Title

Subtitle

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Date/Event

Background

Theorem

(Hochschild-Kostant-Rosenberg) Let A be a regular, commutative algebra over a field k of characteristic 0. Then,

$$(C_\bullet(A,A), b) \xrightarrow{\sim} \Omega^\bullet_{A/k}$$
$$(C^\bullet(A,A), \delta) \xrightarrow{\sim} \wedge^\bullet T_A = \wedge^\bullet(Der_k(A,A)).$$

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Theorem

(Kontsevich, '97) Let $A = C^\infty(M)$ for M a smooth real manifold. Then, there is an L_∞ map

$$(C^{\bullet+1}(A,A),\delta,[,]_{Ger}) \xrightarrow{\sim} (\wedge^{\bullet+1} T_A, d=0, [,]_{SN}).$$

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Theorem

(Tamarkin, '98) Dependent on the choice of a Drinfeld associator, there is a Ger_∞ *map*

$$(C^{\bullet+1}(A,A), \delta, [,]_{Ger}, \cup, \dots) \xrightarrow{\sim} (\wedge^\bullet T_A, d=0, [,]_{SN}, \wedge).$$

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Theorem

(Dolgushev-Tamarkin-Tsygan, '08) There is a $Calc_\infty$ map

$$(C^\bullet(A,A), C_{-\bullet}(A,A)) \xrightarrow{\sim} (\wedge^\bullet T_A, \Omega^\bullet_{A/k}).$$

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Can we use this calculus structure to create a cyclic object?

A cyclic object in dg cocategories

Fix an algebra, A.
Define a dg category, $Hoch(A)$:

A cyclic object in dg cocategories

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Morphisms: $Hoch(A)(f, g) = (C^\bullet(A, {}_fA_g), {}_f\delta_g)$

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$$\begin{aligned} {}_f\delta_g(\phi)(a_1 \otimes \ldots \otimes a_n) =& \epsilon_\phi \Big(f(a_1) \cdot \phi(a_2, \ldots, a_n) + \\ & + \sum_{1 \leq i \leq n-1} (-1)^i \phi(a_1, \ldots, a_i a_{i+1}, \ldots, a_n) + \\ & + (-1)^n \phi(a_1, \ldots, a_{n-1}) \cdot g(a_n) \Big) \\ \epsilon_\phi =& (-1)^{|\phi|+1} \end{aligned}$$

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Composition: cup product on cochains

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Composition: cup product on cochains

$$\begin{tikzcd}
A \arrow[r, bend left=60, "f"] \arrow[r, "g"'] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\phi_1"] \arrow[r, phantom, shift right=5ex, "\Downarrow\phi_2"] & A \arrow[r, squiggly, "compose"] & A \arrow[r, bend left=60, "f"] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, "\overset{\phi_1 \cup \phi_2}{\Downarrow}"] & A
\end{tikzcd}$$

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$$Bar(Hoch(A)) \otimes Bar(Hoch(A)) \xrightarrow{\bullet} Hoch(A)$$

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$$\begin{tikzcd}[column sep=huge]
A \arrow[r, bend left=70, "f_0"] \arrow[r, phantom, bend left=55, "\Downarrow\phi_1"] \arrow[r, bend left=35, "f_1"] \arrow[r, phantom, bend left=18, "\Downarrow\phi_2"] \arrow[r, "f_2"] \arrow[r, phantom, bend right=20, "\vdots"] \arrow[r, bend right=45, "f_{n-1}"] \arrow[r, phantom, bend right=60, "\Downarrow\phi_n"] \arrow[r, bend right=70, "f_n"'] & A
\end{tikzcd}$$

A morphism from f_0 to f_n in Bar(Hoch(A))

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A \arrow[r, bend left=70, "f_0"] \arrow[r, bend left=35, "f_1"] \arrow[r, "f_2"] \arrow[r, bend right=35, "f_{n-1}"'] \arrow[r, bend right=70, "f_n"'] \arrow[r, phantom, shift left=8ex, "\Downarrow\phi_1"] \arrow[r, phantom, shift left=4ex, "\Downarrow\phi_2"] \arrow[r, phantom, shift right=4ex, "\vdots"] \arrow[r, phantom, shift right=9ex, "\Downarrow\phi_n"] & A \arrow[r, bend left, "g_0"] \arrow[r, bend right, "g_1"'] \arrow[r, phantom, "\Downarrow\psi"] & A \arrow[r, mapsto, "\bullet"] & A \arrow[r, bend left=50, "g_0 f_0"] \arrow[r, bend right=50, "g_1 f_n"'] \arrow[r, phantom, "\overset{\psi\{\phi_1,\ldots,\phi_n\}}{\Downarrow}"] & A
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\end{tikzcd}$$

$$\psi\{\phi_1,\dots,\phi_n\}(a_1,\dots,a_q) = \sum_{1\le i_1\le\dots\le i_n\le q} \pm\psi(a_1,\dots,a_{i_1},\phi_1(a_{i_1+1},\dots),\dots,\phi_n(a_{i_n+1},\dots),\dots)$$

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Fix an algebra, A.
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In this context, braces, $\bullet$, give multilinear maps:

$$\begin{tikzcd}
Bar(Hoch(A)) \otimes Bar(Hoch(A)) \arrow[r, "\bullet"] \arrow[dr, dotted, "\bullet"'] & Hoch(A) \\
& Bar(Hoch(A)) \arrow[u, two heads]
\end{tikzcd}$$

Then, (Bar(Hoch(A)), $\bullet$) is an algebra in DGCocats.

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Then, (Bar(Hoch(A)), $\bullet$) is an algebra in DGCocats.
But we have more...

A cyclic object in dg cocategories

A cyclic object in dg cocategories

Fix algebras, $A_0, A_1, \ldots, A_n$.
We will define a dg cocategory $B(A_0 \to A_1 \to \ldots \to A_n \to A_0)$
where $B(A_0 \to A_0) := Bar(Hoch(A_0))$ for n=0.

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Fix algebras, $A_0, A_1, \ldots, A_n$.
We will define a dg cocategory $B(A_0 \to A_1 \to \ldots \to A_n \to A_0)$
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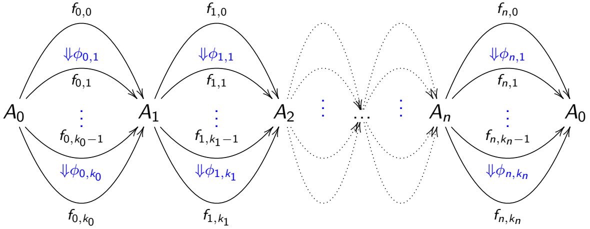
Objects: $\{A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \ldots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_0\}$
A morphism from $(A_0 \xrightarrow{f_{0,0}} \ldots \xrightarrow{f_{n,0}} A_0)$ to $(A_0 \xrightarrow{f_{0,k_0}} \ldots \xrightarrow{f_{n,k_n}} A_0)$:



$\phi_{i,j} \in C^\bullet(A_{i,f_{i,j-1}} {A_{i+1}}_{f_{i,j}})$

A cyclic object in dg cocategories

Example

We have a dg functor

$$\begin{aligned} \hat{\tau}_n : B(A_0 \to \ldots \to A_n \to A_0) &\to B(A_n \to A_0 \to \ldots \to A_n) \\ Obj : \qquad (f_0, \ldots, f_n) &\mapsto (f_n, f_0, \ldots, f_{n-1}) \end{aligned}$$

$$\begin{tikzcd}[column sep=large]
Mor : & A_0 \arrow[r, bend left=80, "f_{0,0}"] \arrow[r, phantom, bend left=60, "\Downarrow\phi_{0,1}"] \arrow[r, bend left=40, "f_{0,1}"'] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=40, "f_{0,k_0-1}"] \arrow[r, phantom, bend right=60, "\Downarrow\phi_{0,k_0}"] \arrow[r, bend right=80, "f_{0,k_0}"'] & A_1 \arrow[r, dotted, bend left=80] \arrow[r, dotted, bend left=40] \arrow[r, phantom, "\vdots"] \arrow[r, dotted, bend right=40] \arrow[r, dotted, bend right=80] & \ldots \arrow[r, dotted, bend left=80] \arrow[r, dotted, bend left=40] \arrow[r, phantom, "\vdots"] \arrow[r, dotted, bend right=40] \arrow[r, dotted, bend right=80] & A_n \arrow[r, bend left=80, "f_{n,0}"] \arrow[r, phantom, bend left=60, "\Downarrow\phi_{n,1}"] \arrow[r, bend left=40, "f_{n,1}"'] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=40, "f_{n,k_n-1}"] \arrow[r, phantom, bend right=60, "\Downarrow\phi_{n,k_n}"] \arrow[r, bend right=80, "f_{n,k_n}"'] & A_0 \arrow[r, phantom, "\mapsto"] & A_n \arrow[r, bend left=80, "f_{n,0}"] \arrow[r, phantom, bend left=60, "\Downarrow\phi_{n,1}"] \arrow[r, bend left=40, "f_{n,1}"'] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=40, "f_{n,k_n-1}"] \arrow[r, phantom, bend right=60, "\Downarrow\phi_{n,k_n}"] \arrow[r, bend right=80, "f_{n,k_n}"'] & A_0 \arrow[r, dotted, bend left=80] \arrow[r, dotted, bend left=40] \arrow[r, phantom, "\vdots"] \arrow[r, dotted, bend right=40] \arrow[r, dotted, bend right=80] & \ldots \arrow[r, dotted, bend left=80] \arrow[r, dotted, bend left=40] \arrow[r, phantom, "\vdots"] \arrow[r, dotted, bend right=40] \arrow[r, dotted, bend right=80] & A_{n-1} \arrow[r, bend left=80, "f_{n-1,0}"] \arrow[r, phantom, bend left=60, "\Downarrow\phi_{n-1,1}"] \arrow[r, bend left=40, "f_{n-1,1}"'] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=40, "f_{n-1,k_{n-1}-1}"] \arrow[r, phantom, bend right=60, "\phi_{n-1,k_{n-1}} \Downarrow"] \arrow[r, bend right=80, "f_{n-1,k_{n-1}}"'] & A_n
\end{tikzcd}$$

A cyclic object in dg cocategories

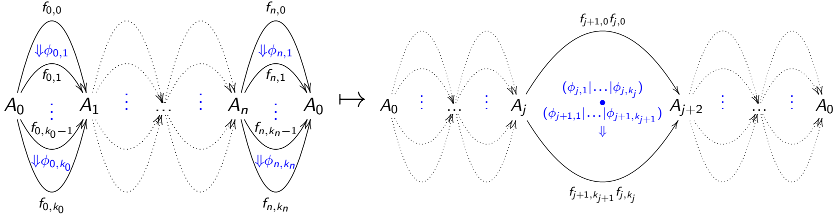
Example

For $n \geq 1, 0 \leq j < n$, we have a dg functor

$$\hat{\delta}_{j,n} : B(A_0 \to \dots \to A_n \to A_0) \to B(A_0 \to \dots \to A_j \to A_{\substack{j+2 \\ (mod\ n+1)}} \to \dots \to A_0)$$

$$Obj : \qquad (f_0, \dots, f_n) \mapsto (f_0, \dots, f_{j+1} f_j, \dots, f_n)$$

Mor :



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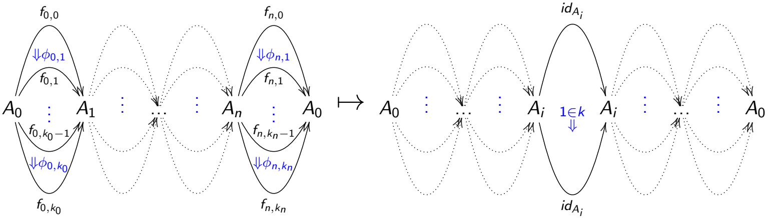
Example

For $n \geq 0, 0 \leq i \leq n$, we have a dg functor

$$\hat{\sigma}_{i,n} : B(A_0 \to \ldots \to A_n \to A_0) \to B(A_0 \to \ldots \to A_i \to A_i \to \ldots \to A_0)$$

$$Obj : \quad (f_0, \ldots, f_n) \mapsto (f_0, \ldots, f_{i-1}, id_{A_i}, f_i, \ldots, f_n)$$

Mor :



A **sheafy**-cyclic object in DGCocat

Definition

Let χ be the category with objects $\{A_0 \to A_1 \to \ldots \to A_n \to A_0\}$ and morphisms compositions of

$$\begin{aligned} \textit{rotations } \tau_n &: \mathcal{A} \mapsto (A_n \to A_0 \to \ldots \to A_n) \\ \textit{coboundaries } \delta_{j,n} &: \mathcal{A} \mapsto (A_0 \to \ldots \to A_j \to A_{j+2\,(mod\; n+1)} \to \ldots \to A_0) \\ \textit{codegeneracies } \sigma_{i,n} &: \mathcal{A} \mapsto (A_0 \to \ldots \to A_i \to A_i \to \ldots \to A_0) \end{aligned}$$

where $\mathcal{A} := (A_0 \to \ldots \to A_n \to A_0)$, subject to the cyclic relations.

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where $\mathcal{A} := (A_0 \to \ldots \to A_n \to A_0)$, subject to the cyclic relations.

Proposition

We have a functor $\chi \to DGCocat$

$$\begin{aligned} \textit{Objects}: &\ (A_0 \to \ldots \to A_n \to A_0) \mapsto B(A_0 \to \ldots \to A_n \to A_0) \\ \textit{Generating morphisms}: &\ \lambda \mapsto \hat{\lambda} \end{aligned}$$

Motivating Question

Each dg cocategory $B(A_0 \to \ldots \to A_n \to A_0)$ has a dg comodule $C(A_0 \to \ldots \to A_n \to A_0)$ (which is a categorified version of the bar complex of Hochschild chains as a module over Hochschild cochains via contraction).

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Motivating Question: Can we extend the sheafy-cyclic structure to the comodules? In other words, can we give a functor

$$\chi \to \mathcal{D} := \{(dg\ cocat,\ dg\ comod)\}?$$

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$$\chi_\infty \to \mathcal{D}_\infty \quad \textit{dg categories}$$

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Motivating Question: Can we extend the sheafy-cyclic structure to the comodules? In other words, can we give an A_∞-functor

~~$\chi \to \mathcal{D} := \{(dg\ cocat, dg\ comod)\}$~~?

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Motivating Question

Rest of this talk: Describe our A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$.

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- Define dg categories χ_∞ and $\mathcal{D}_\infty$
- Define dg comodules $C(A_0 \to \ldots \to A_n \to A_0)$
- Define the A_∞-functor $\mathcal{F}$

Dg categories χ_∞ and $\mathcal{D}_\infty$

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χ_∞:

Objects: same objects as $\chi = \{A_0 \to \ldots \to A_n \to A_0\}$

$\chi_\infty^\bullet(X, Y) := \{\sum a_i v_i : a_i \in k, v_i \in \chi(X, Y)\}$

Dg categories χ_∞ and $\mathcal{D}_\infty$

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$\mathcal{D}_\infty$:

Objects: same objects as $\mathcal{D} = \{(\underset{B}{\text{dg cocategory}}, \underset{C}{\text{dg comodule}})\}$

$$\mathcal{D}_\infty^p\big((B_1, C_1), (B_0, C_0)\big) := \begin{Bmatrix} F : B_1 \to B_0 \textit{ dg functor}, \\ F_! : C_1 \to F^* C_0 \textit{ degree-p linear map} \end{Bmatrix}$$

$$d_{D_\infty}(F, F_!) = (F, [d, F_!] = {\scriptstyle d_{F^*C_0} \circ F_! \pm F_! \circ d_{C_1}})$$

Dg categories χ_∞ and $\mathcal{D}_\infty$

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Objects: same objects as $\chi = \{A_0 \to \ldots \to A_n \to A_0\}$

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$$d_{D_\infty}(F, F_!) = (F, [d, F_!] = d_{F^* C_0} \circ F_! \pm F_! \circ d_{C_1})$$

$F^* C_0$ is the categorified version of co-extension of scalars:

$$F^* C_0 = ker(B_1 \otimes C_0 \rightrightarrows B_1 \otimes B_0 \otimes C_0)$$

Dg comodules over dg cocategories

Definition

A **dg comodule** C over a dg cocategory B consists of the following data:

- for each object $f \in B$, a complex $C^\bullet(f)$, and
- maps of complexes

$$\Delta_C(f) : C^\bullet(f) \to \prod_{g \in Obj(B)} B^\bullet(f,g) \otimes C^\bullet(g).$$

such that the following diagrams for coassociativity and counitality commute:

$$\begin{tikzcd}
C^\bullet(f) \arrow[r, "\Delta_C"] \arrow[d, "\Delta_C"'] & \prod\limits_{g \in Obj(B)} B^\bullet(f,g) \otimes C^\bullet(g) \arrow[d, "id_B \otimes \Delta_C"] \\
\prod\limits_{g \in Obj(B)} B^\bullet(f,g) \otimes C^\bullet(g) \arrow[r, "\Delta_B \otimes id_C"] & \prod\limits_{\substack{g,g' \in \\ Obj(B)}} B^\bullet(f,g) \otimes B^\bullet(g,g') \otimes C^\bullet(g')
\end{tikzcd}$$

$$\begin{tikzcd}
C^\bullet(f) \arrow[r, "\Delta_C"] \arrow[dr, "id"'] & \prod\limits_{g \in Obj(B)} B^\bullet(f,g) \otimes C^\bullet(g) \arrow[d, "\epsilon_B \otimes id_C"] \\
& C^\bullet(f)
\end{tikzcd}$$

Dg comodules over dg cocategories

Fix algebras $A_0, \ldots, A_n$.
Define a dg comodule over $B(A_0 \to \ldots \to A_n \to A_0)$:

$$C(A_0 \to \ldots \to A_n \to A_0)^\bullet(f_{0,0}, \ldots, f_{n,0}) :=$$

$$:= \left\{ \begin{tikzcd}[column sep=large]
A_0 \arrow[r, bend left=70, "f_{0,0}"] \arrow[r, bend left=25, "f_{0,1}"] \arrow[r, bend right=25, "f_{0,k_0}"'] \arrow[r, phantom, bend left=45, "\Downarrow\phi_{0,1}"] \arrow[r, phantom, "\vdots"] \arrow[rrrr, bend right=40, "id_{A_0}"'] \arrow[rrrr, phantom, bend right=30, "\alpha"] & A_1 \arrow[r, dotted, bend left=70] \arrow[r, dotted, bend left=25] \arrow[r, dotted, bend right=25] \arrow[r, phantom, "\vdots"] & \cdots \arrow[r, dotted, bend left=70] \arrow[r, dotted, bend left=25] \arrow[r, dotted, bend right=25] \arrow[r, phantom, "\vdots"] & A_n \arrow[r, bend left=70, "f_{n,0}"] \arrow[r, bend left=25, "f_{n,1}"] \arrow[r, bend right=25, "f_{n,k_n}"'] \arrow[r, phantom, bend left=45, "\Downarrow\phi_{n,1}"] \arrow[r, phantom, "\vdots"] & A_0
\end{tikzcd} = \begin{array}{c} (\phi_{0,1}|\ldots|\phi_{0,k_0}) \otimes \ldots \otimes (\phi_{n,1}|\ldots|\phi_{n,k_n}) \otimes \alpha : \\ s.t.\ \phi_{i,j} \in C^\bullet(A_i, {}_{f_{j-1}}{A_{i+1}}_{f_j}), \\ \alpha \in C_{-\bullet}(A_0, {}_{f_{n,k_n} \ldots f_{0,k_0}}A_0) \end{array} \right\}$$

$$d_{C(A_0 \to \ldots \to A_0)} = d_B \otimes id_{C_{-\bullet}} + id_B \otimes b + \tilde{\iota}$$

Dg comodules over dg cocategories

where $\tilde{\iota}$ is given as follows:

$$\begin{tikzcd}
C(A_0 \to \ldots \to A_0)^\bullet(f_{0,0}, \ldots, f_{n,0}) \arrow[rr, "\tilde{\iota}"] \arrow[drr, "\tilde{\iota}"'] & & C_{-\bullet}(A_0,{}_{f_{n,0}}\ldots{}_{f_{0,0}} A_0) \\
& & C(A_0 \to \ldots \to A_0)^\bullet(f_{0,0}, \ldots, f_{n,0}) \arrow[u, two heads]
\end{tikzcd}$$

Dg comodules over dg cocategories

where $\tilde{\iota}$ is given as follows:

$$\begin{tikzcd}
C(A_0 \to \ldots \to A_0)^\bullet(f_{0,0}, \ldots, f_{n,0}) \arrow[r, "\tilde{\iota}"] \arrow[dr, "\tilde{\iota}"'] & C_{-\bullet}(A_{0,f_{n,0}\ldots f_{0,0}} A_0) \\
 & C(A_0 \to \ldots \to A_0)^\bullet(f_{0,0}, \ldots, f_{n,0}) \arrow[u, two heads]
\end{tikzcd}$$

$$\tilde{\iota}\big((\phi_{0,1}|\ldots|\phi_{0,k_0})\otimes\ldots\otimes(\phi_{n,1}|\ldots|\phi_{n,k_n})\otimes\alpha\big) = \iota_{(\phi_{0,1}|\ldots|\phi_{0,k_0})\bullet\ldots\bullet(\phi_{n,1}|\ldots|\phi_{n,k_n})}\alpha$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

$$\mathcal{A} := (A_0 \to \ldots \to A_n \to A_0)$$

$$\chi_\infty \to \mathcal{D}_\infty$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

$$\mathcal{A} := (A_0 \to \ldots \to A_n \to A_0)$$

$$\begin{aligned} \chi_\infty &\to \mathcal{D}_\infty \\ \mathcal{A} &\mapsto (B(\mathcal{A}), C(\mathcal{A})) \end{aligned}$$

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$$\begin{aligned} \chi_\infty &\to \mathcal{D}_\infty \\ \mathcal{A} &\mapsto (B(\mathcal{A}), C(\mathcal{A})) \end{aligned}$$

Generating Morphisms:

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

$$\mathcal{A} := (A_0 \to \ldots \to A_n \to A_0)$$

$$\chi_\infty \to \mathcal{D}_\infty$$

$$\mathcal{A} \mapsto (B(\mathcal{A}), C(\mathcal{A}))$$

$$\text{Generating Morphisms:} \quad \sigma_{i,n} \mapsto \begin{pmatrix} B(\mathcal{A}) \xrightarrow{\hat{\sigma}_{i,n}} B(\sigma_{i,n}\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{id} \hat{\sigma}^*_{i,n} C(\sigma_{i,n}\mathcal{A}) \end{pmatrix}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

$$\mathcal{A} := (A_0 \to \ldots \to A_n \to A_0)$$

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$$\text{Generating Morphisms:} \quad \sigma_{i,n} \mapsto \begin{pmatrix} B(\mathcal{A}) \xrightarrow{\hat{\sigma}_{i,n}} B(\sigma_{i,n}\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{id} \hat{\sigma}^*_{i,n} C(\sigma_{i,n}\mathcal{A}) \end{pmatrix}$$

$$\delta_{j,n} \mapsto \begin{pmatrix} B(\mathcal{A}) \xrightarrow{\hat{\delta}_{j,n}} B(\delta_{j,n}\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{id} \hat{\delta}^*_{j,n} C(\delta_{j,n}\mathcal{A}) \end{pmatrix}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

$$\mathcal{A} := (A_0 \to \ldots \to A_n \to A_0)$$

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$$\delta_{j,n} \mapsto \begin{pmatrix} B(\mathcal{A}) \xrightarrow{\hat{\delta}_{j,n}} B(\delta_{j,n}\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{id} \hat{\delta}^*_{j,n} C(\delta_{j,n}\mathcal{A}) \end{pmatrix} \quad \tau_n \mapsto \begin{pmatrix} B(\mathcal{A}) \xrightarrow{\hat{\tau}_n} B(\tau_n\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{\tau_{n!}} \hat{\tau}^*_n C(\tau_n\mathcal{A}) \end{pmatrix}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

$$\hat{\tau}_n^* C(\tau_n \mathcal{A})^\bullet(\underbrace{f_0, \dots, f_n}_{\in Obj(B(\mathcal{A}))}) \cong C(\tau_n \mathcal{A})^\bullet(f_n, f_0, \dots, f_{n-1}) \text{ as complexes.}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

$$\hat{\tau}_n^* C(\tau_n\mathcal{A})^\bullet(\underbrace{f_0, \ldots, f_n}_{\in Obj(B(\mathcal{A}))}) \cong C(\tau_n\mathcal{A})^\bullet(f_n, f_0, \ldots, f_{n-1}) \text{ as complexes.}$$

$$\begin{tikzcd}
\cong \{\ A_n \arrow[r, bend left=70, "f_{n,0}"] \arrow[r, bend left=35, "f_{n,1}"] \arrow[r, phantom, shift left=4ex, "\Downarrow\phi_{n,1}"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=35, "f_{n,k_n}"'] \arrow[rrrr, bend right=40, "\alpha", "id_{A_n}"'] & A_0 \arrow[r, dotted, bend left=70] \arrow[r, dotted, bend left=35] \arrow[r, phantom, "\vdots"] \arrow[r, dotted, bend right=35] & \cdots \arrow[r, dotted, bend left=70] \arrow[r, dotted, bend left=35] \arrow[r, phantom, "\vdots"] \arrow[r, dotted, bend right=35] & A_{n-1} \arrow[r, bend left=70, "f_{n-1,0}"] \arrow[r, bend left=35, "f_{n-1,1}"] \arrow[r, phantom, shift left=4ex, "\Downarrow\phi_{n-1,1}"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=35, "f_{n-1,k_{n-1}}"'] & A_n\ \}
\end{tikzcd}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

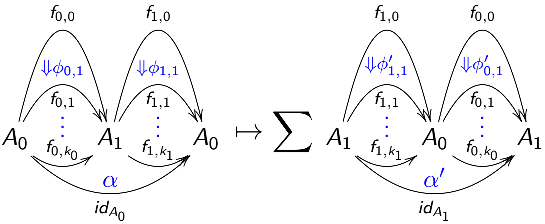
$$\hat{\tau}_n^* C(\tau_n \mathcal{A})^\bullet(\underbrace{f_0, \ldots, f_n}_{\in Obj(B(\mathcal{A}))}) \cong C(\tau_n \mathcal{A})^\bullet(f_n, f_0, \ldots, f_{n-1}) \text{ as complexes.}$$

$$n = 1 : \quad C(A_0 \to A_1 \to A_0)^\bullet(f, g) \xrightarrow{\tau_{1!}} C(A_1 \to A_0 \to A_1)^\bullet(g, f)$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

$$\hat{\tau}_n^* C(\tau_n \mathcal{A})^\bullet(\underbrace{f_0, \ldots, f_n}_{\in Obj(B(\mathcal{A}))}) \cong C(\tau_n \mathcal{A})^\bullet(f_n, f_0, \ldots, f_{n-1}) \text{ as complexes.}$$

$$n = 1 : \quad C(A_0 \to A_1 \to A_0)^\bullet(f, g) \xrightarrow{\tau_{1!}} C(A_1 \to A_0 \to A_1)^\bullet(g, f)$$



An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

$$\hat{\tau}_n^* C(\tau_n\mathcal{A})^\bullet(\underbrace{f_0, \ldots, f_n}_{\in Obj(B(\mathcal{A}))}) \cong C(\tau_n\mathcal{A})^\bullet(f_n, f_0, \ldots, f_{n-1}) \text{ as complexes.}$$

$$n = 1 : \quad C(A_0 \to A_1 \to A_0)^\bullet(f, g) \xrightarrow{\tau_{1!}} C(A_1 \to A_0 \to A_1)^\bullet(g, f)$$

$$\begin{tikzcd}
A_0 \arrow[r, bend left=80, "f_{0,0}"] \arrow[r, phantom, bend left=60, "\Downarrow\phi_{0,1}"] \arrow[r, bend left=40, "f_{0,1}"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=30, "f_{0,k_0}"] \arrow[rr, bend right=60, "\alpha", "id_{A_0}"'] & A_1 \arrow[r, bend left=80, "f_{1,0}"] \arrow[r, phantom, bend left=60, "\Downarrow\phi_{1,1}"] \arrow[r, bend left=40, "f_{1,1}"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=30, "f_{1,k_1}"] & A_0 \arrow[r, phantom, "\mapsto"] & A_1 \arrow[r, "f_{1,0}"] \arrow[rr, bend right=60, "\alpha'", "id_{A_1}"'] & A_0 \arrow[r, "f_{0,0}"] & A_1
\end{tikzcd}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

Solution: Give these maps to cogenerators to define $\tau_{1!}$, then let

$$\begin{aligned}\tau_{n!} : C(\mathcal{A}) &\cong \hat{\delta}_0^{*\,n-1} C(A_0 \to A_n \to A_0) \xrightarrow{\hat{\delta}_0^{*\,n-1}\tau_{1!}} \hat{\delta}_0^{*\,n-1} \hat{\tau}_1^* C(A_n \to A_0 \to A_n) \cong \\ &\cong (\widehat{\tau_1 \delta_0^{n-1}})^* C(A_n \to A_0 \to A_n) \cong (\widehat{\delta_0^{n-1} \tau_n})^* C(A_n \to A_0 \to A_n) \cong \\ &\cong \hat{\tau}_n^* \hat{\delta}_0^{*\,n-1} C(A_n \to A_0 \to A_n) \cong \hat{\tau}_n^* C(\tau_n \mathcal{A}).\end{aligned}$$

$$n = 1 : \quad C(A_0 \to A_1 \to A_0)^\bullet(f, g) \xrightarrow{\tau_{1!}} C(A_1 \to A_0 \to A_1)^\bullet(g, f)$$

$$\begin{tikzcd}
A_0 \arrow[r, bend left=70, "f_{0,0}"] \arrow[r, bend left=35, "f_{0,1}"'] \arrow[r, phantom, bend left=55, "\Downarrow\phi_{0,1}"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=30, "f_{0,k_0}"] \arrow[rr, bend right=50, "\alpha", "id_{A_0}"'] & A_1 \arrow[r, bend left=70, "f_{1,0}"] \arrow[r, bend left=35, "f_{1,1}"'] \arrow[r, phantom, bend left=55, "\Downarrow\phi_{1,1}"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=30, "f_{1,k_1}"] & A_0 \arrow[r, phantom, "\mapsto"] & A_1 \arrow[r, "f_{1,0}"] \arrow[rr, bend right=50, "\alpha'", "id_{A_1}"'] & A_0 \arrow[r, "f_{0,0}"] & A_1
\end{tikzcd}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

Defining $\tau_{1!}$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

Defining $\tau_{1!}$

$$\begin{tikzcd}
A_0 \arrow[r, "f_{0,0}"] \arrow[rr, bend right, "id_{A_0}"', "\alpha"] & A_1 \arrow[r, "f_{1,0}"] & A_0 \arrow[r, phantom, "\overset{?}{\mapsto}"] & A_1 \arrow[r, "f_{1,0}"] \arrow[rr, bend right, "id_{A_1}"', "\alpha'"] & A_0 \arrow[r, "f_{0,0}"] & A_1
\end{tikzcd}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

Defining $\tau_{1!}$

$$\begin{tikzcd}
A_0 \arrow[r, "f_{0,0}"] \arrow[rr, bend right, "\alpha", "id_{A_0}"'] & A_1 \arrow[r, "f_{1,0}"] & A_0 \arrow[r, phantom, "\overset{?}{\mapsto}"] & A_1 \arrow[r, "f_{1,0}"] \arrow[rr, bend right, "\alpha'", "id_{A_1}"'] & A_0 \arrow[r, "f_{0,0}"] & A_1
\end{tikzcd}$$

$$\alpha = a_0 \otimes \ldots \otimes a_n \mapsto \alpha' = f_{0,0}(a_0) \otimes \ldots \otimes f_{0,0}(a_n)$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

Defining $\tau_{1!}$

$$\begin{tikzcd}
A_0 \arrow[r, "f_{0,1}"] \arrow[r, bend left=60, "f_{0,0}"] \arrow[r, phantom, bend left=30, "\Downarrow\phi"] \arrow[rr, bend right=50, "id_{A_0}"', "\alpha"] & A_1 \arrow[r, "f_{1,0}"] & A_0 \arrow[r, phantom, "\overset{?}{\mapsto}"] & A_1 \arrow[r, "f_{1,0}"] \arrow[rr, bend right=50, "id_{A_1}"', "\alpha'"] & A_0 \arrow[r, "f_{0,0}"] & A_1
\end{tikzcd}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

Defining $\tau_{1!}$

$$\begin{tikzcd}
A_0 \arrow[r, "f_{0,1}"] \arrow[r, bend left=60, "f_{0,0}"] \arrow[r, phantom, bend left=30, "\Downarrow\phi"] \arrow[rr, bend right=40, "id_{A_0}"'] \arrow[rr, phantom, bend right=30, "\alpha"] & A_1 \arrow[r, "f_{1,0}"] & A_0 \arrow[r, phantom, "\overset{?}{\mapsto}"] & A_1 \arrow[r, "f_{1,0}"] \arrow[rr, bend right=40, "id_{A_1}"'] \arrow[rr, phantom, bend right=30, "\alpha'"] & A_0 \arrow[r, "f_{0,0}"] & A_1
\end{tikzcd}$$

$$\bar{\tau}_{1!} \circ d(\phi \otimes \alpha) = d \circ \bar{\tau}_{1!}(\phi \otimes \alpha)$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

Defining $\tau_{1!}$

$$\begin{tikzcd}
A_0 \arrow[r, "f_{0,1}"] \arrow[r, bend left=70, "f_{0,0}"] \arrow[r, phantom, shift left=3ex, "\Downarrow\phi"] \arrow[rr, bend right=50, "\alpha", "id_{A_0}"'] & A_1 \arrow[r, "f_{1,0}"] & A_0 \arrow[r, phantom, "\overset{?}{\mapsto}"] & A_1 \arrow[r, "f_{1,0}"] \arrow[rr, bend right=50, "\alpha'", "id_{A_1}"'] & A_0 \arrow[r, "f_{0,0}"] & A_1
\end{tikzcd}$$

$$[b, \bar{\tau}_{1!}](\phi \otimes \alpha) \pm \bar{\tau}_{1!}(\delta\phi \otimes \alpha) = [\bar{\tau}_{1!}, \iota_\phi](\alpha)$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

Defining $\tau_{1!}$

$$\begin{tikzcd}
A_0 \arrow[r, "f_{0,1}"] \arrow[r, bend left=60, "f_{0,0}"] \arrow[r, phantom, bend left=30, "\Downarrow\phi"] \arrow[rr, bend right=40, "\alpha", "id_{A_0}"'] & A_1 \arrow[r, "f_{1,0}"] & A_0 \arrow[r, mapsto, "?"] & A_1 \arrow[r, "f_{1,0}"] \arrow[rr, bend right=40, "\alpha'", "id_{A_1}"'] & A_0 \arrow[r, "f_{0,0}"] & A_1
\end{tikzcd}$$

$$[b, \bar{\tau}_{1!}](\phi \otimes \alpha) \pm \bar{\tau}_{1!}(\delta\phi \otimes \alpha) = [\bar{\tau}_{1!}, \iota_\phi](\alpha)$$

$$\begin{aligned} L_\phi(\alpha) = &\sum_{k \geq 1} \pm a_0 \otimes \ldots \otimes \phi(a_k, \ldots) \otimes a_r \otimes \ldots \otimes a_n + \\ &\sum \pm \phi(a_k, \ldots, a_n, a_0, \ldots) \otimes a_s \otimes \ldots \otimes a_{k-1} \end{aligned}$$

$$[b, L_\phi] \pm L_{\delta\phi} = 0$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

Defining $\tau_{1!}$

$$\begin{tikzcd}
A_0 \arrow[r, bend left=60, "f_{0,0}"] \arrow[r, phantom, bend left=30, "\Downarrow\phi"] \arrow[r, "f_{0,1}"] \arrow[rr, bend right=50, "\alpha", "id_{A_0}"'] & A_1 \arrow[r, "f_{1,0}"] & A_0 \arrow[r, phantom, "\overset{?}{\mapsto}"] & A_1 \arrow[r, "f_{1,0}"] \arrow[rr, bend right=50, "\alpha'", "id_{A_1}"'] & A_0 \arrow[r, "f_{0,0}"] & A_1
\end{tikzcd}$$

$$[b, \bar{\tau}_{1!}](\phi \otimes \alpha) \pm \bar{\tau}_{1!}(\delta\phi \otimes \alpha) = [\bar{\tau}_{1!}, \iota_\phi](\alpha)$$

$$\bar{\tau}_{1!}(\phi \otimes \alpha) = \sum_{k \geq 1} \pm f_{0,0} a_0 \otimes \ldots \otimes \phi(a_k, \ldots) \otimes f_{0,1} a_r \ldots \otimes f_{0,1} a_n +$$

$$\sum \pm \phi(f_{1,0} f_{0,1} a_k, \ldots, f_{1,0} f_{0,1} a_n, a_0, \ldots) \otimes f_{0,1} a_s \otimes \ldots \otimes f_{0,1} a_{k-1}$$

$$[b, \bar{\tau}_{1!}(\phi, -)] \pm \bar{\tau}_{1!}(\delta\phi, -) = [\bar{\tau}_{1!}, \iota_\phi]$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

Defining $\tau_{1!}$

$$\bar{\tau}_{1!}\big((\phi_{0,1}|\ldots|\phi_{0,k_0})\otimes(\phi_{1,1}|\ldots|\phi_{1,k_1})\otimes\alpha\big) =$$

$$= \sum_{\substack{1\leq i\leq n_0\\ i\leq j_1\leq\ldots\leq j_{2k_1}\leq k_0\\ p}} \pm\phi_{0,1}\Big(\begin{matrix} f_{1,0}f_{0,i}a_p,\ldots,f_{1,0}\phi_{0,i+1}(a_*,\ldots), \\ f_{1,0}f_{0,i+1}a_*,\ldots,f_{1,0}\phi_{0,j_1}(a_*,\ldots), \\ f_{1,0}f_{0,j_1}a_*,\ldots,\phi_{1,1}(f_{0,j_1}a_*,\ldots,\phi_{0,j_1+1}(a_*,\ldots),\ldots),\ldots, \\ \phi_{1,k_1}(f_{0,j_{2k_1-1}}a_*,\ldots,\phi_{0,j_{2k_1-1}+1}(a_*,\ldots),\ldots),\ldots,a_0,\ldots \end{matrix}\Big)\otimes$$

$$\otimes f_{0,1}a_*\otimes\ldots\otimes\phi_{0,2}(a_*,\ldots)\otimes f_{0,2}a_*\otimes\ldots\otimes$$

$$\otimes\phi_{0,i}(a_*,\ldots)\otimes f_{0,i}a_*\otimes\ldots f_{0,i}a_{p-1}+$$

$$\Big(\sum\pm f_{0,0}a_0\otimes\ldots\otimes\phi_{0,1}(a_*,\ldots)\otimes\ldots\otimes\phi_{0,n_0}(a_*,\ldots)\otimes$$

$$\otimes f_{0,n_0}a_*\otimes\ldots\otimes f_{0,n_0}a_n \quad \text{if } k_1=0\Big)$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

$$\mathcal{A} := (A_0 \to \ldots \to A_n \to A_0)$$

$$\begin{aligned} \chi_\infty &\to \mathcal{D}_\infty \\ \mathcal{A} &\mapsto (B(\mathcal{A}), C(\mathcal{A})) \\ \text{Generating Morphisms:} \quad \sigma_{i,n} &\mapsto \begin{pmatrix} B(\mathcal{A}) \xrightarrow{\hat{\sigma}_{i,n}} B(\sigma_{i,n}\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{id} \hat{\sigma}^*_{i,n} C(\sigma_{i,n}\mathcal{A}) \end{pmatrix} \end{aligned}$$

$$\delta_{j,n} \mapsto \begin{pmatrix} B(\mathcal{A}) \xrightarrow{\hat{\delta}_{j,n}} B(\delta_{j,n}\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{id} \hat{\delta}^*_{j,n} C(\delta_{j,n}\mathcal{A}) \end{pmatrix} \quad \tau_n \mapsto \begin{pmatrix} B(\mathcal{A}) \xrightarrow{\hat{\tau}_n} B(\tau_n\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{\tau_{n!} = d_0^{*n-1}\tau_{1!}} \hat{\tau}^*_n C(\tau_n\mathcal{A}) \end{pmatrix}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

n=1:

$$\begin{tikzcd}
C(A_0 \to A_1 \to A_0) \arrow[r, "\tau_{1!}"] \arrow[rr, bend right, "id"'] & \hat{\tau}_1^* C(\tau_1(A_0 \to A_1 \to A_0)) \arrow[r, "\hat{\tau}_1^* \tau_{1!}"] & \hat{\tau}_1^{*2} C(\tau_1^2(A_0 \to A_1 \to A_0))
\end{tikzcd}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

n=1:

$$\begin{tikzcd}
C(A_0 \to A_1 \to A_0) \arrow[r, "\tau_{1!}"] \arrow[rr, bend right, "id"'] & \hat{\tau}_1^* C(\tau_1(A_0 \to A_1 \to A_0)) \arrow[r, "\hat{\tau}_1^* \tau_{1!}"] & \hat{\tau}_1^{*2} C(\tau_1^2(A_0 \to A_1 \to A_0))
\end{tikzcd}$$

$$\begin{tikzcd}
A_0 \arrow[r, "f_{0,0}"] \arrow[rr, bend right, "\alpha", "id_{A_0}"'] & A_1 \arrow[r, "f_{1,0}"] & A_0
\end{tikzcd}$$

$$\alpha = a_0 \otimes \ldots \otimes a_n \overset{\tau_{1!}}{\mapsto} f_{0,0}a_0 \otimes \ldots \otimes f_{0,0}a_n \overset{\hat{\tau}^*_{1!}\tau_{1!}}{\mapsto} f_{1,0}f_{0,0}a_0 \otimes \ldots \otimes f_{1,0}f_{0,0}a_n$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

n=1:

$$\begin{tikzcd}
C(A_0 \to A_1 \to A_0) \arrow[r, "\tau_{1!}"] \arrow[rr, bend right, "id"'] & \hat{\tau}_1^* C(\tau_1(A_0 \to A_1 \to A_0)) \arrow[r, "\hat{\tau}_1^* \tau_{1!}"] & \hat{\tau}_1^{*2} C(\tau_1^2(A_0 \to A_1 \to A_0))
\end{tikzcd}$$

$$\begin{tikzcd}
A_0 \arrow[r, "f_{0,0}"] \arrow[rr, bend right, "\alpha", "id_{A_0}"'] & A_1 \arrow[r, "f_{1,0}"] & A_0
\end{tikzcd}$$

$$\alpha = a_0 \otimes \ldots \otimes a_n \overset{\tau_{1!}}{\mapsto} f_{0,0} a_0 \otimes \ldots \otimes f_{0,0} a_n \overset{\hat{\tau}_{1!}^* \tau_{1!}}{\mapsto} f_{1,0} f_{0,0} a_0 \otimes \ldots \otimes f_{1,0} f_{0,0} a_n$$

$$f_{1,0} f_{0,0} \alpha - \alpha = [b, B](\alpha)$$

$$B(a_0 \otimes \ldots \otimes a_n) = \sum_{0 \leq i \leq n} \pm 1 \otimes f_{1,0} f_{0,0} a_i \otimes \ldots \otimes f_{1,0} f_{0,0} a_n \otimes a_0 \otimes \ldots \otimes a_{i-1}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

n=1:

$$\begin{tikzcd}
C(A_0 \to A_1 \to A_0) \arrow[r, "\tau_{1!}"] \arrow[rr, bend right=15, "id"'] & \hat{\tau}_1^* C(\tau_1(A_0 \to A_1 \to A_0)) \arrow[r, "\hat{\tau}_1^* \tau_{1!}"] & \hat{\tau}_1^{*2} C(\tau_1^2(A_0 \to A_1 \to A_0))
\end{tikzcd}$$

$$\begin{tikzcd}
A_0 \arrow[r, bend left=60, "f_{0,0}"] \arrow[r, phantom, bend left=30, "\Downarrow\phi"] \arrow[r, "f_{0,1}"] \arrow[rr, bend right=50, "id_{A_0}"'] \arrow[rr, phantom, bend right=30, "\alpha"] & A_1 \arrow[r, "f_{1,0}"] & A_0 & = \phi \otimes \alpha
\end{tikzcd}$$

$$(\overline{\hat{\tau}^* \tau_{1!} \circ \tau_{1!} - id})(\phi \otimes \alpha) = “f_{1,0} \circ L_\phi(\alpha)”$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

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C(A_0 \to A_1 \to A_0) \arrow[r, "\tau_{1!}"] \arrow[rr, bend right, "id"'] & \hat{\tau}_1^* C(\tau_1(A_0 \to A_1 \to A_0)) \arrow[r, "\hat{\tau}_1^* \tau_{1!}"] & \hat{\tau}_1^{*2} C(\tau_1^2(A_0 \to A_1 \to A_0))
\end{tikzcd}$$

$$(\overline{\hat{\tau}^* \tau_{1!} \circ \tau_{1!} - id})(\phi \otimes \alpha) = \text{``}f_{1,0} \circ L_\phi(\alpha)''$$

$$= \sum_{k \geq 1} \pm f_{1,0} f_{0,0} a_0 \otimes \ldots \otimes f_{1,0} \phi(a_k, \ldots) \otimes f_{1,0} f_{0,1} a_r \ldots \otimes f_{1,0} f_{0,1} a_n +$$

$$\sum \pm f_{1,0} \phi(f_{1,0} f_{0,1} a_k, \ldots, f_{1,0} f_{0,1} a_n, a_0, \ldots) \otimes f_{1,0} f_{0,1} a_s \otimes \ldots \otimes f_{1,0} f_{0,1} a_{k-1}$$

$$\overset{?}{=} [\iota_\phi, B](\alpha) \pm [b, B](\phi \otimes \alpha) \pm B(\delta\phi \otimes \alpha)$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

n=1:

$$\begin{tikzcd}
C(A_0 \to A_1 \to A_0) \arrow[r, "\tau_{1!}"] \arrow[rr, bend right, "id"'] & \hat{\tau}_1^* C(\tau_1(A_0 \to A_1 \to A_0)) \arrow[r, "\hat{\tau}_1^* \tau_{1!}"] & \hat{\tau}_1^{*2} C(\tau_1^2(A_0 \to A_1 \to A_0))
\end{tikzcd}$$

$$(\overline{\hat{\tau}^* \tau_{1!} \circ \tau_{1!} - id})(\phi \otimes \alpha) = ``f_{1,0} \circ L_\phi(\alpha)''$$

$$= \sum_{k \geq 1} \pm f_{1,0} f_{0,0} a_0 \otimes \dots \otimes f_{1,0} \phi(a_k, \dots) \otimes f_{1,0} f_{0,1} a_r \dots \otimes f_{1,0} f_{0,1} a_n +$$

$$\sum \pm f_{1,0} \phi(f_{1,0} f_{0,1} a_k, \dots, f_{1,0} f_{0,1} a_n, a_0, \dots) \otimes f_{1,0} f_{0,1} a_s \otimes \dots \otimes f_{1,0} f_{0,1} a_{k-1}$$

$$\overset{?}{=} [\iota_\phi, B](\alpha) \pm [b, B](\phi \otimes \alpha) \pm B(\delta\phi \otimes \alpha)$$

$$B(\phi \otimes \alpha) = \sum_{0 \leq i \leq n} \pm 1 \otimes f_{1,0} f_{0,0} a_i \otimes \dots \otimes f_{1,0} \phi(a_*, \dots) \otimes f_{1,0} f_{0,1} a_* \otimes \dots \otimes f_{1,0} f_{0,1} a_n \otimes a_0 \otimes \dots \otimes a_{i-1}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

n=1:

$$\begin{aligned}
&B\big((\phi_{0,1}|\dots|\phi_{0,k_0}) \otimes (\phi_{1,1}|\dots|\phi_{1,k_1}) \otimes \alpha\big) = \\
&= \sum_{\substack{0 \leq j_1 \leq \dots \leq j_{2k_1} \leq k_0 \\ p}} \pm 1 \otimes f_{1,0} f_{0,0} a_p \otimes \dots \otimes f_{1,0} \phi_{0,1}(a_*, \dots) \otimes \\
&\qquad \otimes f_{1,0} f_{0,1} a_* \otimes \dots \otimes f_{1,0} \phi_{0,j_1}(a_*, \dots) \otimes \\
&\qquad \otimes f_{1,0} f_{0,j_1} a_* \otimes \dots \otimes \phi_{1,1}(f_{0,j_1} a_*, \dots, \phi_{0,j_1+1}(a_*, \dots), \dots) \otimes \\
&\qquad \otimes \dots \otimes \phi_{1,k_1}(f_{0,j_{2k_1-1}} a_*, \dots, \phi_{0,j_{2k_1-1}+1}(a_*, \dots), \dots) \otimes \dots \otimes \\
&\qquad \otimes a_0 \otimes \dots \otimes a_{p-1}
\end{aligned}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

$$\begin{aligned}
\chi_\infty &\to \mathcal{D}_\infty \\
\mathcal{A} &\mapsto (B(\mathcal{A}), C(\mathcal{A})) \\
\tau_1 &\mapsto \begin{pmatrix} \hat{\tau}_1 : B(\mathcal{A}) \to B(\tau_1 \mathcal{A}) \\ \tau_{1!} : C(\mathcal{A}) \to \hat{\tau}_1^* C(\tau_1 \mathcal{A}) \end{pmatrix} \\
(\tau_1, \tau_1) &\mapsto \begin{pmatrix} id : B(\mathcal{A}) \to B(\mathcal{A}) \\ B : C(\mathcal{A}) \to C(\mathcal{A}) \end{pmatrix} \\
(\tau_1, \tau_1, \tau_1) &\mapsto \begin{pmatrix} \hat{\tau}_1 : B(\mathcal{A}) \to B(\tau_1 \mathcal{A}) \\ 0 : C(\mathcal{A}) \to \hat{\tau}_1^* C(\tau_1 \mathcal{A}) \end{pmatrix} \\
&\vdots
\end{aligned}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

The A_∞ relations mean:

- $\tau_{1!}$ is a map of complexes
- $B^2 = 0$
- The following diagram commutes:

$$\begin{tikzcd}
C(\mathcal{A}) \arrow[r, "\tau_{1!}"] \arrow[d, "B"] & \hat{\tau}_1^* C(\tau_1 \mathcal{A}) \arrow[d, "\hat{\tau}_1^* B"] \\
C(\mathcal{A}) \arrow[r, "\tau_{1!}"'] & \hat{\tau}_1^* C(\tau_1 \mathcal{A})
\end{tikzcd}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

For higher $n > 1$, we want to find a homotopy between "$\tau_{n!}^{n+1}$" and id.

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

For higher $n > 1$, we want to find a homotopy between "$\tau_{n!}^{n+1}$" and *id*. However, it is sufficient to find a homotopy between $\hat{\tau}_n^{*2}\delta_{0,n!} \circ \hat{\tau}_n^{*}\tau_{n!} \circ \tau_{n!}$ and $\hat{\delta}^{*}_{n-1,n}\tau_{n-1!} \circ \delta_{n-1,n!}$.

$$\begin{tikzcd}
{[n]} \arrow[r, "\tau_n"] \arrow[d, "\delta_{n-1,n}"] & {[n]} \arrow[r, "\tau_n"] & {[n]} \arrow[dl, "\delta_{0,n}"] \\
{[n-1]} \arrow[r, "\tau_{n-1}"'] & {[n-1]} &
\end{tikzcd}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

For higher $n > 1$, we want to find a homotopy between "$\tau_{n!}^{n+1}$" and *id*. However, it is sufficient to find a homotopy between $\hat{\tau}_n^{*2}\delta_{0,n!} \circ \hat{\tau}_n^{*}\tau_{n!} \circ \tau_{n!}$ and $\hat{\delta}_{n-1,n}^{*}\tau_{n-1!} \circ \delta_{n-1,n!}$.

$$\begin{tikzcd}
{[n]} \arrow[r, "\tau_n"] \arrow[d, "\delta_{n-1,n}"] & {[n]} \arrow[r, "\tau_n"] & {[n]} \arrow[dl, "\delta_{0,n}"] \\
{[n-1]} \arrow[r, "\tau_{n-1}"'] & {[n-1]} &
\end{tikzcd}$$

Solution: Find such a homotopy, $\mathcal{B}$, for $n = 2$, and use $\hat{\delta}_0^{*n-2}\mathcal{B}$ for $n > 2$.

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

For higher n, the A_∞ relations mean:

- $\tau_{n!}$ is a map of complexes
- $\mathcal{B}^2 = 0$
- We have a pair of homotopic maps:

$$\begin{tikzcd}
C(A_0 \to \ldots \to A_n \to A_0) \arrow[d, bend right, "{\begin{array}{c} \hat{\tau}_n^{*2}\tau_{n!}\circ\hat{\tau}_n^{*}\tau_{n!}\circ\tau_{n!} \\ \text{``apply } \tau_{n!} \text{ 3 times''} \end{array}}"'] \arrow[d, bend left, "{\begin{array}{c} (\delta_{n-2,}\widehat{{}_{n-1}\delta}_{n-1,n})^*\tau_{n-2!} \\ \text{``brace together the last 3 algebras,} \\ \text{then apply } \tau_{n-2!} \text{ once''} \end{array}}"] \\
C(A_{n-2} \to A_{n-1} \to A_n \to A_0 \to \ldots \to A_{n-2}).
\end{tikzcd}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

For higher n, the A_∞ relations mean:

- $\tau_{n!}$ is a map of complexes
- $\mathcal{B}^2 = 0$
- They are homotopic via two different homotopies:

$$\begin{tikzcd}
(\delta_{n-2,}\widehat{{}_{n-1}\delta}_{n-1,n})^*\tau_{n-2!} \arrow[r, "\cong"] & \hat{\delta}^*_{n-1,n}(\hat{\delta}^*_{n-2,n-1}\tau_{n-2!}) \arrow[r, "\hat{\delta}^*_{n-1,n}\mathcal{B}_{n-1}"] & \hat{\delta}^*_{n-1,n}(\hat{\tau}^*_{n-1}\tau_{n-1!} \circ \tau_{n-1!}) \arrow[dd, "\cong"] \\
\text{“brace together } A_{n-2}, A_{n-1}, A_n, \text{ then apply } \tau_{n-2!}\text{”} \arrow[d, "\cong"'] & & \\
(\delta_{n-2,}\widehat{{}_{n-1}\delta}_{n-2,n})^*\tau_{n-2!} \arrow[dd, "\hat{\delta}^*_{n-2,n}\mathcal{B}_{n-1}"'] & & \hat{\tau}^{*2}_n\tau_{n!} \circ \hat{\delta}^*_{n-1,n}\tau_{n-1!} \\
 & & \text{“brace together } A_{n-1}, A_n \text{ and apply } \tau_{n-1!}, \text{ then apply } \tau_{n!}\text{”} \arrow[d, "\tau^{*2}_n\tau_{n!}\circ\mathcal{B}_n"] \\
\hat{\delta}^*_{n-2,n}(\hat{\tau}^*_{n-1}\tau_{n-1!} \circ \tau_{n-1!}) \arrow[r, "\cong"'] & \hat{\tau}^*_n(\hat{\delta}^*_{n-1,n}\tau_{n-1!}) \circ \tau_{n!} \arrow[r, "\hat{\tau}^*_n\mathcal{B}_n\circ\tau_{n!}"] & \hat{\tau}^{*2}_n\tau_{n!} \circ \hat{\tau}^*_n\tau_{n!} \circ \tau_{n!} \\
 & \text{“apply } \tau_{n!}, \text{ then brace together } A_{n-1}, A_{n-2} \text{ and apply } \tau_{n-1!}\text{”} & \text{“apply } \tau_{n!} \text{ three times”}
\end{tikzcd}$$

Summary:

We have a given a “homotopically sheafy-cyclic object in dg cocategories with a dg comodule”, i.e., an A_∞-functor between dg categories χ_∞ and $\mathcal{D}_\infty$.

Time and interest permitting

- Rectify $\mathcal{F}$ to a dg functor
- Give a dg functor $\mathcal{D}_\infty \to \mathcal{E} = \{(\text{dg cat, dg mod})\}$

$$U(\chi_\infty) \xrightarrow{\textit{rectified}} \mathcal{D}_\infty \to \mathcal{E}$$

“A homotopically sheafy-cyclic object in dg categories with a dg module”