B}_{n,m,p}(\vec{\phi} \vec{\psi} \vec{\theta} \alpha)$	9 th row
$\phi_1(\vec{\lambda}(\vec{\phi}_{I_1})\lambda(\vec{\theta})\lambda(\vec{\psi}_{I_1})\lambda(\vec{\phi}_{I_2})\mathbf{a}_3, a_0, \mathbf{a}_1) \otimes \lambda(\vec{\phi}_{I_1 \setminus 1})\mathbf{a}_2$	$b \circ \mathcal{B}_{n,m,p}(\vec{\phi} \vec{\psi} \vec{\theta} \alpha)$	8 th row

Table B.4. Expansion of “standard terms” in Equation B.6 and the terms that cancel them

Expression from ninth Row	Cancels with Extra Term
$\phi_1(\lambda(\vec{\theta}_{K_1})\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_2})[\lambda(\vec{\theta}_{K_3})\lambda(\vec{\psi}_{J_4})\lambda(\vec{\phi}_{I_5})\mathbf{a}_3, a_0, \mathbf{a}_1]) \otimes \lambda(\vec{\phi}_{I_1 \setminus 1})\lambda(\vec{\theta}_{K_2})\lambda(\vec{\psi}_{I_3})\lambda(\vec{\phi}_{I_4})\mathbf{a}_2$	$\phi_1\{\vec{\theta}_{K_1}\}\{\vec{\psi}_{J_1}\}$
$f_0\theta_1(\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_2})[\lambda(\vec{\theta}_{K_2})\lambda(\vec{\psi}_{J_3})\lambda(\vec{\phi}_{I_4})\mathbf{a}_3, a_0, \mathbf{a}_1]) \otimes \lambda(\vec{\phi}_{I_1})\lambda(\vec{\theta}_{K_1 \setminus 1})\lambda(\vec{\psi}_{I_2})\lambda(\vec{\phi}_{I_3})\mathbf{a}_2$	$\mathcal{B}_{n-1, J_2 , K_2 }(\vec{\phi}_{\{2, \dots, n\}} \vec{\psi}_{J_2} \vec{\theta}_{K_2} \alpha)$
$f_0h_0\psi_1(\lambda(\vec{\phi}_{I_2})[\lambda(\vec{\theta}_{K_2})\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_4})\mathbf{a}_3, a_0, \mathbf{a}_1]) \otimes \lambda(\vec{\phi}_{I_1})\lambda(\vec{\theta}_{K_1})\lambda(\vec{\psi}_{J_1 \setminus 1})\lambda(\vec{\phi}_{I_3})\mathbf{a}_2$	$\theta_1\{\vec{\psi}_{J_1}\} \cdot \mathcal{B}_{n, I_2 , p-1}(\vec{\phi} \vec{\psi}_{I_2} \vec{\theta}_{\{2, \dots, p\}} \alpha)$
	$\psi_1 \cdot \mathcal{B}_{n, m-1, p}(\vec{\phi} \vec{\psi}_{\{2, \dots, m\}} \vec{\theta} \alpha)$

Table B.5. Expansion of remaining “ninth row terms” in Equation B.6 and the “extra terms” that cancel them

Proposition B.5. *Let Υ and \mathcal{B} be as defined in the previous propositions. Then, $[\Upsilon, \mathcal{B}] := \Upsilon_{A_1 \bullet A_2, A_0} \mathcal{B}_{A_0, A_1, A_2} - \mathcal{B}_{A_2, A_0, A_1} \Upsilon_{A_0 \bullet A_1, A_2} = 0$. (Note that $[\Upsilon, \mathcal{B}]$ is a map from $C(A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_0)$ to itself.)*

Proof. We show the proposition by direct computation. Since all of the maps are maps of cofree comodules, we only need to check that $\pi_1([\Upsilon, \mathcal{B}]) = 0$ where π_1 denotes projection of the comodule onto cogenerators. We check this directly.

$$\begin{aligned} \pi_1 \circ \Upsilon_{A_1 \bullet A_2, A_0} \mathcal{B}_{A_0, A_1, A_2}(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha) &= v_{|K_1| \leq * \leq |K_1| + |J_1|, |I_1|}(\vec{\psi}_{J_1} \bullet \vec{\theta}_{K_1}|\vec{\phi}_{I_1}|\mathcal{B}_{|I_2|, |J_2|, |K_2|}(\vec{\phi}_{I_2}|\vec{\psi}_{J_2}|\vec{\theta}_{K_2}|\alpha)) \\ &= v_{|K_1| \leq * \leq |K_1| + |J_1|, |I_1|}(\vec{\psi}_{J_1} \bullet \vec{\theta}_{K_1}|\vec{\phi}_{I_1}|1 \otimes \lambda(\vec{\phi}_{I_2})[\lambda(\vec{\theta}_{K_2})\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_3})\mathbf{a}_2, a_0, \mathbf{a}_1]) \\ &= 1 \otimes \lambda(\vec{\theta}_{K_1})\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_1})[\lambda(\vec{\theta}_{K_2})\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_2})\mathbf{a}_2, a_0, \mathbf{a}_1] \end{aligned}$$

$$\begin{aligned} \pi_1 \circ \mathcal{B}_{A_2, A_0, A_1} \Upsilon_{A_0 \bullet A_1, A_2}(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha) &= B_{|K_1|, |I_1|, |J_1|}(\vec{\theta}_{K_1}|\vec{\phi}_{I_1}|\vec{\psi}_{J_1}|v_{|J_2| \leq * \leq |I_2| + |J_2|, |K_2|}(\vec{\phi}_{I_2} \bullet \vec{\psi}_{J_2}|\vec{\theta}_{K_2}|\alpha)) \\ &= 1 \otimes \lambda(\vec{\theta}_{K_1})\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_1})[\lambda(\vec{\theta}_{K_2})\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_2})\mathbf{a}_2, a_0, \mathbf{a}_1] \end{aligned}$$

It's clear that $\pi_1([\Upsilon, \mathcal{B}]) = 0$. □

APPENDIX C

Background on Hochschild chains and cochains

In this section, we give some known constructions on Hochschild chains and cochains for the reader's convenience.

C.1. Standard constructions and notation

Let k be a field of characteristic zero, A a flat unital k -algebra, and M be an A - A -bimodule. Then, we can take $(C_\bullet(A, M), b)$, the (reduced or standard) Hochschild chain complex of A with coefficients in M (see Reference [1], Equation 2.1). When $M = B$ is also an algebra over k with left and right module structure given by two maps of algebras $f : A \rightarrow B$ and $g : A \rightarrow B$, respectively, we may write ${}_f B_g$ to clarify the module structure.

Let k, A, M be as above. We can also take $(C^\bullet(A, M), \delta)$, the (reduced) Hochschild cochain complex of A with coefficients in M (see Reference [1], Equations 2.12-13, 2.19-21). When $M = B$ is an algebra, $(C^\bullet(A, B), \delta, \cup)$ is a dga where the cup product \cup is given in Reference [1], Equation 2.14.

Let $f, g, h : A \rightarrow A$ be maps of algebras. We have a contraction operation of Hochschild cochains and chains, which is a map of complexes:

$$\begin{aligned} \iota : C^p(A, {}_f A_g) \otimes C_{-q}(A, {}_g A_h) &\longrightarrow C_{-(q-p)}(A, {}_f A_h) \\ \phi \otimes a_0 \otimes \cdots \otimes a_q &\mapsto \iota(\phi, a_0 \otimes \cdots \otimes a_q) := \phi \cdot (a_0 \otimes \cdots \otimes a_q) := \\ &:= (-1)^{p(q+1)} \phi(a_{q-p+1}, \dots, a_q) \cdot a_0 \otimes a_1 \otimes \cdots \otimes a_{q-p}. \end{aligned}$$

Finally, we have a “Lie derivative like” operation of Hochschild cochains and chains. Fix an algebra A and let $(\phi_1 \dots \phi_n | \alpha) \in C(A \rightarrow A)(f_0)$ (see Equation 1.1) be the following element

$$(\phi_1 \dots \phi_n | \alpha) = \begin{array}{ccc} & f_0 & \\ & \searrow & \nearrow \\ & \Downarrow \phi_1 & \\ A & \xrightarrow{f_1} & A \\ & \vdots & \\ & f_{n-1} & \\ & \Downarrow \phi_n & \\ & f_n & \\ & \alpha & \\ & id & \end{array}$$

We have a map of complexes

$$C(A \rightarrow A)(f_0) \rightarrow C_{-\bullet}(A, f_0 A)$$

$$\begin{aligned}
(\phi_1 \dots \phi_n | a_1 \otimes \dots \otimes a_p) &\mapsto \lambda(\phi_1 \dots \phi_n) \cdot (a_1 \otimes \dots \otimes a_p) \\
&:= \sum_{0 \leq i_1 \leq \dots \leq i_{2n} \leq p} (-1)^{\sum_{j \geq 1} \substack{j \text{ odd} \\ i_j (\lfloor \phi_{i_{\frac{j+1}{2}}} \rfloor + 1)}} \cdot f_0 a_1 \otimes \dots \otimes f_0 a_{i_1} \otimes \phi_1(a_{i_1+1}, \dots, a_{i_2}) \otimes \\
&\quad \otimes f_1 a_{i_2+1} \otimes \dots \otimes f_1 a_{i_3} \otimes \phi_2(a_{i_3+1}, \dots, a_{i_4}) \otimes \\
&\quad \otimes \dots \otimes \phi_n(a_{i_{2n-1}+1}, \dots, a_{i_{2n}}) \otimes f_n a_{i_{2n}+1} \otimes \dots \otimes f_n a_p.
\end{aligned}$$

C.2. Brace operation on Hochschild cochains

Fix algebras A_0, A_1 and maps of algebras $f_0, f_n : A_0 \rightrightarrows A_1$, $A_0 \leftrightsquigarrow A_1 : g_0, g_m$. We will define a map of complexes called **braces**

$$\begin{aligned}
& C(A_0 \rightarrow A_1 \rightarrow A_0)((f_0, g_0), (f_n, g_m)) \\
&:= \left(\bigoplus_{\substack{i \in \mathbb{N} \\ f_1, \dots, f_i \text{ maps of algebras} \\ f_{i+1} = f_n}} C^\bullet(A_0, f_0 A_1 f_1) \otimes \dots \otimes C^\bullet(A_0, f_i A_1 f_{i+1}) \right) \otimes \\
&\quad \left(\bigoplus_{\substack{j \in \mathbb{N} \\ g_1, \dots, g_j \text{ maps of algebras} \\ g_{j+1} = g_m}} C^\bullet(A_1, g_0 A_0 g_1) \otimes \dots \otimes C^\bullet(A_1, g_j A_0 g_{j+1}) \right) \\
&\quad \downarrow - \bullet - \\
& C(A_0 \rightarrow A_0)(g_0 f_0, g_m f_n) \\
&:= \bigoplus_{\substack{i \in \mathbb{N} \\ h_1, \dots, h_i \text{ maps of algebras} \\ h_{i+1} = g_m f_n}} C^\bullet(A_0, g_0 f_0 A_0 h_1) \otimes \dots \otimes C^\bullet(A_0, h_i A_0 h_{i+1}).
\end{aligned}$$

First, for

$$(\phi_1 \dots \phi_n | 1) = \begin{array}{c} \begin{array}{ccc} & f_0 & \\ \text{---} \nearrow & & \searrow \text{---} \\ & \Downarrow \phi_1 & \\ \text{---} \nearrow & & \searrow \text{---} \\ & f_1 & \\ & \vdots & \\ & f_n & \end{array} & \begin{array}{ccc} & id & \\ \text{---} \nearrow & & \searrow \text{---} \\ & \Downarrow 1 & \\ \text{---} \nearrow & & \searrow \text{---} \\ & id & \end{array} \\ A_0 & A_0 & A_0 \end{array} \quad \text{and} \quad (1 | \phi_1 \dots \phi_n) = \begin{array}{c} \begin{array}{ccc} & f_0 & \\ \text{---} \nearrow & & \searrow \text{---} \\ & \Downarrow \phi_1 & \\ \text{---} \nearrow & & \searrow \text{---} \\ & f_1 & \\ & \vdots & \\ & f_n & \end{array} & \begin{array}{ccc} & id & \\ \text{---} \nearrow & & \searrow \text{---} \\ & \Downarrow 1 & \\ \text{---} \nearrow & & \searrow \text{---} \\ & id & \end{array} \\ A_0 & A_0 & A_0 \end{array},$$

define $(\phi_1 \dots \phi_n | 1) \xrightarrow{\bullet} (\phi_1 \dots \phi_n) \bullet 1 = (\phi_1 \dots \phi_n)$ and $(1 | \phi_1 \dots \phi_n) \xrightarrow{\bullet} 1 \bullet (\phi_1 \dots \phi_n) = (\phi_1 \dots \phi_n)$.

Then, for $n, m \geq 1$, let

$$(\phi_1 \dots \phi_n | \psi_1 \dots \psi_m) = \begin{array}{ccccc} & f_0 & & g_0 & \\ & \curvearrowright & & \curvearrowright & \\ & \Downarrow \phi_1 & & \Downarrow \psi_1 & \\ & f_1 & & g_1 & \\ (\phi_1 \dots \phi_n | \psi_1 \dots \psi_m) = & A_0 & & A_1 & A_0 \\ & \vdots & & \vdots & \\ & f_n & & g_m & \end{array}$$

and define $(\phi_1 \dots \phi_n) \bullet (\psi_1 \dots \psi_m) \in C(A_0 \rightarrow A_0)(g_0 f_0, g_m f_n)$ as follows:

$$\begin{aligned}
(\phi_1 \dots \phi_n) \bullet (\psi_1 \dots \psi_m) &= \sum_{0 \leq i_1 \leq \dots \leq i_{2m} \leq n} (-1)^{\sum_{T \geq 1} T_{\text{odd}} \left((\sum_{t \leq i_T} |\phi_t| + 1)(|\psi_{\frac{T+1}{2}}| + 1) \right)} \\
&\quad \cdot g_0 \phi_1 \otimes \dots \otimes g_0 \phi_{i_1} \otimes \psi_1 \{\phi_{i_1+1} \dots \phi_{i_2}\} \otimes \\
&\quad \otimes g_1 \phi_{i_2+1} \otimes \dots \otimes g_1 \phi_{i_3} \otimes \psi_2 \{\phi_{i_3+1} \dots \phi_{i_4}\} \otimes \\
&\quad \otimes \dots \otimes \psi_m \{\phi_{i_{2m-1}+1} \dots \phi_{i_{2m}}\} \otimes \\
&\quad \otimes g_m \phi_{i_{2m}+1} \otimes \dots \otimes g_m \phi_n \\
\psi_j \{\phi_{i_r+1} \dots \phi_{i_r+s}\} &\in C^\bullet(A_{0, g_{j-1} f_{i_r}} A_{0, g_j f_{i_r+s}}) \\
\psi_j \{\phi_{i_r+1} \dots \phi_{i_r+s}\} (a_1 \otimes \dots \otimes a_p) \\
&= \sum_{0 \leq k_1 \leq \dots \leq k_{2s} \leq p} (-1)^{\sum_{j \text{ odd}} k_j (|\phi_{i_r + \frac{j-1}{2}}| + 1)} \\
&\quad \cdot \psi_j \left(f_{i_r} a_1, \dots, f_{i_r} a_{k_1}, \phi_{i_r+1}(a_{k_1+1}, \dots, a_{k_2}), \right. \\
&\quad \quad f_{i_r+1} a_{k_2+1}, \dots, f_{i_r+1} a_{k_3}, \phi_{i_r+2}(a_{k_3+1}, \dots, a_{k_4}), \\
&\quad \quad , \dots, \phi_{i_r+s}(a_{k_{2s-1}}, \dots, a_{k_{2s}}), \\
&\quad \quad \left. f_{i_r+s} a_{k_{2s}+1}, \dots, f_{i_r+s} a_p \right).
\end{aligned}$$

We follow the sign convention in Reference [1] (Equation 2.25 and Section 4.7.2). In $(\phi_1 \dots \phi_n) \bullet (\psi_1 \dots \psi_m)$, moving ϕ_i past ψ_j introduces a factor of $(-1)^{(|\phi_i|+1)(|\psi_j|+1)}$. Braces are unital, associative maps of complexes (see Reference [1], Proposition 4.7.2). It's also straightforward to check that $\lambda(\psi_1 \dots \psi_m) \lambda(\phi_1 \dots \phi_n) = \lambda((\phi_1 \dots \phi_n) \bullet (\psi_1 \dots \psi_m))$.

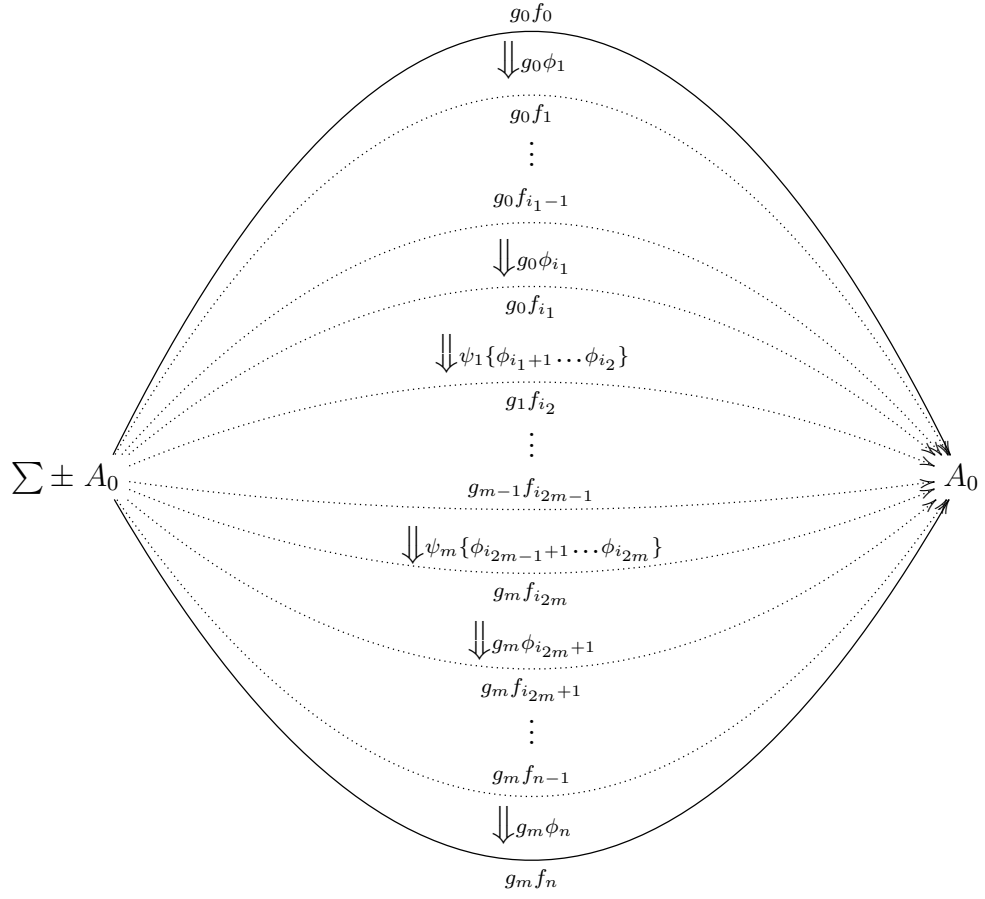


Figure C.1. Picture of the terms in $(\phi_1 \dots \phi_n) \bullet (\psi_1 \dots \psi_m)$