# ABSTRACT

From Noncommutative Calculus to a (Sheafy-)Cyclic Structure

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In 2008, Dolgushev, Tamarkin and Tsygan showed that, for $M$ a smooth real manifold, the pair (Hochschild cochains $C^\bullet(C^\infty(M))$, Hochschild chains $C_\bullet(C^\infty(M))$) is quasi-isomorphic as an $\infty$-calculus algebra to $(\Lambda^\bullet(T_M), \Omega^\bullet(M))$. In this thesis, we use their $\infty$-calculus structure on Hochschild cochains and chains to construct a "homotopically sheafy-cyclic object in dg categories with a dg module", i.e., an $A_\infty$-functor $\chi_\infty \to \mathcal{E}_\infty$. To give a sheafy-cyclic object is to give a dg functor out of $\chi_\infty$, a category fibred over Connes' cyclic category with fibre category $\chi_{\infty[n]}$ = the discrete category on the set of objects {diagrams $A_0 \to \cdots \to A_n \to A_0 : A_i$ is an algebra}. $\mathcal{E}_\infty$ is the dg category of objects small dg categories with a dg module, and morphisms given by a dg functor between dg categories with a (not necessarily graded) linear map from the source module to a pullback along the functor of the target module. Interestingly, in the construction of our $A_\infty$-functor, the formulas we give for cyclically rotating a dg module resemble the Lie derivative of Hochschild cochains on chains, and the homotopy between the image of

$\tau_n^{n+1}$ and the image of identity resembles the $B$ operator, the analogue of the deRham differential on Hochschild chains.

# Nomenclature

$k -$ a fixed ground field of char 0

$1 -$ the unit in (a vector space isomorphic to) $k$

$[1] -$ shift operator on complexes, $C^\bullet[1] = C^{\bullet+1}$

$\Lambda -$ Connes cyclic category, see Appendix A

$\Delta(b) = \sum_{(b)} b_{(1)} \otimes b_{(2)} -$ Sweedler notation for coproducts

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CHAPTER 1

# Introduction

This dissertation builds on work in noncommutative differential geometry, and we start with a brief history of the subject. In 1962, Hochschild, Kostant and Rosenberg gave the first hint that, to build a notion of differential geometry for noncommutative algebras, one should start by studying Hochschild homology and cohomology. Their famous HKR theorem states that, for a regular, commutative algebra over a field of characteristic zero, (1) Hochschild homology is quasi-isomorphic to deRham forms on the algebra (with zero differential), and (2) Hochschild cohomology is quasi-isomorphic to poly-vector fields on the algebra (also with zero differential) (Reference [**3**]).

However, when the algebra is an algebra of smooth functions on a real manifold, we have much more differential-geometric structure. For example, (shifted) poly-vector fields form a dgla with the Schouten-Nijenhuis bracket, an extension of the Lie bracket on vector fields to poly-vector fields; and in 1997, Kontsevich proved that Hochschild co-homology with its known Gerstenhaber bracket is quasi-isomorphic to poly-vector fields, not as dglas, but as $L_\infty$ algebras (Reference [**4**]). Moreover, the associative wedge product satisfies a Leibniz relation with the Schouten-Nijenhuis bracket, making poly-vector fields a Gerstenhaber algebra. In 1998, Tamarkin gave, for each Drinfeld associator, a quasi-isomorphism between Hochschild cochains and poly-vector fields as Gerstenhaber$_\infty$ algebras (Reference [**5**], Theorem 2.1).

But what about differential forms? Poly-vector fields act on forms in two ways: (1) via contraction as a dga, and (2) via the Lie derivative as a dgla. Together, these two actions satisfy Cartan's formula. We can axiomatize all of this structure (a Gerstenhaber algebra and the two actions on a module satsifying relations) into the notion of an algebra over the Calc operad. In 2008, Dolgushev, Tamarkin and Tsygan proved that the pair (Hochschild

cochains, Hochschild chains) are quasi-isomorphic to (poly-vector fields, deRham forms) as $\mathrm{Calc}_\infty$ algebras (Reference [**1**], Corollary 4).

Loosely given, the motivating question of this thesis is: Can we use the $\mathrm{Calc}_\infty$ structure on Hochschild cochains and chains to create a cyclic object? First, we repackage and generalize the $L_\infty$ structure on Hochschild cochains to construct dg cocategories, $B(n)$ (Section 2.2). Then, we show that these dg cocategories form, not a cyclic, but a sheafy-cyclic object (Section 3.1). We introduce dg comodules $C(n)$ over $B(n)$, which are categorified versions of the bar construction of Hochschild chains as a module over Hochschild cochains via contraction (Section 2.3). Given this background, we ask the motivating question in a more technical way: Can we extend the sheafy-cyclic structure on dg cocategories to include the dg comodules?

In Sections 3.3 and 3.4, we establish the theoretical background needed to work with the dg comodules. Then, we give a reasonable candidate for a sheafy-cyclic structure on dg comodules (Section 3.6). Here, we find, surprisingly, that the formulas for cyclically rotating a dg comodule resemble the formulas for the Lie derivative of Hochschild cochains on chains (Equation B.1). Unfortunately, our candidate only respects composition of morphisms up to homotopy. Yet, we are able to give these homotopies explicitly (Section 4.2), and interestingly, they are also familiar formulas–they look like generalizations of the B operator, the analogue of the deRham differential on Hochschild chains (Equation B.7). We show that no higher homotopies are needed (Section 4.3), and repackage our "functor up to homotopy" into an $A_\infty$ functor (Section 4.4). Finally, we apply (categorified) Cobar to get a "homotopically sheafy-cyclic object in dg categories with dg modules" (Section 4.5.2, 3.5).

Appendix A gives the presentation of Connes' cyclic category, $\Lambda$, that we will use throughout this thesis. Appendix C gives background on Hochschild chains and cochains and their calculus structure. Appendix B contains all of our computations.

CHAPTER 2

# $B(n)$ and $C(n)$

## 2.1. Motivation of this chapter

In this chapter, we introduce the main characters/objects of study, $B(n)$ and $C(n)$, $n \in \mathbb{N}$. The $B(n)$'s are dg cocategories constructed using Hochschild cochains. Each $C(n)$ is a dg comodule over $B(n)$ constructed using an action of Hochschild cochains on Hochschild chains. We start with definitions for the less-widely-used concepts, and show that the main characters are conilpotent.

See Appendix C for definitions of known operations on Hochschild chains and cochains as well as our notation and conventions.

## 2.2. Dg cocategories: $B(n)$

### 2.2.1. Background on dg cocategories

**Definition 2.2.1.** A **dg cocategory** is a cocategory enriched over chain complexes. More explicitly, a dg cocategory $B$ consists of the following data:

- A collection of objects denoted $Obj(B)$;

- For each pair of objects, $x, z \in Obj(B)$, a complex $B^\bullet(x, z)$ and a morphism of complexes

$$\Delta_B(x, z) : B^\bullet(x, z) \to \prod_{y \in Obj(B)} B^\bullet(x, y) \otimes B^\bullet(y, z)$$

such that the following diagrams commute (coassociativity):

$$
\begin{array}{ccc}
B^\bullet(x, z) & \xrightarrow{\Delta_B(x,z)} & \prod_{y \in Obj(B)} B^\bullet(x, y) \otimes B^\bullet(y, z) \\
\Big\downarrow{\scriptstyle \Delta_B(x,z)} & & \Big\downarrow{\scriptstyle \prod_y id_{B(x,y)} \otimes \Delta_B(y,z)} \\
\prod_{y \in Obj(B)} B^\bullet(x, y) \otimes B^\bullet(y, z) & \xrightarrow{\prod_y \Delta_B(x,y) \otimes id_{B(y,z)}} & \prod_{y, y' \in Obj(B)} B^\bullet(x, y) \otimes B^\bullet(y, y') \otimes B^\bullet(y', z)
\end{array}
$$

- For each pair of objects, $x, z \in Obj(B)$, a morphism of complexes

$$\epsilon_B(x, z) : B^\bullet(x, z) \to k$$

where $k$ is the ground field considered as a chain complex concentrated in degree 0 and $\epsilon_B(x, z) = 0$ if $x \neq z$, such that the following diagrams commute (counitality):

$$
\begin{array}{ccc}
B^\bullet(x, z) & \xrightarrow{\ \ \Delta_B(x,z)\ \ } & \prod_{y \in Obj(B)} B^\bullet(x, y) \otimes B^\bullet(y, z) \\
\Delta_B(x,z) \downarrow & \searrow^{id} & \downarrow \prod_y \epsilon_B(x,y) \otimes id_{B(y,z)} \\
\prod_{y \in Obj(B)} B^\bullet(x, y) \otimes B^\bullet(y, z) & \xrightarrow[\prod_y id_{B(x,y)} \otimes \epsilon_B(y,z)]{} & B^\bullet(x, z).
\end{array}
$$

We will denote a dg cocategory with its cocomposition and counit as $(B, \Delta_B, \epsilon_B)$. To make the notation more readable, when the meaning is clear, we will omit references to the objects and write $\Delta_B$ instead of $\Delta_B(x, z)$, $\epsilon_B$ instead of $\epsilon_B(x, z)$, and for the differentials on morphisms, $d_B$ instead of $d_B(x, z)$.

**Definition 2.2.2.** A **(dg) functor** $F : A \to B$ between two dg cocategories is a functor between the cocategories satisfying $d_B \circ F(f) = F \circ d_A(f)$ for all morphisms $f$ in $A$.

**Definition 2.2.3.** A **conilpotent** dg cocategory is a dg cocategory $(B, \Delta_B, \epsilon_B)$ satisfying: for each morphism $f : x \to y$ in $B$, there exists $n_f \in \mathbb{N}$ such that $\bar{\Delta}_B^{n_f}(f) = 0$ where

$$
\bar{\Delta}_B(x, z) : B^\bullet(x, z) \to \prod_{y \in Obj(B)} B^\bullet(x, y) \otimes B^\bullet(y, z)
$$

$$
f \mapsto \Delta_B(f) - \sum_{e_x \in \epsilon_B(x,x)^{-1}(1)} e_x \otimes f - \sum_{e_z \in \epsilon_B(z,z)^{-1}(1)} f \otimes e_z.
$$

The following fact follows from the definitions: If $B$ is a conilpotent dg cocategory, then for all $x \in Obj(B)$, $\epsilon_B(x,x)^{-1}(1)$ has exactly one element, which we will denote $e_x$.

### 2.2.2. Structure of $B(n)$

For each sequence of algebras, $A_0, A_1, \cdots, A_n$, we will define a conilpotent dg cocategory, $B(A_0 \rightarrow A_1 \rightarrow \cdots \rightarrow A_n \rightarrow A_0)$. In this chapter, we fix the sequence of algebras, and abbreviate

$$B(n) := B(A_0 \rightarrow A_1 \rightarrow \cdots \rightarrow A_n \rightarrow A_0).$$

**2.2.2.1. Objects.** $B(n)$ has objects tuples $(f_0, f_1, \cdots, f_n)$ where $f_i : A_i \rightarrow A_{i+1 \text{ (mod n+1)}}$, $0 \le i \le n$, are maps of algebras. We can picture an object in $B(n)$ as follows:

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_0$$

**2.2.2.2. Morphisms.** The graded vector space of morphisms in $B(n)$ between two objects, $(f_0, ..., f_n)$ and $(g_0, ..., g_n)$, is

$$Bar(C^\bullet(A_{0,f_0} A_{1g_0})) \otimes Bar(C^\bullet(A_{1,f_1} A_{2g_1})) \otimes \cdots \otimes Bar(C^\bullet(A_{n,f_n} A_{0g_n}))$$

where $Bar(C^\bullet(A,_f B_g))$ is the following complex:

$$Bar(C^\bullet(A,_f B_g)) := Bar_0(C^\bullet(A,_f B_g)) \oplus \bigoplus_{m \geq 1} Bar_m(C^\bullet(A,_f B_g))$$

$$Bar_0(C^\bullet(A,_f B_g)) := k$$

$$Bar_m(C^\bullet(A,_f B_g)) := \bigoplus_{\substack{h_0=f, \\ h_m=g, \\ h_1,\ldots,h_{m-1} \\ \text{algebra maps}}} \begin{array}{c} C^\bullet(A,\, _{h_0}B_{h_1})[1] \otimes C^\bullet(A,\, _{h_1}B_{h_2})[1] \otimes \cdots \otimes \\ \otimes\ C^\bullet(A,\, _{h_{m-1}}B_{h_m})[1] \end{array}$$

$(C^\bullet(A,_{h_i} B_{h_j}),\, _{h_i}\delta_{h_j}) =$ Hochschild cochain complex, see Appendix C

$$d_{Bar(C^\bullet(A,_f B_g))} = \tilde{\delta} + b'$$

$$\tilde{\delta}(\phi_1 \otimes \ldots \otimes \phi_m) = \sum_{1 \leq i \leq m} (-1)^{1+\sum_{j<i} |\phi_i|+1} \phi_1 \otimes \ldots \otimes [_{h_{i-1}}\delta_{h_i}](\phi_i) \otimes \ldots \otimes \phi_m$$

$$b'(\phi_1 \otimes \ldots \otimes \phi_m) = \sum_{1 \leq i \leq m-1} (-1)^{\sum_{j \leq i} |\phi_i|+1} \phi_1 \otimes \ldots \otimes \phi_i \cup \phi_{i+1} \otimes \ldots \otimes \phi_m$$

$\cup =$ cup product on Hochschild cochains, see Appendix C.

(This sign convention is consistent with Reference [**6**], Section 4.6.)

**2.2.2.3. Aside on notation.** When referring to an arbitrary morphism in $B(n)$, we will assume it is a morphism from object $(f_{0,0}, f_{1,0}, \cdots, f_{n,0})$ to object $(f_{0,k_0}, f_{1,k_1}, \cdots, f_{n,k_n})$. We will denote the morphism

$$\phi_{0,1}\ldots\phi_{0,k_0}|\phi_{1,1}\ldots\phi_{1,k_1}|\ldots|\phi_{n,1}\ldots\phi_{n,k_n}$$

where $\phi_{i,j} \in C^\bullet(A_{i,f_{j-1}} A_{i+1 \text{ (mod n+1)}_{f_j}})$. See Figure 2.1 for a picture of this morphism.
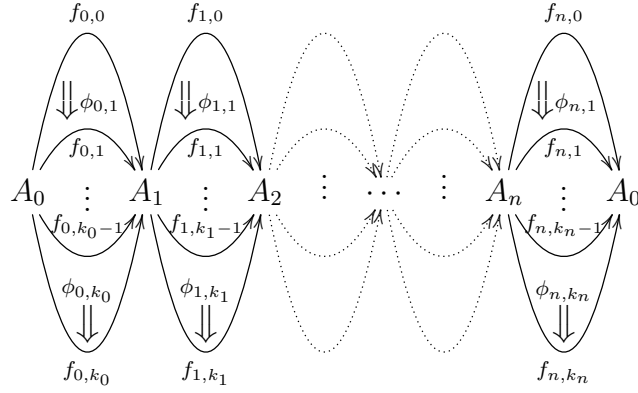
Figure 2.1. A morphism in $B(n)$

from $(f_{0,0}, f_{1,0}, \cdots, f_{n,0})$ to $(f_{0,k_0}, f_{1,k_1}, \cdots, f_{n,k_n})$ where $\phi_{i,j} \in C^\bullet(A_{i,f_{j-1}} A_{i+1 \ (\mathrm{mod}\ n+1)_{f_j}})$

**2.2.2.4. Differential on $B(n)$.** Putting everything together, the differential on

$B(n)((f_{0,0}, ..., f_{n,0}), (f_{0,k_0}, ..., f_{n,k_n}))$ is

$$
d_{B(n)}\big(\phi_{0,1}...\phi_{0,k_0}|...|\phi_{n,1}...\phi_{n,k_n}\big)
$$

$$
= \sum_{0 \le i \le n} (-1)^{\sum\limits_{p<i;q} |\phi_{p,q}|+1} \phi_{0,1}...|...|d_{Bar(C^\bullet(A_i,A_{i+1}))}(\phi_{i,1}...\phi_{i,k_i})|...|...\phi_{n,k_n}
$$

**2.2.2.5. Counit.** Define

$$
\epsilon_{B(n)}((f_{0,0},..., f_{n,0}), (f_{0,k_0}, ..., f_{n,k_n})) : B(n)((f_{0,0}, ..., f_{n,0}), (f_{0,k_0}, ..., f_{n,k_n})) =
$$

$$
= Bar(C^\bullet(A_{0,f_{0,0}} A_{1_{f_{0,k_0}}})) \otimes \cdots \otimes Bar(C^\bullet(A_{0,f_{n,0}} A_{1_{f_{n,k_n}}})) \to
$$

$$
\xrightarrow{project} Bar_0(C^\bullet(A_{0,f_{0,0}} A_{1_{f_{0,k_0}}})) \otimes \cdots \otimes Bar_0(C^\bullet(A_{0,f_{n,0}} A_{1_{f_{n,k_n}}})) \cong k.
$$

**2.2.2.6. Cocomposition.** We have a coassociative map of complexes

$$\Delta_{A,_f B_g} : Bar(C^\bullet(A,_f B_g)) \to \bigoplus_{h:A\to B} Bar(C^\bullet(A,_f B_h)) \otimes Bar(C^\bullet(A,_h B_g))$$

$$\phi_1 \cdots \phi_k \mapsto \sum_{1\leq i\leq k-1} \phi_1 \cdots \phi_i \otimes \phi_{i+1} \cdots \phi_k$$

$$+ e_f \otimes \phi_1 \ldots \phi_k + \phi_1 \ldots \phi_k \otimes e_g$$

where $e_f = 1$ in $Bar_0(C^\bullet(A,_f B_f)) \cong k$. Extend $\Delta_{A,_f B_g}$ to a cocomposition on $B(n)$ by taking (up to signs)

$$\Delta_{B(n)}((f_{0,0}, ..., f_{n,0}), (f_{0,k_0}, ..., f_{n,k_n})) := \Delta_{A_0,_{f_{0,0}} A_1{}_{f_{0,k_0}}} \otimes \cdots \otimes \Delta_{A_0,_{f_{n,0}} A_1{}_{f_{n,k_n}}}.$$

The sign on the term $(\phi_{0,1}...\phi_{0,i_0}|...|\phi_{n,1}...\phi_{n,i_n}) \otimes (\phi_{0,i_0+1}...\phi_{0,k_0}|...|\phi_{n,i_n+1}...\phi_{n,k_n})$ in the cocomposition is:

(2.1)
$$(-1)^{\substack{\sum_{\substack{1\leq p\leq n \\ 1\leq r<p}} (\sum_{1\leq q\leq i_p} |\phi_{p,q}|+1)(\sum_{i_r+1\leq s\leq k_r} |\phi_{r,s}|+1)}}$$

In other words, moving $\phi_{i,j}$ past $\phi_{p,q}$ introduces a factor of $(-1)^{(|\phi_{i,j}|+1)(|\phi_{p,q}|+1)}$. It's clear from the definitions that $(B(n), \Delta_{B(n)}, \epsilon_{B(n)})$ satisfy the diagrams needed to form a dg cocategory. We also see that $B(n)$ is conilpotent:

$$\bar{\Delta}_{B(n)}^{min(k_0,\ldots,k_n)}(\phi_{0,1}...\phi_{0,k_0}|...|\phi_{n,1}...\phi_{n,k_n}) = 0.$$

## 2.3. Dg comodules: $C(n)$

### 2.3.1. Background on dg comodules

**Definition 2.3.1.** A **dg comodule** $C$ over a dg cocategory $B$ consists of the following data:

- for each object $f \in B$, a complex $C^\bullet(f)$, and
- maps of complexes

$$\Delta_C(f) : C^\bullet(f) \to \prod_{g \in Obj(B)} B^\bullet(f,g) \otimes C^\bullet(g).$$

such that the following diagrams for coassociativity and counitality commute:



To simplify notation, we will write $\Delta_C$ instead of $\Delta_C(f)$ when the meaning is clear.

**Example 2.3.1.** *A dg comodule over a dg cocategory $B$ with one object, $*$, is a dg comodule over the counital dg coalgebra $B^\bullet(*, *)$.*

**Definition 2.3.2.** A **morphism of dg comodules** $H : C \to D$ over a dg category $B$ consists of maps of complexes $\left(H_f : C^\bullet(f) \to D^\bullet(f)\right)_{f \in Obj(B)}$ such that for each $f \in Obj(B)$, the following diagram commutes:

$$
\begin{array}{ccc}
C^\bullet(f) & \xrightarrow{\ H_f\ } & D^\bullet(f) \\
\downarrow{\scriptstyle \Delta_C} & & \downarrow{\scriptstyle \Delta_D} \\
\displaystyle\prod_{g \in Obj(B)} B^\bullet(f, g) \otimes C^\bullet(g) & \xrightarrow{\ \prod\limits_g id_B \otimes H_g\ } & \displaystyle\prod_{g \in Obj(B)} B^\bullet(f, g)) \otimes D^\bullet(g).
\end{array}
$$

Again, when the meaning is clear, we may write $H$ instead of $H_f$.

**Definition 2.3.3.** A **conilpotent** dg comodule over a dg cocategory $B$ is a dg comodule $(C, \Delta_C)$ over $B$ satisfying: for each $f \in Obj(B)$ and each element $\alpha \in C^\bullet(f)$, there exists $n_\alpha \in \mathbb{N}$ such that $\bar{\Delta}_f^{n_\alpha}(\alpha) = 0$ where

$$
\bar{\Delta}_C(f) : C^\bullet(f) \to \prod_{g \in Obj(B)} B^\bullet(f, g) \otimes C^\bullet(g)
$$

$$
\alpha \mapsto \Delta_B(\alpha) - \sum_{e_f \in \epsilon_B(f, f)^{-1}(1)} e_f \otimes f.
$$

### 2.3.2. Structure of $C(n)$

**Reminder:** In this chapter, we fix algebras $A_0, A_1, \cdots, A_n$. $C(n)$ and $B(n)$ are short for $C(A_0 \to A_1 \to \cdots \to A_n \to A_0)$ and $B(A_0 \to A_1 \to \cdots \to A_n \to A_0)$, respectively.

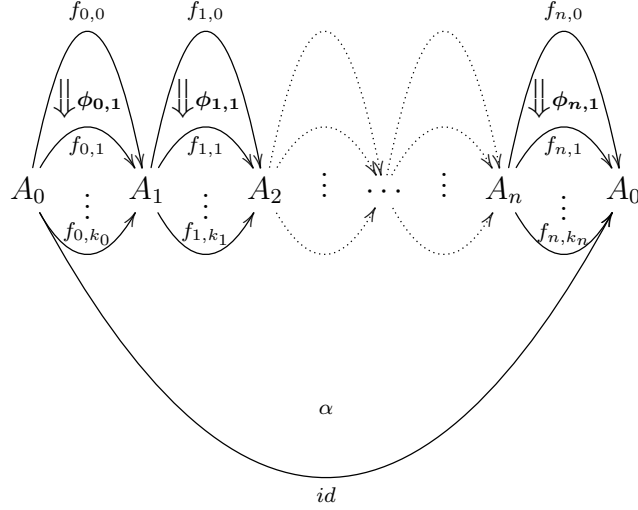We now give dg comodules $C(n)$ over $B(n)$. First, we will describe the graded comodule

Figure 2.2. An element of $C(n)^\bullet(f)$
where $f = (f_{0,0}, f_{1,0}, \cdots, f_{n,0})$, $\phi_{i,j} \in C^\bullet(A_{i,f_{j-1}} A_{i+1 \text{ (mod n+1)}_{f_j}})$, and
$\alpha \in C_{-\bullet}(A_{0,f_n \cdots f_1 f_0} A_{0id})$

structure; then, we will describe the differentials. For an object $f = (f_0, f_1, \cdots, f_n) \in B(n)$, we have

$$(2.2) \qquad C(n)^\bullet(f) = \bigoplus_{g \in Obj(B(n))} B(n)^\bullet(f, g) \otimes C_{-\bullet}(A_{0,comp(g)} A_{0id})$$

where, for $g = (g_0, g_1, \cdots, g_n)$, we write $comp(g) = g_n \circ g_{n-1} \circ \cdots \circ g_0$, and $(C_\bullet(A, B), \ _g b)$ is the Hochschild chain complex (see Appendix C). We will denote a typical element of $C(n)^\bullet(f)$ as

$$\phi_{0,1} ... \phi_{0,k_0} | \phi_{1,1} ... \phi_{1,k_1} | ... | \phi_{n,1} ... \phi_{n,k_n} | \alpha$$

where $\phi_{0,1} \cdots \phi_{n,k_n}$ is a morphism in $B(n)$ (see Section 2.2.2.3) and $\alpha \in C_{-\bullet}(A_{0,f_{k_n} \cdots f_{k_0}} A_0)$. See Figure 2.2 for a picture of a typical element of $C(n)^\bullet(f)$.

**2.3.2.1. Comodule structure.** The comodule maps on $C(n)^\bullet(f)$ are given by the co-composition maps in $B(n)$:

$$C(n)^\bullet(f) \xrightarrow{\Delta_C} \bigoplus_{h \in Obj(B(n))} B(n)^\bullet(f,h) \otimes C(n)^\bullet(h)$$

$$\bigg\| \qquad\qquad\qquad\qquad \bigg\|$$

$$\bigoplus_{g \in Obj(B(n))} B(n)^\bullet(f,g) \otimes C_{-\bullet}(A_{0,g}\,A_{0\,id}) \xrightarrow{\Delta_{B(n)} \otimes id_{C_{-\bullet}}} \bigoplus_{g,h \in Obj(B(n))} \begin{smallmatrix} B(n)^\bullet(f,h) \otimes B(n)^\bullet(h,g) \\ \otimes C_{-\bullet}(A_{0,g}A_{0\,id}) \end{smallmatrix}$$

Because $\Delta_{C(n)}$ is induced by $\Delta_{B(n)}$, we have that $\Delta_{C(n)}$ satisfies coassocitivity and counitality and is conilpotent.

$C(n)$ cofree as a comodule in the sense that a morphism to $C(n)$ is determined by projections to its Hochschild-chains component. More precisely, there is a one-to-one correspondence

(2.3)

$$\left\{ \begin{matrix} \text{maps of comodules} \\ D \to C(n) \text{ over } B(n) \end{matrix} \right\} \xleftrightarrow{1:1} \left\{ \left( \begin{matrix} \text{maps of graded vector spaces} \\ D^\bullet(f) \to C_{-\bullet}(A_{0,f}\,A_{0\,id}) \end{matrix} \right)_{f \in Obj(B(n))} \right\}$$

$$\left( F : D \to C(n) \right) \mapsto \left( \begin{matrix} D(f) \xrightarrow{F_f} C(n)(f) \\ \xrightarrow{project} C_{-\bullet}(A_{0,f}\,A_{0\,id}) \end{matrix} \right)_f$$

$$\left( \begin{matrix} D(f) \xrightarrow{\Delta_D} \bigoplus_{g \in Obj(B(n))} B(n)(f,g) \otimes D(g) \\ \xrightarrow{\oplus_g id_B \otimes F_g} \oplus_g B(n)(f,g) \otimes C_{-\bullet}(A_{0,g}\,A_{0\,id}) \\ \cong C(n)(f) \end{matrix} \right)_f \hookleftarrow \left( D(f) \to C_{-\bullet}(A_{0,f}\,A_{0\,id}) \right)_f$$

**Definition 2.3.4.** We will call elements of

$$T(A_0 \to \ldots \to A_n \to A_0)(f) := C_{-\bullet}(A_{0,f}\,A_0)$$

the **cogenerators** of $C(A_0 \to \ldots A_n \to A_0)(f)$. More generally, we will refer to the set

$T(A_0 \to \ldots \to A_n \to A_0) = \{T(A_0 \to \ldots \to A_n \to A_0)(f) | f \in Obj(B(A_0 \to \ldots \to A_n \to A_0))\}$ as the **cogenerators** of $C(A_0 \to \ldots \to A_n \to A_0)$. When we have fixed a sequence

of algebras, $A_0, \ldots A_n$, we will use $T(n)$ to denote $T(A_0 \to \ldots \to A_n \to A_0)$.

**2.3.2.2. Differential.** The differential $d_{C(n)}$ on $C(n)$ is:

(2.4)

$$d_{C(n)} = \tilde{d}_{B(n)} + \tilde{b} + \mathfrak{I}$$

$$\tilde{d}_{B(n)}\big(\phi_{0,1}\ldots\phi_{0,k_0}|\ldots|\phi_{n,1}\ldots\phi_{n,k_n}|\alpha\big) = d_{B(n)}(\phi_{0,1}\ldots|\ldots|\ldots\phi_{n,k_n})|\alpha$$

$$\tilde{b}\big(\phi_{0,1}\ldots\phi_{0,k_0}|\ldots|\phi_{n,1}\ldots\phi_{n,k_n}|\alpha\big) = (-1)^{\sum_{i,j}|\phi_{i,j}|+1} \phi_{0,1}\ldots|\ldots|\ldots\phi_{n,k_n}|[f_{n,k_n}\ldots f_{0,k_0}b](\alpha)$$

where $d_{B(n)}$ is the differential on $B(n)$, and $\mathfrak{I}$ is a term that captures an action of cochains
on chains described by the equations below:

(2.5)

$$\mathfrak{I} = (id_{B(n)} \otimes \eta_{C(n)}) \circ \Delta_{C(n)}$$

$$\eta_{C(n)}(\phi_{0,1}\ldots\phi_{0,k_0}|\ldots|\phi_{n,1}\ldots\phi_{n,k_n}|\alpha) = \iota_{C(0)}(\pi_{B(0)}((\phi_{0,1}\ldots\phi_{0,k_0}) \bullet \cdots \bullet (\phi_{n,1}\ldots\phi_{n,k_n})), \alpha)$$

$$\bullet = \text{brace operation on cochains, see Section C.2}$$

$$\pi_{B(0)} : B(0)^\bullet(f_{n,1}\ldots f_{0,1}, f_{n,k_n}\ldots f_{0,k_0}) \xrightarrow[\text{component}]{\text{project onto}} C^\bullet(A_{0,f_{n,1}\ldots f_{0,1}} A_{0 f_{n,k_n}\ldots f_{0,k_0}})$$

$$\iota = \text{contraction operation, see Section C}$$

Given Equation 2.4, it's easy to check that we can promote Equation 2.3 to a dg statement:

$$\left\{ \begin{matrix} \text{maps of dg comodules} \\ D \to C(n) \text{ over } B(n) \end{matrix} \right\} \xleftrightarrow{1:1} \left\{ \left( \begin{matrix} \text{maps of complexes} \\ D^\bullet(f) \to C_{-\bullet}(A_{0,f} A_{0id}) \end{matrix} \right)_{f \in Obj(B(n))} \right\}.$$

CHAPTER 3

# Pullbacks, Pushforwards and Adjunctions

## 3.1. A sheafy-cyclic object in dg cocategories

We would like to say that we have a functor from Connes cyclic category $\Lambda$ (see Appendix A for generators and relations) to the category of dg cocategories where $[n] \mapsto B(n)$, but defining $B(n)$ involved choosing a sequence of algebras $A_0, \ldots, A_n$.

Instead, we have the following: Let $X : \Lambda \to Set$ be the functor that sends $[n]$ to the set of diagrams $A_0 \to A_1 \to \cdots \to A_n \to A_0$ where the $A_i$'s are algebras. On generating morphisms in $\Lambda$, $X$ acts as follows: Let $\mathcal{A} = (A_0 \to \ldots \to A_n \to A_0) \in X([n])$.

$$X(\tau_n) : \mathcal{A} \mapsto (A_n \to A_0 \to \ldots \to A_{n-1} \to A_n)$$

$$X(\delta_{j,n}) : \mathcal{A} \mapsto (A_0 \to \ldots \to A_j \longrightarrow A_{j+2 \text{ (mod n+1)}} \to \ldots \to A_n \to A_0)$$

$$X(\sigma_{i,n}) : \mathcal{A} \mapsto \begin{cases} (A_0 \to \ldots \to A_i \to A_i \to \ldots A_n \to A_0) & 1 \leq i \leq n \\ (A_0 \to \ldots \to A_n \to A_0 \to A_0) & i = n+1 \end{cases}$$

It's straightforward to check that $X$ respects composition of morphisms. Now, let $\chi$ be the category with objects given by diagrams $A_0 \to \ldots \to A_n \to A_0$ where the $A_i$'s are algebras and $n \in \mathbb{N}$. Morphisms in $\chi$ are the pointwise images of $X$. In other words, the set of morphisms in $\chi$ is $\{X(\lambda)|_x : \lambda \in \Lambda([n],[m]), x \in X([n])\}$. We will give a functor, $\mathcal{G}$, from $\chi$ to the category of dg cocategories; (this is our sheafy-cyclic object, i.e., a **sheafy-cyclic** object in a category $\mathcal{C}$ is a functor $\chi \to \mathcal{C}$).

### 3.1.1. Aside on notation:

Fix $\lambda : [n] \to [m]$ in $\Lambda$ and $x \in X([n])$. To define $\mathcal{G}$, we will need to define a functor $\mathcal{G}(X(\lambda)|_x) : B(x) \to B(X(\lambda)(x))$. To simplify notation, we will denote $\hat{\lambda} := \mathcal{G}(X(\lambda)|_x)$

and write $\hat{\lambda} : B(x) \to B(\lambda x)$. Technically, we are losing information about the $x$ when we write $\hat{\lambda}$ instead of $\mathcal{G}\big(X(\lambda)|_x\big)$, but we will be clear about the source and target when needed.

### 3.1.2. Definition of $\mathcal{G}$

Now, we will define $\mathcal{G}$. On objects,

$$\mathcal{G} : (A_0 \to \ldots \to A_n \to A_0) \mapsto B(A_0 \to \ldots \to A_n \to A_0)$$

(see Section 2.2 for definition of $B(\cdot)$)

On generating morphisms in $\chi$, set $\mathcal{A} = (A_0 \to \dots \to A_n \to A_0) \in Obj(\chi)$, and define

$$\hat{\tau}_n \mapsto \begin{cases} B(\mathcal{A}) \longrightarrow B(A_n \to A_0 \to \dots \to A_{n-1} \to A_n) \\[2mm] \text{objects: } (f_0, f_1, \dots, f_n) \mapsto (f_n, f_0, \dots, f_{n-1}) \\[2mm] \text{morphisms: } \phi_{0,1}\dots\phi_{0,k_0}|\dots|\phi_{n,1}\dots\phi_{n,k_n} \mapsto \phi_{n,1}\dots\phi_{n,k_n}|\dots|\phi_{n-1,1}\dots\phi_{n-1,k_{n-1}} \end{cases}$$

$$\hat{\delta}_{j,n} \mapsto \begin{cases} B(\mathcal{A}) \longrightarrow B(A_0 \to \dots \to A_j \to A_{j+2 \text{ (mod n+1)}} \to \dots \to A_0) \\[2mm] \text{objects: } (f_0, f_1, \dots, f_n) \mapsto (f_0, \dots, f_{j+1} \circ f_j, \dots, f_n) \\[2mm] \text{morphisms: } \phi_{0,1}\dots\phi_{0,k_0}|\dots|\phi_{n,1}\dots\phi_{n,k_n} \mapsto \\[2mm] \qquad \phi_{0,1}\dots\phi_{0,k_0}|\dots|(\phi_{j,1}\dots\phi_{j,k_j}) \bullet (\phi_{j+1,1}\dots\phi_{j+1,k_{j+1}})|\dots|\phi_{n,1}\dots\phi_{n,k_n} \end{cases}$$

$$\begin{array}{c} \hat{\sigma}_{i,n} \\ {\scriptstyle 1 \le i \le n} \end{array} \mapsto \begin{cases} B(\mathcal{A}) \longrightarrow B(A_0 \to \dots \to A_i \to A_i \to \dots \to A_0) \\[2mm] \text{objects: } (f_0, f_1, \dots, f_n) \mapsto (f_0, \dots, f_{i-1}, id_{A_i}, f_i, \dots, f_n) \\[2mm] \text{morphisms: } \phi_{0,1}\dots\phi_{0,k_0}|\dots|\phi_{n,1}\dots\phi_{n,k_n} \mapsto \\[2mm] \qquad \phi_{0,1}\dots\phi_{0,k_0}|\dots|\phi_{i-1,1}\dots\phi_{i-1,k_{i-1}}|1|\phi_{i,1}\dots\phi_{i,k_i}|\dots|\phi_{n,1}\dots\phi_{n,k_n} \\[2mm] \qquad 1 \in k = \text{degree 0 component of } Bar(C^\bullet(A_i, A_i)) \end{cases}$$

$$\hat{\sigma}_{n+1,n} \mapsto \begin{cases} B(\mathcal{A}) \longrightarrow B(A_0 \to \dots \to A_n \to A_0 \to A_0) \\[2mm] \text{objects: } (f_0, f_1, \dots, f_n) \mapsto (f_0, \dots, f_n, id_{A_0}) \\[2mm] \text{morphisms: } \phi_{0,1}\dots\phi_{0,k_0}|\dots|\phi_{n,1}\dots\phi_{n,k_n} \mapsto \\[2mm] \qquad \phi_{0,1}\dots\phi_{0,k_0}|\dots|\phi_{n,1}\dots\phi_{n,k_n}|1 \\[2mm] \qquad 1 \in k = \text{degree 0 component of } Bar(C^\bullet(A_0, A_0)) \end{cases}$$

It's straightforward to check that $\mathcal{G}$ is a functor (i.e., that composition of morphisms and the relations are preserved). The only facts we need are that $\bullet$ is an associative map of complexes and that $1 \bullet (\phi_0...\phi_k) = (\phi_0...\phi_k) \bullet 1 = (\phi_0...\phi_k)$ where the 1's are in the degree 0 components of $Bar(C^\bullet(A_i, A_i))$ for the appropriate $A_i$ (see Appendix Section C.2).

## 3.2. Motivation of this chapter

We would like to extend the sheafy-cyclic structure in Section 3.1 from $B(\mathcal{A})$ to the pair $(B(\mathcal{A}), C(\mathcal{A}))$ where $\mathcal{A} = (A_0 \to \dots \to A_n \to A_0) \in Obj(\chi)$. However, this presents some complications as $C(\mathcal{A})$ and $C(\lambda\mathcal{A})$ are comodules over different cocategories (where $\lambda$ is a morphism in $\Lambda$ inducing a morphism in $\chi$ with source $\mathcal{A}$). Instead, we will use the functors $\hat{\lambda} : B(\mathcal{A}) \to B(\lambda\mathcal{A})$ from Section 3.1.2 to define pullbacks $\hat{\lambda}^* C(\lambda\mathcal{A})$ and maps $\lambda_! : C(\mathcal{A}) \to \hat{\lambda}^* C(\lambda\mathcal{A})$ of dg comodules over $B(\mathcal{A})$.

First, we will define functors $\hat{\lambda}^*$ from the category of conilpotent dg comodules over $B(\lambda\mathcal{A})$ to the category of conilpotent dg comodules over $B(\mathcal{A})$. Second, we will give $\hat{\lambda}_{\#}$, the left adjoint to $\hat{\lambda}^*$. Finally, we will give explicit maps of dg comodules $\lambda_! : C(\mathcal{A}) \to \hat{\lambda}^* C(\lambda\mathcal{A})$, and apply the adjunction to these maps. The following chapter will formalize the relations between the $\lambda_!$'s.

## 3.3. Pullbacks of dg comodules–theory

Let $\lambda : B_1 \to B_0$ be a functor between conilpotent dg cocategories. In this section, we will define a functor $\lambda^*$ from the category of conilpotent dg comodules over $B_0$ to the category of conilpotent dg comodules over $B_1$. We call $\lambda^*$ "co-extension of scalars".

### 3.3.1. Category-theoretic definition of $\lambda^*$

Let $\lambda$ be as above, and let $C$ be a conilpotent dg comodule over $B_0$. We define $\lambda^* C$ as follows:

$$(3.1) \qquad \lambda^* C := ker\Big(B_1 \otimes_\lambda C \underset{(\mathrm{id}_{B_1} \otimes \lambda \otimes \mathrm{id}_C) \circ (\Delta_{B_1} \otimes \mathrm{id}_C)}{\overset{\mathrm{id}_{B_1} \otimes \Delta_C}{\rightrightarrows}} B_1 \otimes_\lambda B_0 \otimes C\Big)$$

where $B_1 \otimes_\lambda C$ and $B_1 \otimes_\lambda B_0 \otimes C$ are dg comodules over $B_1$ defined below. For $f \in Obj(B_1)$,

$$[B_1 \otimes_\lambda C](f) := \Big( \bigoplus_{h \in Obj(B_1)} B_1^\bullet(f,h) \otimes C^\bullet(\lambda h), \Delta(f) = \bigoplus_h \Delta_{B_1(f,h)} \otimes id_{C(\lambda h)}\Big)$$

$$[B_1 \otimes_\lambda B_0 \otimes C](f) := \Big( \bigoplus_{\substack{h_1 \in Obj(B_1), \\ h_2 \in Obj(B_0)}} B_1^\bullet(f,h_1) \otimes B_0^\bullet(\lambda h_1, h_2) \otimes C^\bullet(h_2),$$

$$\Delta(f) = \bigoplus_{h_1, h_2} \Delta_{B_1(f,h_1)} \otimes id_{B_0(\lambda h_1, h_2)} \otimes id_{C(h_2)}\Big).$$

The names of the maps in Equation 3.1 are also meant to be suggestive. In full detail, for $f \in Obj(B_1)$,

$$[id_{B_1} \otimes \Delta_C](f) := \bigoplus_h id_{B_1(f,h)} \otimes \Delta_C(\lambda h)$$

and

$$[B_1 \otimes_\lambda C](f) \xrightarrow{[\Delta_{B_1} \otimes id_C](f) := \bigoplus_h \Delta_{B_1}(f,h) \otimes id_{C(\lambda h)}} \bigoplus_{h_1, h_2 \in Obj(B_1)} B_1(f, h_1) \otimes B_1(h_1, h_2) \otimes C(\lambda h_2)$$

$$\xrightarrow{[id_{B_1} \otimes \lambda \otimes id_C](f) := \bigoplus_{h_1, h_2} id_{B_1(f,h_1)} \otimes \lambda(h_1, h_2) \otimes id_{C(\lambda h)}} [B_1 \otimes_\lambda B_0 \otimes C](f).$$

That the kernel is well-defined follows formally from the abelianness of the category of chain complexes, but it is also easy to check that the induced differentials from $[B_1 \otimes_\lambda C](f)$ on the kernel are well-defined. Since $\Delta_{\lambda^* C}$ is induced by $\Delta_{B_1}$, we have that $\Delta_{\lambda^* C}$ also satisfies coassociativity, counitality and conilpotency.

Next, we will define $\lambda^*$ on morphisms. Let $F : C \to D$ be a map of conilpotent dg comodules over $B_0$. By the universal property of $\lambda^* D$, we can define a morphism $\lambda^* F : \lambda^* C \to \lambda^* D$ by giving a morphism from $(\lambda^* F)' : \lambda^* C \to B_1 \otimes_\lambda D$ such that the two maps

(3.2)

$$(id_{B_1} \otimes \Delta_D) \circ (\lambda^* F)', (id_{B_1} \otimes \lambda \otimes id_D) \circ (\Delta_{B_1} \otimes id_D) \circ (\lambda^* F)' : \lambda^* C \to B_1 \otimes_\lambda D \rightrightarrows B_1 \otimes_\lambda B_0 \otimes D$$

coincide. We define $(\lambda^* F)'$ as follows:

$$(\lambda^* F)' : \lambda^* C \xrightarrow[inclusion]{canonical} B_1 \otimes_\lambda C \xrightarrow{id_{B_1} \otimes F} B_1 \otimes_\lambda D$$

It's easy to check that the two maps in Equation 3.2 coincide: Let $b \otimes c$ be an arbitrary element of $\lambda^* C(f) \hookrightarrow [B_1 \otimes_\lambda C](f)$. Then,

$$[(id_{B_1} \otimes \Delta_D) \circ (\lambda^* F)'](b \otimes c) = \sum_{(Fc)} b \otimes (Fc)_{(1)} \otimes (Fc)_{(2)}$$

$$= \sum_{(c)} b \otimes Fc_{(1)} \otimes Fc_{(2)} \quad \text{(F is a map of comodules)}$$

$$= [(id_{B_1} \otimes F \otimes F) \circ (id_{B_1} \otimes \Delta_C)](b \otimes c)$$

$$= [(id_{B_1} \otimes F \otimes F) \circ (id_{B_1} \otimes \lambda \otimes id_C) \circ (\Delta_{B_1} \otimes id_C)](b \otimes c)$$

$$(b \otimes c \text{ is in the kernel})$$

$$= \sum_{(b)} b_{(1)} \otimes \lambda b_{(2)} \otimes Fc$$

$$= [(id_{B_1} \otimes \lambda \otimes id_D) \circ (\Delta_{B_1} \otimes id_D) \circ (\lambda^* F)'](b \otimes c).$$

So, $\lambda^* F$ is well-defined. In summary, we have commuting diagrams:

(3.3)

$$
\begin{array}{ccc}
\lambda^* C & \xrightarrow{\substack{\text{canonical} \\ \text{inclusion}}} & B_1 \otimes_\lambda C \\
\lambda^* F \downarrow & & \downarrow {\scriptstyle id_{B_1} \otimes F = \text{ map inducing } \lambda^* F} \\
\lambda^* D & \xrightarrow{\substack{\text{canonical} \\ \text{inclusion}}} & B_1 \otimes_\lambda D
\end{array}
$$

Finally, it is straightforward to see that $\lambda^*$ is a functor, i.e., that $\lambda^*$ preserves composition of morphisms: Let $C \xrightarrow{F} D \xrightarrow{G} E$ be composable morphisms of dg comodules over $B_0$. The maps inducing $\lambda^* F$, $\lambda^* G$ and $\lambda^* (G \circ F)$ are $id_{B_1} \otimes F$, $id_{B_1} \otimes G$ and $id_{B_1} \otimes GF$, respectively. The inducing maps respect composition–$(id_{B_1} \otimes G) \circ (id_{B_1} \otimes F) = id_{B_1} \otimes GF$–and by the commuting diagrams 3.3, the functor $\lambda^*$ does as well.

**Proposition 3.1.** *Let $F : B_2 \to B_1$ and $G : B_1 \to B_0$ be functors between dg cocategories $B_2$, $B_1$ and $B_0$. Let $M$ be a dg comodule over $B_0$. Then,*

$$(GF)^* M \cong F^* G^* M.$$

**Proof.** We will prove the proposition by showing that $F^* G^* M$ satisfies the universal property of $(GF)^* M$. First, let $N$ be a dg comodule over $B_2$ and $H : N \to B_2 \otimes_{GF} M$ be a map of dg comodules such that the two maps

(3.4)

$$(id_{B_2} \otimes GF \otimes id_M) \circ (\Delta_{B_2} \otimes id_M) \circ H, \, (id_{B_2} \otimes \Delta_M) \circ H : N \to B_2 \otimes_{GF} M \rightrightarrows B_2 \otimes_{GF} \otimes B_0 \otimes M$$

coincide. We will show that $H$ determines a map of dg comodules $\tilde{H} : N \to F^* G^* M$. Let $x \in Obj(B_2)$. Define

$$H'_x : N(x) \xrightarrow{H_x} \bigoplus_{y \in Obj(B_2)} B_2(x, y) \otimes M(GFy)$$

$$\xrightarrow{F \otimes id_M} \bigoplus_{y \in Obj(B_2)} B_1(Fx, Fy) \otimes M(GFy)$$

$$\subset [B_1 \otimes_G M](Fx).$$

The image of $H'_x$ lands in $G^* M(Fx)$, a subcomplex of $[B_1 \otimes_G M](Fx)$; checking this is straightforward using the universal property of $G^* M$, the fact that $F$ commutes with the coproducts, and Equation 3.4. So, for each $x \in Obj(B_2)$, we have a map of complexes

$H'_x : N(x) \to G^*M(Fx)$. Now define $\tilde{H}$ as follows:

$$\tilde{H}_x : N(x) \xrightarrow{\Delta_N} \bigoplus_{y \in Obj(B_2)} B_2(x,y) \otimes N(y)$$

$$\xrightarrow{\prod id_{B_2} \otimes H'_y} \bigoplus_{y \in Obj(B_2)} B_2(x,y) \otimes G^*M(Fy)$$

$$\subset [B_2 \otimes_F G^*M](x).$$

Showing that $\tilde{H}$ lands in $G^*F^*M$, a subcomodule of $[B_2 \otimes_F G^*M]$, is also straightforward; we only need that $F$ and $H$ commute with the appropriate coproducts, and that the cocomposition on $B_2$ is coassociative. So, for each $x \in Obj(B_2)$, we have a map $\tilde{H}_x :$ $N(x) \to G^*F^*M(x)$. It's clear that $\tilde{H}$ is a map of dg comodules since all of the maps used to construct $\tilde{H}$ are maps of dg comodules.

Now, let $\tilde{H} : N \to F^*G^*M$ be a map of dg comodules over $B_2$. We will show that $\tilde{H}$ determines a map of dg comodules $H : N \to B_2 \otimes_G FM$ satisfying Equation 3.4. For $x \in Obj(B_2)$, let $H$ be defined as follows:

$$H_x : N(x) \xrightarrow{\tilde{H}_x} F^*G^*M(x)$$

$$\xrightarrow[inclusion]{canonical} \bigoplus_{\substack{y \in Obj(B_2) \\ z_1 \in Obj(B_1)}} B_2(x,y) \otimes B_1(Fy, z_1) \otimes M(Gz_1)$$

$$\xrightarrow{id_{B_2} \otimes \epsilon_{B_1} \otimes id_M} \bigoplus_{y \in Obj(B_2)} B_2(x,y) \otimes M(GFy).$$

The universal property of $G^*M$ implies that $(id_{B_2} \otimes \Delta_M) \circ H$ is equal to:

$$N(x) \xrightarrow{\tilde{H}_x} \bigoplus_{\substack{y \in Obj(B_2) \\ z_1 \in Obj(B_1)}} B_2(x,y) \otimes B_1(Fy, z_1) \otimes M(Gz_1)$$

$$\xrightarrow[\substack{(id_{B_2} \otimes \Delta_{B_1} \otimes id_M)}]{(id_{B_2} \otimes id_{B_1} \otimes G \otimes id_M) \circ} \bigoplus_{\substack{y \in Obj(B_2) \\ y_1, z_1 \in Obj(B_1)}} B_2(x,y) \otimes B_1(Fy, y_1) \otimes B_0(Gy_1, Gz_1) \otimes M(Gz_1)$$

$$\xrightarrow{id_{B_2} \otimes \epsilon_{B_1} \otimes id_{B_0} \otimes id_M} \bigoplus_{\substack{y \in Obj(B_2) \\ z_1 \in Obj(B_1)}} B_2(x,y) \otimes B_0(GFy, Gz_1) \otimes M(Gz_1).$$

On the other hand, the universal property of $F^*$ implies that $(id_{B_2} \otimes GF \otimes id_M) \circ (\Delta_{B_2} \otimes id_M) \circ H$ is equal to:

$$N(x) \xrightarrow{\tilde{H}_x} \bigoplus_{\substack{y \in Obj(B_2) \\ z_1 \in Obj(B_1)}} B_2(x,y) \otimes B_1(Fy, z_1) \otimes M(Gz_1)$$

$$\xrightarrow[\substack{(id_{B_2} \otimes \Delta_{B_1} \otimes id_M)}]{(id_{B_2} \otimes G \otimes id_{B_1} \otimes id_M) \circ} \bigoplus_{\substack{y \in Obj(B_2) \\ y_1, z_1 \in Obj(B_1)}} B_2(x,y) \otimes B_0(GFy, Gy_1) \otimes B_1(y_1, z_1) \otimes M(Gz_1)$$

$$\xrightarrow{id_{B_2} \otimes id_{B_0} \otimes \epsilon_{B_1} \otimes id_M} \bigoplus_{\substack{y \in Obj(B_2) \\ z_1 \in Obj(B_1)}} B_2(x,y) \otimes B_0(GFy, Gz_1) \otimes M(Gz_1).$$

So, the difference between the two maps in Equation 3.4 comes down to the difference between $(\epsilon_{B_1} \otimes G) \circ \Delta_{B_1}$ and $(G \otimes \epsilon_{B_1}) \circ \Delta_{B_1}$. However, by the counitality of $B_1$, both of these maps are equal to $G$. So, $H$ satisfies Equation 3.4. $\square$

**Proposition 3.2.** *Let* $\lambda : B_1 \to B_0$ *be a functor between conilpotent dg cocategories and* $C$ *a conilpotent quasi-cofree dg comodule over* $B_0$. *Then, as comodules,*

$$(3.5) \qquad\qquad \lambda^* C \cong B_1 \otimes_\lambda T$$

*where righthand side is the following cofree comodule over $B_1$:*

$$[B_1 \otimes_\lambda T](f) := \bigoplus_{h \in Obj(B_0)} B_1(f, h) \otimes T(\lambda h)$$

$$T(\lambda h) = \text{cogenerators of } C(\lambda h) \text{ (see Section 2.3.4).}$$

PROOF OF PROPOSITION 3.2. To prove the proposition, we will give maps

$$F : \lambda^* C \rightleftarrows B_1 \otimes_\lambda T : G$$

and show that $F \circ G = id_{B_1 \otimes_\lambda T}$ and $G \circ F = id_{\lambda^* C}$. We define $F$ as follows:

$$F : \lambda^* C \xrightarrow[\text{inclusion}]{\text{canonical}} B_1 \otimes_\lambda C \xrightarrow[\text{cogenerators}]{\text{project onto}} B_1 \otimes_\lambda T.$$

To define $G$, we will give a map $G' : B_1 \otimes_\lambda T \to B_1 \otimes_\lambda C$, and show that the image of $G'$ lands in $\lambda^* C$. We define $G'$ as follows:

$$G'(b \otimes t) = \sum_{(b)} b_{(1)} \otimes \lambda b_{(2)} \cdot t$$

where $b \otimes t \in B_1 \otimes_\lambda T$ and $\lambda b_{(2)} \cdot t$ are elements of the appropriate components of $C$ written in terms of cogenerators.

To prove that the image of $G'$ lands in $\lambda^* C$, we need to show that the two maps

$$(id_{B_1} \otimes \Delta_C) \circ G', \ (id_{B_1} \otimes \lambda \otimes id_C) \circ (\Delta_{B_1} \otimes id_C) \circ G' : B_1 \otimes_\lambda T \to B_1 \otimes_\lambda C \rightrightarrows B_1 \otimes_\lambda B_0 \otimes C$$

coincide. We have

$$[(1 \otimes \Delta_C) \circ G'](b \otimes t) = \sum_{(b), (\lambda b)} b_{(1)} \otimes (\lambda b_{(2)})_{(1)} \otimes (\lambda b_{(2)})_{(2)} \cdot t$$

$$= \sum_{(b)} b_{(1)} \otimes \lambda b_{(2)} \otimes \lambda b_{(3)} \cdot t$$

$$= [(id_{B_1} \otimes \lambda \otimes id_C) \circ (\Delta_{B_1} \otimes id_C) \circ G'](b \otimes t)$$

where the second equality holds since $\lambda$ is a map of cocategories and $\Delta_{B_1}$ is coassociative.

It's clear from the definitions that $F$ and $G$ are maps of comodules and that $F \circ G = id_{B_1 \otimes_\lambda T}$. All that remains is to show that $G \circ F = id_{\lambda^* C}$. Let $\kappa = \Sigma_i b_i \otimes \beta_i \cdot t_i$ be an arbitrary element of $\lambda^* C \hookrightarrow B_1 \otimes_\lambda C$ where $\beta_i \cdot t_i$ are elements of $C$ written in terms of cogenerators. Then,

$$GF(\kappa) = GF(\Sigma_i b_i \otimes \beta_i \cdot t_i) = \sum_{\substack{i, \\ \beta_i = 1, \\ (b_i)}} b_{i(1)} \otimes \lambda b_{i(2)} \cdot t_i.$$

We can divide the terms in $\kappa$ into two groups: (a) terms in which $\beta_i = 1 \in k$ and (b) terms in which $\beta_i \neq 1 \in k$. Likewise, we can divide the terms in $GF(\kappa)$ into (a) terms in which $\lambda b_{i(2)} = 1$ and (b) terms in which $\lambda b_{i(2)} \neq 1$. From the definitions of $F$ and $G$, it's clear that the Group A terms in $\kappa$ are exactly the Group A terms in $GF(\kappa)$.

To show that the Group B terms are the same, let $b_i \otimes \beta_i \cdot t_i$ be an arbitrary Group B term in $\kappa$. Then, there is a term $b_i \otimes \beta_i \otimes t_i$ in $(id_{B_1} \otimes \Delta_C)\kappa$. Since $(id_{B_1} \otimes \Delta_C)\kappa = (id_{B_1} \otimes \lambda \otimes id_C) \circ (\Delta_{B_1} \otimes id_C)\kappa$, there must be a Group A term, $b_{j_i} \otimes t_{j_i}$, in $\kappa$ such that $b_i \otimes \beta_i \otimes t_i$ is one of the terms in the sum $[(id_{B_1} \otimes \lambda \otimes id_C) \circ (\Delta_{B_1} \otimes id_C)](b_{j_i} \otimes t_{j_i}) = \sum_{(b_{j_i})} b_{j_i(1)} \otimes \lambda b_{j_i(2)} \otimes t_{j_i}$. Thus, $b_i \otimes \beta_i \cdot t_i$ is a Group B term in $GF(\kappa)$.

Now let $b_{i(1)} \otimes \lambda b_{i(2)} \cdot t_i$ be an arbitrary Group B term in $GF(\kappa)$. Then, $b_{i(1)} \otimes \lambda b_{i(2)} \otimes t_i$ is a term in $(id_{B_1} \otimes \lambda \otimes id_C) \circ (\Delta_{B_1} \otimes id_C)\kappa = (id_{B_1} \otimes \Delta_C)\kappa$. So, there is a Group B term, $b_{j_i} \otimes \beta_{j_i} \cdot t_{j_i}$, in $\kappa$ such that $b_{i(1)} \otimes \lambda b_{i(2)} \otimes t_i$ is one of the terms in the sum $(id_{B_1} \otimes \Delta_C)(b_{j_i} \otimes \beta_{j_i} \cdot t_{j_i}) = \sum_{(\beta_{j_i})} b_{j_i} \otimes \beta_{j_i(1)} \otimes \beta_{j_i(2)} \cdot t_{j_i}$. Since $t_i$ is a cogenerator, the only term in the sum that could be equal to $b_{i(1)} \otimes \lambda b_{i(2)} \otimes t_i$ is $b_{j_i} \otimes \beta_{j_i} \otimes t_{j_i}$. Thus, $b_{i(1)} \otimes \lambda b_{i(2)} \cdot t_i$ is a Group B term in $\kappa$. $\qquad\square$

### 3.4. Pullbacks of dg comodules–examples

**Example 3.4.1** (Another definition of C(1))**.** *Let* $\lambda = \delta_{0,1} \in \Lambda([1],[0])$. *Fix algebras* $A_0$ *and* $A_1$, *and set*

$$\hat{\lambda} : B(1) := B(A_0 \to A_1 \to A_0) \to B(A_0 \to A_0) =: B(0)$$

$$C(1) := C(A_0 \to A_1 \to A_0)$$

$$C(0) := C(A_0 \to A_0)$$

*(*$\hat{\lambda}$ *is given by braces, see Section 3.1.2.)*

*From Proposition 3.2, we have* $\hat{\delta}_{0,1}^* C(0) \cong [B(1) \otimes_{\hat{\delta}_{0,1}} T(0)]$ *as comodules. We will use the isomorphisms* $F$ *and* $G$ *in Proposition 3.2 to induce a differential on* $[B(1) \otimes_{\hat{\delta}_{0,1}} T(0)]$ *from* $\hat{\delta}_{0,1}^* C(0)$. *Let* $\phi_{0,1}...\phi_{0,k_0}|\phi_{1,1}...\phi_{1,k_1}|\alpha$ *be a typical element of* $[B(1) \otimes_{\hat{\delta}_{0,1}} T(0)](f_{0,0}, f_{1,0})$ *(see Figure 2.2 for notational conventions). Then,*

$$d_{induced}(\phi_{0,1}...\phi_{0,k_0}|\phi_{1,1}...\phi_{1,k_1}|\alpha)$$

$$= F d_{\hat{\delta}_{0,1}^* C(0)} G(\phi_{0,1}...\phi_{0,k_0}|\phi_{1,1}...\phi_{1,k_1}|\alpha)$$

$$= [F \circ (d_{B(1)} \otimes id_{C(0)} + id_{B(1)} \otimes d_{C(0)})]$$

$$\left( \sum_{\substack{1 \le r_0 \le k_0+1 \\ 1 \le r_1 \le k_1+1}} (\phi_{0,1}...\phi_{0,r_0-1}|\phi_{1,1}...\phi_{1,r_1-1}) \otimes ((\phi_{0,r_0}...\phi_{0,k_0}) \bullet (\phi_{1,r_1}...\phi_{1,k_1})|\alpha)) \right)$$

$$= d_{C(1)(f_{0,0},f_{1,0})}(\phi_{0,1}...\phi_{0,k_0}|\phi_{1,1}...\phi_{1,k_1}|\alpha)$$

*where the last equality holds by looking at which terms from $d_{B(1)} \otimes id_{C(0)} + id_{B(1)} \otimes d_{C(0)}$ are non-zero after projecting to cogenerators, and seeing that those are the same terms as in $d_{C(1)}$. So, $\hat{\delta}_{0,1}^* C(0) \cong C(1)$ as dg comodules.*

**Example 3.4.2** (Another definition of $C(n)$). *Let $\lambda = \delta_{0,n} \in \Lambda([n], [n-1])$. Fix algebras $A_0, \ldots A_n$, and set*

$$\hat{\lambda} : B(n) := B(A_0 \to A_1 \to \cdots \to A_n \to A_0) \to B(A_0 \to A_2 \to A_3 \to \cdots \to A_n \to A_0)$$

$$C(n) := C(A_0 \to A_1 \to \cdots \to A_n \to A_0)$$

*($\hat{\lambda}$ is given by bracing the first and second terms, see Section 3.1.2.)*

*Example 3.4.1 shows*

(3.6)
$$C(1) \cong \hat{\delta}_{0,1}^* C(0)$$

*as dg comodules. Given Equation 3.6 above as a base case, we can show by induction that*

$$C(n) \cong \hat{\delta}_{0,n}^* \ldots \hat{\delta}_{0,1}^* C(0)$$

*as dg comodules. Suppose that $C(W_0 \to \cdots \to W_{n-1} \to W_0) \cong \hat{\delta}_{0,n-1}^* \ldots \hat{\delta}_{0,1}^* C(W_0 \to W_0)$ for any choice of algebras $W_0, \ldots W_{n-1}$. (inductive hypothesis). Then, as comodules, we*

*know*

$$\hat{\delta}_{0,n}^* \ldots \hat{\delta}_{0,1}^* C(0) \cong \hat{\delta}_{0,n}^* C(A_0 \to A_2 \to \ldots \to A_n \to A_0)$$

*(inductive hypothesis applied to algebras $A_0, A_2, \ldots, A_n$)*

$$\cong B(n) \otimes_{\hat{\delta}_{0,n}} T(A_0 \to A_2 \to \ldots \to A_n \to A_0) \quad \text{(Proposition 3.2)}$$

$$\cong B(n) \otimes_{\hat{\delta}_{0,n}} T(A_0 \to A_0) \quad \text{(Definition of $T$)}$$

$$\cong C(n) \quad \text{(Definition of $C(n)$)}$$

*where $T(n)$ are the cogenerators of $C(n)$ (see Definition 2.3.4).*

*To show that the differentials coincide, we compute*

$$F d_{\hat{\delta}_{0,n}^* \ldots \hat{\delta}_{0,1}^* C(0)} G(\phi_{0,1} \ldots \phi_{0,k_0} | \ldots | \phi_{n,1} \ldots \phi_{n,k_n} | t)$$

$$= [F \circ (d_{B(n)} \otimes id_{\hat{\delta}_{0,n-1}^* \ldots \hat{\delta}_{0,1}^* C(0)} + id_{B(n)} \otimes d_{\hat{\delta}_{0,n-1}^* \ldots \hat{\delta}_{0,1}^* C(0)})]$$

$$\left( \sum_{\substack{1 \le j \le n \\ 1 \le r_j \le k_j + 1}} (\phi_{0,1} \ldots \phi_{0,r_0-1} | \ldots | \phi_{n,1} \ldots \phi_{n,r_1-1}) \otimes \right.$$

$$\left. ((\phi_{0,r_0} \ldots \phi_{0,k_0}) \bullet (\phi_{1,r_1} \ldots \phi_{1,k_1}) | \phi_{2,r_1} \ldots \phi_{2,k_2} | \ldots | \phi_{n,r_1} \ldots \phi_{n,k_n} | t) \right)$$

$$= [F \circ (d_{B(n)} \otimes id_{C(A_0 \to A_2 \to \ldots \to A_n \to A_0)} + id_{B(n)} \otimes d_{C(A_0 \to A_2 \to \ldots \to A_n \to A_0)})]$$

$$\left( \sum_{\substack{1 \le j \le n \\ 1 \le r_j \le k_j + 1}} (\phi_{0,1} \ldots \phi_{0,r_0-1} | \ldots | \phi_{n,1} \ldots \phi_{n,r_1-1}) \otimes \right.$$

$$\left. ((\phi_{0,r_0} \ldots \phi_{0,k_0}) \bullet (\phi_{1,r_1} \ldots \phi_{1,k_1}) | \phi_{2,r_1} \ldots \phi_{2,k_2} | \ldots | \phi_{n,r_1} \ldots \phi_{n,k_n} | t) \right)$$

*where the last equality holds by the inductive hypothesis. The terms from*

$d_{B(n)} \otimes id_{C(A_0 \to A_2 \to \ldots \to A_n \to A_0)} + id_{B(n)} \otimes d_{C(A_0 \to A_2 \to \ldots \to A_n \to A_0)}$ *that are non-zero after*

*projecting to cogenerators are exactly the terms in $d_{C(n)}$. So, $C(n) \cong \hat{\delta}_{0,n}^* \ldots \hat{\delta}_{0,1}^* C(0)$ as dg comodules.*

**Example 3.4.3** (Yet another description of C(n)). *Choose a sequence of generating coboundaries $\delta_{i_1,1}, \ldots, \delta_{i_n,n}$ with $0 \le i_j \le j - 1$, $1 \le j \le n$, $n > 0$. Then,*

$$\delta_{i_1,1} \circ \cdots \circ \delta_{i_n,n} = \delta_{0,1} \circ \cdots \circ \delta_{0,n} = \text{unique map in } \Delta([n], [0]) \subset \Lambda([n], [0]).$$

*This implies that, as functors on categories of comodules,*

$$\hat{\delta}_{i_n,n}^* \ldots \hat{\delta}_{i_1,1}^* = (\delta_{i_1,1} \overbrace{\circ \cdots \circ} \delta_{i_n,n})^* \quad \text{(Proposition 3.1)}$$

$$= (\delta_{0,1} \overbrace{\circ \cdots \circ} \delta_{0,n})^* \quad \text{(Computation above)}$$

$$= \hat{\delta}_{0,n}^* \ldots \hat{\delta}_{0,1}^* \quad \text{(Proposition 3.1).}$$

*Since braces are associative, the differentials on $\hat{\delta}_{i_n,n}^* \ldots \hat{\delta}_{i_1,1}^* C(A_0 \to A_0)$ and $\hat{\delta}_{0,n}^* \ldots \hat{\delta}_{0,1}^* C(A_0 \to A_0)$ coincide. So, $\hat{\delta}_{i_n,n}^* \ldots \hat{\delta}_{i_1,1}^* C(A_0 \to A_0) \cong \hat{\delta}_{0,n}^* \ldots \hat{\delta}_{0,1}^* C(A_0 \to A_0)$ as dg comodules.*

**Example 3.4.4** (Pullbacks along codegeneracies). *Fix algebras $A_0, \ldots, A_n$ and let $\sigma_{i,n} \in \Lambda([n], [n+1])$, $0 \le i \le n$ be a generating codegeneracy. Set*

$$\hat{\sigma}_{i,n} : B(n) := B(A_0 \to \ldots \to A_n \to A_0) \to B(A_0 \to \cdots \to A_i \to A_i \to \ldots A_n \to A_0)$$

$$C(n) := C(A_0 \to \cdots \to A_n \to A_0)$$

From Proposition 3.2, we know that $\hat{\sigma}_{i,n}^* C(A_0 \to \cdots \to A_i \to A_i \to \ldots A_n \to A_0) \cong$ $B(n) \otimes_{\hat{\sigma}_{i,n}} T(A_0 \to \cdots \to A_i \to A_i \to \ldots A_n \to A_0) \cong C(n)$ as comodules. To show that the differentials coincide, we compute

$$F d_{\hat{\sigma}_{i,n}^* C(A_0 \to \cdots \to A_i \to A_i \to \ldots A_n \to A_0)} G(\phi_{0,1} \ldots \phi_{0,k_0} | \ldots | \phi_{n,1} \ldots \phi_{n,k_n} | t)$$

$$= [F \circ (d_{B(n)} \otimes id_{\hat{\sigma}_{i,n}^* C(A_0 \to \cdots \to A_i \to A_i \to \ldots A_n \to A_0)} + id_{B(n)} \otimes d_{\hat{\sigma}_{i,n}^* C(A_0 \to \cdots \to A_i \to A_i \to \ldots A_n \to A_0)})]$$

$$\left( \sum_{\substack{1 \le j \le n \\ 1 \le r_j \le k_j + 1}} (\phi_{0,1} \ldots \phi_{0,r_0-1} | \ldots | \phi_{n,1} \ldots \phi_{n,r_1-1}) \otimes \right.$$

$$\left. (\phi_{0,r_0} \ldots \phi_{0,k_0} | \ldots | \phi_{0,r_{i-1}} \ldots \phi_{0,k_{i-1}} | 1 | \phi_{0,r_i} \ldots \phi_{0,k_i} | \ldots | \phi_{n,r_1} \ldots \phi_{n,k_n} | t) \right).$$

Since 1 is a unit for braces, the terms from $d_{B(n)} \otimes id_{\hat{\sigma}_{i,n}^* C(A_0 \to \cdots \to A_i \to A_i \to \ldots A_n \to A_0)} + id_{B(n)} \otimes$ $d_{\hat{\sigma}_{i,n}^* C(A_0 \to \cdots \to A_i \to A_i \to \ldots A_n \to A_0)}$ that are non-zero after projecting to cogenerators are exactly the terms in $d_{C(n)}$. So, $\hat{\sigma}_{i,n}^* C(A_0 \to \cdots \to A_i \to A_i \to \ldots A_n \to A_0) \cong C(n)$ as dg comodules.

**Example 3.4.5** (Pullbacks along rotations). *Fix algebras* $A_0, \ldots, A_n$ *and let* $\tau_n \in$ $\Lambda([n], [n])$ *be a generating rotation. Set*

$$\hat{\tau}_n : B(n) := B(A_0 \to \ldots \to A_n \to A_0) \to B(A_n \to A_0 \to \ldots \to A_n)$$

$$C(n) := C(A_0 \to \cdots \to A_n \to A_0)$$

*From Proposition 3.2, we know that* $\hat{\tau}_n^* C(A_n \to A_0 \to \ldots \to A_n) \cong B(n) \otimes_{\hat{\tau}_n} T(A_n \to$ $A_0 \to \ldots \to A_n)$ *as comodules. Unpacking the righthand side, we see that* $B(n) \otimes_{\hat{\tau}_n}$ $T(A_n \to A_0 \to \ldots \to A_n) \cong C(A_n \to A_0 \to \ldots \to A_n)$ *as complexes–the isomorphism is given by* $\hat{\tau}_n \otimes id_T$.

## 3.5. Adjunction between $\lambda^*$ and $\lambda_\#$

In this section, we define $\lambda_\#$, the left adjoint to $\lambda^*$. More precisely, for any functor, $\lambda : B_1 \to B_0$ between conilpotent dg cocategories, we define a functor $\lambda_\#$ from the category of conilpotent dg comodules over $B_1$ to the category of conilpotent dg comodules over $B_0$. The adjunction will be used to show that structures we've established for $(B(n), C(n))$ still exist after we pass from cocategories and comodules to categories and modules by applying (a categorified) $Cobar(-)$ to $(B(n), C(n))$ (see Section 4.5.2). If a lesson of this thesis is that working with cocategories is more tractable than with categories, then the reader may skip this section or save it until s/he is ready for Section 4.5.1.

### 3.5.1. The functors $\lambda_\#$

Let $\lambda : B_1 \to B_0$ be a functor between conilpotent dg cocategories. Let $C$ be a conilpotent dg comodule over $B_1$. We define $\lambda_\# C$ as follows: for $f \in Obj(B_0)$,

$$\lambda_\# C(f) := \Big( \bigoplus_{f' \in \lambda^{-1}f} C^\bullet(f'),$$

$$\Delta_{\lambda_\# C}(f) : \bigoplus_{f' \in \lambda^{-1}f} C^\bullet(f') \xrightarrow{\bigoplus_{f'} \Delta_{C^\bullet}(f')} \bigoplus_{\substack{f' \in \lambda^{-1}f \\ h' \in Obj(B_1)}} B_1^\bullet(f', h') \otimes C^\bullet(h')$$

$$\xrightarrow{\bigoplus_{h', f'} \lambda \otimes id_{C^\bullet(h')}} \bigoplus_{h' \in Obj(B_1)} B_0^\bullet(f, \lambda h') \otimes C^\bullet(h')$$

$$\xrightarrow{include} \bigoplus_{h \in Obj(B_0)} B_0^\bullet(f, h) \otimes \Big( \bigoplus_{h' \in \lambda^{-1}h} C^\bullet(h') \Big) \Big).$$

To check that $\Delta_{\lambda_\# C}$ is well-defined, we need that the image of the first map, $\bigoplus_{f'} \Delta_{C^\bullet}(f')$, is a finite sum. This is true since $C$ being conilpotent implies that the image of $\Delta_{C^\bullet}(f')$

is a finite sum for each $f' \in Obj(B_1)$. If $\lambda^{-1}f$ is empty, we set $\lambda_\# C(f) := 0$. It is straightforward to check that $(\lambda_\# C, \Delta_{\lambda_\# C})$ is coassociative, conilpotent and coaugmented. We will call $\lambda_\#$ "co-restriction of scalars".

Let $F : C \to D$ be map of dg comodules over $B_1$. We define $\lambda_\# F$ as follows:

$$(\lambda_\# F)_f : \lambda_\# C(f) = \bigoplus_{f' \in \lambda^{-1}f} C^\bullet(f') \xrightarrow{\bigoplus_{f' \in \lambda^{-1}f} F_{f'}} \bigoplus_{f' \in \lambda^{-1}f} D^\bullet(f') = \lambda_\# D(f).$$

It's straightforward to check that $\lambda_\#$ is a functor (i.e., respects composition of morphisms).

### 3.5.2. Adjunction

**Proposition 3.3.** *Given a functor between conilpotent dg cocategories, $\lambda : B_1 \to B_0$, let*

$$\lambda^* : \begin{matrix} \text{Category of} \\ \text{conilpotent} \\ \text{dg comodules over } B_0 \end{matrix} \leftrightarrows \begin{matrix} \text{Category of} \\ \text{conilpotent} \\ \text{dg comodules over } B_1 \end{matrix} : \lambda_\#$$

*be the functors defined in Sections 3.3.1 and 3.5.1. Then, $\lambda_\#$ is left adjoint to $\lambda^*$.*

**Remark 3.5.1.** Proposition 3.3 is a categorified co-version of the adjunction between extension of scalars (left) and restriction of scalars (right) for modules over algebras.

Proof of Proposition 3.3. Let $C$ be a conilpotent dg comodule over $B_1$ and $D$ be a dg conilpotent dg comodule over $B_0$. We want to show that

$$Hom_{B_1}(C, \lambda^* D) = Hom_{B_0}(\lambda_\# C, D)$$

as sets.

We will give maps

$$\Phi : Hom_{B_0}(\lambda_\# C,\, D) \leftrightarrows Hom_{B_1}(C,\, \lambda^* D) : \Phi^{-1}$$

satisfying $\Phi \circ \Phi^{-1} = id$ and $\Phi^{-1} \circ \Phi = id$.

First, we define $\Phi$. Let $F$ be a morphism from $\lambda_\# C$ to $D$. By defintion, for $f \in Obj(B_0)$, we have maps of complexes

$$F_f : \bigoplus_{f' \in \lambda^{-1} f} C^\bullet(f') \to D^\bullet(f).$$

Define $\Phi F \in Hom_{B_1}(C,\, \lambda^* D)$ as follows: for $f' \in Obj(B_1)$,

(3.7)
$$\Phi F_{f'} : C^\bullet(f') \xrightarrow{\Delta_C} \bigoplus_{h' \in Obj(B_1)} B_1^\bullet(f', h') \otimes C^\bullet(h')$$
$$\xrightarrow{\bigoplus_{h'} id_{B_1} \otimes F_{\lambda h'}|_{h'}} \bigoplus_{h' \in Obj(B_1)} B_1^\bullet(f', h') \otimes D^\bullet(\lambda h')$$
$$\xrightarrow{include} [B_1 \otimes_\lambda D](f').$$

By the universal property of $\lambda^* D$, this defines a morphism $C \to \lambda^* D$ if the two maps

$$(id_{B_1} \otimes \Delta_D) \circ \Phi F,\ (id_{B_1} \otimes \lambda \otimes id_D) \circ (\Delta_{B_1} \otimes id_D) \circ \Phi F : C \rightrightarrows B_1 \otimes_\lambda B_0 \otimes D$$

coincide. In fact, on $f' \in Obj(B_1)$, both maps are equal to:

$$C^\bullet(f') \xrightarrow{\Delta_C} \bigoplus_{h' \in Obj(B_1)} B_1^\bullet(f', h') \otimes C^\bullet(h')$$

$$\xrightarrow[h']{\bigoplus id_{B_1} \otimes \Delta_C} \bigoplus_{g', h' \in Obj(B_1)} B_1^\bullet(f', g') \otimes B_1^\bullet(g', h') \otimes C^\bullet(h')$$

$$\xrightarrow[h', g']{\bigoplus id_{B_1} \otimes \lambda \otimes 1_C} \bigoplus_{g', h' \in Obj(B_1)} B_1^\bullet(f', g') \otimes B_0^\bullet(\lambda g', \lambda h') \otimes C^\bullet(h')$$

$$\xrightarrow[h', g']{\bigoplus id_{B_1} \otimes id_{B_0} \otimes F_{\lambda h'}|_{h'}} \bigoplus_{g', h' \in Obj(B_1)} B_1^\bullet(f', g') \otimes B_0^\bullet(\lambda g', \lambda h') \otimes D^\bullet(\lambda h').$$

This fact follows from $F$ being a map of comodules. It's also clear that $\Phi F$ commutes with coproducts and differentials. So, we've shown $\Phi F \in Hom_{B_1}(C, \lambda^* D)$.

Second, we define $\Phi^{-1}$. Now, let $F \in Hom_{B_1}(C, \lambda^* D)$. For $f \in Obj(B_0)$, define

$$\Phi^{-1} F_f : \bigoplus_{f' \in \lambda^{-1} f} C^\bullet(f') \xrightarrow[f']{\bigoplus F_{f'}} \bigoplus_{\substack{f' \in \lambda^{-1} f, \\ h' \in Obj(B_1)}} B_1^\bullet(f', h') \otimes D^\bullet(\lambda h')$$

$$\xrightarrow[f', h']{\bigoplus \lambda \otimes id_D} \bigoplus_{h \in Obj(B_0)} B_0^\bullet(f, h) \otimes D^\bullet(h)$$

$$\xrightarrow[h]{\bigoplus \epsilon_{B_0} \otimes id_D} D^\bullet(f).$$

It's clear that $\Phi^{-1} F$ commutes with the differentials. We will show that $\Phi^{-1} F$ is a map of comodules. Figure 3.1 gives a diagram showing that

$$(3.8) \qquad \Delta_D \circ \Phi^{-1} F_f = (\bigoplus_{f', h', r'} \epsilon_{B_0} \lambda \otimes \lambda \otimes id_D) \circ (\bigoplus_{f', h'} \Delta_{B_1} \otimes id_D) \circ (\bigoplus_{f'} F_{f'}).$$

On the other hand, Figure 3.2 gives a diagram showing that

$$(3.9) \quad (id_{B_1} \otimes \Phi^{-1}F) \circ \Delta_{\lambda_\# C} = ( \bigoplus_{f',h',r'} \lambda \otimes \epsilon_{B_0} \lambda \otimes id_D) \circ (\bigoplus_{f',h'} \Delta_{B_1} \otimes id_D) \circ (\bigoplus_{f'} F_{f'}).$$

We see that the righthand sides of Equations 3.8 and 3.9 are the same except for the $B_0$ factor on which $\epsilon_{B_0}$ acts. However, in general, for $\lambda : B_1 \to B_0$ a map of dg cocategories, we have

$$(\lambda \otimes \epsilon_{B_0} \lambda) \circ \Delta_{B_1} = (id_{B_0} \otimes \epsilon_{B_0}) \circ \Delta_{B_0} \circ \lambda \quad (\lambda \text{ commutes with coproduct})$$

$$= id_{B_0} \circ \lambda \quad (\text{definition of cocategory})$$

$$= (\epsilon_{B_0} \otimes id_{B_0}) \circ (\Delta_{B_0}) \circ \lambda \quad (\text{definition of cocategory})$$

$$= (\epsilon_{B_0} \lambda \otimes \lambda) \circ \Delta_{B_1} \quad (\lambda \text{ commutes with coproduct}).$$

So, $(id_{B_1} \otimes \Phi^{-1}F) \circ \Delta_{\lambda_\# C} = \Delta_D \circ \Phi^{-1}F$, and $\Phi^{-1}F \in Hom_{B_0}(\lambda_\# C, D)$.

For $F : C \to \lambda^* D$ a map of dg comodules and $f' \in B_1$, Figure 3.3 shows that $\Phi\Phi^{-1}F_{f'} = F_{f'}$. For $F : \lambda_\# C \to D$ a map of dg comodules and $f \in B_0$, Figure 3.4 shows that $\Phi^{-1}\Phi F_f = F_f$. Thus, we have $\Phi\Phi^{-1} = id$ and $\Phi^{-1}\Phi = id$. $\qquad \square$

Figure 3.1. Commuting diagram involving $\Delta_D \circ \Phi^{-1}F$. The fact that $F : C \to \lambda^*D$ and the universal property of $\lambda^*D$ imply that the diagram commutes.

$\Delta_D \circ \Phi^{-1}F =$ composition of red arrows.

Figure 3.2. Commuting diagram involving $(id_{B_1} \otimes \Phi^{-1}F) \circ \Delta_{\lambda_\# C}$. The fact that $F$ respects coproducts implies that the left square commutes.

$(id_{B_1} \otimes \Phi^{-1}F) \circ \Delta_{\lambda_\# C} = $ composition of red arrows.

Figure 3.3. Commuting diagram involving $\Phi\Phi^{-1}F_{f'}$. The square commutes because $F$ respects coproducts; the composition of the bottom row of horizontal arrows is equal to the identity because $\lambda_{\#}D$ satisfies counitality.

$\Phi\Phi^{-1}F_{f'}$ = composition of red arrows.

$\displaystyle\bigoplus_{f'\in\lambda^{-1}f} C^\bullet(f')$

$\displaystyle\bigoplus_{\substack{f'\in\lambda^{-1}f,\\ g'\in Obj(B_1)}} B_1^\bullet(f',g')\otimes C^\bullet(g')$

$\displaystyle\bigoplus_{\substack{f'\in\lambda^{-1}f,\\ g'\in Obj(B_1)}} B_1^\bullet(f',g')\otimes D\bullet(\lambda g')$

$\displaystyle\bigoplus_{g'\in Obj(B_1)} B_0^\bullet(f,\lambda g')\otimes C^\bullet(g')$

$\displaystyle\bigoplus_{g\in Obj(B_0)} B_0^\bullet(f,g)\otimes D^\bullet(g)$

$D^\bullet(f)$

$D^\bullet(f)$

$\bigoplus_{f'}\Delta_C$

$\bigoplus_{f',g'} id_{B_1}\otimes F_{\lambda g'}|_{g'}$

$\bigoplus_{f',g'} \lambda\otimes id_C$

$\bigoplus_{f',g'} \lambda\otimes id_D$

$\bigoplus_{g'} id_{B_0}\otimes F_{\lambda g'}|_{g'}$

$\bigoplus_{g'}^q \epsilon_{B_0}\otimes id_D$

$F_f$

$\Delta_D$

$id$
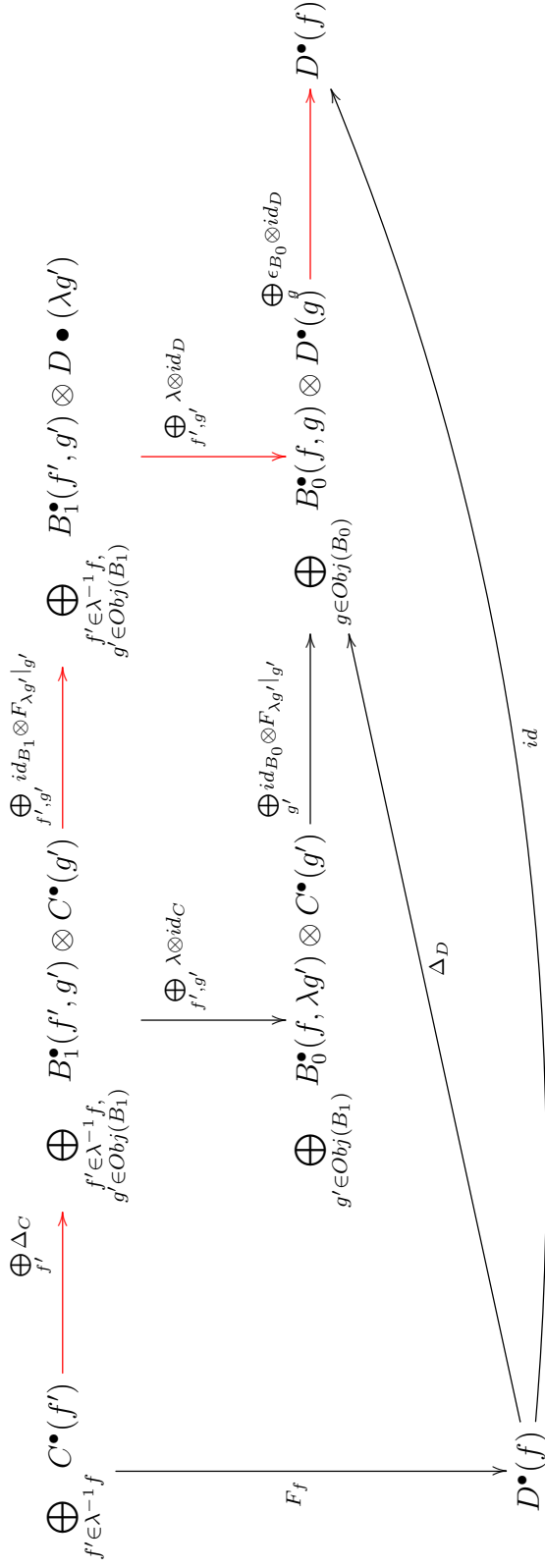
Figure 3.4. Commuting diagram involving $\Phi^{-1}\Phi F_f$. The concave pentagon on the left side commutes because $F$ respects coproducts; the triangle in the bottom right corner commutes because $D$ satisfies counitality.

$\Phi^{-1}\Phi F_f =$ composition of red arrows.

## 3.6. Maps $\lambda_!$

In this section, we give maps $\lambda_! : C(\mathcal{A}) \to \hat{\lambda}^* C(\lambda\mathcal{A})$ of dg comodules over $B(\mathcal{A})$ where $\lambda$ is a generating morphism in $\Lambda$ that induces a morphsim in $\chi$ with source $\mathcal{A} \in Obj(\chi)$. Showing that the $\lambda_!$'s satisfy cyclic relations up to homotopy is the computational heart of this thesis, and will be done in the next chapter. For now, we introduce the $\lambda_!$'s.

Technically, we should write $\lambda_{\mathcal{A}!}$ instead of $\lambda_!$, but we will be clear about the source when needed. In this section, we fix algebras $A_0, ..., A_n$ and set $B(n) := B(A_0 \to ... \to A_n \to A_0)$, $C(n) := C(A_0 \to ... \to A_n \to A_0)$. We define maps $\lambda_! : C(A_0 \to ... \to A_n \to A_0) \to \hat{\lambda}^* C(\lambda(A_0 \to ... \to A_n \to A_0))$.

### 3.6.1. Generating coboundaries $\delta_{j,n!}$ for $n \in \mathbb{N}$, $0 \leq j \leq n-1$

From Example 3.4.3, we know that $C(n) \cong \hat{\delta}^*_{j,n} C(A_0 \to ... \to A_j \longrightarrow A_{j+2} \to ... \to A_0)$. So, define $\delta_{j,n!} : C(n) \xrightarrow{id} C(n) \cong \hat{\delta}^*_{j,n} C(A_0 \to ... \to A_j \longrightarrow A_{j+2} \to ... \to A_0)$.

### 3.6.2. Generating codegeneracies $\sigma_{i,n!}$ for $n \in \mathbb{N}$, $0 \leq i \leq n$

From Example 3.4.4, we know that $C(n) \cong \hat{\sigma}^*_{i,n} C(A_0 \to ... \to A_i \to A_i \to ... \to A_0)$. So, define $\sigma_{i,n!} : C(n) \xrightarrow{id} C(n) \cong \hat{\sigma}^*_{i,n} C(A_0 \to ... \to A_i \to A_i \to ... \to A_0)$.

### 3.6.3. Generating rotations $\tau_{n!}$

**3.6.3.1.** $n = 0$. Let $\tau_{0!} = id : C(0) \xrightarrow{id} C(0) \cong id^* C(0) \cong \hat{\tau}^*_0 C(0)$.

**3.6.3.2.** $n = 1$. We want to define a map of dg comodules over $B(1)$

$$\tau_{1!} : C(1) := C(A_0 \to A_1 \to A_0) \to \hat{\tau}^*_1 C(A_1 \to A_0 \to A_1).$$

Example 3.4.5 describes the structure of $\hat{\tau}_1^* C(A_1 \to A_0 \to A_1)$, which is quasi-cofree over $B(1)$. So, we can define $\tau_{1!}$ by giving maps from $C(1)$ to the cogenerators of $\hat{\tau}_1^* C(A_1 \to A_0 \to A_1)$ and checking that the corresponding map of comodules commutes with the differentials.

More explicitly, for $f = (f_{0,0}, f_{1,0}) \in Obj(B(1))$, we will give $k$-linear maps

$$v^f : C(1)^\bullet(f) \to C_{-\bullet}(A_{1, f_{0,0} f_{1,0}} A_{1 id})$$

$$(\phi_{0,1} ... \phi_{0,n_0} | \phi_{1,1} ... \phi_{0,n_1} | \alpha) \mapsto v^f_{n_0, n_1}(\phi_{0,1} ... \phi_{0,n_0} | \phi_{1,1} ... \phi_{1,n_1} | \alpha).$$

Then, we lift $\{v_f | f \in Obj(B(1))\}$ to a map of comodules in the standard way:

$$
\begin{aligned}
&\tau_{1!f}(\phi_{0,1} ... \phi_{0,n_0} | \phi_{1,1} ... \phi_{1,n_1} | \alpha) \\
(3.10) \quad &= \sum_{\substack{0 \le k_0 \le n_0 \\ 0 \le k_1 \le n_1}} \phi_{0,1} ... \phi_{0,k_1} | \phi_{1,1} ... \phi_{1,k_0} | v^{f_{0,k_1}, f_{1,k_0}}_{n_0 - k_0, n_1 - k_1}(\phi_{0,k_0+1} ... \phi_{0,n_0} | \phi_{1,k_1+1} ... \phi_{1,n_1} | \alpha)
\end{aligned}
$$

(see Figure 2.2 for notation). Finally, we will check by direct computation that $\tau_{1!}$ defined as such commutes with the differentials. To make the exposition smooth, all of this is done in Appendix Proposition B.1.

**3.6.3.3.** $n > 1$. For $n > 1$, we define $\tau_{n!}$ by pulling back $\tau_{1!}$ along $\delta_{0,*}$'s as follows:

$$\tau_{n!} : C(n) \cong (\delta_{0,2} \overbrace{\circ \dots \circ} \delta_{0,n})^* C(A_0 \to A_n \to A_0)$$

$$\xrightarrow{(\delta_{0,2} \overbrace{\circ \dots \circ} \delta_{0,n})^* \tau_{1!}} (\delta_{0,2} \overbrace{\circ \dots \circ} \delta_{0,n})^* \hat{\tau}_1^* C(A_n \to A_0 \to A_n)$$

$$\cong (\tau_1 \circ \delta_{0,2} \overbrace{\circ \dots \circ} \delta_{0,n})^* C(A_n \to A_0 \to A_n)$$

$$\cong (\delta_{1,2} \circ \overbrace{\dots \circ} \delta_{1,n} \circ \tau_n)^* C(A_n \to A_0 \to A_n)$$

$$\cong \hat{\tau}_n^* (\delta_{1,2} \overbrace{\circ \dots \circ} \delta_{1,n})^* C(A_n \to A_0 \to A_n)$$

$$\cong \hat{\tau}_n^* C(A_n \to A_0 \to \dots \to A_n).$$

CHAPTER 4

# A homotopically sheafy-cyclic object

## 4.1. Motivation of this chapter

In Section 3.1, we gave a sheafy-cyclic object in dg cocategories. We would like to extend that construction to a sheafy-cyclic object in the category of dg cocategories with a dg comodule. Namely, we would like to give a functor from $\chi$ to $\mathcal{D}$ where $\mathcal{D}$ the following category:

$$Obj(\mathcal{D}) = \{(B,C)|\text{B is a dg cocategory, C is a dg comodule over B}\}$$

$$\mathcal{D}((B_1, C_1), (B_0, C_0)) = \{(f, f_!)|f : B_1 \to B_0 \text{ is a functor,}$$

$$f_! : C_1 \to f^*C_0 \text{ is a map of dg comodules over } B_1\}$$

$$\mathcal{D}((B_2, C_2), (B_1, C_1)) \times \mathcal{D}((B_1, C_1), (B_0, C_0)) \xrightarrow{composition} \mathcal{D}((B_2, C_2), (B_0, C_0))$$

$$(f, f_!) \times (g, g_!) \mapsto (gf, f^*(g_!) \circ f_!)$$

(Proposition 3.1 implies that composition in $\mathcal{D}$ is associative.)

In Section 3.6, we gave maps $\lambda_! : C(\mathcal{A}) \to \hat{\lambda}^*C(\lambda\mathcal{A})$ where $\lambda$ is a generating morphism in $\Lambda$ that induces a morphism in $\chi$ with source $\mathcal{A} \in Obj(\chi)$. Ideally, $(\hat{\lambda}, \lambda_!)$ would give a functor $\chi \to \mathcal{D}$, however, the $\lambda_!$'s only respect composition up to homotopy. Fortunately, most of the compositions are respected and, for the ones that are not, we have explicit homotopies. Our homotopies commute with the composable $\lambda_!$'s so that no higher homotopies are needed.

In this chapter, we will show which compositions are respected on the nose and which ones need homotopies. We will then give these homotopies and show that no higher

homotopies are needed. These sections are the computational heart of this thesis. Finally, we will repackage this "functor up to homotopy" in more abstract terms.

## 4.2. Homotopies

Here, we will show that the maps of dg comodules given in Section 3.6 satisfy the relations in $\Lambda$ (Equation A.2) up to homotopy. More precisely, we will show that

$$\hat{\delta}_{j,n}^*(\delta_{i,n-1!}) \circ \delta_{j,n!} = \hat{\delta}_{i,n}^*(\delta_{j-1,n-1!}) \circ \delta_{i,n!} \quad 0 \le i < j \le n-1$$

$$\hat{\sigma}_{j,n}^*(\sigma_{i,n+1!}) \circ \sigma_{j,n!} = \hat{\sigma}_{i,n}^*(\sigma_{j+1,n+1!}) \circ \sigma_{i,n!} \quad 0 \le i \le j \le n$$

(4.1a)
$$\hat{\sigma}_{i,n}^*(\delta_{j,n+1!}) \circ \sigma_{i,n!} = \begin{cases} \hat{\delta}_{j-1,n}^*(\sigma_{i,n-1!}) \circ \delta_{j-1,n!} & 0 \le i < j \le n \\[2mm] id & j = i, i-1 \\[2mm] \hat{\delta}_{j,n}^*(\sigma_{i-1,n-1!}) \circ \delta_{j,n!} & 0 \le j < i-1 \le n-1 \end{cases}$$

(4.1b)
$$\hat{\sigma}_{i,n}^*(\tau_{n+1!}) \circ \sigma_{i,n!} = \hat{\tau}_n^*(\sigma_{i+1,n!}) \circ \tau_{n!} \quad 0 \le i \le n-1$$

$$\hat{\delta}_{j,n}^*(\tau_{n-1!}) \circ \delta_{j,n!} = \hat{\tau}_n^*(\delta_{j+1,n!}) \circ \tau_{n!} \quad 0 \le j \le n-1$$

(4.1c)
$$(\widehat{\tau_1\sigma_{0,0}})^*(\delta_{0,1!}) \circ \hat{\sigma}_{0,0}^*(\tau_{1!}) \circ \sigma_{0,0!} = id$$

and

(4.2a)
$$\hat{\tau}_n^{*2}(\delta_{0,n!}) \circ \hat{\tau}_n^*(\tau_{n!}) \circ \tau_{n!} \simeq \hat{\delta}_{n-1,n}^*(\tau_{n-1!}) \circ \delta_{n-1,n!}$$

(4.2b)
$$\hat{\tau}_n^{*n}(\tau_{n!}) \circ \ldots \circ \hat{\tau}_n^*(\tau_{n!}) \circ \tau_{n!} \simeq id$$

(4.2c)
$$\hat{\sigma}_{n,n}^*(\tau_{n+1!}) \circ \sigma_{n,n!}$$
$$\simeq (\widehat{\tau_{n+1}^n\sigma_{0,n}\tau_n})^*(\tau_{n+1!}) \circ \ldots \circ (\widehat{\tau_{n+1}\sigma_{0,n}\tau_n})^*(\tau_{n+1!}) \circ (\widehat{\sigma_{0,n}\tau_n})^*(\tau_{n+1!}) \circ \hat{\tau}_n^*(\sigma_{0,n!}) \circ \tau_{n!}$$

### 4.2.1. Aside on composing $\lambda_!$'s

In this section and the next, we will use the following convention for composing $\lambda_!$'s, which we first illustrate with an example. Suppose we have 3 composable generating morphisms in $\Lambda$: $\lambda_1 \in \Lambda([m], [n])$, $\lambda_2 \in \Lambda([n], [p])$, and $\lambda_3 \in \Lambda([p], [q])$. Fix a sequence of algebras $A_0, ..., A_m$. We can construct the following composition of morphisms of dg comodules over $B(\mathcal{A}) := B(A_0 \to ... \to A_m \to A_0)$:

$$C(\mathcal{A}) \xrightarrow{\lambda_{1,\mathcal{A}!}} \hat{\lambda}_1^* C(\lambda_1 \mathcal{A}) \xrightarrow{\hat{\lambda}_1^*(\lambda_{2,\lambda_1 \mathcal{A}!})} \hat{\lambda}_1^* \hat{\lambda}_2^* C(\lambda_2 \lambda_1 \mathcal{A}) \xrightarrow{\hat{\lambda}_1^* \hat{\lambda}_2^*(\lambda_{3,\lambda_2 \lambda_1 \mathcal{A}!})} \hat{\lambda}_1^* \hat{\lambda}_2^* \hat{\lambda}_3^* C(\lambda_3 \lambda_2 \lambda_1 \mathcal{A}).$$

To simplify the notation, we will write

$$\hat{\lambda}_1^* \hat{\lambda}_2^*(\lambda_{3!}) \circ \hat{\lambda}_1^*(\lambda_{2!}) \circ \lambda_{1!} \quad \text{instead of} \quad \hat{\lambda}_1^* \hat{\lambda}_2^*(\lambda_{3,\lambda_2 \lambda_1 \mathcal{A}!}) \circ \hat{\lambda}_1^*(\lambda_{2,\lambda_1 \mathcal{A}!}) \circ \lambda_{1,\mathcal{A}!}.$$

More generally, when the reader sees $\hat{\lambda}_1^* ... \hat{\lambda}_{r-1}^*(\lambda_{r!}) \circ ... \circ \hat{\lambda}_1^*(\lambda_{2!}) \circ \lambda_{1!}$ for a sequence of composable generating morphisms $\lambda_r \circ ... \circ \lambda_1$ in $\Lambda$, s/he may decode this notation by choosing a sequence of algebras $A_0, ..., A_{m=\text{source of } \lambda_1}$, setting $\mathcal{A} = (A_0 \to ... \to A_m \to A_0)$, and letting the composition of $\lambda_!$'s denote $\hat{\lambda}_1^* ... \hat{\lambda}_{r-1}^*(\lambda_{r,\lambda_{r-1}...\lambda_1 \mathcal{A}!}) \circ ... \circ \hat{\lambda}_1^*(\lambda_{2,\lambda_1 \mathcal{A}!}) \circ \lambda_{1,\mathcal{A}!}$.

### 4.2.2. Strict relations: showing Equations 4.1 hold

Equation 4.1a has three relations. All of the $\sigma_!$'s and $\delta_!$'s in Equation 4.1a are identity maps, so it's clear that these relations hold.

Equation 4.1b has two relations. To show that the first one holds, we have

$$\hat{\sigma}_{i,n}^*(\tau_{n+1!}) \circ \sigma_{i,n!} = \hat{\sigma}_{i,n}^*((\widehat{\delta_{0,2}...\delta_{0,n+1}})^*(\tau_{1!})) \circ \sigma_{i,n!} \quad \text{definition of } \tau_{n+1!}$$

$$= (\widehat{\delta_{0,2}...\delta_{0,n+1}\sigma_{i,n}})^*(\tau_{1!}) \circ \sigma_{i,n!} \quad \text{Proposition 3.1}$$

$$= (\widehat{\delta_{0,2}...\delta_{0,n}})^*(\tau_{1!}) \circ \sigma_{i,n!}$$

$$= \tau_{n!} \circ \sigma_{i,n!} \quad \text{definition of } \tau_{n!}$$

$$= \tau_{n!} \circ id = id \circ \tau_{n!}$$

$$= \hat{\tau}_n^*(\sigma_{i+1,n!}) \circ \tau_{n!}.$$

To show that the second relation holds, the reasoning is the same as above. We have

$$\hat{\delta}_{j,n}^*(\tau_{n-1!}) \circ \delta_{j,n!} = \hat{\delta}_{j,n}^*((\widehat{\delta_{0,2}...\delta_{0,n-1}})^*(\tau_{1!})) \circ \delta_{j,n!}$$

$$= (\widehat{\delta_{0,2}...\delta_{0,n-1}\delta_{j,n}})^*(\tau_{1!}) \circ \delta_{j,n!}$$

$$= \tau_{n!} \circ \delta_{j,n!}$$

$$= \tau_{n!} \circ id = id \circ \tau_{n!}$$

$$= \hat{\tau}_n^*(\delta_{j+1,n!}) \circ \tau_{n!}.$$

Equation 4.1c has one relation. The only map in this relation that is not defined to be an identity map is $\hat{\sigma}_{0,0}^*(\tau_{1!})$. We will compute this map and show that it is also an identity. Let $(\phi_{0,1}...\phi_{0,k_0}|\alpha) \in C(A_0 \to A_0)$ (see Figure 2.2 for notation). By Proposition

3.2,

$$C(A_0 \to A_0) \xrightarrow{\cong} \hat{\sigma}_{0,0}^* C(A_0 \to A_0 \to A_0)$$

$$(\phi_{0,1}...\phi_{0,k_0}|\alpha) \mapsto \sum_{0 \le r_0 \le k_0} (\phi_{0,1}...\phi_{0,r_0}) \otimes (1|\phi_{0,r_0+1}...\phi_{0,k_0}|\alpha).$$

Applying $\hat{\sigma}_{0,0}^*(\tau_{1!})$ to the righthand side, we have

$$\hat{\sigma}_{0,0}^* C(A_0 \to A_0 \to A_0) \xrightarrow{\hat{\sigma}_{0,0}^*(\tau_{1!})} \hat{\sigma}_{0,0}^* \hat{\tau}_1^* C(A_0 \to A_0 \to A_0)$$

$$\sum_{0 \le r_0 \le k_0} (\phi_{0,1}...\phi_{0,r_0}) \otimes (1|\phi_{0,r_0+1}...\phi_{0,k_0}|\alpha) \mapsto \sum_{0 \le r_0 \le s_0 \le k_0} (\phi_{0,1}...\phi_{0,r_0}) \otimes$$

$$(\phi_{0,r_0+1}...\phi_{0,s_0}|1|\tau_1(1|\phi_{0,s_0+1}...\phi_{0,k_0}|\alpha)).$$

The righthand side above is equal to

$$\sum_{0 \le r_0 \le s_0 \le k_0} (\phi_{0,1}...\phi_{0,r_0}) \otimes (\phi_{0,r_0+1}...\phi_{0,s_0}|1|\tau_1(1|\phi_{0,s_0+1}...\phi_{0,k_0}|\alpha))$$

$$= \sum_{0 \le r_0 \le s_0 \le k_0} (\phi_{0,1}...\phi_{0,r_0}) \otimes (\phi_{0,r_0+1}...\phi_{0,s_0}|1|\upsilon_{0,k_0-s_0}(1|\phi_{0,s_0+1}...\phi_{0,k_0}|\alpha))$$

$$\text{(see Proposition B.1 for } \upsilon_{\cdot,\cdot})$$

$$= \sum_{0 \le r_0 \le k_0} (\phi_{0,1}...\phi_{0,r_0}) \otimes (\phi_{0,r_0+1}...\phi_{0,k_0}|1|\alpha) \qquad (\upsilon_{0,>0} = 0)$$

$$\in \hat{\sigma}_{0,0}^* \hat{\tau}_1^* C(A_0 \to A_0 \to A_0).$$

Finally, applying Proposition 3.2 again, we have

$$\hat{\sigma}_{0,0}^* \hat{\tau}_1^* C(A_0 \to A_0 \to A_0) \xrightarrow[\cong]{\text{project onto cogenerators}} C(A_0 \to A_0)$$

$$\sum_{0 \le r_0 \le k_0} (\phi_{0,1}...\phi_{0,r_0}) \otimes (\phi_{0,r_0+1}...\phi_{0,k_0}|1|\alpha) \mapsto (\phi_{0,1}...\phi_{0,k_0}|\alpha).$$

So, we've shown

$$C(A_0 \to A_0) \cong \hat{\sigma}_{0,0}^* C(A_0 \to A_0 \to A_0) \xrightarrow{\hat{\sigma}_{0,0}^*(\tau_{1!})} \hat{\sigma}_{0,0}^* \hat{\tau}_1^* C(A_0 \to A_0 \to A_0) \cong C(A_0 \to A_0)$$

is the identity map.

### 4.2.3. Weak relations: showing Equations 4.2 hold

**4.2.3.1. Showing Equation 4.2a holds.** For $n = 1$, eliminating the identity maps reduces Equation 4.2a to:

$$\hat{\tau}_1^*(\tau_{1!}) \circ \tau_{1!} \simeq id.$$

We prove the above in Appendix Proposition B.2. (In the appendix, $\tau_{1!} = \Upsilon_{A_0,A_1}$, $\hat{\tau}_1^*(\tau_{1!}) = \Upsilon_{A_1,A_0}$, and the homotopy is denoted $B$.)

For $n = 2$, eliminating the identity maps and writing $\tau_{2!}$ in terms of $\tau_{1!}$ reduces Equation 4.2a to:

$$(\widehat{\delta_{0,2}\tau_2})^*(\tau_{1!}) \circ \hat{\delta}_{0,2}^*(\tau_{1!}) \simeq \hat{\delta}_{1,2}^*(\tau_{1!}).$$

We prove the above in Appendix Proposition B.4. (In the appendix, $\hat{\delta}_{0,2}^*(\tau_{1!}) = \Upsilon_{A_0 \bullet A_1, A_2}$, $(\widehat{\delta_{0,2}\tau_2})^*(\tau_{1!}) = \Upsilon_{A_2 \bullet A_0, A_1}$, $\hat{\delta}_{1,2}^*(\tau_{1!}) = \Upsilon_{A_0, A_1 \bullet A_2}$, and the homotopy is denoted $\mathcal{B}$.)

For $n > 2$, we reduce Equation 4.2a to the case when $n = 2$. We have

$$\text{Lefthand side of Equation 4.2a} = \hat{\tau}_n^{*2}(\delta_{0,n!}) \circ \hat{\tau}_n^*(\tau_{n!}) \circ \tau_{n!}$$

$$= id \circ \hat{\tau}_n^*((\widehat{\delta_{0,2}...\delta_{0,n}})^*(\tau_{1!})) \circ \tau_{n!}$$

$$= (\widehat{\delta_{0,2}...\delta_{0,n}\tau_n})^*(\tau_{1!}) \circ \tau_{n!}$$

$$= (\widehat{\delta_{0,2}\tau_2\delta_{0,3}...\delta_{0,n}})^*(\tau_{1!}) \circ \tau_{n!}$$

$$= (\widehat{\delta_{0,2}\tau_2\delta_{0,3}...\delta_{0,n}})^*(\tau_{1!}) \circ (\widehat{\delta_{0,2}...\delta_{0,n}\tau_n})^*(\tau_{1!})$$

$$= (\widehat{\delta_{0,3}...\delta_{0,n}})^*((\widehat{\delta_{0,2}\tau_2})^*(\tau_{1!}) \circ \hat{\delta}_{0,2}^*\tau_{1!})$$

$$\text{Righthand side of Equation 4.2a} = \hat{\delta}_{n-1,n}^*(\tau_{n-1!}) \circ \delta_{n-1,n!}$$

$$= \hat{\delta}_{n-1,n}^*((\widehat{\delta_{0,2}...\delta_{0,n-1}})^*(\tau_{1!})) \circ id$$

$$= (\widehat{\delta_{0,2}...\delta_{0,n-1}\delta_{n-1,n}})^*(\tau_{1!}))$$

$$= (\widehat{\delta_{1,2}\delta_{0,3}...\delta_{0,n}})^*(\tau_{1!})$$

$$= (\widehat{\delta_{0,3}...\delta_{0,n}})^*(\hat{\delta}_{1,2}^*(\tau_{1!})).$$

So, Equation 4.2a $= (\widehat{\delta_{0,3}...\delta_{0,n}})^*($ Equation 4.2a, $n = 2)$. If $\mathcal{B}$ is a homotopy giving Equation 4.2a for $n = 2$, then $(\widehat{\delta_{0,3}...\delta_{0,n}})^*\mathcal{B}$ is a homotopy giving Equation 4.2a for $n > 2$.

**4.2.3.2. Showing Equation 4.2b holds.** We prove this by induction on $n$. For $n = 1$, Equation 4.2b is the same as Equation 4.2a, which we established in the previous section. Now, assume that Equation 4.2b holds for $N = n - 1$. We show that Equation 4.2b holds

for $N = n$ below:

$$\hat{\tau}_n^{*n}(\tau_{n!}) \circ \ldots \circ \hat{\tau}_n^*(\tau_{n!}) \circ \tau_{n!} = \hat{\tau}_n^{*n-1}(\hat{\tau}_n^*\tau_{n!} \circ \tau_{n!}) \circ \hat{\tau}_n^{*n-2}\tau_{n!} \circ \ldots \circ \tau_{n!}$$

$$\simeq \hat{\tau}_n^{*n-1}(\hat{\delta}_{n-1,n}^*\tau_{n-1!}) \circ \hat{\tau}_n^{*n-2}\tau_{n!} \circ \ldots \circ \tau_{n!} \qquad \text{(Equation 4.2a)}$$

$$= (\widehat{\tau_{n-1}^{n-1}\delta_{0,n}})^*\tau_{n-1!}\circ$$

$$\circ \left(\hat{\tau}_n^{*n-2}\hat{\delta}_{n-2,n}^*\tau_{n-1!} \circ \ldots \circ \hat{\tau}_n^*\hat{\delta}_{1,n}^*\tau_{n-1!} \circ \hat{\delta}_{0,n}^*\tau_{n-1!}\right)$$

$$= (\widehat{\tau_{n-1}^{n-1}\delta_{0,n}})^*\tau_{n-1!} \circ \hat{\delta}_{0,n}^*\left(\hat{\tau}_{n-1}^{*n-2}\tau_{n-1!} \circ \ldots \circ \hat{\tau}_{n-1}^*\tau_{n-1!} \circ \tau_{n-1!}\right)$$

$$= \hat{\delta}_{0,n}^*\left(\hat{\tau}_{n-1}^{*n-1}\tau_{n-1!} \circ \ldots \circ \tau_{n-1!}\right)$$

$$\simeq \hat{\delta}_{0,n}^*\left(id\right) \qquad \text{(Inductive hypothesis)}$$

$$= id.$$

**4.2.3.3. Showing Equation 4.2c holds.** By manipulating morphisms in $\Lambda$, we have

$$\text{Righthand side of Equation 4.2c} = \hat{\tau}_n^{*n+1}\tau_{n!} \circ \hat{\tau}_n^{*n}\tau_{n!} \circ \ldots \circ \hat{\tau}_n^*\tau_{n!} \circ \hat{\tau}_n^{*n+1}id \circ \tau_{n!}$$

$$= \tau_{n!} \circ \left(\hat{\tau}_n^{*n}\tau_{n!} \circ \ldots \circ \hat{\tau}_n^*\tau_{n!} \circ \tau_{n!}\right)$$

$$\simeq \tau_{n!} \circ \left(id\right) \quad \text{Equation 4.2b.}$$

On the other hand, we have

$$\text{Lefthand side of Equation 4.2c} = \hat{\sigma}^*_{n,n}(\tau_{n+1!}) \circ id$$

$$= \hat{\sigma}^*_{n,n}(\hat{\delta}^*_{n,n+1}(\tau_{n+1!}))$$

$$= (\widehat{\delta_{n,n+1}\sigma_{n,n}})^*(\tau_{n!})$$

$$= id^*(\tau_{n!}).$$

So, Equation 4.2c holds.

## 4.3. Higher Homotopies

In this section, we show that no higher homotopies are needed. First, we will summarize the maps of comodules that we have already given. Let $\lambda$ be a generating morphism in $\Lambda$ that induces a morphism in $\chi$ with source $\mathcal{A} \in Obj(\chi)$. We have

$$\lambda_! : C(\mathcal{A}) \to \hat{\lambda}^* C(\lambda\mathcal{A}) \quad \text{maps of dg comodules}$$

$$\sigma_! : C(\mathcal{A}) \to \tau^{*2} C(\tau^2 \mathcal{A}) \quad \text{deg -1 map of comodules.}$$

where

$$\sigma_! = \begin{cases} \mathcal{B} \text{ given in Appendix Proposition B.2} & \text{if } \mathcal{A} = (A_0 \to A_1 \to A_0) \\ \mathcal{B} \text{ given in Appendix Proposition B.4} & \text{if } \mathcal{A} = (A_0 \to A_1 \to A_2 \to A_0) \\ (\widehat{\delta_{0,3}...\delta_{0,n}})^* \mathcal{B} & \text{if } \mathcal{A} = (A_0 \to ... \to A_n \to A_0), n > 2. \end{cases}$$

Using the constructions we've given, a typical map between comodules is one that is freely generated by composable pullbacks of $\lambda_!$'s and $\sigma_!$'s. First, we will establish that there are no such maps of degree $\geq 2$. Suppose we have a map $\eta_!$ of degree $\geq 2$. Then, $\eta_!$ must contain at least two (pullbacks of) some $\sigma_!$'s. Each $\sigma_!$ involves inserting a 1 into the first slot of the Hochschild chains component (see Equations B.7, B.9). However, since we are working with reduced chains, any chain with two or more 1's is equal to zero. So, $\eta_! = 0$.

Since there are no maps of degree $\geq 2$, we know from the classical theory of $A_\infty$ algebras that the only need for higher homotopies will arise from the following situation:

For $n \geq 2$, We have two maps of dg comodules

$$(4.3) \qquad C(A_0 \to \ldots \to A_n \to A_0)$$

$$\hat{\tau}_n^{*2}\tau_{n!} \circ \hat{\tau}_n^{*}\tau_{n!} \circ \tau_{n!} \qquad\qquad (\delta_{n-2,n-1}\widehat{\delta}_{n-1,n})^{*}\tau_{n-2!}$$

"apply $\tau_{n!}$ 3 times" $\qquad\qquad$ "brace together the last 3 algebras, then apply $\tau_{n-2!}$ once"

$$C(A_{n-2} \to A_{n-1} \to A_n \to A_0 \to \ldots \to A_{n-2}).$$

These two maps are homotopic via two homotopies: $\hat{\delta}^{*}_{n-1,n}\sigma_! + \tau_n^{*2}\tau_{n!} \circ \sigma_!$ and $\hat{\delta}^{*}_{n-2,n}\sigma_! + \hat{\tau}_n^{*}\sigma_! \circ \tau_{n!}$ (see Figure 4.1). If the two homotopies were different, then their difference would be closed and we would desire a higher homotopy (i.e., a degree -2 map of comodules) between them. However, we will show the two homotopies are the same, so that no higher homotopies are needed.

First, we show that $\hat{\delta}^{*}_{n-1,n}\sigma_! = \hat{\delta}^{*}_{n-2,n}\sigma_!$. We have

$$\hat{\delta}^{*}_{n-1,n}\sigma_! = \mathcal{B}_{A_0\bullet\ldots\bullet A_{n-2}, A_{n-1}\bullet A_n} = \mathcal{B}_{A_0\bullet\ldots\bullet A_{n-1}, A_n} = \hat{\delta}^{*}_{n-2,n}\sigma_!$$

where the second equality holds by definition of $\mathcal{B}$ in Appendix Equation B.9.

Second, we show that $\tau_n^{*2}\tau_{n!} \circ \sigma_! = \hat{\tau}_n^{*}\sigma_! \circ \tau_{n!}$ in Appendix Proposition B.5. In the appendix, $\tau_n^{*2}\tau_{n!} \circ \sigma_! = \Upsilon_{A_1\bullet A_2, A_0}\mathcal{B}_{A_0, A_1, A_2}$ and $\hat{\tau}_n^{*}\sigma_! \circ \tau_{n!} = \mathcal{B}_{A_2, A_0, A_1}\Upsilon_{A_0\bullet A_1, A_2}$.

For $n = 1$, the situation in Equation 4.3 reduces to: We have two maps of dg comodules

$$C(A_0 \to A_1 \to A_0)$$

$$\hat{\tau}_1^{*2}\tau_{1!} \circ \hat{\tau}_1^{*}\tau_{1!} \circ \tau_{1!} \qquad\qquad \tau_{1!}$$

$$C(A_1 \to A_0 \to A_1).$$

$$\big(\widehat{\delta_{n-2,n-1}\delta_{n-1,n}}\big)^*\tau_{n-2!} \xrightarrow{\ \cong\ } \hat{\delta}^*_{n-1,n}\big(\hat{\delta}^*_{n-2,n-1}\tau_{n-2!}\big) \xrightarrow{\ \hat{\delta}^*_{n-1,n}\sigma_!\ } \hat{\delta}^*_{n-1,n}\big(\hat{\tau}^*_{n-1}\tau_{n-1!}\circ\tau_{n-1!}\big)$$

"brace together $A_{n-2}, A_{n-1}, A_n$,
then apply $\tau_{n-2!}$"

$\cong$ (vertical, left)          $\cong$ (vertical, right)

$$\big(\widehat{\delta_{n-2,n-1}\delta_{n-2,n}}\big)^*\tau_{n-2!} \qquad\qquad \hat{\tau}^{*2}_n\tau_{n!}\circ\hat{\delta}^*_{n-1,n}\tau_{n-1!}$$

"brace together $A_{n-1}, A_n$
and apply $\tau_{n-1!}$,
then apply $\tau_{n!}$"

$\hat{\delta}^*_{n-2,n}\sigma_!$ (left vertical)          $\tau^{*2}_n\tau_{n!}\circ\sigma_!$ (right vertical)

$$\hat{\delta}^*_{n-2,n}\big(\hat{\tau}^*_{n-1}\tau_{n-1!}\circ\tau_{n-1!}\big) \xrightarrow{\ \cong\ } \hat{\tau}^*_n\big(\hat{\delta}^*_{n-1,n}\tau_{n-1!}\big)\circ\tau_{n!} \xrightarrow{\ \hat{\tau}^*_n\sigma_!\circ\tau_{n!}\ } \hat{\tau}^{*2}_n\tau_{n!}\circ\hat{\tau}^*_n\tau_{n!}\circ\tau_{n!}$$

"apply $\tau_{n!}$,
then brace together $A_{n-1}, A_{n-2}$
and apply $\tau_{n-1!}$"

"apply $\tau_{n!}$ three times"

Figure 4.1. Two homotopies between $\big(\widehat{\delta_{n-2,n-1}\delta_{n-1,n}}\big)^*\tau_{n-2!}$ and $\hat{\tau}^{*2}_n\tau_{n!}\circ$ $\hat{\tau}^*_n\tau_{n!}\circ\tau_{n!}$
Vertices are maps of dg comodules and arrows are chain homotopies.

$$id\circ\tau_{1!}=\tau_{1!}=\tau_{1!}\circ id$$

$\sigma_{A_1\to A_0\to A_1!}\circ\tau_{1!}$
$=\hat{\tau}^*_1(\sigma_{A_0\to A_1\to A_0!})\circ\tau_{1!}$          $\tau_{1!}\circ\sigma_{A_0\to A_1\to A_0!}$

$$\big(\hat{\tau}^{*2}_1\tau_{1!}\circ\hat{\tau}^*_1\tau_{1!}\big)\circ\tau_{1!}=\hat{\tau}^{*2}_1\tau_{1!}\circ\big(\hat{\tau}^*_1\tau_{1!}\circ\tau_{1!}\big)$$

Figure 4.2. Two homotopies between $\tau_{1!}$ and $\hat{\tau}^{*2}_1\tau_{1!}\circ\hat{\tau}^*_1\tau_{1!}\circ\tau_{1!}$
Vertices are maps of dg comodules and arrows are chain homotopies.

These two maps are homotopic via two homotopies: $\tau_{1!}\circ\sigma_{A_0\to A_1\to A_0!}$ and $\sigma_{A_1\to A_0\to A_1!}\circ\tau_{1!}$ (see Figure 4.2; for clarity, here, we indicate the sources of the $\sigma_!$'s). We show that these two homotopies are the same in Appendix Proposition B.3, so no higher homotopies are needed. In the appendix, $\tau_{1!}\circ\sigma_{A_0\to A_1\to A_0!}=\Upsilon_{A_0,A_1}\circ B_{A_0,A_1}$ and $\sigma_{A_1\to A_0\to A_1!}\circ\tau_{1!}=B_{A_1,A_0}\circ\Upsilon_{A_0,A_1}$.

## 4.4. An $A_\infty$-functor

We can repackage the work of the previous sections into a concise statement: We have constructed an $A_\infty$-functor from $\chi$ to $\mathcal{D}$. This section is devoted to making that concise statement more rigorous. We refer to Reference [**2**], Appendix A, Definitions A.6 and A.8 for the notation and definition of an $A_\infty$-category and $A_\infty$-functor.

First, we must think of $\chi$ and $\mathcal{D}$ as $A_\infty$-categories. Let $\chi_\infty$ be the (usual) category with the same objects as $\chi$ and morphisms linear combinations over $k$ of morphisms in $\chi$. We will think of the morphisms in $\chi_\infty$ as complexes concentrated in degree zero. $\chi_\infty$ is an $A_\infty$-category with $m_2 =$ (usual) composition of morphisms in $\chi_\infty$, $m_1 = m_{\geq 3} = 0$. The relations for an $A_\infty$-category are satisfied because $m_2$ is associative.

Let $\mathcal{D}_\infty$ be the dg category with the same objects as $\mathcal{D}$ and morphisms

$$\mathcal{D}_\infty^\bullet((B_1, C_1), (B_0, C_0)) = \big\{ \big( F : B_1 \to B_0 \quad \text{dg functor,}$$

$$F_! : C_1 \to F^* C_0 \quad \text{map of comodules of degree } \bullet \big) \big\}$$

$$d_{\mathcal{D}_\infty}(F, F_!) = (F, \ d_{F^* C_0} \circ F_! - (-1)^{|F_!|} F_! \circ d_{C_1}).$$

Composition in $\mathcal{D}_\infty$ works like composition in $\mathcal{D}$–we can apply the same formulas to pullback a (not necessarily graded) morphism of comodules. We can think of $\mathcal{D}_\infty$ as an $A_\infty$-category with $m_1 = d_{\mathcal{D}_\infty}$, $m_2 =$ composition of morphisms, $m_{\geq 3} = 0$. For $\mathcal{D}_\infty$, the relations for an $A_\infty$-category are precisely that (1) the differentials square to zero, (2) composition is a map of complexes, and (3) composition is associative.

Now, we will show that the constructions given in the previous sections constitute an $A_\infty$-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$. Still using the notation in Reference [**2**], Definition A.8, we

define $\mathcal{F}$ as follows:

$$f : Obj(\chi_\infty) \to Obj(\mathcal{D}_\infty) \quad \text{map of sets}$$

$$\mathcal{A} \mapsto (B(\mathcal{A}), C(\mathcal{A})) \text{ defined in Chapter 1}$$

$$f_1 : \chi_\infty^\bullet(x_0, x_1) \to \mathcal{D}_\infty^\bullet(fx_0, fx_1) \quad \text{map of graded vector spaces}$$

$$\lambda \mapsto (\hat{\lambda}, \lambda_!) \text{ defined in Sections 3.1, 3.6}$$

$$\text{for } \lambda \text{ a generating morphism in } \Lambda$$

$$id_\mathcal{A} \mapsto (id_{B(\mathcal{A})}, id_{C(\mathcal{A})})$$

$$f_2 : \chi_\infty^\bullet(x_0, x_1) \otimes \chi_\infty^\bullet(x_1, x_2) \to \mathcal{D}_\infty^\bullet(fx_0, fx_2) \quad \text{degree } -1 \text{ map of vector spaces}$$

$$f_{i \geq 3} : \chi_\infty^\bullet(x_0, x_1) \otimes ... \otimes \chi_\infty^\bullet(x_{i-1}, x_i) \to \mathcal{D}_\infty^\bullet(fx_0, fx_i)$$

$$\text{degree } 1 - i \text{ map of vector spaces}, \mu_1 \otimes ... \otimes \mu_i \mapsto 0 \text{ for all morphisms } \mu$$

We will show that the $f_i$'s satisfy the relations in Reference [**2**], Definition A.8.

First, we finish defining $\mathcal{F}$, namely, we must define $f_1(\mu)$ for $\mu$ not a generating morphism in $\Lambda$ as well as for linear combinations of morphisms. We also still need to define $f_2$.

Let $\mu$ be a non-generating morphism in $\Lambda$ that induces a morphsim in $\chi$ with source $\mathcal{A}$. Choose (i.e., fix once and for all) a presentation of $\mu$ as a composition of generating morphisms. Within the chosen presentation, in the following order, (1) replace all occurrences of $\tau_{n-1}\delta_{n-1,n}$ with $\delta_{0,n}\tau_n^2$, (2) replace all $\tau_{n+1}\sigma_{n,n}$ with $\tau_{n+1}^{n+1}\sigma_{0,n}\tau_n$, (3) replace all decompositions of identity maps with identity maps, (4) remove all identity maps if $\mu \neq id$, (5) call this new presentation "the presentation corresponding to $\mu$", denoted

$\mu = \lambda_{\mu,k_\mu} ... \lambda_{\mu,1}$. The presentation corresponding to $\mu$ is not unique (i.e., still depends on the original, chosen presentation). However, letting

$$f_1(\mu) := \big( \hat{\mu} : B(\mathcal{A}) \to B(\mu \mathcal{A})$$

$$\hat{\lambda}^*_{\mu,1} ... \hat{\lambda}^*_{\mu,k_\mu-1}(\lambda_{\mu,k_\mu!}) \circ ... \circ \hat{\lambda}^*_{\mu,1}(\lambda_{\mu,2!}) \circ \lambda_{\mu,1!} : C(\mathcal{A}) \to \hat{\mu}^* C(\mu\mathcal{A}) \big)$$

is well-defined because we have made consistent choices for all of the relations among $\lambda_!$'s that only hold up to homotopy (see Equations 4.2). More explicitly, $f_1(\mu)$ would have been well-defined for any choice of presentation if Equations 4.2 were equalities rather than homotopies. Instead, we choose to define $f_1(\mu)$ via a presentation that only uses the lefthand side of Equation 4.2a and only uses the righthand sides of Equations 4.2c and 4.2b. Finally, extend $f_1$ linearly over $k$ to define $f_1$ for all linear combinations of morphisms in $\chi$.

Before defining $f_2$, let's take a look at an $A_\infty$ relation we expect $f_2$ to satisfy: For $\cdot \xrightarrow{\mu_1} \cdot \xrightarrow{\mu_2} \cdot$ composable morphisms in $\chi$, we expect

$$(4.4) \qquad f_1(\mu_2 \circ \mu_1) = f_1(\mu_2) \circ f_1(\mu_1) + d_{\mathcal{D}_\infty} \circ f_2(\mu_1, \mu_2).$$

Given the definition of $f_1$ above, we require a non-zero $f_2$ only if: (Condition H) the presentation corresponding to $\mu_2$ composed with the presentation corresponding to $\mu_1$ contains, after removing (decompositions of) identity maps except for $\tau_n^{n+1}$, one or more of the following terms: $\tau_{n-1}\delta_{n-1,n}$, $\tau_{n+1}\sigma_{n,n}$, $\tau_n^{n+1}$. If $\mu_1, \mu_2$ satisfy Condition H, homotopies given in Section 4.2.3 can be used to define $f_2$. If $\mu_1, \mu_2$ do not satisfy Condition H, let $f_2(\mu_1, \mu_2) = 0$.

We will give some instructive examples of non-zero $f_2$ that satisfy Equation 4.4.

**Example 4.4.1.** Let $\mu_1 = \delta_{n-1,n}$, $\mu_2 = \tau_{n-1}$. Then, the presentation corresponding to $\mu_2\mu_1$ is $\delta_{0,n}\tau_n^2$. Let $f_2(\mu_1, \mu_2)$ be the homotopy given in Section 4.2.3.1 (also given in Appendix Proposition B.2 or B.4). Then, Equation 4.4 is equivalent to Equation 4.2a.

**Example 4.4.2.** Let $\mu_1 = \sigma_{0,n-1}\delta_{n-1,n}$, $\mu_2 = \tau_{n-1}\delta_{0,n}$. To form the presentation corresponding to $\mu_2\mu_1$, we follow these steps:

$$\tau_{n-1}\delta_{0,n}\sigma_{0,n-1}\delta_{n-1,n} \xrightarrow[\text{of identities}]{\text{remove decompositions}} \tau_{n-1}\delta_{n-1,n} \xrightarrow{\text{replace}} \delta_{0,n}\tau_n^2.$$

On the other hand,

$$f_1(\mu_2)f_1(\mu_1) = (\delta_{0,n}\widehat{\sigma_{0,n-1}\delta_{n-1,n}})^*(\tau_{n-1!}) \circ (\widehat{\sigma_{0,n-1}\delta_{n-1,n}})^*(\delta_{0,n!}) \circ \hat{\delta}_{n-1,n}^*(\sigma_{0,n-1!}) \circ \delta_{n-1,n!}$$

$$= \hat{\delta}_{n-1,n}^*(\tau_{n-1!}) \circ id \circ \delta_{n-1,n!}.$$

So, we can let $f_2(\mu_1, \mu_2)$ be the homotopy given in Section 4.2.3.1, and Equation 4.4 is equivalent to Equation 4.2a.

**Example 4.4.3.** Let $(\mu_1, \mu_2) \in \{(\tau_{n+1}, \sigma_{n,n}), (\tau_n^{n+1-j}, \tau_n^j) : 1 \le j \le n, n \in \mathbb{N}\}$. Let $f_2(\mu_1, \mu_2)$ be the homotopy given in 4.2.3.3 if $\mu_2 = \sigma_{n,n}$ and the homotopy given in 4.2.3.2 if $\mu_2 \ne \sigma_{n,n}$. Then, Equation 4.4 is equivalent to either Equation 4.2c ($\mu_2 = \sigma_{n,n}$) or Equation 4.2b ($\mu_2 \ne \sigma_{n,n}$).

**Example 4.4.4.** *Let* $\mu_1 = \sigma_{n-1,n-1}\delta_{n-1,n}$, $\mu_2 = \tau_n$. *To form the presentation corresponding to* $\mu_2\mu_1$, *we follow these steps:*

$$(\tau_n\sigma_{0,n-1})\delta_{n-1,n} \xrightarrow{\text{replace } (\cdot)} \tau_n^n\sigma_{0,n-1}(\tau_{n-1}\delta_{n-1,n}) \xrightarrow{\text{replace } (\cdot)} \tau_n^n\sigma_{0,n-1}\delta_{0,n}\tau_n^2.$$

*Let* $f_2(\mu_1,\mu_2) = g_1 + g_2$ *where* $g_1 = \hat{\delta}^*_{n-1,n}(\text{homotopy in Section 4.2.3.3}) \circ \delta_{n-1,n!}$ *and* $g_2 = (\widehat{\tau_{n-1}\delta_{n-1,n}})^*\big((\widehat{\tau_n^{n-1}\sigma_{0,n-1}})^*(\tau_{n!}) \circ \ldots \circ \hat{\sigma}^*_{0,n-1}(\tau_{n!}) \circ \sigma_{0,n-1!}\big) \circ (\text{homotopy in Section 4.2.3.1})$. *Then, Equation 4.4 reduces to* $\delta^*_{n-1,n}(\text{Equation 4.2c})$ *and Equation 4.2a.*

Now, we will check that the $f_i$'s we gave satisfy the rest of the relations for an $A_\infty$-functor from Reference [**2**], Definition A.8: For $\cdot \xrightarrow{\mu_1} \cdot \xrightarrow{\mu_2} \cdot \xrightarrow{\mu_3} \cdot \xrightarrow{\mu_4} \cdot$ composable morphisms in $\chi$, we expect

$$(4.5) \qquad\qquad 0 = d_{\mathcal{D}_\infty} \circ f_1(\mu_1)$$

$$(4.6) \qquad f_2(\mu_3, \mu_2 \circ \mu_1) - f_2(\mu_3 \circ \mu_2, \mu_1) = f_2(\mu_3, \mu_2) \circ f_1(\mu_1) - f_1(\mu_3) \circ f_2(\mu_2, \mu_1)$$

$$(4.7) \qquad\qquad 0 = f_2(\mu_4, \mu_3) \circ f_2(\mu_2, \mu_1).$$

Equation 4.5 is satisfied since the $\lambda_!$'s we defined in Section 3.6 are maps of complexes. Equation 4.7 is satisfied since composing two of our degree $-1$ homotopies is always equal to zero (see Section 4.3, first paragraph). Finally, Equation 4.6 boils down to showing that the two homotopies in Figure 4.1 and in Figure 4.2 are the same (see Section 4.3).

Thus, we have an $A_\infty$-functor $\mathcal{F}: \chi_\infty \to \mathcal{D}_\infty$. Applying Reference [**2**], Remark A.27, we can rectify $\mathcal{F}$ to a dg functor $\tilde{\mathcal{F}}: U(\chi_\infty) \to \mathcal{D}_\infty$ where $U(\chi_\infty)$ is the enveloping dg category of $\chi$ (see Reference [**2**], Definition A.25).

## 4.5. A functor to dg categories

In the previous section 4.4, we gave a dg functor from $U(\chi_\infty)$ to $\mathcal{D}_\infty =$ "dg cocategories and dg comodules". We would like to have our functor land in the dg category, $\mathcal{E} =$ "dg categories and dg modules". To do so, we will first give a dg functor $\mathcal{D}_\infty \to \mathcal{D}_1$, which makes use of the adjunction in Proposition 3.3. Then, we will give a dg functor $\Omega : \mathcal{D}_1 \to \mathcal{E}$.

### 4.5.1. Using the adjunction

Let $\mathcal{D}_1$ be the dg category with the same objects as $\mathcal{D}$ and morphisms

$$\mathcal{D}_1^\bullet((B_1, C_1), (B_0, C_0)) = \big\{ \big( F : B_1 \to B_0 \quad \text{dg functor,} $$

$$F_! : F_\# C_1 \to C_0 \quad \text{map of comodules of degree } \bullet \big) \big\}$$

$$d_{\mathcal{D}_\infty}(F, F_!) = (F, \ d_{C_0} \circ F_! - (-1)^{|F_!|} F_! \circ d_{F_\# C_1})$$

with composition

$$\mathcal{D}_1^\bullet((B_2, C_2), (B_1, C_1)) \otimes \mathcal{D}_1^\bullet((B_1, C_1), (B_0, C_0)) \to \mathcal{D}_1^\bullet((B_2, C_2), (B_0, C_0))$$

$$(f, f_!) \otimes (g, g_!) \mapsto (gf, g_! \circ g_\#(f_!)).$$

This composition is well-defined because we can apply the formulas from $g_\#$ to (not necessarily graded) morphisms of comodules. The composition is associative because of the following easy-to-check fact: $g_\# f_\# C = (gf)_\# C$ for $B_2 \xrightarrow{f} B_1 \xrightarrow{g} B_0$ functors of dg cocategories and $C$ a dg comodule over $B_2$.

Now, we define a dg functor

$$Adj : \mathcal{D}_\infty \to \mathcal{D}_1$$

$$\text{on objects: } (B, C) \mapsto (B, C)$$

$$\text{on morphisms: } \left( (B_1, C_1) \xrightarrow{(F, F_!)} (B_0, C_0) \right) \mapsto \left( (B_1, C_1) \xrightarrow{(F, \Phi_F^{-1} F)} (B_0, C_0) \right)$$

where $\Phi_F^{-1} : Hom_{\substack{\text{dg comodules} \\ \text{over } B_1}}(C, F^*D) \to Hom_{\substack{\text{dg comodules} \\ \text{over } B_0}}(F_\# C, D)$ is defined in the proof

of Proposition 3.3 and makes sense as a function on (not necessarily graded) maps of

comodules. To check that $Adj$ commutes with the differentials and respects composition,

we need

$$\Phi_F^{-1} \circ d_{Hom_{B_2}(C_2, F^*C_1)} = d_{Hom_{B_1}(F_\# C_2, C_1)} \circ \Phi_F^{-1}$$

$$\Phi_{GF}^{-1}(F^*G_! \circ F_!) = \Phi_G^{-1}(G_!) \circ G_\#(\Phi_F^{-1}(F_!))$$

$$\text{where } (B_2, C_2) \xrightarrow{(F, F_!)} (B_1, C_1) \xrightarrow{(G, G_!)} (B_0, C_0) \text{ in } \mathcal{D}_\infty.$$

The equations above follow straight-forwardly from the definitions.

### 4.5.2. Applying *Cobar*

In this section, we will use the notion of a dg module over a dg category. This is dual to

a dg comodule over a dg cocategory (Definition 2.3.1). Given a dg functor between dg

categories $F : A_1 \to A_0$, we define "restriction of scalars", $F^*$, a functor from the category

of dg comodules over $A_0$ to the category of dg comodules over $A_1$. For $M_0$ a dg comodule

over $A_0$ and $f \in Obj(B_1)$, $F^* M_0(f) := M_0(Ff)$.

Let $\mathcal{E}$ be the dg category defined below:

$$Obj(\mathcal{E}) = \{(A, M)|\text{A is a dg category, M is a dg module over A}\}$$

$$\mathcal{E}^p((A_1, M_1), (A_0, M_0)) = \{(f, f_!)|f : A_1 \to A_0 \text{ is a dg functor,}$$

$$f_! : M_1 \to f^*M_0 \text{ is a degree-}p \text{ map of modules over } A_1\}$$

$$d_\mathcal{E}(f, f_!) = (f, \ d_{f^*M_0} \circ f_! - (-1)^{|f_!|}f_! \circ d_{C_1})$$

$$\mathcal{E}^\bullet((A_2, M_2), (A_1, M_1)) \times \mathcal{E}^\bullet((A_1, M_1), (A_0, M_0)) \xrightarrow{composition} \mathcal{E}^\bullet((A_2, M_2), (A_0, M_0))$$

$$(f, f_!) \times (g, g_!) \mapsto (gf, f^*(g_!) \circ f_!).$$

We will define a dg functor $\Omega : \mathcal{D}_1 \to \mathcal{E}$. On objects,

$$\Omega(B, C) := (Cobar(B), Cobar(B, C))$$

where the first Cobar is a dg functor from the category of dg cocategories to the category of dg categories, and the second Cobar sends dg comodules over $B$ to dg modules over $Cobar(B)$ (see [**6**], Section 4.6). On morphisms,

$$\mathcal{D}_1 \ni \begin{pmatrix} B_1 \xrightarrow{F} B_0 \\ F_\# C_1 \xrightarrow{F_!} C_0 \end{pmatrix} \mapsto \begin{pmatrix} Cobar(B_1) \xrightarrow{Cobar(F)} Cobar(B_0) \\ Cobar(B_1, C_1) \xrightarrow{\Omega(F_!)} (Cobar(F))^*Cobar(B_0, C_0) \end{pmatrix} \in \mathcal{E}$$

where $\Omega(F_!) : Cobar(B_1, C_1) \to (Cobar(F))^*Cobar(B_0, C_0)$

$$(b_1|...|b_n|c) \mapsto (Fb_1|...|Fb_n|F_!c)$$

for $b_i \in B_1^\bullet(f_{i-1}, f_i), c \in C_1^\bullet(f_n)$, and $f_i \in Obj(B_1), 0 \le i \le n$.

It's straightforward from the definitions to check that $\Omega$ commutes with the differentials and respects composition.

### 4.5.3. The end: putting everything together

We have dg functors

$$U(\chi_\infty) \xrightarrow{\tilde{\mathcal{F}}} \mathcal{D}_\infty \xrightarrow{Adj} \mathcal{D}_1 \xrightarrow{\Omega} \mathcal{E}.$$

This is our homotopically sheafy-cyclic object in dg categories with a trace functor.

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APPENDIX A

# Connes cyclic category, $\Lambda$

Here, we give generators and relations for the cyclic category, $\Lambda$. None of this is new, but we do it to establish notation for the rest of the paper.

$\Lambda$ has objects $\{[n] : n \in \mathbb{N}\}$ and generating morphisms:

$$\text{rotations } \tau_n : [n] \to [n],$$

(A.1) $$\text{coboundaries } \delta_{j,n} : [n] \to [n-1], 0 \leq j \leq n-1,$$

$$\text{codegeneracies } \sigma_{i,n} : [n] \to [n+1], 0 \leq i \leq n$$

subject to relations:

$$\delta_{i,n-1}\delta_{j,n} = \delta_{j-1,n-1}\delta_{i,n} \quad 0 \le i < j \le n-1$$

$$\sigma_{i,n+1}\sigma_{j,n} = \sigma_{j+1,n+1}\sigma_{i,n} \quad 0 \le i \le j \le n$$

$$\delta_{j,n+1}\sigma_{i,n} = \begin{cases} \sigma_{i,n-1}\delta_{j-1,n} & 0 \le i < j \le n \\ id & j = i, i-1 \\ \sigma_{i-1,n-1}\delta_{j,n} & 0 \le j < i-1 \le n-1 \end{cases}$$

(A.2)

$$\tau_{n+1}\sigma_{i,n} = \sigma_{i+1,n}\tau_n \quad 0 \le i \le n-1$$

$$\tau_{n-1}\delta_{j,n} = \delta_{j+1,n}\tau_n \quad 0 \le j \le n-1$$

$$\tau_n^{n+1} = id$$

$$\delta_{0,1}\tau_1\sigma_{0,0} = id$$

$$\tau_{n+1}\sigma_{n,n} = \tau_{n+1}^{n+1}\sigma_{0,n}\tau_n$$

$$\delta_{0,n}\tau_n^2 = \tau_{n-1}\delta_{n-1,n}.$$

Some presentations of $\Lambda$ include an extra coboundary $\delta_{n,n}$ and codegeneracy $\sigma_{n+1,n}$. In terms of our generators, they are $\delta_{n,n} := \delta_{0,n}\tau_n$ and $\sigma_{n+1,n} := \tau_{n+1}^{n+1}\sigma_{0,n}$.

APPENDIX B

# Computations

In this appendix, we give the computational propositions needed to establish the homotopically sheafy-cyclic structure on dg comodules. All the comodules we work with will be cofree, and we will define maps into them by giving maps into cogenerators (see Equation 2.3).

## B.1. Computational notation

For this section's propositions, we establish the following notation:

$A_0, A_1$ fixed algebras

$$(\vec{\phi}|\vec{\psi}|\alpha) := (\phi_1...\phi_n|\psi_1...\psi_m|\alpha)$$



$$= \quad A_0 \quad \vdots \quad A_1 \quad \vdots \quad A_0 \quad \in C(A_0 \to A_1 \to A_0)(g_0 f_0)$$

$$\vec{\phi}_{\{i_1,i_2,...,i_k\}} := \phi_{i_1}\phi_{i_2}...\phi_{i_k}$$

where $\{i_1, i_2, ..., i_k\}$ is an ordered subset of $\{1, ..., n\}$

$$\vec{\phi}_{\{\}} := 1 \in k \cong Bar_0(C^\bullet(A_0, A_1))$$

$$\vec{\psi}_{\{\}} := 1 \in k \cong Bar_0(C^\bullet(A_1, A_0))$$

$$|I| := \text{number of elements in a set } I$$

$$I_1 I_2 := \text{concatenation as ordered sets of possibly-empty sets } I_1 \text{ and } I_2$$

$$\epsilon_{I_1,J_1} := (-1)^{(\sum\limits_{r \in I_1} |\phi_r|+1)(\sum\limits_{s \in J_1} |\psi_s|+1)}$$

when $I_1$, $J_1$ are ordered indexing sets

$$\lambda(\vec{\psi}), \tilde{\delta}, b', b, \psi\{\vec{\phi}\} \cdot \alpha = \text{see Appendix C for operations on Hochschild (co)chains}$$

### B.1.1. Notation for elements of Hochschild chains

Let $a_0 \otimes a_1 \otimes \cdots \otimes a_n$ denote a typical element of $C_{-\bullet}(A, A)$ where $A$ is some algebra. At times, we wish to feed a portion of $a_0 \otimes a_1 \otimes \ldots \otimes a_n$ to a Hochschild cochain (or other map on chains) without specifying the degree of the cochain. To do this, we will rewrite $a_0 \otimes a_1 \otimes \ldots \otimes a_n = a_0 \otimes \mathfrak{a}_1 \otimes \ldots \otimes \mathfrak{a}_r$ where each $\mathfrak{a}_i = a_{j_i} \otimes a_{j_i+1} \otimes \ldots \otimes a_{j_{i+1}-1}$ and $\mathfrak{a}_i$ is an empty chain if $j_i = j_{i+1}$.

For example, if $\phi \in C^2(A, A)$, then we rewrite

$$\sum_{1 \leq i \leq n-1} a_0 \otimes a_1 \otimes \ldots a_{i-1} \otimes \phi(a_i, a_{i+1}) \otimes a_{i+2} \otimes \ldots \otimes a_n = \sum a_0 \otimes \mathfrak{a}_1 \otimes \phi(\mathfrak{a}_2) \otimes \mathfrak{a}_3.$$

If $\mathfrak{a}_1 = a_1 \otimes \ldots \otimes a_p$, then $|\mathfrak{a}_1| = p$. For $a_0 \otimes \mathfrak{a}_1 \otimes \mathfrak{a}_2$, we write $\eta_{\mathfrak{a}_1, \mathfrak{a}_2} = (-1)^{|\mathfrak{a}_1|(|\mathfrak{a}_1|+|\mathfrak{a}_2|)}$.

## B.2. Computational Propositions

**Proposition B.1.** *Let* $\hat{\tau}_1 : B(A_0 \to A_1 \to A_0) \longrightarrow B(A_1 \to A_0 \to A_1)$ *be as defined in Section 3.1. Recall from Example 3.4.5 that* $\hat{\tau}_1^* C(A_1 \to A_0 \to A_0) \cong C(A_1 \to A_0 \to A_1)$ *as complexes. Define a map*

$$\Upsilon_{A_0, A_1} : C(A_0 \to A_1 \to A_0) \to \hat{\tau}_1^* C(A_1 \to A_0 \to A_1)$$

*of comodules over $B(A_0 \to A_1 \to A_0)$ by mapping into cogenerators as follows:*

(B.1)
$$v^{f_0,g_0} : C(A_0 \to A_1 \to A_0)(f_0, g_0) \to \hat{\tau}_1^* C(A_1 \to A_0 \to A_1)(g_0, f_0)$$

(B.2)
$$\cong C(A_1 \to A_0 \to A_1)(g_0, f_0)$$

(B.3)
$$\xrightarrow[\text{cogenerators}]{\text{project onto}} C_{-\bullet}(A_{1,f_0g_0} A_{1id})$$

(B.4)
$$v_{n,m}^{f_0,g_0}(\vec{\phi}|\vec{\psi}|\alpha) = \sum_{\substack{I_1 I_2 = \{2,\cdots,n\} \\ \text{as ordered sets}}} \phi_1(\lambda(\vec{\psi})\lambda(\vec{\phi}_{I_2}) \cdot \mathfrak{a}_3, a_0, \mathfrak{a}_1) \otimes \lambda(\vec{\phi}_{I_1}) \cdot \mathfrak{a}_2$$

(B.5)
$$\left( + f_0 a_0 \otimes \lambda(\vec{\phi})\mathfrak{a}_1 \quad if \ \ m = 0 \right).$$

*Then, $\Upsilon_{A_0,A_1} : C(A_0 \to A_1 \to A_0) \to \hat{\tau}^* C(A_1 \to A_0 \to A_1)$ is a map of dg comodules over $B(A_0 \to A_1 \to A_0)$.*

**Proof.** We must show: (1) $\Upsilon$ is a map of comodules, and (2) $\Upsilon$ commutes with the differentials. (In this proof, we drop the subscripts and write $\Upsilon := \Upsilon_{A_0,A_1}$.)

(1) This proof is standard for cofree comodules. Let $(\vec{\phi}|\vec{\psi}|\alpha)$ be as in the statement of the proposition. We want to show that $\Upsilon$ commutes with the coproducts. On one hand,

$$[(id_B \otimes \Upsilon) \circ \Delta_{C(A_0 \to A_1 \to A_0)}](\vec{\phi}|\vec{\psi}|\alpha)$$

$$= [id_B \otimes \Upsilon] \Big( \sum_{\substack{I_1 I_2 = \{1,2,\cdots,n\} \text{ and} \\ J_1 J_2 = \{1,2,\cdots,m\} \\ \text{as ordered sets}}} \epsilon_{I_2,J_1} \cdot (\vec{\phi}_{I_1}|\vec{\psi}_{J_1}) \otimes (\vec{\phi}_{I_2}|\vec{\psi}_{J_2}|\alpha) \Big)$$

$$= \sum_{\substack{I_1 I_2 I_3 = \{1,2,\cdots,n\} \text{ and} \\ J_1 J_2 J_3 = \{1,2,\cdots,m\} \\ \text{as ordered sets}}} \epsilon_{I_2 I_3, J_1} \cdot \epsilon_{I_3, J_2} \cdot (\vec{\phi}_{I_1}|\vec{\psi}_{J_1}) \otimes (\vec{\phi}_{I_2}|\vec{\psi}_{J_2}) \otimes \upsilon_{|I_3|,|J_3|}(\vec{\phi}_{I_3}|\vec{\psi}_{J_3}|\alpha).$$

On the other hand,

$$[\Delta_{\hat{\tau}^* C(A_1 \to A_0 \to A_1)} \circ \Upsilon](\vec{\phi}|\vec{\psi}|\alpha)$$

$$= \Delta_{\hat{\tau}^* C(A_1 \to A_0 \to A_1)} \Big( \sum_{\substack{I_1 I_2 = \{1,2,\cdots,n\} \text{ and} \\ J_1 J_2 = \{1,2,\cdots,m\} \\ \text{as ordered sets}}} \epsilon_{I_2,J_1} \cdot (\vec{\phi}_{I_1}|\vec{\psi}_{J_1}) \otimes \upsilon_{|I_2|,|J_2|}(\vec{\phi}_{I_2}|\vec{\psi}_{J_2}|\alpha) \Big)$$

$$= \sum_{\substack{I_1 I_2 I_3 = \{1,2,\cdots,n\} \text{ and} \\ J_1 J_2 J_3 = \{1,2,\cdots,m\} \\ \text{as ordered sets}}} \epsilon_{I_2 I_3, J_1} \cdot \epsilon_{I_3, J_2} \cdot (\vec{\phi}_{I_1}|\vec{\psi}_{J_1}) \otimes (\vec{\phi}_{I_2}|\vec{\psi}_{J_2}) \otimes \upsilon_{|I_3|,|J_3|}(\vec{\phi}_{I_3}|\vec{\psi}_{J_3}|\alpha).$$

Clearly $(id_B \otimes \Upsilon) \circ \Delta_{C(A_0 \to A_1 \to A_0)} = \Delta_{\hat{\tau}^* C(A_1 \to A_0 \to A_1)} \circ \Upsilon$.

(2) We will show that $\Upsilon$ commutes with the differentials by direct computation. Since $\Upsilon$ is a map of cofree comodules, we only need to check that $\pi_1 \circ D(\Upsilon) = 0$ where $D(\Upsilon)$ is the differential applied to $\Upsilon$ as a linear map between complexes and $\pi_1$ denotes projection

of a comodule onto its cogenerators. More explicitly, we want to check that

$$v_{n,m}(\tilde{\delta}(\vec{\phi})|\vec{\psi}|\alpha) \ + v_{n,m}(\vec{\phi}|\tilde{\delta}(\vec{\psi})|\alpha) \ + v_{n-1,m}(b'(\vec{\phi})|\vec{\psi}|\alpha) \ + v_{n,m-1}(\vec{\phi}|b'(\vec{\psi})|\alpha) \ +$$

$$v_{n,m}(\vec{\phi}|\vec{\psi}|b(\alpha)) \ + b \circ v_{n,m}(\vec{\phi}|\vec{\psi}|\alpha) \ +$$

$$\sum_{\substack{I_1 I_2 = \{1,\ldots,n\} \\ \text{as ordered sets}}} \epsilon_{I_2,\{1,\ldots,m-1\}} \cdot v_{|I_1|,m-1}(\vec{\phi}_{I_1}|\vec{\psi}_{\{1,\cdots,m-1\}}|\psi_m\{\vec{\phi}_{I_2}\} \cdot \alpha) \ +$$

(B.6)
$$\sum_{\substack{J_1 J_2 = \{1,\ldots,m\} \\ \text{as ordered sets}}} \epsilon_{\{2,\ldots,n\},J_1} \cdot \phi_1\{\psi_{J_1}\} \cdot v_{n-1,|J_2|}(\phi_{\{2,\cdots,n\}}|\psi_{J_2}|\alpha) \ +$$

$$\epsilon_{\{n\},\{1,\ldots,m\}} \cdot v_{n-1,m}(\vec{\phi}_{\{1,\cdots,n-1\}}|\vec{\psi}|\phi_n \cdot \alpha) \ +$$

$$\epsilon_{\{1,\ldots,n\},\{1\}} \cdot \psi_1 \cdot v_{n,m-1}(\vec{\phi}|\vec{\psi}_{\{2,\cdots,m\}}|\alpha)$$

$$= 0.$$

In Equation B.6, we will call the terms in rows 1-2 the "standard terms", and the terms in rows 3-6 the "extra terms".

We compute the sum of the standard terms. In Table B.1, the leftmost column lists the expressions that don't cancel in the sum of the standard terms, the middle column gives the standard term from which the expression comes, and the rightmost column gives the term (extra or standard) that cancels the expression.

All of the terms in Table B.1 cancel, so $\Upsilon$ is a map of complexes. $\qquad\square$

| Expression (Expansion) | Comes from Standard Term in Equation B.6 | Cancelling Term in Equation B.6 |
|---|---|---|
| $f_0\psi_1(\lambda(\vec{\phi}_{I_2})\mathbf{a}_3)\cdot$ <br> $\phi_1(\lambda(\vec{\psi}_{\{2,\cdots,m\}}\lambda(\vec{\phi}_{I_3})\mathbf{a}_4,a_0,\mathbf{a}_1)\otimes\lambda(\vec{\phi}_{I_1})\mathbf{a}_2$ | $v_{n,m}(\delta(\phi_1)\phi_2\cdots\phi_n|\vec{\psi}|\alpha)$ | $f_0\psi_1\cdot v_{n,m-1}(\vec{\phi}|\vec{\psi}_{\{2,\cdots,m\}}|\alpha)$ |
| $\phi_1(\lambda(\vec{\psi}_{\{1,\cdots,m-1\}})\lambda(\vec{\phi}_{I_2})\mathbf{a}_3,$ <br> $\psi_m(\lambda(\vec{\phi}_{I_3})\mathbf{a}_4\cdot a_0,\mathbf{a}_1)\otimes\lambda(\vec{\phi}_{I_1})\mathbf{a}_2$ | $v_{n,m}(\delta(\phi_1)\phi_2\cdots\phi_n|\vec{\psi}|\alpha)$ | $v_{|I_1|,m-1}(\vec{\phi}_{I_1}|\vec{\psi}_{\{1,\cdots,m-1\}}|\psi_m\{\vec{\phi}_{I_2}\}\cdot\alpha)$ |
| $\phi_1(\lambda(\vec{\psi})\lambda(\vec{\phi}_{I_2})\mathbf{a}_3,g_m\phi_n(\mathbf{a}_4)\cdot a_0,\mathbf{a}_1)\otimes$ <br> $\otimes\lambda(\vec{\phi}_{I_1})\mathbf{a}_2$ | $v_{n,m}(\delta(\phi_1)\phi_2\cdots\phi_n|\vec{\psi}|\alpha)$ | $v_{n-1,m}(\vec{\phi}_{\{1,\cdots,n-1\}}|\vec{\psi}|g_m\phi_n\cdot\alpha)$ |
| $\phi_1(\lambda(\vec{\psi})\lambda(\vec{\phi}_{I_2})\mathbf{a}_2)\cdot f_1(a_0)\otimes\lambda(\vec{\phi}_{I_1})\mathbf{a}_1$ | $v_{n,m}(\delta(\phi_1)\phi_2\cdots\phi_n|\vec{\psi}|\alpha)$ | $\phi_1\cdot v_{n-1,0}(\vec{\phi}_{\{2,\cdots,n\}}|\vec{\psi}|\alpha)$ |
| $f_0a_0\cdot\phi_1(\mathbf{a}_1)\otimes\lambda(\vec{\phi}_{\{1,\cdots,n-1\}})\mathbf{a}_2$ | $v_{n,m}(\delta(\phi_1)\phi_2\cdots\phi_n|\vec{\psi}|\alpha)$ <br> if $\vec{\psi}=1$ | $b\circ v_{n,m}(\vec{\phi}|\vec{\psi}|\alpha)$ <br> if $\vec{\psi}=1$ |
| $f_0g_m\phi_n(\mathbf{a}_2)f_0a_0\otimes\lambda(\vec{\phi}_{\{1,\cdots,n-1\}})\mathbf{a}_1$ | $b\circ v_{n,m}(\vec{\phi}|\vec{\psi}|\alpha)$ <br> if $\vec{\psi}=1$ | $v_{n-1,m}(\vec{\phi}_{\{1,\cdots,n-1\}}|\vec{\psi}|g_m\phi_n\cdot\alpha)$ <br> if $\vec{\psi}=1$ |
| $\phi_1(\lambda(\vec{\phi})\lambda(\vec{\phi}_{I_2})\mathbf{a}_4,a_0,\mathbf{a}_1)\cdot\phi_2(\mathbf{a}_2)\otimes\lambda(\vec{\phi}_{I_1})\mathbf{a}_3$ | $b\circ v_{n,m}(\vec{\phi}|\vec{\psi}|\alpha)$ | $v_{n-1,m}(\phi_1\cup\vec{\phi}_2\phi_3\cdots\phi_n|\vec{\psi}|\alpha)$ |
| $\phi_1(\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_2})\mathbf{a}_3)\phi_2(\lambda(\vec{\psi}_{J_2}\lambda(\vec{\phi}_{I_3})\mathbf{a}_3,$ <br> $a_0,\mathbf{a}_1)\otimes\lambda(\vec{\phi}_{I_1})\mathbf{a}_2$ | $v_{n-1,m}(\phi_1\cup\phi_2\phi_3\cdots\phi_n|\vec{\psi}|\alpha)$ | $\phi_1\{\vec{\psi}_{J_1}\}\cdot v_{n-1,|J_2|}(\vec{\phi}_{\{2,\cdots,n\}}|\vec{\psi}_{J_2}|\alpha)$ |
| $f_0\psi_1(\lambda(\vec{\phi}_{I_2})\mathbf{a}_2)\cdot f_0a_0\otimes\lambda(\vec{\phi}_{I_1})\mathbf{a}_1$ | $f_0\psi_1\cdot v_{n,0}(\vec{\phi}|1|\alpha)$ <br> if $\vec{\psi}=\psi_1$ | $v_{|I_1|,0}(\vec{\phi}_{I_1}|1|\psi_1\{\vec{\phi}_{I_2}\}\cdot\alpha)$ <br> if $\vec{\psi}=\psi_1$ |

Table B.1. Expansion of terms in Equation B.6

(Technically, the last term in the middle column is not a standard term, but we include it in the table for convenience.)

**Proposition B.2.** *Let $B_{A_0,A_1} = B : C(A_0 \to A_1 \to A_0) \longrightarrow C(A_0 \to A_1 \to A_0)$ be the map of cofree comodules defined by the following maps to cogenerators:*

$$(B.7) \qquad B_{n,m}(\vec{\phi}|\vec{\psi}|\alpha) = \eta_{\mathfrak{a}_1,\mathfrak{a}_2} \cdot 1 \otimes \lambda(\psi)\lambda(\phi)\mathfrak{a}_2 \otimes a_0 \otimes \mathfrak{a}_1.$$

*Then, $D(B_{A_0,A_1}) = \Upsilon_{A_1,A_0}\Upsilon_{A_0,A_1} - id$ where $\Upsilon$ is defined in Proposition B.1.*

**Proof.** We prove the statement by direct computation. Since all of the maps are maps of cofree comodules, we only need to check that $\pi_1(D(B_{A_0,A_1}) - \Upsilon_{A_1,A_0}\Upsilon_{A_0,A_1} - id) = 0$ where $\pi_1$ denotes projection of the comodule onto cogenerators. More explicitly, for an element $(\vec{\phi}|\vec{\psi}|\alpha)$, we want to check that

$$
\begin{aligned}
&(B.8) \quad B_{n,m}(\tilde{\delta}(\vec{\phi})|\vec{\psi}|\alpha) + B_{n,m}(\vec{\phi}|\tilde{\delta}(\vec{\psi})|\alpha) + B_{n-1,m}(b'(\vec{\phi})|\vec{\psi}|\alpha) + B_{n,m-1}(\vec{\phi}|b'(\vec{\psi})|\alpha) + \\
&B_{n,m}(\vec{\phi}|\vec{\psi}|b(\alpha)) + b \circ B_{n,m}(\vec{\phi}|\vec{\psi}|\alpha) + \\
&\epsilon_{\{n\},\{1,\dots,m\}} \cdot B_{n-1,m}(\vec{\phi}_{\{1,\cdots,n-1\}}|\vec{\psi}_m|\phi_n \cdot \alpha) + \\
&\epsilon_{\{1,\dots,n\},\{1\}} \cdot \psi_1 \cdot B_{n,m-1}(\vec{\phi}|\vec{\psi}_{\{2,\cdots,m\}}|\alpha) + \\
&\sum_{\substack{I_1I_2=\{1,\dots,n\}\\ \text{as ordered setts}}} \epsilon_{I_2,\{1,\dots,m-1\}} \cdot B_{|I_1|,m-1}(\vec{\phi}_{I_1}|\vec{\psi}_{\{1,\cdots,m-1\}}|\psi_m\{\vec{\phi}_{I_2}\} \cdot \alpha) + \\
&\sum_{\substack{J_1J_2=\{1,\dots,m\}\\ \text{as ordered setts}}} \epsilon_{\{2,\dots,n\},J_1} \cdot \phi_1\{\psi_{J_1}\} \cdot B_{n-1,|J_2|}(\phi_{\{2,\cdots,n\}}|\psi_{J_2}|\alpha) - \\
&\sum_{\substack{I_1I_2=\{1,\dots,n\}\\ J_1J_2=\{1,\dots,m\}\\ \text{as ordered setts}}} \epsilon_{I_1,J_2} \cdot \upsilon_{|J_1|,|I_1|}(\vec{\psi}_{J_1}|\vec{\phi}_{I_1}|\upsilon_{|I_2|,|J_2|}(\vec{\phi}_{I_2}|\vec{\psi}_{J_2}|\alpha)) - \pi_1(\vec{\phi}|\vec{\psi}|\alpha) \\
&= 0.
\end{aligned}
$$

We will call the terms in rows 1-2 the "standard terms" in the computation of $D(B_{A_0,A_1})$, and the terms in rows 3-6 the "extra terms" in the computation of $D(B_{A_0,A_1})$. The seventh row is $\pi_1(\Upsilon_{A_1,A_0}\Upsilon_{A_0,A_1} - id)$.

We compute the sum of the standard terms. In Table B.2, the leftmost column lists the expressions that don't cancel in the sum of the standard terms, the middle column gives the standard term from which the expression comes, and the rightmost column gives the extra term that cancels the expression. Table B.3 lists the remaining terms from the seventh row that are not already listed in Table B.2. In Table B.3, the left column lists the remaining expressions that don't cancel in the seventh row, and the right column gives the extra term that cancels the expression.

All of the terms in the tables describing the expansion of equation B.8 cancel, so $D(B_{A_0,A_1}) = \Upsilon_{A_1,A_0}\Upsilon_{A_0,A_1} - id.$ $\square$

| Expression (Expansion) | Comes from Standard Term in Equation B.8 | Cancels with Extra Term in Equation B.8 |
|---|---|---|
| $\psi_1(\lambda(\vec{\phi}_{I_1})\boldsymbol{\alpha}_2)\otimes\lambda(\vec{\psi}_{\{2,\cdots,m\}})\lambda(\vec{\phi}_{I_2})\boldsymbol{\alpha}_3\otimes a_0\otimes\boldsymbol{\alpha}_1$ | $b\circ B_{n,m}(\vec{\phi}\vec{\psi}|\alpha)$ | $\psi_1\{\vec{\phi}_{I_1}\}\cdot B_{|I_2|,m-1}(\vec{\phi}_{I_2}|\vec{\psi}_{\{2,\cdots,m\}}|\alpha)$ |
| $g_0\phi_1(\boldsymbol{\alpha}_2)\otimes\lambda(\vec{\psi})\lambda(\vec{\phi}_{\{2,\cdots,n\}})\boldsymbol{\alpha}_3\otimes a_0\otimes\boldsymbol{\alpha}_1$ | $b\circ B_{n,m}(\vec{\phi}\vec{\psi}|\alpha)$ | $\phi_1\cdot B_{n-1,m}(\vec{\phi}_{\{2,\cdots,n\}}|\vec{\psi}|\alpha)$ |
| $1\otimes\lambda(\vec{\psi})\lambda(\vec{\phi}_{\{1,\cdots,n-1\}})\boldsymbol{\alpha}_2\otimes g_m\phi_n(\boldsymbol{\alpha}_3)\cdot a_0\otimes\boldsymbol{\alpha}_1$ | $b\circ B_{n,m}(\vec{\phi}\vec{\psi}|\alpha)$ | $B_{n-1,m}(\vec{\phi}_{\{1,\cdots,n-1\}}|\vec{\psi}|\phi_n\cdot\alpha)$ |
| $1\otimes\lambda(\vec{\psi}_{\{1,\cdots,m-1\}})\lambda(\vec{\phi}_{I_1})\boldsymbol{\alpha}_2\otimes g_m\psi_m(\lambda(\vec{\phi}_{I_2}\boldsymbol{\alpha}_3))\cdot a_0\otimes\boldsymbol{\alpha}_1$ | $b\circ B_{n,m}(\vec{\phi}\vec{\psi}|\alpha)$ | $B_{|I_1|,m-1}(\vec{\phi}_{I_2}|\vec{\psi}_{\{1,\cdots,m-1\}}|\psi_m\{\vec{\phi}_{I_2}\}\cdot\alpha)$ |
| $g_0f_0a_0\otimes\lambda(\vec{\psi})\lambda(\vec{\phi})\boldsymbol{\alpha}_1$ | $b\circ B_{n,m}(\vec{\phi}\vec{\psi}|\alpha)$ | $v_{|J_1|,|I_1|}(\vec{\psi}_{J_1}|\vec{\phi}_{I_1}|v_{|I_2|,|J_2|}(\vec{\phi}_{I_2}|\vec{\psi}_{J_2}|\alpha))$ |

Table B.2. Expansion of terms in Equation B.8: "standard terms" and the "extra terms" that cancel them

(Technically, the last term in the right column is not an extra term, but we include it in the table for convenience.)

| Expression (Expansion) from Seventh-Row in Equation B.8 | Cancels with Extra Term in Equation B.8 |
|---|---|
| $\psi_1(\lambda(\vec{\phi}_{I_1})\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_4})\boldsymbol{\alpha}_4,\phi_{|I_1|+1}(\lambda(\vec{\psi}_{J_3})\lambda(\vec{\phi}_{I_5})\boldsymbol{\alpha}_5,a_0,\boldsymbol{\alpha}_1),$ $\lambda(\vec{\phi}_{I_2\setminus|I_1|+1})\boldsymbol{\alpha}_2)\otimes\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_3})\boldsymbol{\alpha}_3$ | $\psi_1\{\vec{\phi}_{I_1}\}\cdot B_{|I_2|,m-1}(\vec{\phi}_{I_2}|\vec{\psi}_{\{2,\cdots,m\}}|\alpha)$ |
| $\psi_1(\lambda(\vec{\phi}_{I_1})\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_4})\boldsymbol{\alpha}_4,f_{|I_1|+1}a_0,\lambda(\vec{\phi}_{I_2\setminus|I_1|+1})\boldsymbol{\alpha}_1)\otimes\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_3})\boldsymbol{\alpha}_2$ | $\phi_1\cdot B_{n-1,m}(\vec{\phi}_{\{2,\cdots,n\}}|\vec{\psi}|\alpha)$ |
| $g_0\phi_1(\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_2})\boldsymbol{\alpha}_3,a_0,\boldsymbol{\alpha}_1)\otimes\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_1})\boldsymbol{\alpha}_2$ | $\psi_1\{\vec{\phi}_{I_1}\}\cdot B_{|I_2|,m-1}(\vec{\phi}_{I_2}|\vec{\psi}_{\{2,\cdots,m\}}|\alpha)$ |

Table B.3. Expansion of terms in Equation B.8: remaining "seventh-row terms" and the "extra terms" that cancel them

**Proposition B.3.** *Let* $\Upsilon_{A_0,A_1} : C(A_0 \to A_1 \to A_0) \longrightarrow C(A_1 \to A_0 \to A_1)$ *and*

$B_{A_0,A_1} : C(A_0 \to A_1 \to A_0) \longrightarrow C(A_0 \to A_1 \to A_0)$ *be the maps defined in Propositions*

*B.1 and B.2 above. Then,* $[\Upsilon, B] := \Upsilon_{A_0,A_1} B_{A_0,A_1} - B_{A_1,A_0} \Upsilon_{A_0,A_1} = 0.$

**Proof.** We show that $[\Upsilon, B] = 0$ by direct computation. Since all of the maps are maps of cofree comodules, we only need to check that $\pi_1([\Upsilon, B]) = 0$ where $\pi_1$ denotes projection of the comodule onto cogenerators. We check this directly.

$$\pi_1 \circ \Upsilon_{A_0,A_1} \circ B_{A_0,A_1}(\vec{\phi}|\vec{\psi}|\alpha) = \sum_{\substack{I_1 I_2 = \{1,\dots,n\} \\ J_1 J_2 = \{1,\dots,m\} \\ \text{as ordered sets}}} \epsilon_{I_1,J_2} \cdot v_{|I_1|,|J_1|}(\vec{\phi}_{I_1}|\vec{\psi}_{J_1}|B_{|I_2|,|J_2|}(\vec{\phi}_{I_2}|\vec{\psi}_{J_2}|\alpha))$$

$$= \sum_{\substack{I_1 I_2 = \{1,\dots,n\} \\ J_1 J_2 = \{1,\dots,m\} \\ \text{as ordered sets}}} \begin{array}{c} \epsilon_{I_1,J_2} \cdot \eta_{\mathfrak{a}_1,\mathfrak{a}_2} \cdot \\ v_{|I_1|,|J_1|}(\vec{\phi}_{I_1}|\vec{\psi}_{J_1}|1 \otimes \lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_2})\mathfrak{a}_2, a_0, \mathfrak{a}_1) \end{array}$$

$$= \sum_{\substack{I_1 I_2 = \{1,\dots,n\} \\ J_1 J_2 = \{1,\dots,m\} \\ \text{as ordered sets}}} \epsilon_{I_1,J_2} \cdot \eta_{\mathfrak{a}_1,\mathfrak{a}_2} \cdot 1 \otimes \lambda(\vec{\phi}_{I_1})\big(\lambda(\vec{\psi})\lambda(\vec{\phi}_{I_2})\mathfrak{a}_2, a_0, \mathfrak{a}_1\big)$$

$$\pi_1 \circ B_{A_1,A_0} \circ \Upsilon_{A_0,A_1}(\vec{\phi}|\vec{\psi}|\alpha)$$

$$= \sum_{\substack{I_1 I_2 = \{1,\ldots,n\} \\ J_1 J_2 = \{1,\ldots,m\} \\ \text{as ordered sets}}} \epsilon_{I_1,J_2} \cdot B_{|J_1|,|I_1|}(\vec{\psi}_{J_1}|\vec{\phi}_{I_1}|v_{|I_2|,|J_2|}(\vec{\phi}_{I_2}|\vec{\psi}_{J_2}|\alpha))$$

$$= \sum_{\substack{I_1 I_2 = \{1,\ldots,n\} \\ J_1 J_2 = \{1,\ldots,m\} \\ \text{as ordered sets}}} \epsilon_{I_1,J_2} \cdot B_{|J_1|,|I_1|}\big(\vec{\psi}_{J_1}|\vec{\phi}_{I_1}|\phi_{|I_1|+1}(\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_3})\mathfrak{a}_3, a_0, \mathfrak{a}_1) \otimes \lambda(\vec{\phi}_{I_2 \setminus |I_1|+1})\mathfrak{a}_2 +$$
$$+ a_0 \otimes \lambda(\vec{\phi}_{I_2 \setminus |I_1|+1})\mathfrak{a}_1 \quad \text{if } J_2 = \emptyset\big)$$

$$= \sum_{\substack{I_1 I_2 = \{1,\ldots,n\} \\ J_1 J_2 = \{1,\ldots,m\} \\ \text{as ordered sets}}} \epsilon_{I_1,J_2} \cdot \eta_{\mathfrak{a}_2,\mathfrak{a}_3} \cdot 1 \otimes \lambda(\vec{\phi}_{I_1})\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_3})\mathfrak{a}_3 \otimes \phi_{|I_1|+1}(\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_4})\mathfrak{a}_4, a_0, \mathfrak{a}_1) \otimes$$
$$\otimes \lambda(\vec{\phi}_{I_2 \setminus |I_1|+1})\mathfrak{a}_2 +$$

$$+ \epsilon_{I_1,J_2} \cdot \eta_{\mathfrak{a}_1,\mathfrak{a}_2} \cdot 1 \otimes \lambda(\vec{\phi}_{I_1})\lambda(\vec{\psi})\lambda(\vec{\phi}_{I_3})\mathfrak{a}_2 \otimes a_0 \otimes \lambda(\vec{\phi}_{I_2})\mathfrak{a}_1$$

It's clear that $\pi_1 \circ \Upsilon_{A_0,A_1} \circ B_{A_0,A_1} = \pi_1 \circ B_{A_1,A_0} \circ \Upsilon_{A_0,A_1}$: The final expansion of $\pi_1 \circ \Upsilon_{A_0,A_1} \circ B_{A_0,A_1}$ is the sum of the two terms in the final expansion of $\pi_1 \circ B_{A_1,A_0} \circ \Upsilon_{A_0,A_1}$, which is the sum of terms in which one of the $\phi$'s contains $a_0$ and the terms in which none of the $\phi$'s contains $a_0$). $\qquad \square$

## B.3. More notation

For the next two propositions, we will need some more notation. Set

$$A_0, A_1, A_2 \text{ fixed algebras}$$

$$(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha) := (\phi_1...\phi_n|\psi_1...\psi_m|\theta_1...\theta_r|\alpha)$$



$$\in C(A_0 \to A_1 \to A_2 \to A_0)(h_0 g_0 f_0)$$

$$\epsilon_{I_2,J_1,J_2,K_1} := (-1)^{(\sum\limits_{r\in I_1}|\phi_r|+1)((\sum\limits_{s\in J_1}|\psi_s|+1)+(\sum\limits_{t\in K_1}|\theta_t|+1))} .$$
$$(-1)^{(\sum\limits_{s\in J_2}|\psi_s|+1)(\sum\limits_{t\in K_1}|\theta_t|+1)}$$

when $I_1$, $J_1$, $J_2$, $K_1$, are ordered indexing sets

$$\Upsilon_{A_0\bullet A_1, A_2} : C(A_0 \to A_1 \to A_2 \to A_0) \to \hat{\tau}_2^* C(A_2 \to A_0 \to A_1 \to A_2)$$

$$(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha) \mapsto \Upsilon_{A_0, A_2}(\vec{\phi} \bullet \vec{\psi}|\vec{\theta}|\alpha) \quad \text{map of dg comodules}$$

$$\Upsilon_{A_0, A_1\bullet A_2} : C(A_0 \to A_1 \to A_2 \to A_0) \to \hat{\tau}_2^{*2} C(A_1 \to A_2 \to A_0 \to A_1)$$

$$(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha) \mapsto \Upsilon_{A_0, A_1}(\vec{\phi}|\vec{\psi} \bullet \vec{\theta}|\alpha) \quad \text{map of dg comodules}$$

## B.4. More Propositions

**Proposition B.4.** *Let*

$$\mathcal{B}_{A_0,A_1,A_2} = \mathcal{B} : C(A_0 \to A_1 \to A_2 \to A_0) \to \hat{\tau}_2^{*2} C(A_1 \to A_2 \to A_0 \to A_1)$$

*be a map of comodules over* $B(A_0 \to A_1 \to A_2 \to A_0)$ *determined by the following maps to cogenerators:*

$$\mathcal{B}_{A_0,A_1,A_2}^{f_0,g_0,h_0} : C(A_0 \to A_1 \to A_0)(h_0 g_0 f_0) \to \hat{\tau}_2^{*2} C(A_1 \to A_2 \to A_0 \to A_1)(f_0 h_0 g_0)$$

(B.9)
$$\xrightarrow[\substack{cogenerators}]{\substack{project\ onto}} C_{-\bullet}(A_1, f_0 h_0 g_0 A_{1id})$$

$$\mathcal{B}_{n,m,p}(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha) = \sum_{\substack{I_1 I_2 = \{1,2,\cdots,n\} \\ as\ ordered\ sets}} \eta_{\mathfrak{a}_1,\mathfrak{a}_2} \cdot 1 \otimes \lambda(\vec{\phi}_{I_1}) \big(\lambda(\vec{\theta})\lambda(\vec{\psi})\lambda(\vec{\phi}_{I_2})\mathfrak{a}_2 \otimes a_0 \otimes \mathfrak{a}_1\big)$$

*Then,*

(B.10)
$$D(\mathcal{B}_{A_0,A_1,A_2}) = \Upsilon_{A_2 \bullet A_0, A_1} \circ \Upsilon_{A_0 \bullet A_1, A_2} - \Upsilon_{A_0, A_1 \bullet A_2}.$$

**Proof.** We will show that Equation B.10 holds by direct computation. Since all of the maps are maps of cofree comodules, we only need to check that $\pi_1($ Equation B.10) holds where $\pi_1$ denotes projection of the comodule onto cogenerators. More explicitly, we

want to check that

(B.11)

$$\mathcal{B}_{n,m,p}(\tilde{\delta}(\vec{\phi})|\vec{\psi}|\vec{\theta}|\alpha) \ + \mathcal{B}_{n,m,p}(\vec{\phi}|\tilde{\delta}(\vec{\psi})|\vec{\theta}|\alpha) \ + \mathcal{B}_{n,m,p}(\vec{\phi}|\vec{\psi}|\tilde{\delta}(\vec{\theta})|\alpha) \ +$$

$$\mathcal{B}_{n-1,m,p}(b'(\vec{\phi})|\vec{\psi}|\vec{\theta}|\alpha) \ + \mathcal{B}_{n,m-1,p}(\vec{\phi}|b'(\vec{\psi})|\vec{\theta}|\alpha) \ + \mathcal{B}_{n,m,p-1}(\vec{\phi}|\vec{\psi}|b'(\vec{\theta})|\alpha) \ +$$

$$\mathcal{B}_{n,m,p}(\vec{\phi}|\vec{\psi}|\vec{\theta}|b(\alpha)) \ + b \circ \mathcal{B}_{n,m,p}(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha) \ +$$

$$\sum_{\substack{I_1 I_2 = \{1,\ldots,n\} \\ J_1 J_2 = \{1,\ldots,m\} \\ \text{as ordered sets}}} \epsilon_{I_2, J_1, J_2, \{1,\ldots,p-1\}} \cdot \mathcal{B}_{|I_1|,|J_1|,p-1}(\vec{\phi}_{I_1}|\vec{\psi}_{J_1}|\vec{\theta}_{\{1,\cdots,p-1\}}|\theta_p\{\vec{\psi}_{J_2}\}\{\vec{\phi}_{I_2}\} \cdot \alpha) \ +$$

$$\sum_{\substack{I_1 I_2 = \{1,\ldots,n\} \\ \text{as ordered sets}}} \epsilon_{I_2, \{1,\ldots,m-1\}, \{m\}, \{1,\ldots,p\}} \cdot \mathcal{B}_{|I_1|,m-1,p}(\vec{\phi}_{I_1}|\vec{\psi}_{\{1,\cdots,m-1\}}|\vec{\theta}|\psi_m\{\vec{\phi}_{I_2}\} \cdot \alpha) \ +$$

$$\epsilon_{\{n\},\{1,\ldots,m\},\{\},\{1,\ldots,p\}} \cdot \mathcal{B}_{n-1,m,p}(\vec{\phi}_{\{1,\cdots,n-1\}}|\vec{\psi}_m|\vec{\theta}|\phi_n \cdot \alpha) \ +$$

$$\sum_{\substack{J_1 J_2 = \{1,\ldots,m\} \\ K_1 K_2 = \{1,\ldots,p\} \\ \text{as ordered sets}}} \epsilon_{\{2,\ldots,n\}, J_1, J_2, K_1} \cdot \phi_1\{\vec{\theta}_{K_1}\}\{\vec{\psi}_{J_1}\} \cdot \mathcal{B}_{n-1,|J_2|,|K_2|}(\vec{\phi}_{\{2,\cdots,n\}}|\vec{\psi}_{J_2}|\vec{\theta}_{K_2}|\alpha) \ +$$

$$\sum_{\substack{J_1 J_2 = \{1,\ldots,m\} \\ \text{as ordered sets}}} \epsilon_{\{1,\ldots,n\}, J_1, J_2, \{1\}} \cdot \theta_1\{\vec{\psi}_{J_1}\} \cdot \mathcal{B}_{n,|J_2|,p-1}(\vec{\phi}|\vec{\psi}_{J_2}|\vec{\theta}_{\{2,\cdots,p\}}|\alpha) \ +$$

$$\epsilon_{\{1,\ldots,n\},\{1\},\{2,\ldots,m\},\{\}} \cdot \psi_1 \cdot \mathcal{B}_{n,m-1,p}(\vec{\phi}|\vec{\psi}_{\{2,\cdots,m\}}|\vec{\theta}|\alpha) \ +$$

$$\upsilon_{n, p \le * \le m+p}(\vec{\phi}|\vec{\psi} \bullet \vec{\theta}|\alpha) \ +$$

$$\sum_{\substack{I_1 I_2 = \{1,\ldots,n\} \\ J_1 J_2 = \{1,\ldots,m\} \\ K_1 K_2 = \{1,\ldots,p\} \\ \text{as ordered sets}}} \begin{matrix} \epsilon_{I_2, J_1, J_2, K_1} \cdot \\ \upsilon_{|I_1| \le * \le |I_1|+|K_1|, |J_1|}(\vec{\theta}_{K_1} \bullet \vec{\phi}_{I_1}, \vec{\psi}_{J_1}, \upsilon_{|J_2| \le * \le |I_2|+|J_2|, |K_2|}(\vec{\phi}_{I_2} \bullet \vec{\psi}_{J_2}|\vec{\theta}_{K_2}|\alpha)) \end{matrix}$$

$$= 0.$$

In Equation B.11 above, we call the terms in rows 1-3 the "standard terms" in the computation of $D(\mathcal{B}_{A_0,A_1,A_2})$, and the terms in rows 4-9 the "extra terms" in the computation of $D(\mathcal{B}_{A_0,A_1,A_2})$. The terms in rows 10-11 are $\pi_1$ of the righthand side of Equation B.10; we will call these the "$10^{th}$- and $11^{th}$-row terms".

We compute the sum of the standard terms. In Table B.4, the leftmost column lists the expressions that don't cancel in the sum of the standard terms, the middle column gives the standard term from which the expression comes, and the rightmost column gives the term that cancels the expression. Table B.5 lists the remaining ninth row terms that aren't already listed in Table B.4. In Table B.5, the left column lists the remaining expressions that don't cancel in the ninth row, and the right column gives the extra term that cancels the expression.

All of the terms in the tables describing the expansion of Equation B.11 cancel, so we're done. $\square$

| Expression (Expansion) | Comes from Standard Term in Equation B.11 | Cancelling Term in Equation B.11 |
|---|---|---|
| $1 \otimes \lambda(\vec{\phi}_{I_1})[\lambda(\vec{\theta}_{\{1,\cdots,p-1\}})\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_2})\mathbf{a}_2 \otimes \\ \otimes \theta_p(\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_3})\mathbf{a}_3) \cdot a_0 \otimes \mathbf{a}_1]$ | $b \circ \mathcal{B}_{n,m,p}(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha)$ | $\mathcal{B}_{|I_1|,|J_1|,p-1}(\vec{\phi}_{I_1}|\vec{\psi}_{J_1}|\vec{\theta}_{\{1,\cdots,p-1\}}| \\ \theta_p\{\vec{\psi}_{J_2}\}\{\vec{\phi}_{I_2}\} \cdot \alpha)$ |
| $1 \otimes \lambda(\vec{\phi}_{I_1})[\lambda(\vec{\theta}\lambda(\vec{\psi}_{\{1,\cdots,m-1\}})\lambda(\vec{\phi}_{I_2})\mathbf{a}_2 \otimes \\ \otimes \psi_m(\lambda(\vec{\phi}_{I_3})\mathbf{a}_3) \cdot a_0 \otimes \mathbf{a}_1]$ | $b \circ \mathcal{B}_{n,m,p}(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha)$ | $\mathcal{B}_{|I_1|,m-1,p}(\vec{\phi}_{I_1}|\vec{\psi}_{\{1,\cdots,m-1\}}|\vec{\theta}| \\ \psi_m\{\vec{\phi}_{I_2}\} \cdot \alpha)$ |
| $1 \otimes \lambda(\vec{\phi}_{I_1})[\lambda(\vec{\theta}\lambda(\vec{\psi}\lambda(\phi_{\{1,\cdots,n-1\}})\mathbf{a}_2 \otimes \psi_n(\mathbf{a}_3) \cdot a_0 \otimes \mathbf{a}_1]$ | $b \circ \mathcal{B}_{n,m,p}(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha)$ | $\mathcal{B}_{n-1,m,p}(\phi_{\{1,\cdots,n-1\}}|\vec{\psi}|\vec{\theta}|\phi_n \cdot \alpha)$ |
| $\phi_1(\lambda(\vec{\theta}_{K_1})\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_2})\mathbf{a}_2) \otimes \\ \otimes \lambda(\vec{\phi}_{I_1 1})[\lambda(\vec{\theta}_{K_2})\lambda(\vec{\psi}_{J_3})\lambda(\vec{\phi}_{I_3})\mathbf{a}_3 \otimes a_0 \otimes \mathbf{a}_1]$ | $b \circ \mathcal{B}_{n,m,p}(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha)$ | $\phi_1\{\vec{\theta}_{K_1}\}\{\vec{\psi}_{J_1}\} \cdot \\ \mathcal{B}_{n-1,|J_2|,|K_2|}(\vec{\phi}_{\{2,\cdots,n\}}|\vec{\psi}_{J_2}|\vec{\theta}_{K_2}|\alpha)$ |
| $f_0\theta_1(\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_2})\mathbf{a}_2) \otimes \\ \otimes \lambda(\vec{\phi}_{I_1})[\lambda(\vec{\theta}_{\{2,\cdots,p\}})\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_3})\mathbf{a}_3 \otimes a_0 \otimes \mathbf{a}_1]$ | $b \circ \mathcal{B}_{n,m,p}(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha)$ | $\theta_1\{\vec{\psi}_{J_1}\} \cdot \\ \mathcal{B}_{n,|J_2|,p-1}(\vec{\phi}|\vec{\psi}_{J_2}|\vec{\theta}_{\{2,\cdots,p\}}|\alpha)$ |
| $f_0 h_0 \psi_1(\lambda(\vec{\phi}_{I_2})\mathbf{a}_2) \otimes \\ \otimes \lambda(\vec{\phi}_{I_1})[\lambda(\vec{\theta})\lambda(\vec{\psi}_{\{2,\cdots,m\}})\lambda(\vec{\phi}_{I_3})\mathbf{a}_3 \otimes a_0 \otimes \mathbf{a}_1]$ | $b \circ \mathcal{B}_{n,m,p}(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha)$ | $\psi_1 \cdot \mathcal{B}_{n,m-1,p}(\vec{\phi}|\vec{\psi}_{\{2,\cdots,m\}}|\vec{\theta}|\alpha)$ |
| $f_0 h_0 g_0 \phi_{i_1}(\lambda(\vec{\theta}_{K_2})\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_3})\mathbf{a}_3, a_0, \mathbf{a}_1) \otimes \\ \otimes \lambda(\vec{\phi}_{I_1})\lambda(\vec{\theta}_{K_1})\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_2 \setminus i_1})\mathbf{a}_2$ | $b \circ \mathcal{B}_{n,m,p}(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha)$ | $11^{th}$ row |
| $f_0 h_0 g_0 f_{i_1} a_0 \otimes \lambda(\vec{\phi}_{I_1})\lambda(\vec{\theta})\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_2})\mathbf{a}_1$ | $b \circ \mathcal{B}_{n,m,p}(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha)$ | $11^{th}$ row |
| $\phi_1(\lambda(\vec{\phi}_{I_1})\lambda(\vec{\theta})\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_2})\mathbf{a}_3, a_0, \mathbf{a}_1) \otimes \lambda(\vec{\phi}_{I_1 \setminus 1})\mathbf{a}_2$ | $b \circ \mathcal{B}_{n,m,p}(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha)$ | $10^{th}$ row |

Table B.4. Expansion of terms in Equation B.11: "standard terms" and the terms that cancel them

| Expression (expansion) from $11^{th}$-Row Term in Equation B.11 | Cancels with Extra Term in Equation B.11 |
|---|---|
| $\phi_1(\lambda(\vec{\theta}_{K_1})\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_2})[\lambda(\vec{\theta}_{K_3})\lambda(\vec{\psi}_{J_4})\lambda(\vec{\phi}_{I_5})\boldsymbol{a}_3, a_0, \boldsymbol{a}_1])\otimes$ <br> $\otimes\lambda(\vec{\phi}_{I_1\setminus1})\lambda(\vec{\theta}_{K_2})\lambda(\vec{\psi}_{J_3})\lambda(\vec{\phi}_{I_4})\boldsymbol{a}_2$ | $\phi_1\{\vec{\theta}_{K_1}\}\{\vec{\psi}_{J_1}\}\cdot$ <br> $\mathcal{B}_{n-1,|J_2|,|K_2|}(\vec{\phi}_{\{2,\cdots,n\}}|\vec{\psi}_{J_2}|\vec{\theta}_{K_2}|\alpha)$ |
| $f_0\theta_1(\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_2})[\lambda(\vec{\theta}_{K_2})\lambda(\vec{\psi}_{J_3})\lambda(\vec{\phi}_{I_4})\boldsymbol{a}_3, a_0, \boldsymbol{a}_1])\otimes$ <br> $\otimes\lambda(\vec{\phi}_{I_1})\lambda(\vec{\theta}_{K_1\setminus1})\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_3})\boldsymbol{a}_2$ | $\theta_1\{\vec{\psi}_{J_1}\}\cdot$ <br> $\mathcal{B}_{n,|J_2|,p-1}(\vec{\phi}|\vec{\psi}_{J_2}|\vec{\theta}_{\{2,\cdots,p\}}|\alpha)$ |
| $f_0h_0\psi_1(\lambda(\vec{\phi}_{I_2})[\lambda(\vec{\theta}_{K_2})\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_4})\boldsymbol{a}_3, a_0, \boldsymbol{a}_1]) \otimes \lambda(\vec{\phi}_{I_1})\lambda(\vec{\theta}_{K_1})\lambda(\vec{\psi}_{J_1\setminus1})\lambda(\vec{\phi}_{I_3})\boldsymbol{a}_2$ | $\psi_1\cdot\mathcal{B}_{n,m-1,p}(\vec{\phi}|\vec{\psi}_{\{2,\cdots,m\}}|\vec{\theta}|\alpha)$ |

Table B.5. Expansion of terms in Equation B.11: remaining "$11^{th}$ row terms" and the "extra terms" that cancel them

**Proposition B.5.** *Let* $\Upsilon$ *and* $\mathcal{B}$ *be as defined in the previous propositions. Then,*

$$[\Upsilon, \mathcal{B}] := \Upsilon_{A_1 \bullet A_2, A_0} \mathcal{B}_{A_0, A_1, A_2} - \mathcal{B}_{A_2, A_0, A_1} \Upsilon_{A_0 \bullet A_1, A_2} = 0. \ \textit{(Note that } [\Upsilon, \mathcal{B}] \textit{ is a map from}$$

$C(A_0 \to A_1 \to A_2 \to A_0) \textit{ to itself.)}$

**Proof.** We show the proposition by direct computation. Since all of the maps are maps of cofree comodules, we only need to check that $\pi_1([\Upsilon, \mathcal{B}]) = 0$ where $\pi_1$ denotes projection of the comodule onto cogenerators. We check this directly.

$$\pi_1 \circ \Upsilon_{A_1 \bullet A_2, A_0} \mathcal{B}_{A_0, A_1, A_2}(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha)$$

$$= \sum_{\substack{I_1 I_2 = \{1, \ldots, n\} \\ J_1 J_2 = \{1, \ldots, m\} \\ K_1 K_2 = \{1, \ldots, p\} \\ \text{as ordered sets}}} \epsilon_{I_2, J_1, J_2, K_1} \cdot \upsilon_{|K_1| \leq * \leq |K_1| + |J_1|, |I_1|} (\vec{\psi}_{J_1} \bullet \vec{\theta}_{K_1} | \vec{\phi}_{I_1} | \mathcal{B}_{|I_2|, |J_2|, |K_2|} (\vec{\phi}_{I_2} | \vec{\psi}_{J_2} | \vec{\theta}_{K_2} | \alpha))$$

$$= \sum_{\substack{I_1 I_2 = \{1, \ldots, n\} \\ J_1 J_2 = \{1, \ldots, m\} \\ K_1 K_2 = \{1, \ldots, p\} \\ \text{as ordered sets}}} \begin{array}{l} \epsilon_{I_2, J_1, J_2, K_1} \cdot \eta_{\mathfrak{a}_1, \mathfrak{a}_2} \cdot \\ \upsilon_{|K_1| \leq * \leq |K_1| + |J_1|, |I_1|} (\vec{\psi}_{J_1} \bullet \vec{\theta}_{K_1} | \vec{\phi}_{I_1} | 1 \otimes \lambda(\vec{\phi}_{I_2}) [\lambda(\vec{\theta}_{K_2}) \lambda(\vec{\psi}_{J_2}) \lambda(\vec{\phi}_{I_3}) \mathfrak{a}_2, a_0, \mathfrak{a}_1] \end{array}$$

$$= \sum_{\substack{I_1 I_2 = \{1, \ldots, n\} \\ J_1 J_2 = \{1, \ldots, m\} \\ K_1 K_2 = \{1, \ldots, p\} \\ \text{as ordered sets}}} \epsilon_{I_2, J_1, J_2, K_1} \cdot \eta_{\mathfrak{a}_1, \mathfrak{a}_2} \cdot 1 \otimes \lambda(\vec{\theta}_{K_1}) \lambda(\vec{\psi}_{J_1}) \lambda(\vec{\phi}_{I_1}) [\lambda(\vec{\theta}_{K_2}) \lambda(\vec{\psi}_{J_2}) \lambda(\vec{\phi}_{I_2}) \mathfrak{a}_2, a_0, \mathfrak{a}_1]$$

$$\pi_1 \circ \mathcal{B}_{A_2,A_0,A_1} \Upsilon_{A_0 \bullet A_1,A_2}(\vec{\phi}|\vec{\psi}|\vec{\theta}|\alpha)$$

$$= \sum_{\substack{I_1 I_2 = \{1,\ldots,n\} \\ J_1 J_2 = \{1,\ldots,m\} \\ K_1 K_2 = \{1,\ldots,p\} \\ \text{as ordered sets}}} \epsilon_{I_2,J_1,J_2,K_1} \cdot B_{|K_1|,|I_1|,|J_1|}(\vec{\theta}_{K_1}|\vec{\phi}_{I_1}|\vec{\psi}_{J_1}|v_{|J_2| \leq * \leq |I_2|+|J_2|,|K_2|}(\vec{\phi}_{I_2} \bullet \vec{\psi}_{J_2}|\vec{\theta}_{K_2}|\alpha))$$

$$= \sum_{\substack{I_1 I_2 = \{1,\ldots,n\} \\ J_1 J_2 = \{1,\ldots,m\} \\ K_1 K_2 = \{1,\ldots,p\} \\ \text{as ordered sets}}} \epsilon_{I_2,J_1,J_2,K_1} \cdot \eta_{\mathfrak{a}_1,\mathfrak{a}_2} \cdot 1 \otimes \lambda(\vec{\theta}_{K_1})\lambda(\vec{\psi}_{J_1})\lambda(\vec{\phi}_{I_1})[\lambda(\vec{\theta}_{K_2})\lambda(\vec{\psi}_{J_2})\lambda(\vec{\phi}_{I_2})\mathfrak{a}_2, a_0, \mathfrak{a}_1]$$

It's clear that $\pi_1([\Upsilon, \mathcal{B}]) = 0$. $\qquad\qquad\square$

APPENDIX C

# Background on Hochschild chains and cochains

In this section, we give some known constructions on Hochschild chains and cochains for the reader's convenience.

## C.1. Standard constructions and notation

Let $k$ be a field of characteristic zero, $A$ a flat unital $k$-algebra, and $M$ be an $A$-$A$-bimodule. Then, we can take $(C_\bullet(A, M), b)$, the (reduced or standard) Hochschild chain complex of $A$ with coefficients in $M$ (see Reference [6], Equation 2.1). When $M = B$ is also an algebra over $k$ with left and right module structure given by two maps of algebras $f : A \to B$ and $g : A \to B$, respectively, we may write ${}_f B_g$ to clarify the module structure.

Let $k, A, M$ be as above. We can also take $(C^\bullet(A, M), \delta)$, the (reduced) Hochschild cochain complex of $A$ with coefficients in $M$ (see Reference [6], Equations 2.12-13, 2.19-21). When $M = B$ is an algebra, $(C^\bullet(A, B), \delta, \cup)$ is a dga where the cup product $\cup$ is given in Reference [6], Equation 2.14.

Let $f, g, h : A \to A$ be maps of algebras. We have a contraction operation of Hochschild cochains and chains, which is a map of complexes:

$$\iota : C^p(A, {}_f A_g) \bigotimes C_{-q}(A, {}_g A_h) \longrightarrow C_{-(q-p)}(A, {}_f A_h)$$

$$\phi \bigotimes a_0 \otimes \cdots \otimes a_q \mapsto \iota(\phi, a_0 \otimes \cdots \otimes a_q) := \phi \cdot (a_0 \otimes \cdots \otimes a_q) :=$$

$$:= (-1)^{p(q+1)} \phi(a_{q-p+1}, \ldots, a_q) \cdot a_0 \otimes a_1 \otimes \cdots \otimes a_{q-p}.$$

Finally, we have a "Lie derivative like" operation of Hochschild cochains and chains. Fix an algebra $A$ and let $(\phi_1 \ldots \phi_n | \alpha) \in C(A \to A)(f_0)$ (see Equation 2.2) be the following element

We have a map of complexes

$$C(A \to A)(f_0) \to C_{-\bullet}(A,_{f_0} A)$$

$$(\phi_1...\phi_n | a_1 \otimes ... \otimes a_p) \mapsto \lambda(\phi_1...\phi_n) \cdot (a_1 \otimes ... \otimes a_p)$$

$$:= \sum_{0 \leq i_1 \leq ... \leq i_{2n} \leq p} (-1)^{\sum_{\substack{j \text{ odd} \\ j \geq 1}} i_j (|\phi_{i_{\frac{j+1}{2}}}|+1)} \quad \cdot$$

$$\cdot\, f_0 a_1 \otimes ... \otimes f_0 a_{i_1} \otimes \phi_1(a_{i_1+1}, ..., a_{i_2}) \otimes$$

$$\otimes f_1 a_{i_2+1} \otimes ... \otimes f_1 a_{i_3} \otimes \phi_2(a_{i_3+1}, ..., a_{i_4}) \otimes$$

$$\otimes ... \otimes \phi_n(a_{i_{2n-1}+1}, ..., a_{i_{2n}}) \otimes f_n a_{i_{2n}+1} \otimes ... \otimes f_n a_p.$$

## C.2. Brace operation on Hochschild cochains

Fix algebras $A_0, A_1$ and maps of algebras $f_0, f_n : A_0 \rightrightarrows A_1$, $A_0 \leftleftarrows A_1 : g_0, g_m$. We will define a map of complexes called **braces**

$$C(A_0 \to A_1 \to A_0)((f_0, g_0), (f_n, g_m))$$

$$:= \Big( \bigoplus_{\substack{i \in \mathbb{N} \\ f_1, \ldots, f_i \text{ maps of algebras} \\ f_{i+1} = f_n}} C^\bullet(A_{0, f_0} A_{1 f_1}) \otimes \ldots \otimes C^\bullet(A_{0, f_i} A_{1 f_{i+1}}) \Big) \otimes$$

$$\Big( \bigoplus_{\substack{j \in \mathbb{N} \\ g_1, \ldots, g_j \text{ maps of algebras} \\ g_{j+1} = g_m}} C^\bullet(A_{1, g_0} A_{0 g_1}) \otimes \ldots \otimes C^\bullet(A_{1, g_j} A_{0 g_{j+1}}) \Big)$$

$$\Big\downarrow - \bullet -$$

$$C(A_0 \to A_0)(g_0 f_0, g_m f_n)$$

$$:= \bigoplus_{\substack{i \in \mathbb{N} \\ h_1, \ldots, h_i \text{ maps of algebras} \\ h_{i+1} = g_m f_n}} C^\bullet(A_{0, g_0 f_0} A_{0 h_1}) \otimes \ldots \otimes C^\bullet(A_{0, h_i} A_{0 h_{i+1}}).$$

First, for

define $(\phi_1...\phi_n|1) \overset{\bullet}{\mapsto} (\phi_1...\phi_n) \bullet 1 = (\phi_1...\phi_n)$ and $(1|\phi_1...\phi_n) \overset{\bullet}{\mapsto} 1 \bullet (\phi_1...\phi_n) = (\phi_1...\phi_n)$.

Then, for $n, m \geq 1$, let

$$(\phi_1...\phi_n|\psi_1...\psi_m) = \quad A_0 \quad A_1 \quad A_0$$

and define $(\phi_1...\phi_n) \bullet (\psi_1...\psi_m) \in C(A_0 \to A_0)(g_0 f_0, g_m f_n)$ as follows:

$$(\phi_1...\phi_n) \bullet (\psi_1...\psi_m) = \sum_{0 \leq i_1 \leq ... \leq i_{2m} \leq n} (-1)^{\sum_{\substack{T \text{ odd} \\ T \geq 1}} \left( (\sum_{t \leq i_T} |\phi_t|+1)(|\psi_{\frac{T+1}{2}}|+1) \right)} .$$

$$\cdot \, g_0\phi_1 \otimes ... \otimes g_0\phi_{i_1} \otimes \psi_1\{\phi_{i_1+1}...\phi_{i_2}\} \otimes$$

$$\otimes \, g_1\phi_{i_2+1} \otimes ... \otimes g_1\phi_{i_3} \otimes \psi_2\{\phi_{i_3+1}...\phi_{i_4}\} \otimes$$

$$\otimes \cdots \otimes \psi_m\{\phi_{i_{2m-1}+1}...\phi_{i_{2m}}\} \otimes$$

$$\otimes \, g_m\phi_{i_{2m}+1} \otimes ... \otimes g_m\phi_n$$

$$\psi_j\{\phi_{i_r+1}...\phi_{i_r+s}\} \in C^\bullet(A_{0,g_{j-1}f_{i_r}} \, A_{0g_j f_{i_r+s}})$$

$$\psi_j\{\phi_{i_r+1}...\phi_{i_r+s}\}(a_1 \otimes ... \otimes a_p)$$

$$= \sum_{0 \leq k_1 \leq ... \leq k_{2s} \leq p} (-1)^{\sum_{\substack{j \text{ odd} \\ j \geq 1}} k_j(|\phi_{i_r+\frac{j-1}{2}}|+1)} .$$

$$\cdot \, \psi_j \Bigg( f_{i_r}a_1, ..., f_{i_r}a_{k_1}, \phi_{i_r+1}(a_{k_1+1}, ..., a_{k_2}),$$

$$f_{i_r+1}a_{k_2+1}, ..., f_{i_r+1}a_{k_3}, \phi_{i_r+2}(a_{k_3+1}, ..., a_{k_4}),$$

$$, ..., \phi_{i_r+s}(a_{k_{2s-1}}, ..., a_{k_{2s}}),$$

$$f_{i_r+s}a_{k_{2s}+1}, ..., f_{i_r+s}a_p \Bigg).$$

We follow the sign convention in Reference [6] (Equation 2.25 and Section 4.7.2). In $(\phi_1...\phi_n) \bullet (\psi_1...\psi_m)$, moving $\phi_i$ past $\psi_j$ introduces a factor of $(-1)^{(|\phi_i|+1)(|\psi_j|+1)}$. Braces are unital, associative maps of complexes (see Reference [6], Proposition 4.7.2). It's also straightforward to check that $\lambda(\psi_1...\psi_m)\lambda(\phi_1...\phi_n) = \lambda((\phi_1...\phi_n) \bullet (\psi_1...\psi_m))$.
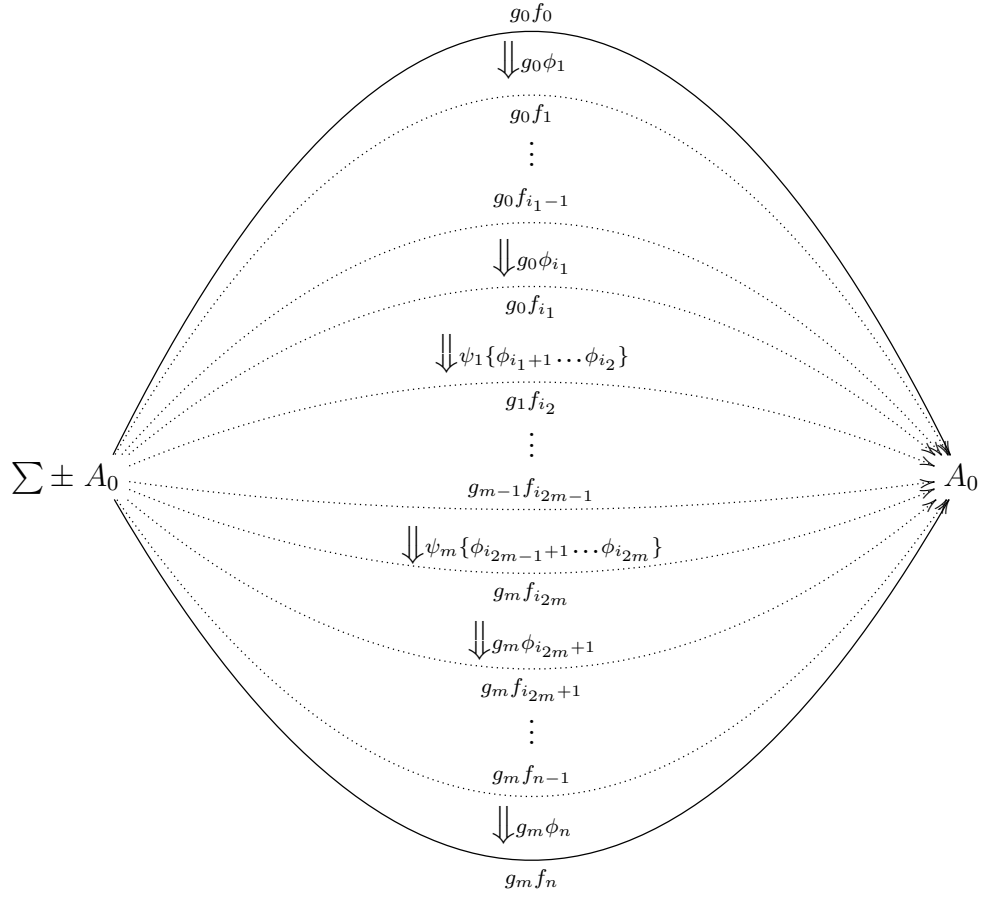
Figure C.1. Picture of the terms in $(\phi_1 \ldots \phi_n) \bullet (\psi_1 \ldots \psi_m)$