Title

Subtitle

Rebecca Wei

Northwestern University

Date/Event

Braces, categorically

Fix an algebra, A.
Define a dg category, $Hoch(A)$:

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Morphisms: $Hoch(A)(f, g) = (C^\bullet(A, {}_fA_g), {}_f\delta_g)$

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$$
\begin{aligned}
{}_f\delta_g(\phi)(a_1 \otimes \ldots \otimes a_n) = &\epsilon_\phi \Bigg(f(a_1) \cdot \phi(a_2, \ldots, a_n) + \\
&+ \sum_{1 \le i \le n-1} (-1)^i \phi(a_1, \ldots, a_i a_{i+1}, \ldots, a_n) + \\
&+ (-1)^n \phi(a_1, \ldots, a_{n-1}) \cdot g(a_n) \Bigg) \\
\epsilon_\phi = &(-1)^{|\phi|+1}
\end{aligned}
$$

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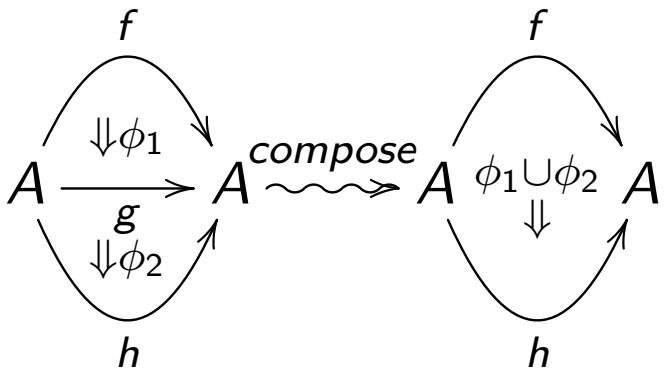

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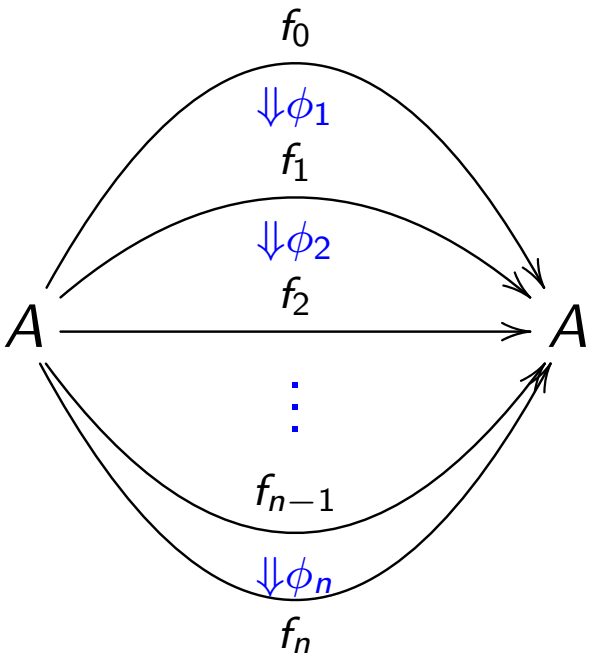

A morphism from f_0 to f_n in Bar(Hoch(A))

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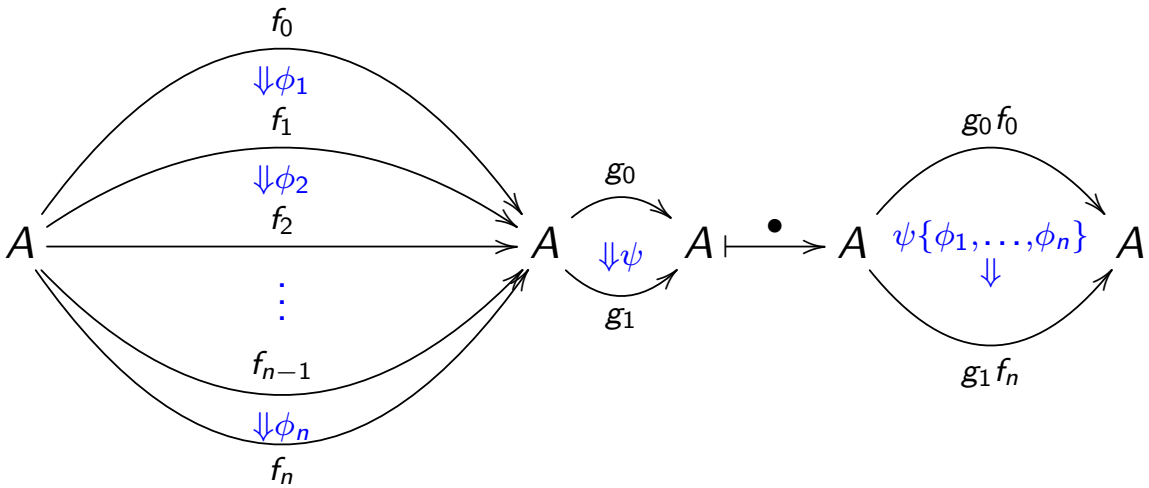

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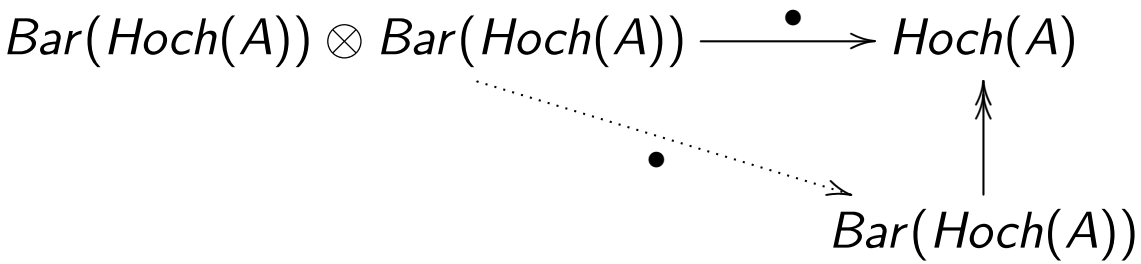
In this context, braces give multilinear maps:



Then, (Bar(Hoch(A)), •) is an algebra in DGCocats.

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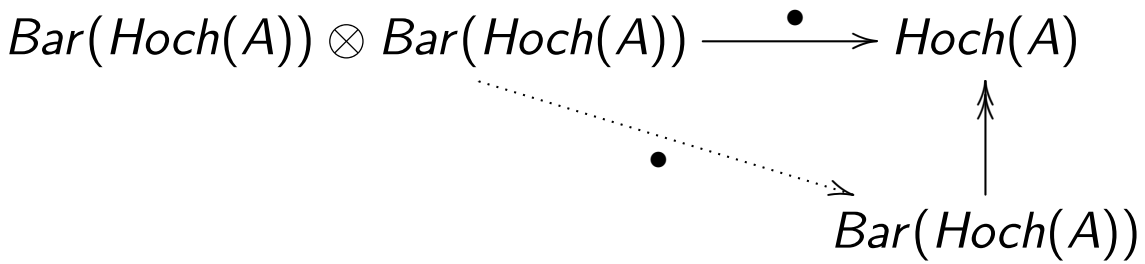

Then, (Bar(Hoch(A)), •) is an algebra in DGCocats.
But we have more...

More structure

More structure

Fix algebras, $A_0, A_1, \ldots, A_n$.
We will define a dg cocategory $B(A_0 \to A_1 \to \ldots \to A_n \to A_0)$
where $B(A_0 \to A_0) := Bar(Hoch(A_0))$ for n=0.

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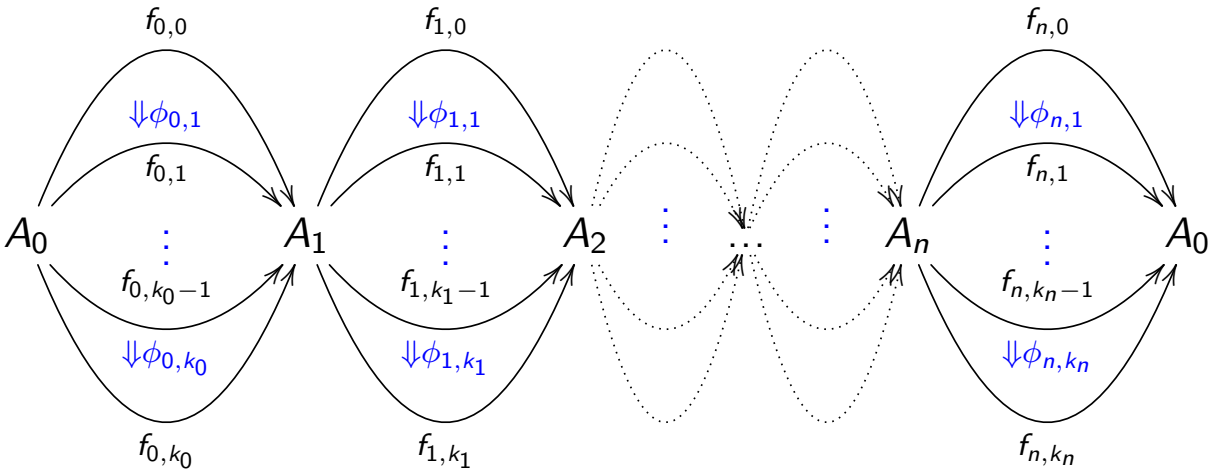
Objects: $\{A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \ldots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_0\}$

A morphism from $(A_0 \xrightarrow{f_{0,0}} \ldots \xrightarrow{f_{n,0}} A_0)$ to $(A_0 \xrightarrow{f_{0,k_0}} \ldots \xrightarrow{f_{n,k_n}} A_0)$:



$\phi_{i,j} \in C^\bullet(A_{i,f_{i,j-1}} A_{i+1 f_{i,j}})$

Structure among the $B(A_0 \to \ldots \to A_n \to A_0)$'s

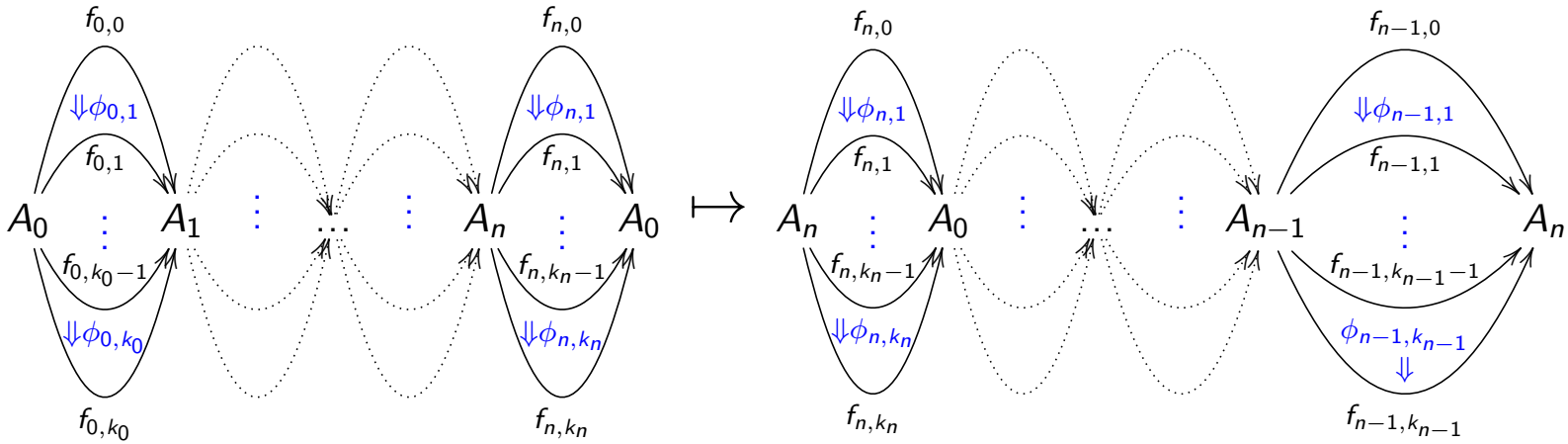
Example

We have a dg functor

$$\hat{\tau}_n : B(A_0 \to \ldots \to A_n \to A_0) \to B(A_n \to A_0 \to \ldots \to A_n)$$

$$Obj : \quad (f_0, \ldots, f_n) \mapsto (f_n, f_0, \ldots, f_{n-1})$$

Mor :



Structure among the $B(A_0 \to \ldots \to A_n \to A_0)$'s

Example

For $n \geq 1, 0 \leq j < n$, we have a dg functor

$$\hat{\delta}_{j,n} : B(A_0 \to \ldots \to A_n \to A_0) \to B(A_0 \to \ldots \to A_j \to A_{\substack{j+2 \\ (mod\ n+1)}} \to \ldots \to A_0)$$

$$Obj : \quad (f_0, \ldots, f_n) \mapsto (f_0, \ldots, f_{j+1} f_j, \ldots, f_n)$$

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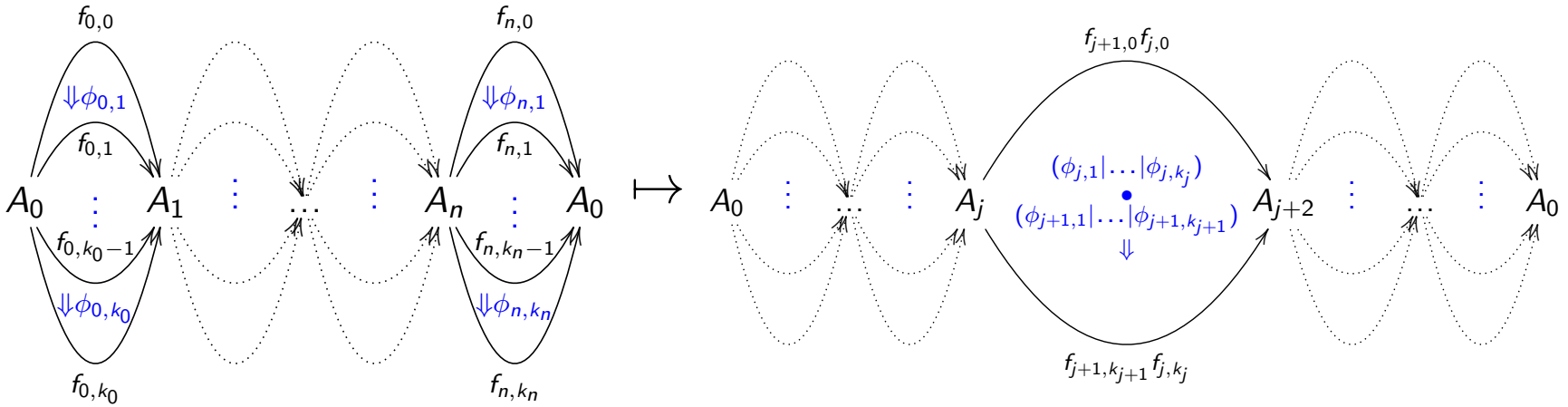

Structure among the $B(A_0 \to \ldots \to A_n \to A_0)$'s

Example

For $n \geq 0, 0 \leq i \leq n$, we have a dg functor

$$\hat{\sigma}_{i,n} : B(A_0 \to \ldots \to A_n \to A_0) \to B(A_0 \to \ldots \to A_i \to A_i \to \ldots \to A_0)$$

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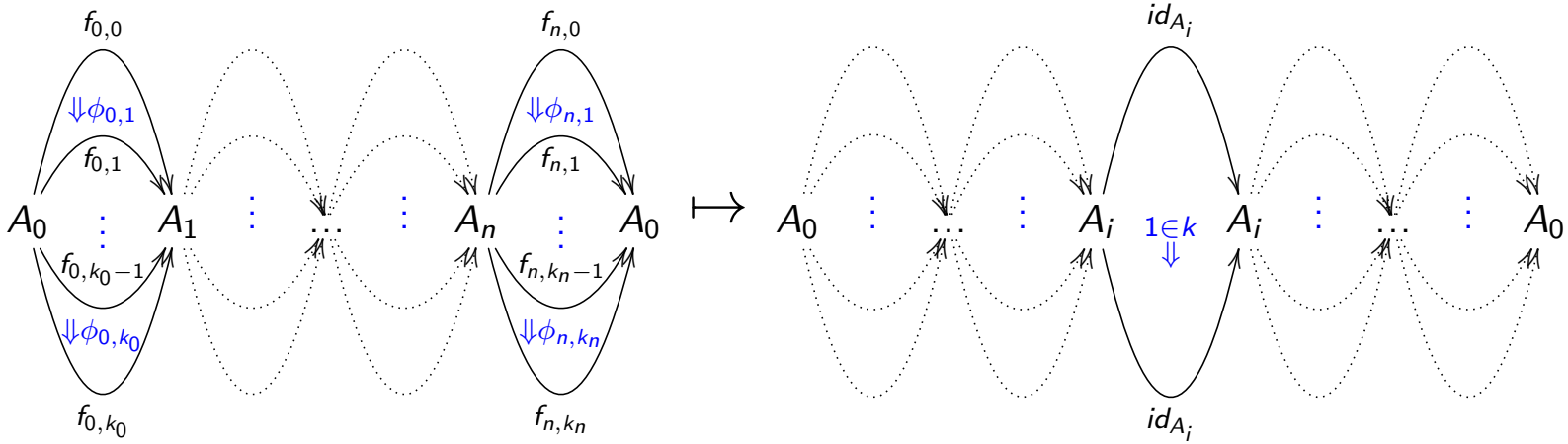

A sheafy-cyclic object in DGCocat

Definition

Let χ be the category with objects $\{A_0 \to A_1 \to \ldots \to A_n \to A_0\}$ and morphisms compositions of

$$\textit{rotations } \tau_n : \mathcal{A} \mapsto (A_n \to A_0 \to \ldots \to A_n)$$
$$\textit{coboundaries } \delta_{j,n} : \mathcal{A} \mapsto (A_0 \to \ldots \to A_j \to A_{j+2\,(\textit{mod } n+1)} \to \ldots \to A_0)$$
$$\textit{codegeneracies } \sigma_{i,n} : \mathcal{A} \mapsto (A_0 \to \ldots \to A_i \to A_i \to \ldots \to A_0)$$

where $\mathcal{A} := (A_0 \to \ldots \to A_n \to A_0)$, subject to the cyclic relations.

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Proposition

We have a functor $\chi \to DGCocat$

$$\textit{Objects}: \ (A_0 \to \ldots \to A_n \to A_0) \mapsto B(A_0 \to \ldots \to A_n \to A_0)$$
$$\textit{Generating morphisms}: \ \lambda \mapsto \hat{\lambda}$$

Motivating Question

Each dg cocategory $B(A_0 \to \ldots \to A_n \to A_0)$ has a dg comodule $C(A_0 \to \ldots \to A_n \to A_0)$ (which is a categorified version of the bar complex of Hochschild chains as a module over Hochschild cochains via contraction).

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Motivating Question: Can we extend the sheafy-cyclic structure to the comodules? In other words, can we give a functor

$$\chi \to \mathcal{D} := \{(dg\ cocat,\ dg\ comod)\}?$$

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$$\sout{\chi \to \mathcal{D} := \{(\textit{dg cocat}, \textit{dg comod})\}?}$$

$$\chi_\infty \to \mathcal{D}_\infty \quad \textit{dg categories}$$

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- Define dg categories χ_∞ and $\mathcal{D}_\infty$
- Define dg comodules $C(A_0 \to \ldots \to A_n \to A_0)$
- Define the A_∞-functor $\mathcal{F}$

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- Define dg categories χ_∞ and $\mathcal{D}_\infty$
- Define dg comodules $C(A_0 \to \ldots \to A_n \to A_0)$
- Define the A_∞-functor $\mathcal{F}$

- Time and interest permitting
  - Rectify $\mathcal{F}$ to a dg functor
  - Give a dg functor $\mathcal{D}_\infty \to \mathcal{E} = \{(\text{dg cat, dg mod})\}$

$$U(\chi_\infty) \xrightarrow{\textit{rectified}} \mathcal{D}_\infty \to \mathcal{E}$$

"A homotopically sheafy-cyclic object in dg categories with a dg module"

Dg categories χ_∞ and $\mathcal{D}_\infty$

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χ_∞:

Objects: same objects as $\chi = \{A_0 \to \ldots \to A_n \to A_0\}$

$\chi_\infty^\bullet(X, Y) := \{\sum a_i v_i : a_i \in k, v_i \in \chi(X, Y)\}$

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Objects: same objects as $\mathcal{D} = \{(\text{dg cocategory, dg comodule})\}$

$$\mathcal{D}_\infty^p((B_1, C_1), (B_0, C_0)) := \left\{ \begin{array}{c} F : B_1 \to B_0 \text{ dg functor} \\ F_! : C_1 \to F^* C_0 \text{ degree-}p \text{ linear map} \end{array} \right\}$$

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$F^* C_0$ is the categorified version of co-extension of scalars:

$$F^* C_0 = ker(B_1 \otimes C_0 \rightrightarrows B_1 \otimes B_0 \otimes C_0)$$

Dg comodules over dg cocategories

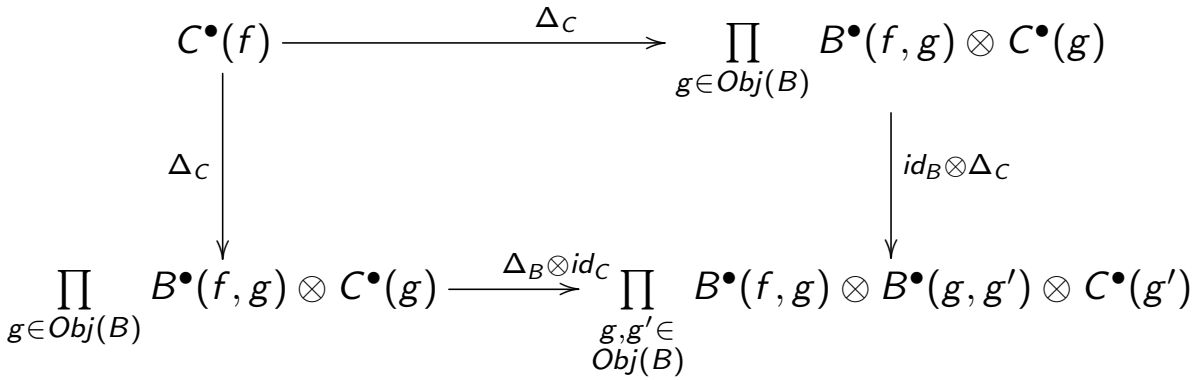
Definition

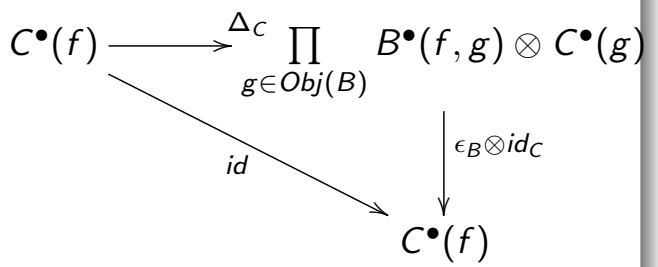
A **dg comodule** C over a dg cocategory B consists of the following data:

- for each object $f \in B$, a complex $C^\bullet(f)$, and
- maps of complexes

$$\Delta_C(f) : C^\bullet(f) \to \prod_{g \in Obj(B)} B^\bullet(f,g) \otimes C^\bullet(g).$$

such that the following diagrams for coassociativity and counitality commute:

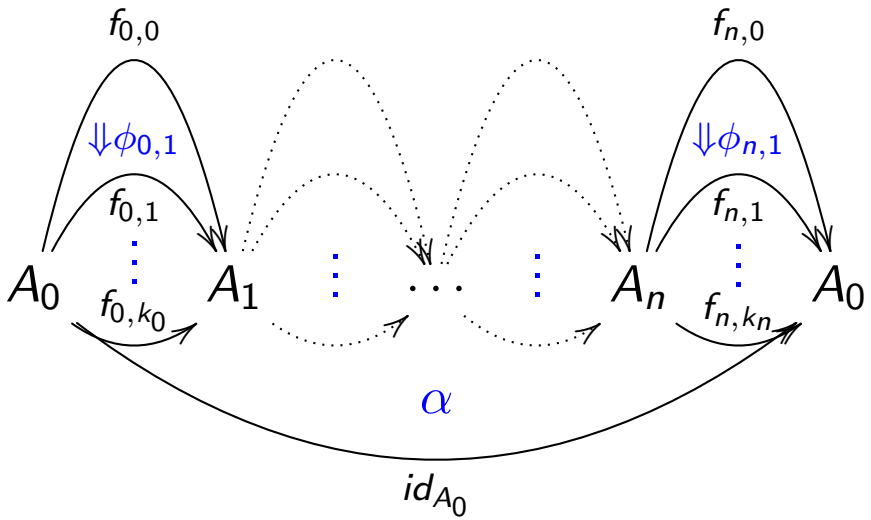




Dg comodules over dg cocategories

Fix algebras $A_0, \ldots, A_n$.
Define a dg comodule over $B(A_0 \to \ldots \to A_n \to A_0)$:

$C(A_0 \to \ldots \to A_n \to A_0)^\bullet(f_{0,0}, \ldots, f_{n,0}) :=$

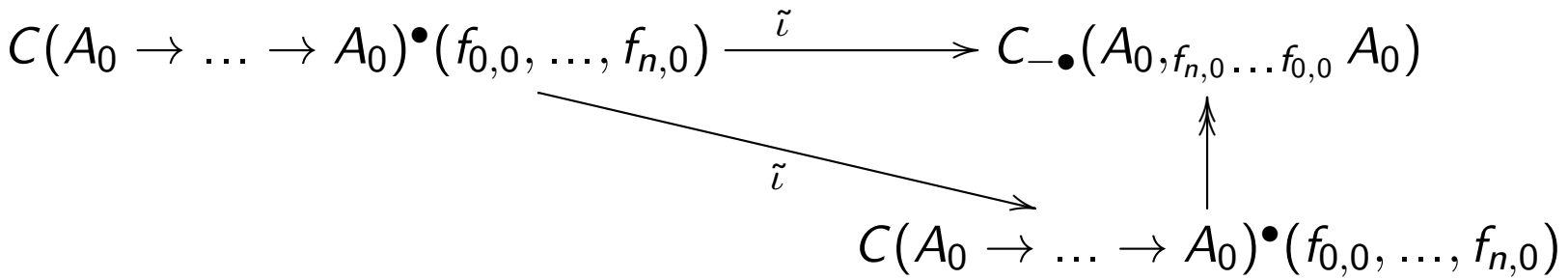
$:= \{$ 

$=$ $(\phi_{0,1}|\ldots|\phi_{0,k_0})\otimes\ldots\otimes(\phi_{n,1}|\ldots|\phi_{n,k_n})\otimes\alpha$:
$s.t.\ \phi_{i,j}\in C^\bullet(A_i, f_{j-1}\, A_{i+1}\, f_j)$, $\}$
$\alpha\in C_{-\bullet}(A_0, f_{n,k_n}\ldots f_{0,k_0}\, A_0)$

$$d_{C(A_0\to\ldots\to A_0)} = d_B \otimes id_{C_{-\bullet}} + id_B \otimes b + \tilde{\imath}$$

Dg comodules over dg cocategories

where $\tilde{\imath}$ is given as follows:



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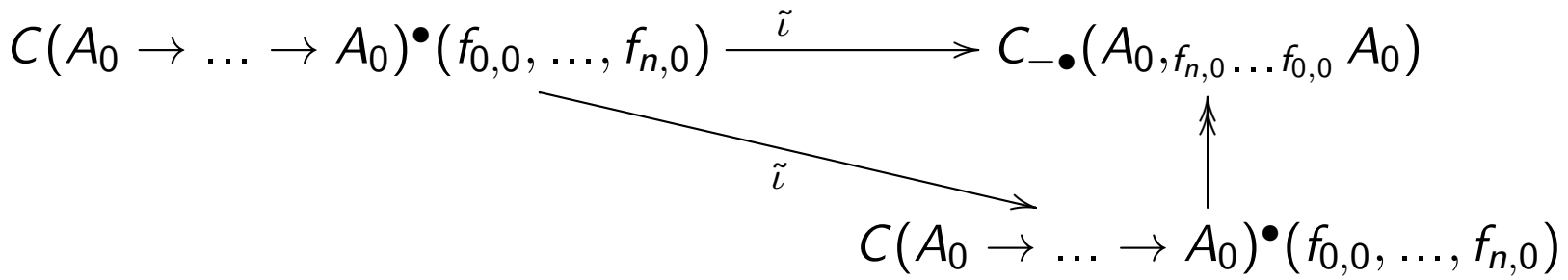

$$\tilde{\imath}\big((\phi_{0,1}|\ldots|\phi_{0,k_0})\otimes\ldots\otimes(\phi_{n,1}|\ldots|\phi_{n,k_n})\otimes\alpha\big) = \iota_{(\phi_{0,1}|\ldots|\phi_{0,k_0})\bullet\ldots\bullet(\phi_{n,1}|\ldots|\phi_{n,k_n})}\alpha$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

$$\mathcal{A} := (A_0 \to \ldots \to A_n \to A_0)$$

$$\chi_\infty \to \mathcal{D}_\infty$$

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Generating Morphisms:

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Generating Morphisms: $$\sigma_{i,n} \mapsto \begin{pmatrix} B(\mathcal{A}) \xrightarrow{\hat{\sigma}_{i,n}} B(\sigma_{i,n}\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{id} \hat{\sigma}_{i,n}^* C(\sigma_{i,n}\mathcal{A}) \end{pmatrix}$$

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An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

$$\hat{\tau}_n^* C(\tau_n \mathcal{A})^\bullet(\underbrace{f_0, \ldots, f_n}_{\in Obj(B(\mathcal{A}))}) \cong C(\tau_n \mathcal{A})^\bullet(f_n, f_0, \ldots, f_{n-1}) \text{ as complexes.}$$

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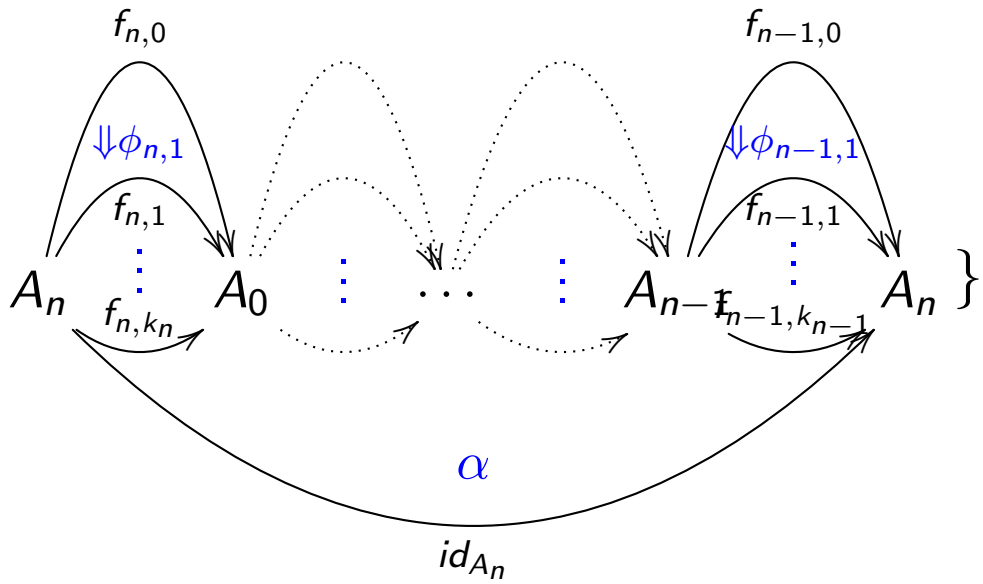

$\}$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

$$\hat{\tau}_n^* C(\tau_n \mathcal{A})^\bullet(\underbrace{f_0, \ldots, f_n}_{\in Obj(B(\mathcal{A}))}) \cong C(\tau_n \mathcal{A})^\bullet(f_n, f_0, \ldots, f_{n-1}) \text{ as complexes.}$$

$$n = 1 : \quad C(A_0 \to A_1 \to A_0)^\bullet(f, g) \xrightarrow{\tau_{1!}} C(A_1 \to A_0 \to A_1)^\bullet(g, f)$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

$$\hat{\tau}_n^* C(\tau_n \mathcal{A})^\bullet(\underbrace{f_0, \ldots, f_n}_{\in Obj(B(\mathcal{A}))}) \cong C(\tau_n \mathcal{A})^\bullet(f_n, f_0, \ldots, f_{n-1}) \text{ as complexes.}$$

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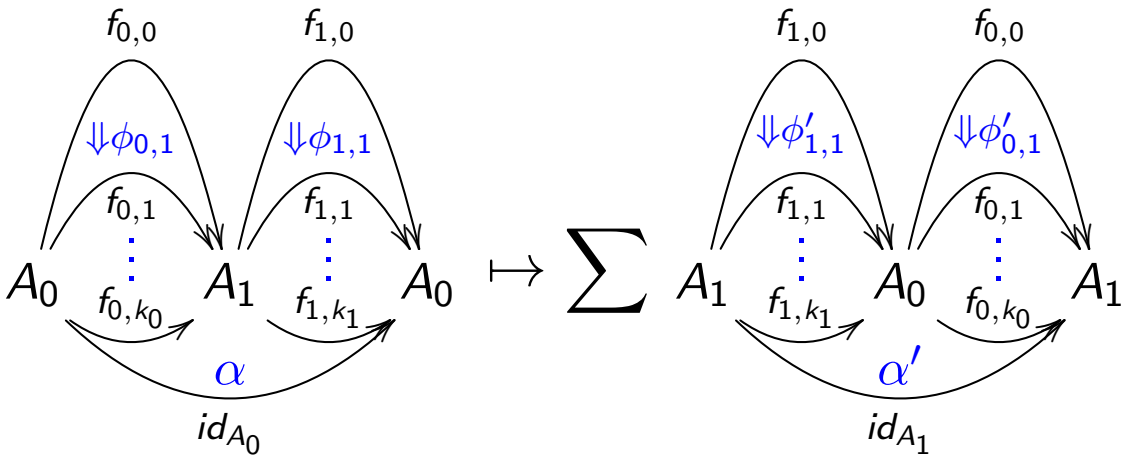
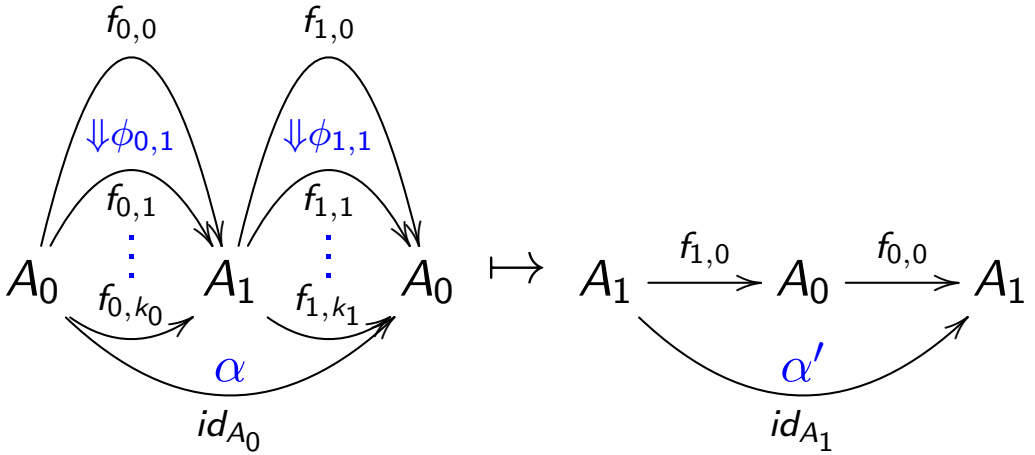


An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

$$\hat{\tau}_n^* C(\tau_n \mathcal{A})^\bullet(\underbrace{f_0, \ldots, f_n}_{\in Obj(B(\mathcal{A}))}) \cong C(\tau_n \mathcal{A})^\bullet(f_n, f_0, \ldots, f_{n-1}) \text{ as complexes.}$$

$$n = 1: \quad C(A_0 \to A_1 \to A_0)^\bullet(f, g) \xrightarrow{\tau_{1!}} C(A_1 \to A_0 \to A_1)^\bullet(g, f)$$

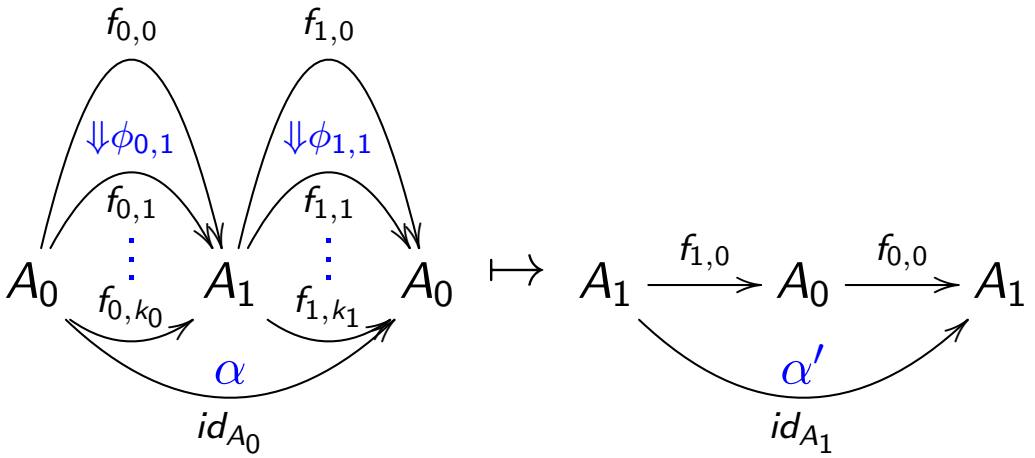


An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

Solution: Give these maps to cogenerators to define $\tau_{1!}$, then let

$$\begin{aligned}\tau_{n!} : C(\mathcal{A}) &\cong \hat{\delta}_0^{*n-1} C\big(A_0 \to A_n \to A_0\big) \xrightarrow{\hat{\delta}_0^{*n-1}\tau_{1!}} \hat{\delta}_0^{*n-1}\hat{\tau}_1^* C\big(A_n \to A_0 \to A_n\big) \cong \\ &\cong \widehat{(\tau_1 \delta_0^{n-1})}^* C\big(A_n \to A_0 \to A_n\big) \cong \widehat{(\delta_0^{n-1}\tau_n)}^* C\big(A_n \to A_0 \to A_n\big) \cong \\ &\cong \hat{\tau}_n^* \hat{\delta}_0^{*n-1} C\big(A_n \to A_0 \to A_n\big) \cong \hat{\tau}_n^* C(\tau_n \mathcal{A}).\end{aligned}$$

$$n = 1: \quad C(A_0 \to A_1 \to A_0)^\bullet(f, g) \xrightarrow{\tau_{1!}} C(A_1 \to A_0 \to A_1)^\bullet(g, f)$$

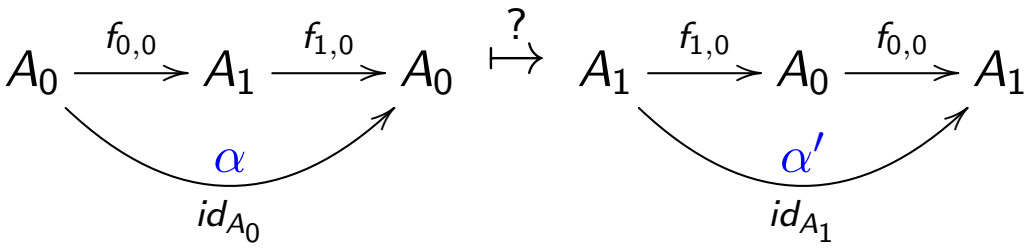


An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

Defining $\tau_{1!}$

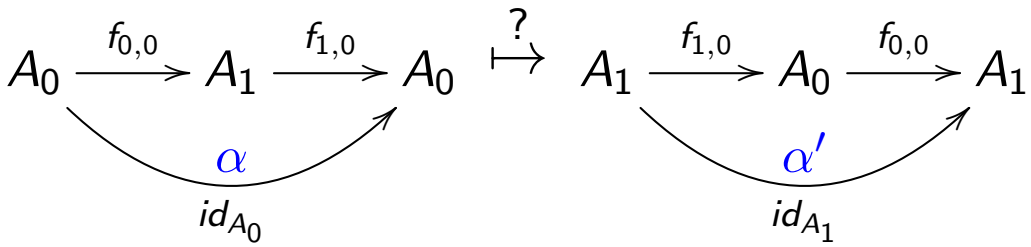
An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

Defining $\tau_{1!}$



An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

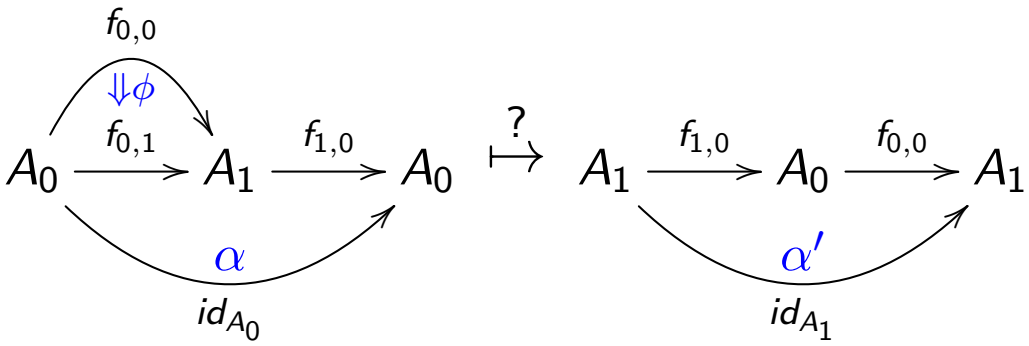
Defining $\tau_{1!}$



$$\alpha = a_0 \otimes \ldots \otimes a_n \mapsto \alpha' = f_{0,0}(a_0) \otimes \ldots \otimes f_{0,0}(a_n)$$

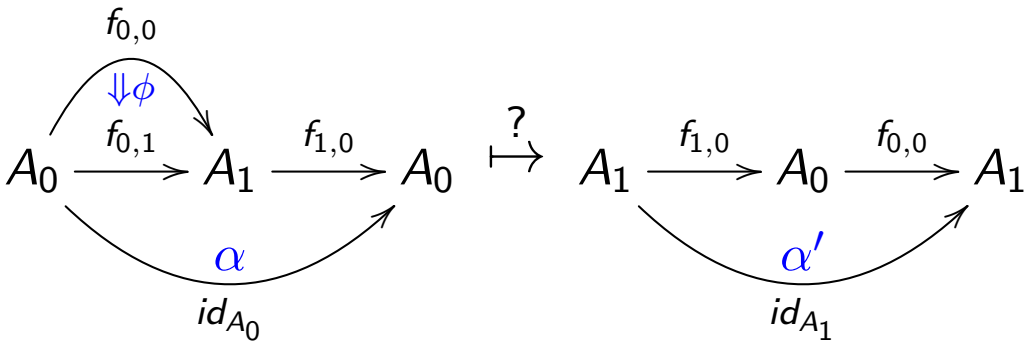
An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

Defining $\tau_{1!}$



An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

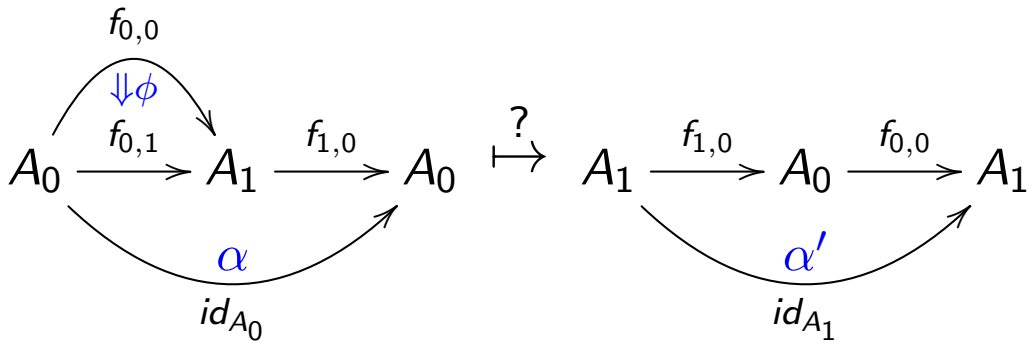
Defining $\tau_{1!}$



$$\bar{\tau}_{1!} \circ d(\phi \otimes \alpha) = d \circ \bar{\tau}_{1!}(\phi \otimes \alpha)$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

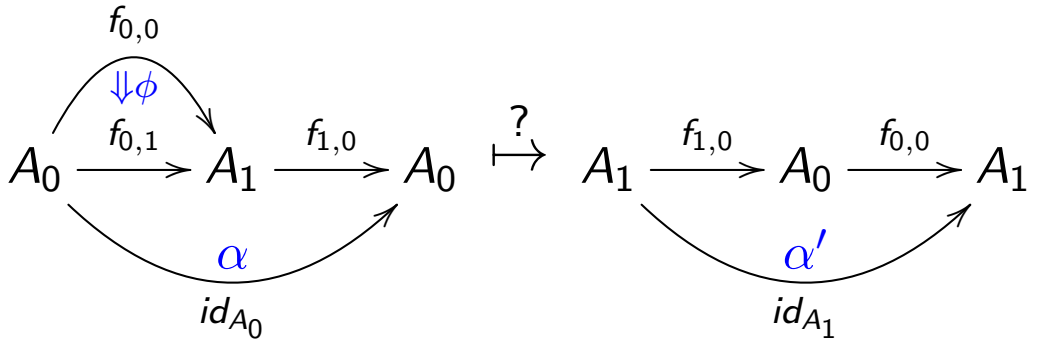
Defining $\tau_{1!}$



$$[b, \bar{\tau}_{1!}](\phi \otimes \alpha) \pm \bar{\tau}_{1!}(\delta\phi \otimes \alpha) = [\bar{\tau}_{1!}, \iota_\phi](\alpha)$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

Defining $\tau_{1!}$



$$[b, \bar{\tau}_{1!}](\phi \otimes \alpha) \pm \bar{\tau}_{1!}(\delta\phi \otimes \alpha) = [\bar{\tau}_{1!}, \iota_\phi](\alpha)$$

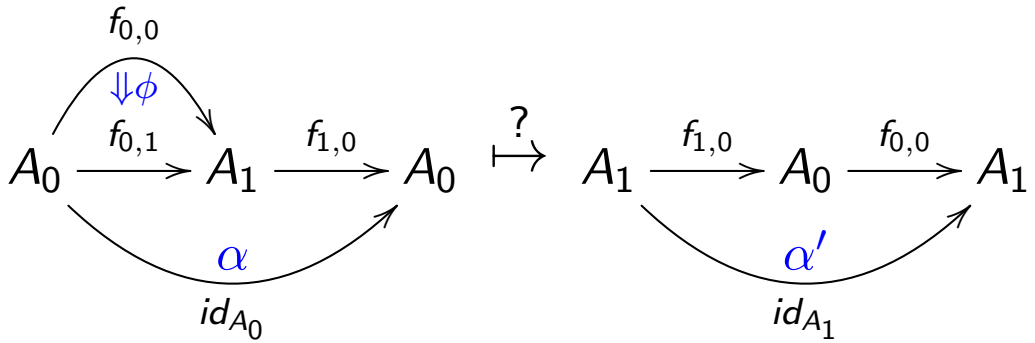
$$L_\phi(\alpha) = \sum_{k \geq 1} \pm a_0 \otimes \ldots \otimes \phi(a_k, \ldots) \otimes a_r \otimes \ldots \otimes a_n +$$

$$\sum \pm \phi(a_k, \ldots, a_n, a_0, \ldots) \otimes a_s \otimes \ldots \otimes a_{k-1}$$

$$[b, L_\phi] \pm L_{\delta\phi} = 0$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

Defining $\tau_{1!}$



$$[b, \bar{\tau}_{1!}](\phi \otimes \alpha) \pm \bar{\tau}_{1!}(\delta\phi \otimes \alpha) = [\bar{\tau}_{1!}, \iota_\phi](\alpha)$$

$$\bar{\tau}_{1!}(\phi \otimes \alpha) = \sum_{k \geq 1} \pm f_{0,0} a_0 \otimes \ldots \otimes \phi(a_k, \ldots) \otimes f_{0,1} a_r \ldots \otimes f_{0,1} a_n +$$

$$\sum \pm \phi(f_{1,0} f_{0,1} a_k, \ldots, f_{1,0} f_{0,1} a_n, a_0, \ldots) \otimes f_{0,1} a_s \otimes \ldots \otimes f_{0,1} a_{k-1}$$

$$[b, \bar{\tau}_{1!}(\phi, -)] \pm \bar{\tau}_{1!}(\delta\phi, -) = [\bar{\tau}_{1!}, \iota_\phi]$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

Defining $\tau_{1!}$

$$\bar{\tau}_{1!}\big((\phi_{0,1}|\dots|\phi_{0,k_0}) \otimes (\phi_{1,1}|\dots|\phi_{1,k_1}) \otimes \alpha\big) =$$

$$= \sum_{\substack{1\le i\le n_0 \\ i\le j_1\le\dots\le j_{2k_1}\le k_0 \\ p}} \pm\phi_{0,1}\Big(\begin{array}{c} f_{1,0}f_{0,i}a_p,\dots,f_{1,0}\phi_{0,i+1}(a_*,\dots), \\ f_{1,0}f_{0,i+1}a_*,\dots,f_{1,0}\phi_{0,j_1}(a_*,\dots), \\ f_{1,0}f_{0,j_1}a_*,\dots,\phi_{1,1}(f_{0,j_1}a_*,\dots,\phi_{0,j_1+1}(a_*,\dots),\dots),\dots, \\ \phi_{1,k_1}(f_{0,j_{2k_1-1}}a_*,\dots,\phi_{0,j_{2k_1-1}+1}(a_*,\dots),\dots),\dots,a_0,\dots \end{array} \Big)\otimes$$

$$\otimes f_{0,1}a_* \otimes \dots \otimes \phi_{0,2}(a_*,\dots) \otimes f_{0,2}a_* \otimes \dots\otimes$$

$$\otimes \phi_{0,i}(a_*,\dots) \otimes f_{0,i}a_* \otimes \dots f_{0,i}a_{p-1}+$$

$$\Big(\sum \pm f_{0,0}a_0 \otimes \dots \otimes \phi_{0,1}(a_*,\dots) \otimes \dots \otimes \phi_{0,n_0}(a_*,\dots)\otimes$$

$$\otimes f_{0,n_0}a_* \otimes \dots \otimes f_{0,n_0}a_n \quad \text{if } k_1 = 0\Big)$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

$$\mathcal{A} := (A_0 \to \ldots \to A_n \to A_0)$$

$$\chi_\infty \to \mathcal{D}_\infty$$
$$\mathcal{A} \mapsto (B(\mathcal{A}), C(\mathcal{A}))$$

Generating Morphisms:

$$\sigma_{i,n} \mapsto \begin{pmatrix} B(\mathcal{A}) \xrightarrow{\hat{\sigma}_{i,n}} B(\sigma_{i,n}\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{id} \hat{\sigma}_{i,n}^* C(\sigma_{i,n}\mathcal{A}) \end{pmatrix}$$

$$\delta_{j,n} \mapsto \begin{pmatrix} B(\mathcal{A}) \xrightarrow{\hat{\delta}_{j,n}} B(\delta_{j,n}\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{id} \hat{\delta}_{j,n}^* C(\delta_{j,n}\mathcal{A}) \end{pmatrix} \qquad \tau_n \mapsto \begin{pmatrix} B(\mathcal{A}) \xrightarrow{\hat{\tau}_n} B(\tau_n\mathcal{A}) \\ C(\mathcal{A}) \xrightarrow{\tau_{n!} = d_0^{*n-1}\tau_{1!}} \hat{\tau}_n^* C(\tau_n\mathcal{A}) \end{pmatrix}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

n=1:

$$C(A_0 \to A_1 \to A_0) \xrightarrow{\tau_{1!}} \hat{\tau}_1^* C(\tau_1(A_0 \to A_1 \to A_0)) \xrightarrow{\hat{\tau}_1^* \tau_{1!}} \hat{\tau}_1^{*2} C(\tau_1^2(A_0 \to A_1 \to A_0))$$

$$\overset{id}{\frown}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

n=1:

$$C(A_0 \to A_1 \to A_0) \xrightarrow{\tau_{1!}} \hat{\tau}_1^* C(\tau_1(A_0 \to A_1 \to A_0)) \xrightarrow{\hat{\tau}_1^* \tau_{1!}} \hat{\tau}_1^{*2} C(\tau_1^2(A_0 \to A_1 \to A_0))$$

$$\overset{id}{\frown}$$

$$A_0 \xrightarrow{f_{0,0}} A_1 \xrightarrow{f_{1,0}} A_0$$

$$\overset{\alpha}{\underset{id_{A_0}}{\frown}}$$

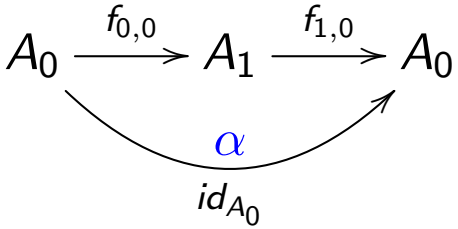
$$\alpha = a_0 \otimes \ldots \otimes a_n \overset{\tau_{1!}}{\mapsto} f_{0,0}a_0 \otimes \ldots \otimes f_{0,0}a_n \overset{\hat{\tau}_{1!}^* \tau_{1!}}{\mapsto} f_{1,0}f_{0,0}a_0 \otimes \ldots \otimes f_{1,0}f_{0,0}a_n$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

n=1:

$$C(A_0 \to A_1 \to A_0) \xrightarrow{\tau_{1!}} \hat{\tau}_1^* C(\tau_1(A_0 \to A_1 \to A_0)) \xrightarrow{\hat{\tau}_1^* \tau_{1!}} \hat{\tau}_1^{*2} C(\tau_1^2(A_0 \to A_1 \to A_0))$$

$$\xrightarrow{\qquad\qquad id \qquad\qquad}$$



$$\alpha = a_0 \otimes \ldots \otimes a_n \overset{\tau_{1!}}{\mapsto} f_{0,0}a_0 \otimes \ldots \otimes f_{0,0}a_n \overset{\hat{\tau}_{1!}^* \tau_{1!}}{\mapsto} f_{1,0}f_{0,0}a_0 \otimes \ldots \otimes f_{1,0}f_{0,0}a_n$$

$$f_{1,0}f_{0,0}\alpha - \alpha = [b, B](\alpha)$$

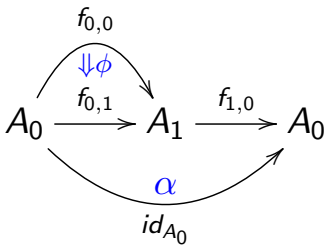
$$B(a_0 \otimes \ldots \otimes a_n) = \sum_{0 \le i \le n} \pm 1 \otimes f_{1,0}f_{0,0}a_i \otimes \ldots \otimes f_{1,0}f_{0,0}a_n \otimes a_0 \otimes \ldots \otimes a_{i-1}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

n=1:

$$C(A_0 \to A_1 \to A_0) \xrightarrow{\tau_{1!}} \hat{\tau}_1^* C(\tau_1(A_0 \to A_1 \to A_0)) \xrightarrow{\hat{\tau}_1^* \tau_{1!}} \hat{\tau}_1^{*2} C(\tau_1^2(A_0 \to A_1 \to A_0))$$

$$\overset{id}{\frown}$$



$$= \phi \otimes \alpha$$

$$(\overline{\hat{\tau}^* \tau_{1!} \circ \tau_{1!} - id})(\phi \otimes \alpha) = \text{“}f_{1,0} \circ L_\phi(\alpha)''$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

n=1:

$$C(A_0 \to A_1 \to A_0) \xrightarrow{\tau_{1!}} \hat{\tau}_1^* C(\tau_1(A_0 \to A_1 \to A_0)) \xrightarrow{\hat{\tau}_1^* \tau_{1!}} \hat{\tau}_1^{*2} C(\tau_1^2(A_0 \to A_1 \to A_0))$$

$$\overset{id}{\frown}$$

$$(\overline{\hat{\tau}^* \tau_{1!} \circ \tau_{1!} - id})(\phi \otimes \alpha) = \text{“}f_{1,0} \circ L_\phi(\alpha)''$$

$$= \sum_{k \geq 1} \pm f_{1,0} f_{0,0} a_0 \otimes \ldots \otimes f_{1,0} \phi(a_k, \ldots) \otimes f_{1,0} f_{0,1} a_r \ldots \otimes f_{1,0} f_{0,1} a_n +$$

$$\sum \pm f_{1,0} \phi(f_{1,0} f_{0,1} a_k, \ldots, f_{1,0} f_{0,1} a_n, a_0, \ldots) \otimes f_{1,0} f_{0,1} a_s \otimes \ldots \otimes f_{1,0} f_{0,1} a_{k-1}$$

$$\overset{?}{=} [\iota_\phi, B](\alpha) \pm [b, B](\phi \otimes \alpha) \pm B(\delta\phi \otimes \alpha)$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

n=1:

$$C(A_0 \to A_1 \to A_0) \xrightarrow{\tau_{1!}} \hat{\tau}_1^* C(\tau_1(A_0 \to A_1 \to A_0)) \xrightarrow{\hat{\tau}_1^* \tau_{1!}} \hat{\tau}_1^{*2} C(\tau_1^2(A_0 \to A_1 \to A_0))$$

$$\overset{id}{\frown}$$

$$\overline{(\hat{\tau}^* \tau_{1!} \circ \tau_{1!} - id)}(\phi \otimes \alpha) = \text{“}f_{1,0} \circ L_\phi(\alpha)''$$

$$= \sum_{k \geq 1} \pm f_{1,0} f_{0,0} a_0 \otimes \ldots \otimes f_{1,0} \phi(a_k, \ldots) \otimes f_{1,0} f_{0,1} a_r \ldots \otimes f_{1,0} f_{0,1} a_n +$$

$$\sum \pm f_{1,0} \phi(f_{1,0} f_{0,1} a_k, \ldots, f_{1,0} f_{0,1} a_n, a_0, \ldots) \otimes f_{1,0} f_{0,1} a_s \otimes \ldots \otimes f_{1,0} f_{0,1} a_{k-1}$$

$$\overset{?}{=} [\iota_\phi, B](\alpha) \pm [b, B](\phi \otimes \alpha) \pm B(\delta\phi \otimes \alpha)$$

$$B(\phi \otimes \alpha) = \sum_{0 \leq i \leq n} \pm 1 \otimes f_{1,0} f_{0,0} a_i \otimes \ldots \otimes f_{1,0} \phi(a_*, \ldots) \otimes f_{1,0} f_{0,1} a_* \otimes \ldots \otimes f_{1,0} f_{0,1} a_n \otimes a_0 \otimes \ldots \otimes a_{i-1}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

n=1:

$$B\big((\phi_{0,1}|\dots|\phi_{0,k_0}) \otimes (\phi_{1,1}|\dots|\phi_{1,k_1}) \otimes \alpha\big) =$$

$$= \sum_{\substack{0 \le j_1 \le \dots \le j_{2k_1} \le k_0 \\ p}} \pm 1 \otimes f_{1,0} f_{0,0} a_p \otimes \dots \otimes f_{1,0}\phi_{0,1}(a_*, \dots) \otimes$$

$$\otimes f_{1,0} f_{0,1} a_* \otimes \dots \otimes f_{1,0}\phi_{0,j_1}(a_*, \dots) \otimes$$

$$\otimes f_{1,0} f_{0,j_1} a_* \otimes \dots \otimes \phi_{1,1}(f_{0,j_1} a_*, \dots, \phi_{0,j_1+1}(a_*, \dots), \dots) \otimes$$

$$\otimes \dots \otimes \phi_{1,k_1}(f_{0,j_{2k_1-1}} a_*, \dots, \phi_{0,j_{2k_1-1}+1}(a_*, \dots), \dots) \otimes \dots \otimes$$

$$\otimes a_0 \otimes \dots \otimes a_{p-1}$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

$$
\begin{aligned}
\chi_\infty &\to \mathcal{D}_\infty \\
\mathcal{A} &\mapsto (B(\mathcal{A}), C(\mathcal{A})) \\
\tau_1 &\mapsto \begin{pmatrix} \hat{\tau}_1 : B(\mathcal{A}) \to B(\tau_1 \mathcal{A}) \\ \tau_{1!} : C(\mathcal{A}) \to \hat{\tau}_1^* C(\tau_1 \mathcal{A}) \end{pmatrix} \\
(\tau_1, \tau_1) &\mapsto \begin{pmatrix} id : B(\mathcal{A}) \to B(\mathcal{A}) \\ B : C(\mathcal{A}) \to C(\mathcal{A}) \end{pmatrix} \\
(\tau_1, \tau_1, \tau_1) &\mapsto \begin{pmatrix} \hat{\tau}_1 : B(\mathcal{A}) \to B(\tau_1 \mathcal{A}) \\ 0 : C(\mathcal{A}) \to \hat{\tau}_1^* C(\tau_1 \mathcal{A}) \end{pmatrix} \\
&\vdots
\end{aligned}
$$

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

The A_∞ relations mean:

An A_∞-functor $\mathcal{F} : \chi_\infty \to \mathcal{D}_\infty$

For higher n, ...