Title

Subtitle

Rebecca Wei

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Date/Event

Braces, categorically

Fix an algebra, A.
Define a dg category, $Hoch(A)$:

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Objects: algebra maps $f : A \to A$

Morphisms: $Hoch(A)(f, g) = (C^\bullet(A, {}_fA_g), {}_f\delta_g)$

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$$\begin{aligned} {}_f\delta_g(\phi)(a_1 \otimes \ldots \otimes a_n) =& \epsilon_\phi \Big(f(a_1) \cdot \phi(a_2, \ldots, a_n) + \\ & + \sum_{1 \le i \le n-1} (-1)^i \phi(a_1, \ldots, a_i a_{i+1}, \ldots, a_n) + \\ & + (-1)^n \phi(a_1, \ldots, a_{n-1}) \cdot g(a_n) \Big) \\ \epsilon_\phi =& (-1)^{|\phi|+1} \end{aligned}$$

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$$
\begin{tikzcd}
A \arrow[r, bend left=60, "f"] \arrow[r, "g"'] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\phi_1"] \arrow[r, phantom, shift right=3ex, "\Downarrow\phi_2"] & A \arrow[r, squiggly, "compose"] & A \arrow[r, bend left=60, "f"] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, "\phi_1 \cup \phi_2 \atop \Downarrow"] & A
\end{tikzcd}
$$

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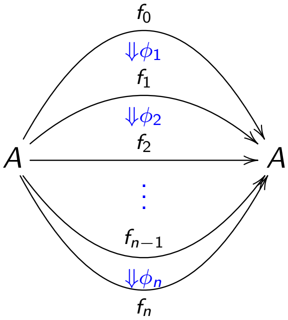

A morphism from f_0 to f_n in Bar(Hoch(A))

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Define a dg category, $Hoch(A)$:

$$Bar(Hoch(A)) \otimes Bar(Hoch(A)) \xrightarrow{\bullet} Hoch(A)$$

$$\begin{tikzcd}
A \arrow[r, bend left=70, "f_0"] \arrow[r, bend left=50, phantom, "\Downarrow\phi_1"] \arrow[r, bend left=35, "f_1"] \arrow[r, bend left=18, phantom, "\Downarrow\phi_2"] \arrow[r, "f_2"] \arrow[r, bend right=20, phantom, "\vdots"] \arrow[r, bend right=45, "f_{n-1}"] \arrow[r, bend right=55, phantom, "\Downarrow\phi_n"] \arrow[r, bend right=70, "f_n"'] & A \arrow[r, bend left, "g_0"] \arrow[r, phantom, "\Downarrow\psi"] \arrow[r, bend right, "g_1"'] & A \arrow[r, mapsto, "\bullet"] & A \arrow[r, bend left=50, "g_0 f_0"] \arrow[r, phantom, "\psi\{\phi_1,\dots,\phi_n\} \Downarrow"] \arrow[r, bend right=50, "g_1 f_n"'] & A
\end{tikzcd}$$

Braces, categorically

Fix an algebra, A.
Define a dg category, $Hoch(A)$:

In this context, braces give multilinear maps:

$$\begin{tikzcd} Bar(Hoch(A)) \otimes Bar(Hoch(A)) \arrow[r, "\bullet"] \arrow[dr, dotted, "\bullet"'] & Hoch(A) \\ & Bar(Hoch(A)) \arrow[u, two heads] \end{tikzcd}$$

Then, (Bar(Hoch(A)), $\bullet$) is an algebra in DGCocats.

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Then, (Bar(Hoch(A)), $\bullet$) is an algebra in DGCocats.
But we have more...

More structure

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Fix algebras, $A_0, A_1, \ldots, A_n$.
We will define a dg cocategory $B(A_0 \to A_1 \to \ldots \to A_n \to A_0)$
where $B(A_0 \to A_0) := Bar(Hoch(A_0))$ for n=0.

More structure

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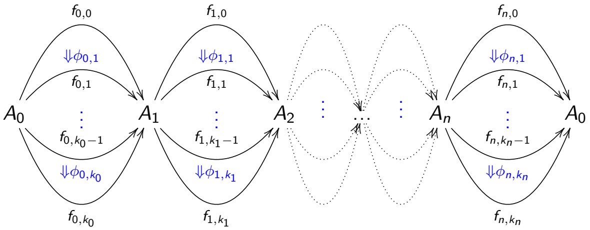
Objects: $\{A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \ldots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_0\}$
A morphism from $(A_0 \xrightarrow{f_{0,0}} \ldots \xrightarrow{f_{n,0}} A_0)$ to $(A_0 \xrightarrow{f_{0,k_0}} \ldots \xrightarrow{f_{n,k_n}} A_0)$:



$\phi_{i,j} \in C^\bullet(A_i{}_{f_{i,j-1}} A_{i+1}{}_{f_{i,j}})$

Structure among the $B(A_0 \to \ldots \to A_n \to A_0)$'s

Example

We have a dg functor

$$\hat{\tau}_n : B(A_0 \to \ldots \to A_n \to A_0) \to B(A_n \to A_0 \to \ldots \to A_n)$$

$$Obj : \qquad (f_0, \ldots, f_n) \mapsto (f_n, f_0, \ldots, f_{n-1})$$

$$Mor : \begin{tikzcd}[column sep=large] A_0 \arrow[r, bend left=70, "f_{0,0}"] \arrow[r, bend left=30, "f_{0,1}"] \arrow[r, bend right=30, "f_{0,k_0-1}"'] \arrow[r, bend right=70, "f_{0,k_0}"'] \arrow[r, phantom, shift left=4ex, "\Downarrow\phi_{0,1}"] \arrow[r, phantom, "\vdots"] \arrow[r, phantom, shift right=4ex, "\Downarrow\phi_{0,k_0}"] & A_1 \arrow[r, dotted, bend left] \arrow[r, dotted, bend right] \arrow[r, phantom, "\vdots"] & \cdots \arrow[r, dotted, bend left] \arrow[r, dotted, bend right] \arrow[r, phantom, "\vdots"] & A_n \arrow[r, bend left=70, "f_{n,0}"] \arrow[r, bend left=30, "f_{n,1}"] \arrow[r, bend right=30, "f_{n,k_n-1}"'] \arrow[r, bend right=70, "f_{n,k_n}"'] \arrow[r, phantom, shift left=4ex, "\Downarrow\phi_{n,1}"] \arrow[r, phantom, "\vdots"] \arrow[r, phantom, shift right=4ex, "\Downarrow\phi_{n,k_n}"] & A_0 \end{tikzcd} \mapsto \begin{tikzcd}[column sep=large] A_n \arrow[r, bend left=70, "f_{n,0}"] \arrow[r, bend left=30, "f_{n,1}"] \arrow[r, bend right=30, "f_{n,k_n-1}"'] \arrow[r, bend right=70, "f_{n,k_n}"'] \arrow[r, phantom, shift left=4ex, "\Downarrow\phi_{n,1}"] \arrow[r, phantom, "\vdots"] \arrow[r, phantom, shift right=4ex, "\Downarrow\phi_{n,k_n}"] & A_0 \arrow[r, dotted, bend left] \arrow[r, dotted, bend right] \arrow[r, phantom, "\vdots"] & \cdots \arrow[r, dotted, bend left] \arrow[r, dotted, bend right] \arrow[r, phantom, "\vdots"] & A_{n-1} \arrow[r, bend left=70, "f_{n-1,0}"] \arrow[r, bend left=30, "f_{n-1,1}"] \arrow[r, bend right=30, "f_{n-1,k_{n-1}-1}"'] \arrow[r, bend right=70, "f_{n-1,k_{n-1}}"'] \arrow[r, phantom, shift left=4ex, "\Downarrow\phi_{n-1,1}"] \arrow[r, phantom, "\vdots"] \arrow[r, phantom, shift right=4ex, "\phi_{n-1,k_{n-1}} \Downarrow"] & A_n \end{tikzcd}$$

Structure among the $B(A_0 \to \ldots \to A_n \to A_0)$'s

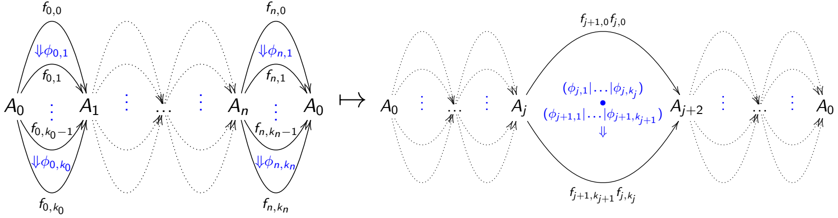
Example

For $n \geq 1, 0 \leq j < n$, we have a dg functor

$$\hat{\delta}_{j,n} : B(A_0 \to \ldots \to A_n \to A_0) \to B\left(A_0 \to \ldots \to A_j \to A_{\substack{j+2 \\ (mod\ n+1)}} \to \ldots \to A_0\right)$$

$$Obj: \qquad (f_0, \ldots, f_n) \mapsto (f_0, \ldots, f_{j+1}f_j, \ldots, f_n)$$

Mor :



Structure among the $B(A_0 \to \ldots \to A_n \to A_0)$'s

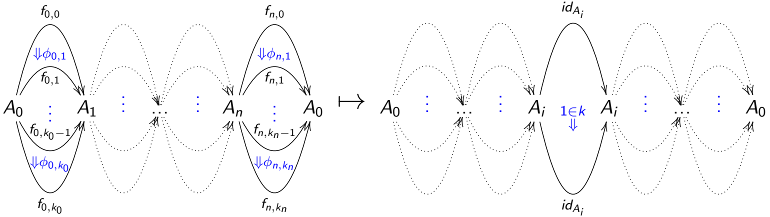
Example

For $n \geq 0, 0 \leq i \leq n$, we have a dg functor

$$\hat{\sigma}_{i,n} : B(A_0 \to \ldots \to A_n \to A_0) \to B(A_0 \to \ldots \to A_i \to A_i \to \ldots \to A_0)$$

$$Obj : \quad (f_0, \ldots, f_n) \mapsto (f_0, \ldots, f_{i-1}, id_{A_i}, f_i, \ldots, f_n)$$

Mor :



A sheafy-cyclic object in DGCocat

Definition

Let χ be the category with objects $\{A_0 \to A_1 \to \ldots \to A_n \to A_0\}$ and morphisms compositions of

$$\begin{aligned} \textit{rotations } \tau_n &: \mathcal{A} \mapsto (A_n \to A_0 \to \ldots \to A_n) \\ \textit{coboundaries } \delta_{j,n} &: \mathcal{A} \mapsto (A_0 \to \ldots \to A_j \to A_{j+2\,(mod\; n+1)} \to \ldots \to A_0) \\ \textit{codegeneracies } \sigma_{i,n} &: \mathcal{A} \mapsto (A_0 \to \ldots \to A_i \to A_i \to \ldots \to A_0) \end{aligned}$$

where $\mathcal{A} := (A_0 \to \ldots \to A_n \to A_0)$, subject to the cyclic relations.

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where $\mathcal{A} := (A_0 \to \ldots \to A_n \to A_0)$, subject to the cyclic relations.

Proposition

We have a functor $\chi \to DGCocat$

$$\begin{aligned} \textit{Objects}: &\ (A_0 \to \ldots \to A_n \to A_0) \mapsto B(A_0 \to \ldots \to A_n \to A_0) \\ \textit{Generating morphisms}: &\ \lambda \mapsto \hat{\lambda} \end{aligned}$$

Motivating Question

Each dg cocategory $B(A_0 \to \ldots \to A_n \to A_0)$ has a dg comodule $C(A_0 \to \ldots \to A_n \to A_0)$ (which is a categorified version of the bar complex of Hochschild chains as a module over Hochschild cochains via contraction).

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Motivating Question: Can we extend the sheafy-cyclic structure to the comodules? In other words, can we give a functor

$$\chi \to \mathcal{D} := \{(dg\ cocat,\ dg\ comod)\}?$$

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$$\chi_\infty \to \mathcal{D}_\infty \quad dg\ categories$$

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Rest of this talk: Describe our A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$.

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- Define dg categories χ_∞ and $\mathcal{D}_\infty$
- Define dg comodules $C(A_0 \to \ldots \to A_n \to A_0)$
- Define the A_∞-functor $\mathcal{F}$

Motivating Question

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- Define dg categories χ_∞ and $\mathcal{D}_\infty$
- Define dg comodules $C(A_0 \to \ldots \to A_n \to A_0)$
- Define the A_∞-functor $\mathcal{F}$

- Time and interest permitting
  - Rectify $\mathcal{F}$ to a dg functor
  - Give a dg functor $\mathcal{D}_\infty \to \mathcal{E} = \{(\text{dg cat, dg mod})\}$

$$U(\chi_\infty) \xrightarrow{\textit{rectified}} \mathcal{D}_\infty \to \mathcal{E}$$

"A homotopically sheafy-cyclic object in dg categories with a dg module"

Dg categories χ_∞ and $\mathcal{D}_\infty$

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χ_∞:

Objects: same objects as $\chi = \{A_0 \to \ldots \to A_n \to A_0\}$

$\chi^\bullet_\infty(X, Y) := \{\sum a_i v_i : a_i \in k, v_i \in \chi(X, Y)\}$

Dg categories χ_∞ and $\mathcal{D}_\infty$

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$\mathcal{D}_\infty$:

Objects: same objects as $\mathcal{D} = \{$(dg cocategory, dg comodule)$\}$

$$\mathcal{D}_\infty^p\big((B_1, C_1), (B_0, C_0)\big) := \left\{ \begin{array}{c} F : B_1 \to B_0 \textit{ dg functor} \\ F_! : C_1 \to F^* C_0 \textit{ degree-p linear map} \end{array} \right\}$$

Dg categories χ_∞ and $\mathcal{D}_\infty$

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$$\mathcal{D}^p_\infty((B_1, C_1), (B_0, C_0)) := \left\{ \begin{matrix} F : B_1 \to B_0 \text{ } dg \text{ } functor \\ F_! : C_1 \to F^* C_0 \text{ } degree\text{-}p \text{ } linear \text{ } map \end{matrix} \right\}$$

$F^* C_0$ is the categorified version of co-extension of scalars:

$$F^* C_0 = ker(B_1 \otimes C_0 \rightrightarrows B_1 \otimes B_0 \otimes C_0)$$

Dg comodules over dg cocategories

Definition

A **dg comodule** C over a dg cocategory B consists of the following data:

- for each object $f \in B$, a complex $C^\bullet(f)$, and
- maps of complexes

$$\Delta_C(f) : C^\bullet(f) \to \prod_{g \in Obj(B)} B^\bullet(f,g) \otimes C^\bullet(g).$$

such that the following diagrams for coassociativity and counitality commute:

$$\begin{tikzcd}
C^\bullet(f) \arrow[r, "\Delta_C"] \arrow[d, "\Delta_C"'] & \prod\limits_{g \in Obj(B)} B^\bullet(f,g) \otimes C^\bullet(g) \arrow[d, "id_B \otimes \Delta_C"] \\
\prod\limits_{g \in Obj(B)} B^\bullet(f,g) \otimes C^\bullet(g) \arrow[r, "\Delta_B \otimes id_C"] & \prod\limits_{\substack{g,g' \in \\ Obj(B)}} B^\bullet(f,g) \otimes B^\bullet(g,g') \otimes C^\bullet(g')
\end{tikzcd}
\qquad
\begin{tikzcd}
C^\bullet(f) \arrow[r, "\Delta_C"] \arrow[dr, "id"'] & \prod\limits_{g \in Obj(B)} B^\bullet(f,g) \otimes C^\bullet(g) \arrow[d, "\epsilon_B \otimes id_C"] \\
& C^\bullet(f)
\end{tikzcd}$$

Dg comodules over dg cocategories

Fix algebras $A_0, \ldots, A_n$.
Define a dg comodule over $B(A_0 \to \ldots \to A_n \to A_0)$:

$$C(A_0 \to \ldots \to A_n \to A_0)^{\bullet}(f_{0,0}, \ldots, f_{n,0}) :=$$

$$:= \left\{ \begin{tikzcd}[column sep=large] A_0 \arrow[r, bend left=70, "f_{0,0}"] \arrow[r, bend left=30, "f_{0,1}"'] \arrow[r, bend right=30, "f_{0,k_0}"'] \arrow[r, phantom, shift left=4ex, "\Downarrow\phi_{0,1}"] \arrow[r, phantom, "\vdots"] \arrow[rrrrr, bend right=40, "\alpha", "id_{A_0}"'] & A_1 \arrow[r, bend left=70, "f_{1,0}"] \arrow[r, bend left=30, "f_{1,1}"'] \arrow[r, bend right=30, "f_{1,k_1}"'] \arrow[r, phantom, shift left=4ex, "\Downarrow\phi_{1,1}"] \arrow[r, phantom, "\vdots"] & A_2 \arrow[r, dotted, bend left=70] \arrow[r, dotted, bend left=30] \arrow[r, dotted, bend right=30] \arrow[r, phantom, "\vdots"] & \cdots \arrow[r, dotted, bend left=70] \arrow[r, dotted, bend left=30] \arrow[r, dotted, bend right=30] \arrow[r, phantom, "\vdots"] & A_n \arrow[r, bend left=70, "f_{n,0}"] \arrow[r, bend left=30, "f_{n,1}"'] \arrow[r, bend right=30, "f_{n,k_n}"'] \arrow[r, phantom, shift left=4ex, "\Downarrow\phi_{n,1}"] \arrow[r, phantom, "\vdots"] & A_0 \end{tikzcd} \right\}$$

$$(\phi_{0,1}|\ldots|\phi_{0,k_0}) \otimes \ldots \otimes (\phi_{n,1}|\ldots|\phi_{n,k_n}) \otimes \alpha$$

$$\phi_{i,j} \in C^{\bullet}({A_i}_{f_{j-1}} {A_{i+1}}_{f_j}), \ \alpha \in C_{-\bullet}({A_0}_{f_n \cdots f_1 f_0} {A_0}_{id})$$