Title

Subtitle

Rebecca Wei

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Date/Event

Braces, categorically

Fix an algebra, A.
Define a dg category, $Hoch(A)$:

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  Objects: algebra maps $f : A \to A$
  Morphisms: $Hoch(A)(f, g) = (C^\bullet(A, {}_fA_g), {}_f\delta_g)$

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$$\begin{aligned}
{}_f\delta_g(\phi)(a_1 \otimes \ldots \otimes a_n) =& \epsilon_\phi \Big(f(a_1) \cdot \phi(a_2, \ldots, a_n) + \\
& + \sum_{1 \le i \le n-1} (-1)^i \phi(a_1, \ldots, a_i a_{i+1}, \ldots, a_n) + \\
& + (-1)^n \phi(a_1, \ldots, a_{n-1}) \cdot g(a_n) \Big) \\
\epsilon_\phi =& (-1)^{|\phi|+1}
\end{aligned}$$

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Composition: cup product on cochains

$$
\begin{tikzcd}
A \arrow[r, bend left=60, "f"] \arrow[r, "g"'] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=3, "\Downarrow\phi_1"] \arrow[r, phantom, shift right=4, "\Downarrow\phi_2"] & A \arrow[r, squiggly, "compose"] & A \arrow[r, bend left=60, "f"] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, "\phi_1 \cup \phi_2 \Downarrow"] & A
\end{tikzcd}
$$

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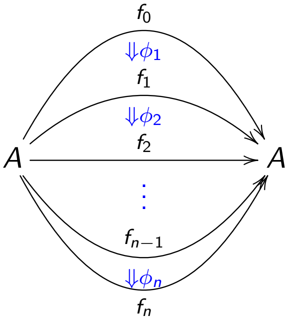

A morphism from f_0 to f_n in Bar(Hoch(A))

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$$\begin{tikzcd}[column sep=huge]
A \arrow[r, bend left=60, "f_0"] \arrow[r, bend left=30, "f_1"] \arrow[r, "f_2"] \arrow[r, bend right=30, "f_{n-1}"'] \arrow[r, bend right=60, "f_n"'] \arrow[r, phantom, shift left=6ex, "\Downarrow\phi_1"] \arrow[r, phantom, shift left=3ex, "\Downarrow\phi_2"] \arrow[r, phantom, shift right=3ex, "\vdots"] \arrow[r, phantom, shift right=6ex, "\Downarrow\phi_n"] & A \arrow[r, bend left, "g_0"] \arrow[r, bend right, "g_1"'] \arrow[r, phantom, "\Downarrow\psi"] & A \arrow[r, mapsto, "\bullet"] & A \arrow[r, bend left=50, "g_0 f_0"] \arrow[r, bend right=50, "g_1 f_n"'] \arrow[r, phantom, "\psi\{\phi_1,\dots,\phi_n\}", "\Downarrow"'] & A
\end{tikzcd}$$

Braces, categorically

Fix an algebra, A.
Define a dg category, $Hoch(A)$:

In this context, braces give multilinear maps:

$$\begin{tikzcd} Bar(Hoch(A)) \otimes Bar(Hoch(A)) \arrow[r, "\bullet"] \arrow[dr, dotted, "\bullet"'] & Hoch(A) \\ & Bar(Hoch(A)) \arrow[u, two heads] \end{tikzcd}$$

Then, (Bar(Hoch(A)), $\bullet$) is an algebra in DGCocats.

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But we have more...

More structure

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Fix algebras, $A_0, A_1, \ldots, A_n$.
We will define a dg cocategory $B(A_0 \to A_1 \to \ldots \to A_n \to A_0)$
where $B(A_0 \to A_0) := Bar(Hoch(A_0))$ for n=0.

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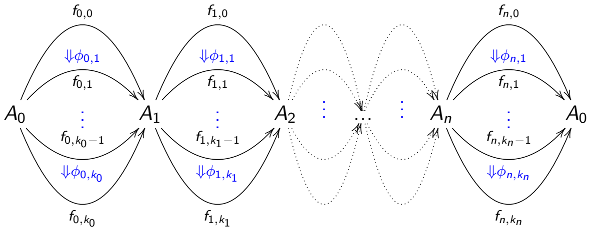
Objects: $\{A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \ldots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_0\}$

A morphism from $(A_0 \xrightarrow{f_{0,0}} \ldots \xrightarrow{f_{n,0}} A_0)$ to $(A_0 \xrightarrow{f_{0,k_0}} \ldots \xrightarrow{f_{n,k_n}} A_0)$:



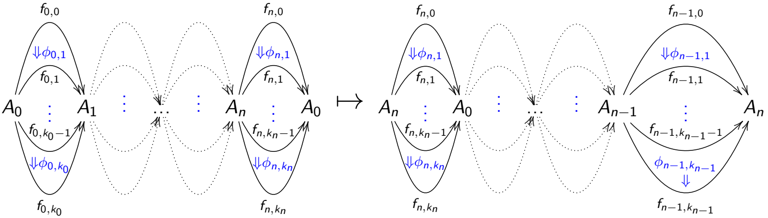
$\phi_{i,j} \in C^\bullet(A_{i,f_{i,j-1}}\, A_{i+1\,f_{i,j}})$

Example

We have a dg functor

$$\begin{aligned}
\hat{\tau}_n : B(A_0 \to \ldots \to A_n \to A_0) &\to B(A_n \to A_0 \to \ldots \to A_n) \\
Obj : \qquad (f_0, \ldots, f_n) &\mapsto (f_n, f_0, \ldots, f_{n-1})
\end{aligned}$$

Mor :



Structure among the $B(A_0 \to \dots \to A_n \to A_0)$'s

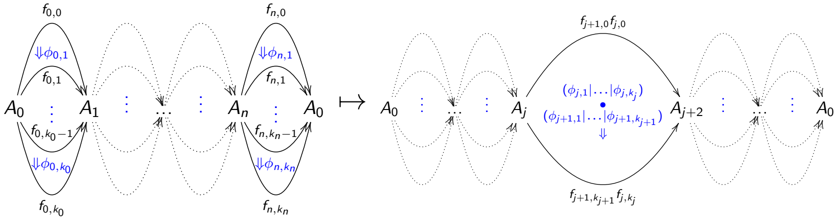
Example

For $n \geq 1, 0 \leq j < n$, we have a dg functor

$$\hat{\delta}_{j,n} : B(A_0 \to \dots \to A_n \to A_0) \to B\left(A_0 \to \dots \to A_j \to A_{\substack{j+2 \\ (mod\ n+1)}} \to \dots \to A_0\right)$$

$$Obj : \qquad (f_0, \dots, f_n) \mapsto (f_0, \dots, f_{j+1} f_j, \dots, f_n)$$

Mor :



Structure among the $B(A_0 \to \ldots \to A_n \to A_0)$'s

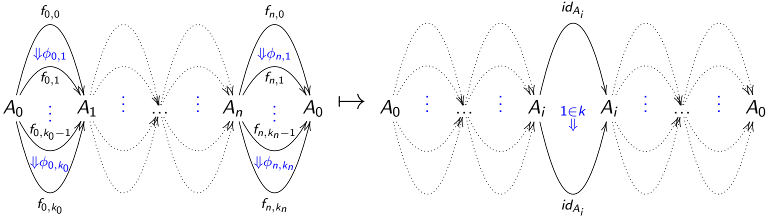
Example

For $n \geq 0, 0 \leq i \leq n$, we have a dg functor

$$\hat{\sigma}_{i,n} : B(A_0 \to \ldots \to A_n \to A_0) \to B(A_0 \to \ldots \to A_i \to A_i \to \ldots \to A_0)$$

$$Obj : \qquad (f_0, \ldots, f_n) \mapsto (f_0, \ldots, f_{i-1}, id_{A_i}, f_i, \ldots, f_n)$$

Mor :



A sheafy-cyclic object in DGCocat

Definition

Let χ be the category with objects $\{A_0 \to A_1 \to \ldots \to A_n \to A_0\}$ and morphisms compositions of

$$\begin{aligned} \textit{rotations } \tau_n &: \mathcal{A} \mapsto (A_n \to A_0 \to \ldots \to A_n) \\ \textit{coboundaries } \delta_{j,n} &: \mathcal{A} \mapsto (A_0 \to \ldots \to A_j \to A_{j+2\,(mod\; n+1)} \to \ldots \to A_0) \\ \textit{codegeneracies } \sigma_{i,n} &: \mathcal{A} \mapsto (A_0 \to \ldots \to A_i \to A_i \to \ldots \to A_0) \end{aligned}$$

where $\mathcal{A} := (A_0 \to \ldots \to A_n \to A_0)$, subject to the cyclic relations.

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where $\mathcal{A} := (A_0 \to \ldots \to A_n \to A_0)$, subject to the cyclic relations.

Proposition

We have a functor $\chi \to DGCocat$

$$\begin{aligned} \textit{Objects}: &\ (A_0 \to \ldots \to A_n \to A_0) \mapsto B(A_0 \to \ldots \to A_n \to A_0) \\ \textit{Generating morphisms}: &\ \lambda \mapsto \hat{\lambda} \end{aligned}$$

Motivating Question

Each dg cocategory $B(A_0 \to \ldots \to A_n \to A_0)$ has a dg comodule $C(A_0 \to \ldots \to A_n \to A_0)$ (which is a categorified version of the bar complex of Hochschild chains as a module over Hochschild cochains via contraction).

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Motivating Question: Can we extend the sheafy-cyclic structure to the comodules? In other words, can we give a functor

$$\chi \to \mathcal{D} := \{(dg\ cocat,\ dg\ comod)\}?$$

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$$\chi_\infty \to \mathcal{D}_\infty \quad \textit{dg categories}$$

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- Define the A_∞-functor $\mathcal{F}$
    - Define $C(A_0 \to \ldots \to A_n \to A_0)$

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- Define dg categories χ_∞ and $\mathcal{D}_\infty$
- Define the A_∞-functor $\mathcal{F}$
  - Define $C(A_0 \to \ldots \to A_n \to A_0)$
- Add-ons
  - Rectify $\mathcal{F}$ to a dg functor
  - Give a dg functor $\mathcal{D}_\infty \to \mathcal{E} = \{(\text{dg cat, dg mod})\}$

$$U(\chi_\infty) \xrightarrow{\textit{rectified}} \mathcal{D}_\infty \to \mathcal{E}$$

"A homotopically sheafy-cyclic object in dg categories with a dg module"

Dg categories χ_∞ and $\mathcal{D}_\infty$

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χ_∞:

Objects: same objects as $\chi = \{A_0 \to \ldots \to A_n \to A_0\}$

$\chi_\infty^\bullet(X, Y) := \{\sum a_i v_i : a_i \in k, v_i \in \chi(X, Y)\}$

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$\mathcal{D}_\infty$:

Objects: same objects as $\mathcal{D} = \{(\text{dg cocategory, dg comodule})\}$

$$\mathcal{D}^p_\infty((B_1, C_1), (B_0, C_0)) = \left\{ \begin{matrix} F : B_1 \to B_0 \textit{ dg functor} \\ F_! : C_1 \to F^* C_0 \textit{ degree-p linear map} \end{matrix} \right\}$$

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$F^* C_0$ is the categorified version of co-extension of scalars:

$$F^* C_0 = ker(B_1 \otimes C_0 \rightrightarrows B_1 \otimes B_0 \otimes C_0)$$