

Title

Subtitle

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Northwestern University

Date/Event

Braces, categorically

Fix an algebra, A .

Define a dg category, $Hoch(A)$:

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Objects: algebra maps $f : A \rightarrow A$

Morphisms: $Hoch(A)(f, g) = (C^\bullet(A, {}_f A_g), {}_f \delta_g)$

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$$\begin{aligned} {}_f \delta_g(\phi)(a_1 \otimes \dots \otimes a_n) = & \epsilon_\phi \left(f(a_1) \cdot \phi(a_2, \dots, a_n) + \right. \\ & + \sum_{1 \leq i \leq n-1} (-1)^i \phi(a_1, \dots, a_i a_{i+1}, \dots, a_n) + \\ & \left. + (-1)^n \phi(a_1, \dots, a_{n-1}) \cdot g(a_n) \right) \\ \epsilon_\phi = & (-1)^{|\phi|+1} \end{aligned}$$

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Composition: cup product on cochains

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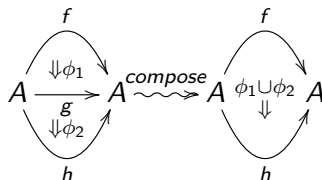
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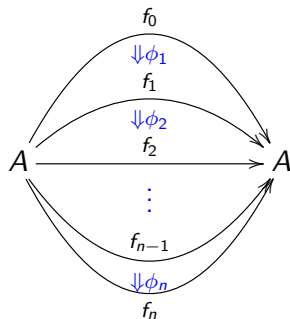
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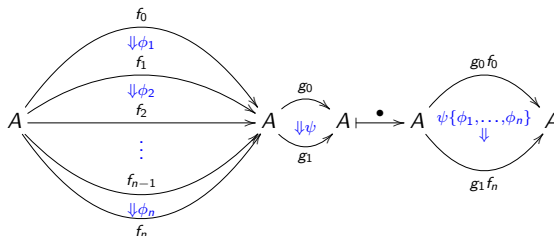
A morphism from f_0 to f_n in $Bar(Hoch(A))$

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In this context, braces give multilinear maps:

$$\begin{array}{ccc} Bar(Hoch(A)) \otimes Bar(Hoch(A)) & \xrightarrow{\bullet} & Hoch(A) \\ & \searrow \bullet & \uparrow \\ & & Bar(Hoch(A)) \end{array}$$

Then, $(Bar(Hoch(A)), \bullet)$ is an algebra in $DGCocats$.

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But we have more...

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Fix algebras, A_0, A_1, \dots, A_n .

We will define a dg cocategory $B(A_0 \rightarrow A_1 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$
where $B(A_0 \rightarrow A_0) := \text{Bar}(\text{Hoch}(A_0))$ for $n=0$.

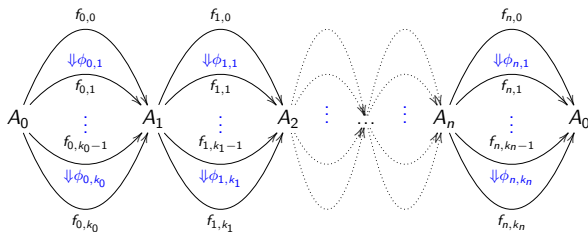
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Objects: $\{A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \dots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_0\}$

A morphism from $(A_0 \xrightarrow{f_{0,0}} \dots \xrightarrow{f_{n,0}} A_0)$ to $(A_0 \xrightarrow{f_{0,k_0}} \dots \xrightarrow{f_{n,k_n}} A_0)$:



$$\phi_{i,j} \in C^\bullet(A_{i,f_{i,j-1}} A_{i+1,f_{i,j}})$$

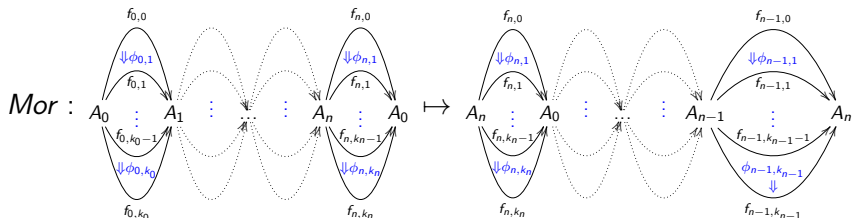
Structure among the $B(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$'s

Example

We have a dg functor

$$\hat{\tau}_n : B(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0) \rightarrow B(A_n \rightarrow A_0 \rightarrow \dots \rightarrow A_n)$$

$$\text{Obj} : (f_0, \dots, f_n) \mapsto (f_n, f_0, \dots, f_{n-1})$$



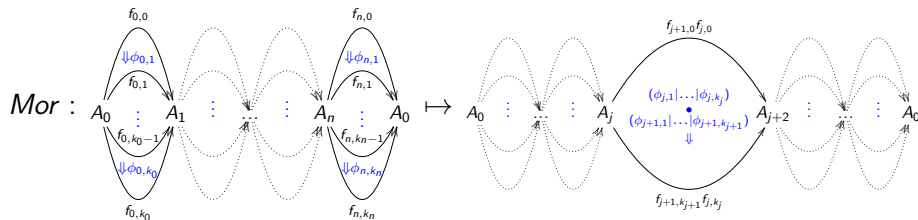
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Example

For $n \geq 1, 0 \leq j < n$, we have a dg functor

$$\hat{\delta}_{j,n} : B(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0) \rightarrow B(A_0 \rightarrow \dots \rightarrow A_j \rightarrow A_{j+2} \rightarrow A_0)$$

$$\text{Obj} : (f_0, \dots, f_n) \mapsto (f_0, \dots, f_{j+1} f_j, \dots, f_n)$$



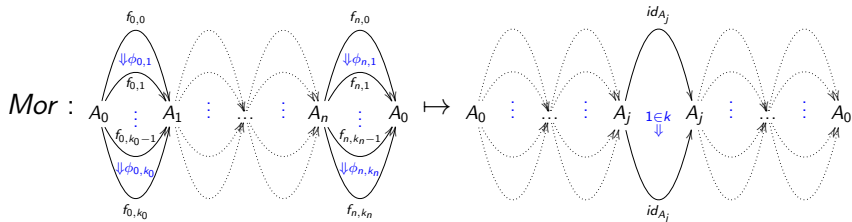
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$$\text{Obj} : (f_0, \dots, f_n) \mapsto (f_0, \dots, f_{j-1}, id_{A_j}, f_j, \dots, f_n)$$



A sheafy-cyclic object in DGCocat

Definition

Let χ be the category with objects $\{A_0 \rightarrow A_1 \rightarrow \dots \rightarrow A_n \rightarrow A_0\}$ and morphisms compositions of

$$\text{rotations } \tau_n : \mathcal{A} \mapsto (A_n \rightarrow A_0 \rightarrow \dots \rightarrow A_n)$$

$$\text{coboundaries } \delta_{j,n} : \mathcal{A} \mapsto (A_0 \rightarrow \dots \rightarrow A_j \rightarrow A_{j+2 \pmod{n+1}} \rightarrow \dots \rightarrow A_0)$$

$$\text{codegeneracies } \sigma_{i,n} : \mathcal{A} \mapsto (A_0 \rightarrow \dots \rightarrow A_i \rightarrow A_i \rightarrow \dots \rightarrow A_0)$$

where $\mathcal{A} := (A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$, subject to the cyclic relations.

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Proposition

We have a functor $\chi \rightarrow DGCocat$

$$\text{Objects} : (A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0) \mapsto B(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$$

$$\text{Generating morphisms} : \lambda \mapsto \hat{\lambda}$$

Each dg cocategory $B(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$ has a dg comodule $C(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$ (which is a categorified version of the bar complex of Hochschild chains as a module over Hochschild cochains via contraction).

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Motivating Question: Can we extend the sheafy-cyclic structure to the comodules? In other words, can we give a functor

$$\chi \rightarrow \mathcal{D} := \{(dg\ cocat, dg\ comod)\}?$$

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