From Noncommutative Calculus to a (Sheafy-)Cyclic Structure

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Date/Event

Background

Theorem

(Hochschild-Kostant-Rosenberg, '62) Let A be a regular, commutative algebra over a field k of characteristic 0. Then,

$$(C_\bullet(A, A), b) \xrightarrow{\sim} \Omega^\bullet_{A/k}$$

$$(C^\bullet(A, A), \delta) \xrightarrow{\sim} \wedge^\bullet T_A = \wedge^\bullet(Der_k(A, A)).$$

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Theorem

(Kontsevich, '97) Let $A = C^\infty(M)$ for M a smooth real manifold. Then, there is an L_∞ map

$$(C^{\bullet+1}(A, A), \delta, [,]_{Ger}) \xrightarrow{\sim} (\wedge^{\bullet+1} T_A, d = 0, [,]_{SN}).$$

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Theorem

(Tamarkin, '98) Dependent on the choice of a Drinfeld associator, there is a Ger_∞ map

$$(C^{\bullet+1}(A,A), \delta, [,]_{Ger}, \cup, ...) \xrightarrow{\sim} (\wedge^\bullet T_A, d = 0, [,]_{SN}, \wedge).$$

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Theorem

(Dolgushev-Tamarkin-Tsygan, '08) There is a $Calc_\infty$ map

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Can we use this calculus structure to create a cyclic object?

A cyclic object in dg cocategories

Fix an algebra, A.
Define a dg category, $Hoch(A)$.

A cyclic object in dg cocategories

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Objects: algebra maps $f : A \to A$

Morphisms: $Hoch(A)(f, g) = (C^\bullet(A, {}_f A_g), {}_f \delta_g)$
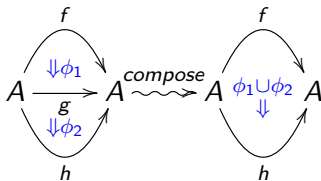
A cyclic object in dg cocategories

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$$
{}_f\delta_g(\phi)(a_1 \otimes ... \otimes a_n) = \epsilon_\phi \bigg(f(a_1) \cdot \phi(a_2, ..., a_n) +
$$
$$
+ \sum_{1 \leq i \leq n-1} (-1)^i \phi(a_1, ..., a_i a_{i+1}, ..., a_n) +
$$
$$
+ (-1)^n \phi(a_1, ..., a_{n-1}) \cdot g(a_n) \bigg)
$$
$$
\epsilon_\phi = (-1)^{|\phi|+1}
$$

A cyclic object in dg cocategories

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   Composition: cup product on cochains

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Fix an algebra, A.
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Take Bar(Hoch(A)).

$$Bar : DGCat \rightarrow DGCocat$$

A cyclic object in dg cocategories

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$$Bar : DGCat \to DGCocat$$

Bar(Hoch(A)) has the same objects as Hoch(A).
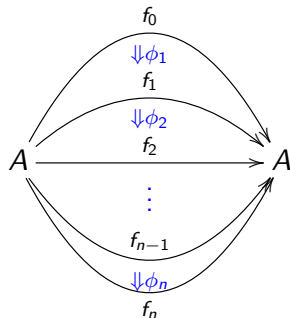
A morphism from f_0 to f_n in Bar(Hoch(A))

A cyclic object in dg cocategories

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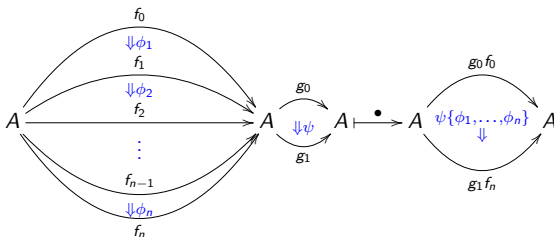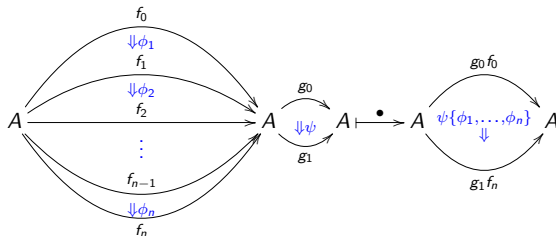$$Bar(Hoch(A)) \otimes Bar(Hoch(A)) \xrightarrow{\;\bullet\;} Hoch(A)$$

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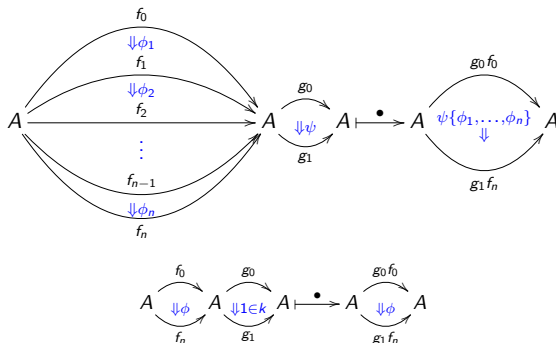
A cyclic object in dg cocategories

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$$Bar(Hoch(A)) \otimes Bar(Hoch(A)) \xrightarrow{\;\bullet\;} Hoch(A)$$

$$\psi\{\phi_1,\ldots,\phi_n\}(a_1,\ldots,a_q) = \sum_{1 \leq i_1 \leq \cdots \leq i_n \leq q} \pm \psi(a_1,\ldots,a_{i_1},\phi_1(a_{i_1+1},\ldots),\ldots,\phi_n(a_{i_n+1},\ldots),\ldots)$$

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A cyclic object in dg cocategories

Fix an algebra, A.
Define a dg category, $Hoch(A)$.
Take Bar(Hoch(A)).

In this context, braces, $\bullet$, give multilinear maps:

$$Bar(Hoch(A)) \otimes Bar(Hoch(A)) \xrightarrow{\;\;\bullet\;\;} Hoch(A)$$

$$\bullet$$

$$Bar(Hoch(A))$$

Then, (Bar(Hoch(A)), $\bullet$) is an algebra in DGCocats.

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Then, (Bar(Hoch(A)), $\bullet$) is an algebra in DGCocats.
But we have more...

A cyclic object in dg cocategories

A cyclic object in dg cocategories

Fix algebras, $A_0, A_1, ..., A_n$.

We will define a dg cocategory $B(A_0 \to A_1 \to ... \to A_n \to A_0)$

where $B(A_0 \to A_0) := Bar(Hoch(A_0))$ for n=0.

A cyclic object in dg cocategories

Fix algebras, $A_0, A_1, ..., A_n$.

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   where $B(A_0 \to A_0) := Bar(Hoch(A_0))$ for n=0.

Objects: $\{A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} ... \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_0\}$

A morphism from $(A_0 \xrightarrow{f_{0,0}} ... \xrightarrow{f_{n,0}} A_0)$ to $(A_0 \xrightarrow{f_{0,k_0}} ... \xrightarrow{f_{n,k_n}} A_0)$:
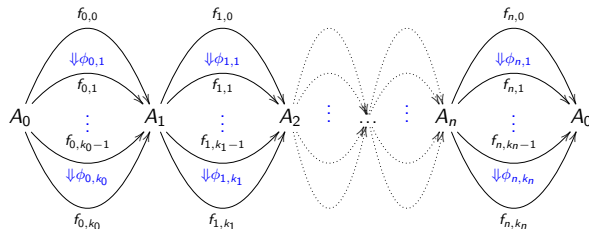
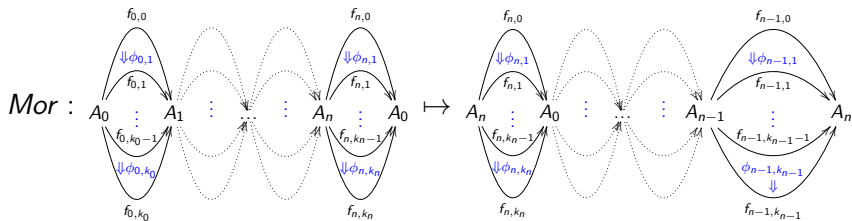$$\phi_{i,j} \in C^\bullet(A_{i,f_{i,j-1}} A_{i+1_{f_{i,j}}})$$

A cyclic object in dg cocategories

Example

We have a dg functor

$$\hat{\tau}_n : B(A_0 \to \ldots \to A_n \to A_0) \to B(A_n \to A_0 \to \ldots \to A_n)$$

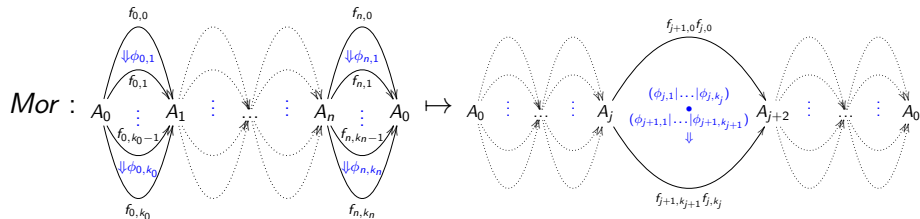$$Obj : \qquad\qquad (f_0, \ldots, f_n) \mapsto (f_n, f_0, \ldots, f_{n-1})$$

A cyclic object in dg cocategories

Example

For $n \geq 1, 0 \leq j < n$, we have a dg functor

$$\hat{\delta}_{j,n} : B(A_0 \to \ldots \to A_n \to A_0) \to B(A_0 \to \ldots \to A_j \to A_{\substack{j+2 \\ (mod\, n+1)}} \to \ldots \to A_0)$$

$$Obj : \qquad\qquad (f_0, \ldots, f_n) \mapsto (f_0, \ldots, f_{j+1} f_j, \ldots, f_n)$$

$Mor :$

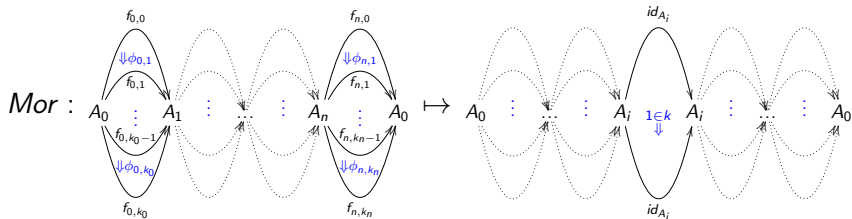# A cyclic object in dg cocategories

Example

For $n \geq 0, 0 \leq i \leq n$, we have a dg functor

$$\hat{\sigma}_{i,n} : B(A_0 \to \ldots \to A_n \to A_0) \to B(A_0 \to \ldots \to A_i \to A_i \to \ldots \to A_0)$$

$$Obj : \qquad\qquad (f_0, \ldots, f_n) \mapsto (f_0, \ldots, f_{i-1}, id_{A_i}, f_i, \ldots, f_n)$$

Mor :

A **sheafy**-cyclic object in DGCocat

Definition

Let χ be the category with objects $\{A_0 \to A_1 \to ... \to A_n \to A_0\}$ and morphisms compositions of

$$\text{rotations } \tau_n : \mathcal{A} \mapsto (A_n \to A_0 \to ... \to A_n)$$

$$\text{coboundaries } \delta_{j,n} : \mathcal{A} \mapsto (A_0 \to ... \to A_j \to A_{j+2\,(mod\ n+1)} \to ... \to A_0)$$

$$\text{codegeneracies } \sigma_{i,n} : \mathcal{A} \mapsto (A_0 \to ... \to A_i \to A_i \to ... \to A_0)$$

where $\mathcal{A} := (A_0 \to ... \to A_n \to A_0)$, subject to the cyclic relations.

A **sheafy**-cyclic object in DGCocat

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Proposition

We have a functor $\chi \to DGCocat$

$$Objects : \ (A_0 \to \ldots \to A_n \to A_0) \mapsto B(A_0 \to \ldots \to A_n \to A_0)$$

$$Generating \ morphisms : \ \lambda \mapsto \hat{\lambda}$$

Motivating Question

Each dg cocategory $B(A_0 \to \ldots \to A_n \to A_0)$ has a dg comodule $C(A_0 \to \ldots \to A_n \to A_0)$ (which is a categorified version of the bar complex of Hochschild chains as a module over Hochschild cochains via contraction).

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Motivating Question: Can we extend the sheafy-cyclic structure to the comodules? In other words, can we give a functor

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$$\chi_\infty \to \mathcal{D}_\infty \quad \text{dg categories}$$

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Motivating Question: Can we extend the sheafy-cyclic structure to the comodules? In other words, can we give ~~an~~ an A_∞-functor

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Rest of this talk: Describe our A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$.

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- Define dg categories χ_∞ and $\mathcal{D}_\infty$
- Define dg comodules $C(A_0 \to \dots \to A_n \to A_0)$
- Define the A_∞-functor $\mathcal{F}$

Dg categories χ_∞ and $\mathcal{D}_\infty$

χ_∞:

Objects: same objects as $\chi = \{A_0 \to ... \to A_n \to A_0\}$

$\chi_\infty^\bullet(X, Y) := \{\sum a_i v_i : a_i \in k, v_i \in \chi(X, Y)\}$

Dg categories χ_∞ and $\mathcal{D}_\infty$

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$\mathcal{D}_\infty$:

    Objects: same objects as $\mathcal{D} = \{(\underset{B}{\mathrm{dg\ cocategory}}, \underset{C}{\mathrm{dg\ comodule}})\}$

$$\mathcal{D}_\infty^p((B_1, C_1), (B_0, C_0)) := \left\{ \begin{array}{c} F : B_1 \to B_0 \text{ } dg \text{ } functor, \\ F_! : C_1 \to F^* C_0 \text{ } degree\text{-}p \text{ } linear \text{ } map \end{array} \right\}$$

$$d_{D_\infty}(F, F_!) = (F, [d, F_!] = d_{F^* C_0} \circ F_! \pm F_! \circ d_{C_1})$$

Dg categories χ_∞ and $\mathcal{D}_\infty$

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  Objects: same objects as $\chi = \{A_0 \to \ldots \to A_n \to A_0\}$
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$$\mathcal{D}_\infty^p((B_1, C_1), (B_0, C_0)) := \left\{ \begin{array}{c} F : B_1 \to B_0 \text{ dg functor,} \\ F_! : C_1 \to F^* C_0 \text{ degree-}p \text{ linear map} \end{array} \right\}$$

$$d_{D_\infty}(F, F_!) = (F, [d, F_!] = {}_{d_{F^* C_0} \circ F_! \pm F_! \circ d_{C_1}})$$

$F^* C_0$ is the categorified version of co-extension of scalars:

$$F^* C_0 = ker(B_1 \otimes C_0 \rightrightarrows B_1 \otimes B_0 \otimes C_0)$$

Dg comodules over dg cocategories

Definition

A **dg comodule** C over a dg cocategory B consists of the following data:

- for each object $f \in B$, a complex $C^\bullet(f)$, and
- maps of complexes

$$\Delta_C(f) : C^\bullet(f) \to \prod_{g \in Obj(B)} B^\bullet(f,g) \otimes C^\bullet(g).$$

such that the following diagrams for coassociativity and counitality commute:

$$
\begin{array}{ccc}
C^\bullet(f) & \xrightarrow{\Delta_C} & \displaystyle\prod_{g \in Obj(B)} B^\bullet(f,g) \otimes C^\bullet(g) \\
\Big\downarrow{\scriptstyle \Delta_C} & & \Big\downarrow{\scriptstyle id_B \otimes \Delta_C} \\
\displaystyle\prod_{g \in Obj(B)} B^\bullet(f,g) \otimes C^\bullet(g) & \xrightarrow{\Delta_B \otimes id_C} & \displaystyle\prod_{\substack{g,g' \in \\ Obj(B)}} B^\bullet(f,g) \otimes B^\bullet(g,g') \otimes C^\bullet(g')
\end{array}
$$

$$
\begin{array}{ccc}
C^\bullet(f) & \xrightarrow{\Delta_C} & \displaystyle\prod_{g \in Obj(B)} B^\bullet(f,g) \otimes C^\bullet(g) \\
& \searrow{\scriptstyle id} & \Big\downarrow{\scriptstyle \epsilon_B \otimes id_C} \\
& & C^\bullet(f)
\end{array}
$$

Fix algebras $A_0, ..., A_n$.
Define a dg comodule over $B(A_0 \to ... \to A_n \to A_0)$:

$$C(A_0 \to ... \to A_n \to A_0)^\bullet(f_{0,0}, ..., f_{n,0}) :=$$

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$$:= \{ \quad = \quad
\begin{array}{c}
(\phi_{0,1}|...|\phi_{0,k_0}) \otimes ... \otimes (\phi_{n,1}|...|\phi_{n,k_n}) \otimes \alpha: \\
s.t.\ \phi_{i,j} \in C^\bullet(A_{i,f_{j-1}} A_{i+1_{f_j}}), \\
\alpha \in C_{-\bullet}(A_{0,f_{n,k_n}} ... f_{0,k_0} A_0)
\end{array}
\quad \}$$

Dg comodules over dg cocategories

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$$:= \{ \quad = \quad \begin{matrix} (\phi_{0,1}|...|\phi_{0,k_0})\otimes...\otimes(\phi_{n,1}|...|\phi_{n,k_n})\otimes\alpha: \\ s.t. \ \phi_{i,j}\in C^\bullet(A_{i,f_{j-1}}A_{i+1_{f_j}}), \\ \alpha\in C_{-\bullet}(A_{0,f_{n,k_n}}...f_{0,k_0}A_0) \end{matrix} \quad \}$$

$$d_{C(A_0\to...\to A_0)} = d_B \otimes id_{C_{-\bullet}} + id_B \otimes b + \tilde{\iota}$$

Dg comodules over dg cocategories

where $\tilde{\iota}$ is given as follows:

$$C(A_0 \to \ldots \to A_0)^\bullet(f_{0,0}, \ldots, f_{n,0}) \xrightarrow{\ \tilde{\iota}\ } C_{-\bullet}(A_0,_{f_{n,0}\ldots f_{0,0}} A_0)$$

$\tilde{\iota}$

extend as a
coderivation

$$C(A_0 \to \ldots \to A_0)^\bullet(f_{0,0}, \ldots, f_{n,0})$$

Dg comodules over dg cocategories

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$$C(A_0 \to \ldots \to A_0)^\bullet(f_{0,0}, \ldots, f_{n,0})$$

$$\tilde{\iota}\big((\phi_{0,1}|\ldots|\phi_{0,k_0}) \otimes \ldots \otimes (\phi_{n,1}|\ldots|\phi_{n,k_n}) \otimes \alpha\big) = \iota_{(\phi_{0,1}|\ldots|\phi_{0,k_0})\bullet \ldots \bullet (\phi_{n,1}|\ldots|\phi_{n,k_n})} \alpha$$

$$\iota_\phi(a_0 \otimes \ldots a_p) = \pm \phi(a_{d+1}, \ldots, a_p) \cdot a_0 \otimes a_1 \otimes \ldots \otimes a_d \quad \text{where } |\phi| = n - d$$

$$\mathcal{A} := (A_0 \to ... \to A_n \to A_0)$$

$$\chi_\infty \to \mathcal{D}_\infty$$

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$$\chi_\infty \to \mathcal{D}_\infty$$
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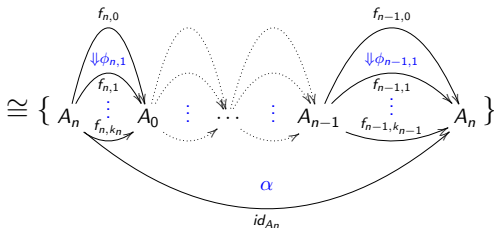$$\mathcal{A} \mapsto (B(\mathcal{A}), C(\mathcal{A}))$$

Generating Morphisms: $\quad \sigma_{i,n} \mapsto \begin{pmatrix} B(\mathcal{A}) \xrightarrow{\hat{\sigma}_{i,n}} B(\sigma_{i,n}\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{id} \hat{\sigma}_{i,n}^* C(\sigma_{i,n}\mathcal{A}) \end{pmatrix}$

$$\delta_{j,n} \mapsto \begin{pmatrix} B(\mathcal{A}) \xrightarrow{\hat{\delta}_{j,n}} B(\delta_{j,n}\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{id} \hat{\delta}_{j,n}^* C(\delta_{j,n}\mathcal{A}) \end{pmatrix} \quad \tau_n \mapsto \begin{pmatrix} B(\mathcal{A}) \xrightarrow{\hat{\tau}_n} B(\tau_n\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{\tau_{n!}} \hat{\tau}_n^* C(\tau_n\mathcal{A}) \end{pmatrix}$$
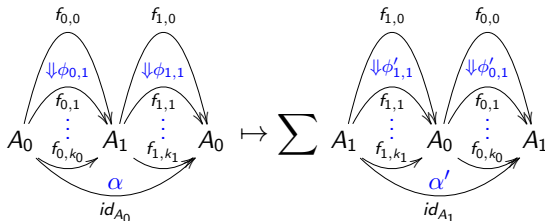
$$\hat{\tau}_n^* C(\tau_n \mathcal{A})^\bullet (\underbrace{f_0, \ldots, f_n}_{\in Obj(B(\mathcal{A}))}) \cong C(\tau_n \mathcal{A})^\bullet(f_n, f_0, \ldots, f_{n-1}) \text{ as complexes.}$$
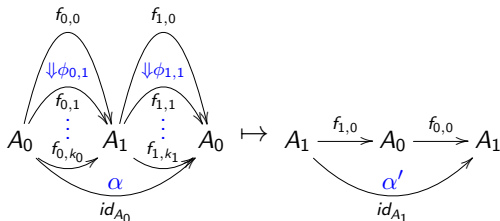
$$\hat{\tau}_n^* C(\tau_n \mathcal{A})^\bullet (\underbrace{f_0, ..., f_n}_{\in Obj(B(\mathcal{A}))}) \cong C(\tau_n \mathcal{A})^\bullet (f_n, f_0, ..., f_{n-1}) \text{ as complexes.}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

$$\hat{\tau}_n^* C(\tau_n \mathcal{A})^\bullet(\underbrace{f_0, ..., f_n}_{\in Obj(B(\mathcal{A}))}) \cong C(\tau_n \mathcal{A})^\bullet(f_n, f_0, ..., f_{n-1}) \text{ as complexes.}$$

$$n = 1: \quad C(A_0 \to A_1 \to A_0)^\bullet(f, g) \xrightarrow{\tau_{1!}} C(A_1 \to A_0 \to A_1)^\bullet(g, f)$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

$$\hat{\tau}_n^* C(\tau_n \mathcal{A})^\bullet(\underbrace{f_0, ..., f_n}_{\in Obj(B(\mathcal{A}))}) \cong C(\tau_n \mathcal{A})^\bullet(f_n, f_0, ..., f_{n-1}) \text{ as complexes.}$$

$$n = 1 : \quad C(A_0 \to A_1 \to A_0)^\bullet(f, g) \xrightarrow{\tau_{1!}} C(A_1 \to A_0 \to A_1)^\bullet(g, f)$$

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$$n = 1: \quad C(A_0 \to A_1 \to A_0)^\bullet(f, g) \xrightarrow{\tau_{1!}} C(A_1 \to A_0 \to A_1)^\bullet(g, f)$$
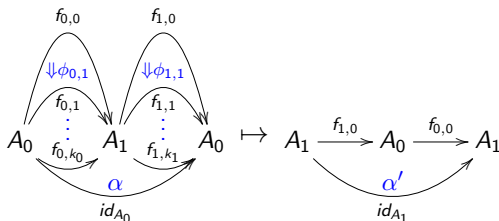
An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

Strategy: Give these maps to cogenerators to define $\tau_{1!}$, then let

$$\tau_{n!} : C(\mathcal{A}) \cong \hat{\delta}_0^{*n-1} C\left(_{A_0 \to A_n \to A_0}\right) \xrightarrow{\hat{\delta}_0^{*n-1}\tau_{1!}} \hat{\delta}_0^{*n-1}\hat{\tau}_1^* C\left(_{A_n \to A_0 \to A_n}\right) \cong$$

$$\cong (\widehat{\tau_1 \delta_0^{n-1}})^* C\left(_{A_n \to A_0 \to A_n}\right) \cong (\widehat{\delta_0^{n-1} \tau_n})^* C\left(_{A_n \to A_0 \to A_n}\right) \cong$$

$$\cong \hat{\tau}_n^* \hat{\delta}_0^{*n-1} C\left(_{A_n \to A_0 \to A_n}\right) \cong \hat{\tau}_n^* C(\tau_n \mathcal{A}).$$

$$n = 1 : \quad C(A_0 \to A_1 \to A_0)^\bullet(f, g) \xrightarrow{\tau_{1!}} C(A_1 \to A_0 \to A_1)^\bullet(g, f)$$

Defining $\tau_{1!}$

Defining $\tau_{1!}$

$$A_0 \xrightarrow{f_{0,0}} A_1 \xrightarrow{f_{1,0}} A_0 \overset{?}{\mapsto} A_1 \xrightarrow{f_{1,0}} A_0 \xrightarrow{f_{0,0}} A_1$$

$A_0 \xrightarrow{f_{0,0}} A_1 \xrightarrow{f_{1,0}} A_0$ with α, id_{A_0}

$A_1 \xrightarrow{f_{1,0}} A_0 \xrightarrow{f_{0,0}} A_1$ with α', id_{A_1}

Defining $\tau_{1!}$

$$A_0 \xrightarrow{f_{0,0}} A_1 \xrightarrow{f_{1,0}} A_0 \xmapsto{?} A_1 \xrightarrow{f_{1,0}} A_0 \xrightarrow{f_{0,0}} A_1$$

$$\underset{id_{A_0}}{\overset{\alpha}{\curvearrowright}} \qquad \underset{id_{A_1}}{\overset{\alpha'}{\curvearrowright}}$$

$$\alpha = a_0 \otimes \ldots \otimes a_n \mapsto \alpha' = f_{0,0}(a_0) \otimes \ldots \otimes f_{0,0}(a_n)$$

Defining $\tau_{1!}$

Defining $\tau_{1!}$

$$\overline{\tau_{1!} \circ d}(\phi \otimes \alpha) = \overline{d \circ \tau_{1!}}(\phi \otimes \alpha)$$

Defining $\tau_{1!}$

$$[b, \bar{\tau}_{1!}](\phi \otimes \alpha) \pm \bar{\tau}_{1!}(\delta\phi \otimes \alpha) = [\bar{\tau}_{1!}, \iota_\phi](\alpha)$$

Defining $\tau_{1!}$

$$[b, \bar\tau_{1!}](\phi \otimes \alpha) \pm \bar\tau_{1!}(\delta\phi \otimes \alpha) = [\bar\tau_{1!}, \iota_\phi](\alpha)$$

$$L_\phi(\alpha) = \sum_{k \geq 1} \pm a_0 \otimes \ldots \otimes \phi(a_k, \ldots) \otimes a_r \otimes \ldots \otimes a_n +$$

$$\sum \pm \phi(a_k, \ldots, a_n, a_0, \ldots) \otimes a_s \otimes \ldots \otimes a_{k-1}$$

$$[b, L_\phi] \pm L_{\delta\phi} = 0$$

Defining $\tau_{1!}$

$$[b, \bar{\tau}_{1!}](\phi \otimes \alpha) \pm \bar{\tau}_{1!}(\delta\phi \otimes \alpha) = [\bar{\tau}_{1!}, \iota_\phi](\alpha)$$

$$\bar{\tau}_{1!}(\phi \otimes \alpha) = \sum_{k \geq 1} \pm f_{0,0} a_0 \otimes \ldots \otimes \phi(a_k, \ldots) \otimes f_{0,1} a_r \ldots \otimes f_{0,1} a_n +$$

$$\sum \pm \phi(f_{1,0} f_{0,1} a_k, \ldots, f_{1,0} f_{0,1} a_n, a_0, \ldots) \otimes f_{0,1} a_s \otimes \ldots \otimes f_{0,1} a_{k-1}$$

$$[b, \bar{\tau}_{1!}(\phi, -)] \pm \bar{\tau}_{1!}(\delta\phi, -) = [\bar{\tau}_{1!}, \iota_\phi]$$

An A_∞-functor $\mathcal{F}: \chi_\infty \to \mathcal{D}_\infty$

Defining $\tau_{1!}$

$$\bar{\tau}_{1!}\big((\phi_{0,1}|\ldots|\phi_{0,k_0}) \otimes (\phi_{1,1}|\ldots|\phi_{1,k_1}) \otimes \alpha\big) =$$

$$= \sum_{\substack{1 \leq i \leq n_0 \\ i \leq j_1 \leq \ldots \leq j_{2k_1} \leq k_0 \\ p}} \pm \phi_{0,1}\big(\begin{smallmatrix} f_{1,0}f_{0,i}a_p,\ldots,f_{1,0}\phi_{0,i+1}(a_*,\ldots), \\ f_{1,0}f_{0,i+1}a_*,\ldots,f_{1,0}\phi_{0,j_1}(a_*,\ldots), \\ f_{1,0}f_{0,j_1}a_*,\ldots,\phi_{1,1}(f_{0,j_1}a_*,\ldots,\phi_{0,j_1+1}(a_*,\ldots),\ldots),\ldots, \\ \phi_{1,k_1}(f_{0,j_{2k_1}-1}a_*,\ldots,\phi_{0,j_{2k_1}-1+1}(a_*,\ldots),\ldots),\ldots,a_0,\ldots \end{smallmatrix}\big) \otimes$$

$$\otimes f_{0,1}a_* \otimes \ldots \otimes \phi_{0,2}(a_*,\ldots) \otimes f_{0,2}a_* \otimes \ldots \otimes$$

$$\otimes \phi_{0,i}(a_*,\ldots) \otimes f_{0,i}a_* \otimes \ldots f_{0,i}a_{p-1} +$$

$$\bigg(\sum \pm f_{0,0}a_0 \otimes \ldots \otimes \phi_{0,1}(a_*,\ldots) \otimes \ldots \otimes \phi_{0,n_0}(a_*,\ldots) \otimes$$

$$\otimes f_{0,n_0}a_* \otimes \ldots \otimes f_{0,n_0}a_n \quad \text{if } k_1 = 0\bigg)$$

$$\mathcal{A} := (A_0 \to ... \to A_n \to A_0)$$

$$\chi_\infty \to \mathcal{D}_\infty$$
$$\mathcal{A} \mapsto (B(\mathcal{A}), C(\mathcal{A}))$$

Generating Morphisms: $\sigma_{i,n} \mapsto \begin{pmatrix} B(\mathcal{A}) \xrightarrow{\hat{\sigma}_{i,n}} B(\sigma_{i,n}\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{id} \hat{\sigma}^*_{i,n} C(\sigma_{i,n}\mathcal{A}) \end{pmatrix}$

$\delta_{j,n} \mapsto \begin{pmatrix} B(\mathcal{A}) \xrightarrow{\hat{\delta}_{j,n}} B(\delta_{j,n}\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{id} \hat{\delta}^*_{j,n} C(\delta_{j,n}\mathcal{A}) \end{pmatrix}$ $\quad \tau_n \mapsto \begin{pmatrix} B(\mathcal{A}) \xrightarrow{\hat{\tau}_n} B(\tau_n\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{\tau_{n!} = \delta_0^{*n-1}\tau_{1!}} \hat{\tau}^*_n C(\tau_n\mathcal{A}) \end{pmatrix}$

n=1:

$$C(A_0 \to A_1 \to A_0) \xrightarrow{\tau_{1!}} \hat{\tau}_1^* C(\tau_1(A_0 \to A_1 \to A_0)) \xrightarrow{\hat{\tau}_1^* \tau_{1!}} \hat{\tau}_1^{*2} C(\tau_1^2(A_0 \to A_1 \to A_0))$$

$$\downarrow{\text{id}}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

n=1:

$$C(A_0 \to A_1 \to A_0) \xrightarrow{\tau_{1!}} \hat{\tau}_1^* C(\tau_1(A_0 \to A_1 \to A_0)) \xrightarrow{\hat{\tau}_1^* \tau_{1!}} \hat{\tau}_1^{*2} C(\tau_1^2(A_0 \to A_1 \to A_0))$$

with the curved arrow labeled id.

$$A_0 \xrightarrow{f_{0,0}} A_1 \xrightarrow{f_{1,0}} A_0$$

with the curved arrow labeled α above and id_{A_0} below.

$$\alpha = a_0 \otimes \ldots \otimes a_n \overset{\tau_{1!}}{\mapsto} f_{0,0}a_0 \otimes \ldots \otimes f_{0,0}a_n \overset{\hat{\tau}_{1!}^* \tau_{1!}}{\mapsto} f_{1,0}f_{0,0}a_0 \otimes \ldots \otimes f_{1,0}f_{0,0}a_n$$

n=1:

$$C(A_0 \to A_1 \to A_0) \xrightarrow{\tau_{1!}} \hat{\tau}_1^* C(\tau_1(A_0 \to A_1 \to A_0)) \xrightarrow{\hat{\tau}_1^* \tau_{1!}} \hat{\tau}_1^{*2} C(\tau_1^2(A_0 \to A_1 \to A_0))$$

$$\underset{id}{\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad}}$$

$$A_0 \xrightarrow{f_{0,0}} A_1 \xrightarrow{f_{1,0}} A_0$$

$$\underset{id_{A_0}}{\underbrace{\qquad\quad}} \overset{\alpha}{}$$

$$\alpha = a_0 \otimes \ldots \otimes a_n \overset{\tau_{1!}}{\mapsto} f_{0,0} a_0 \otimes \ldots \otimes f_{0,0} a_n \overset{\hat{\tau}_{1!}^* \tau_{1!}}{\mapsto} f_{1,0} f_{0,0} a_0 \otimes \ldots \otimes f_{1,0} f_{0,0} a_n$$

$$f_{1,0} f_{0,0} \alpha - \alpha = [b, B](\alpha)$$

$$B(a_0 \otimes \ldots \otimes a_n) = \sum_{0 \le i \le n} \pm 1 \otimes f_{1,0} f_{0,0} a_i \otimes \ldots \otimes f_{1,0} f_{0,0} a_n \otimes a_0 \otimes \ldots \otimes a_{i-1}$$

n=1:

$$C(A_0 \to A_1 \to A_0) \xrightarrow{\tau_{1!}} \hat{\tau}_1^* C(\tau_1(A_0 \to A_1 \to A_0)) \xrightarrow{\hat{\tau}_1^* \tau_{1!}} \hat{\tau}_1^{*2} C(\tau_1^2(A_0 \to A_1 \to A_0))$$

with id arrow from the first term to the last term.

$$A_0 \xrightarrow[f_{0,1}]{\overset{f_{0,0}}{\underset{\Downarrow \phi}{}}} A_1 \xrightarrow{f_{1,0}} A_0 = \phi \otimes \alpha$$

with α over id_{A_0}

$$(\overline{\hat{\tau}^* \tau_{1!} \circ \tau_{1!} - id})(\phi \otimes \alpha) = \text{``} f_{1,0} \circ L_\phi(\alpha) \text{''}$$

An A_∞-functor $\mathcal{F}: \chi_\infty \to \mathcal{D}_\infty$

n=1:

$$C(A_0 \to A_1 \to A_0) \xrightarrow{\tau_{1!}} \hat{\tau}_1^* C(\tau_1(A_0 \to A_1 \to A_0)) \xrightarrow{\hat{\tau}_1^* \tau_{1!}} \hat{\tau}_1^{*2} C(\tau_1^2(A_0 \to A_1 \to A_0))$$

$$\xrightarrow{id}$$

$$(\overline{\hat{\tau}^* \tau_{1!} \circ \tau_{1!} - id})(\phi \otimes \alpha) = \text{``} f_{1,0} \circ L_\phi(\alpha)\text{''}$$

$$= \sum_{k \geq 1} \pm f_{1,0} f_{0,0} a_0 \otimes \ldots \otimes f_{1,0}\phi(a_k, \ldots) \otimes f_{1,0}f_{0,1}a_r \ldots \otimes f_{1,0}f_{0,1}a_n +$$

$$\sum \pm f_{1,0}\phi(f_{1,0}f_{0,1}a_k, \ldots, f_{1,0}f_{0,1}a_n, a_0, \ldots) \otimes f_{1,0}f_{0,1}a_s \otimes \ldots \otimes f_{1,0}f_{0,1}a_{k-1}$$

$$\stackrel{?}{=} [\iota_\phi, B](\alpha) \pm [b, B](\phi \otimes \alpha) \pm B(\delta\phi \otimes \alpha)$$

n=1:

$$C(A_0 \to A_1 \to A_0) \xrightarrow{\tau_{1!}} \hat{\tau}_1^* C(\tau_1(A_0 \to A_1 \to A_0)) \xrightarrow{\hat{\tau}_1^* \tau_{1!}} \hat{\tau}_1^{*2} C(\tau_1^2(A_0 \to A_1 \to A_0))$$

$$id$$

$$(\overline{\hat{\tau}^* \tau_{1!} \circ \tau_{1!} - id})(\phi \otimes \alpha) = "f_{1,0} \circ L_\phi(\alpha)"$$

$$= \sum_{k \geq 1} \pm f_{1,0} f_{0,0} a_0 \otimes \ldots \otimes f_{1,0} \phi(a_k, \ldots) \otimes f_{1,0} f_{0,1} a_r \ldots \otimes f_{1,0} f_{0,1} a_n +$$

$$\sum \pm f_{1,0} \phi(f_{1,0} f_{0,1} a_k, \ldots, f_{1,0} f_{0,1} a_n, a_0, \ldots) \otimes f_{1,0} f_{0,1} a_s \otimes \ldots \otimes f_{1,0} f_{0,1} a_{k-1}$$

$$\stackrel{?}{=} [\iota_\phi, B](\alpha) \pm [b, B](\phi \otimes \alpha) \pm B(\delta\phi \otimes \alpha)$$

$$B(\phi \otimes \alpha) = \sum_{0 \leq i \leq n} \pm 1 \otimes f_{1,0} f_{0,0} a_i \otimes \ldots \otimes f_{1,0} \phi(a_*, \ldots) \otimes f_{1,0} f_{0,1} a_* \otimes \ldots \otimes f_{1,0} f_{0,1} a_n \otimes a_0 \otimes \ldots \otimes a_{i-1}$$

n=1:

$$B\big((\phi_{0,1}|\ldots|\phi_{0,k_0}) \otimes (\phi_{1,1}|\ldots|\phi_{1,k_1}) \otimes \alpha\big) =$$

$$= \sum_{\substack{0 \le j_1 \le \ldots \le j_{2k_1} \le k_0 \\ p}} \pm 1 \otimes f_{1,0} f_{0,0} a_p \otimes \ldots \otimes f_{1,0} \phi_{0,1}(a_*, \ldots) \otimes$$

$$\otimes f_{1,0} f_{0,1} a_* \otimes \ldots \otimes f_{1,0} \phi_{0,j_1}(a_*, \ldots) \otimes$$

$$\otimes f_{1,0} f_{0,j_1} a_* \otimes \ldots \otimes \phi_{1,1}(f_{0,j_1} a_*, \ldots, \phi_{0,j_1+1}(a_*, \ldots), \ldots) \otimes$$

$$\otimes \ldots \otimes \phi_{1,k_1}(f_{0,j_{2k_1-1}} a_*, \ldots, \phi_{0,j_{2k_1-1}+1}(a_*, \ldots), \ldots) \otimes \ldots \otimes$$

$$\otimes a_0 \otimes \ldots \otimes a_{p-1}$$

$$\chi_\infty \to \mathcal{D}_\infty$$
$$\mathcal{A} \mapsto (B(\mathcal{A}), C(\mathcal{A}))$$
$$\tau_1 \mapsto \begin{pmatrix} \hat{\tau}_1 : B(\mathcal{A}) \to B(\tau_1 \mathcal{A}) \\ \tau_{1!} : C(\mathcal{A}) \to \hat{\tau}_1^* C(\tau_1 \mathcal{A}) \end{pmatrix}$$
$$(\tau_1, \tau_1) \mapsto \begin{pmatrix} id : B(\mathcal{A}) \to B(\mathcal{A}) \\ B : C(\mathcal{A}) \to C(\mathcal{A}) \end{pmatrix}$$
$$(\tau_1, \tau_1, \tau_1) \mapsto \begin{pmatrix} \hat{\tau}_1 : B(\mathcal{A}) \to B(\tau_1 \mathcal{A}) \\ 0 : C(\mathcal{A}) \to \hat{\tau}_1^* C(\tau_1 \mathcal{A}) \end{pmatrix}$$
$$\vdots$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

The A_∞ relations mean:

- $\tau_{1!}$ is a map of complexes
- $B^2 = 0$
- The following diagram commutes:

$$
\begin{array}{ccc}
C(\mathcal{A}) & \xrightarrow{\tau_{1!}} & \hat{\tau}_1^* C(\tau_1 \mathcal{A}) \\
\downarrow{\scriptstyle B} & & \downarrow{\scriptstyle \hat{\tau}_1^* B} \\
C(\mathcal{A}) & \xrightarrow[\tau_{1!}]{} & \hat{\tau}_1^* C(\tau_1 \mathcal{A})
\end{array}
$$

For higher $n > 1$, we want to find a homotopy between "$\tau_{n!}^{n+1}$" and id.

For higher $n > 1$, we want to find a homotopy between "$\tau_{n!}^{n+1}$" and id. However, it is sufficient to find a homotopy between $\hat{\tau}_n^{*2}\delta_{0,n!} \circ \hat{\tau}_n^*\tau_{n!} \circ \tau_{n!}$ and $\hat{\delta}_{n-1,n}^*\tau_{n-1!} \circ \delta_{n-1,n!}$.

$$
\begin{array}{ccc}
[n] \xrightarrow{\ \tau_n\ } [n] \xrightarrow{\ \tau_n\ } [n] \\
\downarrow{\scriptstyle \delta_{n-1,n}} \qquad\qquad \swarrow{\scriptstyle \delta_{0,n}} \\
[n-1] \xrightarrow[\ \tau_{n-1}\ ]{} [n-1]
\end{array}
$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

For higher $n > 1$, we want to find a homotopy between "$\tau_{n!}^{n+1}$" and id. However, it is sufficient to find a homotopy between $\hat{\tau}_n^{*2} \delta_{0,n!} \circ \hat{\tau}_n^* \tau_{n!} \circ \tau_{n!}$ and $\hat{\delta}_{n-1,n}^* \tau_{n-1!} \circ \delta_{n-1,n!}$.

$$
\begin{array}{ccc}
[n] & \xrightarrow{\ \tau_n\ } [n] \xrightarrow{\ \tau_n\ } & [n] \\
\downarrow{\scriptstyle \delta_{n-1,n}} & & \\
[n-1] & \xrightarrow[\ \tau_{n-1}\ ]{} [n-1] &
\end{array}
$$

with $\delta_{0,n}$ arrow.

Strategy: Find such a homotopy, $\mathcal{B}$, for $n = 2$, and use $\hat{\delta}_0^{*n-2} \mathcal{B}$ for $n > 2$.

$$\chi_\infty \to \mathcal{D}_\infty$$

$$\mathcal{A} \mapsto (B(\mathcal{A}), C(\mathcal{A}))$$

$$\mu = \tau_{n-1} \circ \delta_{n-1,n} = \delta_{0,n} \circ \tau_n^2 \mapsto \begin{pmatrix} \hat{\tau}_{n-1} \circ \hat{\delta}_{n-1,n} = \hat{\delta}_{0,n} \circ \hat{\tau}_n^2 \\ \hat{\tau}_n^{*2} \delta_{0,n!} \circ \hat{\tau}_n^* \tau_{n!} \circ \tau_{n!} \end{pmatrix}$$

$$(\delta_{0,n}, \tau_n^2) \mapsto \begin{pmatrix} \hat{\delta}_{0,n} \circ \hat{\tau}_n^2 \\ 0 \end{pmatrix}$$

$$(\tau_{n-1}, \delta_{n-1,n}) \mapsto \begin{pmatrix} \hat{\tau}_{n-1} \circ \hat{\delta}_{n-1,n} \\ \mathcal{B} \end{pmatrix}$$

$$(\tau_{n-1}, \delta_{n-1,n}, \lambda) \mapsto \begin{pmatrix} \hat{\tau}_{n-1} \circ \hat{\delta}_{n-1,n} \circ \hat{\lambda} \\ 0 \end{pmatrix}$$

$$\vdots$$

For $n > 1$, the A_∞ relations mean:

- $\tau_{n!}$ is a map of complexes
- $\mathcal{B}^2 = 0$
- We have a pair of homotopic maps:

$$C(A_0 \to \ldots \to A_n \to A_0)$$

$\hat{\tau}_n^{*2}\tau_{n!} \circ \hat{\tau}_n^{*}\tau_{n!} \circ \tau_{n!}$

"apply $\tau_{n!}$ 3 times"

$(\delta_{n-2,n-1}\widehat{\delta_{n-1,n}})^*\tau_{n-2!}$

"brace together the last 3 algebras,
then apply $\tau_{n-2!}$ once"

$$C(A_{n-2} \to A_{n-1} \to A_n \to A_0 \to \ldots \to A_{n-2}).$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

For $n > 1$, the A_∞ relations mean:

- $\tau_{n!}$ is a map of complexes
- $\mathcal{B}^2 = 0$
- They are homotopic via two homotopies:

Summary:

We have a given a "homotopically sheafy-cyclic object in dg cocategories with a dg comodule", i.e., an A_∞-functor $\chi_\infty \to \mathcal{D}_\infty$.

Time and interest permitting: "A homotopically sheafy-cyclic object in dg categories with a dg module"

We have a given a "homotopically sheafy-cyclic object in dg cocategories with a dg comodule", i.e., an A_∞-functor $\chi_\infty \to \mathcal{D}_\infty$.

Time and interest permitting: "A homotopically sheafy-cyclic object in dg categories with a dg module"

- Rectify $\mathcal{F}$ to a dg functor $U(\chi_\infty) \to \mathcal{D}_\infty$

$$Hom_{A_\infty - Cat}(\chi_\infty, \mathcal{D}_\infty) = Hom_{DGCat}(U(\chi_\infty), \mathcal{D}_\infty)$$

Summary:

We have a given a "homotopically sheafy-cyclic object in dg cocategories with a dg comodule", i.e., an A_∞-functor $\chi_\infty \to \mathcal{D}_\infty$.

Time and interest permitting: "A homotopically sheafy-cyclic object in dg categories with a dg module"

- Rectify $\mathcal{F}$ to a dg functor $U(\chi_\infty) \to \mathcal{D}_\infty$
- Give a dg functor $Adj : \mathcal{D}_\infty \to \mathcal{D}_1$

$$\mathcal{D}_1^\bullet\big((B_1, C_1), (B_0, C_0)\big) = \left\{ \begin{array}{c} F : B_1 \to B_0 \text{ dg functor} \\ F_! : F_\# C_1 \to C_0 \text{ linear map} \\ \text{"co-restriction of scalars"} \end{array} \right\}$$

$$Hom_{B_1}(C_1, F^* C_0) = Hom_{B_0}(F_\# C_1, C_0)$$

Summary:

We have a given a "homotopically sheafy-cyclic object in dg cocategories with a dg comodule", i.e., an A_∞-functor $\chi_\infty \to \mathcal{D}_\infty$.

Time and interest permitting: "A homotopically sheafy-cyclic object in dg categories with a dg module"

- Rectify $\mathcal{F}$ to a dg functor $U(\chi_\infty) \to \mathcal{D}_\infty$
- Give a dg functor $Adj : \mathcal{D}_\infty \to \mathcal{D}_1$
- Give a dg functor $Cobar : \mathcal{D}_1 \to \mathcal{E}$

$$Obj(\mathcal{E}) = \{(\text{dg category}, \text{dg module})\}$$

$$\mathcal{E}^\bullet\big((A_1, M_1), (A_0, M_0)\big) = \left\{ \begin{array}{c} F : A_1 \to A_0 \text{ dg functor} \\ F_! : M_1 \to F^* M_0 \text{ linear map} \\ \text{``restriction of scalars''} \end{array} \right\}$$

Summary:

We have a given a "homotopically sheafy-cyclic object in dg cocategories with a dg comodule", i.e., an A_∞-functor $\chi_\infty \to \mathcal{D}_\infty$.

Time and interest permitting: "A homotopically sheafy-cyclic object in dg categories with a dg module"

- Rectify $\mathcal{F}$ to a dg functor $U(\chi_\infty) \to \mathcal{D}_\infty$
- Give a dg functor $Adj : \mathcal{D}_\infty \to \mathcal{D}_1$
- Give a dg functor $Cobar : \mathcal{D}_1 \to \mathcal{E}$

$$U(\chi_\infty) \xrightarrow{\text{rectified } \mathcal{F}} \mathcal{D}_\infty \xrightarrow{Adj} \mathcal{D}_1 \xrightarrow{Cobar} \mathcal{E}$$

Thank you!