Title

Subtitle

Rebecca Wei

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Date/Event

Braces, categorically

Fix an algebra, A.
Define a dg category, $Hoch(A)$:

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Objects: algebra maps $f : A \to A$

Morphisms: $Hoch(A)(f, g) = (C^\bullet(A, {}_fA_g), {}_f\delta_g)$

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$$
\begin{aligned}
{}_f\delta_g(\phi)(a_1 \otimes \ldots \otimes a_n) =& \epsilon_\phi \Big(f(a_1) \cdot \phi(a_2, \ldots, a_n) + \\
& + \sum_{1 \leq i \leq n-1} (-1)^i \phi(a_1, \ldots, a_i a_{i+1}, \ldots, a_n) + \\
& + (-1)^n \phi(a_1, \ldots, a_{n-1}) \cdot g(a_n) \Big) \\
\epsilon_\phi =& (-1)^{|\phi|+1}
\end{aligned}
$$

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  Composition: cup product on cochains

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Composition: cup product on cochains

$$\begin{tikzcd}[column sep=large] A \arrow[r, bend left=70, "f"] \arrow[r, "g"'] \arrow[r, bend right=70, "h"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\phi_1"] \arrow[r, phantom, shift right=4ex, "\Downarrow\phi_2"] & A \arrow[r, squiggly, "compose"] & A \arrow[r, bend left=70, "f"] \arrow[r, bend right=70, "h"'] \arrow[r, phantom, "\phi_1 \cup \phi_2", "\Downarrow"'] & A \end{tikzcd}$$

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Fix an algebra, A.
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$$\begin{tikzcd}[column sep=huge]
A \arrow[r, bend left=70, "f_0"] \arrow[r, bend left=35, "f_1"] \arrow[r, "f_2"] \arrow[r, bend right=45, "f_{n-1}"] \arrow[r, bend right=70, "f_n"'] & A
\end{tikzcd}$$

$\Downarrow\phi_1$ $\Downarrow\phi_2$ $\vdots$ $\Downarrow\phi_n$

A morphism from f_0 to f_n in Bar(Hoch(A))

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\end{tikzcd}$$

Braces, categorically

Fix an algebra, A.
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In this context, braces give multilinear maps:

$$\begin{tikzcd} Bar(Hoch(A)) \otimes Bar(Hoch(A)) \arrow[r, "\bullet"] \arrow[dr, dotted, "\bullet"'] & Hoch(A) \\ & Bar(Hoch(A)) \arrow[u, two heads] \end{tikzcd}$$

Then, (Bar(Hoch(A)), $\bullet$) is an algebra in DGCocats.

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Then, (Bar(Hoch(A)), $\bullet$) is an algebra in DGCocats.
But we have more...

More structure

More structure

Fix algebras, $A_0, A_1, \dots, A_n$.
We will define a dg cocategory $B(A_0 \to A_1 \to \dots \to A_n \to A_0)$
where $B(A_0 \to A_0) := Bar(Hoch(A_0))$ for n=0.

More structure

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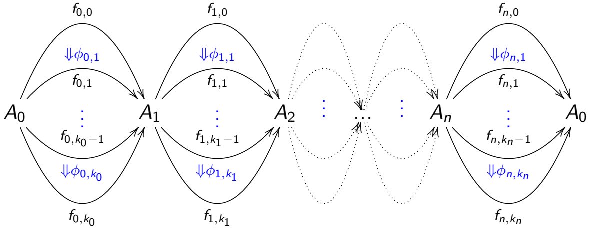
We will define a dg cocategory $B(A_0 \to A_1 \to \ldots \to A_n \to A_0)$
where $B(A_0 \to A_0) := Bar(Hoch(A_0))$ for n=0.

Objects: $\{A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \ldots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_0\}$
A morphism from $(A_0 \xrightarrow{f_{0,0}} \ldots \xrightarrow{f_{n,0}} A_0)$ to $(A_0 \xrightarrow{f_{0,k_0}} \ldots \xrightarrow{f_{n,k_n}} A_0)$:



$\phi_{i,j} \in C^\bullet(A_{i,f_{i,j-1}} A_{i+1\,f_{i,j}})$

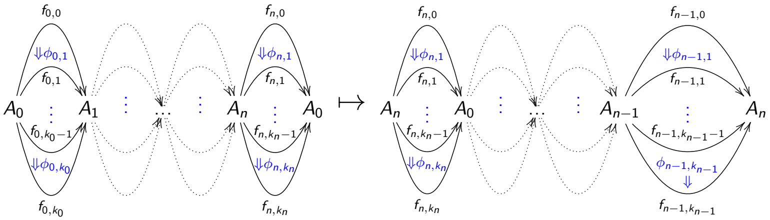
Example

We have a dg functor

$$\hat{\tau}_n : B(A_0 \to \ldots \to A_n \to A_0) \to B(A_n \to A_0 \to \ldots \to A_n)$$

$$Obj : \quad (f_0, \ldots, f_n) \mapsto (f_n, f_0, \ldots, f_{n-1})$$

Mor :



Structure among the $B(A_0 \to \ldots \to A_n \to A_0)$'s

Example

For $n \geq 1, 0 \leq j < n$, we have a dg functor

$$\hat{\delta}_{j,n} : B(A_0 \to \ldots \to A_n \to A_0) \to B\left(A_0 \to \ldots \to A_j \to A_{\substack{j+2 \\ (mod\ n+1)}} \to \ldots \to A_0\right)$$

$$Obj : \qquad (f_0, \ldots, f_n) \mapsto (f_0, \ldots, f_{j+1} f_j, \ldots, f_n)$$

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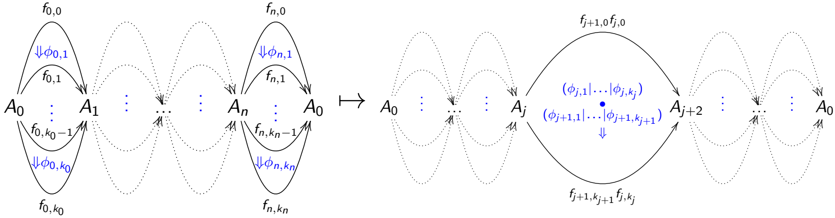


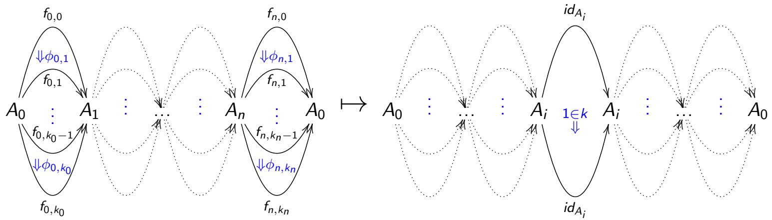
Structure among the $B(A_0 \to \ldots \to A_n \to A_0)$'s

Example

For $n \geq 0, 0 \leq i \leq n$, we have a dg functor

$$\begin{aligned}
\hat{\sigma}_{i,n} : B(A_0 \to \ldots \to A_n \to A_0) &\to B\big(A_0 \to \ldots \to A_i \to A_i \to \ldots \to A_0\big) \\
Obj : \qquad (f_0, \ldots, f_n) &\mapsto (f_0, \ldots, f_{i-1}, id_{A_i}, f_i, \ldots, f_n)
\end{aligned}$$

Mor :



A sheafy-cyclic object in DGCocat

Definition

Let χ be the category with objects $\{A_0 \to A_1 \to \ldots \to A_n \to A_0\}$ and morphisms compositions of

$$\begin{aligned} \textit{rotations } \tau_n &: \mathcal{A} \mapsto (A_n \to A_0 \to \ldots \to A_n) \\ \textit{coboundaries } \delta_{j,n} &: \mathcal{A} \mapsto (A_0 \to \ldots \to A_j \to A_{j+2\,(mod\ n+1)} \to \ldots \to A_0) \\ \textit{codegeneracies } \sigma_{i,n} &: \mathcal{A} \mapsto (A_0 \to \ldots \to A_i \to A_i \to \ldots \to A_0) \end{aligned}$$

where $\mathcal{A} := (A_0 \to \ldots \to A_n \to A_0)$, subject to the cyclic relations.

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where $\mathcal{A} := (A_0 \to \ldots \to A_n \to A_0)$, subject to the cyclic relations.

Proposition

We have a functor $\chi \to DGCocat$

$$\begin{aligned} Objects: \; (A_0 \to \ldots \to A_n \to A_0) &\mapsto B(A_0 \to \ldots \to A_n \to A_0) \\ Generating\; morphisms: \; \lambda &\mapsto \hat{\lambda} \end{aligned}$$

Motivating Question

Each dg cocategory $B(A_0 \to \ldots \to A_n \to A_0)$ has a dg comodule $C(A_0 \to \ldots \to A_n \to A_0)$ (which is a categorified version of the bar complex of Hochschild chains as a module over Hochschild cochains via contraction).

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Motivating Question: Can we extend the sheafy-cyclic structure to the comodules? In other words, can we give a functor

$$\chi \to \mathcal{D} := \{(dg\ cocat,\ dg\ comod)\}?$$

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$$\chi_\infty \to \mathcal{D}_\infty \quad dg\ categories$$

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Motivating Question: Can we extend the sheafy-cyclic structure to the comodules? In other words, can we give an A_∞-functor

$$\cancel{\chi \to \mathcal{D} := \{(dg\ cocat, dg\ comod)\}}?$$

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Rest of this talk: Describe our A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$.

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- Define dg categories χ_∞ and $\mathcal{D}_\infty$
- Define dg comodules $C(A_0 \to \ldots \to A_n \to A_0)$
- Define the A_∞-functor $\mathcal{F}$

Motivating Question

Rest of this talk: Describe our A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$.

- Define dg categories χ_∞ and $\mathcal{D}_\infty$
- Define dg comodules $C(A_0 \to \ldots \to A_n \to A_0)$
- Define the A_∞-functor $\mathcal{F}$
- Time and interest permitting
  - Rectify $\mathcal{F}$ to a dg functor
  - Give a dg functor $\mathcal{D}_\infty \to \mathcal{E} = \{(\text{dg cat, dg mod})\}$

$$U(\chi_\infty) \xrightarrow{\textit{rectified}} \mathcal{D}_\infty \to \mathcal{E}$$

"A homotopically sheafy-cyclic object in dg categories with a dg module"

Dg categories χ_∞ and $\mathcal{D}_\infty$

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χ_∞:

Objects: same objects as $\chi = \{A_0 \to \dots \to A_n \to A_0\}$

$\chi_\infty^\bullet(X, Y) := \{\sum a_i v_i : a_i \in k, v_i \in \chi(X, Y)\}$

Dg categories χ_∞ and $\mathcal{D}_\infty$

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$\mathcal{D}_\infty$:

Objects: same objects as $\mathcal{D} = \{$(dg cocategory, dg comodule)$\}$

$$\mathcal{D}_\infty^p((B_1, C_1), (B_0, C_0)) := \left\{ \begin{array}{c} F : B_1 \to B_0 \textit{ dg functor} \\ F_! : C_1 \to F^* C_0 \textit{ degree-}p \textit{ linear map} \end{array} \right\}$$

Dg categories χ_∞ and $\mathcal{D}_\infty$

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$$\mathcal{D}_\infty^p\big((B_1, C_1), (B_0, C_0)\big) := \left\{ \begin{matrix} F : B_1 \to B_0 \text{ } dg \text{ } functor \\ F_! : C_1 \to F^* C_0 \text{ } degree\text{-}p \text{ } linear \text{ } map \end{matrix} \right\}$$

$F^* C_0$ is the categorified version of co-extension of scalars:

$$F^* C_0 = ker(B_1 \otimes C_0 \rightrightarrows B_1 \otimes B_0 \otimes C_0)$$

Dg comodules over dg cocategories

Definition

A **dg comodule** C over a dg cocategory B consists of the following data:

- for each object $f \in B$, a complex $C^\bullet(f)$, and
- maps of complexes

$$\Delta_C(f) : C^\bullet(f) \to \prod_{g \in Obj(B)} B^\bullet(f,g) \otimes C^\bullet(g).$$

such that the following diagrams for coassociativity and counitality commute:

$$\begin{tikzcd} C^\bullet(f) \arrow[r, "\Delta_C"] \arrow[d, "\Delta_C"'] & \prod_{g \in Obj(B)} B^\bullet(f,g) \otimes C^\bullet(g) \arrow[d, "id_B \otimes \Delta_C"] \\ \prod_{g \in Obj(B)} B^\bullet(f,g) \otimes C^\bullet(g) \arrow[r, "\Delta_B \otimes id_C"] & \prod_{\substack{g,g' \in \\ Obj(B)}} B^\bullet(f,g) \otimes B^\bullet(g,g') \otimes C^\bullet(g') \end{tikzcd}$$

$$\begin{tikzcd} C^\bullet(f) \arrow[r, "\Delta_C"] \arrow[dr, "id"'] & \prod_{g \in Obj(B)} B^\bullet(f,g) \otimes C^\bullet(g) \arrow[d, "\epsilon_B \otimes id_C"] \\ & C^\bullet(f) \end{tikzcd}$$

Dg comodules over dg cocategories

Fix algebras $A_0, \dots, A_n$.
Define a dg comodule over $B(A_0 \to \dots \to A_n \to A_0)$:

$$C(A_0 \to \dots \to A_n \to A_0)^\bullet(f_{0,0}, \dots, f_{n,0}) :=$$

$$:= \left\{ \begin{tikzcd}[column sep=large] A_0 \arrow[r, bend left=70, "f_{0,0}"] \arrow[r, bend left=30, "f_{0,1}"'] \arrow[r, bend right=30, "f_{0,k_0}"'] \arrow[r, phantom, "\vdots"] \arrow[r, phantom, shift left=4ex, "\Downarrow\phi_{0,1}"] \arrow[rrrr, bend right=40, "id_{A_0}"'] \arrow[rrrr, phantom, shift right=6ex, "\alpha"] & A_1 \arrow[r, dotted, bend left=70] \arrow[r, dotted, bend left=30] \arrow[r, dotted, bend right=30] \arrow[r, phantom, "\vdots"] & \cdots \arrow[r, dotted, bend left=70] \arrow[r, dotted, bend left=30] \arrow[r, dotted, bend right=30] \arrow[r, phantom, "\vdots"] & A_n \arrow[r, bend left=70, "f_{n,0}"] \arrow[r, bend left=30, "f_{n,1}"'] \arrow[r, bend right=30, "f_{n,k_n}"'] \arrow[r, phantom, "\vdots"] \arrow[r, phantom, shift left=4ex, "\Downarrow\phi_{n,1}"] & A_0 \end{tikzcd} = \begin{array}{c} (\phi_{0,1}|\dots|\phi_{0,k_0}) \otimes \dots \otimes (\phi_{n,1}|\dots|\phi_{n,k_n}) \otimes \alpha : \\ s.t.\ \phi_{i,j} \in C^\bullet(A_i, {}_{f_{j-1}}A_{i+1}{}_{f_j}), \\ \alpha \in C_{-\bullet}(A_0, {}_{f_{n,k_n} \dots f_{0,k_0}}A_0) \end{array} \right\}$$

$$d_{C(A_0 \to \dots \to A_0)} = d_B \otimes id_{C_{-\bullet}} + id_B \otimes b + \tilde{\iota}$$

Dg comodules over dg cocategories

where $\tilde{\iota}$ is given as follows:

$$\begin{tikzcd} C(A_0 \to \ldots \to A_0)^\bullet(f_{0,0}, \ldots, f_{n,0}) \arrow[r, "\tilde{\iota}"] \arrow[dr, "\tilde{\iota}"'] & C_{-\bullet}(A_0 {}_{f_{n,0}} \ldots {}_{f_{0,0}} A_0) \\ & C(A_0 \to \ldots \to A_0)^\bullet(f_{0,0}, \ldots, f_{n,0}) \arrow[u, two heads] \end{tikzcd}$$

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$$\tilde{\iota}\big((\phi_{0,1}|\ldots|\phi_{0,k_0})\otimes\ldots\otimes(\phi_{n,1}|\ldots|\phi_{n,k_n})\otimes\alpha\big) = \iota_{(\phi_{0,1}|\ldots|\phi_{0,k_0})\bullet\ldots\bullet(\phi_{n,1}|\ldots|\phi_{n,k_n})}\alpha$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

$$\mathcal{A} := (A_0 \to \ldots \to A_n \to A_0)$$

$$\chi_\infty \to \mathcal{D}_\infty$$

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$$\begin{aligned} \chi_\infty &\to \mathcal{D}_\infty \\ \mathcal{A} &\mapsto (B(\mathcal{A}), C(\mathcal{A})) \end{aligned}$$

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Generating Morphisms:

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

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$$\begin{aligned} \chi_\infty &\to \mathcal{D}_\infty \\ \mathcal{A} &\mapsto (B(\mathcal{A}), C(\mathcal{A})) \\ \text{Generating Morphisms:} \quad \sigma_{i,n} &\mapsto \begin{pmatrix} B(\mathcal{A}) \xrightarrow{\hat{\sigma}_{i,n}} B(\sigma_{i,n}\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{id} \hat{\sigma}^*_{i,n} C(\sigma_{i,n}\mathcal{A}) \end{pmatrix} \end{aligned}$$

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Generating Morphisms: $\sigma_{i,n} \mapsto \begin{pmatrix} B(\mathcal{A}) \xrightarrow{\hat{\sigma}_{i,n}} B(\sigma_{i,n}\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{id} \hat{\sigma}^*_{i,n} C(\sigma_{i,n}\mathcal{A}) \end{pmatrix}$

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$$\delta_{j,n} \mapsto \begin{pmatrix} B(\mathcal{A}) \xrightarrow{\hat{\delta}_{j,n}} B(\delta_{j,n}\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{id} \hat{\delta}^*_{j,n} C(\delta_{j,n}\mathcal{A}) \end{pmatrix} \quad \tau_n \mapsto \begin{pmatrix} B(\mathcal{A}) \xrightarrow{\hat{\tau}_n} B(\tau_n\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{\tau_{n!}} \hat{\tau}^*_n C(\tau_n\mathcal{A}) \end{pmatrix}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

$$\hat{\tau}_n^* C(\tau_n \mathcal{A})^\bullet(\underbrace{f_0, \ldots, f_n}_{\in Obj(B(\mathcal{A}))}) \cong C(\tau_n \mathcal{A})^\bullet(f_n, f_0, \ldots, f_{n-1}) \text{ as complexes.}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

$$\hat{\tau}_n^* C(\tau_n\mathcal{A})^\bullet(\underbrace{f_0, \ldots, f_n}_{\in Obj(B(\mathcal{A}))}) \cong C(\tau_n\mathcal{A})^\bullet(f_n, f_0, \ldots, f_{n-1}) \text{ as complexes.}$$

$$\cong \left\{ \begin{tikzcd} A_n \arrow[r, bend left=70, "f_{n,0}"] \arrow[r, bend left=40, "f_{n,1}"] \arrow[r, phantom, bend left=55, "\Downarrow\phi_{n,1}"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=30, "f_{n,k_n}"'] \arrow[rrrr, bend right=40, "id_{A_n}"'] \arrow[rrrr, phantom, bend right=30, "\alpha"] & A_0 \arrow[r, dotted, bend left=70] \arrow[r, dotted, bend left=40] \arrow[r, phantom, "\vdots"] \arrow[r, dotted, bend right=30] & \cdots \arrow[r, dotted, bend left=70] \arrow[r, dotted, bend left=40] \arrow[r, phantom, "\vdots"] \arrow[r, dotted, bend right=30] & A_{n-1} \arrow[r, bend left=70, "f_{n-1,0}"] \arrow[r, bend left=40, "f_{n-1,1}"] \arrow[r, phantom, bend left=55, "\Downarrow\phi_{n-1,1}"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=30, "f_{n-1,k_{n-1}}"'] & A_n \end{tikzcd} \right\}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

$$\hat{\tau}_n^* C(\tau_n \mathcal{A})^\bullet(\underbrace{f_0, \ldots, f_n}_{\in Obj(B(\mathcal{A}))}) \cong C(\tau_n \mathcal{A})^\bullet(f_n, f_0, \ldots, f_{n-1}) \text{ as complexes.}$$

$$n = 1 : \quad C(A_0 \to A_1 \to A_0)^\bullet(f, g) \xrightarrow{\tau_{1!}} C(A_1 \to A_0 \to A_1)^\bullet(g, f)$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

$$\hat{\tau}_n^* C(\tau_n \mathcal{A})^\bullet(\underbrace{f_0, \ldots, f_n}_{\in Obj(B(\mathcal{A}))}) \cong C(\tau_n \mathcal{A})^\bullet(f_n, f_0, \ldots, f_{n-1}) \text{ as complexes.}$$

$$n = 1 : \quad C(A_0 \to A_1 \to A_0)^\bullet(f, g) \xrightarrow{\tau_{1!}} C(A_1 \to A_0 \to A_1)^\bullet(g, f)$$

$$\begin{tikzcd}[column sep=large]
A_0 \arrow[r, bend left=80, "f_{0,0}"] \arrow[r, bend left=40, "f_{0,1}"] \arrow[r, phantom, bend left=60, "\Downarrow\phi_{0,1}"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=30, "f_{0,k_0}"'] \arrow[rr, bend right=50, "id_{A_0}"'] \arrow[rr, phantom, bend right=30, "\alpha"] & A_1 \arrow[r, bend left=80, "f_{1,0}"] \arrow[r, bend left=40, "f_{1,1}"] \arrow[r, phantom, bend left=60, "\Downarrow\phi_{1,1}"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=30, "f_{1,k_1}"'] & A_0
\end{tikzcd}
\mapsto \sum
\begin{tikzcd}[column sep=large]
A_1 \arrow[r, bend left=80, "f_{1,0}"] \arrow[r, bend left=40, "f_{1,1}"] \arrow[r, phantom, bend left=60, "\Downarrow\phi'_{1,1}"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=30, "f_{1,k_1}"'] \arrow[rr, bend right=50, "id_{A_1}"'] \arrow[rr, phantom, bend right=30, "\alpha'"] & A_0 \arrow[r, bend left=80, "f_{0,0}"] \arrow[r, bend left=40, "f_{0,1}"] \arrow[r, phantom, bend left=60, "\Downarrow\phi'_{0,1}"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=30, "f_{0,k_0}"'] & A_1
\end{tikzcd}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

$$\hat{\tau}_n^* C(\tau_n \mathcal{A})^\bullet(\underbrace{f_0, \ldots, f_n}_{\in Obj(B(\mathcal{A}))}) \cong C(\tau_n \mathcal{A})^\bullet(f_n, f_0, \ldots, f_{n-1}) \text{ as complexes.}$$

$$n = 1: \quad C(A_0 \to A_1 \to A_0)^\bullet(f, g) \xrightarrow{\tau_{1!}} C(A_1 \to A_0 \to A_1)^\bullet(g, f)$$

$$\begin{tikzcd}
A_0 \arrow[r, bend left=80, "f_{0,0}"] \arrow[r, bend left=40, "f_{0,1}"'] \arrow[r, phantom, bend left=60, "\Downarrow\phi_{0,1}"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=30, "f_{0,k_0}"] \arrow[rr, bend right=50, "id_{A_0}"'] \arrow[rr, phantom, bend right=35, "\alpha"] & A_1 \arrow[r, bend left=80, "f_{1,0}"] \arrow[r, bend left=40, "f_{1,1}"'] \arrow[r, phantom, bend left=60, "\Downarrow\phi_{1,1}"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=30, "f_{1,k_1}"] & A_0
\end{tikzcd} \mapsto \sum \begin{tikzcd}
A_1 \arrow[r, "f_{1,0}"] \arrow[rr, bend right=50, "id_{A_1}"'] \arrow[rr, phantom, bend right=35, "\alpha'"] & A_0 \arrow[r, "f_{0,0}"] & A_1
\end{tikzcd}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

Solution: Give these maps to cogenerators to define $\tau_{1!}$, then let

$$\begin{aligned}
\tau_{n!} : C(\mathcal{A}) &\cong \hat{\delta}_0^{*\,n-1} C\big(A_0 \to A_n \to A_0\big) \xrightarrow{\hat{\delta}_0^{*\,n-1}\tau_{1!}} \hat{\delta}_0^{*\,n-1}\hat{\tau}_1^{*} C\big(A_n \to A_0 \to A_n\big) \cong \\
&\cong \big(\widehat{\tau_1 \delta_0^{n-1}}\big)^{*} C\big(A_n \to A_0 \to A_n\big) \cong \big(\widehat{\delta_0^{n-1}\tau_n}\big)^{*} C\big(A_n \to A_0 \to A_n\big) \cong \\
&\cong \hat{\tau}_n^{*}\hat{\delta}_0^{*\,n-1} C\big(A_n \to A_0 \to A_n\big) \cong \hat{\tau}_n^{*} C(\tau_n \mathcal{A}).
\end{aligned}$$

$$n = 1 : \qquad C(A_0 \to A_1 \to A_0)^\bullet(f, g) \xrightarrow{\tau_{1!}} C(A_1 \to A_0 \to A_1)^\bullet(g, f)$$

$$\begin{tikzcd}
A_0 \arrow[r, bend left=70, "f_{0,0}"] \arrow[r, phantom, bend left=45, "\Downarrow\phi_{0,1}"] \arrow[r, bend left=30, "f_{0,1}"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=20, "f_{0,k_0}"] \arrow[rr, bend right=50, "\alpha", "id_{A_0}"'] & A_1 \arrow[r, bend left=70, "f_{1,0}"] \arrow[r, phantom, bend left=45, "\Downarrow\phi_{1,1}"] \arrow[r, bend left=30, "f_{1,1}"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=20, "f_{1,k_1}"] & A_0
\end{tikzcd}
\mapsto \sum
\begin{tikzcd}
A_1 \arrow[r, "f_{1,0}"] \arrow[rr, bend right=50, "\alpha'", "id_{A_1}"'] & A_0 \arrow[r, "f_{0,0}"] & A_1
\end{tikzcd}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

Defining $\tau_{1!}$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

Defining $\tau_{1!}$

$$\begin{tikzcd} A_0 \arrow[r, "f_{0,0}"] \arrow[rr, bend right, "id_{A_0}"', "\alpha"] & A_1 \arrow[r, "f_{1,0}"] & A_0 \end{tikzcd} \overset{?}{\mapsto} \sum \begin{tikzcd} A_1 \arrow[r, "f_{1,0}"] \arrow[rr, bend right, "id_{A_1}"', "\alpha'"] & A_0 \arrow[r, "f_{0,0}"] & A_1 \end{tikzcd}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

Defining $\tau_{1!}$

$$\begin{tikzcd} A_0 \arrow[r, "f_{0,0}"] \arrow[rr, bend right, "id_{A_0}"'] & A_1 \arrow[r, "f_{1,0}"] \arrow[d, phantom, "\color{blue}\alpha"] & A_0 \end{tikzcd} \overset{?}{\mapsto} \sum \begin{tikzcd} A_1 \arrow[r, "f_{1,0}"] \arrow[rr, bend right, "id_{A_1}"'] & A_0 \arrow[r, "f_{0,0}"] \arrow[d, phantom, "\color{blue}\alpha'"] & A_1 \end{tikzcd}$$

$$\alpha = a_0 \otimes \ldots \otimes a_n \mapsto \alpha' = f_{0,0}(a_0) \otimes \ldots \otimes f_{0,0}(a_n)$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

Defining $\tau_{1!}$

$$\begin{tikzcd}
A_0 \arrow[r, bend left=60, "f_{0,0}"] \arrow[r, phantom, bend left=30, "\Downarrow\phi"] \arrow[r, "f_{0,1}"] \arrow[rr, bend right=40, "\alpha", "id_{A_0}"'] & A_1 \arrow[r, "f_{1,0}"] & A_0 \arrow[r, mapsto, "?"] & \sum A_1 \arrow[r, "f_{1,0}"] \arrow[rr, bend right=40, "\alpha'", "id_{A_1}"'] & A_0 \arrow[r, "f_{0,0}"] & A_1
\end{tikzcd}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

Defining $\tau_{1!}$

$$\begin{tikzcd}
A_0 \arrow[r, bend left=60, "f_{0,0}"] \arrow[r, phantom, shift left=3, "\Downarrow\phi"] \arrow[r, "f_{0,1}"] \arrow[rr, bend right=50, "id_{A_0}"'] \arrow[rr, phantom, shift right=4, "\alpha"] & A_1 \arrow[r, "f_{1,0}"] & A_0 \arrow[r, mapsto, "?"] & \sum A_1 \arrow[r, "f_{1,0}"] \arrow[rr, bend right=50, "id_{A_1}"'] \arrow[rr, phantom, shift right=4, "\alpha'"] & A_0 \arrow[r, "f_{0,0}"] & A_1
\end{tikzcd}$$

$$\bar{\tau}_{1!} \circ d(\phi \otimes \alpha) = d \circ \bar{\tau}_{1!}(\phi \otimes \alpha)$$

Defining $\tau_{1!}$

$$\begin{tikzcd} A_0 \arrow[r, bend left=70, "f_{0,0}"] \arrow[r, "f_{0,1}"] \arrow[r, phantom, bend left=35, "\color{blue}\Downarrow\phi"] \arrow[rr, bend right=50, "id_{A_0}"'] \arrow[rr, phantom, bend right=30, "\color{blue}\alpha"] & A_1 \arrow[r, "f_{1,0}"] & A_0 \end{tikzcd} \overset{?}{\mapsto} \sum \begin{tikzcd} A_1 \arrow[r, "f_{1,0}"] \arrow[rr, bend right=50, "id_{A_1}"'] \arrow[rr, phantom, bend right=30, "\color{blue}\alpha'"] & A_0 \arrow[r, "f_{0,0}"] & A_1 \end{tikzcd}$$

$$[b, \bar{\tau}_{1!}](\phi \otimes \alpha) \pm \bar{\tau}_{1!}(\delta\phi \otimes \alpha) = [\bar{\tau}_{1!}, \iota_\phi](\alpha)$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

Defining $\tau_{1!}$

$$\begin{tikzcd}
A_0 \arrow[r, "f_{0,1}"] \arrow[r, bend left=60, "f_{0,0}"] \arrow[r, phantom, shift left=3, "\Downarrow\phi"] \arrow[rr, bend right=50, "id_{A_0}"'] \arrow[rr, phantom, shift right=3, "\alpha"] & A_1 \arrow[r, "f_{1,0}"] & A_0
\end{tikzcd}
\overset{?}{\mapsto} \sum
\begin{tikzcd}
A_1 \arrow[r, "f_{1,0}"] \arrow[rr, bend right=50, "id_{A_1}"'] \arrow[rr, phantom, shift right=3, "\alpha'"] & A_0 \arrow[r, "f_{0,0}"] & A_1
\end{tikzcd}$$

$$[b, \bar{\tau}_{1!}](\phi \otimes \alpha) \pm \bar{\tau}_{1!}(\delta\phi \otimes \alpha) = [\bar{\tau}_{1!}, \iota_\phi](\alpha)$$

$$L_\phi(\alpha) = \sum_{k \geq 1} \pm a_0 \otimes \ldots \otimes \phi(a_k, \ldots) \otimes a_r \otimes \ldots \otimes a_n +$$

$$\sum \pm \phi(a_k, \ldots, a_n, a_0, \ldots) \otimes a_s \otimes \ldots \otimes a_{k-1}$$

$$[b, L_\phi] \pm L_{\delta\phi} = 0$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

Defining $\tau_{1!}$

$$
\begin{tikzcd}
A_0 \arrow[r, bend left=60, "f_{0,0}"] \arrow[r, "f_{0,1}"'] \arrow[r, phantom, bend left=30, "\Downarrow\phi"] \arrow[rr, bend right=40, "\alpha", "id_{A_0}"'] & A_1 \arrow[r, "f_{1,0}"] & A_0
\end{tikzcd}
\overset{?}{\mapsto} \sum
\begin{tikzcd}
A_1 \arrow[r, "f_{1,0}"] \arrow[rr, bend right=40, "\alpha'", "id_{A_1}"'] & A_0 \arrow[r, "f_{0,0}"] & A_1
\end{tikzcd}
$$

$$[b, \bar{\tau}_{1!}](\phi \otimes \alpha) \pm \bar{\tau}_{1!}(\delta\phi \otimes \alpha) = [\bar{\tau}_{1!}, \iota_\phi](\alpha)$$

$$\bar{\tau}_{1!}(\phi \otimes \alpha) = \sum_{k \geq 1} \pm f_{0,0} a_0 \otimes \ldots \otimes \phi(a_k, \ldots) \otimes f_{0,1} a_r \ldots \otimes f_{0,1} a_n +$$

$$\sum \pm \phi(f_{1,0} f_{0,1} a_k, \ldots, f_{1,0} f_{0,1} a_n, a_0, \ldots) \otimes f_{0,1} a_s \otimes \ldots \otimes f_{0,1} a_{k-1}$$

$$[b, \bar{\tau}_{1!}(\phi, -)] \pm \bar{\tau}_{1!}(\delta\phi, -) = [\bar{\tau}_{1!}, \iota_\phi]$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

Defining $\tau_{1!}$

$$\bar{\tau}_{1!}\big((\phi_{0,1}|\ldots|\phi_{0,k_0}) \otimes (\phi_{1,1}|\ldots|\phi_{1,k_1}) \otimes \alpha\big) =$$

$$= \sum_{\substack{1\le i\le n_0 \\ i\le j_1\le\ldots\le j_{2k_1}\le k_0 \\ p}} \pm\phi_{0,1}\left(\begin{matrix} f_{1,0}f_{0,i}a_p,\ldots,f_{1,0}\phi_{0,i+1}(a_*,\ldots), \\ f_{1,0}f_{0,i+1}a_*,\ldots,f_{1,0}\phi_{0,j_1}(a_*,\ldots), \\ f_{1,0}f_{0,j_1}a_*,\ldots,\phi_{1,1}(f_{0,j_1}a_*,\ldots,\phi_{0,j_1+1}(a_*,\ldots),\ldots),\ldots, \\ \phi_{1,k_1}(f_{0,j_1}a_*,\ldots,\phi_{0,j_{2k_1-1}+1}(a_*,\ldots),\ldots),\ldots,a_0,\ldots \end{matrix}\right)\otimes$$

$$\otimes\, f_{0,1}a_* \otimes \ldots \otimes \phi_{0,2}(a_*,\ldots) \otimes f_{0,2}a_* \otimes \ldots\otimes$$

$$\otimes\, \phi_{0,i}(a_*,\ldots) \otimes f_{0,i}a_* \otimes \ldots f_{0,i}a_{p-1} +$$

$$\Big(\sum \pm f_{0,0}a_0 \otimes \ldots \otimes \phi_{0,1}(a_*,\ldots) \otimes \ldots \otimes \phi_{0,n_0}(a_*,\ldots)\otimes$$

$$\otimes\, f_{0,n_0}a_* \otimes \ldots \otimes f_{0,n_0}a_n \quad \text{if } k_1 = 0\Big)$$