Homotopy Algebras for Operads

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Abstract

We present a definition of homotopy algebra for an operad, and explore its consequences.

The paper should be accessible to topologists, category theorists, and anyone acquainted with operads. After a review of operads and monoidal categories, the definition of homotopy algebra is given. Specifically, suppose that $\mathcal{M}$ is a monoidal category in which it makes sense to talk about algebras for some operad P. Then our definition says what a homotopy P-algebra in $\mathcal{M}$ is, provided only that some of the morphisms in $\mathcal{M}$ have been marked out as 'homotopy equivalences'.

The bulk of the paper consists of examples of homotopy algebras. We show that any loop space is a homotopy monoid, and, in fact, that any n-fold loop space is an n-fold homotopy monoid in an appropriate sense. We try to compare weakened algebraic structures such as A_∞-spaces, A_∞-algebras and non-strict monoidal categories to our homotopy algebras, with varying degrees of success. We also prove results on 'change of base', e.g. that the classifying space of a homotopy monoidal category is a homotopy topological monoid. Finally, we reflect on the advantages and disadvantages of our definition, and on how the definition really ought to be replaced by a more subtle ∞-categorical version.

This paper is long (100 pages), but a taste of it can be got from the introductory paper [Lei4] (8 pages), which does not use operads.

Contents

Introduction

A pressing concern in mathematics is to find a coherent theory of weakened algebraic structures. In topology this need was apparent quite early on, with the work of Boardman and Vogt (amongst others) on homotopy-invariant algebraic structures; however, this only covered 'algebraic structures' in quite a narrow sense, and basically only in the context of topological spaces. Another aspect is the recent push to develop a workable theory of weak n-categories (perhaps most famously advocated in the Grothendieck manuscript 'Pursuing Stacks' [Gro]), which has so far resulted in a multiplicity of proposed definitions whose relationships to one another remain mysterious. In the last five years or so there has been a flood of publications involving weakened or up-to-homotopy algebraic structures of some sort: in algebraic geometry, topology, category theory, quantum algebra, deformation theory, in the 'operad renaissance', and even in mirror symmetry; there have been far more developments than I can even attempt to list.

What follows is a contribution to the theory of weakened algebraic structures. More specifically, it is a definition, in an appropriate context, of a *homotopy algebra* for an operad. In other words, given an operad, one can consider its algebras; the definition given here allows one to consider also its 'weak algebras' or 'algebras up to homotopy'. There have been general definitions of this kind made before, as is more fully discussed in Chapter 6. There are also some very popular notions of 'homotopy algebra' for specific operads: for instance, A_∞-algebras, strong homotopy Lie algebras, and special Γ-spaces (to take a random selection). But I think that the strength of the present definition lies in its generality. Roughly speaking, suppose that P is an operad and $\mathcal{M}$ a monoidal category of some sort, so that it makes sense to talk about P-algebras in $\mathcal{M}$. Then the only extra ingredient we will need in order to define homotopy P-algebras in $\mathcal{M}$ is the knowledge of which morphisms in $\mathcal{M}$ are 'homotopy equivalences'. So, for example, in order to talk about homotopy topological algebras we only need to have before us the monoidal category of spaces together with the knowledge of which continuous maps are homotopy equivalences; to talk about homotopy differential graded algebras we only need to know which maps of chain complexes are chain homotopy equivalences; to talk about homotopy categorical algebras we only need to know which functors are equivalences of categories. In particular, we do not need to know what a 'homotopy' between maps is, or anything about resolutions, fibrations, cylinders, etc. Note also that the definition works in any monoidal category, not just in those (like the category of spaces) where the monoidal structure is given by cartesian product.

I hope that this paper will be accessible to both category theorists and topologists, and in fact to anyone acquainted with operads. Although the main examples come from topology and related areas, the spirit of this work is fairly conceptual and category-theoretic. With luck, I have included enough background material that no-one will be put off too rapidly. In particular, there is a short introduction to operads, and this ought to give a rough idea of what is going on to those who have not met them before.

While on the subject of different readerships, I should say a couple of things about loop spaces, which will mentioned frequently. Given a topological space B with basepoint, the *loop space* of B is the space of all based loops in B; that is, it is the set of basepoint-preserving maps from the circle S^1 into B, endowed with a suitable topology. Thus the connected-components of the loop space of B are the elements of the fundamental group of B. Because loops can be composed, and composition of loops is associative up to homotopy, any loop space is a 'topological monoid up to homotopy'. What exactly this means—and there are subtleties concerning higher homotopies which I have not mentioned—was the subject of a great deal of work by topologists. (A summary can be found in [Ad].) Two of the most popular ways of saying 'topological monoid up to homotopy' precisely are 'A_∞-space' and 'special Δ-space', both of which will be discussed later. For those who know all about loop spaces already, I should add that the phrase 'loop space' will be used to mean a space *homeomorphic* to the space of loops in some space B, not merely homotopy-equivalent to such a space.

The idea behind this paper is very simple, so I am slightly embarrassed that it has turned out at 100 pages. Maybe I can reassure the reader that this is mostly for good reasons. For a start, the pace is slow and the margins wide. Also, there is a lengthy preliminary chapter on the basics of operads and monoidal categories, which can be skipped over by many readers. The main definition (of homotopy algebra for an operad) is then made very quickly; what takes up the bulk of the paper is that there are lots of examples. These are the 'good reasons' why the paper is long. A 'bad reason' is that a certain amount of unwieldy calculation is present, although this is mostly sketched rather than done explicitly. As discussed in the final chapter, I think that while this computational effort is inevitable with today's technology, it may be possible to give a more conceptual account when the theory of weak n-categories is better developed.

The origin of the idea

The way I came to the idea behind this paper is as follows. Being a 'historical' explanation of a mathematical idea, it does not represent the cleanest approach, but perhaps it will be helpful. Readers without a background in topology, especially, should not let it put them off the rest of this work.

I had been reading Segal's paper [Seg2] defining Γ-spaces (nowadays usually called 'special Γ-spaces'), which are a precise formulation of the idea of up-to-homotopy topological commutative monoids. A (special) Γ-space is a contravariant functor from Γ to the category of topological spaces, with certain additional properties which need not concern us for now; what intrigued me was the process by which the concept of 'commutative monoid' gave rise to the category Γ.

Explicitly, the objects of Γ are all finite sets, and a map from S to T in Γ is a function θ from S to the set of subsets of T, such that $\theta(i) \cap \theta(j) = \emptyset$ when $i \neq j$. The idea is that the morphisms $\theta : \{1, \ldots, m\} \longrightarrow \{1, \ldots, n\}$ in Γ are the maps $A^n \longrightarrow A^m$ which exist for a 'generic' commutative monoid A; thus θ corresponds to the map

$$\begin{array}{ccc} A^n & \longrightarrow & A^m \\ (a_1, \ldots, a_n) & \longmapsto & (b_1, \ldots, b_m) \end{array}$$

with $b_j = \sum_{i \in \theta(j)} a_i$. But on closer inspection, this idea is rather strange. It says, for instance, that a typical map $A^8 \longrightarrow A^3$ arising 'purely because A is a commutative monoid' is something like

$$(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) \longmapsto (a_3 + a_4 + a_8, a_5, a_2 + a_6).$$

Note that on the right-hand side, none of the terms a_i are repeated (because of the restriction $\theta(i) \cap \theta(j) = \emptyset$), but some of the terms are omitted altogether (namely, a_1 and a_7). So each a_i can be used 0 times or 1 time, but not 2 or more times. It is hard to see in what context this would be reasonable. On the one hand, if we are discussing commutative monoids in the category of sets or of topological spaces, then the maps $A^n \longrightarrow A^m$ for a generic commutative monoid A are the $m \times n$ matrices of natural numbers: an $m \times n$ matrix X corresponds to the map

$$\begin{array}{ccc} A^n & \longrightarrow & A^m \\ (a_1, \ldots, a_n) & \longmapsto & (b_1, \ldots, b_m) \end{array}$$

with $b_j = \sum_{i=1}^{n} X_{ji} a_i$. In this context, each a_i can be used k times for any $k \geq 0$. On the other hand, suppose we are discussing commutative monoids in the category of abelian groups, of graded abelian groups, or of topological abelian groups (i.e. commutative rings, commutative graded rings, or commutative topological rings), so that we no longer have product-projections $A^{\otimes n} \longrightarrow A$. Then a map $A^{\otimes n} \longrightarrow A^{\otimes m}$ for a generic commutative monoid A is simply a function $\phi : \{1, \ldots, n\} \longrightarrow \{1, \ldots, m\}$, corresponding to the map

$$\begin{array}{ccc} A^{\otimes n} & \longrightarrow & A^{\otimes m} \\ a_1 \otimes \cdots \otimes a_n & \longmapsto & b_1 \otimes \cdots \otimes b_m \end{array}$$

where $b_j = \sum_{i \in \phi^{-1}(j)} a_i$. In this case, each a_i is used precisely once. The category Γ does not fit either situation: it is neither fish nor fowl.

Categorical logic, which has plenty to say on how an algebraic theory gives rise to a category, does not provide an immediate answer either. On the one hand, a commutative monoid in a category $\mathcal{M}$ with finite products is essentially a finite-product-preserving functor

$$\mathbf{Matr}_{\mathbb{N}} \longrightarrow \mathcal{M},$$

where $\mathbf{Matr}_{\mathbb{N}}$ is the category whose objects are $0, 1, 2, \ldots$ and whose morphisms $n \longrightarrow m$ are the $m \times n$ matrices of natural numbers. On the other hand, a commutative monoid in a symmetric monoidal category $\mathcal{M}$ is essentially a map

$$\Phi \longrightarrow \mathcal{M}$$

of symmetric monoidal categories, where Φ is the category of finite sets and functions, with monoidal structure given by disjoint union (as the tensor) and the empty set (as the unit). Neither of these categories, $\mathbf{Matr}_{\mathbb{N}}$ or Φ, is the same as Γ.

So, at first it is rather difficult to understand how the category Γ arises from the theory of commutative monoids. The answer to the puzzle comes in the realization that a contravariant functor from Γ to spaces can be described in an alternative but equivalent way: namely, as a 'colax symmetric monoidal functor' from Φ to spaces. Later, 'colax symmetric monoidal functor' will be defined properly; for now, all that matters is that it is one of the various possible notions of a map between symmetric monoidal categories.

We now have two important facts:

a. the theory of commutative monoids naturally gives rise to the category Φ

b. a special Γ-space (i.e. a homotopy topological commutative monoid) can be defined as a colax symmetric monoidal functor $\Phi \longrightarrow$ (spaces) with certain additional properties.

Both of these facts are ripe for generalization. In (a), all that matters about commutative monoids is that they are the algebras for a certain operad; thus if P is any operad, there is a symmetric monoidal category $\widehat{P}$ which plays the same role in relation to P-algebras as Φ does in relation to commutative monoids. In (b), there is nothing special about the symmetric monoidal category of spaces except that certain of its morphisms can be distinguished as 'homotopy equivalences'; this is what is used to define the 'additional properties' referred to. So the scene is set: given any operad P, and any symmetric monoidal category $\mathcal{M}$ in which some of the morphisms are called 'homotopy equivalences', we ought to be able to define a *homotopy P-algebra in $\mathcal{M}$* as a colax symmetric monoidal functor $\widehat{P} \longrightarrow \mathcal{M}$ with certain additional properties. This is, in fact, what we will do.

Layout

The paper is laid out as follows. In Chapter 1, Preliminaries, we cover the basic facts of operads and monoidal categories, and how the two concepts are connected. There are no new ideas here. The first 'proper' chapter, 2, gives the definition of homotopy algebra for an operad. Chapter 3 explores homotopy monoids and homotopy semigroups, including how these relate to special Γ-spaces and Δ-spaces, to monoidal categories in the traditional non-strict sense, to A_∞-spaces, and to A_∞-algebras; this chapter also contains a proof that any loop space is a homotopy topological monoid. In Chapter 4 we look at some other examples of homotopy algebras, including the homotopy algebraic structure on an iterated loop space, and make a closer examination of homotopy algebras in the category **Cat** of categories. Chapter 5 is on 'change of environment' or 'change of base', and includes such results as 'the classifying space of a monoidal category is a homotopy topological monoid', as well as a way of explaining 'why' the higher homotopy groups of a space are abelian. We finish with a discussion (Chapter 6) of various general issues arising in the course of the paper, including homotopy invariance, the relation of this work to higher-dimensional category theory, and the pros and cons of our definition of homotopy algebra. There is also a glossary, listing most of the notation we have introduced.

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Chapter 1

Preliminaries

This paper makes fundamental use of two basic concepts: operads and monoidal categories. In this preliminary chapter we review operads and monoidal categories, and look at the connection between the two. To do this it is useful, although not essential, to look also at multicategories ('coloured operads'). We will also need to be able to do everything in the enriched context—that is, in the context where operations form structures more complex than mere sets—and so we review enrichment for both categories and operads.

Everything in this chapter is old hat, and as such it might not be obvious why it needed to be written. My reason is that although the topics covered (operads, multicategories, monoidal categories and enrichment) form a natural unit, not everyone who knows some of it knows all of it. Specifically, I hope that this paper will be read both by some who would describe themselves as topologists and some who self-define as category theorists; but in category theory it is (sadly) not widely appreciated that an operad is a very natural categorical structure, while topologists are perhaps not so conversant with multicategories and enrichment. So I have included sketches of all these ideas.

That said, experts will be able to skip lightly over this chapter, pausing perhaps to pick up some notation and terminology. (Note, in particular, the terminology concerning symmetric *vs.* non-symmetric operads (1.2), the context in which algebras are taken (1.2), and the notation $\widehat{P}$ for the free monoidal category on an operad P (1.6).) There is a glossary at the end of the paper containing the names of commonly-used categories and operads.

We start with a review of monoidal categories (1.1), operads (1.2), and multicategories (1.3). Enriched categories and operads are covered in 1.4 and 1.5. Finally (1.6) we look at how to form the free monoidal category on an operad.

Miscellaneous notation If $\mathcal{C}$ is a category and A, B are objects of $\mathcal{C}$ then $\mathcal{C}(A, B)$ means $\mathrm{Hom}_{\mathcal{C}}(A, B)$. The opposite (dual) category of $\mathcal{C}$ is written $\mathcal{C}^{\mathrm{op}}$;

thus $\mathcal{C}^{\mathrm{op}}(B,A) = \mathcal{C}(A,B)$. If

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{C}' \arrow[r, bend left, "F'"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\alpha'"] & \mathcal{C}''
\end{tikzcd}$$

is a diagram of categories, functors and natural transformations, then we write $\alpha' * \alpha$ for the composite natural transformation

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F'\circ F"] \arrow[r, bend right, "G'\circ G"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{C}''.
\end{tikzcd}$$

The symbol $\cong$ denotes isomorphism, whereas $\simeq$ means equivalence (between categories, spaces, etc.).

0 is an element of the natural numbers, $\mathbb{N}$.

1.1 Monoidal Categories

We will consider monoidal categories $\mathcal{M}$, not necessarily symmetric, in which the monoidal product is written $\otimes$ and the unit object is written I. The associativity and unit isomorphisms will go nameless, as will the symmetries $A \otimes B \longrightarrow B \otimes A$ in symmetric monoidal categories.

A strict monoidal category is one in which the associativity and unit isomorphisms are actually identities. The coherence theorem for monoidal categories states that any monoidal category is equivalent (in a suitable sense) to a strict monoidal category (see [JS]). This justifies leaving out the brackets in expressions such as $A \otimes B \otimes C$, which we will often do. In the symmetric case, the corresponding coherence result is that every symmetric monoidal category is equivalent to a symmetric strict monoidal category. Note that the word 'strict' qualifies 'monoidal' but not 'symmetric' in the term 'symmetric strict monoidal category': we can force the tensor product to satisfy strict associativity and unit laws, but not to be strictly commutative.

Examples

a. Let $\mathcal{M}$ be any category in which finite products exist. By choosing a particular product $A \times B$ for each pair (A, B) of objects, and a particular terminal object 1, one obtains a symmetric monoidal category $(\mathcal{M}, \times, 1)$ in a natural way. A monoidal category arising in this way is called a *cartesian monoidal category.*

b. **Cat** is the category of (small) categories and functors. The usual (cartesian) product $\times$ and the terminal category **1** make **Cat** into a symmetric monoidal category.

c. **Set**, the category of sets and functions, with cartesian product $\times$ and the one-element set 1, forms a symmetric monoidal category.

d. **Mod** = $\mathbf{Mod}_R$ is the category of left modules over a fixed commutative ring R. It has a symmetric monoidal structure given by $\otimes$ and R.

e. **GrMod** = $\mathbf{GrMod}_R$ is the category of $\mathbb{Z}$-graded R-modules with the usual $\otimes$,

$$(A \otimes B)_n = \bigoplus_{p+q=n} A_p \otimes B_q,$$

and unit object R (abusing notation slightly) given by

$$R_n = \begin{cases} R & \text{if } n = 0 \\ 0 & \text{if } n \neq 0. \end{cases}$$

There are (at least) two possible symmetries on **GrMod**: we can define $\gamma : A \otimes B \longrightarrow B \otimes A$ either by $\gamma(a \otimes b) = b \otimes a$ or by $\gamma(a \otimes b) = (-1)^{pq} b \otimes a$, for $a \in A_p$ and $b \in B_q$. We shall generally use the latter.

f. **ChCx** = $\mathbf{ChCx}_R$ is the category of $\mathbb{Z}$-graded chain complexes of R-modules, that is, of diagrams

$$\cdots \longrightarrow A_1 \xrightarrow{d} A_0 \xrightarrow{d} A_{-1} \longrightarrow \cdots$$

of R-modules with $d \circ d = 0$. The tensor and unit object (also denoted R) are the usual ones, and the symmetry is $\gamma(a \otimes b) = (-1)^{pq} b \otimes a$ for $a \in A_p, b \in B_q$. (This time there is no choice; the other γ mentioned for **GrMod** isn't a chain map.)

g. **Top** is the category of topological spaces and continuous maps, with symmetric monoidal structure given by cartesian product. At various points we will actually need to use some cartesian closed version of the category of spaces, i.e. a version where it is possible to form function spaces. In these situations **Top** will mean the category of compactly generated Hausdorff spaces and continuous maps. This category carries a symmetric monoidal structure by virtue of having products, but these products are not the same as those in the category of all topological spaces. In what follows, the issue of which version of **Top** is appropriate is usually swept under the carpet.

h. $\mathbf{Top}_*$ is the category of based spaces, whose objects are topological spaces with basepoint and whose maps are continuous basepoint-preserving functions. As with **Top** above, sometimes we will really mean 'compactly generated Hausdorff space' rather than 'space'. Two (symmetric) monoidal structures on $\mathbf{Top}_*$ will be of interest to us. Firstly, there is that given by the product $\times$ in $\mathbf{Top}_*$ and the one-point space 1. Secondly, there is the wedge product $\vee$ (join two spaces by their basepoints), whose unit is also 1.

i. Φ is the skeletal category of finite sets: its objects are the finite sets $n = \{0, \ldots, n-1\}$ for each integer $n \geq 0$, and its maps are all functions. This is equivalent to the category of all finite sets. Addition (disjoint union) provides a monoidal product, with unit object the empty set 0. Then Φ becomes a symmetric monoidal category.

j. So far all the examples of monoidal categories have had a symmetry on them. This one does not. Let Δ be the category whose objects are the finite sets n as in (i) for $n \geq 0$, and whose maps are the order-preserving functions with respect to the obvious total order on each n. Thus Δ is equivalent to the category of all finite totally ordered sets. Note that the empty set 0 is an object of Δ, so that Δ is one object bigger than the category usually denoted Δ by topologists. Our Δ becomes a monoidal category via $+$ and 0, but there is no symmetry. (The identity maps $m + n \longrightarrow n + m$ don't provide one, because they don't form a natural transformation.)

There are various notions of a map between monoidal categories, and the distinction is important in this work. I will therefore give a formal definition of monoidal functor, and of monoidal transformation too.

Definition 1.1.1 *Let $\mathcal{L}$ and $\mathcal{M}$ be monoidal categories. A* monoidal functor *$(X, \xi) : \mathcal{L} \longrightarrow \mathcal{M}$ consists of a functor $X : \mathcal{L} \longrightarrow \mathcal{M}$ together with isomorphisms*

$$\begin{array}{rccc} \xi_{A,B} : & X(A \otimes B) & \longrightarrow & X(A) \otimes X(B) \\ \xi_0 : & X(I) & \longrightarrow & I, \end{array}$$

the former natural in $A, B \in \mathcal{L}$, such that the following diagrams commute for all $A, B, C \in \mathcal{L}$:

$$\begin{tikzcd}
X(A \otimes B \otimes C) \arrow[r, "\xi_{A \otimes B, C}"] \arrow[d, "\xi_{A, B \otimes C}"'] & X(A \otimes B) \otimes X(C) \arrow[d, "\xi_{A,B} \otimes 1"] \\
X(A) \otimes X(B \otimes C) \arrow[r, "1 \otimes \xi_{B,C}"'] & X(A) \otimes X(B) \otimes X(C)
\end{tikzcd}$$

$$\begin{tikzcd}
X(I \otimes A) \arrow[r, "\xi_{I,A}"] \arrow[dr, "\sim"'] & X(I) \otimes X(A) \arrow[d, "\xi_0 \otimes 1"] \\
& I \otimes X(A)
\end{tikzcd}
\qquad
\begin{tikzcd}
X(A) \otimes X(I) \arrow[d, "1 \otimes \xi_0"'] & X(A \otimes I) \arrow[l, "\xi_{A,I}"'] \arrow[dl, "\sim"] \\
X(A) \otimes I. &
\end{tikzcd}$$

If $\mathcal{L}$ and $\mathcal{M}$ are symmetric monoidal categories then a symmetric monoidal functor *$\mathcal{L} \longrightarrow \mathcal{M}$ consists of an (X, ξ) as above, satisfying the additional axiom* *that*

$$\begin{tikzcd} X(A \otimes B) \arrow[r, "\xi_{A,B}"] \arrow[d, "\sim"'] & X(A) \otimes X(B) \arrow[d, "\sim"] \\ X(B \otimes A) \arrow[r, "\xi_{B,A}"] & X(B) \otimes X(A) \end{tikzcd}$$

commutes for each $A, B \in \mathcal{L}$.

Note that the maps $\xi_{A,B}$ and ξ_0 are required to be isomorphisms; thus tensor and unit are preserved up to coherent isomorphism.

Definition 1.1.2 *Let* $\mathcal{L}$ *and* $\mathcal{M}$ *be (symmetric or not) monoidal categories, and let*

$$(W, \omega), (X, \xi) : \mathcal{L} \longrightarrow \mathcal{M}$$

be (symmetric or not) monoidal functors. A monoidal transformation

$$\sigma : (W, \omega) \longrightarrow (X, \xi)$$

is a natural transformation $\sigma : W \longrightarrow X$ *such that the following coherence diagrams commute* ($A, B \in \mathcal{L}$)*:*

$$\begin{tikzcd} W(A \otimes B) \arrow[r, "\sigma_{A \otimes B}"] \arrow[d, "\omega_{A,B}"'] & X(A \otimes B) \arrow[d, "\xi_{A,B}"] \\ W(A) \otimes W(B) \arrow[r, "\sigma_A \otimes \sigma_B"'] & X(A) \otimes X(B) \end{tikzcd} \qquad \begin{tikzcd} W(I) \arrow[r, "\sigma_I"] \arrow[d, "\omega_0"'] & X(I) \arrow[d, "\xi_0"] \\ I \arrow[r, equal] & I. \end{tikzcd}$$

Thus if $\mathcal{L}$ and $\mathcal{M}$ are monoidal categories, there is a category $\mathbf{Mon}(\mathcal{L}, \mathcal{M})$ of monoidal functors from $\mathcal{L}$ to $\mathcal{M}$ and monoidal transformations. Similarly, if $\mathcal{L}$ and $\mathcal{M}$ are symmetric monoidal categories then there's a category $\mathbf{SMon}(\mathcal{L}, \mathcal{M})$ of symmetric monoidal functors and monoidal transformations.

1.2 Operads

In this section are the definitions of operad and of algebra for an operad, with numerous examples. The reader is probably very familiar with the definitions; nevertheless, I have included them to establish my terminology for operads and my context for algebras, both of which are slightly non-standard.

Definition 1.2.1 *a. A* non-symmetric operad P *consists of a sequence* $(P(n))_{n\in\mathbb{N}}$ *of sets, together with an element* $1 \in P(1)$ *and a function*

$$\begin{array}{ccc} P(n) \times P(k_1) \times \cdots \times P(k_n) & \longrightarrow & P(k_1 + \cdots + k_n) \\ (\theta, \theta_1, \ldots, \theta_n) & \longmapsto & \theta\circ(\theta_1, \ldots, \theta_n) \end{array}$$

for each $n, k_1, \ldots, k_n \geq 0$*, satisfying unit and associativity axioms.*

b. A symmetric operad *consists of a non-symmetric operad* P *together with a right action of the symmetric group* S_n *on* $P(n)$*, for each* n*, satisfying compatibility laws.*

Points to note

- The exact axioms can be found in [May2].
- By default, our operads will be operads of sets: that is, each $P(n)$ is a set rather than a space or an abelian group, etc. Later (1.5) we will consider these more sophisticated kinds of $P(n)$.
- We give equal emphasis to symmetric operads (usually just called 'operads' in the literature) and non-symmetric operads (also called 'non-Σ operads'). We will generally regard the non-symmetric case as the more basic, and the symmetric case as an elaboration of it. The term *operad* on its own will refer to both cases equally.
- Operads always have a unit element $1 \in P(1)$. There is no requirement (unlike in [May1, 1.1]) that $P(0)$ has only one element.

Definition 1.2.2 *a. Let* P *be a non-symmetric operad and let* $\mathcal{M}$ *be a monoidal category. An* algebra for P in $\mathcal{M}$ *(or a* P-algebra in $\mathcal{M}$*) consists of an object* A *of* $\mathcal{M}$ *together with a function*

$$P(n) \longrightarrow \mathcal{M}(A^{\otimes n}, A)$$

for each n*, written* $\theta \longmapsto \overline{\theta}$ *and satisfying some axioms.*

b. Let P *be a symmetric operad and let* $\mathcal{M}$ *be a symmetric monoidal category. An* algebra for P in $\mathcal{M}$ *consists of an algebra for* P *in* $\mathcal{M}$ *in the sense of (a), satisfying further axioms concerning symmetries.*

The axioms can be found in [May2]. There is an obvious notion of a map of algebras, and we thus obtain the category $\mathbf{Alg}(P, \mathcal{M})$ of P-algebras in $\mathcal{M}$. (We will always regard an operad P as either being symmetric or being non-symmetric; however, the notation $\mathbf{Alg}(P, \mathcal{M})$ does not reveal which.)

Examples

a. Let **Obj** be the unique non-symmetric operad with

$$\mathbf{Obj}(n) = \begin{cases} 1 & \text{if } n = 1 \\ \emptyset & \text{otherwise.} \end{cases}$$

Then an algebra for **Obj** in a monoidal category $\mathcal{M}$ is merely an object of $\mathcal{M}$, and in fact $\mathbf{Alg}(\mathbf{Obj}, \mathcal{M}) \cong \mathcal{M}$. Similarly, let **SObj** be the *symmetric* operad defined by the same formula as **Obj**; then $\mathbf{Alg}(\mathbf{SObj}, \mathcal{M}) \cong \mathcal{M}$ for any symmetric monoidal category $\mathcal{M}$.

b. Let **Mon** be the unique non-symmetric operad with $\mathbf{Mon}(n) = 1$ for all $n \geq 0$. Then an algebra for **Mon** in a monoidal category $\mathcal{M}$ is simply a monoid in $\mathcal{M}$: that is, an object A of $\mathcal{M}$ equipped with maps $m : A \otimes A \longrightarrow A$ and $e : I \longrightarrow A$ such that m is associative and e is a unit for m. So:

- $\mathbf{Alg}(\mathbf{Mon}, \mathbf{Set})$ is the category of monoids in the usual sense
- $\mathbf{Alg}(\mathbf{Mon}, \mathbf{Cat})$ is the category of *strict* monoidal categories
- $\mathbf{Alg}(\mathbf{Mon}, \mathbf{Mod}_R)$ is the category of algebras over the commutative ring R; when we speak of algebras (or graded algebras, etc.) over a ring, we always mean *unital* algebras
- $\mathbf{Alg}(\mathbf{Mon}, \mathbf{GrMod}_R)$ is the category of graded R-algebras
- $\mathbf{Alg}(\mathbf{Mon}, \mathbf{ChCx}_R)$ is the category of differential graded R-algebras
- $\mathbf{Alg}(\mathbf{Mon}, \mathbf{Top})$ is the category of topological monoids
- $\mathbf{Alg}(\mathbf{Mon}, \mathbf{Top}_*)$ is also (isomorphic to) the category of topological monoids.

c. Let **Sem** be the unique non-symmetric operad with

$$\mathbf{Sem}(n) = \begin{cases} 1 & \text{if } n \geq 1 \\ \emptyset & \text{if } n = 0. \end{cases}$$

Then an algebra for **Sem** in a monoidal category $\mathcal{M}$ is a semigroup in $\mathcal{M}$, that is, an object A of $\mathcal{M}$ equipped with an associative binary operation $A \otimes A \longrightarrow A$. So

- $\mathbf{Alg}(\mathbf{Sem}, \mathbf{Set})$ is the category of semigroups in the usual sense
- $\mathbf{Alg}(\mathbf{Sem}, \mathbf{Mod}_R)$ is the category of non-unital R-algebras, and similarly **GrMod** and **ChCx** for graded and differential graded non-unital algebras
- $\mathbf{Alg}(\mathbf{Sem}, \mathbf{Top})$ is the category of topological semigroups
- An object of $\mathbf{Alg}(\mathbf{Sem}, \mathbf{Top}_*)$ is a topological semigroup with a distinguished idempotent element; note that this is not necessarily a topological monoid.

d. Let **CMon** be the unique symmetric operad with $\mathbf{CMon}(n) = 1$ for all n (this being the symmetric analogue of the non-symmetric operad **Mon** in (b)). Then a **CMon**-algebra in a symmetric monoidal category $\mathcal{M}$ is a commutative monoid in $\mathcal{M}$; all but one of the examples of $\mathcal{M}$ given in (b) can be repeated here, with the word 'commutative' inserted each time. The exception is **Cat**: a commutative monoid in **Cat** is a strict monoidal category in which $x \otimes y = y \otimes x$ and $f \otimes g = g \otimes f$ for all objects x, y and morphisms f, g. Such 'strictly symmetric strict monoidal categories' are very rare in nature; see the comments on coherence at the start of Section 1.1.

e. Let **CSem** be defined by the same formula as **Sem** in (c), but now regarded as a symmetric operad (in the only possible way). Then a **CSem**-algebra in a symmetric monoidal category $\mathcal{M}$ is a commutative semigroup in $\mathcal{M}$.

f. Let **Pt** be the unique non-symmetric operad defined by

$$\mathbf{Pt}(n) = \begin{cases} 1 & \text{if } n = 0 \text{ or } n = 1 \\ \emptyset & \text{if } n \geq 2. \end{cases}$$

Then a **Pt**-algebra in a monoidal category $\mathcal{M}$ is a pointed object of $\mathcal{M}$, i.e. an object A of $\mathcal{M}$ together with a map $I \longrightarrow A$. So:

- $\mathbf{Alg}(\mathbf{Pt}, \mathbf{Set})$ is the category of pointed sets (i.e. sets with a distinguished element)
- $\mathbf{Alg}(\mathbf{Pt}, \mathbf{Top}) \cong \mathbf{Alg}(\mathbf{Pt}, \mathbf{Top}_*) \cong \mathbf{Top}_*$
- $\mathbf{Alg}(\mathbf{Pt}, \mathbf{Mod}_R)$ is the category in which an object is an R-module with a chosen element, and a morphism is a homomorphism preserving chosen elements
- Similarly, an object of $\mathbf{Alg}(\mathbf{Pt}, \mathbf{GrMod})$ is a graded module A together with a chosen element of A_0
- An object of $\mathbf{Alg}(\mathbf{Pt}, \mathbf{ChCx})$ is a chain complex A together with a chosen cycle in A_0
- An object of $\mathbf{Alg}(\mathbf{Pt}, \mathbf{Cat})$ is a (small) category with a chosen object.

A symmetric operad **SPt** can be defined by the same formula as **Pt**, and

$$\mathbf{Alg}(\mathbf{SPt}, \mathcal{M}) \cong \mathbf{Alg}(\mathbf{Pt}, \mathcal{M})$$

for any symmetric monoidal category $\mathcal{M}$.

g. There is a symmetric operad **Sym** in which $\mathbf{Sym}(n)$ is S_n, the nth symmetric group. The unit for this operad is uniquely determined and the symmetric group actions are by translation; a description of the composition is more lengthy, but can be found, effectively, in [May2, 1.1.1(c)]. If $\mathcal{M}$ is any symmetric monoidal category then $\mathbf{Alg}(\mathbf{Sym}, \mathcal{M})$ is the category of monoids in $\mathcal{M}$, which category we have also described via the non-symmetric operad **Mon** in (b).

h. Let us say that a *monoid with involution* in a symmetric monoidal category $\mathcal{M}$ is a monoid A in $\mathcal{M}$ together with a map $(\ )^* : A \longrightarrow A$, satisfying commuting diagrams corresponding to the equations

$$1^* = 1, \qquad (a \cdot b)^* = b^* \cdot a^*.$$

(For instance, any group becomes a monoid with involution in **Set**, by defining $a^* = a^{-1}$.) There is an operad **Inv** such that **Inv**-algebras are monoids with involution, in any symmetric monoidal category $\mathcal{M}$. More explicitly,

$$\mathbf{Inv}(n) = C_2^n \times S_n$$

where C_2 is the cyclic group of order 2; the description of the rest of the operad structure is omitted.

i. Fix a monoid G (in **Set**). Then there is a non-symmetric operad $\mathbf{Act}_G$, with

$$\mathbf{Act}_G(n) = \begin{cases} G & \text{if } n = 1 \\ \emptyset & \text{otherwise.} \end{cases}$$

Composition and identity in $\mathbf{Act}_G$ are given by multiplication and identity in G. An algebra for $\mathbf{Act}_G$ in a monoidal category $\mathcal{M}$ is a *left G-object* in $\mathcal{M}$: that is, an object A of $\mathcal{M}$ equipped with a map $g \cdot - : A \longrightarrow A$ for each $g \in G$, satisfying the usual axioms for an action. E.g.:

- $\mathbf{Alg}(\mathbf{Act}_G, \mathbf{Set})$ is the category of left G-sets
- $\mathbf{Alg}(\mathbf{Act}_G, \mathbf{Mod}_R)$ is the category of R-linear representations of G.

There is also a *symmetric* operad $\mathbf{SAct}_G$, defined by the same formula as $\mathbf{Act}_G$. If $\mathcal{M}$ is a symmetric monoidal category (as in the two examples just mentioned) then

$$\mathbf{Alg}(\mathbf{SAct}_G, \mathcal{M}) \cong \mathbf{Alg}(\mathbf{Act}_G, \mathcal{M}).$$

1.3 Multicategories

We shall occasionally make passing reference to multicategories. Multicategories are the same as 'coloured operads' or 'typed operads', if we ignore the issue of whether or not there are symmetric group actions, and they are a very natural generalization of operads. As far as I can tell, multicategories were actually invented a little earlier than operads, their first applications being in logic, linguistics and computer science rather than in topology. (See Lambek's original paper [Lam].)

So, a *multicategory* P (or 'non-symmetric multicategory', for emphasis) consists of

- a collection $\mathrm{ob}(P)$ of objects

- for each $n \geq 0$ and $a_1, \ldots, a_n, a \in \mathrm{ob}(P)$, a set $P(a_1, \ldots, a_n; a)$, whose elements are written $\theta : a_1, \ldots, a_n \longrightarrow a$
- for each $a \in \mathrm{ob}(P)$, an 'identity' element 1_a of $P(a; a)$
- 'composition' functions

$$\begin{aligned} P(a_1, \ldots, a_n; a) \times P(a_1^1, \ldots, a_1^{k_1}; a_1) \times \cdots \times P(a_n^1, \ldots, a_n^{k_n}; a_n) \\ \longrightarrow P(a_1^1, \ldots, a_n^{k_n}; a). \end{aligned}$$

Associative and unit laws must be obeyed. The exact definition can be found in [Lam, p. 103].

A *symmetric multicategory* is a multicategory P together with a function

$$\begin{array}{ccc} P(a_1, \ldots, a_n; a) & \longrightarrow & P(a_{\sigma(1)}, \ldots, a_{\sigma(n)}; a), \\ \theta & \longmapsto & \theta.\sigma \end{array}$$

for each $a_1, \ldots, a_n, a \in \mathrm{ob}(P)$ and $\sigma \in S_n$, satisfying further axioms.

A one-object multicategory is precisely an operad (in both the non-symmetric and symmetric flavours): if the single object of the multicategory P is called a, then the corresponding operad P' has

$$P'(n) = P(\underbrace{a, \ldots, a}_{n}; a).$$

We shall subsequently write the operad and the multicategory both as P.

This provides one source of examples of multicategories. Another comes from monoidal categories: if $\mathcal{M}$ is a monoidal category then there is a multicategory $\overline{\mathcal{M}}$ with the same objects as $\mathcal{M}$, and with

$$\overline{\mathcal{M}}(a_1, \ldots, a_n; a) = \mathcal{M}(a_1 \otimes \cdots \otimes a_n, a).$$

Alternatively, we could obtain a multicategory by taking only some of the objects of $\mathcal{M}$; if we take only one then we obtain the familiar endomorphism operads. If $\mathcal{M}$ is a *symmetric* monoidal category, then $\overline{\mathcal{M}}$ becomes a symmetric multicategory.

We can now rephrase the definition of algebra. An algebra in a monoidal category $\mathcal{M}$ for a non-symmetric operad P is simply a map $P \longrightarrow \overline{\mathcal{M}}$ of multicategories; similarly for the symmetric version. More generally, we can define an algebra in $\mathcal{M}$ for a *multicategory* P as a multicategory map $P \longrightarrow \overline{\mathcal{M}}$.

One more example of a multicategory will be referred to later (Chapter 6). Fix an operad P—say non-symmetric, for simplicity. Then there is a 2-object multicategory (a '2-coloured operad') $\mathbf{Map}_P$, with the property that a $\mathbf{Map}_P$-algebra in a monoidal category $\mathcal{M}$ consists of a pair (A_0, A_1) of P-algebras in $\mathcal{M}$ together with a P-algebra map $A_0 \longrightarrow A_1$. Explicitly, define the objects of $\mathbf{Map}_P$ to be $\{0, 1\}$, and define the 'hom-sets' of $\mathbf{Map}_P$ by

$$\begin{aligned} \mathbf{Map}_P(a_1, \ldots, a_n; 0) &= \begin{cases} P(n) & \text{if } a_1 = \cdots = a_n = 0 \\ \emptyset & \text{otherwise,} \end{cases} \\ \mathbf{Map}_P(a_1, \ldots, a_n; 1) &= P(n) \text{ for all } a_1, \ldots, a_n \in \{0, 1\}. \end{aligned}$$

Composition and identities are defined as in P.

1.4 Enriched Categories

The core ideas of this paper do not depend at all on the idea of *enrichment*, in which 'hom-sets' (and similar things) are not in fact sets but some richer kind of structure. However, the core ideas (such as the definition of homotopy algebra) can all be extended to the enriched setting; we will repeatedly say 'here's an idea; now here it is again in the enriched setting'. (Indeed, the enriched setting is where some of the most interesting examples occur.) This process of extension begins in the next two sections, in which we discuss enriched categories and enriched operads.

Let $\mathcal{V}$ be a monoidal category. A *category enriched in* $\mathcal{V}$, or $\mathcal{V}$*-enriched category*, $\mathcal{C}$, consists of a class $\mathrm{ob}(\mathcal{C})$ (the *objects* of $\mathcal{C}$), an object $\mathcal{C}[A, B]$ of $\mathcal{V}$ for each $A, B \in \mathrm{ob}(\mathcal{C})$, and then morphisms in $\mathcal{V}$ representing composition and identities in $\mathcal{C}$. The full definition is laid out in [Bor, 6.2.1].

Examples

a. A **Set**-enriched category is just a category.

b. Let $\mathcal{V}$ be any symmetric monoidal category which is closed, in the sense that there is a functor

$$[—, —] : \mathcal{V}^{\mathrm{op}} \times \mathcal{V} \longrightarrow \mathcal{V}$$

such that

$$\mathcal{V}(U \otimes V, W) \cong \mathcal{V}(U, [V, W])$$

naturally in $U, V, W \in \mathcal{V}$. Then we obtain a $\mathcal{V}$-enriched category, which we also call $\mathcal{V}$, by putting $\mathcal{V}[U, V] = [U, V]$.

c. A particular example of (b) is **Set**, where $[U, V]$ is the set of functions from U to V.

d. Another example of (b) is $\mathbf{Mod}_R$, where $[U, V]$ is the usual module of homomorphisms $U \longrightarrow V$.

e. Another is **GrMod**; this time $[U, V]_n$ is the module of degree n maps from U to V (i.e. families of homomorphisms $(U_k \longrightarrow V_{k+n})_{k \in \mathbb{Z}}$).

f. Another is **Top**, recalling (1.1(g)) that we can form the function space $[U, V]$ because our spaces are assumed to be compactly generated Hausdorff. (The square bracket notation does not mean homotopy classes of maps, as sometimes it does in the literature.)

g. Another is **Cat**, where $[U, V]$ is the usual category of functors from U to V and natural transformations.

h. Let $\mathcal{V} = \mathbf{GrMod}$. Then there is a $\mathcal{V}$-enriched category **ChCx**, in which the objects are chain complexes and if U and V are chain complexes, $(\mathbf{ChCx}[U, V])_n$ is the module whose elements are the degree n chain maps from U to V.

Any $\mathcal{V}$-enriched category has an underlying (**Set**-enriched) category. Formally this is obtained by applying the 'change of base' $\mathcal{V}(I, \text{—}) : \mathcal{V} \longrightarrow \mathbf{Set}$. Informally it's clear enough what's going on in each of our examples of $\mathcal{V}$: e.g. if $\mathcal{C}$ is a **Top**-enriched category, one obtains an ordinary category simply by forgetting the topology on each $\mathcal{C}[A, B]$ and regarding it as a mere set. In the case $\mathcal{V} = \mathbf{GrMod}$ it's not quite so obvious; the analogous process is to take each graded module $\mathcal{C}[A, B]$ and extract from it the set $(\mathcal{C}[A, B])_0$ which is its degree 0 part. Thus the underlying category of the **GrMod**-enriched category **ChCx** (see (h)) is the category we called **ChCx** in 1.1(f).

Now suppose that $\mathcal{V}$ is a *symmetric* monoidal category. Then it's possible to define what a *$\mathcal{V}$-enriched monoidal category* is, and similarly a *$\mathcal{V}$-enriched symmetric monoidal category*, a *$\mathcal{V}$-enriched (symmetric) monoidal functor*, and a *$\mathcal{V}$-enriched monoidal transformation*. (The details might be in the encyclopaedic [Kel]; I could not locate a copy.) A **Set**-enriched (symmetric) monoidal category is just a (symmetric) monoidal category, and similarly functors and transformations. All the examples (b)–(h) of $\mathcal{V}$-enriched categories are in fact $\mathcal{V}$-enriched symmetric monoidal categories in an obvious way: in (b), for instance, any symmetric monoidal closed category is a symmetric monoidal category enriched in itself, and in (h), **ChCx** is a **GrMod**-enriched symmetric monoidal category.

1.5 Enriched Operads

We now move on to enrichment of operads. We could, more generally, talk about enriched multicategories, but will not; it is in that context that the term 'enrichment' is most evidently appropriate.

So, let $\mathcal{V}$ be a symmetric monoidal category. Then *$\mathcal{V}$-enriched operads* (symmetric and non-symmetric) are defined just as ordinary operads were, except that the sets $P(n)$ are now objects of $\mathcal{V}$, cartesian product $\times$ becomes $\otimes$ (the tensor product in $\mathcal{V}$), and the identity element of $P(1)$ is now a map $I \longrightarrow P(1)$ in $\mathcal{V}$. So a **Set**-enriched operad is an operad as defined above (1.2.1), a **Top**-enriched operad is a 'topological operad', a **ChCx**-enriched operad is what is known as a 'differential graded operad', and so on; all of this in both symmetric and non-symmetric flavours.

We can discuss algebras too. If P is a $\mathcal{V}$-enriched non-symmetric operad and $\mathcal{M}$ a $\mathcal{V}$-enriched monoidal category, then a *P-algebra in $\mathcal{M}$* is an object A of $\mathcal{M}$ together with a map $P(n) \longrightarrow \mathcal{M}[A^{\otimes n}, A]$ in $\mathcal{V}$ for each n, satisfying suitable axioms. The symmetric case is similar. In both cases, the P-algebras in $\mathcal{M}$ form a category $\mathbf{Alg}_{\mathcal{V}}(P, \mathcal{M})$ —or just $\mathbf{Alg}(P, \mathcal{M})$, for simplicity.

Examples

a. When $\mathcal{V} = \mathbf{Set}$, this is the definition of algebra given above (1.2.2).

b. Let $\mathcal{V} = \mathbf{Top}$ and let G be a topological monoid: then there is a **Top**-enriched non-symmetric operad $\mathbf{Act}_G$, defined by the formula of Example 1.2(i). An $\mathbf{Act}_G$-algebra in **Top** is a space with a continuous left action by G. The same applies to the symmetric version, $\mathbf{SAct}_G$. Alternatively, we can use **Cat** instead of **Top** and take G to be a strict monoidal category.

c. Let $\mathcal{V} = \mathbf{GrMod}$ and $\mathcal{M} = \mathbf{ChCx}$, as in 1.4(h); let G be a graded algebra. Then, as in (b), there is a **GrMod**-enriched non-symmetric operad $\mathbf{Act}_G$. An algebra for $\mathbf{Act}_G$ in **ChCx** is a chain complex with a left action by the graded algebra G.

d. Let $\mathcal{V}$ be **Ab** (abelian groups) and let $\mathcal{M}$ be $\mathbf{Mod}_R$, which is a $\mathcal{V}$-enriched symmetric monoidal category in a natural way. There is an **Ab**-enriched symmetric operad **Lie** with the property that $\mathbf{Alg}_{\mathbf{Ab}}(\mathbf{Lie}, \mathbf{Mod}_R)$ is the category of Lie algebras over R. See [Kap, 2.2], [GK, 1.3.9], [May3], or [KSV, 1.5] for more details on **Lie**.

e. Let $\mathcal{V} = \mathbf{GrAb}(= \mathbf{GrMod}_{\mathbb{Z}})$ and let $\mathcal{M} = \mathbf{GrMod}_R$. There is a **GrAb**-enriched symmetric operad **GrLie** such that $\mathbf{Alg}_{\mathbf{GrAb}}(\mathbf{GrLie}, \mathbf{GrMod}_R)$ is the category of graded Lie algebras over R. By a 'graded Lie algebra' I mean a graded module A together with a binary operation of degree -1 —that is, a family of homomorphisms

$$[—,—] : A_p \otimes A_q \longrightarrow A_{p+q-1}$$

—satisfying suitable identities. See 4.1.1 for further details.

(If we want the bracket to be of degree 0 then we can get away with taking $\mathcal{V}$ to be the more simple category **Ab** instead: just change $\mathcal{M}$ to $\mathbf{GrMod}_R$ in Example (d).)

f. Taking $\mathcal{M} = \mathbf{ChCx}$ in (e), $\mathbf{Alg}(\mathbf{GrLie}, \mathbf{ChCx})$ is the category of differential graded Lie algebras.

g. Let $\mathcal{V} = \mathbf{GrAb}$ and $\mathcal{M} = \mathbf{GrMod}_R$. There's a certain **GrAb**-enriched symmetric operad **Ger**, such that $\mathbf{Alg}_{\mathbf{GrAb}}(\mathbf{Ger}, \mathbf{GrMod}_R)$ is the category of Gerstenhaber algebras over R. A Gerstenhaber algebra (see [Vor]) is by definition a graded module which is both a graded-commutative algebra and a graded Lie algebra, with the two structures being compatible.

Aside: the definition of algebra

Suppose we have a $\mathcal{V}$-enriched operad P, and wish to discuss P-algebras in some kind of monoidal category $\mathcal{M}$. In order to do this it isn't actually necessary for $\mathcal{M}$ to be enriched in $\mathcal{V}$. For instance, if $\mathcal{M} = \mathcal{V}$ is the category of *all* topological spaces, not necessarily compactly generated Hausdorff, then one can still define a 'P-algebra in $\mathcal{M}$' sensibly: it's an object A of $\mathcal{M}$ together with a suitable family of maps

$$P(n) \times A^n \longrightarrow A.$$

More generally, suppose that $\mathcal{M}$ is an (ordinary) monoidal category and that $\mathcal{V}$ 'acts' on $\mathcal{M}$, in the sense that there is a functor

$$\begin{array}{ccc} \mathcal{V} \times \mathcal{M} & \longrightarrow & \mathcal{M}, \\ (V, A) & \longmapsto & V \cdot A \end{array}$$

with suitable properties. Then one can define a P-algebra in $\mathcal{M}$ as an object A of $\mathcal{M}$ together with maps

$$P(n) \cdot A^{\otimes n} \longrightarrow A$$

satisfying axioms. For instance, any symmetric monoidal category $\mathcal{V}$ (closed or not) acts on itself, and so one has a notion of P-algebras in $\mathcal{V}$. If $\mathcal{V}$ *is* closed then the two notions of P-algebra in $\mathcal{V}$, one given by enrichment and the other by action, coincide.

So we now have two possible contexts for forming a category of algebras: $\mathcal{M}$ can either be enriched in $\mathcal{V}$ or acted on by $\mathcal{V}$. How are we to combine the two? An obvious answer is to stipulate that $\mathcal{M}$ is a monoidal category and that there is a given functor

$$H : \mathcal{V}^{\mathrm{op}} \times \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \longrightarrow \mathbf{Set}$$

with suitable properties. If $\mathcal{M}$ is enriched in $\mathcal{V}$ then H arises as

$$(V, A, B) \longmapsto \mathcal{V}(V, \mathcal{M}[A, B]),$$

and if $\mathcal{M}$ is acted on by $\mathcal{V}$ then H arises as

$$(V, A, B) \longmapsto \mathcal{M}(V \cdot A, B).$$

For a general H, one ought to be able to define a category $\mathbf{Alg}(P, \mathcal{M})$ in a sensible way.

However, we do not take these thoughts any further in this work. By good luck, and with the aid of devices such as compactly generated spaces, enrichment suffices to cover all the examples that have come to mind.

1.6 The Free Monoidal Category on an Operad

This last preliminary section explains how an operad gives rise to a strict monoidal category. This process was probably first described by Boardman and Vogt; an account can also be found in the book of Adams. (See [BV] and [Ad, p. 42].)

Here we offer three different descriptions of the construction: the first abstract, the last concrete, and the second somewhere in between. Then, after some examples, we prove a crucial property of the construction, which provides an alternative description of algebras for an operad and is a conceptual stepping-stone to the definition of homotopy algebra.

Our aim, then, is to take a (symmetric or non-symmetric) operad P and construct from it a (symmetric or not) strict monoidal category $\widehat{P}$.

First Description Recall from 1.3 that any strict monoidal category L has an 'underlying' multicategory $\overline{L}$. This defines a functor

$$\overline{(\ )} : \text{(strict monoidal categories)} \longrightarrow \text{(multicategories)},$$

which happens to have a left adjoint $\widehat{(\ )}$. A non-symmetric operad P is just a one-object multicategory, and so we obtain from P a strict monoidal category $\widehat{P}$. The same goes in the symmetric case.

(This adjunction is discussed in more depth and generality in [Lei1, 4.3].)

Second Description Let P be a non-symmetric operad; again, what follows can be repeated with the obvious changes for symmetric operads. Then $\widehat{P}$ can be constructed as the free strict monoidal category containing a P-algebra. Consider, by way of comparison, the category Δ of Example 1.1(j). Δ can be described as the free strict monoidal category containing a monoid: for it is generated (as a strict monoidal category) by the objects and morphisms

$$0 \xrightarrow{e} 1 \xleftarrow{m} 1+1,$$

subject to the usual monoid axioms on m and e. (See [Mac, VII.5.1].) To build our category $\widehat{P}$, we must first of all put into it an object A, the underlying object of the P-algebra it contains; then, since $\widehat{P}$ is a monoidal category, it must contain the nth tensor power $A^{\otimes n}$ for each $n \geq 0$; then, since A is meant to be a P-algebra in $\widehat{P}$, there must be a morphism $A^{\otimes n} \longrightarrow A$ in $\widehat{P}$ for each element of $P(n)$; and so on.

In the Third Description below, which is the one we will actually use, the object $A^{\otimes n}$ of $\widehat{P}$ is written merely as n, and $\otimes$ is therefore written as $+$.

Third Description Concretely, let P be a non-symmetric operad, and define a monoidal category $\widehat{P}$ as follows. The objects are the natural numbers $0, 1, \ldots$, and the monoidal structure on the objects is addition, with unit 0. The homsets are given by

$$\widehat{P}(m,n) = \coprod_{m_1+\cdots+m_n=m} P(m_1) \times \cdots \times P(m_n)$$

for $m, n \in \mathbb{N}$. Thus if $\theta_1 \in P(m_1), \ldots, \theta_n \in P(m_n)$ and $m_1 + \cdots + m_n = m$, there is an element $(\theta_1, \ldots, \theta_n)$ of $\widehat{P}(m,n)$. The tensor product of morphisms is defined by

$$(\theta_1, \ldots, \theta_n) \otimes (\theta'_1, \ldots, \theta'_{n'}) = (\theta_1, \ldots, \theta_n, \theta'_1, \ldots \theta'_{n'}).$$

It remains to describe the identity and composition in $\widehat{P}$ (and to check all the axioms). The identity element of $\widehat{P}(m,m)$ is $(1, \ldots, 1)$, consisting of m copies of the unit 1 of P. For composition, take $\phi_1 \in P(k_1), \ldots, \phi_m \in P(k_m)$ with $k_1 + \cdots + k_m = k$, so that $(\phi_1, \ldots, \phi_m) \in \widehat{P}(k,m)$, and take $(\theta_1, \ldots, \theta_n) \in \widehat{P}(m,n)$ as above. Since $m_1 + \cdots + m_n = m$, we may rewrite the sequence $(k_1, \ldots, k_m)$ as $(k_1^1, \ldots, k_1^{m_1}, \ldots, k_n^1, \ldots, k_n^{m_n})$, and similarly $(\phi_1, \ldots, \phi_m)$ as $(\phi_1^1, \ldots, \phi_1^{m_1}, \ldots, \phi_n^1, \ldots, \phi_n^{m_n})$. Thus $\phi_i^j \in P(k_i^j)$. We then have

$$\begin{aligned} \theta_1 \circ (\phi_1^1, \ldots, \phi_1^{m_1}) &\in P(k_1^1 + \cdots + k_1^{m_1}), \ldots, \\ \theta_n \circ (\phi_n^1, \ldots, \phi_n^{m_n}) &\in P(k_n^1 + \cdots + k_n^{m_n}), \end{aligned}$$

and the composite $(\theta_1, \ldots, \theta_m) \circ (\phi_1, \ldots, \phi_n)$ is defined as

$$(\theta_1 \circ (\phi_1^1, \ldots, \phi_1^{m_1}), \ldots, \theta_n \circ (\phi_n^1, \ldots, \phi_n^{m_n})).$$

This is indeed an element of $\widehat{P}(k,n)$, since

$$(k_1^1+\cdots+k_1^{m_1})+\cdots+(k_n^1+\cdots+k_n^{m_n}) = k_1+\cdots+k_m = k.$$

Similarly, let P be a symmetric operad: then there is an associated symmetric strict monoidal category, described shortly, which we also write as $\widehat{P}$. This is possibly an abuse of language, because $\widehat{P}$ is different depending on whether P is considered with or without its symmetric structure, but I hope that context will always make things clear.

The objects of $\widehat{P}$ are again the natural numbers, with monoidal structure given by $+$ and 0. The homsets are given by

$$\widehat{P}(m,n) = \coprod_{f\in\Phi(m,n)} P(f^{-1}\{0\})\times\cdots\times P(f^{-1}\{n-1\}).$$

Here Φ is (a skeleton of) the category of finite sets, defined in 1.1(i), and we write $P(S)$ to mean $P(s)$ when s is the cardinality of a finite set S. Thus if we replace Φ by Δ in this formula, we obtain the definition of $\widehat{P}(m,n)$ in the non-symmetric version. If $f\in\Phi(m,n)$ and if

$$\theta_1\in P(f^{-1}\{0\}),\ldots,\theta_n\in P(f^{-1}\{n-1\})$$

then the corresponding element of $\widehat{P}(m,n)$ is written $(f;\theta_1,\ldots,\theta_n)$; the tensor of morphisms is defined by

$$(f;\theta_1,\ldots,\theta_n)\otimes(f';\theta'_1,\ldots,\theta'_{n'}) = (f+f';\theta_1,\ldots,\theta_n,\theta'_1,\ldots,\theta'_{n'}).$$

The symmetry on $\widehat{P}$ is given by the element $(t_{m,n};1,\ldots,1)$ of $\widehat{P}(m+n,n+m)$, where $t_{m,n}\in\Phi(m+n,n+m)$ adds n to the first m elements of $m+n$ and subtracts m from the last n, and where there are $m+n$ copies of $1=1_P$ after the semicolon. The identity morphism $1_m\in\widehat{P}(m,m)$ is given by the identities in Φ and in P; composition in $\widehat{P}$ is described more or less as in the non-symmetric version, but with some slightly intricate manipulation of permutations. (A closely related but slightly different construction is detailed in [MT, 4.1], and this gives an impression of the method involved. In the formula there for $\widehat{\mathcal{C}}(m,n)$, the first $\prod$ should be a $\coprod$.)

Examples

a. Let **Mon** be the non-symmetric operad whose algebras are monoids, as in 1.2(b). Then $\widehat{\mathbf{Mon}}$ is Δ, the skeleton of the category of finite totally ordered sets (defined in 1.1(j)), with $+$ and 0 as its monoidal structure.

b. Similarly, take the non-symmetric operad **Sem** of 1.2(c): then $\widehat{\mathbf{Sem}}$ is Δ_{surj}, the subcategory of Δ consisting of all its objects n ($n\geq 0$) but only the *surjective* order-preserving maps.

c. As some kind of dual to (b), $\widehat{\mathbf{Pt}} = \Delta_{\mathrm{inj}}$, where $\mathbf{Pt}$ is the non-symmetric operad of 1.2(f) and Δ_{inj} is the subcategory of Δ made up from injective maps.

d. Let **CMon** be the symmetric operad for commutative monoids (1.2(d)): then $\widehat{\mathbf{CMon}}$ is Φ, the skeleton of the category of finite sets (defined in 1.1(i)).

e. As in (b) and (c), $\widehat{\mathbf{CSem}} = \Phi_{\mathrm{surj}}$ and $\widehat{\mathbf{SPt}} = \Phi_{\mathrm{inj}}$, where Φ_{surj} and Φ_{inj} are defined in the obvious way.

f. If G is a monoid and $\mathbf{Act}_G$ the non-symmetric operad whose algebras are left G-objects (1.2(i)) then $\widehat{\mathbf{Act}_G}$ is the monoidal category with objects $0, 1, \ldots$ and

$$\widehat{\mathbf{Act}_G}(m,n) = \begin{cases} G^n & \text{if } m = n \\ \emptyset & \text{otherwise.} \end{cases}$$

Composition and identities are as in G, and tensor of morphisms is juxtaposition.

If we take the symmetric operad $\mathbf{SAct}_G$ instead, then

$$\widehat{\mathbf{SAct}_G}(m,n) = \begin{cases} G^n \times S_n & \text{if } m = n \\ \emptyset & \text{otherwise.} \end{cases}$$

In the Second Description we characterized $\widehat{P}$ as 'the free (symmetric) monoidal category containing a P-algebra'. This was meant in a syntactic sense: $\widehat{P}$ is generated by certain objects and morphisms subject to certain equations. But this characterization can also be interpreted as a universal property: if $\mathcal{M}$ is any (symmetric) strict monoidal category then P-algebras in $\mathcal{M}$ correspond one-to-one with (symmetric) strict monoidal functors $\widehat{P} \longrightarrow \mathcal{M}$. In fact, the correspondence extends to the non-strict situation, as stated in the following important result.

Theorem 1.6.1 *a. Let P be a non-symmetric operad and $\mathcal{M}$ a monoidal category. Then there is an equivalence of categories*

$$\mathbf{Alg}(P, \mathcal{M}) \simeq \mathbf{Mon}(\widehat{P}, \mathcal{M}).$$

b. Let P be a symmetric operad and $\mathcal{M}$ a symmetric monoidal category. Then there is an equivalence of categories

$$\mathbf{Alg}(P, \mathcal{M}) \simeq \mathbf{SMon}(\widehat{P}, \mathcal{M}).$$

Sketch Proof For (a), take a P-algebra in $\mathcal{M}$, consisting of an object A of $\mathcal{M}$ and a map $\overline{\theta} : A^{\otimes n} \longrightarrow A$ for each $\theta \in P(n)$. Then there arises a functor $X : \widehat{P} \longrightarrow \mathcal{M}$ given by setting $X(n) = A^{\otimes n}$ on objects, and by setting

$$X(\theta_1, \ldots, \theta_n) = \overline{\theta_1} \otimes \cdots \otimes \overline{\theta_n} : A^{\otimes m_1} \otimes \cdots \otimes A^{\otimes m_n} \longrightarrow A^{\otimes n}$$

for any $\theta_1 \in P(m_1), \ldots, \theta_n \in P(m_n)$ making up a map $m_1 + \cdots + m_n \longrightarrow n$ in $\widehat{P}$. This functor X has a natural monoidal structure.

Conversely, take a monoidal functor $(X, \xi) : \widehat{P} \longrightarrow \mathcal{M}$. Put $A = X(1)$. The components of ξ fit together (in exactly one way) to produce an isomorphism $\xi^{(n)} : X(n) \xrightarrow{\sim} X(1)^{\otimes n}$, for any given n. Thus if $\theta \in P(n)$ then we may define $\overline{\theta}$ to be the composite

$$A^{\otimes n} \xrightarrow{(\xi^{(n)})^{-1}} X(n) \xrightarrow{X(\theta)} X(1) = A,$$

where we are regarding θ as an element of $\widehat{P}(n, 1)$. This defines an algebra structure on A.

We have to prove that two categories are equivalent, and have shown how to pass from an object of either category to an object of the other. These processes extend to morphisms in a straightforward way, and the two functors so defined are mutually inverse up to natural isomorphism.

Part (b) is just a more elaborate version of (a). The trickiest moment is in obtaining the functor $X : \widehat{P} \longrightarrow \mathcal{M}$ arising from a P-algebra $\mathcal{M}$: one must use some permutations to define X on morphisms, and then check that X really is a functor. $\square$

Examples

g. In the case of the non-symmetric operad **Mon**, the Theorem says that the category of monoids in a monoidal category $\mathcal{M}$ is equivalent to the category of monoidal functors $\Delta \longrightarrow \mathcal{M}$. This is very well-known: see [Mac, VII.5.1].

h. Similarly, in the symmetric case, taking $P = \mathbf{CMon}$ tells us that a commutative monoid in a symmetric monoidal category $\mathcal{M}$ is essentially the same thing as a symmetric monoidal functor $\Phi \longrightarrow \mathcal{M}$.

We finish by observing that the theory above can be generalized in two directions. Firstly, 'operad' can be replaced by 'multicategory' (= 'coloured operad') in the Theorem, provided that $\widehat{P}$ is defined correctly (for which see the First Description above). We shall not need this generalization.

Secondly, we can extend to the situation where all the operads and monoidal categories concerned are enriched in a suitable symmetric monoidal category $\mathcal{V}$.

Thus if $\mathcal{L}$ and $\mathcal{M}$ are $\mathcal{V}$-enriched monoidal categories, there is an (ordinary) category $\mathbf{Mon}(\mathcal{L}, \mathcal{M})$ of $\mathcal{V}$-enriched monoidal functors $\mathcal{L} \longrightarrow \mathcal{M}$ and $\mathcal{V}$-enriched monoidal transformations, and similarly $\mathbf{SMon}(\mathcal{L}, \mathcal{M})$ in the symmetric case: see 1.4. Now assume that $\mathcal{V}$ has finite coproducts and that $\otimes$ distributes over them, as is the case for the $\mathcal{V}$ in each of Examples 1.4(c)–(g). Then for any $\mathcal{V}$-enriched operad P, a $\mathcal{V}$-enriched (symmetric) monoidal category $\widehat{P}$ can be defined just as in the non-enriched version above (in the Third Description), replacing $\times$ by $\otimes$ and 1 by I, and reading $\coprod$ as coproduct in $\mathcal{V}$. Theorem 1.6.1 follows, with P and $\mathcal{M}$ both $\mathcal{V}$-enriched. Thus a P-algebra in $\mathcal{M}$ is essentially the same thing as a $\mathcal{V}$-enriched (symmetric) monoidal functor $\widehat{P} \longrightarrow \mathcal{M}$.

Chapter 2

The Definition of Homotopy Algebra

The path to defining homotopy algebras is now clear. The key is Theorem 1.6.1, which gave an alternative description of an algebra for an operad: namely, if P is an operad and $\mathcal{M}$ a monoidal category, then a P-algebra in $\mathcal{M}$ is a functor $X : \widehat{P} \longrightarrow \mathcal{M}$ together with a coherent family of isomorphisms

$$X(m+n) \longrightarrow X(m) \otimes X(n), \qquad X(0) \longrightarrow I.$$

To define 'homotopy P-algebra in $\mathcal{M}$', we simply take this description and change the word 'isomorphisms' to 'homotopy equivalences': and that is our definition.

In order for this to make sense, we must of course have some notion of what a 'homotopy equivalence' in $\mathcal{M}$ is. Since in a naked monoidal category there is no *a priori* notion of homotopy, this is tagged on as extra structure. Thus we consider a monoidal category $\mathcal{M}$ equipped with a class of its morphisms, called the 'homotopy equivalences'; and $\mathcal{M}$ equipped with this extra structure is a suitable environment in which to define homotopy algebras for an operad. (Naturally enough, we insist that the class of homotopy equivalences obeys a few axioms such as closure under composition, so that the definition behaves reasonably.)

This method of capturing the notion of homotopy is very crude, and consequently the definition of homotopy algebra is a crude one. Taking $\mathcal{M}$ to be the monoidal category of topological spaces, for instance, we have simply recorded which continuous maps are homotopy equivalences. We have no notion of what it means for one map to be homotopy-inverse to another, or for two maps to be homotopic, or for two homotopies to be homotopic, and so on. Thus we are missing out a vast amount of the homotopy theory of spaces—all we have is the '1-dimensional trace' of a whole ∞-category of information—and we should therefore expect our definition of homotopy algebra to have certain shortcomings.

On the other hand, the definition has some advantages. Simplicity is one. Another is historical precedent: a homotopy topological commutative monoid will turn out to be exactly the same as a (special) Γ-space in the sense of Segal's paper [Seg2]. (I call this an 'advantage' because Γ-spaces have already been well explored, thus providing a firm attachment between the present definition and established topology.) Moreover, despite the fact that our definition is only a dim reflection of the fully glorious (and as yet unformulated) ∞-categorical definition, it can at least be seen as the first rung on a ladder leading up to this ideal.

A further discussion of how the definition fits into the big picture, including some more on ∞-categories, can be found in the conclusion, Chapter 6.

This chapter is laid out as follows. In 2.1 we make precise the notion of a monoidal category with a class of equivalences (the 'environment' in which homotopy algebras are taken), and run through some examples. Section 2.2 consists of the definition of homotopy algebra. In 2.3 we take a first look at some examples: both some rather trivial ones, and sketches of some more substantial ones which are the subject of later chapters. All of this so far is for the non-enriched case (i.e. for operads P in which each $P(n)$ is a mere set, with no extra structure); but in 2.4 we extend the definition to the enriched setting.

2.1 The Environment

Definition 2.1.1 *A* monoidal category with equivalences *is a monoidal category $\mathcal{M}$ equipped with a subclass $\mathcal{E}$ of the morphisms in $\mathcal{M}$, whose elements are called* equivalences *or* homotopy equivalences*, such that the following properties hold:*

E1 *any isomorphism is an equivalence*

E2 *if $h = g \circ f$ is a composite of morphisms in $\mathcal{M}$, and if any two of f, g, h are equivalences, then so is the third*

E3 *if $A \xrightarrow{f} B$ and $A' \xrightarrow{f'} B'$ are equivalences then so is $A \otimes A' \xrightarrow{f \otimes f'} B \otimes B'$.*

If $\mathcal{M}$ is a symmetric monoidal category, then $\mathcal{M}$ together with $\mathcal{E}$ forms a symmetric monoidal category with equivalences*. In both cases, we call $\mathcal{E}$ a* class of equivalences in $\mathcal{M}$*.*

Examples

a. Let $\mathcal{M}$ be any monoidal category and let $\mathcal{E}$ be the class of isomorphisms in $\mathcal{M}$. This is (by **E1**) the smallest possible class of equivalences in $\mathcal{M}$.

b. Dually, if $\mathcal{M}$ is any monoidal category then taking $\mathcal{E}$ to be all morphisms in $\mathcal{M}$ gives the largest possible class of equivalences in $\mathcal{M}$.

c. $\mathcal{M}$ is $(\mathbf{Cat}, \times, \mathbf{1})$, and equivalences are equivalences of categories: that is, those functors G for which there exists some functor F with $F \circ G \cong 1$ and $G \circ F \cong 1$. (F is called a *pseudo-inverse* to G).

d. $\mathcal{M}$ is $(\mathbf{Top}, \times, 1)$, and equivalences are homotopy equivalences.

e. $\mathcal{M}$ is $(\mathbf{Top}_*, \times, 1)$, and equivalences are homotopy equivalences relative to basepoints.

f. Example (e) can be repeated with the wedge product $\vee$ in place of $\times$.

g. $\mathcal{M}$ is $(\mathbf{ChCx}, \otimes, R)$ (see 1.1(f)); equivalences are chain homotopy equivalences.

h. The reader might be wondering whether, in (g), we could have taken quasi-isomorphisms in place of chain homotopy equivalences. (A chain map is called a *quasi-isomorphism* if it induces an isomorphism on each homology group.) Axioms **E1** and **E2** are easily verified, but **E3** is more demanding. Consider the commutative square

$$\begin{tikzcd}
H_\bullet(A) \otimes H_\bullet(A') \arrow[r, "f_* \otimes f'_*"] \arrow[d] & H_\bullet(B) \otimes H_\bullet(B') \arrow[d] \\
H_\bullet(A \otimes A') \arrow[r, "(f \otimes f')_*"'] & H_\bullet(B \otimes B')
\end{tikzcd}$$

of graded modules, in which the vertical maps are the natural ones. We know that the map along the top is an isomorphism, and would like to conclude that the map along the bottom is an isomorphism. This will be true if the vertical maps are also isomorphisms, which in turn is true if the ground ring R is a field (by the Künneth Theorem, [Wei, 3.6.3]). So if R is a field then the quasi-isomorphisms form a class of equivalences in $\mathbf{ChCx}_R$.

However, this gives us little or nothing more than Example (g): for when one is working over a field, quasi-isomorphisms are (almost?) the same thing as chain homotopy equivalences. More precisely, I am informed that they are the same thing if either the field is of characteristic 0, or if the complexes concerned are 0 in negative degrees; I do not know if the statement is true in complete generality. Whatever the truth, quasi-isomorphisms will not be mentioned again.

i. Similarly, in (d) and (e) we can take weak homotopy equivalences rather than homotopy equivalences as long as we work over a field, but this does not seem to provide a significant generalization.

j. If $\mathcal{E}$ is a class of equivalences in a monoidal category $\mathcal{M}$, then $\mathcal{E}$ is also a class of equivalences in the opposite category $\mathcal{M}^{\mathrm{op}}$.

k. Let $\mathcal{N}$ be a monoidal 2-category—that is, a **Cat**-enriched monoidal category. Then $\mathcal{N}$ consists of *0-cells* (or *objects*), *1-cells*, and *2-cells*, together with various ways of composing them, a tensor product, and a unit object. There is a notion of a 1-cell $G : A \longrightarrow B$ in $\mathcal{N}$ being an *equivalence*: namely, if there exists a 1-cell $F : B \longrightarrow A$, an invertible 2-cell between $G \circ F$ and 1_B, and an invertible 2-cell between $F \circ G$ and 1_A. The underlying (1-)category $\mathcal{M}$ of $\mathcal{N}$, formed by the 0-cells and 1-cells, is then a monoidal category with equivalences. A typical example is **Cat** itself: see (c). In fact, Examples (d)–(g) all arise in this way too, as we shall see in Sections 3.4 and 3.5.

2.2 The Definition

In order to make the definition of a homotopy algebra for an operad, we will need a notion of map between monoidal categories which is more general than the notion of monoidal functor (1.1.1). The notion is that of a *colax monoidal functor*, and the definition is obtained from Definition 1.1.1 simply by replacing the word 'isomorphisms' with 'maps'. We get the definition of *colax symmetric monoidal functor* from 1.1.1 in the same way.

Thus in a (symmetric) colax monoidal functor $(X, \xi) : \mathcal{L} \longrightarrow \mathcal{M}$, we have coherence maps

$$\xi_0 : X(I) \longrightarrow I, \qquad \xi_{A,B} : X(A \otimes B) \longrightarrow X(A) \otimes X(B)$$

$(A, B \in \mathcal{L})$. We shall also refer in passing to *lax (symmetric) monoidal functors*, in which these maps ξ go in the opposite direction. One explanation of the terminology is that a lax monoidal functor from $\mathcal{L}$ to $\mathcal{M}$ induces a functor from the category of monoids in $\mathcal{L}$ to the category of monoids in $\mathcal{M}$, whereas a *co*lax monoidal functor induces a functor between the categories of *co*monoids.

Note that a monoidal functor is a special kind of colax monoidal functor, not vice-versa. Thus the role of the adjective ('colax') is contrary to normal English usage. Note also that the definition of monoidal transformation (1.1.2) makes sense for colax monoidal functors in general.

We now present the main definition of this paper.

Definition 2.2.1 *a. Let P be a non-symmetric operad and let $\mathcal{M}$ be a monoidal category with equivalences. A* homotopy P-algebra in $\mathcal{M}$ *is a colax monoidal functor $(X, \xi) : \widehat{P} \longrightarrow \mathcal{M}$ in which ξ_0 and each $\xi_{m,n}$ $(m, n \in \mathbb{N})$ are equivalences. (Here $\widehat{P}$ is the monoidal category of Section 1.6.)*

b. Let P be a symmetric operad and let $\mathcal{M}$ be a symmetric monoidal category with equivalences. A homotopy P-algebra in $\mathcal{M}$ *is a colax symmetric monoidal functor $(X, \xi) : \widehat{P} \longrightarrow \mathcal{M}$ in which ξ_0 and each $\xi_{m,n}$ $(m, n \in \mathbb{N})$ are equivalences. (Here $\widehat{P}$ is the* symmetric *monoidal category of 1.6.)*

In both symmetric and non-symmetric cases, a *map of homotopy P-algebras* is a monoidal transformation, and the homotopy P-algebras in $\mathcal{M}$ thus form a category $\mathbf{HtyAlg}(P, \mathcal{M})$.

2.3 Brief Examples

Most of the rest of this paper consists of examples of homotopy algebras. Each non-trivial example takes a while to explain, so for now we just present the trivial cases and briefly sketch out the more substantial examples.

a. Suppose that the only equivalences in $\mathcal{M}$ are the isomorphisms, as in 2.1(a). Then a homotopy P-algebra in $\mathcal{M}$ is essentially just a P-algebra in $\mathcal{M}$, by Theorem 1.6.1. In symbols,

$$\mathbf{Alg}(P, \mathcal{M}) \simeq \mathbf{Mon}(\widehat{P}, \mathcal{M}) = \mathbf{HtyAlg}(P, \mathcal{M}).$$

So in an $\mathcal{M}$ with 'no interesting homotopy', homotopy algebras are just algebras. This holds in both the symmetric and the non-symmetric case.

b. Take any P and $\mathcal{M}$ as in Definition 2.2.1 ((a) or (b)). Then by axiom **E1** for a class of equivalences, any P-algebra is a homotopy P-algebra. More precisely, there is an inclusion as shown:

$$\mathbf{Alg}(P, \mathcal{M}) \simeq \mathbf{Mon}(\widehat{P}, \mathcal{M}) \hookrightarrow \mathbf{HtyAlg}(P, \mathcal{M}).$$

c. Let $P = \mathbf{Obj}$ (see 1.2(a)) and let $\mathcal{M}$ be any monoidal category with equivalences. An $\mathbf{Obj}$-algebra in $\mathcal{M}$ is just an object of $\mathcal{M}$; what is a homotopy $\mathbf{Obj}$-algebra? Roughly speaking, when $\mathcal{M} = \mathbf{Top}$, for instance, a homotopy $\mathbf{Obj}$-algebra consists of a space A together with a homotopy model for each power A^n of A.

In detail, $\widehat{\mathbf{Obj}}$ is the discrete category $\mathbb{N}$ (all morphisms are identities), so a colax monoidal functor $\widehat{\mathbf{Obj}} \longrightarrow \mathcal{M}$ consists of a sequence $X(0), X(1), \ldots$ of objects of $\mathcal{M}$, together with maps

$$\xi_{m,n} : X(m+n) \longrightarrow X(m) \otimes X(n), \qquad \xi_0 : X(0) \longrightarrow I$$

satisfying coherence axioms. These axioms guarantee that for each sequence $k_1, \ldots k_n$ (with $n \geq 0$, $k_i \geq 0$), there is a unique map

$$\xi_{k_1,\ldots,k_n} : X(k_1 + \cdots k_n) \longrightarrow X(k_1) \otimes \cdots \otimes X(k_n)$$

built up from the $\xi_{m,n}$'s and ξ_0. (The notations ξ_0 and $\xi_{k_1,\ldots,k_n}$ conflict, but this should not cause serious problems.) In particular, taking all the k_i's to be 1 yields a canonical map

$$\xi^{(n)} : X(n) \longrightarrow X(1)^{\otimes n}.$$

A homotopy $\mathbf{Obj}$-algebra is a colax monoidal functor (X, ξ) as above, with the property that ξ_0 and each $\xi_{m,n}$ are equivalences. Using the axioms on equivalences (2.1.1), this property can be restated in two different ways: that each $\xi_{k_1,\ldots,k_n}$ is an equivalence, or that each $\xi^{(n)} : X(n) \longrightarrow X(1)^{\otimes n}$ is an equivalence.

Throughout this work we think of $X(1)$ as the 'base object' of a homotopy algebra (X, ξ), and in fact we have already encountered this idea in the context of genuine algebras (see the proof of 1.6.1). If P is any operad, $\mathcal{M}$ a (symmetric) monoidal category, and A an object of $\mathcal{M}$, we will write '*A is a homotopy P-algebra*' to mean that there is a homotopy P-algebra (X, ξ) with $X(1) \cong A$.

d. A similar analysis can be made of homotopy algebras for the symmetric operad **SObj** (defined in 1.2(a)). In this case the maps $\xi_{m,n}$ are compatible with the symmetries in $\mathcal{M}$, in the sense of Definition 1.1.1. Hence the maps $\xi^{(n)} : X(n) \longrightarrow X(1)^{\otimes n}$ are also compatible with the symmetries, in the obvious sense.

e. Let $\mathcal{M} = \mathbf{Cat}$ and let $P = \mathbf{Mon}$ (1.1(b) and 1.2(b)). A P-algebra in $\mathcal{M}$ is a monoid in **Cat**, that is, a strict monoidal category. Homotopy algebras are meant to be some weakened version of genuine algebras, so a homotopy monoid (= homotopy **Mon**-algebra) in **Cat** ought to be something comparable to a (non-strict) monoidal category. Similarly, a homotopy **CMon**-algebra in **Cat** should be something along the lines of a (non-strict) symmetric monoidal category. We look at these weakened notions of monoidal category in Sections 3.3 and 4.4. In particular, we will see that a homotopy commutative monoid in **Cat** is exactly what Segal called a Γ-category in [Seg2]. (We will call these things *special* Γ-categories instead, following the more popular terminology.)

f. A prime example of something which ought to be a homotopy monoid is a loop space. It is, as is proved in 3.2. More precisely, we prove that for any based space B there is a homotopy **Mon**-algebra (X, ξ) in $(\mathbf{Top}, \times, 1)$ with $X(1)$ isomorphic to the space of based loops in B; cf. the remarks at the end of Example (c) above.

g. Taking $\mathcal{M}$ to be $(\mathbf{Top}, \times, 1)$ again, we might well expect a homotopy monoid in $\mathcal{M}$ to be something like an A_∞-space (as defined in [Sta1]). Both concepts are, after all, meant to provide an up-to-higher-homotopy version of topological monoid. Section 3.4 provides a partial comparison between the two. More accurately, the comparison is between A_∞-spaces and homotopy *semigroups* in $\mathbf{Top}_*$ (the category of based spaces (2.1(e))): there is a slightly delicate issue concerning spaces with or without basepoint and semigroups with or without unit, which is explained there.

   We will also see in Section 3.1 that a homotopy commutative monoid in **Top** is precisely a Γ-space (as defined in [Seg2]), or 'special Γ-space' in the alternative terminology. It is from this re-definition of (special) Γ-space that the general definition of homotopy algebra descended.

h. Let $\mathcal{M}$ be the category **ChCx** of chain complexes, with usual tensor and homotopy equivalences, as in 2.1(g). Then a monoid in $\mathcal{M}$ is a differential graded algebra, so a homotopy monoid in $\mathcal{M}$ should be something comparable to an A_∞-algebra (as defined in [Sta2]). A comparison of sorts is made in Section 3.5.

i. The following example suggests that the definition of homotopy algebra does not encompass as much as we might like. Fix a monoid G, let $\mathbf{Act}_G$ be the non-symmetric operad of Example 1.2(i), and let $\mathcal{M}$ be any monoidal category. An $\mathbf{Act}_G$-algebra in $\mathcal{M}$ is an object of $\mathcal{M}$ equipped with a left action by G, and we might therefore expect a homotopy $\mathbf{Act}_G$-algebra to be an object with an 'action up to homotopy', so that laws like $g \cdot (g' \cdot x) = (gg') \cdot x$ only hold in some weak sense. This is not the case. For by the description of $\widehat{\mathbf{Act}_G}$ in 1.6(f), a homotopy $\mathbf{Act}_G$-algebra consists of a sequence $X(0), X(1), \ldots$ of objects of $\mathcal{M}$, with a *strict* action of G^n on $X(n)$ for each n, and homotopy equivalences $\xi_{m,n}, \xi_0$ (as in (c)) which preserve the G^n-actions. In particular, the 'base' object $X(1)$ has a strict action by G, which might be a disappointment in cases such as **Cat** and **Top**. This matter is discussed further in Chapter 6, together with the related matter of homotopy invariance.

By way of advertisement, one can perform various 'changes of environment': for instance, the classifying-space functor $B : \mathbf{Cat} \longrightarrow \mathbf{Top}$ induces a map from homotopy monoids in **Cat** to homotopy monoids in **Top**. This is the subject of Chapter 5.

2.4 The Enriched Version

Homotopy algebras can be defined in the enriched context too (1.4, 1.5).

First of all, we need an enriched version of 'monoidal categories with equivalences'. Fix a symmetric monoidal category $\mathcal{V}$. Then a *class of equivalences* in a $\mathcal{V}$-enriched monoidal category $\mathcal{M}$ is simply a class of equivalences in the underlying monoidal category $|\mathcal{M}|$ of $\mathcal{M}$.

Examples

a. Let $\mathcal{V} = \mathbf{Ab}$. Then a $\mathcal{V}$-enriched monoidal category with equivalences is an **Ab**-enriched monoidal category $\mathcal{M}$ together with a sub*set* (not necessarily a sub*group*) of $\mathcal{M}[A, B]$, for each $A, B \in \mathcal{M}$, whose elements are called the 'equivalences' from A to B, satisfying the axioms **E1**–**E3** of Definition 2.1.1.

b. Let $\mathcal{V} = \mathbf{GrMod}$ and $\mathcal{M} = \mathbf{ChCx}$, as in 1.4(h). The underlying (ordinary) monoidal category $|\mathcal{M}|$ of $\mathcal{M}$ is the category also denoted by **ChCx**, and the (degree 0) chain homotopy equivalences provide a class of equivalences in $|\mathcal{M}|$ (by 2.1(g)) and hence in $\mathcal{M}$.

I would now like to define homotopy algebras in the enriched setting. It may be that the reader has no head or stomach for the niceties of enriched category theory, in which case he should jump straight to the examples. Otherwise, take a $\mathcal{V}$-enriched operad P (symmetric or not) and a $\mathcal{V}$-enriched (symmetric) monoidal category with equivalences, $\mathcal{M}$. A homotopy P-algebra in $\mathcal{M}$ will be defined as a $\mathcal{V}$-enriched colax (symmetric) monoidal functor

$$(X, \xi) : \widehat{P} \longrightarrow \mathcal{M},$$

where $\widehat{P}$ is as at the end of 1.6, satisfying certain conditions. These conditions should say 'the components of ξ are equivalences'. Now, the components of ξ are maps

$$\begin{array}{rccc} \xi_{m,n}: & I & \longrightarrow & \mathcal{M}[X(M+n), X(m) \otimes X(n)], \\ \xi_0: & I & \longrightarrow & \mathcal{M}[X(0), I] \end{array}$$

in $\mathcal{V}$, which means exactly that $\xi_{m,n}$ is a map $X(m+n) \longrightarrow X(m) \otimes X(n)$ in $|\mathcal{M}|$, and similarly ξ_0 is a map $X(0) \longrightarrow I$ in $|\mathcal{M}|$. So the following makes sense:

Definition 2.4.1 *Let $\mathcal{V}$, P and $\mathcal{M}$ be as above. A* homotopy P-algebra in $\mathcal{M}$ *is a $\mathcal{V}$-enriched colax (symmetric) monoidal functor*

$$(X, \xi) : \widehat{P} \longrightarrow \mathcal{M}$$

in which the components $\xi_{m,n}$, ξ_0 of ξ are equivalences ($m, n \in \mathbb{N}$).

A *map of homotopy P-algebras* is a $\mathcal{V}$-enriched monoidal transformation, and we thus obtain a category $\mathbf{HtyAlg}(P, \mathcal{M})$.

Examples

c. When $\mathcal{V} = \mathbf{Set}$, this reduces to the ordinary, non-enriched definition of homotopy algebras.

d. Suppose that the only equivalences in $\mathcal{M}$ are the isomorphisms. Then by an enriched version of Theorem 1.6.1, a homotopy P-algebra in $\mathcal{M}$ is essentially just a P-algebra in $\mathcal{M}$. Compare Example 2.3(a).

e. Take any $\mathcal{V}$, P and $\mathcal{M}$ as in Definition 2.4.1. Then, just as in Example 2.3(b), any genuine P-algebra is a homotopy P-algebra.

f. Referring back to Examples 1.5(b) and 2.3(i), if $\mathcal{V} = \mathcal{M} = \mathbf{Top}$ and G is a topological monoid then a homotopy $\mathbf{Act}_G$-algebra in $\mathcal{M}$ gives rise to a *strict* continuous action of G on a space. The same goes for the other $\mathcal{V}$'s, $\mathcal{M}$'s and G's in Examples 1.5(b), (c).

g. In 1.5(e) we defined a **GrAb**-enriched operad **GrLie**, and observed that a **GrLie**-algebra in **ChCx** is a differential graded Lie algebra. A homotopy **GrLie**-algebra in **ChCx** might therefore be called a 'homotopy d.g. Lie algebra'. It is natural to want to compare this definition with that of L_∞-algebras (also known as strong homotopy Lie algebras—see [KSV] and [LM]); however, I have not made such a comparison. We come back to homotopy Lie algebras in 4.1.1.

h. Similarly, a homotopy **Ger**-algebra in **ChCx** is some kind of ‘homotopy Gerstenhaber algebra’. **Ger** is defined in 1.5(g), and homotopy Gerstenhaber algebras are returned to in 4.1.2.

Chapter 3

Homotopy Monoids and Semigroups

Just about the simplest algebraic theory is the theory of monoids, and just about the simplest operad is **Mon**. In this chapter we look at homotopy monoids—that is, homotopy **Mon**-algebras—and at homotopy commutative monoids, homotopy semigroups and homotopy commutative semigroups. (Recall that a monoid is by definition a semigroup with unit.) Despite the simplicity of monoids, homotopy monoids and homotopy semigroups provide some of the most interesting and important examples of homotopy algebras.

The first section (3.1) is devoted to showing that homotopy topological commutative monoids are exactly the same as special Γ-spaces, and various similar results. For a topologist of a certain kind, this should help to put the ideas of this paper back onto home turf. But I have tried to write this paper for both topologists and category theorists; and in terms of category theory the result is perhaps not all that interesting. I will now sketch the ideas of Section 3.1, and if the reader judges them not worthy of further attention then she can ignore 3.1 altogether, without causing a problem in understanding later sections.

So, the basic idea of 3.1 runs as follows. Let Φ be the skeletal category of finite sets (1.1(i)). It just so happens that there is a category Γ such that for any category $\mathcal{M}$ with finite products,

$$\text{colax monoidal functors } (\Phi, +, 0) \longrightarrow (\mathcal{M}, \times, 1)$$

correspond one-to-one with functors $\Gamma^{\mathrm{op}} \longrightarrow \mathcal{M}$. This is merely a representability result and is not surprising or particularly interesting from a categorical viewpoint (although the fact that Γ has a simple direct description is not so obvious). A homotopy **CMon**-algebra in $\mathcal{M}$ is a special kind of colax monoidal functor from $(\Phi, +, 0)$ to $(\mathcal{M}, \times, 1)$, supposing now that $\mathcal{M}$ is equipped with a class of equivalences; in other words, it is a special kind of functor $\Gamma^{\mathrm{op}} \longrightarrow \mathcal{M}$. Such special functors are called 'special Γ-spaces', and this formulation of the notion of homotopy topological commutative monoid has been used since around 1970 (see [Seg2], [Ad], [And]).

The second section (3.2) sets out our first major example of a homotopy algebra: any loop space is a homotopy topological monoid. In Chapter 4 we will also exhibit iterated loop spaces as homotopy-algebraic structures, but this is left alone for now. (See page 3 for the definition of loop space.)

The last three sections (3.3, 3.4, 3.5) each provide a comparison between other notions of weakened or up-to-homotopy algebraic structure and the present definition of homotopy algebra. Respectively:

- any homotopy monoid in **Cat** gives rise to a monoidal category
- any homotopy semigroup in $\mathbf{Top}_*$ gives rise to an A_4-space
- any homotopy semigroup in **ChCx** gives rise to an A_4-algebra

In the last two, I conjecture that 'A_4' can be replaced by 'A_∞'. In all three, I would like to understand how or whether we might also pass in the opposite direction (A_∞-algebras giving rise to homotopy semigroups in **ChCx**, for instance), but at present I do not. In a later chapter (4.4) we describe a converse process for the first case, **Cat**, but there are still many unanswered questions there.

The strategy for these last three sections is to do the hard work only once. Having done (or rather, asserted that we could do) many tedious calculations to show that a homotopy monoid in **Cat** gives rise to a monoidal category, we can observe that the proof is repeatable in any monoidal 2-category; and from this the results on A_4-spaces and A_4-algebras follow quite quickly. It is perhaps only the limitations of higher-dimensional category theory today which prevent the A_4's from becoming A_∞'s.

For those unfamiliar with A_n-spaces and A_n-algebras, the definitions were made by Stasheff in his 1963 papers [Sta1] and [Sta2]. A_n-spaces are to be thought of as topological monoids up to higher homotopy, and A_n-algebras as differential graded algebras up to higher homotopy; the higher n is, the higher the levels of homotopy go.

3.1 Γ-Objects

This section is divided into three. The first part concerns homotopy commutative monoids and special Γ-objects, and the last part is the non-symmetric version of this, on homotopy monoids and special simplicial objects. In the middle is an 'aside' outlining a result of category theory which makes the proofs in the first and last parts much easier. Fans of abstract methods might like to read the aside first, but others can safely ignore it.

Those who do not already know what a Γ-space is are warned (as above) that they might not find this section very interesting, and might prefer to jump to 3.2.

The symmetric case: Γ-objects

Let Γ be the category defined in [Seg2]. It is most easily described by saying that Γ^{op} is (a skeleton of) the category of finite based sets: thus its objects are $[n] = \{0, 1, \ldots, n\}$ for $n \geq 0$, and a map $[m] \longrightarrow [n]$ in Γ^{op} is a function $g : [m] \longrightarrow [n]$ such that $g(0) = 0$.

Given categories $\mathcal{L}$ and $\mathcal{M}$, we write $[\mathcal{L}, \mathcal{M}]$ for the usual functor category. If $(\mathcal{L}, \otimes, I)$ and $(\mathcal{M}, \otimes, I)$ are symmetric monoidal categories then

$$\mathbf{SColax}((\mathcal{L}, \otimes, I), (\mathcal{M}, \otimes, I))$$

denotes the category of colax symmetric monoidal functors from $(\mathcal{L}, \otimes, I)$ to $(\mathcal{M}, \otimes, I)$ and monoidal transformations.

In this section, the objects of Φ will be written as $\mathbf{0}, \mathbf{1}, \ldots$ rather than $0, 1, \ldots$, for clarity. Thus $\mathbf{n}$ and $[n-1]$ are both n-element sets.

Proposition 3.1.1 *Let $\mathcal{M}$ be a category with finite products. Then there is an isomorphism of categories*

$$\mathbf{SColax}((\Phi, +, \mathbf{0}), (\mathcal{M}, \times, 1)) \cong [\Gamma^{\mathrm{op}}, \mathcal{M}].$$

Proof This is a direct corollary of the general category-theoretic Proposition 3.1.5, under 'Aside' below. Alternatively, a direct argument can be used, as follows.

Given a colax symmetric monoidal functor

$$(X, \xi) : (\Phi, +, \mathbf{0}) \longrightarrow (\mathcal{M}, \times, 1),$$

we must define a functor $Y : \Gamma^{\mathrm{op}} \longrightarrow \mathcal{M}$. On objects, take $Y[n] = X(\mathbf{n})$. To define Y on morphisms, first note the following:

- for each $m \geq 0$, there is a map $\eta_m : \mathbf{m} \longrightarrow \mathbf{1} + \mathbf{m}$ in Φ given by $\eta_m(i) = 1 + i$
- for each map $g : [m] \longrightarrow [n]$ in Γ^{op}, there is a corresponding map $g : \mathbf{1} + \mathbf{m} \longrightarrow \mathbf{1} + \mathbf{n}$ in Φ

Now if $g : [m] \longrightarrow [n]$ is a map in Γ^{op}, let $Y(g)$ be the composite

$$X(\mathbf{m}) \xrightarrow{X(\eta_m)} X(\mathbf{1} + \mathbf{m}) \xrightarrow{X(g)} X(\mathbf{1} + \mathbf{n}) \xrightarrow{\xi^2_{1,n}} X(\mathbf{n})$$

where $\xi^2_{1,n}$ is the second component of $\xi_{1,n}$. This defines a functor $Y : \Gamma^{\mathrm{op}} \longrightarrow \mathcal{M}$.

Conversely, take a functor $Y : \Gamma^{\mathrm{op}} \longrightarrow \mathcal{M}$. Define a functor $X : \Phi \longrightarrow \mathcal{M}$ by $X(\mathbf{n}) = Y[n]$ on objects; if $f : \mathbf{m} \longrightarrow \mathbf{n}$ is a morphism in Φ then define

$$X(f) = (Y[m] \xrightarrow{Y[g]} Y[n])$$

where the map $[g] : [m] \longrightarrow [n]$ in Γ^{op} is given by

$$g(i) = \begin{cases} 0 & \text{if } i = 0 \\ 1 + f(i-1) & \text{if } 1 \leq i \leq m. \end{cases}$$

To define ξ, obviously ξ_0 is the unique map $X(\mathbf{0}) \longrightarrow 1$. For the $\xi_{m,n}$'s, first define maps

$$[m] \xleftarrow{\pi^1_{m,n}} [m+n] \xrightarrow{\pi^2_{m,n}} [n]$$

in Γ^{op} (for $m, n \geq 0$) by

$$\begin{aligned} \pi^1_{m,n}(i) &= \begin{cases} i & \text{if } 0 \leq i \leq m \\ 0 & \text{if } m+1 \leq i \leq m+n, \end{cases} \\ \pi^2_{m,n}(i) &= \begin{cases} 0 & \text{if } 0 \leq i \leq m \\ i-m & \text{if } m+1 \leq i \leq m+n. \end{cases} \end{aligned}$$

Then define

$$\xi_{m,n} : X(\mathbf{m}+\mathbf{n}) \longrightarrow X(\mathbf{m}) \times X(\mathbf{n})$$

to be

$$(Y(\pi^1_{m,n}), Y(\pi^2_{m,n})) : Y[m+n] \longrightarrow Y[m] \times Y[n].$$

After performing all the checks we see that the two processes are mutually inverse, and that they can be extended to apply to transformations too. Thus we obtain the required isomorphism of categories. □

We are really only interested in those colax symmetric monoidal functors

$$(X, \xi) : (\Phi, +, \mathbf{0}) \longrightarrow (\mathcal{M}, \times, 1)$$

in which the components of ξ are equivalences—i.e. the homotopy commutative monoids in $(\mathcal{M}, \times, 1)$. The following result says what the corresponding condition is on functors $\Gamma^{\mathrm{op}} \longrightarrow \mathcal{M}$. In its statement, the maps $\pi^1_{m,n}$ and $\pi^2_{m,n}$ are as in the proof of Proposition 3.1.1, and if $0 \leq j < n$ then the map

$$\rho^n_j : [n] \longrightarrow [1]$$

is given by

$$\rho^n_j(i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Recall from 1.1(a) that a 'cartesian monoidal category' is one in which the monoidal structure is given by cartesian product and terminal object.

Proposition 3.1.2 *Let $(\mathcal{M}, \times, 1)$ be a cartesian monoidal category with equivalences. Let*

$$(X, \xi) : (\Phi, +, \mathbf{0}) \longrightarrow (\mathcal{M}, \times, 1)$$

be a colax symmetric monoidal functor, and let

$$Y : \Gamma^{\mathrm{op}} \longrightarrow \mathcal{M}$$

be the functor corresponding to (X, ξ) under Proposition 3.1.1. The following conditions are equivalent:

a. (X, ξ) *is a homotopy commutative monoid*

b. *for each* $m, n \geq 0$, *the map*

$$(Y(\pi^1_{m,n}), Y(\pi^2_{m,n})) : Y[m+n] \longrightarrow Y[m] \times Y[n]$$

is an equivalence, and so is the unique map $Y[0] \longrightarrow 1$

c. *for each* $n \geq 0$, *the map*

$$(Y(\rho^n_0), \ldots, Y(\rho^n_{n-1})) : Y[n] \longrightarrow Y[1]^n$$

is an equivalence.

Proof (a) $\Leftrightarrow$ (b) is immediate from the second half of the proof of Proposition 3.1.1. An easy induction, again using this half of the proof, shows that

$$(Y(\rho^n_0), \ldots, Y(\rho^n_{n-1})) : Y[n] \longrightarrow Y[1]^n$$

equals

$$\xi^{(n)} : X(\mathbf{n}) \longrightarrow X(\mathbf{1})^n :$$

so by the comments in 2.3(c), we have (a) $\Leftrightarrow$ (c). □

A functor $Y : \Gamma^{\mathrm{op}} \longrightarrow \mathcal{M}$ satisfying the equivalent conditions (b) and (c) will be called a *special* Γ*-object* in $\mathcal{M}$, and the category of such, with natural transformations as morphisms, will be written

$$\mathbf{Special}(\Gamma^{\mathrm{op}}, \mathcal{M}).$$

(A Γ-*object* in $\mathcal{M}$ is any old functor from Γ^{op} to $\mathcal{M}$.) We then have:

Corollary 3.1.3 *Let* $(\mathcal{M}, \times, 1)$ *be a cartesian monoidal category with equivalences. Then there is an isomorphism of categories*

$$\mathbf{HtyAlg}(\mathbf{CMon}, \mathcal{M}) \cong \mathbf{Special}(\Gamma^{\mathrm{op}}, \mathcal{M}).$$

□

In the original paper [Seg2], the cases $\mathcal{M} = \mathbf{Top}$ and $\mathcal{M} = \mathbf{Cat}$ are considered. (I should re-iterate that what are called Γ-spaces and Γ-categories in [Seg2] are called special Γ-spaces and special Γ-categories here.) In these cases we have:

Corollary 3.1.4

a. *homotopy topological commutative monoids are the same as special* Γ*-spaces*

b. *homotopy symmetric monoidal categories are the same as special* Γ*-categories.*

□

Aside: a general result

Proposition 3.1.1 is in fact a special case of the following category-theoretic result. Anyone who understands the statement will probably have little trouble in supplying a proof; I hope to write it up separately. For the definition of Kleisli category see [Mac, VI.5].

Proposition 3.1.5 *Let $(\mathcal{L}, \otimes, I)$ be a symmetric monoidal category with a terminal object 1, and let $\mathcal{L}_{1\otimes —}$ be the Kleisli category for the monad $1 \otimes —$ on $\mathcal{L}$. For any category $\mathcal{M}$ with finite products, there is an isomorphism of categories*

$$\mathbf{SColax}((\mathcal{L}, \otimes, I), (\mathcal{M}, \times, 1)) \cong [\mathcal{L}_{1\otimes —}, \mathcal{M}].$$

□

Taking $(\mathcal{L}, \otimes, I) = (\Phi, +, \mathbf{0})$ we obtain Proposition 3.1.1 immediately: for $\mathcal{L}_{1\otimes —}$ is the skeletal category of finite based sets, Γ^{op}.

There is a non-symmetric version of Proposition 3.1.5 too, as follows. Here **Colax** denotes the category of colax monoidal functors.

Proposition 3.1.6 *Let $(\mathcal{L}, \otimes, I)$ be a monoidal category with a terminal object 1, and let $\mathcal{L}_{1\otimes — \otimes 1}$ be the Kleisli category for the monad $1 \otimes — \otimes 1$ on $\mathcal{L}$. For any category $\mathcal{M}$ with finite products, there is an isomorphism of categories*

$$\mathbf{Colax}((\mathcal{L}, \otimes, I), (\mathcal{M}, \times, 1)) \cong [\mathcal{L}_{1\otimes — \otimes 1}, \mathcal{M}].$$

□

To apply this to the case of (non-commutative) monoids, take $(\mathcal{L}, \otimes, I) = (\Delta, +, \mathbf{0})$. Then the Kleisli category $\Delta_{1+—+1}$ is a skeleton of the category of *finite strict intervals*, that is, of finite totally ordered sets with distinct greatest and least elements. But it is well-known that this category is isomorphic to $(\Delta^+)^{\mathrm{op}}$ (defined below), and this gives us Proposition 3.1.7. (The isomorphism

$$(\Delta^+)^{\mathrm{op}} \cong (\text{finite strict intervals})$$

can be written in either direction as $\mathrm{Hom}(—, [1])$, i.e. $[1]$ is a 'schizophrenic object'. A few more details can be found in [Joy].)

We have extracted maximum use from Propositions 3.1.5 and 3.1.6, in the following sense: we can't apply 3.1.5 with $\mathcal{L} = \widehat{P}$ for any P other than **CMon**, since $\mathcal{L}$ is required to have a terminal object, and similarly **Mon** for Proposition 3.1.6. On the other hand, some general categorical arguments guarantee that whether or not $\mathcal{L}$ has a terminal object, there exists a category $\mathcal{L}'$ with the property that $\mathcal{L}_{1\otimes —}$ (or $\mathcal{L}_{1\otimes — \otimes 1}$) has in Proposition 3.1.5 (or 3.1.6). (I hope to explain this properly elsewhere.) So, for any operad P, there is *some* category $(\widehat{P})'$ playing the role that Γ^{op} did for **CMon** and that $(\Delta^+)^{\mathrm{op}}$ did for **Mon**. However, $(\widehat{P})'$ might not always be easy to describe.

The non-symmetric case: simplicial objects

Returning to the main exposition, let Δ^+ be the "topologists' simplicial category", that is, the category whose objects are $[n] = \{0, 1, \ldots, n\}$ for $n \geq 0$, and whose morphisms are order-preserving functions. So Δ^+ is obtained from Δ by removing the object 0 and renaming the objects which remain. For emphasis I will now write the objects of Δ as $\mathbf{0}, \mathbf{1}, \ldots$ rather than $0, 1, \ldots$; hence $\mathbf{n}$ and $[n-1]$ are both n-element ordered sets.

By definition, a simplicial object in a category $\mathcal{M}$ is a functor from $(\Delta^+)^{\mathrm{op}}$ to $\mathcal{M}$. The next result gives an alternative definition of simplicial object.

Proposition 3.1.7 *Let $\mathcal{M}$ be a category with finite products. Then there is an isomorphism of categories*

$$\mathbf{Colax}((\Delta, +, \mathbf{0}), (\mathcal{M}, \times, 1)) \cong [(\Delta^+)^{\mathrm{op}}, \mathcal{M}].$$

Remark **Colax** denotes the category of colax monoidal functors and monoidal transformations.

Sketch Proof A colax monoidal functor

$$(X, \xi) : (\Delta, +, \mathbf{0}) \longrightarrow (\mathcal{M}, \times, 1)$$

consists of a functor $X : \Delta \longrightarrow \mathcal{M}$ together with maps

$$X(\mathbf{m}) \xleftarrow{\xi^1_{m,n}} X(\mathbf{m} + \mathbf{n}) \xrightarrow{\xi^2_{m,n}} X(\mathbf{n})$$

for each $m, n \geq 0$, satisfying certain axioms. These axioms imply that all the $\xi^i_{m,n}$'s can be built up as composites of maps

$$\xi^1_{k,1} : X(\mathbf{k} + \mathbf{1}) \longrightarrow X(\mathbf{k}), \qquad \xi^2_{1,k} : X(\mathbf{1} + \mathbf{k}) \longrightarrow X(\mathbf{k}).$$

Hence a colax monoidal functor from $(\Delta, +, \mathbf{0})$ to $(\mathcal{M}, \times, 1)$ is a functor $X :$ $\Delta \longrightarrow \mathcal{M}$ together with a pair of maps

$$X(\mathbf{k} + \mathbf{1}) \rightrightarrows X(\mathbf{k})$$

for each $k \geq 0$, satisfying certain axioms. This data can be depicted as

$$\begin{tikzcd} X(\mathbf{0}) \arrow[r, shift left=2, leftarrow, dotted] \arrow[r] \arrow[r, shift right=2, leftarrow, dotted] & X(\mathbf{1}) \arrow[r, shift left=4, leftarrow, dotted] \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] \arrow[r, shift right=4, leftarrow, dotted] & X(\mathbf{2}) & \cdots \end{tikzcd}$$

where the solid lines are the image under X of the face and degeneracy maps in Δ, and the dotted lines are $\xi^1_{k,1}$ and $\xi^2_{1,k}$. But this diagram looks just like the usual picture of a simplicial object: so we define a functor $Y : (\Delta^+)^{\mathrm{op}} \longrightarrow \mathcal{M}$ by

$$\begin{array}{rcl} Y[n] & = & X(\mathbf{n}) \\ Y(\sigma_i^n) & = & X(\delta_i^{n-1}) \\ Y(\delta_i^n) & = & \begin{cases} \xi^1_{n-1,1} & \text{if } i = 0 \\ X(\sigma_{i-1}^{n-1}) & \text{if } 1 \leq i \leq n-1 \\ \xi^2_{1,n-1} & \text{if } i = n \end{cases} \end{array}$$

where σ_i^n are the degeneracy maps in Δ, and δ_i^n the face maps, with notation as in [Mac, VII.5]. □

Remark I want to emphasize the difference between the categories Δ and Δ^+. For our purposes, the only interesting relation between the two is that stated in the Proposition: a colax monoidal functor from Δ to a cartesian monoidal category $\mathcal{M}$ is the same as a functor from $(\Delta^+)^{\mathrm{op}}$ to $\mathcal{M}$. The fact that Δ^+ is Δ with one object removed can be regarded as nothing more than a distracting coincidence.

We are mostly interested in a certain subset of the colax monoidal functors from Δ to $\mathcal{M}$: the homotopy monoids. The next result shows what property of a functor $\Delta^+ \longrightarrow \mathcal{M}$ corresponds to the colax monoidal functor being a homotopy monoid.

To state it we need some more notation. For $m, n \geq 0$, define maps

$$[m] \xrightarrow{\alpha^1_{m,n}} [m+n] \xleftarrow{\alpha^2_{m,n}} [n]$$

in Δ^+ by $\alpha^1_{m,n}(i) = i$ and $\alpha^2_{m,n}(i) = m + i$. For $0 \leq j < n$, define a map

$$\beta_j^n : [1] \longrightarrow [n]$$

by $\beta_j^n(i) = i + j$ for $i = 0, 1$.

Proposition 3.1.8 *Let $\mathcal{M}$ be a cartesian monoidal category with equivalences. Let*

$$(X, \xi) : (\Delta, +, \mathbf{0}) \longrightarrow (\mathcal{M}, \times, 1)$$

be a colax monoidal functor, and let

$$Y : (\Delta^+)^{\mathrm{op}} \longrightarrow \mathcal{M}$$

be the functor corresponding to (X, ξ) under Proposition 3.1.7. The following conditions are equivalent:

a. *(X, ξ) is a homotopy monoid*

b. *for each $m, n \geq 0$, the map*

$$(Y(\alpha^1_{m,n}), Y(\alpha^2_{m,n})) : Y[m+n] \longrightarrow Y[m] \times Y[n]$$

is an equivalence, and so is the unique map $Y[0] \longrightarrow 1$

c. for each $n \geq 0$, the map

$$(Y(\beta_0^n), \ldots, Y(\beta_{n-1}^n)) : Y[n] \longrightarrow Y[1]^n$$

is an equivalence.

Proof (a) $\Leftrightarrow$ (b): From the definition of Y in the proof of Proposition 3.1.7, it is easy to show that $Y(\alpha_{m,n}^1) = \xi_{m,n}^1$, and similarly $\alpha_{m,n}^2$. Hence

$$(Y(\alpha_{m,n}^1), Y(\alpha_{m,n}^2)) = \xi_{m,n}.$$

Also, ξ_0 is the unique map from $X(\mathbf{0}) = Y[0]$ to 1.

(a) $\Leftrightarrow$ (c): Similarly, an easy induction on n shows that

$$(Y(\beta_0^n), \ldots, Y(\beta_{n-1}^n)) = \xi^{(n)} : X(\mathbf{n}) \longrightarrow X(\mathbf{1})^n,$$

where $\xi^{(n)}$ is the map defined in 2.3(c). But by the comments in 2.3(c), (X, ξ) is a homotopy monoid if and only if each $\xi^{(n)}$ is an equivalence. $\square$

A functor $Y : (\Delta^+)^{\mathrm{op}} \longrightarrow \mathcal{M}$ satisfying the equivalent conditions (b) and (c) will be called a *special simplicial object* in $\mathcal{M}$. (When $\mathcal{M} = \mathbf{Top}$ the usual name is 'special Δ-space'; the 'Δ' used in this name is our Δ^+.) Write

$$\mathbf{Special}((\Delta^+)^{\mathrm{op}}, \mathcal{M})$$

for the category of special simplicial objects in $\mathcal{M}$ and natural transformations. Then we immediately have:

Corollary 3.1.9 *Let $(\mathcal{M}, \times, 1)$ be a cartesian monoidal category with equivalences. There is an isomorphism of categories*

$$\mathbf{HtyAlg}(\mathbf{Mon}, \mathcal{M}) \cong \mathbf{Special}((\Delta^+)^{\mathrm{op}}, \mathcal{M}).$$

$\square$

In particular, this holds for $\mathcal{M} = \mathbf{Cat}$ and $\mathcal{M} = \mathbf{Top}$, in which contexts special simplicial objects are best known (see [And], for example).

We have spent this section showing that homotopy commutative monoids are the same as special Γ-objects, and homotopy monoids the same as special simplicial objects. *But this is only true when $\mathcal{M}$ is cartesian.* If the tensor $\otimes$ in a symmetric monoidal category $\mathcal{M}$ is not the cartesian (categorical) product, or if the unit I is not the terminal object, then the two categories

$$[\Gamma^{\mathrm{op}}, \mathcal{M}], \qquad \mathbf{SColax}((\Phi, +, \mathbf{0}), (\mathcal{M}, \otimes, I))$$

are in general nowhere near equivalent. Similarly, the two categories

$$[(\Delta^+)^{\mathrm{op}}, \mathcal{M}], \qquad \mathbf{Colax}((\Delta, +, \mathbf{0}), (\mathcal{M}, \otimes, I))$$

are quite different. Moreover, the definition of 'special' (Propositions 3.1.2(b),(c) and 3.1.8(b),(c)) depend on the product in $\mathcal{M}$ being the cartesian product, and there is no obvious way to express it for an arbitrary monoidal structure on $\mathcal{M}$.

In Section 3.5, on A_∞-algebras, we will see our first significant example of homotopy algebras in a non-cartesian monoidal category.

3.2 Loop Spaces

Our main result is that any loop space is a homotopy monoid. More exactly:

Theorem 3.2.1 *There is a functor*

$$\Omega : \mathbf{Top}_* \longrightarrow \mathbf{HtyAlg}(\mathbf{Mon}, \mathbf{Top})$$

which sends a based space B to a homotopy monoid ΩB, with

$$(\Omega B)(1) \cong \mathbf{Top}_*(S^1, B).$$

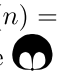
The heart of the proof is that the circle S^1 is a 'homotopy comonoid':

Lemma 3.2.2 *There is a homotopy monoid*

$$(W, \omega) : (\Delta, +, 0) \longrightarrow (\mathbf{Top}_*^{\mathrm{op}}, \vee, 1)$$

with $W(1) = S^1$.

Proof For each $n \geq 0$, let $(\Delta^n / \sim)$ denote the standard n-simplex Δ^n with its $n+1$ vertices collapsed to a single point, and this point declared the basepoint. Informally, (W, ω) can be described as follows:

- $W(n) = \Delta^n / \sim$, e.g. $W(0)$ is a single point, $W(1) = S^1$, and $W(2)$ looks like 
- W is defined on morphisms by the standard face and degeneracy maps of simplices
- ω is defined by face maps, e.g.

  $$\omega_{1,1} : W(1) \vee W(1) \longrightarrow W(2)$$

  is the evident inclusion

   $\hookrightarrow$ ,

  which is a homotopy equivalence.

Formally, it's easiest to employ the description of homotopy monoids given in Proposition 3.1.8. So, first consider the usual functor from Δ^+ to $\mathbf{Top}$, mapping $[n]$ to Δ^n and defined on morphisms by face and degeneracy maps. Since all the face and degeneracy maps take vertices to vertices, this functor induces another functor

$$Y : \Delta^+ \longrightarrow \mathbf{Top}_*$$

with $Y[n] = \Delta^n / \sim$. We may also view Y as a functor

$$Y : (\Delta^+)^{\mathrm{op}} \longrightarrow \mathbf{Top}_*^{\mathrm{op}};$$

$\vee$ and 1 are respectively coproduct and initial object in $\mathbf{Top}_*$, so they are product and terminal object in $\mathbf{Top}_*^{\mathrm{op}}$. So by Proposition 3.1.7, Y corresponds to a colax monoidal functor

$$(W, \omega) : (\Delta, +, 0) \longrightarrow (\mathbf{Top}_*^{\mathrm{op}}, \vee, 1)$$

with $W(1) = Y[1] = S^1$. To see that (W, ω) is a homotopy monoid we must first of all check that the unique map $Y[0] \longrightarrow 1$ is a homotopy equivalence, and then check that the map

$$(\Delta^m / \sim) \vee (\Delta^n / \sim) \longrightarrow \Delta^{m+n} / \sim$$

induced by the two maps

$$[m] \xleftarrow{\alpha^1_{m,n}} [m+n] \xrightarrow{\alpha^2_{m,n}} [n]$$

in Δ^+ is also a homotopy equivalence (see Proposition 3.1.8(b)). The first check is trivial, and for the second it is easy to construct a homotopy inverse. (From the conceptual angle, note however that there is no canonical choice of a homotopy inverse: see the picture of $\omega_{1,1}$ above, for instance.) $\Box$

To prove Theorem 3.2.1, observe that if B is a based space then

$$\mathbf{Top}_*(—, B) : (\mathbf{Top}_*^{\mathrm{op}}, \vee, 1) \longrightarrow (\mathbf{Top}, \times, 1)$$

is a monoidal functor, since $\vee$ is the coproduct in $\mathbf{Top}_*$ and 1 is the initial object. Observe also that $\mathbf{Top}_*(—, B)$ preserves homotopy equivalences. Hence the composite

$$(\Delta, +, 0) \xrightarrow{(W,\omega)} (\mathbf{Top}_*^{\mathrm{op}}, \vee, 1) \xrightarrow{\mathbf{Top}_*(—,B)} (\mathbf{Top}, \times, 1)$$

is a homotopy monoid, which we call ΩB. Moreover, any map $f : B \longrightarrow B'$ induces a monoidal transformation

$$\mathbf{Top}_*(—, B) \longrightarrow \mathbf{Top}_*(—, B')$$

and therefore a map $\Omega B \longrightarrow \Omega B'$; this makes Ω into a functor

$$\mathbf{Top}_* \longrightarrow \mathbf{HtyAlg}(\mathbf{Mon}, \mathbf{Top}).$$

Finally,

$$(\Omega B)(1) = \mathbf{Top}_*(W(1), B) = \mathbf{Top}_*(S^1, B),$$

as required.

(In 5.2(e) we'll see a neater way to express the proof of the Theorem: $\mathbf{Top}_*(—, B)$ is a 'change of environment'. The proof that a loop space is a special simplicial object was apparently first found by Segal.)

Let us pause for a moment to examine the homotopy monoid structure with which we have just equipped loop spaces, and in particular at how the composition of two loops is handled.

Fix a space B, and write (X, ξ) for the homotopy monoid ΩB. We have

$X(2) = \mathbf{Top}_*($$, B),$

and the pieces of (X, ξ) relevant to binary composition are the maps

$$\begin{tikzcd} & X(2) \arrow[dl, "\xi_{1,1}"', "\simeq"] \arrow[dr, "X(!)"] & \\ X(1)^2 & & X(1). \end{tikzcd}$$

This diagram is

$$\begin{tikzcd} & \mathbf{Top}_*(\quad, B) \arrow[dl, "\simeq"'] \arrow[dr] & \\ \mathbf{Top}_*(\quad, B) = \mathbf{Top}_*(S^1, B)^2 & & \mathbf{Top}_*(S^1, B), \end{tikzcd} \qquad (*)$$

where the map on the left is restriction to the two inner circles and the map on the right is restriction to the outer circle. All of the data making up (X, ξ), and in particular the maps in $(*)$, is constructed canonically from B: no arbitrary choices have been made. In contrast, there is no canonical map

$$\mathbf{Top}_*(S^1, B)^2 \longrightarrow \mathbf{Top}_*(S^1, B)$$

defining 'composition': although the obvious and customary choice is to use the map described by the instruction 'travel each loop at double speed', this appears to have no particular advantage or special algebraic status compared to any other choice. Since the usual formulation of the idea of homotopy topological monoid, A_∞-spaces, does entail this arbitrary choice of a composition law for loops, one might regard this as an ideological virtue of the definition presented here.

In Section 3.4 below we make a down-to-earth comparison between A_∞-spaces and homotopy topological monoids.

This section closes with three further remarks on loop spaces. Firstly, we have shown that any loop space is a homotopy topological monoid in the sense of being a homotopy algebra for the non-symmetric operad **Mon**. But the symmetric operad **Sym** (1.2(g)) also has the property that algebras for it (in any symmetric monoidal category, such as **Top**) are monoids, so we might also try to show that any loop space is a homotopy monoid in the sense of being a homotopy **Sym**-algebra. I believe this to be true, using a colax symmetric monoidal functor (W, ω) with $W(n) = \Delta^n / \sim$ again, but I am not sure. It seems that a homotopy **Mon**-algebra is not automatically a homotopy **Sym**-algebra; some extra input is required.

Secondly, recall from 3.1 that a special Γ-space is the same thing as a homotopy topological commutative monoid. The relationship between special

Γ-spaces and infinite loop spaces (spectra) is well-explored (see [Seg2], [Ad], [MT] etc.), and we may say 'any infinite loop space is a homotopy commutative monoid'. So we have a homotopy algebra structure on n-fold loop spaces for $n = 1$ and $n = \infty$. Thirdly, then, how about $1 < n < \infty$? The answer is that any n-fold loop space is an 'n-fold homotopy monoid'—but for an explanation of that, the reader will have to wait until Section 4.2.

3.3 Monoidal Categories

Another major example of homotopy algebras is that of homotopy **Mon**-algebras in **Cat**, which I shall call *homotopy monoidal categories*. This example was introduced in 2.3(e). We have three different kinds of structure in front of us: strict monoidal categories, (non-strict) monoidal categories, and homotopy monoidal categories. In this section a partial comparison is made between the last two.

The philosophical difference between monoidal categories and homotopy monoidal categories can be put like this. In an (ordinary) monoidal category such as $(\mathbf{Ab}, \otimes, \mathbb{Z})$, tensor is an operation with definite and precise values: that is, if A and B are abelian groups then there is assigned an abelian group $A \otimes B$, not just defined up to isomorphism, but with actual, specific, elements. Of course, we do not care precisely what these elements are; there are various definitions of $\otimes$ which are all naturally isomorphic, and this is all that matters in practice. So there is some degree of artificial choice in the definition of tensor, but it is at least an honest functor. Similar comments apply to the cartesian product $\times$ of sets. In contrast, homotopy monoidal categories allow a certain amount of fuzziness: there is not an actual operation 'tensor' (as we shall see), but the substitute for 'tensor' is intended to avoid artificial choices. Applied to **Ab**, for instance, the rough idea is to record all quadruples (A, B, u, C) where $u : A \times B \longrightarrow C$ is a bilinear map with the usual universal property, but without choosing any preferred C for each A and B. Thus one could speak of C as 'a tensor' of A and B, but one would never speak of 'the tensor'.

This idea is well known (if not deeply understood) in higher-dimensional category theory: see, for instance, [Her], [Baez], [BD], [HMP], [Joy]. Within category theory it was perhaps first taken seriously by Makkai in his study of anafunctors (see [Mak]), which arise implicitly throughout the present work. Homotopy monoidal categories resemble closely what Makkai called anamonoidal categories. The idea was also exploited in topology by Segal.

In this section we show how a homotopy monoidal category gives rise to a monoidal category, and similarly for symmetric monoidal categories. We then notice that this result generalizes effortlessly to an arbitrary monoidal 2-category (in place of **Cat**). This is very trivial, but will enable us later to read off results about A_n-spaces and A_n-algebras.

There is also a converse process in the case of **Cat**: any monoidal category gives rise to a homotopy monoidal category. This is more mysterious and less well-developed than the process the first way round, and the overall comparison between monoidal categories and homotopy monoidal categories is therefore incomplete. We do not describe this converse process here, but instead leave it until a later section (4.4) where it is done in more generality.

How a homotopy monoidal category give rise to a monoidal category

A homotopy monoidal category consists of a functor $C : \Delta \longrightarrow \mathbf{Cat}$ (previously called X) together with equivalences of categories

$$\begin{array}{rccc} \xi_{m,n} : & C(m+n) & \longrightarrow & C(m) \times C(n) \\ \xi_0 : & C(0) & \longrightarrow & \mathbf{1} \end{array}$$

$(m, n \geq 0)$ fitting together nicely. We regard $C(1)$ as the 'base category' of the homotopy monoidal category (C, ξ), recalling Example 2.3(c).

Proposition 3.3.1 *A homotopy monoidal category gives rise to a monoidal category.*

Proof Take a homotopy monoidal category (C, ξ) in $\mathbf{Cat}$, and construct from it a monoidal category as follows.

Underlying category: $C(1)$.

Tensor: What we want to define is a functor

$$\otimes : C(1) \times C(1) \longrightarrow C(1);$$

what we actually have are functors

$$\begin{tikzcd} & C(2) \arrow[dl, "\xi_{1,1}"', "\simeq"] \arrow[dr, "C(!)"] & \\ C(1) \times C(1) & & C(1) \end{tikzcd}$$

where ! is the unique map $2 \longrightarrow 1$ in Δ. So for each m and n, choose (arbitrarily) a pseudo-inverse $\psi_{m,n}$ to $\xi_{m,n}$, and define $\otimes$ as the composite

$$C(1) \times C(1) \xrightarrow{\psi_{1,1}} C(2) \xrightarrow{C(!)} C(1).$$

Associativity isomorphisms: The next piece of data we need is a natural isomorphism between $\otimes\circ(\otimes \times 1)$ and $\otimes\circ(1 \times \otimes)$. To see why such an isomorphism should exist, consider what would happen if the $\psi_{m,n}$'s were *genuine* inverses to the $\xi_{m,n}$'s. Then the $\psi_{m,n}$'s would satisfy the same coherence and naturality axioms as the $\xi_{m,n}$'s (with the arrows reversed), and this would guarantee that all sensible diagrams built up out of $\psi_{m,n}$'s commuted. Hence $\otimes$ would be strictly associative. As it is, $\psi_{m,n}$ is only inverse to $\xi_{m,n}$ up to isomorphism, and correspondingly $\otimes$ is associative up to isomorphism.

In practice, let us choose (at random) natural isomorphisms

$$\eta_{m,n} : 1 \xrightarrow{\sim} \xi_{m,n} \circ \psi_{m,n}, \qquad \varepsilon_{m,n} : \psi_{m,n} \circ \xi_{m,n} \xrightarrow{\sim} 1$$

for each m and n. Then a natural isomorphism

$$\alpha : \otimes \circ (\otimes \times 1) \xrightarrow{\sim} \otimes \circ (1 \times \otimes)$$

can be built up from the $\eta_{m,n}$'s and $\varepsilon_{m,n}$'s. The exact formula for α is rather complicated, and only included for the record. Most readers will therefore want to skip the next paragraph and ignore Figure 3A.

In order to define α, consider the diagram at the top of Figure 3A. The composites around the outside are $\otimes \circ (\otimes \times 1)$ and $\otimes \circ (1 \times \otimes)$, so we must find natural isomorphisms inside each of the four inner squares. The bottom-right square, in which σ_0, σ_1 are the two surjections $3 \longrightarrow 2$ in Δ, is genuinely commutative. In each of the other three squares, imagine taking each arrow labelled by a ψ, reversing its direction, and changing the ψ to a ξ. The imaginary square would then be genuinely commutative in each case, which means that the actual square is commutative up to isomorphism. This is the thought behind the formula for α given in the rest of Figure 3A. (For the usage of $*$, see page 7.)

Pentagon: We must now check that the associativity isomorphism just defined satisfies the famous pentagon coherence axiom. This asserts the commutativity of a certain diagram built up from components of α, that is, built up from $\eta_{m,n}$'s and $\varepsilon_{m,n}$'s. However, this diagram does *not* commute, which is perhaps unsurprising since $\eta_{m,n}$ and $\varepsilon_{m,n}$ were chosen independently.

But all is not lost: for recall the result that if

$$\begin{aligned} F : A \longrightarrow B, \quad & G : B \longrightarrow A, \\ \sigma : 1 \xrightarrow{\sim} G \circ F, \quad & \tau : F \circ G \xrightarrow{\sim} 1 \end{aligned}$$

is an equivalence of categories, then τ can be exchanged for another natural isomorphism τ' so that (F, G, σ, τ') is both an adjunction and an equivalence (see [Mac, IV.4.1]). So when we chose the natural isomorphisms $\eta_{m,n}$ and $\varepsilon_{m,n}$ above, we could have done it so that $(\psi_{m,n}, \xi_{m,n}, \eta_{m,n}, \varepsilon_{m,n})$ was an adjunction. Assume that we did so. Then this being an adjunction says that certain basic diagrams involving $\eta_{m,n}$ and $\varepsilon_{m,n}$ commute (namely, the diagrams for the triangle identities [Mac, IV.1(9)]): and that is enough to ensure that the pentagon commutes.

$$
\begin{tikzcd}
C(1)^3 \arrow[r, "\psi_{1,1} \times 1"] \arrow[d, "1 \times \psi_{1,1}"'] & C(2) \times C(1) \arrow[r, "C(!) \times 1"] \arrow[d, "\psi_{2,1}"] & C(1)^2 \arrow[d, "\psi_{1,1}"] \\
C(1) \times C(2) \arrow[r, "\psi_{1,2}"] \arrow[d, "1 \times C(!)"'] & C(3) \arrow[r, "C(\sigma_0)"] \arrow[d, "C(\sigma_1)"] & C(2) \arrow[d, "C(!)"] \\
C(1)^2 \arrow[r, "\psi_{1,1}"'] & C(2) \arrow[r, "C(!)"'] & C(1)
\end{tikzcd}
$$

$$
\begin{aligned}
& \otimes\circ(\otimes \times 1) \\
= \;& C(!)\circ\psi_{1,1}\circ[C(!) \times 1]\circ[\psi_{1,1} \times 1] \\
\xrightarrow{1*\eta_{2,1}*1} \;& C(!)\circ\psi_{1,1}\circ[C(!) \times 1]\circ\xi_{2,1}\circ\psi_{2,1}\circ[\psi_{1,1} \times 1] \\
= \;& C(!)\circ\psi_{1,1}\circ\xi_{1,1}\circ C(\sigma_0)\circ\psi_{2,1}\circ[\psi_{1,1} \times 1] \\
\xrightarrow{1*\varepsilon_{1,1}*1} \;& C(!)\circ C(\sigma_0)\circ\psi_{2,1}\circ[\psi_{1,1} \times 1] \\
\xrightarrow{1*[1\times\eta_{1,1}]} \;& C(!)\circ C(\sigma_0)\circ\psi_{2,1}\circ[\psi_{1,1} \times 1]\circ[1 \times \xi_{1,1}]\circ[1 \times \psi_{1,1}] \\
\xrightarrow{1*\eta_{1,2}*1} \;& C(!)\circ C(\sigma_0)\circ\psi_{2,1}\circ[\psi_{1,1} \times 1]\circ[1 \times \xi_{1,1}]\circ\xi_{1,2}\circ\psi_{1,2}\circ[1 \times \psi_{1,1}] \\
= \;& C(!)\circ C(\sigma_0)\circ\psi_{2,1}\circ[\psi_{1,1} \times 1]\circ[\xi_{1,1} \times 1]\circ\xi_{2,1}\circ\psi_{1,2}\circ[1 \times \psi_{1,1}] \\
\xrightarrow{1*[\varepsilon_{1,1}\times 1]*1} \;& C(!)\circ C(\sigma_0)\circ\psi_{2,1}\circ\xi_{2,1}\circ\psi_{1,2}\circ[1 \times \psi_{1,1}] \\
\xrightarrow{1*\varepsilon_{2,1}*1} \;& C(!)\circ C(\sigma_0)\circ\psi_{1,2}\circ[1 \times \psi_{1,1}] \\
= \;& C(!)\circ C(\sigma_1)\circ\psi_{1,2}\circ[1 \times \psi_{1,1}] \\
\xrightarrow{1*\varepsilon_{1,1}^{-1}*1} \;& C(!)\circ\psi_{1,1}\circ\xi_{1,1}\circ C(\sigma_1)\circ\psi_{1,2}\circ[1 \times \psi_{1,1}] \\
= \;& C(!)\circ\psi_{1,1}\circ[1 \times C(!)]\circ\xi_{1,2}\circ\psi_{1,2}\circ[1 \times \psi_{1,1}] \\
\xrightarrow{1*\eta_{1,2}^{-1}*1} \;& C(!)\circ\psi_{1,1}\circ[1 \times C(!)]\circ[1 \times \psi_{1,1}] \\
= \;& \otimes\circ(1 \times \otimes).
\end{aligned}
$$

Figure 3A: Formula for the associativity isomorphism α

Any reader who followed the construction of α will see that the pentagon involves 40 terms of the form $\eta_{m,n}$ or $\varepsilon_{m,n}$. Checking that it commutes is therefore an appreciable task, but in the absence of higher technology there is no alternative.

Unit: So far we have only mentioned binary tensor, and not units. To construct the unit object I of $C(1)$, choose a pseudo-inverse ψ_0 to the equivalence of categories $\xi_0 : C(0) \longrightarrow \mathbf{1}$ (in other words, pick an object of $C(0)$), and define $I \in C(1)$ as (the image of) the composite

$$\mathbf{1} \xrightarrow{\psi_0} C(0) \xrightarrow{C(!)} C(1).$$

Unit isomorphisms: We need left and right unit isomorphisms

$$\lambda_a : I \otimes a \xrightarrow{\sim} a, \qquad \rho_a : a \otimes I \xrightarrow{\sim} a$$

natural in $a \in C_0$. To define them, choose natural isomorphisms

$$\eta_0 : 1 \xrightarrow{\sim} \xi_0 \circ \psi_0, \qquad \varepsilon_0 : \psi_0 \circ \xi_0 \xrightarrow{\sim} 1$$

in such a way that $(\psi_0, \xi_0, \eta_0, \varepsilon_0)$ is an adjunction (ψ_0 left adjoint to ξ_0); this is possible by the result referred to under 'Pentagon' above. (In fact, it's not strictly necessary to use that general result, since the involvement of the category $\mathbf{1}$ makes the situation trivial; but the argument from general principles is conceptually cleaner.) Then λ and ρ can be built up from $\eta_{m,n}$'s, $\varepsilon_{m,n}$'s, η_0 and ε_0. For the record only, λ is defined in Figure 3B, which can be explained in the same way that Figure 3A was.

Triangle: The final check is that the triangle axiom holds; this is the 'other' coherence axiom for monoidal categories, along with the pentagon. It is built up out of λ, ρ and α, hence out of $\eta_{m,n}$'s, $\varepsilon_{m,n}$'s, η_0 and ε_0, and commutes for the same reason that the pentagon commutes. $\square$

The statement of the Proposition is rather vague. What the proof actually consists of is a construction, involving arbitrary choices, of a monoidal category from a homotopy monoidal category. Soon (Theorem 3.3.3) we will give an exact statement capturing what we have done, and at the same time we will see that making the arbitrary choices differently only affects the resulting monoidal category up to isomorphism. To achieve this we consider the functoriality of the construction.

The category $\mathbf{HtyAlg}(P, \mathcal{M})$ was defined with 'strict' maps of homotopy algebras as its morphisms. Thus if (X, ξ) and (X', ξ') are homotopy P-algebras

$$
\begin{tikzcd}
\mathbf{1} \times C(1) \arrow[r, "\psi_0 \times 1"] \arrow[dr, "i"'] & C(0) \times C(1) \arrow[r, "C(!) \times 1"] \arrow[d, "\psi_{0,1}"] & C(1) \times C(1) \arrow[d, "\psi_{1,1}"] \\
 & C(1) \arrow[r, "C(\delta_0)"] \arrow[dr, "1"'] & C(2) \arrow[d, "C(!)"] \\
 & & C(1)
\end{tikzcd}
$$

$$
\begin{aligned}
& \otimes \circ (I \times 1) \\
= \; & C(!) \circ \psi_{1,1} \circ [C(!) \times 1] \circ \xi_{0,1} \circ \psi_{0,1} \circ [\psi_0 \times 1] \\
\xrightarrow{1 * \eta_{0,1} * 1} \; & C(!) \circ \psi_{1,1} \circ [C(!) \times 1] \circ \xi_{0,1} \circ \psi_{0,1} \circ \xi_{0,1} \circ \psi_{0,1} \circ [\psi_0 \times 1] \\
= \; & C(!) \circ \psi_{1,1} \circ \xi_{1,1} \circ C(\delta_0) \circ \psi_{0,1} \circ \xi_{0,1} \circ \psi_{0,1} \circ [\psi_0 \times 1] \\
\xrightarrow{1 * \varepsilon_{1,1} * 1} \; & C(!) \circ C(\delta_0) \circ \psi_{0,1} \circ [\psi_0 \times 1] \\
= \; & \psi_{0,1} \circ [\psi_0 \times 1] \\
= \; & \psi_{0,1} \circ [\psi_0 \times 1] \circ [\xi_0 \times 1] \circ \xi_{0,1} \circ i \\
\xrightarrow{1 * [\varepsilon_0 \times 1] * 1} \; & \psi_{0,1} \circ \xi_{0,1} \circ i \\
\xrightarrow{\varepsilon_{0,1} * 1} \; & i
\end{aligned}
$$

Figure 3B: Formula for the left unit isomorphism λ. The functor i is the canonical isomorphism, and δ_0 is the map from 1 to 2 with image $\{1\}$.

in $\mathcal{M}$, and σ a map $(X,\xi) \longrightarrow (X',\xi')$ in $\mathbf{HtyAlg}(P,\mathcal{M})$, then the diagram

$$\begin{tikzcd}
X(m) \arrow[r, "\sigma_m"] \arrow[d, "X(\Theta)"'] & X'(m) \arrow[d, "X'(\Theta)"] \\
X(n) \arrow[r, "\sigma_n"'] & X'(n)
\end{tikzcd}$$

in $\mathcal{M}$ commutes (for $\Theta \in \widehat{P}(m,n)$), as do the squares of Definition 1.1.2. If $\mathcal{M}$ is a category like **Top**, **ChCx** or **Cat** where it is meaningful to speak of a diagram commuting 'up to homotopy' or 'up to isomorphism', then one can consider a more relaxed kind of morphism of homotopy algebras. But, of course, this is not meaningful for a general $\mathcal{M}$ in our theory, since all we know about $\mathcal{M}$ is which maps in it are 'homotopy equivalences'. The general point about weak morphisms of homotopy algebras is returned to in Chapter 6.

The maps in $\mathbf{HtyAlg}(\mathbf{Mon},\mathbf{Cat})$ should, therefore, not be considered as being as weak or lax as ordinary monoidal functors. Nevertheless, any map in $\mathbf{HtyAlg}(\mathbf{Mon},\mathbf{Cat})$ certainly ought to give rise to a monoidal functor, and this is what the next result says.

Proposition 3.3.2 *A map of homotopy monoidal categories gives rise to a monoidal functor. That is, if (C,ξ) and (C',ξ') are homotopy monoids in* **Cat***, and if $C(1)$ and $C'(1)$ are monoidal categories constructed from them as in Proposition 3.3.1, then any map $(C,\xi) \longrightarrow (C',\xi')$ induces (canonically) a monoidal functor $C(1) \longrightarrow C'(1)$.*

Proof Let $\sigma : (C,\xi) \longrightarrow (C',\xi')$ be a monoidal transformation. Let $\psi_{m,n}$ and ψ_0 be the (arbitrarily-chosen) functors used in the construction of (C,ξ), and $\eta_{m,n}, \eta_0, \varepsilon_{m,n}, \varepsilon_0$ the natural transformations, and similarly $\psi'_{m,n}$ etc. for (C',ξ').

We now construct a monoidal functor from $C(1)$ to $C'(1)$. The functor part is $\sigma_1 : C(1) \longrightarrow C'(1)$. For the rest of the structure we need isomorphisms $\sigma_1(a \otimes b) \xrightarrow{\sim} \sigma_1(a) \otimes' \sigma_1(b)$ (natural in $a, b \in C(1)$) and $\sigma_1(I) \xrightarrow{\sim} I'$. The first can be extracted from the diagram

$$\begin{tikzcd}
C(1) \times C(1) \arrow[r, "\psi_{1,1}"] \arrow[d, "\sigma_1 \times \sigma_1"'] & C(2) \arrow[r, "C(!)"] \arrow[d, "\sigma_2"] & C(1) \arrow[d, "\sigma_1"] \\
C'(1) \times C'(1) \arrow[r, "\psi'_{1,1}"'] & C'(2) \arrow[r, "C'(!)"'] & C'(1),
\end{tikzcd}$$

in which the right-hand square commutes and the left-hand square commutes up to isomorphism. The second arises similarly from the diagram

$$
\begin{tikzcd}
\mathbf{1} \arrow[r, "\psi_0"] \arrow[d, equal] & C(0) \arrow[r, "C(!)"] \arrow[d, "\sigma_0"] & C(1) \arrow[d, "\sigma_1"] \\
\mathbf{1} \arrow[r, "\psi'_0"'] & C'(0) \arrow[r, "C'(!)"'] & C'(1).
\end{tikzcd}
$$

Once these coherence isomorphisms have been written down explicitly, it is just a matter of checking the axioms. $\Box$

This construction preserves composites and identities, and so by making a large number of non-canonical choices we obtain a functor

$$\mathbf{HtyMonCat} \longrightarrow \mathbf{MonCat}.$$

Here and in what follows,

$$\mathbf{HtyMonCat} = \mathbf{HtyAlg}(\mathbf{Mon}, \mathbf{Cat})$$

and **MonCat** is the category of (small) monoidal categories and monoidal functors.

In order to state the result more precisely, and to get a *canonical* functor, let us introduce a new category, $\mathbf{Hty\widetilde{MonCat}}$. An object of $\mathbf{Hty\widetilde{MonCat}}$ is a homotopy monoidal category (C, ξ) together with a functor

$$\psi_{m,n} : C(m) \times C(n) \longrightarrow C(m+n)$$

and natural isomorphisms

$$\eta_{m,n} : 1 \xrightarrow{\sim} \xi_{m,n} \circ \psi_{m,n} \qquad \varepsilon_{m,n} : \psi_{m,n} \circ \xi_{m,n} \longrightarrow 1$$

obeying the triangle identities, for each m and n, and similarly $\psi_0, \eta_0, \varepsilon_0$. (Thus $(\psi_{m,n}, \xi_{m,n}, \eta_{m,n}, \varepsilon_{m,n})$ is an adjoint equivalence, as is $(\psi_0, \xi_0, \eta_0, \varepsilon_0)$.) A map

$$(C, \xi, \psi, \eta, \varepsilon) \longrightarrow (C', \xi', \psi', \eta', \varepsilon')$$

in $\mathbf{Hty\widetilde{MonCat}}$ is just a monoidal transformation $(C, \xi) \longrightarrow (C', \xi')$.

There is a canonical functor

$$\mathbf{Hty\widetilde{MonCat}} \longrightarrow \mathbf{HtyMonCat}$$

(forget ψ, η and ε) which is full, faithful and surjective on objects. Hence $\mathbf{Hty\widetilde{MonCat}}$ and **HtyMonCat** are equivalent. Our main result can now be stated as follows:

Theorem 3.3.3 *There is a canonical functor*

$$\mathbf{Hty}\widetilde{\mathbf{MonCat}} \longrightarrow \mathbf{MonCat}.$$

□

The *canonical* functors which have entered our discussion can be arranged in a diagram,

$$\begin{tikzcd} & \mathbf{Hty}\widetilde{\mathbf{MonCat}} \arrow[dl, "\simeq"'] \arrow[dr] & \\ \mathbf{HtyMonCat} & & \mathbf{MonCat}. \end{tikzcd}$$

By choosing a pseudo-inverse to the left-hand functor, one obtains a functor from **HtyMonCat** to **MonCat**, as we had in Proposition 3.3.1. But there is no *canonical* pseudo-inverse, and no *canonical* such functor. In the language of [Mak], this diagram depicts an anafunctor from **HtyMonCat** to **MonCat**.

Theorem 3.3.3 has the following corollary, which says that although the construction in Proposition 3.3.1 involves arbitrary choices, these choices do not affect the outcome significantly.

Corollary 3.3.4 *Let (C, ξ) be a homotopy monoidal category, let D be a monoidal category arising from (C, ξ) as in the proof of Proposition 3.3.1, and let D' be another monoidal category arising in this way via different choices. Then D and D' are isomorphic objects of* **MonCat**.

Proof Let $(\psi, \eta, \varepsilon)$ be the choices for D, and $(\psi', \eta', \varepsilon')$ those for D'. Then $(C, \xi, \psi, \eta, \varepsilon)$ and $(C, \xi, \psi', \eta', \varepsilon')$ are isomorphic objects of $\mathbf{Hty}\widetilde{\mathbf{MonCat}}$, so their images in **MonCat** under the functor of Theorem 3.3.3 are also isomorphic. In other words, $D \cong D'$ in **MonCat**. □

I want to finish this part with two remarks. First of all, in a brutally honest world the Propositions above should be called conjectures: I have not checked every detail. Secondly, the entire theory above can be repeated for *symmetric* monoidal categories, using homotopy algebras in **Cat** for the symmetric operad **CMon**. This extension should be absolutely straightforward. In fact, we have already seen (3.1) that homotopy symmetric monoidal categories are the same as the Γ-categories defined in [Seg2], which we call special Γ-categories here.

A Generalization

At no point in our discussion of monoidal categories have we mentioned their objects and morphisms. To be a little more accurate, we *have* mentioned them now and then (e.g. the 'a' in λ_a and ρ_a in the proof of Proposition 3.3.1), but only as a matter of linguistic convenience. Essentially, the discussion took place purely in terms of categories, functors, natural transformations and products. Indeed, only the purely formal properties of products were used—we did not exploit their universal property at all.

It follows almost instantly that all the results above hold in any monoidal 2-category, not merely in **Cat**. What exactly this means will be clear to readers experienced in 2-categories, but I will explain it now.

The term 'monoidal 2-category' is defined in Example 2.1(k); for an account of 2-categories in general, see [KS] or [KV]. Recall also from 2.1(k) that any monoidal 2-category $(\mathcal{N}, \otimes, I)$ has an underlying monoidal category with equivalences, which we will call $|\mathcal{N}|$. So, on the one hand, we have homotopy monoids in $|\mathcal{N}|$. On the other hand, we have the concept of a *weak monoid* (also known as a *pseudo-monoid*) in $\mathcal{N}$. Weak monoids are defined so that a weak monoid in **Cat** is a monoidal category in the traditional sense: thus a weak monoid in $\mathcal{N}$ consists of

- an object A of $\mathcal{N}$
- 1-cells $t : A \otimes A \longrightarrow A$, $i : I \longrightarrow A$
- invertible 2-cells

$$\alpha : t\circ(1 \times t) \longrightarrow t\circ(t \times 1),$$
$$\lambda : t\circ(i \times 1) \longrightarrow 1, \qquad \rho : t\circ(1 \times i) \longrightarrow 1$$

satisfying pentagon and triangle axioms.

Weak maps of weak monoids are defined in a similar style, so that a weak map of weak monoids in **Cat** is a monoidal functor.

The arguments concerning homotopy monoidal categories then give us at once:

Proposition 3.3.5 *Let $\mathcal{N}$ be a monoidal 2-category and $|\mathcal{N}|$ the associated monoidal category with equivalences. Then there is a (non-canonical) functor*

$$\mathbf{HtyAlg}(\mathbf{Mon}, |\mathcal{N}|) \longrightarrow (\text{weak monoids and weak maps in } \mathcal{N}).$$

$\square$

Again, this result can be rephrased to eliminate the element of arbitrary choice.

There is just one point where the generalization might not be quite clear, and this concerns adjoint equivalences in $\mathcal{N}$. In [Mac, IV.4.1] it is shown that any equivalence of categories 'might as well' be an adjoint equivalence, but it is not obvious from the proof that this is in fact a general 2-categorical result.

To state the result we need some definitions. Take 0-cells A and B of $\mathcal{N}$, 1-cells

$$A \underset{g}{\overset{f}{\rightleftarrows}} B,$$

and 2-cells

$$\eta : 1 \longrightarrow g\circ f, \qquad \varepsilon : f\circ g \longrightarrow 1.$$

Then $(f, g, \eta, \varepsilon)$ is called an *adjunction* in $\mathcal{N}$ if the triangle identities ([Mac, IV.1(9)]) are satisfied, an *equivalence* if η and ε are both invertible, and an *adjoint equivalence* if both an adjunction and an equivalence. These definitions have the usual meaning when $\mathcal{N} = \mathbf{Cat}$; in the case of adjunctions, f is left adjoint to g.

Lemma 3.3.6 *Let $\mathcal{N}$ be a 2-category and let*

$$A \underset{g}{\overset{f}{\rightleftarrows}} B, \qquad \eta : 1 \xrightarrow{\sim} g\circ f, \qquad \varepsilon : f\circ g \xrightarrow{\sim} 1$$

be an equivalence in $\mathcal{N}$. Then there is a 2-cell $\varepsilon' : f\circ g \xrightarrow{\sim} 1$ such that $(f, g, \eta, \varepsilon')$ is an adjoint equivalence.

Remark When $\mathcal{N}$ is the 2-category of topological spaces, continuous maps, and homotopy classes of homotopies, this result is known as Vogt's Lemma (see [KP, IV.1.14] and [Vogt]). The general result is probably due to Street, and has existed as folklore (at least) since the 1970s.

Proof Take ε' to be the composite

$$fg \xrightarrow{\varepsilon^{-1}fg} fgfg \xrightarrow{f\eta^{-1}g} fg \xrightarrow{\varepsilon} 1$$

and check the triangle identities (a tricky but elementary exercise). □

Thus we have the result we need on adjoint equivalences, and obtain Proposition 3.3.5. As usual, the same (probably) goes for the commutative case.

One final observation will be useful later. By leaving out all mention of i, λ and ρ in the definition of weak monoid, we obtain the definition of *weak semigroup*. By leaving out all mention of i, λ and ρ in the proof of 3.3.5, we also obtain:

Proposition 3.3.7 *Let $\mathcal{N}$ be a monoidal 2-category and $|\mathcal{N}|$ the associated monoidal category with equivalences. Then there is a (non-canonical) functor*

$$\mathbf{HtyAlg}(\mathbf{Sem}, |\mathcal{N}|) \longrightarrow (\text{weak semigroups in } \mathcal{N}).$$

□

This result will be employed in the next two sections, on A_∞-spaces and A_∞-algebras.

3.4 A_∞-Spaces

The main result of this section is that any homotopy semigroup in $\mathbf{Top}_*$ gives rise to an A_4-space. This is an almost immediate consequence of the results of the previous section. There are also some conjectures on how this result might extend to give A_∞-algebras, and on related matters. But before I address any of this, there is a small matter of basepoints which needs clearing up.

Recall that a semigroup is a set with an associative binary operation, and a monoid is a semigroup with a two-sided unit; recall also that $\mathbf{Top}$ is the category of spaces and $\mathbf{Top}_*$ the category of spaces with basepoint. One might casually imagine that a semigroup in $\mathbf{Top}_*$ is the same thing as a monoid in $\mathbf{Top}$: after all, it's just a matter of whether the special point is regarded as part of the topological data (the basepoint) or the algebraic data (the unit). But this is not the case, as we saw in Example 1.2(c): a semigroup in $\mathbf{Top}_*$ is a topological semigroup together with a distinguished idempotent element, which need not be a unit.

If we look at the original definition of an A_∞-space ([Sta1]) then we can see that conceptually, an A_∞-space is an up-to-homotopy version of a semigroup in $\mathbf{Top}_*$, rather than of a monoid in $\mathbf{Top}$. This manifests itself in several ways. Firstly, any semigroup A in $\mathbf{Top}_*$ is naturally an A_∞-space (with trivial structure in dimensions 3 and above); the basepoint of A does not need to be a unit. Secondly, it is only homotopy associativity which is considered (think of the title of [Sta1]!), and not homotopy unit laws. Thirdly, in the definition of monoidal category one has both the pentagon and the triangle coherence laws, whereas the associahedra used to define 'A_∞-space' only include the pentagon, again revealing the semigroupal flavour of A_∞-spaces.

It therefore seems appropriate to compare A_∞-spaces with homotopy semigroups in $\mathbf{Top}_*$, rather than with homotopy monoids in $\mathbf{Top}$: and this is what we do here.

Theorem 3.4.1 *Any homotopy semigroup (X, ξ) in $\mathbf{Top}_*$ gives rise to an A_4-space, whose underlying space is $X(1)$.*

Proof There is a 2-category $\mathbf{Top}_*$ whose objects and 1-cells are the same as those of the (1-)category $\mathbf{Top}_*$, and whose 2-cells are homotopy classes of homotopies. An equivalence in this 2-category is just a homotopy equivalence. (All homotopies mentioned here must respect basepoints.) Moreover, cartesian product $\times$ and the one-point space 1 make $(\mathbf{Top}_*, \times, 1)$ into a monoidal 2-category. So by Proposition 3.3.7, a homotopy semigroup (X, ξ) in $\mathbf{Top}_*$ gives rise to a weak semigroup in $\mathbf{Top}_*$ with underlying space $X(1)$.

Now we only have to see that a weak semigroup in the monoidal 2-category $\mathbf{Top}_*$ gives rise to an A_4-space. (In fact they are more or less the same thing, as the argument reveals.) I shall only do this informally. A weak semigroup in $\mathbf{Top}_*$ consists of a based space (A, a_0), a multiplication

$$m_2 = t : A \times A \longrightarrow A$$

with $m_2(a_0, a_0) = a_0$, and a 2-cell α between the two maps

$$A^3 \underset{m_2 \circ (1 \times m_2)}{\overset{m_2 \circ (m_2 \times 1)}{\rightrightarrows}} A,$$

such that α satisfies the pentagon axiom. Now a 2-cell in $\mathbf{Top}_*$ is a homotopy class of homotopies, so we may pick a representative m_3 of α. The pentagon axiom then says that a certain pair of homotopies h, h' (built up from m_3's) belong to the same homotopy class. Choose a homotopy m_4 between h and h': then m_4 is essentially a map

$$K_4 \times A^4 \longrightarrow A$$

where K_4 is the solid pentagon (as in [Sta1]), and the data (A, a_0, m_2, m_3, m_4) thus describes an A_4-space. $\square$

This proof uses 2-category theory, but as far as I know the number 2 has only one special property: it is the largest value of n for which n-category theory is currently well-understood. I hope that in the near future (weak) n-category theory will be viable, and it will be possible to show

- that a homotopy semigroup in an ∞-category $\mathcal{N}$ gives rise to a 'weak semigroup' in $\mathcal{N}$ (where 'weak' is meant in an ∞-categorical sense), and
- that a weak semigroup in the ∞-category $\mathbf{Top}_*$ is (more or less) an A_∞-space.

So I conjecture that in the Theorem, '4' can be replaced by '∞'. (Of course, the conjecture might be provable without use of higher-dimensional categories.) Naturally this should extend to morphisms too: a map of homotopy semigroups ought to give rise to an A_∞-map.

As for the converse—A_∞-spaces giving rise to homotopy semigroups—I do not know; see 4.4 for a similar question with categories in place of spaces.

3.5 A_∞-Algebras

In the previous section we showed that a homotopy semigroup in $\mathbf{Top}_*$ gives rise to an A_4-space, by

- considering $\mathbf{Top}_*$ as a monoidal 2-category,
- employing the result (3.3.7) that a homotopy semigroup in a monoidal 2-category $\mathcal{N}$ gives rise to a weak semigroup in $\mathcal{N}$, then
- seeing that a weak semigroup in $\mathbf{Top}_*$ is an A_4-space.

In this section we do exactly the same for A_4-algebras, replacing $\mathbf{Top}_*$ by $\mathbf{ChCx}$. So, a homotopy semigroup in $\mathbf{ChCx}$ gives rise to an A_4-algebra, and I conjecture that this process can be extended to give an A_∞-algebra. (See [Sta2] for the definitions of A_n-algebra and A_∞-algebra.)

This section is laid out as follows. First we look at how $\mathbf{ChCx}$ can be made into a monoidal 2-category, in which the (2-categorical) equivalences are just the chain homotopy equivalences. Then we examine weak semigroups in $\mathbf{ChCx}$, and show them to be essentially the same thing as A_4-algebras. Proposition 3.3.7 then applies (as it did for spaces), so we have a proof that homotopy differential graded non-unital algebras give rise to A_4-algebras. We close with a concrete description of this process.

Our first task is to describe the monoidal 2-category $\mathbf{ChCx}$. Objects are chain complexes and 1-cells are (degree 0) chain maps. 2-cells are homotopy classes of chain homotopies: but what does that mean? Take chain complexes A and B, chain maps $f, g : A \longrightarrow B$, and chain homotopies $s, t : f \longrightarrow g$, as shown:

$$\begin{tikzcd} A \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] & B. \arrow[from=U, to=D, phantom, "s\Downarrow\,\Downarrow t"] \end{tikzcd}$$

Then a *homotopy* (or *secondary homotopy*) $\gamma : s \longrightarrow t$ consists of a map

$$\gamma : A_{p-2} \longrightarrow B_p$$

for each $p \in \mathbb{Z}$, such that

$$d\gamma - \gamma d = t - s.$$

(Note the sign on the left-hand side.) We then say that the homotopies s and t are *homotopic*, and being homotopic is an equivalence relation. Later (page 62) we will address the question of why this is a reasonable definition of secondary homotopy.

We've now defined the 0-cells, 1-cells and 2-cells of the prospective monoidal 2-category $\mathbf{ChCx}$. Composition of 1-cells is done in the usual way, 'vertical' composition of 2-cells

$$\begin{tikzcd} A \arrow[r, bend left=60, "f"{name=U}] \arrow[r, "g"{name=M}] \arrow[r, bend right=60, "h"'{name=D}] & B \arrow[from=U, to=M, phantom, "\Downarrow s"] \arrow[from=M, to=D, phantom, "\Downarrow t"] \end{tikzcd}$$

is by addition (i.e. $t \circ s = t + s$), and identities work similarly. The horizontal composite $s' * s$ of 2-cells

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\downarrow s"] & A' \arrow[r, bend left, "f'"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\downarrow s'"] & A'' \end{tikzcd}$$

is defined by

$$(s' * s)(a) = s'g(a) + f's(a)$$

for $a \in A_p$. The interchange law for 2-categories, which says that

$$(t' \circ s') * (t \circ s) = (t' * t) \circ (s' * s)$$

for all suitable s, s', t, t', does hold, but only because we have used *homotopy classes* of chain homotopies; it does not hold at the level of ordinary ('primary') homotopies. (A related issue is that $s' * s$ could equally well have been defined by

$$(s' * s)(a) = g's(a) + s'f(a);$$

the choice of one over the other is quite non-canonical.)

So we now have a 2-category **ChCx**, and the next step is to endow it with a monoidal structure. The tensor $\otimes$ of chain complexes and the unit chain complex R are as usual (see 1.1(f)), and the tensor of chain maps

$$A \xrightarrow{f} B, \qquad A' \xrightarrow{f'} B'$$

is given by the obvious formula. The tensor of chain homotopies

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\downarrow s"] & B, \end{tikzcd} \qquad \begin{tikzcd} A' \arrow[r, bend left, "f'"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\downarrow s'"] & B' \end{tikzcd}$$

is given by

$$(s \otimes s')(a \otimes a') = s(a) \otimes f'(a') + (-1)^p g(a) \otimes s'(a')$$

for $a \in A_p$ and $a' \in A'_{p'}$. Once again, this is one of two equally appropriate formulae, but up to secondary homotopy they are the same.

Finally, then, we have a monoidal 2-category **ChCx**. It is clear that an equivalence inside this 2-category (as defined before Lemma 3.3.6) is just a chain homotopy equivalence. The usual symmetry (1.1(f)) in fact makes **ChCx** into a *symmetric* monoidal 2-category, but we shall not need to use this fact.

Aside: secondary chain homotopies

Earlier I promised to explain why the definition of homotopy of chain homotopies is a reasonable one. The situation can be viewed as follows. Let A and B be topological spaces and let U be the unit interval $[0,1]$. Then a homotopy between maps $f, g : A \longrightarrow B$ is, of course, a map

$$s : U \times A \longrightarrow B$$

with $s(0, \text{—}) = f$ and $s(1, \text{—}) = g$. Next, let s and t be two homotopies between f and g; a homotopy between the homotopies s and t is a map

$$\gamma : U \times U \times A \longrightarrow B$$

satisfying

$$\begin{array}{ll} \gamma(0, \text{—}, \text{—}) = s, & \gamma(1, \text{—}, \text{—}) = t, \\ \gamma(k, 0, \text{—}) = f, & \gamma(k, 1, \text{—}) = g \end{array}$$

for all $k \in U$.

The point is that both of these descriptions can be expressed diagrammatically in the monoidal category $(\mathbf{Top}, \times, 1)$, with the aid of the maps

$$i_0, i_1 : 1 \longrightarrow U, \qquad j : U \longrightarrow 1,$$

where i_0 and i_1 have respective values 0 and 1. Now let's mimic this in the monoidal category $(\mathbf{ChCx}, \otimes, R)$: take U to be the chain complex

$$\begin{array}{cccccccc} \cdots & 0 & \longrightarrow & R & \longrightarrow & R \oplus R & \longrightarrow & 0 & \cdots \\ & & & r & \longmapsto & (-r, r) & & & \end{array}$$

with R in degree 1 and $R \oplus R$ in degree 0, define $i_0, i_1 : R \longrightarrow U$ by the first and second inclusions of R into $R \oplus R$, and define $j : U \longrightarrow R$ by the addition map from $R \oplus R$ to R. A *homotopy* between chain maps can then be defined as a suitable map $U \otimes A \longrightarrow B$, as in the topological case, and this turns out to be equivalent to the usual definition of chain homotopy. A *secondary homotopy* can similarly be defined as a map

$$U \otimes U \otimes A \longrightarrow B$$

satisfying suitable 'boundary conditions', and, with a significant amount of calculation, this turns out to be equivalent to the very simple definition given originally.

Further thoughts of this kind are laid out in [KP, III.3].

Returning to the main flow, we have exhibited **ChCx** as a monoidal 2-category and now wish to look at weak semigroups in it. Such a thing consists of a chain complex A, a (degree 0) chain map

$$m_2 = t : A^{\otimes 2} \longrightarrow A,$$

and a homotopy class α of chain homotopies

$$\begin{tikzcd} A^{\otimes 3} \arrow[r, bend left, "m_2\circ(m_2\otimes 1)"] \arrow[r, bend right, "m_2\circ(1\otimes m_2)"'] \arrow[r, phantom, "\downarrow"] & A, \end{tikzcd} \qquad (\dagger)$$

satisfying the pentagon axiom. Choose a representative m_3 of the class α. Then m_3 is a family of homomorphisms

$$(A^{\otimes 3})_p \longrightarrow A_{p+1}$$

($p \in \mathbb{Z}$), and the fact that m_3 is a homotopy between the two maps in ($\dagger$) says that

$$\begin{aligned} & d(m_3(a_1,a_2,a_3)) \\ & + m_3(da_1,a_2,a_3) + (-1)^{p_1} m_3(a_1,da_2,a_3) + (-1)^{p_1+p_2} m_3(a_1,a_2,da_3) \\ & = \quad -m_2(m_2(a_1,a_2),a_3) + m_2(a_1,m_2(a_2,a_3)) \end{aligned}$$

for all $a_1 \in A_{p_1}$, $a_2 \in A_{p_2}$ and $a_3 \in A_{p_3}$. Finally, the fact that α satisfies the pentagon identity means that there is a secondary homotopy m_4 between a certain pair of homotopies built up as composites of m_3's. Thus m_4 is a family of homomorphisms

$$(A^{\otimes 4})_p \longrightarrow A_{p+2}$$

($p \in \mathbb{Z}$), and the equation '$d\gamma - \gamma d = t - s$' in the definition of secondary homotopy says that for $a_1 \in A_{p_1}$, $a_2 \in A_{p_2}$, $a_3 \in A_{p_3}$ and $a_4 \in A_{p_4}$,

$$\begin{aligned} & d(m_4(a_1,a_2,a_3,a_4)) - m_4(da_1,a_2,a_3,a_4) - (-1)^{p_1} m_4(a_1,da_2,a_3,a_4) \\ & - (-1)^{p_1+p_2} m_4(a_1,a_2,da_3,a_4) - (-1)^{p_1+p_2+p_3} m_4(a_1,a_2,a_3,da_4) \\ & \quad = \quad -m_3(m_2(a_1,a_2),a_3,a_4) + m_3(a_1,m_2(a_2,a_3),a_4) - m_3(a_1,a_2,m_2(a_3,a_4)) \\ & \qquad\quad + m_2(m_3(a_1,a_2,a_3),a_4) + (-1)^{p_1} m_2(a_1,m_3(a_2,a_3,a_4)). \end{aligned} \qquad (\ddagger)$$

So the structure (A, m_2, m_3, m_4) is precisely an A_4-algebra. We therefore have:

Theorem 3.5.1 *Any homotopy differential graded non-unital algebra* (X, ξ) *gives rise to an* A_4*-algebra, whose underlying chain complex is* $X(1)$. □

Let us now look more directly at how a homotopy semigroup

$$(X, \xi) : (\Delta_{\text{surj}}, +, 0) \longrightarrow (\mathbf{ChCx}, \otimes, R)$$

leads to an A_4-algebra A. (See 1.6(b) for the definition of Δ_{surj}.) First of all, $A = X(1)$. Secondly, choose a chain homotopy inverse $\psi_{m,n}$ to each $\xi_{m,n}$, as shown:

$$X(m+n) \underset{\psi_{m,n}}{\overset{\xi_{m,n}}{\rightleftarrows}} X(m) \otimes X(n).$$

Then $m_2 : A^{\otimes 2} \longrightarrow A$ is defined as the composite

$$X(1) \otimes X(1) \xrightarrow{\psi_{1,1}} X(2) \xrightarrow{X(!)} X(1)$$

where ! is the unique map $2 \longrightarrow 1$ in Δ_{surj}. To describe m_3, consider the diagram at the top of Figure 3A (page 50), with C's changed to X's and $\times$'s to $\otimes$'s. For the same reasons as given then, each inner square of the diagram—hence the whole diagram—commutes up to homotopy. This says that $m_2 \circ (m_2 \otimes 1)$ and $m_2 \circ (1 \otimes m_2)$ are chain-homotopic, and indeed we can construct a particular such homotopy, m_3.

Using the formula in 3A, m_3 is a sum of 8 terms. This means that the right-hand side of equation (‡) is a sum of 40 terms, and finding an m_4 to satisfy it would be an enormous task if attempted from cold. However, the general method of Proposition 3.3.7 provides an m_4 automatically.

Just as for A_n-spaces, I know of no reason why the process should stop at A_4, and I conjecture that there are also maps m_5, m_6, $\ldots$ making $X(1)$ into an A_∞-algebra. Similarly, it is plausible that maps of homotopy semigroups in **ChCx** give rise to A_∞-maps. Bearing in mind that **ChCx** is a *symmetric* monoidal category, it might also be possible to do the same things for homotopy d.g. *commutative* non-unital algebras: such a structure might give rise to a C_4-algebra, and perhaps a C_∞-algebra.

Chapter 4

Other Examples of Homotopy Algebras

The previous chapter covered homotopy monoids and homotopy semigroups in some detail. In this chapter we look at various other examples of homotopy algebras, and develop some further theory concerning homotopy algebras in general.

Section 4.1 is an assortment of examples of homotopy algebras: homotopy graded Lie and Gerstenhaber algebras (both of which are homotopy algebras in the enriched sense), homotopy monoids-with-involution, and 'homotopy homotopy algebras' (e.g. homotopy L_∞-algebras). The whole section raises more questions than it answers, and in particular poses a concrete question concerning Hochschild cochains (4.1.2).

Section 4.2 fulfils a promise made in Chapter 3: to put a natural homotopy-algebraic structure on an n-fold loop space, for any $1 \leq n < \infty$. In order to do this we have to develop a theory of 'homotopy $(P_1, \ldots, P_n)$-algebras' for any family (P_i) of operads, which we do briefly. An n-fold loop space is then an 'n-fold homotopy monoid', that is, a homotopy $(\underbrace{\mathbf{Mon}, \ldots, \mathbf{Mon}}_{n})$-algebra.

The final two sections, 4.3 and 4.4, are about comparing different notions of weakened algebraic structure. Any monoidal 2-category has an underlying monoidal category with equivalences (as we saw in Example 2.1(k)), one example being **Cat**. In 4.3 we formulate a notion of 'weak P-algebra' in any monoidal 2-category, and extend the method of Chapter 3 to show that any homotopy P-algebra gives rise to a weak P-algebra. One naturally wants to know whether it is possible to go in the opposite direction too (weak algebras giving homotopy algebras); I cannot provide an answer in general, but 4.4 shows how this is possible in the case of **Cat**.

4.1 Miscellaneous Examples

4.1.1 Graded Lie algebras

So far we have not paid very much attention to the enriched setting: we have said how to *define* homotopy algebras for an enriched operad (2.4), but have not done much by way of examples. As compensation, we now examine in detail the definition of a homotopy graded Lie algebra. That is, we examine homotopy **GrLie**-algebras in **ChCx**.

So, take the **GrAb**-enriched symmetric operad **GrLie** (1.5(e)), which is generated by an element

$$[\ ,\ ] \in (\mathbf{GrLie}(2))_{-1}$$

subject to equations

$$\begin{aligned} [\ ,\ ] + [\ ,\ ].\tau &= 0 \\ [[\ ,\ ],\ ] + [[\ ,\ ],\ ].\sigma + [[\ ,\ ],\ ].\sigma^2 &= 0 \end{aligned}$$

where $\tau \in S_2$ is a 2-cycle and $\sigma \in S_3$ is a 3-cycle.

Consequently, an algebra for **GrLie** in $\mathbf{GrMod}_R$ is a graded R-module A together with a binary operation of degree -1, satisfying the equations

$$\begin{aligned} [a,b] + (-1)^{pq}[b,a] &= 0 \\ (-1)^{rp}[[a,b],c] + (-1)^{pq}[[b,c],a] + (-1)^{qr}[[c,a],b] &= 0 \end{aligned}$$

for $a \in A_p, b \in A_q, c \in A_r$. The signs arise from the symmetry map in $\mathbf{GrMod}_R$ (see 1.1(e)). So as expected, a **GrLie**-algebra is a graded Lie algebra in the usual sense.

From this point on, most of what we have to say about **GrLie** applies equally to all **GrAb**-enriched symmetric operads.

Homotopy algebras are defined via the **GrAb**-enriched symmetric strict monoidal category $\widehat{\mathbf{GrLie}}$, whose objects are the natural numbers and whose 'hom-objects' are the graded abelian groups

$$\widehat{\mathbf{GrLie}}(m,n) = \bigoplus_{f\in\Phi(m,n)} \mathbf{GrLie}(f^{-1}\{0\}) \otimes \cdots \otimes \mathbf{GrLie}(f^{-1}\{n-1\}).$$

(It's not hard to see that $\mathbf{GrLie}(0) = 0$, that $\mathbf{GrLie}(n)$ is concentrated in degree $1-n$ for $n \geq 1$, and that $\widehat{\mathbf{GrLie}}(m,n)$ is therefore concentrated in degree $n-m$; but this doesn't matter for the present account.)

Much as in Example 1.4(h), $\mathbf{ChCx} = \mathbf{ChCx}_R$ can be viewed as a **GrAb**-enriched symmetric monoidal category, with $\mathbf{ChCx}[C,D]$ being the graded abelian group whose degree k part is the abelian group of all degree k chain maps from C to D. The ordinary category underlying this **GrAb**-enriched category is the usual **ChCx**, in which $\mathbf{ChCx}(C,D)$ is the set of all degree 0 chain maps $C \longrightarrow D$. A homotopy **GrLie**-algebra in **ChCx** consists of a **GrAb**-enriched symmetric monoidal functor

$$(X, \xi) : \widehat{\mathbf{GrLie}} \longrightarrow \mathbf{ChCx}$$

such that the components of ξ are homotopy equivalences. Taking this apart further, a homotopy graded Lie algebra consists of

- a sequence $X(0), X(1), \ldots$ of chain complexes
- a chain homotopy equivalence $\xi_0 : X(0) \longrightarrow R$ (where R is the unit chain complex—see 1.1(f))
- a chain homotopy equivalence

  $$\xi_{n,n'} : X(n+n') \longrightarrow X(n) \otimes X(n')$$

  for each $n, n' \geq 0$
- for each map $f : m \longrightarrow n$ of finite sets and each

  $$\theta_1 \in (\mathbf{GrLie}(f^{-1}\{0\}))_{p_1}, \ldots, \theta_n \in (\mathbf{GrLie}(f^{-1}\{n-1\}))_{p_n},$$

  a chain map

  $$X(f; \theta_1, \ldots, \theta_n) : X(m) \longrightarrow X(n)$$

  of degree $p_1 + \cdots + p_n$.

(Note that $X(n) \simeq X(1)^{\otimes n}$ as usual.) This data satisfies various axioms. The expression $X(f; \theta_1, \ldots, \theta_n)$ preserves addition of θ_i's, and X preserves composites of the morphisms $(f; \theta_1, \ldots, \theta_n)$ in $\widehat{\mathbf{GrLie}}$, and similarly preserves identities. The maps ξ_0 and $\xi_{n,n'}$ obey the usual coherence axioms for a colax monoidal functor (given in 1.1.1). Finally, $\xi_{n,n'}$ is natural in n and n': if $f : m \longrightarrow n$ and $f' : m' \longrightarrow n'$ are maps of finite sets, and if

$$\begin{aligned}
&\theta_1 \in (\mathbf{GrLie}(f^{-1}\{0\}))_{p_1}, \ldots, \theta_n \in (\mathbf{GrLie}(f^{-1}\{n-1\}))_{p_n},\\
&\theta'_1 \in (\mathbf{GrLie}(f'^{-1}\{0\}))_{p'_1}, \ldots, \theta'_{n'} \in (\mathbf{GrLie}(f'^{-1}\{n'-1\}))_{p'_{n'}},
\end{aligned}$$

then the diagram

$$\begin{tikzcd}
X(m+m') \arrow[r, "\xi_{m,m'}"] \arrow[d, "{X(f+f'; \theta_1, \ldots, \theta_n, \theta'_1, \ldots, \theta'_{n'})}"'] & X(m) \otimes X(m') \arrow[d, "{X(f; \theta_1, \ldots, \theta_n) \otimes X(f'; \theta'_1, \ldots, \theta'_{n'})}"] \\
X(n+n') \arrow[r, "\xi_{n,n'}"] & X(n) \otimes X(n')
\end{tikzcd}$$

commutes. The horizontal maps here are of degree 0, and the vertical maps are of degree $(p_1 + \cdots + p_n + p'_1 + \cdots + p'_{n'})$.

(In fact, since **GrLie**(k) is concentrated in degree $1-k$, one might as well consider $X(f; \theta_1, \ldots, \theta_n)$ only when

$$p_1 = 1 - |f^{-1}\{0\}|, \ldots, p_n = 1 - |f^{-1}\{n-1\}|.)$$

This concludes the description of homotopy graded Lie algebras. It would be interesting to compare these structures with L_∞-algebras, just as homotopy d.g. algebras were compared with A_∞-algebras in 3.4.

4.1.2 Gerstenhaber algebras

We have a category **HtyAlg**(**Ger**, **ChCx**) of homotopy Gerstenhaber algebras, where **Ger** is the symmetric **GrAb**-enriched operad of 1.5(g). Various other notions of 'homotopy Gerstenhaber algebra' are laid out in the paper of this title by Voronov, and once again a comparison would be nice but is not attempted. In particular, Deligne's Conjecture implies that the Hochschild cochain complex $C^\bullet(A)$ of an associative algebra A is a 'homotopy Gerstenhaber algebra' in any reasonable sense of the phrase. An interesting challenge, therefore, is to prove directly that $C^\bullet(A)$ is a homotopy Gerstenhaber algebra in our sense of the phrase. A sub-challenge, which does not involve ideas of enrichment, is to show that $C^\bullet(A)$ is a homotopy **CMon**-algebra, i.e. a homotopy d.g. commutative algebra: for any homotopy Gerstenhaber algebra is certainly a homotopy **CMon**-algebra. Roughly speaking, this means:

Problem Given an associative algebra A over R, find a functor X from Φ (the skeletal category of finite sets) to $\mathbf{ChCx}_R$ (the category of chain complexes over R), such that

- $X(1) \cong C^\bullet(A)$
- there is a canonical chain homotopy equivalence

$$X(0) \longrightarrow R$$

- there is a canonical chain homotopy equivalence

$$X(m+n) \longrightarrow X(m) \otimes X(n)$$

  for each $m, n \geq 0$.

(For an account of Deligne's Conjecture, see [Kon]. Following a tangled history of proofs and refutations, it appears that the Conjecture is now a Theorem.)

4.1.3 Monoids with involution

We have shown that any loop space is a homotopy topological monoid (3.2), in the sense of being a homotopy algebra in **Top** for the non-symmetric operad

Mon. We have also suggested (p. 46) that a loop space is a homotopy monoid in the sense of being a homotopy algebra for the symmetric operad **Sym**. Since a loop can be travelled backwards, any loop space is also acted on (strictly) by the 2-element group C_2. Combining the two structures, one would therefore expect any loop space to be a homotopy monoid-with-involution: that is, a homotopy **Inv**-algebra in $(\mathbf{Top}, \times, 1)$, where **Inv** is the symmetric operad defined in 1.2(h). I do not know whether this is, in fact, true. It would be enough to show that S^1 is a homotopy monoid-with-involution in $(\mathbf{Top}_*^{\mathrm{op}}, \vee, 1)$, i.e. to construct a suitable colax symmetric monoidal functor

$$(W, \omega) : (\widehat{\mathbf{Inv}}, +, 0) \longrightarrow (\mathbf{Top}_*^{\mathrm{op}}, \vee, 1)$$

with $W(1) = S^1$; for then we could compose with the functor $\mathbf{Top}_*(\text{—}, B)$, as in 3.2.

4.1.4 Homotopy homotopy algebras

I do not know what to make of the following bizarre family of examples. Consider, for instance, the notion of a homotopy Lie algebra: that is, a chain complex which is a graded Lie algebra 'up to higher homotopy'. The present work formalizes this idea as a homotopy **GrLie**-algebra in **ChCx**; on the other hand, the usual way to formalize it is as an L_∞-algebra. An L_∞-algebra is an algebra in **ChCx** for a certain **GrAb**-enriched operad, L_∞. But this means that we can also consider *homotopy* L_∞-algebras in **ChCx**, i.e. the category

$$\mathbf{HtyAlg}(L_\infty, \mathbf{ChCx}).$$

A homotopy L_∞-algebra is then a 'homotopy homotopy Lie algebra'. Similarly, one might consider homotopy A_∞-, B_∞-, C_∞-, G_∞-, ... algebras, which are all 'homotopy homotopy algebras'. It would perhaps be desirable to show that in some sense, any homotopy homotopy algebra is in fact a mere homotopy algebra: 'working up to homotopy is idempotent'.

4.2 Iterated Loop Spaces

We have already seen that any loop space is a homotopy monoid in **Top**. We have also seen that a homotopy commutative monoid in **Top** is the same thing as a special Γ-space, and the relation between special Γ-spaces and infinite loop spaces is well-established (see e.g. [Ad]). One might therefore ask what structure an n-fold loop space carries, in our theory, when $1 < n < \infty$. This section provides an answer.

Before going into detail, here is a sketch of the ideas. Ultimately we will show that any n-fold loop space 'has n commuting homotopy monoid structures on it', or is 'an n-fold homotopy monoid'. To state this precisely we need some general definitions. Given operads (P_i), a '$(P_1, \ldots, P_n)$-algebra' in a symmetric monoidal category $\mathcal{M}$ is an object A of $\mathcal{M}$ which is an algebra for each of the operads P_i, in such a way that the algebra structures commute with each other. An extension of Theorem 1.6.1 says that $(P_1, \ldots, P_n)$-algebras in $\mathcal{M}$ are essentially the same as multi-monoidal functors

$$\widehat{P_1} \times \cdots \times \widehat{P_n} \longrightarrow \mathcal{M},$$

where 'multi-monoidal functors' are to monoidal functors as multilinear maps are to linear maps. A 'homotopy $(P_1, \ldots, P_n)$-algebra' is defined as a *colax* multi-monoidal functor (X, ξ) in which the components of ξ are equivalences. We will show that the n-sphere S^n is a homotopy $(\mathbf{Mon}, \ldots, \mathbf{Mon})$-algebra in $(\mathbf{Top}_*^{\mathrm{op}}, \vee, 1)$, and it follows easily that any n-fold loop space is a homotopy $(\mathbf{Mon}, \ldots, \mathbf{Mon})$-algebra in $(\mathbf{Top}, \times, 1)$.

In detail: let $n \geq 0$, let $\mathcal{L}_1, \ldots, \mathcal{L}_n$ be monoidal categories, and let $\mathcal{M}$ be a symmetric monoidal category. A *colax multi-monoidal functor*[1]

$$(X, \xi) : \mathcal{L}_1 \times \cdots \times \mathcal{L}_n \longrightarrow \mathcal{M}$$

consists of

- a functor $X : \mathcal{L}_1 \times \cdots \times \mathcal{L}_n \longrightarrow \mathcal{M}$
- a map
$$\xi_{\ldots, L_{i-1}, 0, L_{i+1}, \ldots} : X(\ldots, L_{i-1}, I, L_{i+1}, \ldots) \longrightarrow I$$
for each $i \in \{1, \ldots n\}$ and $L_j \in \mathcal{L}_j$ $(j \neq i)$
- a map
$$\begin{aligned} \xi_{\ldots, L_{i-1}, (L_i, L_i'), L_{i+1}, \ldots} : X(\ldots, L_{i-1}, L_i \otimes L_i', L_{i+1}, \ldots) \longrightarrow \\ X(\ldots, L_{i-1}, L_i, L_{i+1}, \ldots) \otimes X(\ldots, L_{i-1}, L_i', L_{i+1}, \ldots) \end{aligned}$$
for each $i \in \{1, \ldots, n\}$, $L_j \in \mathcal{L}_j$, and $L_i' \in \mathcal{L}_i$.

These maps are required to be natural in all the components L_j and to obey the usual colax monoidal functor axioms componentwise, so that for each i and $L_1, \ldots, L_{i-1}, L_{i+1}, \ldots, L_n$, the pair

$$(X(\ldots, L_{i-1}, —, L_{i+1}, \ldots), \xi_{\ldots, L_{i-1}, —, L_{i+1}, \ldots})$$

forms a colax monoidal functor $\mathcal{L}_i \longrightarrow \mathcal{M}$. They are also required to commute with one another, which means that if $1 \leq i < j \leq n$ then the four diagrams in Figure 4A commute. In the figure, $\mathbf{X}(J, K)$ is an abbreviation for

$$X(L_1, \ldots, L_{i-1}, J, L_{i+1}, \ldots, L_{j-1}, K, L_{j+1}, \ldots, L_n),$$

and the diagrams are required to commute for all $L_k \in \mathcal{L}_k$ $(1 \leq k \leq n)$ and $L_i' \in \mathcal{L}_i, L_j' \in \mathcal{L}_j$. All the arrows in the diagrams are the obvious components of ξ, except for the three labelled as isomorphisms. There is an obvious notion

[1] with apologies for the name

$$
\begin{tikzcd}
\mathbf{X}(L_i \otimes L_i', L_j \otimes L_j') \arrow[r] \arrow[dd] & \mathbf{X}(L_i, L_j \otimes L_j') \otimes \mathbf{X}(L_i', L_j \otimes L_j') \arrow[d] \\
& \mathbf{X}(L_i, L_j) \otimes \mathbf{X}(L_i, L_j') \otimes \mathbf{X}(L_i', L_j) \otimes \mathbf{X}(L_i', L_j') \arrow[d, "\sim"] \\
\mathbf{X}(L_i \otimes L_i', L_j) \otimes \mathbf{X}(L_i \otimes L_i', L_j') \arrow[r] & \mathbf{X}(L_i, L_j) \otimes \mathbf{X}(L_i', L_j) \otimes \mathbf{X}(L_i, L_j') \otimes \mathbf{X}(L_i', L_j')
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbf{X}(L_i \otimes L_i', I) \arrow[r] \arrow[d] & \mathbf{X}(L_i, I) \otimes \mathbf{X}(L_i', I) \arrow[d] \\
I \arrow[r, "\sim"'] & I \otimes I
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{X}(I, L_j \otimes L_j') \arrow[r] \arrow[d] & I \arrow[d, "\sim"] \\
\mathbf{X}(I, L_j) \otimes \mathbf{X}(I, L_j') \arrow[r] & I \otimes I
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbf{X}(I, I) \arrow[r, shift left, "\xi_{\ldots, I, \ldots, 0, \ldots}"] \arrow[r, shift right, "\xi_{\ldots, 0, \ldots, I, \ldots}"'] & I
\end{tikzcd}
$$

Figure 4A: Commutativity axioms for a colax multi-monoidal functor.

of *monoidal transformation* between colax multi-monoidal functors.

Now let $P_1, \ldots, P_n$ be non-symmetric operads and let $\mathcal{M}$ be a symmetric monoidal category with equivalences. A *homotopy* $(P_1, \ldots, P_n)$*-algebra in* $\mathcal{M}$ is a colax multi-monoidal functor

$$(X, \xi) : \widehat{P_1} \times \cdots \times \widehat{P_n} \longrightarrow \mathcal{M}$$

such that each component of ξ is an equivalence. With monoidal transformations as maps, this gives a category $\mathbf{HtyAlg}(P_1, \ldots, P_n; \mathcal{M})$ of homotopy $(P_1, \ldots, P_n)$-algebras in $\mathcal{M}$.

To see why this is a reasonable definition, consider first 'genuine' $(P_1, \ldots, P_n)$-algebras. If P and Q are two operads, $\mathcal{M}$ a symmetric monoidal category, and A an object of $\mathcal{M}$ endowed with both P-algebra and Q-algebra structures, let us say that the two algebra structures *commute* if for all $\theta \in P(m)$ and $\chi \in Q(k)$, the diagram

$$\begin{tikzcd}
 & (A^{\otimes m})^{\otimes k} \arrow[r, "(\overline{\theta})^{\otimes k}"] & A^{\otimes k} \arrow[dd, "\overline{\chi}"] \\
(A^{\otimes k})^{\otimes m} \arrow[ur, "\text{symmetry}"] \arrow[d, "(\overline{\chi})^{\otimes m}"'] & & \\
A^{\otimes m} \arrow[rr, "\overline{\theta}"'] & & A
\end{tikzcd}$$

commutes. If $P_1, \ldots, P_n$ are operads and $\mathcal{M}$ a symmetric monoidal category, a $(P_1, \ldots, P_n)$*-algebra in* $\mathcal{M}$ is an object A of $\mathcal{M}$ with the structure of a P_i-algebra for each $i \in \{1, \ldots n\}$, such that the P_i-algebra and P_j-algebra structures commute whenever $i \neq j$. One thus obtains a category $\mathbf{Alg}(P_1, \ldots, P_n; \mathcal{M})$ of $(P_1, \ldots, P_n)$-algebras in $\mathcal{M}$.

For example, a $(\mathbf{Sem}, \mathbf{Sem})$-algebra in $\mathbf{Set}$ is a set A equipped with two associative binary operations, $\cdot$ and $*$, satisfying the 'interchange law':

$$(a * b) \cdot (a' * b') = (a \cdot a') * (b \cdot b').$$

Now, take a symmetric monoidal category $\mathcal{M}$ and define the equivalences in $\mathcal{M}$ to be just the isomorphisms. Let $P_1, \ldots, P_n$ be non-symmetric operads. Then $\mathbf{HtyAlg}(P_1, \ldots, P_n; \mathcal{M})$ is $\mathbf{Mon}(P_1, \ldots, P_n; \mathcal{M})$, the category of *multi-monoidal functors* $P_1 \times \cdots \times P_n \longrightarrow \mathcal{M}$: that is, those colax multi-monoidal functors (X, ξ) in which all the components of ξ are isomorphisms. Theorem 1.6.1 can be generalized to give an equivalence of categories

$$\mathbf{Mon}(P_1, \ldots, P_n; \mathcal{M}) \simeq \mathbf{Alg}(P_1, \ldots, P_n; \mathcal{M}),$$

with 1.6.1 being the case $n = 1$; the algebra corresponding to a multi-monoidal functor (X, ξ) has $X(1, \ldots, 1)$ as its underlying object. Hence when $\mathcal{M}$ has only trivial equivalences,

$$\mathbf{HtyAlg}(P_1, \ldots, P_n; \mathcal{M}) \simeq \mathbf{Alg}(P_1, \ldots, P_n; \mathcal{M}),$$

just as in the case $n = 1$.

(All of these definitions can be repeated, with minor modifications, for the case of *symmetric* operads P_i. Moreover, there is no need for the family (P_i) of operads to be finite; everything above works just as well for infinite families.)

Aside An alternative way of making the definitions is to observe that $(\mathbf{S})\mathbf{Colax}(P, \mathcal{M})$ is naturally a symmetric monoidal category with equivalences, for any (symmetric) operad P and symmetric monoidal category with equivalences $\mathcal{M}$. (Tensor and equivalences are defined pointwise.) In fact, the subcategory $\mathbf{HtyAlg}(P, \mathcal{M})$ is also a symmetric monoidal category with equivalences. So we could define

$$\mathbf{HtyAlg}(Q, P; \mathcal{M}) = \mathbf{HtyAlg}(Q, \mathbf{HtyAlg}(P, \mathcal{M}))$$

for any operads Q and P; and we could iterate this idea in order to define homotopy $(P_1, \ldots, P_n)$-algebras for $n > 2$. This definition is equivalent to the one given above.

We can now return to iterated loop spaces. Our result is:

Theorem 4.2.1 *Any n-fold loop space is an n-fold homotopy monoid. That is, if B is a space with basepoint then there is a homotopy $\underbrace{(\mathbf{Mon}, \ldots, \mathbf{Mon})}_{n}$-algebra (X, ξ) in $(\mathbf{Top}, \times, 1)$ with $X(1, \ldots, 1) = \mathbf{Top}_*(S^n, B)$.*

Proof First we show that S^n is an n-fold homotopy comonoid, i.e. a homotopy $(\mathbf{Mon}, \ldots, \mathbf{Mon})$-algebra in $(\mathbf{Top}_*^{\mathrm{op}}, \vee, 1)$. Let

$$(W, \omega) : (\Delta, +, 0) \longrightarrow (\mathbf{Top}_*^{\mathrm{op}}, \vee, 1)$$

be the colax monoidal functor of Lemma 3.2.2, exhibiting S^1 as a homotopy comonoid. Also let

$$Z : \mathbf{Top}_*^n \longrightarrow \mathbf{Top}_*$$

be the n-fold smash product,

$$Z(A_1, \ldots, A_n) = A_1 \wedge \cdots \wedge A_n.$$

Observe that since $\wedge$ distributes over $\vee$, Z naturally becomes a multi-monoidal functor

$$(Z, \zeta) : (\mathbf{Top}_*, \vee, 1)^n \longrightarrow (\mathbf{Top}_*, \vee, 1);$$

observe moreover that Z preserves homotopy equivalences. Assembling all of this, we get a composite

$$(\Delta, +, 0)^n \xrightarrow{(W,\omega)^n} (\mathbf{Top}_*^{\mathrm{op}}, \vee, 1)^n \xrightarrow{(Z,\zeta)} (\mathbf{Top}_*^{\mathrm{op}}, \vee, 1),$$

and this is a colax multi-monoidal functor (S, σ) in which the components of σ are equivalences. Thus (S, σ) defines an n-fold homotopy comonoid, and

$$\begin{aligned} S(1, \ldots, 1) &= W(1) \wedge \cdots \wedge W(1) \\ &= S^1 \wedge \cdots \wedge S^1 \\ &= S^n \end{aligned}$$

as required.

To finish the proof we simply use the observation made in 3.2 that $\mathbf{Top}_*(—, B)$ defines a homotopy-preserving monoidal functor

$$(\mathbf{Top}_*^{\mathrm{op}}, \vee, 1) \longrightarrow (\mathbf{Top}, \times, 1).$$

Composing this with

$$(S, \sigma) : (\Delta, +, 0)^n \longrightarrow (\mathbf{Top}_*^{\mathrm{op}}, \vee, 1)$$

yields an n-fold homotopy monoid

$$(\widehat{\mathbf{Mon}}, +, 0)^n = (\Delta, +, 0)^n \longrightarrow (\mathbf{Top}, \times, 1),$$

whose value at $(1, \ldots, 1)$ is $\mathbf{Top}_*(S^n, B)$. $\square$

The Theorem swiftly implies that the higher homotopy groups of a space are abelian, as we see in 5.2(g).

4.3 Inside a Monoidal 2-Category

In 3.3 we saw how a homotopy monoidal category gave rise to an ordinary monoidal category, and generalized as follows: if $\mathcal{N}$ is any monoidal 2-category and $|\mathcal{N}|$ the associated monoidal category with equivalences, then a homotopy monoid in $|\mathcal{N}|$ gives rise to a weak monoid in $\mathcal{N}$. (This allowed us to deduce comparison results involving A_n-spaces and A_n-algebras.)

Here we show that this process works not just for homotopy monoids but for homotopy algebras for any operad P. Thus if $\mathcal{N}$ is a monoidal 2-category, there is a concept of 'weak P-algebra' in $\mathcal{N}$, and any homotopy P-algebra in $|\mathcal{N}|$ gives rise to one of these. Most of the section will in fact be devoted to defining weak algebras. Seasoned category theorists can probably imagine the kind of definition this is. Once the definition is made, the actual result is quite easily proved.

So, let P be a non-symmetric operad and $\mathcal{N}$ a monoidal 2-category. Let $|\mathcal{N}|$ denote the underlying monoidal category of $\mathcal{N}$. If A and B are objects of $\mathcal{N}$ then there is a category $\mathcal{N}(A, B)$, whose objects are the 1-cells $A \longrightarrow B$ in $\mathcal{N}$ and whose morphisms are 2-cells; there is also a mere set $|\mathcal{N}|(A, B)$, whose elements are the 1-cells $A \longrightarrow B$. A P-algebra in $|\mathcal{N}|$ consists of an object A

together with a function $P(n) \longrightarrow |\mathcal{N}|(A^{\otimes n}, A)$ for each n, satisfying axioms; by weakening the axioms on these functions we arrive at the following definition. A *weak P-algebra in $\mathcal{N}$* consists of

- an object A of $\mathcal{N}$
- for each $n \in \mathbb{N}$ and $\theta \in P(n)$, a 1-cell
$$\overline{\theta} : A^{\otimes n} \longrightarrow A$$
in $\mathcal{N}$
- for each
$$\theta \in P(n), \theta_1 \in P(k_1), \ldots, \theta_n \in P(k_n),$$
an invertible 2-cell
$$\begin{tikzcd} A^{\otimes(k_1+\cdots+k_n)} \arrow[r, bend left=50, "\overline{\theta}\circ(\overline{\theta_1}\otimes\cdots\otimes\overline{\theta_n})"] \arrow[r, bend right=50, "\overline{\theta\circ(\theta_1,\ \ldots\ ,\theta_n)}"'] \arrow[r, phantom, "\Downarrow\wr"] & A \end{tikzcd}$$
in N, called $\gamma_{\theta;\theta_1,\ldots,\theta_n}$
- an invertible 2-cell
$$\begin{tikzcd} A \arrow[r, bend left=50, "1_A"] \arrow[r, bend right=50, "\overline{1_P}"'] \arrow[r, phantom, "\Downarrow\wr"] & A \end{tikzcd}$$
in $\mathcal{N}$, called ι.

Then γ and ι are required to satisfy coherence axioms looking like associativity and identity laws, as detailed in Figure 4B.

If A and B are weak P-algebras in $\mathcal{N}$ then a *weak map* from A to B consists of

- a 1-cell $f : A \longrightarrow B$
- for each $\theta \in P(n)$, an invertible 2-cell
$$\begin{tikzcd} A^{\otimes n} \arrow[r, bend left=50, "\overline{\theta}\circ f^{\otimes n}"] \arrow[r, bend right=50, "f\circ\overline{\theta}"'] \arrow[r, phantom, "\Downarrow\wr"] & B \end{tikzcd}$$
in $\mathcal{N}$, called ϕ_θ.

$$
\begin{tikzcd}
\overline{\theta}\circ(\overline{\theta_1}\otimes\cdots\otimes\overline{\theta_n})\circ(\overline{\theta_1^1}\otimes\cdots\otimes\overline{\theta_n^{k_n}}) \arrow[r, equal] \arrow[d] & \overline{\theta}\circ(\theta_1\circ(\overline{\theta_1^1}\otimes\cdots\otimes\overline{\theta_1^{k_1}})\otimes\cdots\otimes\theta_n\circ(\overline{\theta_n^1}\otimes\cdots\otimes\overline{\theta_n^{k_n}})) \arrow[d] \\
\overline{\theta\circ(\theta_1,\ldots,\theta_n)}\circ(\overline{\theta_1^1}\otimes\cdots\otimes\overline{\theta_n^{k_n}}) \arrow[d] & \overline{\theta}\circ(\overline{\theta_1\circ(\theta_1^1,\ldots,\theta_1^{k_1})}\otimes\cdots\otimes\overline{\theta_n\circ(\theta_n^1,\ldots,\theta_n^{k_n})}) \arrow[d] \\
\overline{\theta\circ(\theta_1,\ldots,\theta_n)\circ(\theta_1^1,\ldots,\theta_n^{k_n})} \arrow[r, equal] & \overline{\theta\circ(\theta_1\circ(\theta_1^1,\ldots,\theta_1^{k_1}),\ldots,\theta_n\circ(\theta_n^1,\ldots,\theta_n^{k_n}))}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\overline{\theta}\circ(1_A\otimes\cdots\otimes 1_A) \arrow[r, equal] \arrow[d] & \overline{\theta} \arrow[d, equal] \\
\overline{\theta}\circ(\overline{1_P}\otimes\cdots\otimes\overline{1_P}) \arrow[r] & \overline{\theta\circ(1_P,\ldots,1_P)}
\end{tikzcd}
\qquad
\begin{tikzcd}
1_A\circ\overline{\theta} \arrow[r, equal] \arrow[d] & \overline{\theta} \arrow[d, equal] \\
\overline{1_P}\circ\overline{\theta} \arrow[r] & \overline{1_P\circ\theta}
\end{tikzcd}
$$

Figure 4B: Coherence axioms for a weak P-algebra: these diagrams must commute. Here $\theta \in P(n)$, $\theta_i \in P(k_i)$, and all arrows are part(s) of γ or ι.

$$
\begin{tikzcd}
\overline{\theta}\circ(\overline{\theta_1}\otimes\cdots\otimes\overline{\theta_n})\circ f^{\otimes(k_1+\cdots+k_n)} \arrow[r] \arrow[d, equal] & \overline{\theta\circ(\theta_1,\,\ldots\,,\theta_n)}\circ f^{\otimes(k_1+\cdots+k_n)} \arrow[dddd] \\
\overline{\theta}\circ(\overline{\theta_1}\circ f^{\otimes k_1}\otimes\cdots\otimes\overline{\theta_n}\circ f^{\otimes k_n}) \arrow[d] & \\
\overline{\theta}\circ(f\circ\overline{\theta_1}\otimes\cdots\otimes f\circ\overline{\theta_n}) \arrow[d, equal] & \\
\overline{\theta}\circ f^{\otimes n}\circ(\overline{\theta_1}\otimes\cdots\otimes\overline{\theta_n}) \arrow[d] & \\
f\circ\overline{\theta}\circ(\overline{\theta_1}\otimes\cdots\otimes\overline{\theta_n}) \arrow[r] & f\circ\overline{\theta\circ(\theta_1,\,\ldots\,,\theta_n)}
\end{tikzcd}
$$

$$
\begin{tikzcd}
1_B\circ f \arrow[r] \arrow[d, equal] & \overline{1_P}\circ f \arrow[dd] \\
f \arrow[d, equal] & \\
f\circ 1_A \arrow[r] & f\circ\overline{1_P}
\end{tikzcd}
$$

Figure 4C: Coherence axioms for a weak map of weak P-algebras. Vertical arrows come from components of ϕ, and horizontal arrows from γ or ι.

Of course, ϕ is required to satisfy coherence axioms, as shown in Figure 4C. We thus arrive at a category $\mathbf{WkAlg}(P, \mathcal{N})$ of weak P-algebras in $\mathcal{N}$.

(Australian category theorists have used this style of definition extensively in the study of two-dimensional algebra: see [BKP], for instance.)

As a motivating example, consider $P = \mathbf{Mon}$ and $\mathcal{N} = \mathbf{Cat}$. A weak **Mon**-algebra in **Cat** consists of a category A and a functor

$$\otimes_n : A^n \longrightarrow A$$

for each $n \geq 0$, which we think of as n-fold tensor, together with some coherence data. Morally this is the same thing as a monoidal category in the traditional sense, the only difference being that the traditional definition gives a special role to the values 0 and 2 of n. In fact, the categories $\mathbf{WkAlg}(\mathbf{Mon}, \mathbf{Cat})$ and **MonCat** (= monoidal categories and monoidal functors) are equivalent. I hope to write this result up sometime soon; meanwhile, some related considerations can be found in [Lei3, 4.4] and [Lei2, p. 8]. More generally, $\mathbf{WkAlg}(\mathbf{Mon}, \mathcal{N})$ is equivalent to the category of weak monoids in $\mathcal{N}$ (as defined on page 56) for any monoidal 2-category $\mathcal{N}$.

We now have the language in which to state and prove the main result. Just as in the case of homotopy monoids in **Cat**, there are some issues concerning arbitrary choices, which can be dealt with as they were then; we do not give them further attention here.

Theorem 4.3.1 *Let P be a non-symmetric operad, let $\mathcal{N}$ be a monoidal 2-category, and let $|\mathcal{N}|$ be the associated monoidal category with equivalences (as in 2.1(k)). Then there is a functor*

$$\mathbf{HtyAlg}(P, |\mathcal{N}|) \longrightarrow \mathbf{WkAlg}(P, \mathcal{N})$$

sending (X, ξ) to a weak algebra with underlying object $X(1)$.

Sketch Proof Take a homotopy P-algebra in $|\mathcal{N}|$,

$$(X, \xi) : (\widehat{P}, +, 0) \longrightarrow (|\mathcal{N}|, \otimes, I).$$

For each $n, k_1, \ldots, k_n \in \mathbb{N}$, there is a 1-cell

$$\xi_{k_1,\ldots,k_n} : X(k_1 + \cdots + k_n) \longrightarrow X(k_1) \otimes \cdots \otimes X(k_n),$$

as in Example 2.3(c). By Lemma 3.3.6, we can choose a 1-cell $\psi_{k_1,\ldots,k_n}$ and 2-cells $\eta_{k_1,\ldots,k_n}, \varepsilon_{k_1,\ldots,k_n}$ so that

$$(\psi_{k_1,\ldots,k_n}, \xi_{k_1,\ldots,k_n}, \eta_{k_1,\ldots,k_n}, \varepsilon_{k_1,\ldots,k_n})$$

is an adjoint equivalence. When $k_1 = \cdots = k_n$, we write $\psi_{k_1,\ldots,k_n}$ as $\psi^{(n)}$.

A weak P-algebra structure on the object $X(1)$ of $\mathcal{N}$ can now be defined as follows: if $\theta \in P(n)$, then $\overline{\theta}$ is the composite

$$X(1)^{\otimes n} \xrightarrow{\psi^{(n)}} X(n) \xrightarrow{X(\theta)} X(1),$$

and the invertible 2-cells γ and ι are built up from $\eta_{k_1,\ldots,k_n}$'s and $\varepsilon_{k_1,\ldots,k_n}$'s. The process for maps works similarly. $\square$

Once again all of this ought to be repeatable, *mutatis mutandis*, in the symmetric case.

4.4 Inside Cat

We have looked at how a homotopy algebra gives rise to a weak algebra in various different contexts (3.3, 3.4, 3.5, 4.3), and it is natural to wonder whether there is a converse process. This section provides a partial answer: we show how a weak P-algebra in **Cat** naturally gives rise to a homotopy P-algebra in **Cat**, for any operad P.

Before explaining how this works, let me say some things about the current incompleteness of this line of thought. Firstly, I do not know how to repeat the construction in any environment other than **Cat** (e.g. in an arbitrary monoidal 2-category). Secondly, I have only tried to make all the proper checks in the case of non-symmetric operads P; the symmetric case is largely undone. Thirdly, I have not looked seriously at whether a weak map of weak algebras gives rise to a map of homotopy algebras (and in this connection, see the remarks just before Proposition 3.3.2). Finally, and perhaps most importantly, I have not investigated the relation between the two processes

$$(\text{weak } P\text{-algebras}) \rightleftarrows (\text{homotopy } P\text{-algebras})$$

described in this and the previous section. Do they, for instance, form an adjunction, or an equivalence, or a 'weak equivalence' of some kind?

The idea behind this section goes as follows. Suppose that $P = \mathbf{CMon}$, so that our task is to define a function

$$(\text{symmetric monoidal categories}) \longrightarrow (\text{homotopy symmetric monoidal categories}).$$

For instance, given the symmetric monoidal category $(\mathbf{Ab}, \otimes, \mathbb{Z})$, we want to define a homotopy commutative monoid

$$(X, \xi) : (\Phi, +, 0) \longrightarrow (\mathbf{Cat}, \times, 1)$$

with $X(1) \simeq \mathbf{Ab}$. Recalling the introduction to Section 3.3, let us define:

- $X(1) = \mathbf{Ab}$

- $X(2)$ has objects all quadruples (L_1, L_2, u, M) in which L_1, L_2, M are abelian groups and $u : L_1 \times L_2 \longrightarrow M$ is a bilinear map with the universal property for a tensor product
- the functor $X(!) : X(2) \longrightarrow X(1)$ (induced by the map $! : 2 \longrightarrow 1$ in Φ) sends (L_1, L_2, u, M) to M
- the equivalence $\xi_{1,1} : X(2) \longrightarrow X(1) \times X(1)$ sends (L_1, L_2, u, M) to (L_1, L_2).

Continuing to build the definition in this way, we would obtain a homotopy symmetric monoidal category (X, ξ). More truthfully, we would obtain one had I not made certain simplifications in the descriptions of $X(1)$ and $X(2)$: they are inaccurate in various respects, e.g. we have not paid enough attention to the unit for the tensor; but they serve to convey the main idea. This explanation comes from Segal's paper [Seg2].

Now let me try to explain the idea for general operads P; I will concentrate on the case of non-symmetric operads. Let A be a weak P-algebra, and attempt to construct a homotopy P-algebra (X, ξ). With **Ab** above, an object of $X(2)$ was effectively an object (L_1, L_2) of $\mathbf{Ab} \times \mathbf{Ab}$ together with an isomorphism

$$j : L_1 \otimes L_2 \xrightarrow{\sim} M$$

between the 'official' tensor product $L_1 \otimes L_2$ (which is part of the structure of the monoidal category $(\mathbf{Ab}, \otimes, \mathbb{Z})$) and the 'putative' tensor product M of L_1, L_2 (which is the image of (L_1, L_2, u, M) under $X(!)$). So an object of $X(2)$ consists of an object (L_1, L_2) of $\mathbf{Ab}^2$ together with data specifying 'what happens to (L_1, L_2) under every operation of **CMon**'. Inspired by this, an object of $X(n)$ will consist of an object $(a_1, \ldots, a_n)$ of A^n, together with an object of A for every substring $(a_{d+1}, \ldots, a_{m+d})$ of $(a_1, \ldots, a_n)$ and every m-ary operation $\theta \in P(m)$. Call this object $\theta\langle a_{d+1}, \ldots, a_{m+d}\rangle$; it plays the role of M in the **Ab** example. Then there must also be isomorphisms such as

$$\begin{aligned} \overline{\theta}(a_{d+1}, \ldots, a_{m+d}) &\cong \theta\langle a_{d+1}, \ldots, a_{m+d}\rangle, \\ a_{d+1} &\cong 1_P\langle a_{d+1}\rangle, \end{aligned}$$

and a choice of such isomorphisms is also included in the data for the object of $X(n)$. A proper description of $X(n)$ is given in the proof of the Proposition below.

(Since we are treating the case of *non*-symmetric operads, it is not in the spirit of things to take arbitrary subsets of $\{1, \ldots, n\}$ without regard to order, which is why we restrict to 'substrings' $\{d+1, \ldots, m+d\}$. See also the comments at the end of the section.)

We now come to the main result.

Proposition 4.4.1 *Let P be a non-symmetric operad. Then any weak P-algebra in* **Cat** *gives rise to a homotopy P-algebra in* **Cat**. *More precisely, given a weak P-algebra in* **Cat** *with underlying category A, there is an associated homotopy P-algebra (X, ξ) in* **Cat** *with $X(1) \simeq A$.*

Remark This process is canonical, in that the construction of (X, ξ) from A does not involve arbitrary choices. Contrast the converse process (Theorem 4.3.1).

Proof Take a weak P-algebra in **Cat**, consisting of a category A, 1-cells $\overline{\theta}$, and 2-cells γ and ι, as in the definition (4.3). We construct a homotopy P-algebra $(X, \xi) : \widehat{P} \longrightarrow \mathbf{Cat}$.

To do this, we first we need to construct a category $X(n)$ for each $n \geq 0$. An *object* of $X(n)$ is a tuple

$$(a_1, \ldots, a_n, \Omega, g, i),$$

where

- $a_1, \ldots, a_n$ are objects of A
- Ω is a function assigning an object $\Omega(\theta, d)$ of A to each $m \in \mathbb{N}$, $\theta \in P(m)$, and $d \in \mathbb{N}$ with $m + d \leq n$; it is convenient to write $\Omega(\theta, d)$ as $\theta\langle a_{d+1}, \ldots, a_{m+d}\rangle$ (which must be interpreted as a purely formal expression)
- g is a family of isomorphisms

  $$g_{\theta,\theta_1,\ldots,\theta_n,d} : \overline{\theta}(\theta_1\langle a_{d+1}, \ldots, a_{k_1+d}\rangle, \ \ldots\ , \theta_m\langle a_{k_1+\cdots+k_{m-1}+d+1}, \ldots, a_{k_1+\cdots+k_m+d}\rangle) \xrightarrow{\sim} (\theta\circ(\theta_1, \ \ldots\ , \theta_m))\langle a_{d+1}, \ldots, a_{k_1+\cdots+k_m+d}\rangle$$

  in A, one for each $m, k_1, \ldots, k_m \in \mathbb{N}$, $\theta \in P(m)$, $\theta_1 \in P(k_1)$, $\ldots$, $\theta_m \in P(k_m)$, and $d \in \mathbb{N}$ with $k_1 + \cdots + k_m + d \leq n$
- i is a family of isomorphisms

  $$i_d : a_d \xrightarrow{\sim} 1_P\langle a_d\rangle,$$

  one for each $d \in \{1, \ldots, n\}$,

and g and i satisfy coherence axioms looking like those in Figure 4B (page 76), replacing some of the γ's and ι's in the Figure with g's and i's. A *morphism*

$$f : (a_1, \ldots, a_n, \Omega, g, i) \longrightarrow (a'_1, \ldots, a'_n, \Omega', g', i')$$

in $X(n)$ consists of

- maps $f_1 : a_1 \longrightarrow a'_1$, $\ldots$, $f_n : a_n \longrightarrow a'_n$ in A
- a map

  $$f_{\theta,d} : \theta\langle a_{d+1}, \ldots, a_{m+d}\rangle \longrightarrow \theta\langle a'_{d+1}, \ldots, a'_{m+d}\rangle$$

  (that is, $f_{\theta,d} : \Omega(\theta, d) \longrightarrow \Omega'(\theta, d)$) for each $m \in \mathbb{N}$, $\theta \in P(m)$ and $d \in \mathbb{N}$ with $m + d \leq n$, such that the $f_{\theta,d}$'s commute with the g's and the i's. With the obvious composition and identities, $X(n)$ forms a category.

Now that $X(n)$ is defined, there is only one sensible way to define the rest of the data for (X, ξ), and I will just sketch it.

To define X on morphisms, take a map $\Psi : n \longrightarrow p$ in $\widehat{P}$, which consists of an expression $n = n_1 + \cdots + n_p$ together with elements

$$\psi_1 \in P(n_1), \ldots, \psi_p \in P(n_p).$$

Take an object $(a_1, \ldots, a_n, \Omega, g, i)$ of $X(n)$. Then there is a resulting object

$$(X\Psi)(a_1, \ldots, a_n, \Omega, g, i) = (b_1, \ldots, b_p, \widetilde{\Omega}, \widetilde{g}, \widetilde{i})$$

of $X(p)$, in which

$$(b_1, \ldots, b_p) = (\psi_1\langle a_1, \ldots, a_{n_1}\rangle, \ldots, \psi_p\langle a_{n_1+\cdots+n_{p-1}+1}, \ldots, a_{n_1+\cdots+n_p}\rangle).$$

The data for ξ consists of a pair of maps

$$X(n) \xleftarrow{\xi^1_{n,n'}} X(n+n') \xrightarrow{\xi^2_{n,n'}} X(n')$$

for each $n, n' \in \mathbb{N}$. The image under $\xi^1_{n,n'}$ of an object

$$(a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+n'}, \Omega, g, i)$$

of $X(n+n')$ is of the form $(a_1, \ldots, a_n, ?, ?, ?)$, and the image under $\xi^2_{n,n'}$ is of the form $(a_{n+1}, \ldots, a_{n+n'}, ?, ?, ?)$.

Once all the details are filled in, we arrive at a colax monoidal functor

$$(X, \xi) : \widehat{P} \longrightarrow \mathbf{Cat}.$$

The remaining tasks are to show that this is in fact a homotopy P-algebra—that is, the maps $\xi_{n,n'}$ and ξ_0 are equivalences—and that $X(1) \simeq A$. We do both these things at once by considering the forgetful functors

$$\begin{array}{cccc} U_n : & X(n) & \longrightarrow & A^n \\ & (a_1, \ldots, a_n, \Omega, g, i) & \longmapsto & (a_1, \ldots, a_n). \end{array}$$

Evidently, the squares

$$\begin{tikzcd} X(n+n') \arrow[r, "\xi_{n,n'}"] \arrow[d, "U_{n+n'}"'] & X(n) \times X(n') \arrow[d, "U_n \times U_{n'}"] \\ A^{n+n'} \arrow[r, "\sim"'] & A^n \times A^{n'} \end{tikzcd} \qquad \begin{tikzcd} X(0) \arrow[r, "\xi_0"] \arrow[d, "U_0"'] & \mathbf{1} \arrow[d, equal] \\ A^0 \arrow[r, "\sim"'] & \mathbf{1} \end{tikzcd}$$

both commute, so we will be finished if we can show that each functor U_n is an equivalence. And this in turn is easily accomplished: for U_n has a canonical pseudo-inverse, sending $(a_1, \ldots, a_n)$ to an object $(a_1, \ldots, a_n, \Omega, g, i)$ of $X(n)$. Here

$$\Omega(\theta, d) = \overline{\theta}(a_{d+1}, \ldots, a_{m+d}),$$

or put another way,

$$\theta\langle a_{d+1}, \ldots, a_{m+d}\rangle = \overline{\theta}(a_{d+1}, \ldots, a_{m+d}),$$

and g and i are respectively defined by γ and ι. □

I believe that the Proposition can be repeated for the case of symmetric operads. To do this one would replace substrings $(d+1, \ldots, m+d)$ of $(1, \ldots, n)$ with arbitrary non-repeating sequences $(d_1, \ldots, d_m)$, with $d_i \in \{1, \ldots, n\}$; but I have not verified this yet.

Chapter 5

Change of Environment

Throughout this work we have discussed homotopy P-algebras in $\mathcal{M}$ for a *fixed* operad P and a *fixed* monoidal category $\mathcal{M}$ with equivalences. In this short chapter we look at what happens when the 'environment' $\mathcal{M}$ is varied. In other words, we look at how a suitable map $\mathcal{L} \longrightarrow \mathcal{M}$ induces a functor

$$\mathbf{HtyAlg}(P, \mathcal{L}) \longrightarrow \mathbf{HtyAlg}(P, \mathcal{M}),$$

for any operad P.

It is not difficult to say precisely how this process works (see 5.1 below), but first let me try to explain why such a result is plausible.

From a formal point of view, a homotopy P-algebra in $\mathcal{L}$ is a certain kind of map $\widehat{P} \longrightarrow \mathcal{L}$, so composing with the right kind of map $\mathcal{L} \longrightarrow \mathcal{M}$ ought to yield a homotopy P-algebra in $\mathcal{M}$.

From another point of view, consider, for instance, the path-components functor

$$\pi_0 : \mathbf{Top} \longrightarrow \mathbf{Set}.$$

Since π_0 preserves products, and 'group objects' can be formed in any category in which products exist, π_0 induces a functor

$$\text{(topological groups)} \longrightarrow \text{(groups)}.$$

More generally, if we have fixed an algebraic theory (groups, in this case) then any functor $\mathcal{C} \longrightarrow \mathcal{D}$ of the right kind will induce a functor

$$\text{(algebras in } \mathcal{C}) \longrightarrow \text{(algebras in } \mathcal{D}).$$

Since a homotopy P-algebra is some kind of algebraic structure (albeit a rather loose kind), we might expect the same principle to apply; it does.

From a third point of view, topologists will expect results such as 'the classifying space of a monoidal category is a homotopy monoid'. This is indeed the case in our theory, as long as we read 'monoidal category' as 'homotopy monoidal category': see Example 5.2(d).

We could also change operad: a map $P \longrightarrow Q$ of operads induces a functor

$$\mathbf{HtyAlg}(P, \mathcal{M}) \longleftarrow \mathbf{HtyAlg}(Q, \mathcal{M}).$$

But this will not be discussed here.

Section 5.1 sets out exactly how a 'change of environment' induces a functor between categories of homotopy algebras, and Section 5.2 lists some examples.

5.1 The Principle

Definition 5.1.1 *a. Let $\mathcal{L}$ and $\mathcal{M}$ be monoidal categories with equivalences. A* homotopy monoidal functor *$\mathcal{L} \longrightarrow \mathcal{M}$ is a colax monoidal functor (F, ϕ) such that*

- *each component $\phi_0, \phi_{m,n}$ of ϕ is an equivalence in $\mathcal{M}$*
- *if f is an equivalence in $\mathcal{L}$ then $F(f)$ is an equivalence in $\mathcal{M}$.*

b. Homotopy symmetric monoidal functors *are defined by changing 'monoidal' to 'symmetric monoidal' throughout part (a).*

Note that if P is an operad and the (symmetric) monoidal category $\widehat{P}$ is equipped with isomorphisms as its equivalences, then a homotopy P-algebra in $\mathcal{L}$ is exactly a homotopy (symmetric) monoidal functor $\widehat{P} \longrightarrow \mathcal{L}$. Note also that the composite of two homotopy (symmetric) monoidal functors is a homotopy (symmetric) monoidal functor. Hence a homotopy (symmetric) monoidal functor $\mathcal{L} \longrightarrow \mathcal{M}$ induces a functor

$$\mathbf{HtyAlg}(P, \mathcal{L}) \longrightarrow \mathbf{HtyAlg}(P, \mathcal{M}).$$

This simple piece of theory is the basis of this chapter, the remainder of which consists of examples.

5.2 Examples

a. Suppose that $\mathcal{M}$ is a monoidal category and that $\mathcal{E}$ and $\mathcal{E}'$ are both classes of equivalences in $\mathcal{M}$, with $\mathcal{E}' \subseteq \mathcal{E}$. Then the identity is a homotopy monoidal functor $(\mathcal{M}, \mathcal{E}') \longrightarrow (\mathcal{M}, \mathcal{E})$. Thus if P is any operad then there is an induced functor

$$\mathbf{HtyAlg}(P, (\mathcal{M}, \mathcal{E}')) \longrightarrow \mathbf{HtyAlg}(P, (\mathcal{M}, \mathcal{E}))$$

(with what I hope is self-explanatory notation), and this is the obvious inclusion. In particular, if $\mathcal{E}' = \{\text{isomorphisms}\}$ then this is the inclusion

$$\mathbf{Alg}(P, \mathcal{M}) \longrightarrow \mathbf{HtyAlg}(P, (\mathcal{M}, \mathcal{E})).$$

b. Let **Toph** be the category whose objects are topological spaces and whose morphisms are homotopy classes of continuous maps. Let P be an operad. The weakest possible meaning of the phrase 'homotopy topological P-algebra' is 'P-algebra in **Toph**': e.g. a 'homotopy topological semigroup' in this weakest sense is just a space A with a binary operation which is associative up to homotopy. Any homotopy P-algebra in the sense of this paper certainly gives rise to one of these very weak structures. Formally, let $Q : \mathbf{Top} \longrightarrow \mathbf{Toph}$ be the quotient functor (which is the identity on objects). Equip **Toph** with just the isomorphisms as its equivalences. Then Q becomes a homotopy symmetric monoidal functor

$$(\mathbf{Top}, \times, 1) \longrightarrow (\mathbf{Toph}, \times, 1),$$

so for any operad P there is an induced functor

$$\mathbf{HtyAlg}(P, \mathbf{Top}) \longrightarrow \mathbf{HtyAlg}(P, \mathbf{Toph}) \simeq \mathbf{Alg}(P, \mathbf{Toph}).$$

c. Example (b) can be repeated with chain complexes in place of spaces, or with categories in place of spaces (with natural isomorphism classes of functors), or indeed with the objects of any monoidal 2-category.

d. Let $B : \mathbf{Cat} \longrightarrow \mathbf{Top}$ be the classifying-space functor (see [Seg1]). Then B preserves products and sends equivalences to homotopy equivalences, so there is an induced functor

$$B : \mathbf{HtyAlg}(P, \mathbf{Cat}) \longrightarrow \mathbf{HtyAlg}(P, \mathbf{Top})$$

for any operad P. (Here **Top** is equipped with the cartesian monoidal structure.) For instance, let C be a homotopy monoidal category, i.e. a homotopy monoid in **Cat**: then BC is a homotopy topological monoid. Similarly, the classifying space BC of a homotopy symmetric monoidal category is a homotopy topological commutative monoid. This symmetric version is exactly Segal's observation (in [Seg2, §2]) that the classifying space of a (special) Γ-category is a (special) Γ-space.

e. If B is a fixed space with basepoint then

$$\mathbf{Top}_*(—, B) : (\mathbf{Top}_*^{\mathrm{op}}, \vee, 1) \longrightarrow (\mathbf{Top}, \times, 1)$$

is a homotopy monoidal functor, as observed in 3.2. So there is in particular an induced functor

$$\mathbf{HtyAlg}(\mathbf{Mon}, (\mathbf{Top}_*^{\mathrm{op}}, \vee, 1)) \longrightarrow \mathbf{HtyAlg}(\mathbf{Mon}, (\mathbf{Top}, \times, 1)).$$

This is effectively the argument we used in 3.2 to show that the homotopy comonoid structure on S^1 gave a homotopy monoid structure on the loop space $\mathbf{Top}_*(S^1, B)$.

f. In Section 4.2 we used the n-fold smash product

$$\wedge : (\mathbf{Top}_*, \vee, 1)^n \longrightarrow (\mathbf{Top}_*, \vee, 1),$$

which is a 'homotopy multi-monoidal functor' in the obvious sense of the phrase. It translates the homotopy comonoid structure on S^1 (or rather, n copies of this structure) into an n-fold homotopy comonoid structure on $S^1 \wedge \cdots \wedge S^1 = S^n$.

g. The path-components functor $\pi_0 : \mathbf{Top} \longrightarrow \mathbf{Set}$ preserves products and sends homotopy equivalences to isomorphisms, and so induces a functor

$$\mathbf{HtyAlg}(P, \mathbf{Top}) \longrightarrow \mathbf{Alg}(P, \mathbf{Set}).$$

For instance, the path-components of any topological monoid form a monoid.

More generally, if $P_1, \ldots, P_n$ are operads then π_0 induces a functor

$$\pi_0 : \mathbf{HtyAlg}(P_1, \ldots, P_n; \mathbf{Top}) \longrightarrow \mathbf{Alg}(P_1, \ldots, P_n; \mathbf{Set})$$

(see 4.2 for the notation). We saw in 4.2.1 that any n-fold loop space $\mathbf{Top}_*(S^n, B)$ has the structure of an n-fold homotopy monoid: thus

$$\pi_n(B) = \pi_0(\mathbf{Top}_*(S^n, B))$$

is an n-fold monoid in $\mathbf{Set}$. But the Eckmann-Hilton argument[1] says that if a pair of monoid structures on a set commute with each other then they are identical and commutative: so

$$\mathbf{Alg}(\underbrace{\mathbf{Mon}, \ldots, \mathbf{Mon}}_{n}; \mathbf{Set}) = \begin{cases} \mathbf{Set} & \text{if } n = 0 \\ \text{(monoids)} & \text{if } n = 1 \\ \text{(commutative monoids)} & \text{if } n \geq 2. \end{cases}$$

This means that the nth homotopy $\pi_n(B)$ of a space B is a set when $n = 0$, a monoid when $n = 1$, and a commutative monoid when $n \geq 2$. Of course, we know that these monoids are actually groups, and so our result implies that the higher homotopy groups are abelian.

h. Let $\Pi_1 : \mathbf{Top} \longrightarrow \mathbf{Cat}$ be the functor assigning to a space its fundamental groupoid. Then Π_1 is a homotopy monoidal functor

$$(\mathbf{Top}, \times, 1) \longrightarrow (\mathbf{Cat}, \times, 1)$$

by virtue of preserving products: so, for example, the fundamental groupoid of a loop space is a homotopy monoidal category. Similarly, the fundamental groupoid of a special Γ-space is a special Γ-category. The fundamental groupoid of an n-fold loop space is an n-fold homotopy monoidal

[1] $a \cdot b = (a * 1) \cdot (1 * b) = (a \cdot 1) * (1 \cdot b) = a * b = (1 \cdot a) * (b \cdot 1) = (1 * b) \cdot (a * 1) = b \cdot a$

category, i.e. a homotopy $(\mathbf{Mon}, \ldots, \mathbf{Mon})$-algebra in **Cat**. I have not investigated n-fold homotopy monoidal categories, but it would be interesting to see how they compare to braided monoidal categories when $n = 2$, and to the iterated monoidal categories of Balteanu, Fiedorowicz, Schwänzl and Vogt for general n (see [JS] and [BFSV] respectively).

i. Next is a non-example. One of the main features of the definition of A_∞-algebra in Stasheff's original paper [Sta2] is that the chain complex $C_\bullet(A)$ of an A_∞-space A is an A_∞-algebra. We can attempt to mimic this here, by trying to give the singular chains functor $C_\bullet : \mathbf{Top} \longrightarrow \mathbf{ChCx}$ the structure of a homotopy monoidal functor. If this is possible then there is an induced functor

$$\mathbf{HtyAlg}(\mathbf{Mon}, \mathbf{Top}) \longrightarrow \mathbf{HtyAlg}(\mathbf{Mon}, \mathbf{ChCx}),$$

so that the chains of a homotopy topological monoid form a homotopy d.g. algebra. However, it appears to be impossible.

To get an idea of the issues at hand, let us see how $C_\bullet$ is naturally a *lax* monoidal functor (as defined on page 29). This basically means that for spaces X and Y and $p, q \in \mathbb{N}$ there is a canonical map

$$C_p(X) \otimes C_q(Y) \longrightarrow C_{p+q}(X \times Y),$$

that is,

$$R\langle \mathbf{Top}(\Delta^p, X) \times \mathbf{Top}(\Delta^q, Y)\rangle \longrightarrow R\langle \mathbf{Top}(\Delta^{p+q}, X \times Y)\rangle$$

where R is the ground ring, $R\langle S\rangle$ is the free R-module on a set S, and Δ^r is the standard r-simplex. This map is induced by the composite

$$\mathbf{Top}(\Delta^p, X) \times \mathbf{Top}(\Delta^q, Y) \xrightarrow{\times} \mathbf{Top}(\Delta^p \times \Delta^q, X \times Y) \xrightarrow{f^*} \mathbf{Top}(\Delta^{p+q}, X \times Y)$$

where, in turn, $f : \Delta^{p+q} \longrightarrow \Delta^p \times \Delta^q$ is defined by the first degeneracy map $\Delta^{p+q} \longrightarrow \Delta^p$ and the last degeneracy map $\Delta^{p+q} \longrightarrow \Delta^q$.

(If this is right then it contradicts the suggestion of Kontsevich, in [Kon, 2.2], that one needs to use cubical rather than simplicial chains in order to make $C_\bullet$ into a lax monoidal functor.)

Hence $C_\bullet$ naturally induces a functor

$$\mathbf{Alg}(P, \mathbf{Top}) \longrightarrow \mathbf{Alg}(P, \mathbf{ChCx})$$

for any operad P. So, for instance, the chains of a *genuine* topological monoid form a *genuine* d.g. algebra. But this has come from $C_\bullet$ being a *lax* monoidal functor, and what we need in order to obtain an induced functor on homotopy algebras is its dual, a *colax* monoidal functor. As far as I know, there is no suitable colax structure.

j. The homology functor $H_\bullet : \mathbf{ChCx}_R \longrightarrow \mathbf{GrMod}_R$ sends chain homotopy equivalences to isomorphisms, for any commutative ring R. It is also a monoidal functor, i.e. preserves $\otimes$ and unit up to coherent isomorphism, provided that R is a field (by the Künneth Theorem, [Wei, 3.6.3]). So if R is a field then there is an induced functor

$$\mathbf{HtyAlg}(P, \mathbf{ChCx}_R) \longrightarrow \mathbf{Alg}(P, \mathbf{GrMod}_R)$$

for any operad P. Hence the homology over a field of a homotopy d.g. algebra forms a graded algebra, and similarly for commutative algebras, non-unital algebras, Lie algebras, Gerstenhaber algebras, etc.

Chapter 6

Final Thoughts

This has been a long paper, and despite having covered many points, there are still various loose ends and unanswered questions. I hope it will therefore be useful for me to give a summary of how things stand.

First is a list of things done, and then things conspicuously undone. Staying negative in tone, there is next a section on homotopy invariance. More optimistically, the view is then put forward that our definition of homotopy algebra is just a 1-dimensional approximation to an infinite-dimensional ideal, and that the distance between approximation and ideal is what causes many of our difficulties. Also discussed, briefly, is the matter of how our definition relates to other definitions of homotopy algebra.

What We've Done, and What We Haven't

The main achievements of this paper are as follows.

General definition The principal point of the paper is, of course, to give a definition of homotopy algebra for an operad which works in a very general context. We have done this, and once one has understood the process of forming the free monoidal category $\widehat{P}$ on an operad P, the definition is extremely simple.

Special Γ-spaces and Δ-spaces We have shown that a homotopy topological monoid is precisely a 'special Δ-space' (or special simplicial space, in our terminology), and similarly that a homotopy topological commutative monoid is precisely a special Γ-space. Indeed, it was a reformulation of the definition of special Γ-space which led me to the general definition. The advantage of this reformulation is that it allows generalization: to an arbitrary operad P (not just **Mon** or **CMon**), and to monoidal categories which, unlike **Top**, are not cartesian. The reformulation also clarifies the role of Γ from a conceptual point of view, and clarifies the interplay of Δ and Δ^+.

Loop spaces A major example of our definition is that any loop space is a homotopy topological monoid, and, in fact, that any n-fold loop space is an n-fold homotopy topological monoid. To express the latter statement we had to develop (in brief) a theory of homotopy algebras for several operads simultaneously. Completing the picture is the fact that any infinite loop space is a special Γ-space, i.e. a homotopy topological commutative monoid.

Change of environment A suitable map $\mathcal{L} \longrightarrow \mathcal{M}$ of monoidal categories leads, unsurprisingly, to a way of passing from homotopy algebras in $\mathcal{L}$ to homotopy algebras in $\mathcal{M}$. The most interesting applications presented here are topological: the classifying space of a homotopy (symmetric) monoidal category is a homotopy topological (commutative) monoid, and the fundamental groupoid of an n-fold loop space is an n-fold homotopy monoidal category. It also provides an explanation of why the higher homotopy groups of a space are abelian, and why the first homotopy group is not, and why the zeroth is only a set.

Comparisons There are various other notions of weakened or up-to-homotopy algebraic structure in the literature. We have made some partial comparisons between a small number of these and our definition. The result which encompasses most of our comparisons is that a homotopy P-algebra in a monoidal 2-category gives rise to a weak P-algebra (4.3). It follows that a homotopy monoidal category gives rise to a monoidal category (non-strict, in the traditional sense), a homotopy semigroup in the category of based spaces gives rise to an A_4-space, and a homotopy differential graded non-unital algebra to an A_4-algebra. In the opposite direction, we have also shown how to obtain a homotopy monoidal category from a (traditional) monoidal category.

Clarification Aside from the specific points listed above, I hope that this paper has succeeded in clarifying some general points concerning homotopy algebras.

Firstly, we have seen that in order to state our definition of homotopy algebra in $\mathcal{M}$, it is only necessary to have knowledge of which morphisms in $\mathcal{M}$ are 'homotopy equivalences'; knowledge of what it means for two maps to be 'homotopic', or what a 'homotopy' between maps is, etc., is not required.

Secondly, I have tried to draw attention to the distinction between the canonical and the non-canonical, especially in the sections on loop spaces and homotopy monoidal categories (3.2 and 3.3). For instance, since there is no canonical recipe for forming the tensor product of two abelian groups, there is no canonical functor $\otimes : \mathbf{Ab}^2 \longrightarrow \mathbf{Ab}$; similarly, there is no canonical way of composing two based loops in a space. In this connection we repeatedly see diagrams of the shape

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

in which the left-hand map is an equivalence, as a substitute for a map $X \longrightarrow Y$. Such diagrams appear in many parts of mathematics: to take a fairly random selection, [Thomas], [Mak], [Ad, p. 51], [May1].

This is what we've done. On the other hand, the ideas presented in this paper raise many questions crying out to be answered. I believe that the central definition of homotopy algebra is fundamentally crude and cannot be formulated satisfactorily until there is a decent theory of weak ∞-categories; but that will be discussed later, and for now I will limit myself to noting some specific shortcomings.

Comparisons The comparison results presented here are blatantly incomplete. For a start, we showed that homotopy differential graded non-unital algebras give rise to A_4-algebras, but were not able to show that they give A_∞-algebras; and similarly A_∞-spaces. More seriously, the comparison results are almost all of the form 'a homotopy algebra in our sense gives rise to a homotopy algebra in someone else's sense', rarely the other way round. The exception is when we are taking algebras in **Cat** (4.4); but even then, it is not clear whether the two processes are in any sense mutually inverse or adjoint.

Maps We have discussed homotopy algebras at length, but not homotopy maps between homotopy algebras. Just before Proposition 3.3.2 we suggested how a homotopy map between homotopy algebras might look in a category such as **Top** where there is a notion of two maps being homotopic. Another possibility, which makes sense in any monoidal category with equivalences, is to define a homotopy map of homotopy P-algebras as a homotopy $\mathbf{Map}_P$-algebra. Here $\mathbf{Map}_P$ is the multicategory (coloured operad) of 1.3, for which a genuine algebra is a pair of P-algebras with a map between them. This is a pleasing definition of homotopy map, but raises further questions when one thinks about composing them.

Examples We are a little short on actual examples of homotopy algebras, mostly for the reasons mentioned under 'Comparisons' just above. Even if one has in mind an object which one suspects ought to be a homotopy algebra for a certain operad P, it takes creative effort to endow the object with the structure of a homotopy P-algebra. In the terminology of [Ad, p. 60], one has to create a lot of flab. An example of this is the Problem of 4.1.2: how to endow the Hochschild cochain complex of an associative algebra with the structure of a homotopy d.g. commutative algebra.

This completes the summary of things done and undone.

Homotopy Invariance

Earlier we came across a disturbing feature of the definition of homotopy algebra, in Example 2.3(i). This was that if G is a monoid and $\mathbf{Act}_G$ the non-symmetric operad whose algebras are objects with a strict action by G, then a homotopy $\mathbf{Act}_G$-algebra (X, ξ) specifies, amongst other things, a *strict* action of G on the 'base object' $X(1)$.

From this example we can see that our homotopy P-algebras are not 'homotopy invariant algebraic structures'; at least, not if we regard a homotopy P-algebra (X, ξ) as being a structure on the object $X(1)$. That is to say, suppose that (X, ξ) is a homotopy algebra for some operad P, in some (symmetric or not) monoidal category $\mathcal{M}$ with equivalences, and suppose that we have a homotopy equivalence $X(1) \longrightarrow A$, or $A \longrightarrow X(1)$, in $\mathcal{M}$. 'Homotopy invariance' says that there is an induced homotopy P-algebra (W, ω) with $W(1) = A$. In the case $P = \mathbf{Act}_G$ and $\mathcal{M} = \mathbf{Top}$, this implies that if A is homotopy equivalent to a (strict) G-space then there is an induced *strict* action of G on A. Plainly that is not true, so homotopy invariance fails.

This is worrying: homotopy invariance is an attribute which a good theory of homotopy-algebraic structures ought to have. (Boardman and Vogt's book [BV] and Markl's paper [Mar2] say much more on why it is desirable.) In the next section, I will suggest in vague terms an ∞-categorical version of our definition of homotopy algebra which *would* be homotopy invariant; but for now, here is a result which is perhaps the closest we have to homotopy invariance for the definition as it stands.

Proposition 6.0.1 *Let P be a (symmetric) operad and $\mathcal{M}$ a (symmetric) monoidal category. Let*

$$(W, \omega), (X, \xi) : \widehat{P} \longrightarrow \mathcal{M}$$

be two colax (symmetric) monoidal functors, and let

$$\sigma : (W, \omega) \longrightarrow (X, \xi)$$

be a monoidal transformation which is a 'homotopy equivalence', in the sense that each component σ_n is a homotopy equivalence in $\mathcal{M}$. Then (W, ω) is a homotopy P-algebra if and only if (X, ξ) is.

Proof Simply apply the axioms for a class of equivalences (2.1.1) to the commuting diagrams in the definition of monoidal transformation (1.1.2). $\square$

∞-Categories

A monoidal category with equivalences is a very simple device, and it has already been argued that it is really too simple (in the introduction to Chapter 2).

Top and **ChCx**, for example, naturally form monoidal ∞-categories, but when we treated them as monoidal categories with equivalences we threw away all information about cells of dimension 2 and above, except for retaining knowledge of which 1-cells are equivalences in an ∞-categorical sense. In Chapter 3 we gave them marginally more respect by treating them as monoidal 2-categories, but this is still a long way from appreciating their true ∞-categorical nature. This ignorant behaviour is, of course, excused by the fact that there is not yet a well-developed theory of (weak) ∞-categories.

When such a theory has evolved, it should be possible to make the following definition. Let P be a symmetric or non-symmetric operad with, for simplicity, each $P(n)$ being just a set. Let $\mathcal{M}$ be a (symmetric) monoidal ∞-category. Then a 'homotopy P-algebra in $\mathcal{M}$' is simply a (symmetric) monoidal ∞-functor $\widehat{P} \longrightarrow \mathcal{M}$. Here the monoidal category $\widehat{P}$ is made into a monoidal ∞-category by saying that the only k-cells for $k \geq 2$ are the identities, and a 'monoidal ∞-functor' is meant to preserve tensor, composition and identities up to equivalence in the weakest ∞-categorical sense.

In down-to-earth terms, this higher-dimensional structure would make differences of the following kind. Take, for example, a monoid G. With the definition of homotopy algebra used in this paper, the base object $X(1)$ of a homotopy $\mathbf{Act}_G$-algebra (X, ξ) is in fact a strict G-object, which means that there is a map $\alpha_g : X(1) \longrightarrow X(1)$ for each $g \in G$, such that the diagrams

$$\begin{tikzcd} X(1) \arrow[r, "\alpha_{g'}"] \arrow[dr, "\alpha_{gg'}"'] & X(1) \arrow[d, "\alpha_g"] \\ & X(1) \end{tikzcd} \qquad \begin{tikzcd} X(1) \arrow[r, shift left, "1"] \arrow[r, shift right, "\alpha_1"'] & X(1) \end{tikzcd}$$

commute (strictly). It is not possible that they might only have to commute 'up to homotopy': this simply does not make sense in an arbitrary monoidal category with equivalences. But with an ∞-categorical definition of homotopy algebra they would not strictly commute: instead, there would be a 2-cell filling in each of these diagrams (e.g. a homotopy, if we were working in **Top**). Moreover, these 2-cells would obey coherence laws—not strictly, but up to a 3-cell; and so on. Thus the effect of an ∞-categorical definition would be to weaken further the original definition of homotopy algebra.

Another place where this weakening effect would be seen is in maps of homotopy algebras: such a thing would naturally be defined as a weak monoidal transformation, with 'weak' meant in an ∞-categorical sense, and this would be a respectable notion of a homotopy map of homotopy algebras. Thus the squares involved in the discussion of maps just before Proposition 3.3.2 would commute only up to coherent equivalence. Similarly, homotopy invariance should work perfectly well when we use the ∞-categorical definition.

So, we have been using throughout a 1-dimensional approximation to an infinite-dimensional ideal. This is just about the roughest approximation possible, but still it has given us plenty to chew on. In particular, homotopy algebras as we defined them are weak enough to include loop spaces and monoidal categories, and gain a badge of historical respectability by having as particular cases special Γ-spaces and special simplicial spaces.

A similar situation where one 'should' use an ∞-category, but instead substitutes a more crude structure, is described in Hinich's paper [Hin, 1.3].

Other Definitions of Homotopy Algebra

Various other notions of homotopy algebra exist in the literature, and I have not attempted anything like a systematic comparison. The notions I am aware of are the homotopy-invariant algebraic structures in Boardman and Vogt's book [BV], the strong homotopy algebras of [Lada], the minimal models of Markl ([Mar1], [Mar2]), and the A_∞, B_∞, C_∞, E_∞, G_∞ and L_∞ structures developed by many people: see, for instance, [Ad], [KSV], [LM], [Sta1], [Sta2], [Vor]. (My knowledge of this literature is not very thorough, so I hope that no-one will be offended by omissions.) The definition in this paper has the advantage of working in a more general context than any of the others, as far as I know; but of course it has certain disadvantages too.

It seems to me that the comparison between different definitions of homotopy algebra is very much like the comparison between different definitions of weak n-category, of which there are currently about a dozen. In both situations there are essentially two approaches. We discussed in Section 3.2 how the description of a loop space as an A_∞-space is essentially different from the description of a loop space as a homotopy monoid in our sense: to describe it as an A_∞-space we have to make an arbitrary choice concerning how to compose two loops, and so have a specific but non-canonical composition law; to describe it as a homotopy monoid we need make no artificial choices at all, but there is no actual preferred composition law. Similarly, the various proposed definitions of weak n-category split into those which insist that for suitable cells f and g there should be a definite composite $h = g \circ f$, and those in which one could only ever say 'h is **a** composite of g and f', there being perhaps many possible composites of g and f, all equally valid. We have already seen this distinction for monoidal categories, at the beginning of 3.3, and indeed a monoidal category is precisely a weak 2-category with only one object. Examples of the first ('algebraic') approach are the definitions of weak n-category proposed by Batanin ([Bat], [Str2]), by Penon [Pen], and by me [Lei2], and the 'classical' definitions of monoidal category, bicategory [Bén] and tricategory [GPS]. Examples of the second approach are the definitions of Baez and Dolan [BD], Hermida, Makkai and Power [HMP], Joyal [Joy], Street [Str1], and Tamsamani [Tam]. To date no-one knows very much about how the various definitions relate to one another. It may be that some clues lie in the body of material on 'uniqueness of delooping machines', e.g. [MT], [Thmsn] and [SV]; it may also be that the situation for n-categories is substantially more challenging.

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Glossary

A brief description of each term is given; for operads, the description says what the algebras are. Section or page numbers refer to where the term was defined or first mentioned. Some terms have meanings at two or more different levels, in which case there is more than one section number.

Categories

$\mathcal{C}^{\mathrm{op}}$	opposite category	p. 7
$\widehat{P}$	free monoidal category on operad	1.6
$\lvert\mathcal{N}\rvert$	underlying monoidal category with equivalences	4.3
$\mathbf{1}$	terminal category	1.1(b)
Ab	abelian groups	1.5(d)
$\mathbf{Alg}(P, \mathcal{M})$	algebras	1.2.2, 1.5
Cat	categories	1.1(b), 1.4(g), 2.1(k)
$\mathbf{ChCx} = \mathbf{ChCx}_R$	chain complexes	1.1(f), 1.4(h), 3.5
$\mathbf{Colax}(\mathcal{L}, \mathcal{M})$	colax monoidal functors	3.1
GrAb	graded abelian groups	1.5(e)
$\mathbf{GrMod} = \mathbf{GrMod}_R$	graded modules	1.1(e), 1.4(e)
$\mathbf{HtyAlg}(P, \mathcal{M})$	homotopy algebras	2.2.1, 2.4.1
HtyMonCat	homotopy monoidal categories	p. 54
$\mathbf{Hty}\widetilde{\mathbf{Mon}}\mathbf{Cat}$	modified **HtyMonCat**	p. 54
$\mathbf{Mod} = \mathbf{Mod}_R$	modules	1.1(d), 1.4(d)
$\mathbf{Mon}(\mathcal{L}, \mathcal{M})$	monoidal functors	1.1.1, 1.6
MonCat	monoidal categories	p. 54
$\mathbf{SColax}(\mathcal{L}, \mathcal{M})$	colax symmetric monoidal functors	3.1
Set	sets	1.1(c)
$\mathbf{SMon}(\mathcal{L}, \mathcal{M})$	symmetric monoidal functors	1.1.1, 1.6
$\mathbf{Special}(\Gamma^{\mathrm{op}}, \mathcal{M})$	special Γ-objects	3.1
$\mathbf{Special}((\Delta^{+})^{\mathrm{op}}, \mathcal{M})$	special simplicial objects	3.1
Top	topological spaces	1.1(g), 1.4(f)
$\mathbf{Top}_*$	based spaces	1.1(h), 3.4
Toph	spaces and homotopy classes of maps	5.1(b)
$\mathbf{WkAlg}(P, \mathcal{M})$	weak algebras	4.3

Γ	opposite of finite based sets	3.1
Δ	finite totally ordered sets	1.1(j)
Δ_{inj}	injective part of Δ	1.6(c)
Δ_{surj}	surjective part of Δ	1.6(b)
Δ^+	non-empty finite totally ordered sets	p. 41
Φ	finite sets	1.1(i)
Φ_{inj}	injective part of Φ	1.6(e)
Φ_{surj}	surjective part of Φ	1.6(e)

Operads

$\mathbf{Act}_G$	G-objects	1.2(i), 1.5(b),(c)
CMon	commutative monoids	1.2(d)
CSem	commutative semigroups	1.2(e)
Ger	Gerstenhaber algebras	1.5(g)
GrLie	graded Lie algebras	1.5(e), 4.1.1
Inv	monoids with involution	1.2(h)
Lie	Lie algebras	1.5(d)
$\mathbf{Map}_P$	P-algebra maps	1.3
Mon	monoids	1.2(b)
Obj	objects	1.2(a)
Pt	pointed objects	1.2(f)
$\mathbf{SAct}_G$	G-objects	1.2(i), 1.5(b)
Sem	semigroups	1.2(c)
SObj	objects	1.2(a)
SPt	pointed objects	1.2(f)
Sym	monoids	1.2(g)

Other

$\simeq$	equivalence	p. 7
$\cong$	isomorphism	p. 7
$*$	horizontal composite	p. 7
$\vee$	wedge product	1.1(h)
$\wedge$	smash product	4.2.1
$\mathcal{C}(A, B)$	hom-set	p. 7
S^n	n-sphere	4.2
S_n	nth symmetric group	1.2.1
Δ^n	n-simplex	3.2
Ω	loop space functor	3.2