

Rebecca Wei

Northwestern University

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Outline

- **Question:** What do algebras form?
- **Answer 1:** A category in categories (HH^0)
- **Derived Answer 1:** A category in dg cocategories (Hochschild cochains...)
- **Answer 2:** A 2-category with a trace functor (HH_0)
- **Derived Answer 2:** A category in dg cocategories with a trace functor (Hochschild chains...)

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- **Derived Answer 2:** A category in dg cocategories with a trace functor (Hochschild chains...) **up to homotopy**

Answer 1: Algebras form a **2-category**.

- Objects: algebras A, B, \dots
- 1-Morphisms: bimodules ${}_A M_B$
- 1-Composition: ${}_A M_B \otimes_B {}_B N_C$
- 2-Morphisms: morphisms of bimodules

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$$\begin{aligned} \{\text{maps of bimodules } {}_f B \rightarrow_g B\} &\cong Z_A({}_g B_f) \cong HH^0(A, {}_g B_f) \\ M &\mapsto M(1) \\ (M_b : b' &\mapsto b \cdot b') \leftarrow b \end{aligned}$$

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Can we use Hochschild cohomology or cochains instead of HH^0 ?

Derived Answer 1: Algebras form a category in dg cocategories.

- Objects: algebras A, B, \dots
- Morphisms: a dg cocategory $Bar(Hoch(A, B))$
- Composition:
 - $: Bar(Hoch(A, B)) \otimes Bar(Hoch(B, C)) \rightarrow Bar(Hoch(A, C))$
associative map of dg cocategories

Defining $Bar(Hoch(A, B))$

- ① $Hoch(A, B)$ is a dg category with
- Objects: algebra maps $f : A \rightarrow B$
 - Morphisms: $Hoch(A)(f, g) = (C^\bullet(A, {}_f B_g), {}_f \delta_g)$
 - Composition: cup product on cochains

$$\phi \in C^p(A, {}_f B_g)$$

$$\psi \in C^q(A, {}_g B_h)$$

$$(\phi \cup \psi)(a_1, \dots, a_{p+q}) = \pm \phi(a_1, \dots, a_p) \psi(a_{p+1}, \dots, a_q)$$

Defining $\text{Bar}(\text{Hoch}(A, B))$

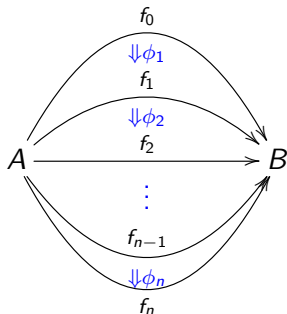
- ① $\text{Hoch}(A, B)$ is a dg category with
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- ② $\text{Bar} : \text{DGCat} \rightarrow \text{DGCocat}$

Defining $\text{Bar}(\text{Hoch}(A, B))$

- 1 $\text{Hoch}(A, B)$ is a dg category with
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- 2 $\text{Bar} : \text{DGCat} \rightarrow \text{DGCocat}$

$\text{Bar}(\text{Hoch}(A, B))$ has the same objects as $\text{Hoch}(A, B)$.



A morphism from f_0 to f_n in $\text{Bar}(\text{Hoch}(A, B))$

$$\Delta(\phi_1 \dots \phi_n) = \sum_{0 \leq i \leq n} \pm \phi_1 \dots \phi_i \otimes \phi_{i+1} \dots \phi_n$$

$$|\phi_1 \dots \phi_n| = \sum_{1 \leq i \leq n} |\phi_i| - n$$

$$d_{\text{Bar}(\text{Hoch}(A, B))} = \tilde{d}_{\text{Hoch}(A, B)} + d_{\cup}$$

Derived Answer 1: Algebra form a category in dg cocategories.

- Objects: algebras A, B, \dots
- Morphisms: a dg cocategory $Bar(Hoch(A, B))$
- Composition:
 - $: Bar(Hoch(A, B)) \otimes Bar(Hoch(B, C)) \rightarrow Bar(Hoch(A, C))$
associative map of dg cocategories

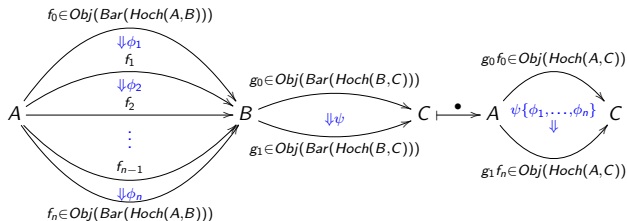
$\text{Bar}(\text{Hoch}(A, C))$ is the cofree dg cocategory on $\text{Hoch}(A, C)$:

$$\begin{array}{ccc} \text{Bar}(\text{Hoch}(A, B)) \otimes \text{Bar}(\text{Hoch}(B, C)) & \xrightarrow{\bullet} & \text{Hoch}(A, C) \\ & \searrow \bullet & \uparrow \\ & & \text{Bar}(\text{Hoch}(A, C)) \end{array}$$

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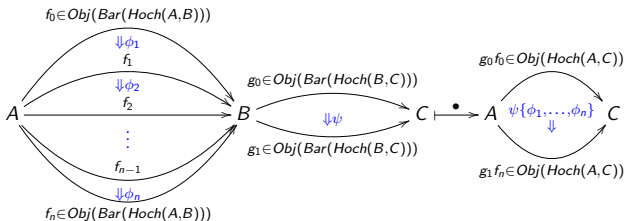
\bullet
 \nearrow
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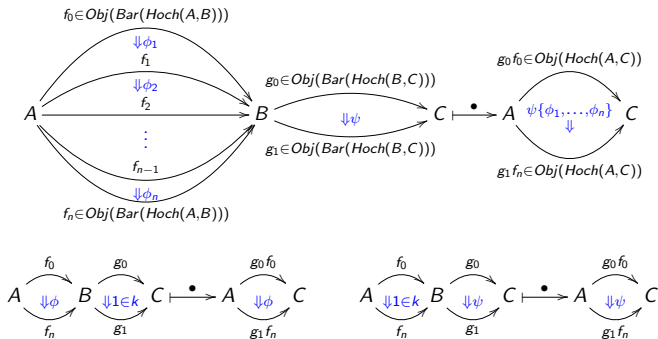


$$\psi\{\phi_1, \dots, \phi_n\}(a_1, \dots, a_q) = \sum \pm \psi(f_0 a_1, \dots, f_0 a_{i_1}, \phi_1(a_{i_1+1}, \dots), f_1 a_*, \dots, f_1 a_*, \\ \phi_2(a_*, \dots), f_2 a_*, \dots, f_n a_q)$$

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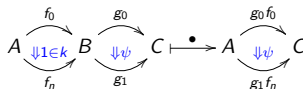
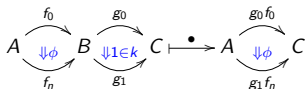
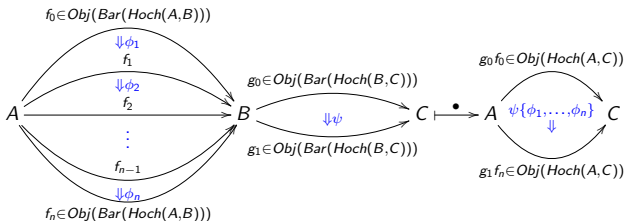
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Braces are associative. (...)

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Answer 2: Algebras form a 2-category with a trace functor

Definition

(Kaledin): A trace functor on a 2-category \mathcal{C} is:

- for each $A \in \text{Obj}(\mathcal{C})$, a functor $TR_A : \mathcal{C}(A, A) \rightarrow k - \text{mod}$
- for each pair $A, B \in \text{Obj}(\mathcal{C})$, a natural transformation $\tau_!(A, B)$

$$\begin{array}{ccc} \mathcal{C}(A, B) \otimes \mathcal{C}(B, A) & \xrightarrow{\tau = \text{flip}} & \mathcal{C}(B, A) \otimes \mathcal{C}(A, B) \\ \downarrow m & & \downarrow m \\ \mathcal{C}(A, A) & \xrightarrow{\tau_!(\vec{A}, B)} & \mathcal{C}(B, B) \\ \swarrow TR_A & & \nwarrow TR_B \\ & k - \text{mod} & \end{array}$$

- such that $\tau_!(B, A) \circ \tau_!(C, B) \circ \tau_!(A, C) = id$

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$$\begin{array}{ccccc}
 & & \mathcal{C}(C, A) \otimes \mathcal{C}(A, C) \otimes \mathcal{C}(B, C) & & \\
 & \nearrow \tau & \downarrow & \nwarrow \tau & \\
 \mathcal{C}(A, B) \otimes \mathcal{C}(B, C) \otimes \mathcal{C}(C, A) & \xleftarrow{\tau_!(\overleftarrow{B}, A)} & & \xrightarrow{\tau_!(\overrightarrow{C}, B)} & \mathcal{C}(B, C) \otimes \mathcal{C}(C, A) \otimes \mathcal{C}(A, B) \\
 & \searrow \tau_!(\overrightarrow{A}, C) & \downarrow & \swarrow \tau_!(\overleftarrow{C}, B) & \\
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Answer 2: Algebras form 2-category with a trace functor

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 $TR_A : \text{bimodule } {}_A M_A \mapsto M/[A, M] \cong HH_0(A, M)$

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 \downarrow TR_A & & \downarrow TR_B \\
 k - \text{mod} & & k - \text{mod}
 \end{array}$$

$\tau_!(A, B)$

$$\frac{{}_A M_B \otimes_B {}_B N_A}{[A, M \otimes_B N]} \xrightarrow{\tau_!(A, B)} \frac{{}_B N_A \otimes_A {}_A M_B}{[B, N \otimes_A M]}$$

$$m \otimes n \mapsto n \otimes m$$

such that

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Can we use Hochschild homology or chains instead of HH_0 to extend this to a trace functor on the category in dg cocategories?

Massaging the definition of a trace functor

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- for each $A \in \text{Obj}(\mathcal{C})$, a functor $TR_A : \mathcal{C}(A, A) \rightarrow k\text{-mod}$

$$\mathcal{C}(A, A)(f, g) \otimes_k TR_A(g) \rightarrow TR_A(f)$$

- for each pair $A, B \in \text{Obj}(\mathcal{C})$, a natural transformation $\tau_!(A, B)$

$$\begin{array}{ccc} \mathcal{C}(A, B) \otimes \mathcal{C}(B, A) & \xrightarrow{\tau} & \mathcal{C}(B, A) \otimes \mathcal{C}(A, B) \\ m \downarrow & & m \downarrow \\ \mathcal{C}(A, A) & \xrightarrow{\tau_!(A, B)} & \mathcal{C}(B, B) \\ TR_A \swarrow & & \nwarrow TR_B \\ & k\text{-mod} & \end{array}$$

- such that $\tau_!(B, A) \circ \tau_!(C, B) \circ \tau_!(A, C) = id$

Massaging the definition of a trace functor

Definition

(Kaledin): A trace functor on a **category in k -linear categories** \mathcal{C} is:

- for each $A \in \text{Obj}(\mathcal{C})$, a functor $TR_A : \mathcal{C}(A, A) \rightarrow k\text{-mod}$

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- such that $\tau_!(B, A) \circ \tau_!(C, B) \circ \tau_!(A, C) = id$

Massaging the definition of a trace functor

Definition

(Kaledin): A trace functor on a category in k -linear categories \mathcal{C} is:

- for each $A \in \text{Obj}(\mathcal{C})$, a left module $T(A)$ over $\mathcal{C}(A, A)$

$$\mathcal{C}(A, A)(f, g) \otimes_k TR_A(g) \rightarrow TR_A(f)$$

- for each pair $A, B \in \text{Obj}(\mathcal{C})$, a natural transformation $\tau_!(A, B)$

$$\begin{array}{ccc} \mathcal{C}(A, B) \otimes \mathcal{C}(B, A) & \xrightarrow{\tau} & \mathcal{C}(B, A) \otimes \mathcal{C}(A, B) \\ m \downarrow & & m \downarrow \\ \mathcal{C}(A, A) & \xrightarrow{\tau_!(A, B)} & \mathcal{C}(B, B) \\ TR_A \swarrow & & \nwarrow TR_B \\ & k\text{-mod} & \end{array}$$

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Massaging the definition of a trace functor

Definition

(Kaledin): A trace functor on a category in k -linear categories \mathcal{C} is:

- for each $A \in \text{Obj}(\mathcal{C})$, a left module TR_A over $\mathcal{C}(A, A)$

$$\mathcal{C}(A, A)(f, g) \otimes_k TR_A(g) \rightarrow TR_A(f)$$

- for each pair $A, B \in \text{Obj}(\mathcal{C})$, a map of modules $\tau_!(A, B) : m^* TR_A \rightarrow \tau^* m^* TR_B$ over $\mathcal{C}(A, B) \otimes \mathcal{C}(B, A)$

$$\begin{array}{ccc}
 \mathcal{C}(A, B) \otimes \mathcal{C}(B, A) & \xrightarrow{\tau} & \mathcal{C}(B, A) \otimes \mathcal{C}(A, B) \\
 \downarrow m & & \downarrow m \\
 \mathcal{C}(A, A) & \xrightarrow{\tau_!(A, B)} & \mathcal{C}(B, B) \\
 \downarrow TR_A & & \downarrow TR_B \\
 k\text{-mod} & & k\text{-mod}
 \end{array}$$

- such that $\tau_!(B, A) \circ \tau_!(C, B) \circ \tau_!(A, C) = id$

Massaging the definition of a trace functor

Definition

(Kaledin): A trace functor on a category in k -linear categories \mathcal{C} is:

- for each $A \in \text{Obj}(\mathcal{C})$, a left module TR_A over $\mathcal{C}(A, A)$

$$\mathcal{C}(A, A)(f, g) \otimes_k TR_A(g) \rightarrow TR_A(f)$$

- for each pair $A, B \in \text{Obj}(\mathcal{C})$, a map of modules $\tau_!(A, B) : m^* TR_A \rightarrow \tau^* m^* TR_B$ over $\mathcal{C}(A, B) \otimes \mathcal{C}(B, A)$

$$\begin{array}{ccc}
 \mathcal{C}(A, B) \otimes \mathcal{C}(B, A) & \xrightarrow{\tau} & \mathcal{C}(B, A) \otimes \mathcal{C}(A, B) \\
 m \downarrow & \tau_!(A, B) & \downarrow m \\
 \mathcal{C}(A, A) & & \mathcal{C}(B, B) \\
 TR_A \searrow & & \swarrow TR_B \\
 & k\text{-mod} &
 \end{array}$$

- such that $\tau^{*2} \tau_!(B, A) \circ \tau^* \tau_!(C, B) \circ \tau_!(A, C) = id$

Definition

Let \mathcal{C} be a category in k -linear categories. Let $\chi(\mathcal{C})$ be the k -linear category with

- Objects =
 $\{A \rightarrow A, A \rightarrow B \rightarrow A, A \rightarrow B \rightarrow C \rightarrow A : A, B, C \in \text{Obj}(\mathcal{C})\}$
- Morphisms = {linear combinations of compositions of rotations τ_i s.t. $\tau_i^{i+1} = id, i = 0, 1, 2$ }.

Definition

Let \mathcal{C} be a category in k -linear categories. Let $\chi(\mathcal{C})$ be the k -linear category with

- Objects =
 $\{A \rightarrow A, A \rightarrow B \rightarrow A, A \rightarrow B \rightarrow C \rightarrow A : A, B, C \in \text{Obj}(\mathcal{C})\}$
- Morphisms = $\{\text{linear combinations of compositions of rotations } \tau_i \text{ s.t. } \tau_i^{i+1} = \text{id}, i = 0, 1, 2\}.$

A trace functor on a category \mathcal{C} in k -linear categories gives a functor

$$\chi(\mathcal{C}) \rightarrow \mathcal{D} = \{(k\text{-linear category, module})\}$$

$$(A \rightarrow A) \mapsto (\mathcal{C}(A, A), T(A))$$

$$(A \rightarrow B \rightarrow A) \mapsto (\mathcal{C}(A, B) \otimes \mathcal{C}(B, A), m^* T(A))$$

$$(A \rightarrow B \rightarrow C \rightarrow A) \mapsto (\mathcal{C}(A, B) \otimes \mathcal{C}(B, C) \otimes \mathcal{C}(C, A), m^{*2} T(A))$$

$$\tau_1 : (A \rightarrow B \rightarrow A) \rightarrow (B \rightarrow A \rightarrow B) \mapsto \tau_1(A, B)$$

$$\tau_2 : (A \rightarrow B \rightarrow C \rightarrow A) \rightarrow (C \rightarrow A \rightarrow B \rightarrow C) \mapsto m^* \tau_1(A, C), \quad m : \mathcal{C}(A, B) \otimes \mathcal{C}(B, C) \rightarrow \mathcal{C}(A, C)$$

$$\tau_1^2 = \text{id}, \tau_2^3 = \text{id} \mapsto \text{relations in the definition of trace functor}$$

Definition

Let \mathcal{C} be a category in k -linear categories. Let $\chi(\mathcal{C})$ be the k -linear category with

- Objects =
 $\{A \rightarrow A, A \rightarrow B \rightarrow A, A \rightarrow B \rightarrow C \rightarrow A : A, B, C \in \text{Obj}(\mathcal{C})\}$
- Morphisms = {linear combinations of compositions of rotations τ_i s.t. $\tau_i^{i+1} = id, i = 0, 1, 2$ }. Why stop at $n=2$? What about δ, σ ?

A trace functor on a category \mathcal{C} in k -linear categories gives a functor

$$\chi(\mathcal{C}) \rightarrow \mathcal{D} = \{(k\text{-linear category, module})\}$$

$$(A \rightarrow A) \mapsto (\mathcal{C}(A, A), T(A))$$

$$(A \rightarrow B \rightarrow A) \mapsto (\mathcal{C}(A, B) \otimes \mathcal{C}(B, A), m^* T(A))$$

$$(A \rightarrow B \rightarrow C \rightarrow A) \mapsto (\mathcal{C}(A, B) \otimes \mathcal{C}(B, C) \otimes \mathcal{C}(C, A), m^{*2} T(A))$$

$$\tau_1 : (A \rightarrow B \rightarrow A) \rightarrow (B \rightarrow A \rightarrow B) \mapsto \tau_1(A, B)$$

$$\tau_2 : (A \rightarrow B \rightarrow C \rightarrow A) \rightarrow (C \rightarrow A \rightarrow B \rightarrow C) \mapsto m^* \tau_1(A, C), \quad m : \mathcal{C}(A, B) \otimes \mathcal{C}(B, C) \rightarrow \mathcal{C}(A, C)$$

$$\tau_1^2 = id, \tau_2^3 = id \mapsto \text{relations in the definition of trace functor}$$

Definition

Let \mathcal{C} be a category in dg cocategories. Let $\chi_\infty(\mathcal{C})$ be the dg category with

- Objects = $\{A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0 : A_i \in \text{Obj}(\mathcal{C}), n \geq 0\}$
- Morphisms = $\{\text{linear combinations of compositions of}$

rotations $\tau_n : \mathcal{A} \mapsto (A_n \rightarrow A_0 \rightarrow \dots \rightarrow A_n)$

coboundaries $\delta_{j,n} : \mathcal{A} \mapsto (A_0 \rightarrow \dots \rightarrow A_j \rightarrow A_{j+2 \pmod{n+1}} \rightarrow \dots \rightarrow A_0)$

codegeneracies $\sigma_{i,n} : \mathcal{A} \mapsto (A_0 \rightarrow \dots \rightarrow A_i \rightarrow A_i \rightarrow \dots \rightarrow A_0)$

where $\mathcal{A} := (A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$, subject to the cyclic relations $\}[0]$

Definition

Let \mathcal{D}_∞ be the dg category with

- Objects = $\{(\underset{B}{\text{dg cocategory}}, \underset{C}{\text{dg comodule}})\}$
- Morphisms:

$$\mathcal{D}_\infty^p((B_1, C_1), (B_0, C_0)) := \left\{ \begin{array}{l} F : B_1 \rightarrow B_0 \text{ dg functor,} \\ F_! : C_1 \rightarrow F^* C_0 \text{ degree-}p \text{ linear map} \end{array} \right\}$$
$$d_{\mathcal{D}_\infty}(F, F_!) = (F, [d, F_!] = d_{F^* C_0} \circ F_! \pm F_! \circ d_{C_1})$$

Definition

Let \mathcal{D}_∞ be the dg category with

- Objects = $\{(\text{dg cocategory}_{\underset{B}{B}}, \text{dg comodule}_{\underset{C}{C}})\}$
- Morphisms:

$$\mathcal{D}_\infty^p((B_1, C_1), (B_0, C_0)) := \left\{ \begin{array}{l} F : B_1 \rightarrow B_0 \text{ dg functor,} \\ F_! : C_1 \rightarrow F^*C_0 \text{ degree-}p \text{ linear map} \end{array} \right\}$$

$$d_{\mathcal{D}_\infty}(F, F_!) = (F, [d, F_!] = d_{F^*C_0} \circ F_! \pm F_! \circ d_{C_1})$$

For us, F^*C_0 is the categorified version of co-extension of scalars:

$$F^*C_0 = \ker(B_1 \otimes C_0 \rightrightarrows B_1 \otimes B_0 \otimes C_0)$$

Question: Can we give a dg functor

$$\chi_{\infty}(\mathcal{C}) \rightarrow \mathcal{D}_{\infty}$$

$$\text{where } (A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0) \mapsto \begin{pmatrix} B(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0) := \\ := \text{Bar}(\text{Hoch}(A_0, A_1)) \otimes \dots \otimes \text{Bar}(\text{Hoch}(A_n, A_0)), \\ C(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0) \end{pmatrix} ?$$

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Rest of this talk:

- Define dg comodules $C(A_0 \rightarrow \dots \rightarrow A_0)$ using Hochschild chains
- Describe the A_∞ -functor, and in particular the role of homotopies

Definition

A **dg comodule** C over a dg cocategory B consists of the following data:

- for each object $f \in B$, a complex $C^\bullet(f)$, and
- maps of complexes

$$\Delta_C(f) : C^\bullet(f) \rightarrow \prod_{g \in \text{Obj}(B)} B^\bullet(f, g) \otimes C^\bullet(g).$$

such that the following diagrams for coassociativity and counitality commute:

$$\begin{array}{ccc}
 C^\bullet(f) & \xrightarrow{\Delta_C} & \prod_{g \in \text{Obj}(B)} B^\bullet(f, g) \otimes C^\bullet(g) \\
 \Delta_C \downarrow & & \downarrow id_B \otimes \Delta_C \\
 \prod_{g \in \text{Obj}(B)} B^\bullet(f, g) \otimes C^\bullet(g) & \xrightarrow{\Delta_B \otimes id_C} & \prod_{g, g' \in \text{Obj}(B)} B^\bullet(f, g) \otimes B^\bullet(g, g') \otimes C^\bullet(g')
 \end{array}
 \qquad
 \begin{array}{ccc}
 C^\bullet(f) & \xrightarrow{\Delta_C} & \prod_{g \in \text{Obj}(B)} B^\bullet(f, g) \otimes C^\bullet(g) \\
 id \searrow & & \downarrow \epsilon_B \otimes id_C \\
 & & C^\bullet(f)
 \end{array}$$

Fix algebras A_0, \dots, A_n . Let $\mathcal{A} = (A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$
 Define a dg comodule $C(\mathcal{A})$ over $B(\mathcal{A})$:

$$C(\mathcal{A})^\bullet(\underbrace{A_0 \xrightarrow{f_{0,0}} \dots \rightarrow A_n \xrightarrow{f_{n,0}} A_0}_{\in \text{Obj}(B(\mathcal{A}))}) :=$$

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[illegible]

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$$:= \{ A_0 \xrightarrow[f_{0,k_0}]{f_{0,1}} A_1 \xrightarrow[\dots]{\vdots} A_n \xrightarrow[f_{n,k_n}]{f_{n,1}} A_0 = (\phi_{0,1} | \dots | \phi_{0,k_0}) \otimes \dots \otimes (\phi_{n,1} | \dots | \phi_{n,k_n}) \otimes \alpha : \\ s.t. \phi_{i,j} \in C^\bullet(A_i, f_{j-1} A_{i+1} f_j), \\ \alpha \in C_{-\bullet}(A_0, f_{n,k_n} \dots f_{0,k_0} A_0) \\ \text{Diagram labels: } f_{0,0}, \Downarrow \phi_{0,1}, f_{0,1}, \vdots, f_{0,k_0}, f_{n,0}, \Downarrow \phi_{n,1}, f_{n,1}, \vdots, f_{n,k_n}, id_{A_0}, \alpha$$

$$d_{C(A_0 \rightarrow \dots \rightarrow A_0)} = d_B \otimes id_{C_{-\bullet}} + id_B \otimes b + \tilde{t}$$

where $\tilde{\iota}$ is given as follows:

$$\begin{array}{ccc}
 C(A_0 \rightarrow \dots \rightarrow A_0)^\bullet(f_{0,0}, \dots, f_{n,0}) & \xrightarrow{\tilde{\iota}} & C_{-\bullet}(A_0, f_{n,0} \dots f_{0,0} A_0) \\
 & \searrow \tilde{\iota} \text{ extend as a coderivation} & \uparrow \\
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 \end{array}$$

$$\begin{aligned}
 \tilde{\iota}((\phi_{0,1} | \dots | \phi_{0,k_0}) \otimes \dots \otimes (\phi_{n,1} | \dots | \phi_{n,k_n}) \otimes \alpha) &= \iota(\phi_{0,1} | \dots | \phi_{0,k_0}) \bullet \dots \bullet (\phi_{n,1} | \dots | \phi_{n,k_n}) \alpha \\
 \iota_\phi(a_0 \otimes \dots \otimes a_p) &= \pm \phi(a_{d+1}, \dots, a_p) \cdot a_0 \otimes a_1 \otimes \dots \otimes a_d \quad \text{where } |\phi| = p - d
 \end{aligned}$$

Give an A_∞ -functor

$$\chi_\infty(\mathcal{C}) \rightarrow \mathcal{D}_\infty$$

$$\mathcal{A} := (A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0) \mapsto \begin{pmatrix} B(\mathcal{A}) \\ C(\mathcal{A}) \end{pmatrix}$$

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Generating Morphisms: $\sigma_{i,n} \mapsto \begin{pmatrix} B(\mathcal{A}) \xrightarrow{\hat{\sigma}_{i,n}} B(\sigma_{i,n}\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{id} \hat{\sigma}_{i,n}^* C(\sigma_{i,n}\mathcal{A}) \end{pmatrix}$

$$\delta_{j,n} \mapsto \begin{pmatrix} B(\mathcal{A}) \xrightarrow{\hat{\delta}_{j,n}} B(\delta_{j,n}\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{id} \hat{\delta}_{j,n}^* C(\delta_{j,n}\mathcal{A}) \end{pmatrix} \quad \tau_n \mapsto \begin{pmatrix} B(\mathcal{A}) \xrightarrow{\hat{\tau}_n} B(\tau_n\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{\tau_n!} \hat{\tau}_n^* C(\tau_n\mathcal{A}) \end{pmatrix}$$