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# Outline

- **Question:** What do algebras form?

- **Answer 1:** A category in categories ($HH^0$)
- **Derived Answer 1:** A category in dg cocategories (Hochschild cochains...)

- **Answer 2:** A 2-category with a trace functor ($HH_0$)
- **Derived Answer 2:** A category in dg cocategories with a trace functor (Hochschild chains...)

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- **Answer 2:** A 2-category with a trace functor ($HH_0$)
- **Derived Answer 2:** A category in dg cocategories with a trace functor (Hochschild chains...) up to homotopy

**Answer 1:** Algebras form a **2-category**.

- Objects: algebras $A, B, \dots$
- 1-Morphisms: bimodules ${}_AM_B$
- 1-Composition: ${}_AM_B \otimes_B {}_BN_C$
- 2-Morphisms: morphisms of bimodules

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$$\begin{aligned} HH^0(A, {}_gB_f) \cong Z_A({}_gB_f) &\mapsto \{\text{maps of bimodules } {}_fB \to_g B\} \\ b &\mapsto (M_b : b' \mapsto b \cdot b') \end{aligned}$$

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Can we use Hochschild cohomology or cochains instead of $HH^0$?

**Derived Answer 1:** Algebra form a category in dg cocategories.

- Objects: algebras $A, B, \ldots$
- Morphisms: a dg cocategory $Bar(Hoch(A, B))$
- Composition:
  $\bullet : Bar(Hoch(A, B)) \otimes Bar(Hoch(B, C)) \to Bar(Hoch(A, C))$
  associative map of dg cocategories

## Defining $Bar(Hoch(A, B))$

1. $Hoch(A, B)$ is a dg category with
   - Objects: algebra maps $f : A \to B$
   - Morphisms: $Hoch(A)(f, g) = (C^\bullet(A, {}_fB_g), {}_f\delta_g)$
   - Composition: cup product on cochains

$$\begin{aligned} \phi &\in C^p(A,_f B_g) \\ \psi &\in C^q(A,_g B_h) \\ (\phi \cup \psi)(a_1, \ldots, a_{p+q}) &= \pm\phi(a_1, \ldots, a_p)\psi(a_{p+1}, \ldots, a_q) \end{aligned}$$

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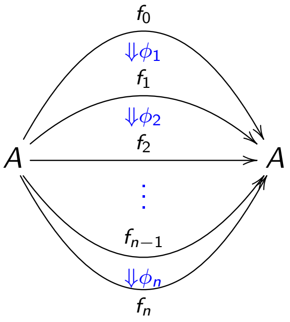
2. $Bar : DGCat \to DGCocat$
   Bar(Hoch(A)) has the same objects as Hoch(A).




A morphism from $f_0$ to $f_n$ in Bar(Hoch(A))

$$\Delta(\phi_1 \dots \phi_n) = \sum_{0 \le i \le n} \pm \phi_1 \dots \phi_i \otimes \phi_{i+1} \dots \phi_n$$

$$|\phi_1 \dots \phi_n| = \sum_{1 \le i \le n} |\phi_i| - n$$

$$d_{Bar(Hoch(A))} = \tilde{d}_{Hoch(A)} + d_\cup$$

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**Answer 2:** A 2-category with a trace functor

### Definition

(Kaledin): A <u>trace functor</u> on a 2-category $\mathcal{C}$ is:

- for each $A \in Obj(\mathcal{C})$, a functor $TR_A : \mathcal{C}(A, A) \to k - mod$
- for each pair $A, B \in Obj(\mathcal{C})$, a natural transformation $\tau_!(A, B)$

$$
\begin{tikzcd}
\mathcal{C}(A,B) \otimes \mathcal{C}(B,A) \arrow[rr, "\tau = flip"] \arrow[d, "m"'] & & \mathcal{C}(B,A) \otimes \mathcal{C}(A,B) \arrow[d, "m"] \\
\mathcal{C}(A,A) \arrow[dr, "TR_A"'] \arrow[rr, phantom, "\tau_!(\overrightarrow{A},B)"] & & \mathcal{C}(B,B) \arrow[dl, "TR_B"] \\
& k - mod &
\end{tikzcd}
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- such that $\tau_!(B, A) \circ \tau_!(C, B) \circ \tau_!(A, C) = id$

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 & \mathcal{C}(C,A)\otimes\mathcal{C}(A,C)\otimes\mathcal{C}(B,C) \arrow[dr, "\tau"] \arrow[dd] \arrow[d, phantom, "\tau_!(\overset{\Leftarrow}{B},A)"] & \\
\mathcal{C}(A,B)\otimes\mathcal{C}(B,C)\otimes\mathcal{C}(C,A) \arrow[ur, "\tau"] \arrow[dr] \arrow[r, phantom, "\tau_!(\overset{\Rightarrow}{A},C)"] & & \mathcal{C}(B,C)\otimes\mathcal{C}(C,A)\otimes\mathcal{C}(A,B) \arrow[ll, bend right, "\tau"'] \arrow[dl] \arrow[l, phantom, "\tau_!(\overset{\Rightarrow}{C},B)"] \\
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\end{tikzcd}$$

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  $TR_A : \text{bimodule } {}_AM_A \mapsto M/[A, M] = HH_0(A, M)$

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\end{tikzcd}$$

$${}_AM_B \otimes_B {}_BN_A/[A, M \otimes_B N] \xrightarrow{\tau_!(A,B)} {}_B N_A \otimes_A {}_AM_B/[B, N \otimes_A M]$$
$$m \otimes n \mapsto n \otimes m$$

such that $\tau_!(B,A) \circ \tau_!(C,B) \circ \tau_!(A,C) = id$

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- for each pair $A, B \in Obj(\mathcal{C})$, a map of modules $\tau_!(A, B) : m^* T(A) \to \tau^* m^* T(B)$ over $\mathcal{C}(A, B) \otimes \mathcal{C}(B, A)$

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\end{tikzcd}$$

- such that $\tau^{*2}\tau_!(B, A) \circ \tau^* \tau_!(C, B) \circ \tau_!(A, C) = id$

## Definition

Let $\mathcal{C}$ be a category in $k$-linear categories. Let $\chi(\mathcal{C})$ be the $k$-linear category with

- Objects $=$ $\{A \to A,\ A \to B \to A,\ A \to B \to C \to A : A, B, C \in Obj(\mathcal{C})\}$
- Morphisms $=$ $\{$linear combinations of compositions of rotations $\tau_i$ s.t. $\tau_i^{i+1} = id,\ i = 0, 1, 2\}$.

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A <u>trace functor</u> on $\mathcal{C}$ gives a functor

$$\chi(\mathcal{C}) \to \mathcal{D} = \{(k\text{-lin category, module})\}$$

$$\begin{aligned}
(A \to A) &\mapsto (\mathcal{C}(A, A), T(A)) \\
(A \to B \to A) &\mapsto (\mathcal{C}(A, B) \otimes \mathcal{C}(B, A), m^* T(A)) \\
(A \to B \to C \to A) &\mapsto (\mathcal{C}(A, B) \otimes \mathcal{C}(B, C) \otimes \mathcal{C}(C, A), m^{*2} T(A)) \\
\tau_1 : (A \to B \to A) \to (B \to A \to B) &\mapsto \tau_!(A, B) \\
\tau_2 : (A \to B \to C \to A) \to (C \to A \to B \to C) &\mapsto m^* \tau_!(A, C) \\
\tau_1^2 = id,\ \tau_2^3 = id &\mapsto \text{relations in definition of trace functor}
\end{aligned}$$

### Definition

Let $\mathcal{C}$ be a category in $k$-linear categories. Let $\chi(\mathcal{C})$ be the $k$-linear category with

- Objects $=$ $\{A \to A,\ A \to B \to A,\ A \to B \to C \to A : A, B, C \in Obj(\mathcal{C})\}$
- Morphisms $=$ {linear combinations of compositions of rotations $\tau_i$ s.t. $\tau_i^{i+1} = id,\ i = 0, 1, 2\}$. Why stop at n=2? What about $\delta$, $\sigma$?

A <u>trace functor</u> on $\mathcal{C}$ gives a functor

$$\chi(\mathcal{C}) \to \mathcal{D} = \{(k\text{-lin category, module})\}$$

$$(A \to A) \mapsto (\mathcal{C}(A, A), T(A))$$
$$(A \to B \to A) \mapsto (\mathcal{C}(A, B) \otimes \mathcal{C}(B, A), m^* T(A))$$
$$(A \to B \to C \to A) \mapsto (\mathcal{C}(A, B) \otimes \mathcal{C}(B, C) \otimes \mathcal{C}(C, A), m^{*2} T(A))$$
$$\tau_1 : (A \to B \to A) \to (B \to A \to B) \mapsto \tau_!(A, B)$$
$$\tau_2 : (A \to B \to C \to A) \to (C \to A \to B \to C) \mapsto m^* \tau_!(A, C)$$
$$\tau_1^2 = id,\ \tau_2^3 = id \mapsto \text{relations in definition of trace functor}$$

## Definition

Let $\mathcal{C}$ be a category in dg cocategories. Let $\chi_\infty(\mathcal{C})$ be the dg category with

- Objects $= \{A_0 \to \ldots \to A_n \to A_0 : A_i \in Obj(\mathcal{C}), n \geq 0\}$
- Morphisms $= \{$linear combinations of compositions of

$$\begin{aligned} \textit{rotations } \tau_n &: \mathcal{A} \mapsto (A_n \to A_0 \to \ldots \to A_n) \\ \textit{coboundaries } \delta_{j,n} &: \mathcal{A} \mapsto (A_0 \to \ldots \to A_j \to A_{j+2\,(mod\; n+1)} \to \ldots \to A_0) \\ \textit{codegeneracies } \sigma_{i,n} &: \mathcal{A} \mapsto (A_0 \to \ldots \to A_i \to A_i \to \ldots \to A_0) \end{aligned}$$

where $\mathcal{A} := (A_0 \to \ldots \to A_n \to A_0)$, subject to the cyclic relations$\}[0]$

## Definition

Let $\mathcal{D}_\infty$ be the dg category with

- Objects $= \{(\underset{B}{\text{dg cocategory}}, \underset{C}{\text{dg comodule}})\}$
- Morphisms:

$$\mathcal{D}^p_\infty\big((B_1, C_1), (B_0, C_0)\big) := \left\{ \begin{matrix} F : B_1 \to B_0 \textit{ dg functor}, \\ F_! : C_1 \to F^* C_0 \textit{ degree-p linear map} \end{matrix} \right\}$$

$$d_{D_\infty}(F, F_!) = (F, [d, F_!] = {\scriptstyle d_{F^*C_0} \circ F_! \pm F_! \circ d_{C_1}})$$

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$$F^* C_0 = ker(B_1 \otimes C_0 \rightrightarrows B_1 \otimes B_0 \otimes C_0)$$

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Can we give a dg functor $\chi(\mathcal{C}) \to \mathcal{D}_\infty$?

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No, but we can give an $A_\infty$-functor.