

What do algebras form?

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Outline

- **Question:** What do algebras form?
- **Answer 1:** A category in categories (HH^0)
- **Derived Answer 1:** A category in dg cocategories (Hochschild cochains...)
- **Answer 2:** A 2-category with a trace functor (HH_0)
- **Derived Answer 2:** A category in dg cocategories with a trace functor (Hochschild chains...)

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Answer 1: Algebras form a **2-category**.

- Objects: algebras A, B, \dots
- 1-Morphisms: bimodules ${}_A M_B$
- 1-Composition: ${}_A M_B \otimes_B {}_B N_C$
- 2-Morphisms: morphisms of bimodules

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- Objects: algebras A, B, \dots
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- 2-Morphisms:

$$\begin{aligned} \{\text{maps of bimodules } {}_f B \rightarrow_g B\} &\cong Z_A({}_g B_f) \cong HH^0(A, {}_g B_f) \\ M &\mapsto M(1) \\ (M_b : b' &\mapsto b \cdot b') \leftarrow b \end{aligned}$$

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Can we use Hochschild cohomology or cochains instead of HH^0 ?

Derived Answer 1: Algebras form a category in dg cocategories.

- Objects: algebras A, B, \dots
- Morphisms: a dg cocategory $Bar(Hoch(A, B))$
- Composition:
 - $: Bar(Hoch(A, B)) \otimes Bar(Hoch(B, C)) \rightarrow Bar(Hoch(A, C))$
associative map of dg cocategories

Defining $Bar(Hoch(A, B))$

① $Hoch(A, B)$ is a dg category with

- Objects: algebra maps $f : A \rightarrow B$
- Morphisms: $Hoch(A)(f, g) = (C^\bullet(A, {}_f B_g), {}_f \delta_g)$
- Composition: cup product on cochains

$$\phi \in C^p(A, {}_f B_g)$$

$$\psi \in C^q(A, {}_g B_h)$$

$$(\phi \cup \psi)(a_1, \dots, a_{p+q}) = \pm \phi(a_1, \dots, a_p) \psi(a_{p+1}, \dots, a_q)$$

Defining $\text{Bar}(\text{Hoch}(A, B))$

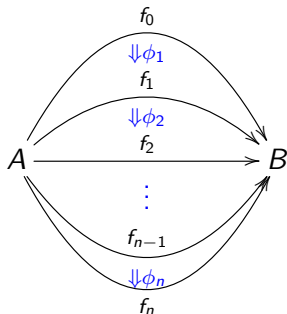
- ① $\text{Hoch}(A, B)$ is a dg category with
 - Objects: algebra maps $f : A \rightarrow B$
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Defining $\text{Bar}(\text{Hoch}(A, B))$

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 - Composition: cup product on cochains

- ② $\text{Bar} : \text{DGCat} \rightarrow \text{DGCocat}$

$\text{Bar}(\text{Hoch}(A, B))$ has the same objects as $\text{Hoch}(A, B)$.



A morphism from f_0 to f_n in $\text{Bar}(\text{Hoch}(A, B))$

$$\Delta(\phi_1 \dots \phi_n) = \sum_{0 \leq i \leq n} \pm \phi_1 \dots \phi_i \otimes \phi_{i+1} \dots \phi_n$$

$$|\phi_1 \dots \phi_n| = \sum_{1 \leq i \leq n} |\phi_i| - n$$

$$d_{\text{Bar}(\text{Hoch}(A, B))} = \tilde{d}_{\text{Hoch}(A, B)} + d_{\cup}$$

Derived Answer 1: Algebra form a category in dg cocategories.

- Objects: algebras A, B, \dots
- Morphisms: a dg cocategory $Bar(Hoch(A, B))$
- Composition:
 - $: Bar(Hoch(A, B)) \otimes Bar(Hoch(B, C)) \rightarrow Bar(Hoch(A, C))$
associative map of dg cocategories

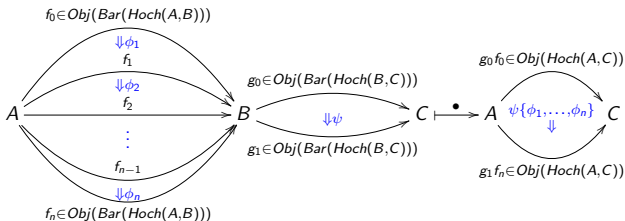
$\text{Bar}(\text{Hoch}(A, C))$ is the cofree dg cocategory on $\text{Hoch}(A, C)$:

$$\begin{array}{ccc} \text{Bar}(\text{Hoch}(A, B)) \otimes \text{Bar}(\text{Hoch}(B, C)) & \xrightarrow{\bullet} & \text{Hoch}(A, C) \\ & \searrow \bullet & \uparrow \\ & & \text{Bar}(\text{Hoch}(A, C)) \end{array}$$

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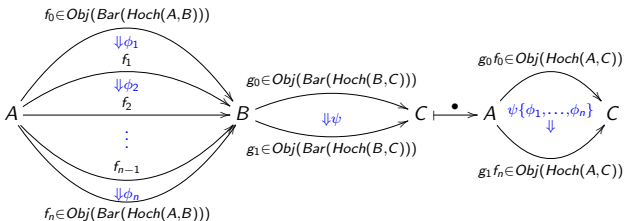
$$\text{Bar}(\text{Hoch}(A, B)) \otimes \text{Bar}(\text{Hoch}(B, C)) \xrightarrow{\bullet} \text{Hoch}(A, C)$$

$$\text{Bar}(\text{Hoch}(A, C)) \xrightarrow{\bullet} \text{Hoch}(A, C)$$



$Bar(Hoch(A, C))$ is the cofree dg cocomonoid on $Hoch(A, C)$:

$$\begin{array}{ccc} \text{Bar}(\text{Hoch}(A, B)) \otimes \text{Bar}(\text{Hoch}(B, C)) & \xrightarrow{\bullet} & \text{Hoch}(A, C) \\ & \searrow \text{dotted} & \uparrow \\ & \bullet & \text{Bar}(\text{Hoch}(A, C)) \end{array}$$

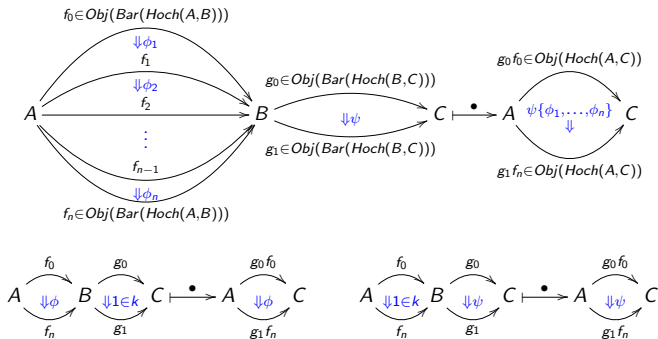


$$\psi\{\phi_1, \dots, \phi_n\}(a_1, \dots, a_q) = \sum \pm \psi(f_0 a_1, \dots, f_0 a_{i_1}, \phi_1(a_{i_1+1}, \dots), f_1 a_*, \dots, f_1 a_*, \\ \phi_2(a_*, \dots), f_2 a_*, \dots, f_n a_q)$$

$\text{Bar}(\text{Hoch}(A, C))$ is the cofree dg cocategory on $\text{Hoch}(A, C)$:

$$\text{Bar}(\text{Hoch}(A, B)) \otimes \text{Bar}(\text{Hoch}(B, C)) \xrightarrow{\bullet} \text{Hoch}(A, C)$$

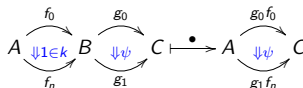
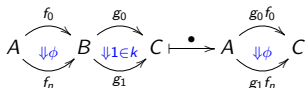
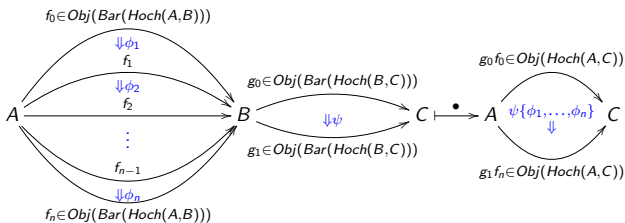
\bullet
 \nearrow
 $\text{Bar}(\text{Hoch}(A, C)) \xrightarrow{\quad} \text{Hoch}(A, C)$



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Braces are associative. (Getzler-Jones; Voronov-Gerstenhaber, Lyubashenko-Manzyuk; Keller)

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- **Question:** What do algebras form?
- **Answer 1:** A category in categories (HH^0)
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- Brief background on non-commutative calculus
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Non-commutative calculus

Theorem

(Hochschild-Kostant-Rosenberg, '62) Let A be a regular, commutative algebra over a field k of characteristic 0. Then,

$$(C_{\bullet}(A, A), b) \xrightarrow{\sim} \Omega_{A/k}^{\bullet}$$

$$(C^{\bullet}(A, A), \delta) \xrightarrow{\sim} \wedge^{\bullet} T_A = \wedge^{\bullet}(\operatorname{Der}_k(A, A)).$$

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Theorem

(Kontsevich, '97) Let $A = C^\infty(M)$ for M a smooth real manifold. Then, there is an L_∞ map

$$(C^{\bullet+1}(A, A), \delta, [,]_{\operatorname{Ger}}) \xrightarrow{\sim} (\wedge^{\bullet+1} T_A, d = 0, [,]_{SN}).$$

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Theorem

(Tamarkin, '98) Dependent on the choice of a Drinfeld associator, there is a $\operatorname{Ger}_{\infty}$ map

$$(C^{\bullet+1}(A, A), \delta, [,]_{\operatorname{Ger}}, \cup, \dots) \xrightarrow{\sim} (\wedge^{\bullet} T_A, d = 0, [,]_{SN}, \wedge).$$

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Theorem

(Dolgushev-Tamarkin-Tsygan, '08) There is a $\operatorname{Calc}_{\infty}$ map

$$(C^{\bullet}(A, A), C_{-\bullet}(A, A)) \xrightarrow{\sim} (\wedge^{\bullet} T_A, \Omega_{A/k}^{\bullet}).$$

Non-commutative calculus

Theorem

(Hochschild-Kostant-Rosenberg, '62) Let A be a regular, commutative algebra over a field k of characteristic 0. Then,

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Answer 2: Algebras form a 2-category with a trace functor

Definition

(Kaledin): A trace functor on a 2-category \mathcal{C} is:

- for each $A \in \text{Obj}(\mathcal{C})$, a functor $TR_A : \mathcal{C}(A, A) \rightarrow k\text{-mod}$
- for each pair $A, B \in \text{Obj}(\mathcal{C})$, a natural transformation $\tau_!(A, B)$

$$\begin{array}{ccc} \mathcal{C}(A, B) \otimes \mathcal{C}(B, A) & \xrightarrow{\tau = \text{flip}} & \mathcal{C}(B, A) \otimes \mathcal{C}(A, B) \\ \downarrow m & & \downarrow m \\ \mathcal{C}(A, A) & \xrightarrow{\tau_!(\vec{A}, B)} & \mathcal{C}(B, B) \\ \swarrow TR_A & & \nwarrow TR_B \\ & k\text{-mod} & \end{array}$$

- such that $\tau_!(B, A) \circ \tau_!(C, B) \circ \tau_!(A, C) = id$

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$$\begin{array}{ccccc}
 & & \mathcal{C}(C, A) \otimes \mathcal{C}(A, C) \otimes \mathcal{C}(B, C) & & \\
 & \nearrow \tau & \downarrow & \nwarrow \tau & \\
 \mathcal{C}(A, B) \otimes \mathcal{C}(B, C) \otimes \mathcal{C}(C, A) & \xleftarrow{\tau_!(\overleftarrow{B}, A)} & & \xrightarrow{\tau_!(\overrightarrow{C}, B)} & \mathcal{C}(B, C) \otimes \mathcal{C}(C, A) \otimes \mathcal{C}(A, B) \\
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 & & k - \text{mod} & &
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Answer 2: Algebras form 2-category with a trace functor

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- for each $A \in \text{Obj}(\mathcal{C})$, a functor $TR_A : \mathcal{C}(A, A) \rightarrow k - \text{mod}$
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 \downarrow m & & \downarrow m \\
 \mathcal{C}(A, A) & \xrightarrow{\tau_!(A, B)} & \mathcal{C}(B, B) \\
 \downarrow TR_A & & \downarrow TR_B \\
 k - \text{mod} & & k - \text{mod}
 \end{array}$$

$\tau_!(A, B)$

$$\frac{{}_A M_B \otimes_B {}_B N_A}{[A, M \otimes_B N]} \xrightarrow{\tau_!(A, B)} \frac{{}_B N_A \otimes_A {}_A M_B}{[B, N \otimes_A M]}$$

$m \otimes n \mapsto n \otimes m$

such that

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$m \otimes n \mapsto n \otimes m$

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Can we use Hochschild homology or chains instead of HH_0 to extend this to a trace functor on the category in dg cocategories?

Massaging the definition of a trace functor

Definition

(Kaledin): A trace functor on a 2-category \mathcal{C} is:

- for each $A \in \text{Obj}(\mathcal{C})$, a functor $TR_A : \mathcal{C}(A, A) \rightarrow k\text{-mod}$

$$\mathcal{C}(A, A)(f, g) \otimes_k TR_A(g) \rightarrow TR_A(f)$$

- for each pair $A, B \in \text{Obj}(\mathcal{C})$, a natural transformation $\tau_!(A, B)$

$$\begin{array}{ccc} \mathcal{C}(A, B) \otimes \mathcal{C}(B, A) & \xrightarrow{\tau} & \mathcal{C}(B, A) \otimes \mathcal{C}(A, B) \\ m \downarrow & & m \downarrow \\ \mathcal{C}(A, A) & \xrightarrow{\tau_!(A, B)} & \mathcal{C}(B, B) \\ TR_A \swarrow & & \nwarrow TR_B \\ & k\text{-mod} & \end{array}$$

- such that $\tau_!(B, A) \circ \tau_!(C, B) \circ \tau_!(A, C) = id$

Massaging the definition of a trace functor

Definition

(Kaledin): A trace functor on a category in k -linear categories \mathcal{C} is:

- for each $A \in \text{Obj}(\mathcal{C})$, a functor $TR_A : \mathcal{C}(A, A) \rightarrow k\text{-mod}$

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- for each pair $A, B \in \text{Obj}(\mathcal{C})$, a natural transformation $\tau_!(A, B)$

$$\begin{array}{ccc} \mathcal{C}(A, B) \otimes \mathcal{C}(B, A) & \xrightarrow{\tau} & \mathcal{C}(B, A) \otimes \mathcal{C}(A, B) \\ \downarrow m & & \downarrow m \\ \mathcal{C}(A, A) & \xrightarrow{\tau_!(A, B)} & \mathcal{C}(B, B) \\ \searrow TR_A & & \swarrow TR_B \\ & k\text{-mod} & \end{array}$$

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Massaging the definition of a trace functor

Definition

(Kaledin): A trace functor on a category in k -linear categories \mathcal{C} is:

- for each $A \in \text{Obj}(\mathcal{C})$, a left module $T(A)$ over $\mathcal{C}(A, A)$

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- such that $\tau_!(B, A) \circ \tau_!(C, B) \circ \tau_!(A, C) = id$

Massaging the definition of a trace functor

Definition

(Kaledin): A trace functor on a category in k -linear categories \mathcal{C} is:

- for each $A \in \text{Obj}(\mathcal{C})$, a left module TR_A over $\mathcal{C}(A, A)$

$$\mathcal{C}(A, A)(f, g) \otimes_k TR_A(g) \rightarrow TR_A(f)$$

- for each pair $A, B \in \text{Obj}(\mathcal{C})$, a map of modules $\tau_!(A, B) : m^* TR_A \rightarrow \tau^* m^* TR_B$ over $\mathcal{C}(A, B) \otimes \mathcal{C}(B, A)$

$$\begin{array}{ccc}
 \mathcal{C}(A, B) \otimes \mathcal{C}(B, A) & \xrightarrow{\tau} & \mathcal{C}(B, A) \otimes \mathcal{C}(A, B) \\
 \downarrow m & & \downarrow m \\
 \mathcal{C}(A, A) & \xrightarrow{\tau_!(A, B)} & \mathcal{C}(B, B) \\
 \downarrow TR_A & & \downarrow TR_B \\
 k\text{-mod} & & k\text{-mod}
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Massaging the definition of a trace functor

Definition

(Kaledin): A trace functor on a category in k -linear categories \mathcal{C} is:

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$$\mathcal{C}(A, A)(f, g) \otimes_k TR_A(g) \rightarrow TR_A(f)$$

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 \downarrow m & & \downarrow m \\
 \mathcal{C}(A, A) & \xrightarrow{\tau_!(A, B)} & \mathcal{C}(B, B) \\
 \downarrow TR_A & & \downarrow TR_B \\
 & k\text{-mod} &
 \end{array}$$

- such that $\tau^{*2} \tau_!(B, A) \circ \tau^* \tau_!(C, B) \circ \tau_!(A, C) = id$

Massaging the definition of a trace functor

Definition

(Kaledin): A trace functor on a **category in dg cocategories** \mathcal{C} is:

- for each $A \in \text{Obj}(\mathcal{C})$, a **left module** $T(A)$ over $\mathcal{C}(A, A)$

$$\mathcal{C}(A, A)(f, g) \otimes_k TR_A(g) \rightarrow TR_A(f)$$

- for each pair $A, B \in \text{Obj}(\mathcal{C})$, a **map of modules**
 $\tau_!(A, B) : m^* T(A) \rightarrow \tau^* m^* T(B)$ over $\mathcal{C}(A, B) \otimes \mathcal{C}(B, A)$

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 \mathcal{C}(A, A) & \xrightarrow{\tau_!(A, B)} & \mathcal{C}(B, B) \\
 \downarrow TR_A & & \downarrow TR_B \\
 & k\text{-mod} &
 \end{array}$$

- such that $\tau^{*2} \tau_!(B, A) \circ \tau^* \tau_!(C, B) \circ \tau_!(A, C) = id$

Massaging the definition of a trace functor

Definition

(Kaledin): A trace functor on a **category in dg cocategories** \mathcal{C} is:

- for each $A \in \text{Obj}(\mathcal{C})$, a **left dg comodule** $T(A)$ over $\mathcal{C}(A, A)$

$$\coprod_{g \in \text{Obj}(\mathcal{C})} \mathcal{C}(A, A)^{\bullet}(f, g) \otimes_k TR_A^{\bullet}(g) \leftarrow TR_A^{\bullet}(f)$$

- for each pair $A, B \in \text{Obj}(\mathcal{C})$, a **map of modules**
 $\tau_!(A, B) : m^* T(A) \rightarrow \tau^* m^* T(B)$ over $\mathcal{C}(A, B) \otimes \mathcal{C}(B, A)$

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 \mathcal{C}(A, B) \otimes \mathcal{C}(B, A) & \xrightarrow{\tau} & \mathcal{C}(B, A) \otimes \mathcal{C}(A, B) \\
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 \mathcal{C}(A, A) & \xrightarrow{\tau_!(A, B)} & \mathcal{C}(B, B) \\
 TR_A \searrow & & \swarrow TR_B \\
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Definition

(Kaledin): A trace functor on a **category in dg cocategories** \mathcal{C} is:

- for each $A \in \text{Obj}(\mathcal{C})$, a **left dg comodule** $T(A)$ over $\mathcal{C}(A, A)$

$$\coprod_{g \in \text{Obj}(\mathcal{C})} \mathcal{C}(A, A)^{\bullet}(f, g) \otimes_k TR_A^{\bullet}(g) \xleftarrow{\quad} TR_A^{\bullet}(f)$$

- for each pair $A, B \in \text{Obj}(\mathcal{C})$, a **map of dg comodules** $\tau_!(A, B) : m^* T(A) \rightarrow \tau^* m^* T(B)$ over $\mathcal{C}(A, B) \otimes \mathcal{C}(B, A)$

$$\begin{array}{ccc} \mathcal{C}(A, B) \otimes \mathcal{C}(B, A) & \xrightarrow{\tau} & \mathcal{C}(B, A) \otimes \mathcal{C}(A, B) \\ m \downarrow & & \downarrow m \\ \mathcal{C}(A, A) & \xrightarrow{\tau_!(A, B)} & \mathcal{C}(B, B) \\ TR_A \swarrow & & \nwarrow TR_B \\ & k - mod & \end{array}$$

- such that $\tau^{*2} \tau_!(B, A) \circ \tau^* \tau_!(C, B) \circ \tau_!(A, C) = id$

Unraveling the definition of a trace functor

Definition

Let \mathcal{C} be a category in dg cocategories. Let $\chi(\mathcal{C})$ be the dg category with

- Objects = $\{A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0 : A_i \in \text{Obj}(\mathcal{C}), n \geq 0\}$
- Morphisms = $\{\text{linear combinations of compositions of}$

rotations $\tau_n : \mathcal{A} \mapsto (A_n \rightarrow A_0 \rightarrow \dots \rightarrow A_n)$

coboundaries $\delta_{j,n} : \mathcal{A} \mapsto (A_0 \rightarrow \dots \rightarrow A_j \rightarrow A_{j+2 \pmod{n+1}} \rightarrow \dots \rightarrow A_0)$

codegeneracies $\sigma_{i,n} : \mathcal{A} \mapsto (A_0 \rightarrow \dots \rightarrow A_i \rightarrow A_i \rightarrow \dots \rightarrow A_0)$

where $\mathcal{A} := (A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$, subject to the cyclic relations $\}[0]$

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A trace functor on a category \mathcal{C} in dg cocategories will give: a dg functor from $\chi(\mathcal{C}) \rightarrow \mathcal{D}$.

Unraveling the definition of a trace functor

Definition

Let \mathcal{D} be the dg category with

- Objects = $\{(\underset{B}{\text{dg cocategory}}, \underset{C}{\text{dg comodule}})\}$
- Morphisms:

$$\mathcal{D}^p((B_1, C_1), (B_0, C_0)) := \left\{ \begin{array}{l} F : B_1 \rightarrow B_0 \text{ dg functor,} \\ F_! : C_1 \rightarrow F^*C_0 \text{ degree-}p \text{ linear map} \end{array} \right\}$$
$$d_{\mathcal{D}}(F, F_!) = (F, [d, F_!] = d_{F^*C_0} \circ F_! \pm F_! \circ d_{C_1})$$

Unraveling the definition of a trace functor

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For us, F^*C_0 is the categorified version of co-extension of scalars:

$$F^*C_0 = \ker(B_1 \otimes C_0 \xrightarrow[1 \otimes \Delta_{C_0}]{(1 \otimes F \otimes 1)(\Delta_{B_1} \otimes 1)} B_1 \otimes B_0 \otimes C_0)$$

Unraveling the definition of a trace functor

Let \mathcal{C} be a category in dg cocategories. A trace functor on \mathcal{C} gives a dg functor

$$\chi(\mathcal{C}) \rightarrow \mathcal{D}$$

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$$\delta_{j,n} \mapsto \left(\begin{array}{c} \dots \otimes \mathcal{C}(A_j, A_{j+1}) \otimes \mathcal{C}(A_{j+1}, A_{j+2}) \otimes \dots \xrightarrow{\delta_{j,n}=m} \dots \otimes \mathcal{C}(A_j, A_{j+2}) \otimes \dots \\ m^{*n} T(A_0) \xrightarrow{\delta_{j,n}=id} \hat{\delta}_{j,n}^* m^{*n-1} T(A_0) \cong (m^{n-1} \hat{\delta}_{j,n})^* T(A_0) \end{array} \right)$$

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Unraveling the definition of a trace functor

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$\tau_n^{n+1} = id$ is preserved:

- $n=2$ cocycle relation,
- $n > 2$ pullback of cocycle relation,
- $n=1$ cocycle relation for $A, B, C = B$ and the fact that $\sigma_{1,1}!$ is an identity map

Unraveling the definition of a trace functor

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Functor is DG: $\delta_{j,n!} = id$, $\sigma_{i,n!} = id$, $\tau_{n!} = m^{*n-1} \tau_1$ are maps of DG comodules.

Question: Can we give a dg functor

$$\chi(\mathcal{C}) \rightarrow \mathcal{D}$$

$$\text{where } (A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0) \mapsto \left(\begin{array}{c} B(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0) := \\ := \text{Bar}(\text{Hoch}(A_0, A_1)) \otimes \dots \otimes \text{Bar}(\text{Hoch}(A_n, A_0)), \\ C(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0) := m^{*n} T(A_0) \end{array} \right) ?$$

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Answer: No, but we can give an A_∞ -functor.

This will imply that algebras form a category in dg cocategories with a trace functor [up to homotopy](#).

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Rest of this talk:

- Define dg comodules $C(A_0 \rightarrow \dots \rightarrow A_0)$ using Hochschild chains
- Describe the A_∞ -functor: $\tau_{1!}, \tau_{n!}^{n+1} = m^{*n-1} \tau_{1!} \sim id$

Fix algebras A_0, \dots, A_n . Let $\mathcal{A} = (A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0)$.
 Define a dg comodule $C(\mathcal{A})$ over $B(\mathcal{A})$:

$$C(\mathcal{A})^\bullet(\underbrace{A_0 \xrightarrow{f_{0,0}} \dots \rightarrow A_n \xrightarrow{f_{n,0}} A_0}_{\in \text{Obj}(B(\mathcal{A}))}) :=$$

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$$:= \{ A_0 \xrightarrow[f_{0,k_0}]{f_{0,1}} A_1 \xrightarrow[\dots]{\vdots} A_n \xrightarrow[f_{n,k_n}]{f_{n,1}} A_0 = (\phi_{0,1} | \dots | \phi_{0,k_0}) \otimes \dots \otimes (\phi_{n,1} | \dots | \phi_{n,k_n}) \otimes \alpha : \\ s.t. \phi_{i,j} \in C^\bullet(A_i, f_{j-1} A_{i+1} f_j), \\ \alpha \in C_{-\bullet}(A_0, f_{n,k_n} \dots f_{0,k_0} A_0) \\ \text{Diagram labels: } f_{0,0}, \Downarrow \phi_{0,1}, f_{0,1}, \vdots, f_{0,k_0}, f_{n,0}, \Downarrow \phi_{n,1}, f_{n,1}, \vdots, f_{n,k_n}, id_{A_0}, \alpha$$

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[illegible]

$$d_{C(A_0 \rightarrow \dots \rightarrow A_0)} = d_B \otimes id_{C_{-\bullet}} + id_B \otimes b + \tilde{t}$$

where $\tilde{\iota}$ is given as follows:

$$\begin{array}{ccc}
 C(A_0 \rightarrow \dots \rightarrow A_0)^\bullet(f_{0,0}, \dots, f_{n,0}) & \xrightarrow{\tilde{\iota}} & C_{-\bullet}(A_0, f_{n,0} \dots f_{0,0} A_0) \\
 & \searrow \tilde{\iota} \text{ extend as a coderivation} & \uparrow \\
 & C(A_0 \rightarrow \dots \rightarrow A_0)^\bullet(f_{0,0}, \dots, f_{n,0}) &
 \end{array}$$

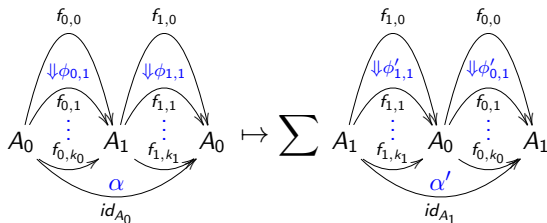
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$$\begin{aligned}
 \tilde{\iota}((\phi_{0,1} | \dots | \phi_{0,k_0}) \otimes \dots \otimes (\phi_{n,1} | \dots | \phi_{n,k_n}) \otimes \alpha) &= \iota(\phi_{0,1} | \dots | \phi_{0,k_0}) \bullet \dots \bullet (\phi_{n,1} | \dots | \phi_{n,k_n}) \alpha \\
 \iota_\phi(a_0 \otimes \dots \otimes a_p) &= \pm \phi(a_{d+1}, \dots, a_p) \cdot a_0 \otimes a_1 \otimes \dots \otimes a_d \quad \text{where } |\phi| = p - d
 \end{aligned}$$

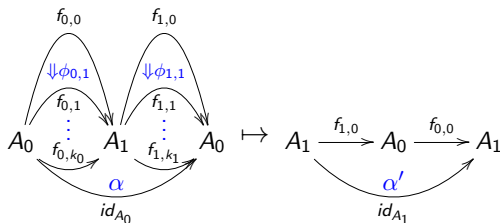
Defining $\tau_1!$

$$C(A_0 \rightarrow A_1 \rightarrow A_0)^{\bullet}(f, g) \xrightarrow{\tau_1!} C(A_1 \rightarrow A_0 \rightarrow A_1)^{\bullet}(g, f)$$



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$$\begin{array}{ccc}
 A_0 \xrightarrow{f_{0,0}} A_1 \xrightarrow{f_{1,0}} A_0 & \xrightarrow{?} & A_1 \xrightarrow{f_{1,0}} A_0 \xrightarrow{f_{0,0}} A_1 \\
 \text{\textcolor{blue}{\alpha}} \text{ (curved arrow)} & & \text{\textcolor{blue}{\alpha'}} \text{ (curved arrow)} \\
 id_{A_0} & & id_{A_1}
 \end{array}$$

Defining $\tau_1!$

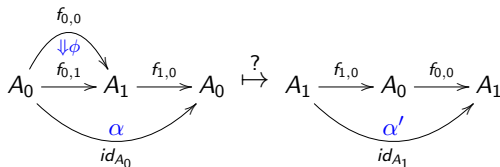
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 \end{array}$$

$$\alpha = a_0 \otimes \dots \otimes a_n \mapsto \alpha' = f_{0,0}(a_0) \otimes \dots \otimes f_{0,0}(a_n)$$

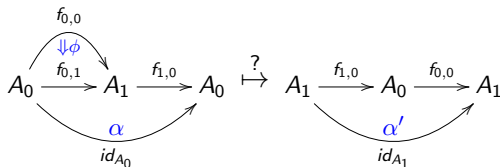
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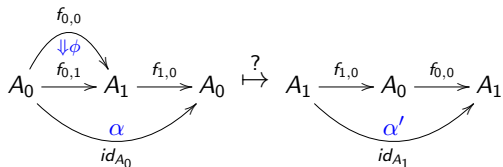
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$$\overline{\tau_1! \circ d}(\phi \otimes \alpha) = \overline{d \circ \tau_1!}(\phi \otimes \alpha)$$

Defining $\tau_{1!}$

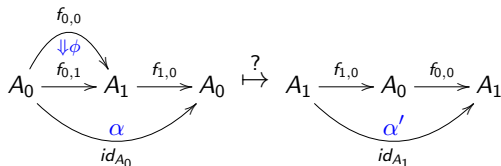
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$$[b, \bar{\tau}_{1!}](\phi \otimes \alpha) \pm \bar{\tau}_{1!}(\delta\phi \otimes \alpha) = [\bar{\tau}_{1!}, \iota_\phi](\alpha)$$

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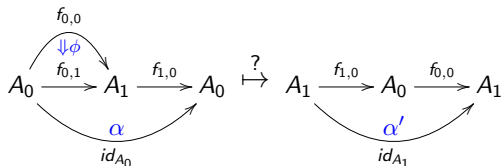
$$L_\phi(\alpha) = \sum_{k \geq 1} \pm a_0 \otimes \dots \otimes \phi(a_k, \dots) \otimes a_r \otimes \dots \otimes a_n +$$

$$\sum \pm \phi(a_k, \dots, a_n, a_0, \dots) \otimes a_s \otimes \dots \otimes a_{k-1}$$

$$[b, L_\phi] \pm L_{\delta\phi} = 0$$

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$$C(A_0 \rightarrow A_1 \rightarrow A_0)^{\bullet}(f, g) \xrightarrow{\tau_{1!}} C(A_1 \rightarrow A_0 \rightarrow A_1)^{\bullet}(g, f)$$



$$[b, \bar{\tau}_{1!}](\phi \otimes \alpha) \pm \bar{\tau}_{1!}(\delta\phi \otimes \alpha) = [\bar{\tau}_{1!}, \iota_\phi](\alpha)$$

$$\begin{aligned} \bar{\tau}_{1!}(\phi \otimes \alpha) = & \sum_{k \geq 1} \pm f_{0,0} a_0 \otimes \dots \otimes \phi(a_k, \dots) \otimes f_{0,1} a_r \dots \otimes f_{0,1} a_n + \\ & \sum \pm \phi(f_{1,0} f_{0,1} a_k, \dots, f_{1,0} f_{0,1} a_n, a_0, \dots) \otimes f_{0,1} a_s \otimes \dots \otimes f_{0,1} a_{k-1} \end{aligned}$$

Defining $\tau_1!$

$$\begin{aligned}
 & \bar{\tau}_1!((\phi_{0,1}|\dots|\phi_{0,k_0}) \otimes (\phi_{1,1}|\dots|\phi_{1,k_1}) \otimes \alpha) = \\
 &= \sum_{\substack{1 \leq i \leq j_1 \leq \dots \leq j_{2k_1} \leq k_0, \\ p}} \pm \phi_{0,1} \left(\begin{array}{c} f_{1,0} f_{0,i} a_p, \dots, f_{1,0} \phi_{0,i+1}(a_*, \dots), \\ f_{1,0} f_{0,i+1} a_*, \dots, f_{1,0} \phi_{0,j_1}(a_*, \dots), \\ \phi_{1,k_1}(f_{0,j_{2k_1}-1} a_*, \dots, \phi_{0,j_{2k_1}-1+1}(a_*, \dots), \dots), \dots, a_0, \dots \end{array} \right) \otimes \\
 & \quad \otimes f_{0,1} a_* \otimes \dots \otimes \phi_{0,2}(a_*, \dots) \otimes f_{0,2} a_* \otimes \dots \otimes \\
 & \quad \otimes \phi_{0,i}(a_*, \dots) \otimes f_{0,i} a_* \otimes \dots f_{0,i} a_{p-1} + \\
 & \quad \left(\sum \pm f_{0,0} a_0 \otimes \dots \otimes \phi_{0,1}(a_*, \dots) \otimes \dots \otimes \phi_{0,n_0}(a_*, \dots) \otimes \right. \\
 & \quad \left. \otimes f_{0,n_0} a_* \otimes \dots \otimes f_{0,n_0} a_n \quad \text{if } k_1 = 0 \right)
 \end{aligned}$$

First homotopy: $\tau_{1!}^2 \sim id$

$$C(A_0 \rightarrow A_1 \rightarrow A_0) \xrightarrow{\tau_{1!}} \hat{\tau}_1^* C(A_1 \rightarrow A_0 \rightarrow A_1) \xrightarrow{\hat{\tau}_1^* \tau_{1!}} \hat{\tau}_1^{*2} C(A_0 \rightarrow A_1 \rightarrow A_0)$$

$\text{-----} id \text{-----}$

First homotopy: $\tau_{1!}^2 \sim id$

$$C(A_0 \rightarrow A_1 \rightarrow A_0) \xrightarrow{\tau_{1!}} \hat{\tau}_1^* C(A_1 \rightarrow A_0 \rightarrow A_1) \xrightarrow{\hat{\tau}_1^* \tau_{1!}} \hat{\tau}_1^{*2} C(A_0 \rightarrow A_1 \rightarrow A_0)$$

\searrow
 id

$$A_0 \xrightarrow{f_{0,0}} A_1 \xrightarrow{f_{1,0}} A_0$$

α
 id_{A_0}

$$\alpha = a_0 \otimes \dots \otimes a_n \xrightarrow{\tau_{1!}} f_{0,0} a_0 \otimes \dots \otimes f_{0,0} a_n \xrightarrow{\hat{\tau}_{1!}^* \tau_{1!}} f_{1,0} f_{0,0} a_0 \otimes \dots \otimes f_{1,0} f_{0,0} a_n$$

First homotopy: $\tau_{1!}^2 \sim id$

$$C(A_0 \rightarrow A_1 \rightarrow A_0) \xrightarrow{\tau_{1!}} \hat{\tau}_1^* C(A_1 \rightarrow A_0 \rightarrow A_1) \xrightarrow{\hat{\tau}_1^* \tau_{1!}} \hat{\tau}_1^{*2} C(A_0 \rightarrow A_1 \rightarrow A_0)$$

$\text{---} id \text{---}$

$$A_0 \xrightarrow{f_{0,0}} A_1 \xrightarrow{f_{1,0}} A_0$$

α
 id_{A_0}

$$\alpha = a_0 \otimes \dots \otimes a_n \xrightarrow{\tau_{1!}} f_{0,0} a_0 \otimes \dots \otimes f_{0,0} a_n \xrightarrow{\hat{\tau}_{1!}^* \tau_{1!}} f_{1,0} f_{0,0} a_0 \otimes \dots \otimes f_{1,0} f_{0,0} a_n$$

$$f_{1,0}f_{0,0}\alpha - \alpha = [b, B](\alpha)$$

$$B(a_0 \otimes \dots \otimes a_n) = \sum_{0 \leq i \leq n} \pm 1 \otimes f_{1,0} f_{0,0} a_i \otimes \dots \otimes f_{1,0} f_{0,0} a_n \otimes a_0 \otimes \dots \otimes a_{i-1}$$

First homotopy: $\tau_{1!}^2 \sim id$

$$C(A_0 \rightarrow A_1 \rightarrow A_0) \xrightarrow{\tau_{1!}} \hat{\tau}_1^* C(A_1 \rightarrow A_0 \rightarrow A_1) \xrightarrow{\hat{\tau}_1^* \tau_{1!}} \hat{\tau}_1^{*2} C(A_0 \rightarrow A_1 \rightarrow A_0)$$

\curvearrowright
 id

$$\begin{aligned} & B((\phi_{0,1} | \dots | \phi_{0,k_0}) \otimes (\phi_{1,1} | \dots | \phi_{1,k_1}) \otimes \alpha) = \\ &= \sum_{\substack{0 \leq j_1 \leq \dots \leq j_{2k_1} \leq k_0 \\ p}} \pm 1 \otimes f_{1,0} f_{0,0} a_p \otimes \dots \otimes f_{1,0} \phi_{0,1}(a_*, \dots) \otimes \\ & \quad \otimes f_{1,0} f_{0,1} a_* \otimes \dots \otimes f_{1,0} \phi_{0,j_1}(a_*, \dots) \otimes \\ & \quad \otimes f_{1,0} f_{0,j_1} a_* \otimes \dots \otimes \phi_{1,1}(f_{0,j_1} a_*, \dots, \phi_{0,j_1+1}(a_*, \dots), \dots) \otimes \\ & \quad \otimes \dots \otimes \phi_{1,k_1}(f_{0,j_{2k_1}-1} a_*, \dots, \phi_{0,j_{2k_1}-1+1}(a_*, \dots), \dots) \otimes \dots \otimes \\ & \quad \otimes a_0 \otimes \dots \otimes a_{p-1} \end{aligned}$$

In the language of A_∞ -functors

$$\chi(\mathcal{C}) \rightarrow \mathcal{D}$$

$$\mathcal{A} := (A_0 \rightarrow A_1 \rightarrow A_0) \mapsto \begin{pmatrix} B(\mathcal{A}), \\ C(\mathcal{A}) \end{pmatrix}$$

$$\tau_1 \mapsto \begin{pmatrix} \hat{\tau}_1 : B(\mathcal{A}) \rightarrow B(\tau_1 \mathcal{A}) \\ \tau_{1!} : C(\mathcal{A}) \rightarrow \hat{\tau}_1^* C(\tau_1 \mathcal{A}) \end{pmatrix}$$

$$(\tau_1, \tau_1) \mapsto \begin{pmatrix} id : B(\mathcal{A}) \rightarrow B(\mathcal{A}) \\ B : C(\mathcal{A}) \rightarrow C(\mathcal{A}) \end{pmatrix}$$

$$(\tau_1, \tau_1, \tau_1) \mapsto \begin{pmatrix} \hat{\tau}_1 : B(\mathcal{A}) \rightarrow B(\tau_1 \mathcal{A}) \\ 0 : C(\mathcal{A}) \rightarrow \hat{\tau}_1^* C(\tau_1 \mathcal{A}) \end{pmatrix}$$

\vdots

The A_∞ relations mean:

- $\tau_{1!}$ is a map of complexes
- $B^2 = 0$
- The following diagram commutes:

$$\begin{array}{ccc} C(\mathcal{A}) & \xrightarrow{\tau_{1!}} & \hat{\tau}_1^* C(\tau_1 \mathcal{A}) \\ \downarrow B & & \downarrow \hat{\tau}_1^* B \\ C(\mathcal{A}) & \xrightarrow{\tau_{1!}} & \hat{\tau}_1^* C(\tau_1 \mathcal{A}) \end{array}$$

For higher $n > 1$, we want to find a homotopy between " $\tau_{n!}^{n+1}$ " and id .

For higher $n > 1$, we want to find a homotopy between “ $\tau_{n!}^{n+1}$ ” and id . However, it is sufficient to find a homotopy between $\hat{\tau}_n^{*2} \delta_{0,n!} \circ \hat{\tau}_n^* \tau_{n!} \circ \tau_{n!}$ and $\hat{\delta}_{n-1,n}^* \tau_{n-1!} \circ \delta_{n-1,n!}$.

$$\begin{array}{ccccc}
 [n] & \xrightarrow{\tau_n} & [n] & \xrightarrow{\tau_n} & [n] \\
 \downarrow \delta_{n-1,n} & & & & \swarrow \delta_{0,n} \\
 [n-1] & \xrightarrow{\tau_{n-1}} & [n-1] & &
 \end{array}$$

For higher $n > 1$, we want to find a homotopy between “ $\tau_{n!}^{n+1}$ ” and id . However, it is sufficient to find a homotopy between $\hat{\tau}_n^{*2} \delta_{0,n!} \circ \hat{\tau}_n^* \tau_{n!} \circ \tau_n!$ and $\hat{\delta}_{n-1,n}^* \tau_{n-1!} \circ \delta_{n-1,n!}$.

$$\begin{array}{ccccc}
 [n] & \xrightarrow{\tau_n} & [n] & \xrightarrow{\tau_n} & [n] \\
 \downarrow \delta_{n-1,n} & & & & \swarrow \delta_{0,n} \\
 [n-1] & \xrightarrow{\tau_{n-1}} & [n-1] & &
 \end{array}$$

Strategy: Find such a homotopy, \mathcal{B} , for $n = 2$, and use $\hat{\delta}_0^{*n-2} \mathcal{B}$ for $n > 2$.

$$\chi(\mathcal{C}) \rightarrow \mathcal{D}$$

$$\mathcal{A} \mapsto (B(\mathcal{A}), C(\mathcal{A}))$$

$$\mu = \tau_{n-1} \circ \delta_{n-1,n} = \delta_{0,n} \circ \tau_n^2 \mapsto \begin{pmatrix} \hat{\tau}_{n-1} \circ \hat{\delta}_{n-1,n} = \hat{\delta}_{0,n} \circ \hat{\tau}_n^2 \\ \hat{\tau}_n^{*2} \delta_{0,n!} \circ \hat{\tau}_n^* \tau_{n!} \circ \tau_{n!} \end{pmatrix}$$

$$(\delta_{0,n}, \tau_n^2) \mapsto \begin{pmatrix} \hat{\delta}_{0,n} \circ \hat{\tau}_n^2 \\ 0 \end{pmatrix}$$

$$(\tau_{n-1}, \delta_{n-1,n}) \mapsto \begin{pmatrix} \hat{\tau}_{n-1} \circ \hat{\delta}_{n-1,n} \\ \mathcal{B} \end{pmatrix}$$

$$(\tau_{n-1}, \delta_{n-1,n}, \lambda) \mapsto \begin{pmatrix} \hat{\tau}_{n-1} \circ \hat{\delta}_{n-1,n} \circ \hat{\lambda} \\ 0 \end{pmatrix}$$

$$\vdots$$

For $n > 1$, the A_∞ relations mean:

- $\tau_{n!}$ is a map of complexes
- $\mathcal{B}^2 = 0$
- We have a pair of homotopic maps:

$$\begin{array}{ccc}
 C(A_0 \rightarrow \dots \rightarrow A_n \rightarrow A_0) & & \\
 \hat{\tau}_n^{*2} \tau_{n!} \circ \hat{\tau}_n^* \tau_{n!} \circ \tau_{n!} & \begin{array}{c} \curvearrowright \\ \downarrow \end{array} & (\widehat{\delta_{n-2, n-1} \delta_{n-1, n}})^* \tau_{n-2!} \\
 \text{"apply } \tau_{n!} \text{ 3 times"} & & \text{"brace together the last 3 algebras,} \\
 & & \text{then apply } \tau_{n-2!} \text{ once"} \\
 C(A_{n-2} \rightarrow A_{n-1} \rightarrow A_n \rightarrow A_0 \rightarrow \dots \rightarrow A_{n-2}). & &
 \end{array}$$

For $n > 1$, the A_∞ relations mean:

- $\tau_{n!}$ is a map of complexes
- $\mathcal{B}^2 = 0$
- They are homotopic via two homotopies:

$$\begin{array}{ccc}
 (\widehat{\delta_{n-2,n-1} \delta_{n-1,n}})^* \tau_{n-2!} & \xrightarrow{\cong} & \hat{\delta}_{n-1,n}^* (\hat{\delta}_{n-2,n-1}^* \tau_{n-2!}) \xrightarrow{\hat{\delta}_{n-1,n}^* \mathcal{B}_{n-1}} \hat{\delta}_{n-1,n}^* (\hat{\tau}_{n-1}^* \tau_{n-1!} \circ \tau_{n-1!}) \\
 \text{"brace together } A_{n-2}, A_{n-1}, A_n, \text{ then apply } \tau_{n-2!}" & & \\
 \downarrow \cong & & \downarrow \cong \\
 (\widehat{\delta_{n-2,n-1} \delta_{n-1,n}})^* \tau_{n-2!} & & \hat{\tau}_n^{*2} \tau_{n!} \circ \hat{\delta}_{n-1,n}^* \tau_{n-1!} \\
 \downarrow \hat{\delta}_{n-2,n}^* \mathcal{B}_{n-1} & & \text{"brace together } A_{n-1}, A_n \text{ and apply } \tau_{n-1!}, \text{ then apply } \tau_{n!}" \\
 & & \downarrow \tau_n^{*2} \tau_{n!} \circ \mathcal{B}_n \\
 \hat{\delta}_{n-2,n}^* (\hat{\tau}_{n-1}^* \tau_{n-1!} \circ \tau_{n-1!}) & \xrightarrow{\cong} & \hat{\tau}_n^* (\hat{\delta}_{n-1,n}^* \tau_{n-1!}) \circ \tau_{n!} \xrightarrow{\hat{\tau}_n^* \mathcal{B}_n \circ \tau_{n!}} \hat{\tau}_n^{*2} \tau_{n!} \circ \hat{\tau}_n^* \tau_{n!} \circ \tau_{n!} \\
 & \text{"apply } \tau_{n!}, \text{ then brace together } A_{n-1}, A_{n-2} \text{ and apply } \tau_{n-1!}" & \text{"apply } \tau_{n!} \text{ three times"}
 \end{array}$$

Summary: We have a given an A_∞ -functor $\chi(\mathcal{C}) \rightarrow \mathcal{D}$, which implies that algebras form a [category in dg cocategories with a trace up to homotopy](#).

To get a category in [dg categories](#) with a trace up to homotopy, apply (categorified) $\mathrm{Cobar}(-)$.

Thank you!